The normalization: a new algorithm, implementation and comparisons

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Introduction

We present a new algorithm for computing the normalization $\bar{R}$ of a reduced affine ring R, together with some remarks on efficiency based on our experience with an implementation of this algorithm in SINGULAR (cf. [GPS]).

Our method to compute $\bar{R}$ (the integral closure of R in its total ring of fractions) is based on a criterion for normality, due to Grauert and Remmert [GR], which was rediscovered in [J]. The criterion states that $R = \bar{R}$ if and only if the canonical map $R \longrightarrow \mathrm{Hom}_R(J, J)$ is an isomorphism, where J denotes a reduced ideal such that its zero set contains the non–normal locus of $\mathrm{Spec}\, R$. In general this map is only injective and we obtain an inclusion of rings,

$$R \subset \mathrm{Hom}_R(J, J) \subset \bar{R}.$$

Our method is to present $\mathrm{Hom}_R(J, J)$ as an affine ring R_1, which is of type $R[T_1, \ldots, T_s]$ modulo an ideal generated by linear and quadratic relations in the T_i.

We continue in the same manner with R_1 and obtain a sequence of rings

$$R = R_0 \subsetneqq R_1 \subsetneqq \cdots \subsetneqq R_k = \mathrm{Hom}_{R_k}(J_k, J_k),$$

such that $R_k = \bar{R}$ by the criterion of Grauert and Remmert (the algorithm must stop, since $\bar{R}$ is finite over R).

At the end of this paper we describe several special cases which allow us to do some steps in the algorithm more efficiently. Examples show that these refinements may be essential to the ability to compute the normalization.

The normalization provides a decomposition of R into the normalization of the prime components, in particular, it computes the number of irreducible components. Our algorithm does not need any prime decomposition, but we may, of course, first make such a decomposition and then normalize the components. The last table shows that this is sometimes useful, but there are also examples where the normalization is easily computed, the prime decomposition, however, is not computable in a reasonable time.

It is clear that this algorithm applies to the case where R is the localization of an affine ring with respect to a general monomial ordering (for example, the localization at a maximal ideal) as described in [GP].

Criterion for normality

Here we describe the algorithm, mentioned in the introduction. Other algorithms were given, for example, by Seidenberg [Se], Stolzenberg [St], Gianni, Trager [GT] and Vasconcelos [V].

The algorithm is based on the following criterion for normality due to Grauert and Remmert [GR]:

Proposition *Let R be a noetherian reduced ring and J be a radical ideal containing a non–zero divisor such that the zero set of J, $V(J)$ contains the non– normal locus of Spec(R). Then R is normal if and only if $R = \mathrm{Hom}_R(J,J)$.*

Proof. Let $\bar{R}$ be the normalization of R. Then we have the canonical inclusions

$$R \subset \mathrm{Hom}_R(J,J) \subset \bar{R}$$
$$r \rightsquigarrow \varphi_r, \qquad \varphi \rightsquigarrow \frac{\varphi(x)}{x}$$

where φ_r is the map defined by the multiplication with r and x is a non–zero divisor of J. It is easy to see that $\frac{\varphi(x)}{x}$ is independent of the choice of x. Since J is finitely generated, the characteristic polynomial of φ defines an integral relation and, therefore, $\dfrac{\varphi(x)}{x} \in \bar{R}$.

Now we claim that $\mathrm{Hom}_R(J,J) = \mathrm{Hom}_R(J,R) \cap \bar{R}$. Indeed, let $h \in \bar{R}$ and $h^n + a_{n-1}h^{k-1} + \cdots + a_0 = 0$, $a_i \in R$. If $h \in \mathrm{Hom}_R(J,R)$, that is $hJ \subset R$, then for all $g \in J$ we have $hg \in R$ and

$$(gh)^n + ga_{n-1}(gh)^{n-1} + \cdots + g^n a_0 = 0$$

which implies $(gh)^n \in J$. But J is a radical ideal and, therefore, $gh \in J$, which implies $h \in \mathrm{Hom}_R(J,J)$.

Now we are prepared for the proof of the proposition.

One implication is trivial. To prove the other, assume $R = \mathrm{Hom}_R(J,J)$. This implies $R = \mathrm{Hom}_R(J,R) \cap \bar{R}$.

Let $h = \frac{f}{g} \in \bar{R}$, it suffices to show that $hJ \subset R$. Let $\Delta = \{P \in \mathrm{Spec}(R) | h \notin R_P\}$, then obviously, Δ is contained in the non–normal locus of Spec(R) and, therefore, by assumption $\Delta \subseteq V(J)$. Δ can be defined by the ideal $C = \{u \in R | hu \in R\}$. By the abstract Nullstellensatz, we obtain $\sqrt{C} \supseteq J$. Now, by definition $h \cdot C \subseteq R$ and we may choose an integer d such that $R \supseteq h\sqrt{C}^d \supseteq hJ^d$. Assume that d is minimal such that $R \supseteq hJ^d$. If $d > 1$, then there exists an $a \in J^{d-1}$ such that $ha \notin R$ but $ha \in \bar{R}$ and $haJ \subset R$. This implies, because of $\mathrm{Hom}_R(J,J) = \mathrm{Hom}_R(J,R) \cap \bar{R}$, that $ha \in J \subseteq R$, which gives a contradiction. Therefore, $d = 1$ and $hJ \subseteq R$, which proves the proposition. □

Remark: Let J, R be as in the proposition and x a non–zero divisor of J. We saw in the beginning of the proof that

1) $xJ : J = x \cdot \mathrm{Hom}_R(J, J)$

and, consequently,

2) $R = \mathrm{Hom}_R(J, J)$ if and only if $xJ : J \subseteq \langle x \rangle$.

3) Let $u_0 = x, u_1, \ldots, u_s$ be generators of $xJ : J$ as R–module. Because $\mathrm{Hom}_R(J, J)$ is a ring we have $\frac{s(s+1)}{2}$ relations $\frac{u_i}{x} \cdot \frac{u_j}{x} = \sum\limits_{k=0}^{s} \xi_k^{ij} \frac{u_k}{x}$, $s \geq i \geq j \geq 1$, $\xi_k^{ij} \in R$ in $\frac{1}{x}(xJ : J)$. Together with the linear relations, the syzygies between $u_0, \ldots, u_s$, they define the ring structure of $\mathrm{Hom}_R(J, J)$:

$$\begin{array}{ccc} R[T_1, \ldots, T_s] & \twoheadrightarrow & \mathrm{Hom}_R(J, J) \\ T_i & \rightsquigarrow & \dfrac{u_i}{x}. \end{array}$$

The kernel of this map is the ideal generated by $T_iT_j - \sum\limits_{k=0}^{s} \xi_k^{ij} T_k \quad (T_0 = 1)$ and $\sum\limits_{k=0}^{s} \eta_k T_k$ such that $\sum\limits_{k=0}^{s} \eta_k u_k = 0$.

The normalization algorithm

Now we are prepared to give the normalization algorithm:

Input: a radical ideal $I \subseteq K[x_1, \ldots, x_n]$.
Output: s polynomial rings $R_1, \ldots, R_s$ and s prime ideals $I_1 \subset R_1, \ldots, I_s \subset R_s$ and s maps $\pi_i : R \to R_i$, such that the induced map $\pi : K[x_1, \ldots, x_n]/I \to R_1/I_1 \times \cdots \times R_s/I_s$ is the normalization of $K[x_1, \ldots, x_n]/I$

normal(I[, inform])

Additional information by the user (respectively by the algorithm) can be given in the optional list inform, as for instance,

– I defines an an isolated singularity
– some elements of the radical of the non–normal locus, which are already known

- Result = $\emptyset$

- compute idempotents of $K[x_1,\dots,x_n]/I$.
  This is optional and gives just the information about the normalization as splitting into a direct sum of rings.
  Assume $K[x_1,\dots,x_n]/I = K[x_1,\dots,x_n]/I_1 \times \cdots \times K[x_1,\dots,x_n]/I_s$.

- For $i = 1$ to s do

  - compute $J =$ singular locus of I_i
  - choose $f \in J \smallsetminus I_i$ and compute $I_i : f$ to check whether f is a zero divisor
  - if $I_i : f \underset{\neq}{\supseteq} I_i$

    Result = Result $\cup$ normal$(I_i : (I_i : f))\cup$ normal$(I_i : f)$
    (Notice that $\sqrt{I_i, f} = I_i : (I_i : f)$ in this situation.)

    else

    If we have an isolated singularity at $0 \in K^n$ then $J = (x_1, \dots, x_n)$.
    In general, if J_0 is the radical of the singular locus of a normalization loop before, given by the list inform, then $J = \sqrt{I_i, f + J_0}$

    else

    $J = \sqrt{I_i, f}$
    $H = fJ : J$
    if $H = \langle f \rangle$
    Result = Result $\cup\{K[x_1, \dots, x_n], I_i, \mathrm{id}\}$
    else
    assume $H = fJ : J = \langle f, u_1, \dots, u_s \rangle$
    then compute an ideal L,
    $L \subseteq K[x_1, \dots, x_n, T_1, \dots, T_s]$
    (as described in the remark above) such that
    $K[x_1, \dots, x_n, T_1, \dots, T_s]/L \xrightarrow{\sim} \mathrm{Hom}(J, J)$
    $T_i \rightsquigarrow \frac{u_i}{f}$,
    let $\varphi : K[x_1, \dots, x_n] \to K[x_1, \dots, x_n, T_1, \dots, T_s]$
    be the inclusion.
    $S =$ normal(L),
    compose the maps of S with φ.
    Result = Result $\cup S$

- return Result

It remains to give an algorithm to compute the idempotents.

We shall explain this for the case when the input ideal I is (weighted) homogeneous with strictly positive weights.

An idempotent e, that is, $e^2 - e \in I$, has to be homogeneous of degree 0. Therefore it will not occur in the first loop.

It may occur after one normalization loop in $\mathrm{Hom}(J,J) \simeq K[x_1,\dots,x_n,T_1,\dots,T_s]/L$ because some of the generators may have the same degree.

Let $T \subseteq \{T_1,\dots,T_s\}$ be the subset of variables of degree 0.

Then $L \cap K[T]$ is zero–dimensional because $T_j^2 - \sum \xi_k^{jj} T_k \in L \cap K[T]$ for all $T_j \in T$ (the weights are ≥ 0 and, therefore, $\xi_k^{jj} \in K,\ T_k \in T$).

For this situation there is an easy algorithm:

Input: $I \subseteq K[x_1,\dots,x_n]$ a (weighted) homogeneous radical ideal, $\deg(x_1) = \dots = \deg(x_k) = 0, \deg(x_i) > 0$ for $i > k$, $I \cap K[x_1,\dots,x_k]$ being 0–dimensional.
Output: ideals $I_1,\dots,I_s$ such that $K[x_1,\dots,x_n]/I = K[x_1,\dots,x_n]/I_1 \times \dots \times K[x_1,\dots,x_n]/I_s$ and $I \cap K[x_1,\dots,x_k] = \cap(I_v \cap K[x_1,\dots,x_k])$ is the prime decomposition

Idempotents(I)

- Result $= \emptyset$
- compute $J = I \cap K[x_1,\dots,x_k]$
- compute $J = P_1 \cap \dots \cap P_s$ the (0–dimensional) prime decomposition.
- For $i = 1$ to s do
  - choose $g_i \neq 0$ in $\bigcap\limits_{v \neq i} P_v$
  - Result = Result $\cup \{I : g_i\}$
- return Result

Implementation and comparisons

The above algorithm has been implemented by the authors. The implementation in SINGULAR is available as SINGULAR library normal.lib.

To have an efficient version of the normalization algorithm, we had to take care of several special cases and tricks for the implementation:

1) If the variety defined in a certain normalization loop has an isolated singularity (let us say at the origin), the varieties arising in the following loops will have an isolated singularity, too. Therefore, there is no longer any need to compute the singular locus: its radical is $(x_1,\dots,x_n)$.

2) If we have computed some radical containing the non–normal locus as zero–set, then we add it in the next step to the corresponding ideal. The "old" radical elements turned out to be very helpful in speeding up the computations.

3) Similar to the property 1), equidimensionality is kept. Then we only have to compute the equidimensional radical of the ideal containing the non–normal locus, which is faster.

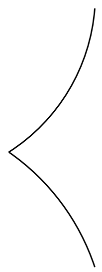
4) There are examples which show that it is faster to make a primary decomposition (for example, à la Gianni, Trager, Zacharias) before the normalization and then computing the normalization without it.

5) Similar to the property 1), irreducibility is kept. Then a check for non–zero divisors is not necessary.

6) Similar to the property 1), Cohen–Macaulayness is kept. Then a test for regularity in codimension 2 as criterion for normality is very useful.

7) It is also useful to try to reduce the number of variables after having computed the ring structure of $\mathrm{Hom}(J, J)$ by substituting those which can be expressed by other variables.

We illustrate the algorithm by computing the normalization of the cuspidal plane cubic:



$$R = K[x, y]/y^2 - x^3$$

- Radical of the singular locus : $J = (x, y)$
- $R \subsetneqq \mathrm{Hom}_R(J, J) = (1, \frac{y}{x})$
- the linear relations are $x^2 - yT_1, y - xT_1$
  the quadratic relation is $T_1^2 - x$
  and, therefore

$$\mathrm{Hom}_R(J, J) = R[T_1]/(x^2 - yT_1, y - xT_1, T_1^2 - x)$$

- reducing the number of variables by $y = xT_1,\ x = T_1^2$ we obtain $\bar{R} = K[T_1]$ and as map

$$\begin{array}{ccc} R & \rightarrow & K[t_1], \\ x & \rightsquigarrow & T_1^2, \\ y & \rightsquigarrow & T_1^3. \end{array}$$

Examples

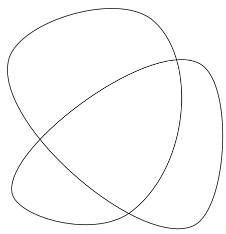
1) Example of Huneke (cf. [V])

   $5abcde - a^5 - b^5 - c^5 - d^5 - e^5,$
   $ab^3c + bc^3d + a^3be + cd^3e + ade^3,$
   $a^2bc^2 + b^2cd^2 + a^2d^2e + ab^2e^2 + c^2de^2,$
   $abc^5 - b^4c^2d - 2a^2b^2cde + ac^3d^2e - a^4de^2 + bcd^2e^3 + abe^5,$
   $ab^2c^4 - b^5cd - a^2b^3de + 2abc^2d^2e + ad^4e^2 - a^2bce^3 - cde^5,$
   $a^3b^2cd - bc^2d^4 + ab^2c^3e - b^5de - d^6e + 3abcd^2e^2 - a^2be^4 - de^6,$
   $a^4b^2c - abc^2d^3 - ab^5e - b^3c^2de - ad^5e + 2a^2bcde^2 + cd^2e^4,$
   $b^6c + bc^6 + a^2b^4e - 3ab^2c^2de + c^4d^2e - a^3cde^2 - abd^3e^2 + bce^5$

   This is a non–minimal abelian surface of degree 15 in ¶^{4} which is linked $(5,5)$ to a Horrocks–Mumford surface.

2) $x^2y^2 + x^2z^2 + y^2z^2$

   This is a 3–nodal quartic:



3) Example of Sturmfels

   The radical of the ideal generated by the 2×2 permanents of a generic 3×3–matrix.

   $bv + su,$
   $bw + tu,$
   $sw + tv,$
   $by + sx,$
   $bz + tx,$
   $sz + ty,$

$uy + vx,$
$uz + wx,$
$vz + wy,$
$bvz.$

4) $wy - vz,$
$vx - uy,$
$tv - sw,$
$su - bv,$
$tuy - bvz.$

5) Example of Riemenschneider

The radical of the base space of the versal deformation of the first "general" cyclic quotient singularity of embedding dimension 6.

$xz,$
$vx,$
$ux,$
$su,$
$qu,$
$txy,$
$stx,$
$qtx,$
$uv^2z - uwz,$
$uv^3 - uvw,$
$puv^2 - puw.$

6) The intersection of the ideals of example 2) and 4).

The examples illustrate the remarks made at the beginning of this section:

Example 1) takes several hours if the information that it is an isolated singularity is not given to the next loop.

Example 2) takes several minutes if the "old" radical is not used in the next loop.

Example 3) and 5) are much faster if a primary decomposition is performed prior to the normalization (using the algorithm of Gianni, Trager, Zacharias).

We obtained the following timings for our implementation in SINGULAR (cf. [GPS]) (in seconds) on an HP740, where * means that the computation took more than one hour:

	computation of the radical	computation of the minimal associated primes	prime decomposition	prime decomposition and normalization	normalization	number of components
1	42	*	*	*	7	1
2	1	1	1	1	1	1
3	3	7	9	11	*	15
4	1	2	2	30	28	3
5	1	2	9	3	33	4
6	2	4	4	32	22	4

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