Some Aspects of Brieskorn's Mathematical Work

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Lieber Egbert, dear colleagues and friends!

I am very happy that this conference on singularities at this wonderful Mathematisches Forschungsinstitut Oberwolfach can take place on the occasion of Egbert Brieskorn's 60th birthday, which was a little bit more than a week ago. There have been several conferences on singularities in Oberwolfach – but this is certainly a special one.

In this conference we have a special day, the "Brieskorn-day" today, and I am especially happy that this day became possible.

As you can clearly see from the programme, the today's speakers are Brieskorn's teacher Prof. Hirzebruch, four of Brieskorn's students and, of course, Prof. Brieskorn himself.

I am particularly grateful to Prof. Hirzebruch that when I asked him whether he could give a talk at this occasion, he did not hesitate but immediately said yes. He will give us many personal and exciting details of the wonderful discovery of the relation between exotic spheres and singularities.

I should also like to thank very much Heidrun Brieskorn, Matthias Kreck and Joseph Steenbrink for preparing a music programme for tonight. They immediately started to exercise when they arrived (even before!). I am sure that we shall have some wonderful music tonight. Thank you very much!

Last but not least I want to thank one person especially for his participation at this conference: it is Brieskorn himself. Actually, you can believe me that it was not a trivial task to convince him to come. As many of you know, Brieskorn does not like ceremonies like this, in particular if they concern his own person. Probably he already thinks I should stop talking now. In some sense I would like to agree. On the other hand, I am convinced that Egbert Brieskorn deserves this special day in honour of the person and of his mathematical work which was important

- for his students
- for the unfolding of singularity theory, and
- for mathematics as a whole.

Before I start to talk about some aspects of Brieskorn's mathematical work, let me mention that Brieskorn is a person whose interest, knowledge and activities reach far beyond mathematics.

He loves music and, by the way, knows a lot about the theory of music. He is definitely a very political person with strong opinions. He was actively engaged in the peace movement and is still engaged in projects for saving the environment. As you know, during the past years, he has become a semiprofessional historian in connection with the life and the work of Felix Hausdorff. Actually, he is the editor of the book "Felix Hausdorff zum Gedächtnis I". And – who is surprised – there will soon be a second volume with Brieskorn's biography of Felix Hausdorff.

The later work of Brieskorn, with important historical and philosophical contributions, as well as his textbooks is, however, not subject of this short overview.

Now let me start with a very short review of part of Brieskorn's mathematical work.

As you will see, the talks of today are all related to some mathematical theory which nowadays is a grown-up theory, but where, sometime at the beginning, there was a discovery of Brieskorn or a development of a germ of a theory by Brieskorn.

Moreover, as I shall try to explain, in all of Brieskorn's work you see the idea of unity of mathematics. Brieskorn's work is led by the idea to combine different mathematical structures, different mathematical categories.

Historically speaking the following different structures are involved. (I hope this will not be too schematic but casts some light on his work):

differential	—	*analytic*	(exotic spheres)
resolution	—	*deformation*	(simultaneous resolutions of ADE singularities)
Lie groups	—	*equations*	(construction of singularities from the corresponding simple Lie groups)
transcendental	—	*algebraic*	(construction of the local Gauß-Manin-connection)
continuous	—	*discrete*	(generalized Braid groups, Milnor lattices and Dynkin diagrams)

It is quite interesting to notice that perhaps in almost all cases these different structures correspond to the two parts of our brain, as Arnol'd explained in his talk yesterday.

Already in his first paper, which, as far as I know, emerged from his dissertation and which has the title

> **"Ein Satz über die komplexen Quadriken"**, Math. Annalen 155 (1964),

he proves:

Let X be a complex n-dimensional Kähler manifold diffeomorphic to Q_n (n-dimensional projective quadric), then

(i) n odd $\Rightarrow$ X is biholomorphic to Q_n

(ii) n even, $n \neq 2$ $\Rightarrow$ $c_1(X) = \pm ng$ ($H^2(X,\mathbb{Z}) \cong \mathbb{Z}g$, g positive) and if $c_1(X) = ng$, then X is biholomorphic to Q_n.

(If $n = 2$ then $X = \mathbb{P}^1 \times \mathbb{P}^1$ has infinitely many different analytic structures, the Σ_{2m} of Hirzebruch)

This was an exact analogue of a previous theorem of Hirzebruch and Kodaira about the complex projective space. (An earlier announcement of the result appeared 1961 in the Notices of the AMS.)

The next paper

> **"Über holomorphe $\mathbb{P}_n$-Bündel über $\mathbb{P}_1$"**, Math. Ann. 157 (1965)

treats the same question and gives a complete answer (including the Hirzebruch Σ-surfaces).

His next paper was

> **"Examples of singular normal complex spaces which are topological manifolds"**, Proc. Nat. Acad. Sci. 55 (1966).

This paper contains already the Brieskorn singularity

$$X \,:\, z_0^3 + z_1^2 + \cdots + z_n^2 = 0\,.$$

He proves: *if $n \geq 4$, n odd, then X is a topological manifold.*

The result at that time was a big surprise, since in 1961 Mumford had published in his well-known paper "The topology of normal singularities of an algebraic surface and a criterion for simplicity", Publ. Math. IHES 8 (1961), that such phenomena are not possible for surfaces.

Now, I should like to switch to the paper

> **"Beispiele zur Differentialtopologie von Singularitäten"**, Inventiones Math. 2 (1966).

The results of this paper were a sensation in the mathematical world.

Brieskorn showed that the just discovered exotic spheres (by Kervaire and Milnor) appear as neighbourhood boundaries of singularities and, therefore, can be described by real algebraic equations! As an example, I should like to mention that

$$\{x_1^{2k-1} + x_2^3 + x_3^2 + x_4^2 + x_5^2 = 0\} \cap S^9\,, \quad k = 1, \dots, 28,$$

represent all 28 different differentiable structures on the topological 7-sphere.

As far as I understood, the story of discovery of this result was also very exciting and Hirzebruch, who was himself involved in this discovery, will tell us very interesting details.

The next two papers

> **"Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen"**, Math. Ann. 166 (1966),

and

> **"Die Auflösung der rationalen Singularitäten holomorpher Abbildungen"**, Math. Ann. 178 (1968),

contain a proof of the fact that the rational double points admit a simultaneous resolution (after base change), i.e., let X, S be smooth, $\dim X = 3$, $\dim S = 1$ and $f : X \to S$ be a morphism such that $\mathrm{Sing}\,(f) = \{x\}$ and $(X_{f(x)}, x)$ is a RDP, then there exists a diagram

$$\begin{array}{ccc} X' & \xrightarrow{\psi} & X \\ f' \downarrow & & \downarrow f \\ T & \xrightarrow[\varphi]{} & S \end{array}$$

where X', T are smooth, $\mathrm{Sing}\,(f') = \emptyset$, φ is a smooth covering of S with $\varphi^{-1}(f(x)) = \{t\}$, ψ is proper, surjective and $\psi|_{X'_t}$ is a resolution of the singularities of $X_{f(x)}$.

Moreover, he describes all simultaneous resolutions in terms of invariants of the group G, defining the quotient singularity of type A_k, D_k or E_6, E_7, E_8. These two papers and the next one had an enormous influence on the deformation theory of 2-dimensional singularities as well as on the minimal model programme for 3-folds.

The paper

> **"Rationale Singularitäten komplexer Flächen"**, Inv. Math. 4 (1968),

contains a description of the resolution of quotient singularities and a proof that

$$\mathbb{C}\{x, y, z\}/(x^2 + y^3 + z^5)$$

is the only 2-dimensional factorial analytic ring which is not regular. The rational surface singularities and, in particular, Brieskorn's work about these play an important role in the subsequent deformation theory of surface singularities. I just mention Riemenschneider, Wahl and later, in connection with the minimal model programme of Mori, Kollar, Reid and others.

One of Brieskorn's shortest papers is certainly one of his most important ones:

> **"Singular elements of semi-simple Algebraic Groups"**, Intern. Congress Math. (1970).

In this famous paper Brieskorn shows how to construct the singularity of type ADE directly from the simple complex Lie group of the same type. Moreover, he constructs the whole semi-universal deformation.

At the end of that paper Brieskorn says:

> *"Thus we see that there is a relation between exotic spheres, the icosahedron and E_8."*

which expresses explicitly Brieskorn's idea of the unity of mathematics. But he continues:

> *"But I still do not understand why the regular polyhedra come in."*

I think that even today there is some mystery in these connections of such different parts of mathematics.

Peter Slodowy, who himself developed the theory of singularities and algebraic groups further, will give us a talk about this fascinating subject.

In 1970 Brieskorn published

> **"Die Monodromie der isolierten Singularitäten von Hyperflächen"**, Manuscripta Math. 2 (1970).

In this paper he constructed the local Gauß-Manin connection of an isolated hypersurface singularity.

This construction gave an algebraic method to compute the characteristic polynomial of the monodromy, and in this way combined topological and algebraic structures. In his paper Brieskorn proves that the eigenvalues of monodromy are roots of unity by a really very beautiful argument using the solution to Hilbert's 7th problem.

This was the time when I was a student in Göttingen, and it was my task to generalize his paper to complete intersections in my Diplomarbeit and later in my dissertation. Much later the work of Brieskorn was taken up by Scherk and

Steenbrink and especially by Morihiko Saito who made a tremendous machinery out of it. Also Claus Herling continued Brieskorn's work and applied it to obtain theorems of Torelli type for singularities. He will explain to us the magic Brieskorn lattice $H_0^{''}$.

Now, let me mention Brieskorn's work about "continuous versus discrete structures" which concerns (generalized) braid groups and actions of these.

In his paper

> **"Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe"**, Inventiones Math. 12 (1971),

Brieskorn shows that the fundamental group of E_{reg}/W of regular orbits of a complex reflection group W has a presentation with generators g_s, $s \in S$, and relations

$$g_s g_t g_s \cdots = g_t g_s g_t \cdots$$

where both sides have m_{st} factors and where (m_{st}) is a Coxeter matrix. These groups are generalized braid groups and were called Artin-groups by Brieskorn and Saito in

> **"Artin-Gruppen und Coxeter-Gruppen"**, Inventiones Math. 17 (1972).

In that paper these groups were studied from a combinatorial point of view and the authors solved the word problem and the conjugation problem.

The connection to singularity theory comes from the fact that for W of type A_n, D_n, E_6, E_7, E_8 the space E_{reg}/W is the complement of the discriminant of the semi-universal deformation of a simple singularity of the same type. This follows from Brieskorn's work "Singular elements of semi-simple algebraic groups" at the International Congress in Nice.

Now, the classical braid group of n strings B_n acts on the set of "distinguished bases" of the Milnor lattice. In his later work Brieskorn and several of his students worked on this subject and Brieskorn expresses at several places that the understanding of this action should be essential for understanding the geometry of the versal unfolding.

The first step is to understand the deformation relations between singularities within a fixed modality class.

The classification of isolated hypersurface singularities with respect to their modality by V.I. Arnol'd is certainly one of the most important achievements of singularity theory. The adjacencies (deformation relations) between these singularities are important as well and still the subject of research articles.

In the paper

"Die Hierarchie der 1-modularen Singularitäten", Manuscripta Math. 27 (1979),

Brieskorn gives all possible deformation relations among Arnold's list of 1-modular (unimodal) singularities. The knowledge of all deformation relations is important by itself but is particularly interesting because of the different other characterizations of this class of singularities.

The deformation relations among the unimodular singularities were related by Brieskorn to a theory which seems to be really far away from singularity theory, the theory of partial compactifications of bounded symmetric domains. This is done in the survey article

"The unfolding of exceptional singularities", Nova Acta Leopoldina 52 (1981).

In the introduction, Brieskorn describes the fascinating relation between these apparently unconnected theories:

"On the one hand we have the deformation theory of the singularities in the boundary layer. It has three strata – corresponding to the simply elliptic singularities, the cusps and the exceptional singularities. And it has three stems, corresponding to the tetrahedron, the octahedron, and the icosahedron. On the other hand, we have the three quadratic forms $E_k \perp U \perp U$ obtained from the three exceptional forms E_6, E_7, E_8 by adding two hyperbolic planes. These three forms correspond to the three stems. To each of the forms is associated a bounded symmetric domain D of type IV and two unbounded realizations belonging to the 0- and 1-dimensional boundary components F_0 and F_1 of D. Corresponding to these there are canonically defined arithmetic quotients $D/\Gamma, D/Z_\Gamma(F_0)$ and $D/Z_\Gamma(F_1)$ and their partial Baily-Borel compactifications identify with the deformation spaces associated to the singularities in the three strata: exceptional singularities, cusps, and simply elliptic singularities."

Still investigating the unimodal singularities which constitute, after the zero-modal (or simple or ADE) singularities, the next class in Arnold's hierarchy of singularities, Brieskorn gives a very fine and detailed study of the Milnor lattice of the 14 exceptional unimodal singularities in

"Die Milnorgitter der exzeptionellen unimodularen Singularitäten", Bonner Math. Schriften 150 (1983).

The Milnor lattice is the integral middle homology together with the quadratic intersection form with respect to a distinguished basis. It is an arithmetic coding of (part of) the geometry of the versal unfolding and it is a great challenge to see to what extent it reflects the essential features of this geometry. This problem which is embedded in a whole programme is again considered in the paper

"Milnor lattices and Dynkin diagrams", Proc. of Symp. in Pure Math. 40 (1983).

Brieskorn poses the question

> *"To which extent is this subtle geometry (the geometry of the unfolding of a singularity) reflected in the invariants associated to these singularities?"*

In the words of Arnol'd, this programme is the attempt to build a bridge between the two parts of our brain.

By the work of Gusein-Zade and Ebeling we understand a lot more, but I guess we are still far away from a complete understanding of the relation between the continuous and the discrete structure of a singularity.

Wolfgang Ebeling will talk on these themes.

The last paper I should like to mention is

"Automorphic Sets and Braids and Singularities", Contemporary Mathematics 78 (1988).

In this paper Brieskorn gives a survey on the action of the braid group on the set of distinguished bases of an isolated singularity. Moreover, he introduces the general concept of an automorphic set Δ, which unifies many investigations about the action of the braid group.

His statement in the introduction of that paper

> *"The beauty of braids is that they make ties between so many different parts of mathematics, combinatorial theory, number theory, group theory, algebra, topology, geometry and analysis, and, last but not least, singularities."*

shows very clearly Brieskorn's strong belief in the unity of mathematics.

Now I come to the end of my talk. I hope I could explain some aspects of Brieskorn's mathematical work and point out that the idea or the wish to show the unity of mathematics, apart from its different realizations, was perhaps one of the leading principles of Brieskorn's work.

I should like to finish with a citation. In January 1992 there was a special colloquium in honour of Felix Hausdorff in Bonn. Brieskorn started his talk by citing Hausdorff. Hausdorff had spoken the following words at the grave of the mathematician Eduard Study and had cited Friedrich Nietsche from "Zaratustra" with the words.

"Trachte ich denn nach dem Glück?
Ich trachte nach meinem Werke."

(Do I aim for happiness? I do aim for my work.)

Brieskorn said, that this was certainly Hausdorff's leading principle and I should like to add, Brieskorn's too.