

Four characterizations of scalar-type operators with spectrum in a half-line

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Abstract

C^0 -scalar-type spectrality criteria for operators A , whose resolvent set contains the negative reals, are provided. The criteria are given in terms of growth conditions on the resolvent of A and the semigroup generated by A . These criteria characterize scalar-type operators on the Banach space X , if and only if X has no subspace isomorphic to the space of complex null-sequences.

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0 Introduction

This paper is written in the spirit of Kantorovitz' [14] work on the characterization of scalar-type spectral operators with spectrum in $[0, \infty)$, and the reader is referred to this article for a brief introduction into the history and the importance of this subject.

A possibly unbounded linear operator A on a Banach space X is called a scalar-type spectral operator on $[0, \infty)$ if there exists a strongly countably additive spectral measure E on the Borel subsets of $[0, \infty)$ so that

$$D(A) = \{x \in X : \lim_{n \rightarrow \infty} \int_0^n t E(dt)x \text{ exists}\}$$

and

$$Ax = \lim_{n \rightarrow \infty} \int_0^n t E(dt)x \quad \text{for } x \in D(A).$$

Let us denote by $C_0[0, \infty)$ the space of complex-valued continuous functions on $[0, \infty)$ which vanish at infinity. By X we always mean a complex Banach space, X^* stands for the dual space of X and U_X denotes the unit ball of X . The space of bounded linear operators on X is denoted by $\mathbf{L}(X)$. The main result of this paper is the following characterization of scalar-type spectral operators on $[0, \infty)$.

THEOREM 1. *Let X be a Banach space which has no subspace isomorphic to the space c_0 of complex-valued null-sequences. Let A be a linear operator on X , whose resolvent set contains the half-line $(-\infty, 0)$. Then A is spectral of scalar-type on $[0, \infty)$ if and only if one of the following conditions is fulfilled:*

(i) *A is of C^0 -scalar-type on $[0, \infty)$; that means there exists a continuous algebra homomorphism $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X)$ with $\Phi(\rho_s) = (s + A)^{-1}$ for $s > 0$, where $\rho_s(t) = 1/(s + t)$.*

(ii) *There exists $M_1 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$*

$$\sup_{k=1,2,\dots} c_k \int_0^\infty t^{k-1} |x^* A^k (t + A)^{-2k} x| dt \leq M_1,$$

where $c_k = \frac{(2k-1)!}{k!(k-2)!}$ if $k \geq 2$ and $c_1 = 1$.

(iii) *There exists $M_2 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$*

$$\sup_{k=0,1,\dots} \sum_{m=0}^k \binom{k}{m} |x^* A^m (1 + A)^{-k} x| \leq M_2.$$

(iv) *$-A$ generates a strongly continuous semigroup $(U(t))_{t \geq 0}$ with $U(t)X \subseteq D(A)$ for every $t > 0$, and there exists $M_3 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$*

$$\sup_{k=1,2,\dots} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} |x^* A^k U(t)x| dt \leq M_3.$$

Note that the assumption $U(t)X \subseteq D(A)$ for all $t > 0$ implies $U(t)X \subseteq D(A^n)$ for every natural number n (see e.g. [1], Proposition 1.1.10).

Dowson [10] showed that bounded scalar-type spectral operators in weakly complete Banach spaces can be characterized by criterion (i), and Kantorovitz and deLaubenfels [7] were the first who recognized that this equivalence, for possibly unbounded operators, is true in any Banach space not containing c_0 . That (ii) is equivalent to A being of scalar-type on $[0, \infty)$, provided the underlying Banach space is reflexive, is due to Kantorovitz [14]. The proof of Kantorovitz' criterion is based on Widders [18] characterization of Stieltjes transforms. Widders characterization of Stieltjes transforms was also used by Ricker [15] to derive a description of scalar-type operators in quasi-complete locally convex spaces. Criteria (iii) and (iv) are related to deLaubenfels work on scalar-type spectral operators on Banach lattices [2] and on cyclic spaces [3]. The proof of these criteria uses respectively characterizations of complex-valued moment sequences and Laplace transforms. Some other characterizations of possibly unbounded operators on reflexive spaces being of scalar-type on $[0, \infty)$ may be found in [3], [6] and [13].

In the first section of this paper we study C^0 -scalar-type operators. Especially, it is shown that A is of C^0 -scalar-type on $[0, \infty)$ if and only if A fulfills condition (ii) or (iii) of Theorem 1, and if, in addition, A generates a strongly continuous semigroup then also condition (iv) is equivalent to A being of C^0 -scalar-type. In the second section we show that the class of scalar-type operators on X and C^0 -scalar-type operators on X coincide in exactly those Banach spaces X which have no subspace isomorphic to c_0 .

Before we start with our study of C^0 -scalar-type operators let us present the following results of Widders' which are the main tools for our investigations.

Let k be a natural number. If $f : (0, \infty) \rightarrow \mathbf{C}$ is infinitely often differentiable let

$$S_k[f](t) = d_k (-t)^{k-1} \frac{d^{2k-1}}{dt^{2k-1}} (t^k f(t)) \quad \text{for } 0 < t < \infty,$$

where $d_k = \frac{1}{k!(k-2)!}$ if $k \geq 2$ and $d_1 = 1$, and let

$$L_k[f](t) = \frac{(-1)^k}{k!} f^{(k)} \left(\frac{k}{t} \right) \left(\frac{k}{t} \right)^{k+1} \quad \text{for } 0 < t < \infty.$$

The formal operators S_k and L_k are respectively called the Widder inversion operators for the Stieltjes and the Laplace transform.

If $\mu = (\mu_n)_{n=0,1,\dots}$ is a complex-valued sequence then

$$\Lambda_k[\mu]_m = \binom{k}{m} (-1)^{k-m} \Delta^{k-m}[\mu]_m \quad \text{for } k, m = 0, 1, \dots \text{ with } m \leq k,$$

where Δ denotes the difference operator $\Delta[\mu]_n = \mu_{n+1} - \mu_n$.

Let I be a closed interval which is bounded from below by $b = \inf I$. A complex-valued function ϕ of bounded variation on I is said to be normalized if $\phi(t) = (\phi(t^-) + \phi(t^+))/2$ for all t in the interior of I and if $\phi(b) = 0$. The total variation of ϕ is denoted by $\text{Var}(\phi)$.

THEOREM 2. *Let $f : (0, \infty) \rightarrow \mathbf{C}$ be an infinitely often differentiable function and μ a complex-valued sequence.*

- (i) *(Widder inversion of the Stieltjes transform) There exists a unique complex-valued normalized function ϕ of bounded variation on $[0, \infty)$ with $f(s) = \int_0^\infty \frac{1}{s+t} d\phi(t)$ for $0 < s < \infty$ if and only if*

$$M = \sup_{k=1,2,\dots} \int_0^\infty |S_k[f](t)| dt < \infty. \quad (1)$$

In this case $M \leq \text{Var}(\phi) \leq M + |A|$, where $A = \lim_{t \rightarrow 0^+} tf(t)$.

- (ii) *(Widder inversion of the Laplace transform) There exists a unique complex-valued normalized function ϕ of bounded variation on $[0, \infty)$ with $f(s) = \int_0^\infty e^{-st} d\phi(t)$ for $0 < s < \infty$ if and only if*

$$M = \sup_{k=1,2,\dots} \int_0^\infty |L_k[f](t)| dt < \infty. \quad (2)$$

In this case $M \leq \text{Var}(\phi) \leq M + |B|$ where $B = \lim_{t \rightarrow \infty} f(t)$.

- (iii) *(Widder characterization of moment sequences) There exists a unique complex-valued normalized function ϕ of bounded variation on $[0, 1]$ with $\mu_n = \int_0^1 t^n d\phi(t)$ for $n = 0, 1, \dots$ if and only if*

$$M = \sup_{k=0,1,\dots} \sum_{m=0}^k |\Lambda_k[\mu]_m| < \infty.$$

In this case $M \leq 2\text{Var}(\phi) \leq 2M$.

For proofs of these statements we refer to Widder [18], Theorem VIII.16, Theorem VII.12a, and Theorem III.2b, respectively. Widder stated the results for real-valued functions and sequences, but the complex case is a trivial consequence. The estimates follow easily from the proofs in the cited reference. Note that the existence of the limits A in (i) and B in (ii) follows from the condition (1) and (2), respectively.

1 Characterization of C^0 -scalar-type operators with spectrum in $[0, \infty)$

In the sequel linear mappings between (complex) vector spaces are called operators. Let A be a bounded operator on X and D a bounded set of complex numbers. Following deLaubenfels [2] we say that A is of C^0 -scalar-type on D if there exists a continuous algebra homomorphism Φ defined on the space of continuous complex-valued functions $C(D)$ into $\mathbf{L}(X)$ with $\Phi(\tau_0) = \text{Id}$ and $\Phi(\tau_1) = A$, where $\tau_n(z) = z^n$ for $n = 0, 1, \dots$ and $z \in D$. We want to extend this definition to possibly unbounded operators with spectrum in the half-line $[0, \infty)$. For this reason we make the following observation.

LEMMA 3. *If A is a bounded operator on X and if $D \subseteq \mathbf{C}$ is closed and bounded then A is of C^0 -scalar-type on D if and only if $\sigma(A) \subseteq D$ and there exists a continuous algebra homomorphism $\Phi : C(D) \rightarrow \mathbf{L}(X)$ with $\Phi\rho_z = (z + A)^{-1}$ for all $z \notin -D$.*

Proof. If A is of C^0 -scalar-type on D with corresponding algebra homomorphism Φ then, since D is closed, $\rho_z \in C(D)$ for all $z \notin -D$, and

$$(z + A)\Phi(\rho_z) = \Phi((z\tau_0 + \tau_1)\rho_z) = \Phi(\tau_0) = \text{Id},$$

and similarly $\Phi(\rho_z)(z + A) = \text{Id}$. Hence $-z$ is contained in the resolvent set of A and $(z + A)^{-1} = \Phi(\rho_z)$. Conversely, if $\Phi : C(D) \rightarrow \mathbf{L}(X)$ is a continuous algebra homomorphism with $\Phi(\rho_z) = (z + A)^{-1}$ for all $z \notin -D$ then

$$\Phi(\tau_0)(z + A)^{-1} = \Phi(\rho_z) = (z + A)^{-1}$$

whence $\Phi(\tau_0) = \text{Id}$. Moreover,

$$(z + A)^{-1}(z + \Phi(\tau_1)) = \Phi(\rho_z(z\tau_0 + \tau_1)) = \Phi(\tau_0) = \text{Id}$$

whence $\Phi(\tau_1) = A$. ≡

Motivated by this lemma we define C^0 -scalar-type operators on $[0, \infty)$ as follows:

DEFINITION 4. *A possibly unbounded operator A on X is of C^0 -scalar-type on $[0, \infty)$ if $\sigma(A) \subseteq [0, \infty)$ and there exists a continuous algebra homomorphism $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X)$ with $\Phi(\rho_z) = (z + A)^{-1}$ for all $z \in \mathbf{C} \setminus (-\infty, 0]$.*

The next Lemma shows that A is of C^0 -scalar-type on $[0, \infty)$ if and only if A admits a $C_0[0, \infty)$ functional calculus in the sense of deLaubenfels [4]. Moreover, if A is densely defined, then A is of C^0 -scalar-type in the sense of Definition 4 if and only if A is C^0 -scalar in the sense of deLaubenfels [2] (see Remark 7 below).

LEMMA 5. *An operator A on X is of C^0 -scalar-type on $[0, \infty)$ if and only if its resolvent set contains the half-line $(-\infty, 0)$ and there exists a continuous algebra homomorphism $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X)$ with $\Phi\rho_s = (s + A)^{-1}$ for all $0 < s < \infty$.*

Proof. That the first assertion implies the second is trivial. So let $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X)$ be a continuous algebra homomorphism with $\Phi(\rho_s) = (s + A)^{-1}$ for all $0 < s < \infty$. Let $z \in \mathbf{C} \setminus (-\infty, 0]$. If $x \in D(A)$ then

$$\begin{aligned} (1 + A)^{-1}\Phi(\rho_z)(z + A)x &= \Phi(\rho_z)(1 + A)^{-1}((z - 1) + (1 + A))x \\ &= \Phi(\rho_z)((z - 1)\rho_1 + 1)x \\ &= \Phi(\rho_1)x = (1 + A)^{-1}x, \end{aligned}$$

whence $\Phi(\rho_z)(z + A)x = x$. If $x \in X$ then

$$\begin{aligned} \Phi(\rho_z)x &= \Phi(\rho_z)((1 - z) + (z + A))(1 + A)^{-1}x \\ &= (1 + A)^{-1}((1 - z)\Phi(\rho_z)x + x) \in D(A). \end{aligned}$$

Applying $z + A = (z - 1) + (1 + A)$ to both sides of the last equation yields

$$(z + A)\Phi(\rho_z)x = (z - 1)\Phi(\rho_z)x + (1 - z)\Phi(\rho_z)x + x = x.$$

Consequently, z is contained in the resolvent set of A and $\Phi(\rho_z) = (z + A)^{-1}$. ≡

If I is a closed interval with $\inf I = 0$ and if ϕ is a complex-valued function of bounded variation on I , then, for $t \geq 0$, let $\text{Var}_\phi(t)$ be the variation of ϕ in the interval $[0, t)$, provided t is not a right end point of I . If $I = [0, a]$, then $\text{Var}_\phi(a)$ is the variation of ϕ on the whole interval $[0, a]$. The function $\text{Var}_\phi : I \rightarrow \mathbf{R}$ is of bounded variation on I , the variation of ϕ and Var_ϕ are equal, and for every $f \in C_0(I)$

$$\int_I f(t) d\phi(t) \leq \int_I |f(t)| d\text{Var}_\phi(t).$$

This is proven for example in [17].

THEOREM 6. *Let A be an operator on X whose resolvent set contains the half-line $(-\infty, 0)$. Then A is of C^0 -scalar-type on $[0, \infty)$ if and only if one of the following conditions (i) or (ii) is fulfilled:*

(i) *There exists $M_1 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$*

$$\sup_{k=1,2,\dots} c_k \int_0^\infty t^{k-1} |x^* A^k (t + A)^{-2k} x| dt \leq M_1,$$

where $c_k = \frac{(2k-1)!}{k!(k-2)!}$ if $k \geq 2$ and $c_1 = 1$.

(ii) *There exists $M_2 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$*

$$\sup_{k=0,1,\dots} \sum_{m=0}^k \binom{k}{m} |x^* A^m (1 + A)^{-k} x| \leq M_2.$$

If, in addition, A is densely defined then A is of C^0 -scalar-type on $[0, \infty)$ if and only if:

(iii) *$-A$ generates a C_0 -semigroup $(U(t))_{t \geq 0}$ such that $U(t)X \subseteq D(A)$ for every $t > 0$, and there exists $M_3 > 0$ so that for all $x \in U_X$ and $x^* \in U_{X^*}$*

$$\sup_{k=1,2,\dots} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} |x^* A^k U(t)x| dt \leq M_3.$$

Proof. (i) Let A be of C^0 -scalar-type on $[0, \infty)$ with corresponding algebra homomorphism Φ . Given $x \in X$, $x^* \in X^*$ and $f \in C_0[0, \infty)$ we put $l[x, x^*]f = x^*\Phi(f)x$. Then $l[x, x^*]$ is a continuous linear functional on $C_0[0, \infty)$ with $\|l[x, x^*]\| \leq \|\Phi\| \cdot \|x\| \cdot \|x^*\|$. So we may apply the Riesz representation theorem which says that there exists a complex-valued function $\phi[x, x^*]$ of bounded variation on $[0, \infty)$ with $\text{Var}(\phi[x, x^*]) \leq M \|x\| \cdot \|x^*\|$, and so that $l[x, x^*](f) = \int_0^1 f(u) d\phi[x, x^*](u)$ for all $f \in C_0[0, \infty)$. We then have

$$t^{k-1}x^*A^k(t+A)^{-2k}x = t^{k-1} \int_0^\infty u^k(t+u)^{-2k} d\phi[x, x^*](u).$$

Hence, by changing the order of integration and integrating by parts we get

$$\begin{aligned} & c_k \int_0^\infty t^{k-1} |x^*A^k(t+A)^{-2k}x| dt \\ & \leq c_k \int_0^\infty \int_0^\infty t^{k-1} u^k (t+u)^{-2k} d\text{Var}_{\phi[x, x^*]}(u) dt \\ & = c_k \int_0^\infty \int_0^\infty s^{k-1} (s+1)^{-2k} ds d\text{Var}_{\phi[x, x^*]}(u) \\ & = \frac{k-1}{k} \text{Var}(\phi[x, x^*]) \leq \|\Phi\| \cdot \|x\| \cdot \|x^*\|. \end{aligned}$$

Conversely, assume that (i) holds. For the proof that A is of C^0 -scalar-type on $[0, \infty)$, we need Widders inversion theorem for the Stieltjes transform. Therefore, given $x \in X$ and $x^* \in X^*$ we calculate $S_k[x^*Rx]$ where $R(t) = (t+A)^{-1}$. We have for every $k = 1, 2, \dots$

$$\begin{aligned} & \frac{d^{2k-1}}{dt^{2k-1}} \left(t^k (t+A)^{-1} \right) \\ & = \sum_{l=0}^{2k-1} \binom{2k-1}{l} \frac{d^l}{dt^l} (t^k) \frac{d^{2k-1-l}}{dt^{2k-1-l}} \left((t+A)^{-1} \right) \\ & = \sum_{l=0}^k \binom{2k-1}{l} \frac{k!(2k-1-l)!}{(k-l)!} t^{k-l} (-1)^{2k-1-l} (t+A)^{-(2k-l)} \\ & = (2k-1)! (-1)^{k-1} \sum_{l=0}^k \binom{k}{l} (-t)^{k-l} (t+A)^l (t+A)^{-2k} \\ & = (2k-1)! (-1)^{k-1} A^k (t+A)^{-2k} \end{aligned}$$

whence

$$S_k[x^*Rx](t) = c_k t^{k-1} x^* A^k (t+A)^{-2k} x.$$

Hence by Widders theorem x^*Rx is the Stieltjes transform of a unique normalized function $\phi[x, x^*]$ of bounded variation on $[0, \infty)$ with $\text{Var}(\phi[x, x^*]) \leq M \|x\| \cdot \|x^*\| + |A[x, x^*]|$ where $A[x, x^*] = \lim_{t \rightarrow 0^+} tx^*R(t)x$. For every $t > 0$ the operator $tR(t)$ is bounded and

$$|x^*tR(t)x| = \left| \int_0^\infty \frac{t}{t+u} d\phi[x, x^*](u) \right| \leq \text{Var}(\phi[x, x^*])$$

for every $x \in X$ and $x^* \in X^*$. Hence $\bar{A} = \sup_{0 < t < \infty} \|tR(t)\| < \infty$ by the uniform boundedness principle. Consequently,

$$\text{Var}(\phi[x, x^*]) \leq \bar{M} \|x\| \cdot \|x^*\|,$$

where $\bar{M} = M + \bar{A}$. Now let $\Phi(f)x$ be the linear functional on X^* which assigns to every $x^* \in X^*$ the complex number $\int_0^\infty f(t) d\phi[x, x^*](t)$. Then $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X, X^{**})$ is linear and we have the estimate

$$|\langle \Phi(f)x, x^* \rangle| \leq \|f\| \text{Var}(\phi[x, x^*]) \leq \bar{M} \|f\| \cdot \|x\| \cdot \|x^*\|.$$

So we infer $\|\Phi\| \leq \bar{M}$. Moreover, we know $\Phi(\rho_s) = (s + A)^{-1} \in \mathbf{L}(X)$ because

$$\langle \Phi(\rho_s)x, x^* \rangle = \int_0^\infty \frac{1}{s+t} d\phi[x, x^*](t) = x^*(s + A)^{-1}x.$$

Since the set $\{\rho_s : 0 < s < \infty\}$ is a total subset of $C_0[0, \infty)$ we infer $\Phi(f) \in \mathbf{L}(X)$ for every $f \in C_0[0, \infty)$, that means $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X)$. We now claim that Φ is an algebra homomorphism. By the resolvent equality we have for all $0 < r, s < \infty$ with $r \neq s$

$$\begin{aligned} \Phi(\rho_r \rho_s) &= (r - s)^{-1} \Phi(\rho_s - \rho_r) \\ &= (r - s)^{-1} \left((s + A)^{-1} - (r + A)^{-1} \right) = (r + A)^{-1} (s + A)^{-1}, \end{aligned}$$

and

$$\begin{aligned} \Phi(\rho_s^2) &= \lim_{h \rightarrow 0} \Phi(h^{-1}(\rho_s - \rho_{s+h})) \\ &= \lim_{h \rightarrow 0} h^{-1} \left((s + A)^{-1} - (s + h + A)^{-1} \right) = (s + A)^{-2}. \end{aligned}$$

So we may conclude $\Phi(fg) = \Phi(f)\Phi(g)$ for all $f, g \in C_0[0, \infty)$ because the set $\{\rho_s : 0 < s < \infty\}$ is total in $C_0[0, \infty)$.

(ii) Let again Φ be the continuous algebra homomorphism corresponding to A . Given $x \in X$, $x^* \in X^*$ and $f \in C[0, 1]$ put $l[x, x^*]f = x^*\Phi(f)x$. Then $l[x, x^*]$ is a continuous linear functional on $C[0, 1]$ with $\|l[x, x^*]\| \leq \|\Phi\| \cdot \|x\| \cdot \|x^*\|$. So we may apply the Riesz representation theorem in order to get a complex-valued function $\phi[x, x^*]$ of bounded variation on $[0, 1]$ which represents $l[x, x^*]$, and with $\text{Var}(\phi[x, x^*]) = \|l[x, x^*]\|$. Consequently,

$$\begin{aligned} \sum_{m=0}^k \binom{k}{m} |x^* A^m (1+A)^{-k} x| &= \sum_{m=0}^k \binom{k}{m} \left| \int_0^\infty t^m (1+t)^{-k} d\phi[x, x^*](t) \right| \\ &\leq \int_0^\infty \sum_{m=0}^k \binom{k}{m} t^m (1+t)^{-k} d\text{Var}_{\phi[x, x^*]}(t) \\ &= \text{Var}(\phi[x, x^*]) \leq \|\Phi\| \cdot \|x\| \cdot \|x^*\|. \end{aligned}$$

Thus, if A is of C^0 -scalar-type on $[0, \infty)$, then (ii) follows.

Assume now that (ii) is fulfilled. We show first that this assumption implies that $(1+A)^{-1}$ is of C -scalar-type on $[0, 1]$. To this end put $\mu_n = (1+A)^{-n}$. Then by induction it follows that $\Delta^k \mu_n = (-1)^k A^k (1+A)^{-(k+n)}$ for $k, n = 0, 1, \dots$, whence

$$\Lambda_k[\mu]_m = \binom{k}{m} A^{k-m} (1+A)^{-k} \quad \text{for } k, m = 0, 1, \dots \text{ with } k \geq m.$$

Hence (ii) implies for every $x \in X$ and $x^* \in X^*$

$$\sup_{k=0,1,\dots} \sum_{m=0}^k |\Lambda_k[x^* \mu x]_m| \leq M \|x\| \cdot \|x^*\|.$$

So we may conclude from Widders Theorem 2 (iii) that there exists a unique complex-valued normalized function $\phi[x, x^*]$ of bounded variation on $[0, 1]$ with $\text{Var}(\phi[x, x^*]) \leq M \|x\| \cdot \|x^*\|$ and so that

$$x^* \mu_n x = \int_0^1 t^n d\phi[x, x^*](t).$$

Since

$$\left| \int_0^1 f(t) d\phi[x, x^*](t) \right| \leq \|f\|_\infty \text{Var}(\phi[x, x^*]) \leq M \|f\|_\infty \|x\| \cdot \|x^*\|,$$

and since $\phi[x, x^*]$ depends linearly on x and x^* , we can define a continuous linear operator $\Psi : C[0, 1] \rightarrow \mathbf{L}(X, X^{**})$ by

$$\langle \Psi(f)x, x^* \rangle = \int_0^\infty f(t) d\phi[x, x^*](t).$$

We then have for every $n = 0, 1, \dots$

$$\langle \Psi(\tau_n)x, x^* \rangle = \int_0^1 t^n d\phi[x, x^*](t) = x^*(1 + A)^{-n}x,$$

whence $\Psi(\tau_n) = (1 + A)^{-n} \in \mathbf{L}(X)$. Since the set $\{\tau_n : n = 0, 1, \dots\}$ is total in $C[0, 1]$ we conclude that Ψ is an operator from $C[0, 1]$ into $\mathbf{L}(X)$. Moreover, for every $l, n = 0, 1, \dots$ there holds

$$\Psi(\tau_l \tau_n) = \Psi(\tau_{l+n}) = (1 + A)^{-(l+n)} = (1 + A)^{-l}(1 + A)^{-n} = \Psi(\tau_l)\Psi(\tau_n).$$

This equation shows that the bounded operator Ψ , in addition, is an algebra homomorphism, because the set $\{\tau_n : n = 0, 1, \dots\}$ is total in $C[0, 1]$. Note that $\Psi(\tau_0) = (1 + A)^{-0} = \text{Id}$ and $\Psi(\tau_1) = (1 + A)^{-1}$, so that $(1 + A)^{-1}$ is of C^0 -scalar-type on $[0, 1]$.

To show that A is of C^0 -scalar-type on $[0, \infty)$ put $\Gamma f(t) = f(1/t - 1)$ for $0 < t \leq 1$ and $\Gamma f(0) = 0$. Then $\Gamma : C_0[0, \infty) \rightarrow C[0, 1]$ is a continuous algebra homomorphism. Hence $\Phi = \Psi \circ \Gamma : C_0[0, \infty) \rightarrow \mathbf{L}(X)$ also is a continuous algebra homomorphism, and we have for every $0 < s < \infty$

$$\begin{aligned} \Phi(\rho_s) &= \Psi(\Gamma(\rho_s)) = \Psi(\tau_1(1 + (s - 1)\tau_1)^{-1}) \\ &= (1 + A)^{-1} \left(1 + (s - 1)(1 + A)^{-1}\right)^{-1} = (s + A)^{-1}. \end{aligned}$$

It follows, by Lemma 5, that A is of C^0 -scalar-type on $[0, \infty)$.

(iii) Let A be of C^0 -scalar-type on $[0, \infty)$ with corresponding algebra homomorphism Φ and assume A to be densely defined. Since

$$\|\lambda(\lambda + A)^{-1}\| = \|\Phi(\lambda\rho_\lambda)\| \leq \|\Phi\| \|\lambda\rho_\lambda\|_\infty \leq \|\Phi\|$$

for all $\lambda \in \mathbf{C}$ with $|\arg(\lambda)| \leq 3\pi/4$, Theorem 5.3 in [12] tells us that $-A$ generates an analytic semigroup. Especially, $D(A^n) \subseteq U(t)X$ for all $t > 0$ and

$$U^{(n)}(t) = (-1)^n A^n U(t).$$

By $\varepsilon_t \in C_0[0, \infty)$, $t > 0$, we denote the function $\varepsilon_t(u) = e^{-ut}$. Put $V(t) = \Phi(\varepsilon_t)$ for $t > 0$ and let $V(0) = \text{Id}$. We show next that $U = V$.

Given $x \in X$ and $x^* \in X^*$ let $\phi[x, x^*]$ be the complex-valued normalized function of bounded variation on $[0, \infty)$ which represents the linear functional $x^*\Phi x$ on $C_0[0, \infty)$. Then we have for $s > 0$

$$\begin{aligned} \int_0^\infty e^{-st} x^* V(t)x \, dt &= \int_0^\infty e^{-st} x^* \Phi(\varepsilon_t)x \, dt \\ &= \int_0^\infty e^{-st} \int_0^\infty e^{-tu} d\phi[x, x^*](u) \, dt \\ &= \int_0^\infty \int_0^\infty e^{-t(s+u)} \, dt d\phi[x, x^*](u) \\ &= \int_0^\infty \frac{1}{s+u} d\phi[x, x^*](u) \\ &= x^* \Phi(\rho_s)x = x^*(s+A)^{-1}x. \end{aligned}$$

Hence the function $s \mapsto (s+A)^{-1}$ is the Laplace transform of V . Since $-A$ generates the C_0 -semigroup U we know, by [12] p. 17, that $s \mapsto (s+A)^{-1}$ is the Laplace transform of U , too. By the injectivity of the Laplace transform, $V(t) = U(t)$ follows for all $t > 0$, since U and V are strongly continuous in $(0, \infty)$. Especially, $U^{(n)}(t) = (-1)^n \Phi(\tau^n \varepsilon_t)$ for all $t > 0$.

Now, by Widders Theorem 2 (ii) the inequality

$$\sup_{k=1,2,\dots} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} |x^* A^k U(t)x| \, dt \leq \|\Phi\| \cdot \|x\| \cdot \|x^*\|$$

follows, because $L_k[U](t) = \frac{1}{k!} A^k U(k/t)(k/t)^{k+1}$. Up to this point we proved that (iii) follows, if A is a densely defined operator of C^0 -scalar-type on $[0, \infty)$.

Conversely, assume that A is densely defined and (iii) holds. Since $T(t)X$ is contained in $D(A)$ for every $t > 0$ it follows, by [1] Proposition 1.1.10, that U is infinitely often differentiable in $t > 0$, $U(t)X \subseteq D(A^n)$, and there holds $U^{(n)}(t) = (-1)^n A^n U(t)$. Hence condition (iii) is equivalent to

$$\sup_{k=1,2,\dots} \int_0^\infty |x^* L_k[U](t)x| \, dt \leq M \|x\| \cdot \|x^*\|.$$

So Widders theorem implies that there exists for every $x \in X$ and $x^* \in X^*$ a unique complex valued normalized function $\phi[x, x^*]$ on $[0, \infty)$ with

$\text{Var}(\phi[x, x^*]) \leq M \|x\| \cdot \|x^*\| + |B[x, x^*]|$, where $B[x, x^*] = \lim_{t \rightarrow \infty} x^*U(t)x$, and so that

$$x^*U(t)x = \int_0^\infty e^{-tu} d\phi[x, x^*](u) \quad \text{for all } 0 < t < \infty.$$

We deduce from this equation $\sup_{t>0} |x^*U(t)x| \leq \text{Var}(\phi[x, x^*])$. Thus the uniform boundedness principle implies $\overline{B} = \sup_{t>0} \|U(t)\| < \infty$ and we infer $\text{Var}(\phi[x, x^*]) \leq \overline{M} \|x\| \cdot \|x^*\|$ where $\overline{M} = M + \overline{B}$. It follows that $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X, X^{**})$ defined by $\langle \Phi(f)x, x^* \rangle = \int_0^\infty f(t) d\phi[x, x^*](t)$ is a continuous linear operator with $\Phi(\varepsilon_t) = U(t) \in \mathbf{L}(X)$. Since the set $\{\varepsilon_t : 0 < t < \infty\}$ is total in $C_0[0, \infty)$ we infer $\Phi f \in \mathbf{L}(X)$ for all $f \in C_0[0, \infty)$.

From $\Phi(\varepsilon_s) = U(s)$, and from the fact that U is generated by $-A$, we deduce

$$\begin{aligned} x^*(s + A)^{-1}x &= \int_0^\infty e^{-st} x^*U(t)x dt = \int_0^\infty e^{-st} \int_0^\infty e^{-tu} d\phi[x, x^*](u) dt \\ &= \int_0^\infty \frac{1}{s + u} d\phi[x, x^*](u) = x^*\Phi(\rho_s)x. \end{aligned}$$

Hence $\Phi(\rho_s) = (s + A)^{-1}$. That Φ is an algebra homomorphism follows as in the proof of (i). Consequently, A is of C^0 -scalar-type on $[0, \infty)$. \equiv

REMARK 7. (i) If A is a bounded operator with spectrum contained in some interval $[m, M]$ then A is of C^0 -scalar-type on $[m, M]$ if and only if $\sup \|p(A)\| < \infty$ where the supremum is taken over all polynomials p on $[m, M]$ with $\sup_{m \leq t \leq M} |p(t)| \leq 1$. This follows immediately from the observation that the polynomials are dense in $C[m, M]$.

Following an idea of Schäfer [16] we can give a second characterization of bounded operators being of C^0 -scalar-type on $[m, M]$, which uses Widders characterization of moment sequences. It is clear that A is of C^0 -scalar-type on $[m, M]$ if and only if $B = (M - m)^{-1}(m - A)$ is of C^0 -scalar-type on $[0, 1]$. Now consider the operator-valued sequence $\mu_n = B^n$, $n = 0, 1, 2, \dots$. Then $\Lambda_k[\mu]_m = \binom{k}{m} B^m (1 - B)^{k-m}$ for $m, k = 0, 1, 2, \dots$ with $m \leq k$. Now applying Widders characterization of moment sequences it can be seen as in the proof of Theorem 6 (ii) that B is of C^0 -scalar-type on $[0, 1]$ if and only if there exists $M > 0$ such that

$$\sup_{k=0,1,\dots} \sum_{m=0}^k \binom{k}{m} |x^* B^m (1 - B)^{k-m} x| \leq M \|x\| \cdot \|x^*\| \quad \text{for all } x \in X, x^* \in X^*.$$

(ii) deLaubenfels [2] defined an operator A to be C^0 -scalar on $[0, \infty)$ if $-A$ generates a uniformly bounded semigroup and $(1 + A)^{-1}$ is of C^0 -scalar-type on $[0, 1]$. From the proof of Theorem 6 (iii) we see that a densely defined operator A with $(-\infty, 0) \subseteq \rho(A)$ is of C^0 -scalar-type on $[0, \infty)$ in our sense if and only if A is C^0 -scalar on $[0, \infty)$ in the sense of deLaubenfels.

We note that not every operator of C^0 -scalar-type on $[0, \infty)$ generates a C_0 -semigroup. Consider the operator A on $C_0[0, \infty)$ with $D(A) = \{v \in C_0[0, \infty) : \lim_{t \rightarrow 0^+} v(t)/t \text{ exists}\}$ and $Av(t) = v(t)/t$ for $t > 0$ and $Av(0) = \lim_{t \rightarrow 0^+} v(t)/t$. Since $(z + A)^{-1}v(t) = t/(tz + 1)v(t)$ we see that every $z \in \mathbf{C} \setminus [0, \infty)$ is contained in the resolvent set of A . Indeed, $\sigma(A) = [0, \infty)$. Moreover, A is of C^0 -scalar-type on $[0, \infty)$, because the operator $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X)$ defined by $\Phi(f)v(t) = f(1/t)v(t)$ if $t > 0$, and $\Phi(f)v(0) = 0$, is a continuous algebra homomorphism with $\Phi(\rho_s)v(t) = t/(ts + 1)v(t)$. But A does not generate a C_0 -semigroup, because A is not densely defined.

(iii) The proof of Theorem 6 shows, that for densely defined operators the condition (iii) in this theorem is equivalent to

(iv) $-A$ generates an analytic semigroup U , and there exists $M_4 > 0$ so that for all $x \in U_x$ and $x^* \in U_{X^*}$

$$\sup_{k=1,2,\dots} \frac{1}{(k-1)!} \int_0^\infty t^{k-1} |x^* A^k U(t)x| dt \leq M_4.$$

2 Characterization of scalar-type operators

In this section we prove the main theorem. The key for the proof is the relationship between vector measures and operators on spaces of continuous functions. For a deeper discussion of this relationship we refer the reader to the monograph of Diestel and Uhl [8]. Here we only recall some basic facts concerning the representation of operators on spaces of continuous functions.

If Σ is a σ -algebra and $F : \Sigma \rightarrow X$ has the property

$$F\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} F(B_k)$$

whenever (B_k) is a sequence of pairwise disjoint members of Σ , then F is called a countably additive vector measure on Σ . If Σ is the Borel σ -algebra

on a locally compact Hausdorff space then we call F weakly regular if $x^* \circ F$ is a regular complex Borel measure for every $x^* \in X^*$.

The following theorem would be a combination of results presented in [8] if we considered operators on $C(K)$, where K is a compact Hausdorff space. But for our investigation of scalar-type operators we have to study operators on $C_0[0, \infty)$.

THEOREM 8. *If $T : C_0[0, \infty) \rightarrow X$ is a bounded operator, then T is weakly compact if and only if T can be represented by a unique weakly regular countably additive vector measure F on $[0, \infty)$; that means $Tf = \int_0^\infty f(t) F(dt)$ for $f \in C_0[0, \infty)$.*

Especially, if X has no subspace isomorphic to c_0 , then every bounded operator from $C_0[0, \infty)$ into X can be represented by a unique weakly regular countably additive vector-measure F .

Proof. In this proof we identify the dual space of $C_0[0, \infty)$ with the space of regular Borel measures on $[0, \infty)$ endowed with the variation norm. This is possible by the Riesz representation theorem. Thus every Borel function g on $[0, \infty)$ can be identified with a member in the second dual of $C_0[0, \infty)$ by putting $\langle g, \nu \rangle = \int_0^\infty g(t) \nu(dt)$ for all regular Borel measures ν . Especially, this identification is possible for characteristic functions χ_E of Borel sets E .

If $T : C_0[0, \infty) \rightarrow X$ is weakly compact then the second dual T^{**} of T takes all its values in X . Given a Borel set B let $F(B) = T^{**}(\chi_B)$. Then we have for every $x^* \in X^*$

$$x^*F(B) = \langle \chi_B, T^*x^* \rangle = T^*x^*(B).$$

Here we identify $T^*x^* \in C_0[0, \infty)^*$ with a regular Borel measure on $[0, \infty)$. Hence F is weakly regular. Moreover, it follows that F is weakly countably additive, and by the Orlicz-Pettis theorem [8], Corollary I.4.4, we infer that F is countably additive. Since for all $x^* \in X^*$

$$x^*Tf = T^*x^*(f) = \int_0^\infty f(t) x^*F(dt)$$

we see that T is represented by F .

Conversely, if there exists a countably additive vector measure F on $[0, \infty)$ which represents T , then define $T_n f = \int_0^n f(t) F(dt)$ for every natural number n . The operators T_n from $C_0[0, \infty)$ into X are weakly compact, by

the Bartle-Dunford-Schwartz Theorem [8], Theorem VI.2.5., and they are converging in norm to the operator T . Since the operator ideal of weakly compact operators is closed with respect to the operator norm, the weak compactness of T is established.

Now assume that c_0 is not contained in X and let T be a bounded operator from $C_0[0, \infty)$ into X . To prove the second part of the theorem, by the first part it is sufficient to establish the weak compactness of T . We extend T continuously to an operator $T_l : C_l[0, \infty) \rightarrow X$ by $T_l(f) = T(f - f_\infty)$ where $f_\infty(t) = \lim_{s \rightarrow \infty} f(s)$ for all $0 \leq t < \infty$. Define $\Gamma : C[0, 1] \rightarrow C_l[0, \infty)$ by $\Gamma f(t) = f(1/(1+t))$ and let $S = T_l \circ \Gamma : C[0, 1] \rightarrow X$. Since c_0 is not contained in X the operator S is weakly compact by [8], Theorem VI.2.15. Hence T_l is also weakly compact because Γ is continuously invertible. Now the weak compactness of T follows from the equality of the ranges of T and T_l . \equiv

If $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X)$ is a bounded operator, let $\Phi[x]f = \Phi(f)x$ for $x \in X$ and $f \in C_0[0, \infty)$. We call the operators $\Phi[x] : C_0[0, \infty) \rightarrow X$ the components of Φ .

THEOREM 9. *Let A be a linear operator on X with $(-\infty, 0) \subseteq \rho(A)$. Then A is of scalar-type on $[0, \infty)$ if and only if A is of C^0 -scalar-type on $[0, \infty)$ and the components $\Phi[x]$ of the corresponding algebra homomorphism Φ are weakly compact for every $x \in X$.*

Especially, if X has no subspace isomorphic to c_0 then A is of scalar-type on $[0, \infty)$ if and only if A is of C^0 -scalar-type on $[0, \infty)$.

REMARK 10 For bounded A , the first half of this theorem is shown in [10], Theorem 4.6.24, and the second half first appeared in [9]. For possibly unbounded A the second half of Theorem 9 is an immediate consequence of [6], Theorem 3.3 and Corollary 3.5.

Proof. (of Theorem 9.) If A is a scalar-type operator with spectral measure E on $[0, \infty)$ then it is well known that $\Phi f = \int_0^\infty f(t) E(dt)$ defines a continuous algebra homomorphism $\Phi : C_0[0, \infty) \rightarrow \mathbf{L}(X)$ with $\Phi(\rho_s) = (s + A)^{-1}$. This is proven for example in [11]. Moreover, for every $x \in X$ the X -valued measure $E[x]$ with $E[x](B) = E(B)x$ is countably additive by assumption. Since $\Phi[x]f = \int_0^\infty f(t) E[x](dt)$ it follows, by Theorem 8, that $\Phi[x]$ is weakly compact.

Conversely, assume A is of C^0 -scalar-type on $[0, \infty)$ with corresponding algebra homomorphism Φ . If Φ has weakly compact components or if X has no subspace isomorphic to c_0 then, by Theorem 8, for every $x \in X$ there exists a countably additive X -valued measure $E[x]$ on $[0, \infty)$ with $\Phi[x]f = \int_0^\infty f(t) E[x](dt)$. Given a Borel measurable set B in $[0, \infty)$ we now define $E(B) : X \rightarrow X$ by $E(B)x = E[x](B)$. Then $E(B)$ is linear and, by [8], Proposition I.1.11 and Theorem I.1.13,

$$\|E(B)\| = \sup_{\|x\| \leq 1} \|E[x](B)\| \leq \sup_{\|x\| \leq 1} \|\Phi[x]\| \leq \|\Phi\|. \quad (3)$$

Hence E is an operator-valued, strongly countably additive bounded measure defined on the Borel subsets of $[0, \infty)$ with

$$(s + A)^{-1}x = \int_0^\infty \frac{1}{s + t} E(dt)x.$$

Moreover, we have $\|t(t + A)^{-1}\| = \|\Phi(t\rho_t)\| \leq \|\Phi\|$.

Now proceed as in Kantorovitz' proof of Theorem 1.1. in [14] to show that E is a spectral measure for A with

$$D(A) = \{x \in X : \lim_{n \rightarrow \infty} \int_0^n tE(dt)x \text{ exists}\}$$

and $Ax = \lim_{n \rightarrow \infty} \int_0^n tE(dt)x$ for $x \in D(A)$. ≡

The proof of the main theorem now follows from Theorem 6 combined with Theorem 9.

Proof. (of Theorem 1.) Let A be of scalar-type on $[0, \infty)$. Then A is of C^0 -scalar-type on $[0, \infty)$; that means assertion (i) of Theorem 1 holds. Moreover, $-A$, by deLaubenfels [2], generates a C_0 -semigroup, whence A has to be densely defined. Hence the assertions (ii), (iii) and (iv) of Theorem 1 follow from Theorem 6 and Theorem 9.

If, conversely, one of the assertions Theorem 1 (i), (ii), (iii) or (iv) holds then, by Theorem 6, A is of C^0 -scalar-type on $[0, \infty)$. Since X has no subspace isomorphic to c_0 , the operator A is of scalar-type on $[0, \infty)$, by Theorem 9. ≡

We finally note, that the condition $c_0 \not\subseteq X$ in Theorem 1 cannot be omitted; more precisely:

THEOREM 11. *If X contains an isomorphic copy of c_0 , then there exists a bounded operator A on X which is of C^0 -scalar-type, but not of scalar-type. Especially, this operator A fulfills each of the conditions of Theorem 1 (i)-(iv), but it is not a scalar-type operator.*

The first part of this theorem is proved in [9] (see also [5]) and the second part is an immediate consequence of Theorem 6.

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