New asymptotics in the geometry of equisingular families of curves

Gert-Martin Greuel
Universität Kaiserslautern
Fachbereich Mathematik
Erwin-Schrödinger-Straße
D – 67663 Kaiserslautern

Christoph Lossen
Universität Kaiserslautern
Fachbereich Mathematik
Erwin-Schrödinger-Straße
D – 67663 Kaiserslautern

Eugenii Shustin
Tel Aviv University
School of Mathematical Sciences
Ramat Aviv
ISR – Tel Aviv 69978

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Introduction

Let D be a divisor on the smooth projective surface Σ and denote by $V = V_{|D|}(S_1, \ldots, S_r)$ the variety of irreducible curves $C \in |D|$ having exactly r singularities of (topological or analytic) types $S_1, \ldots, S_r$. We say that V has the *T-property* at $C \in V$, if the conditions imposed by the individual singularities of C are independent (or transversal), that is, if V is smooth of the expected codimension.

For $\Sigma = \mathbb{P}^2$ it is well-known that the T-property holds for families of nodal curves (cf. [S]). But for more complicated singularities (beginning with cusps) there are examples, where the T-property fails ([Wa1, Lu, Lu1, Sh5]). On the other hand, various sufficient conditions for the T-property were found. The classical result is that the family $V_d(n \cdot A_1, k \cdot A_2)$ of irreducible plane curves of degree d with n nodes and k cusps has the T-property if

$$k < 3d. \qquad (0.1)$$

For arbitrary singularities several generalizations and extensions of (0.1) were found (cf. [GK, GL, Sh, Sh1, V]). They are all of such a form that the sum of certain invariants of the singularities is bounded from above by a linear function in d, whereas the number of singularities might be of order d^2. In [Sh3, Sh4] the T-property was shown when the total Milnor number was bounded by a quadratic function in d. These bounds have been improved in [Sh5] stating that $V_d(S_1, \ldots, S_r)$ has the T-property if

$$\sum_{i=1}^{r} (\tau'(S_i) + 2)^4 \le (d+6)^2, \tag{0.2}$$

where $\tau' = \tau$ (the Tjurina number) for an analytic type and $\tau' = \tau - \mathrm{mod}$ for a topological type (mod denotes the modality in the sense of [AGV]). In particular, for curves with n nodes and k cusps the T-property holds if

$$\frac{72}{5} n + 36k \le (d+6)^2. \tag{0.3}$$

Moreover, in [Sh5] E. Shustin constructs a series of irreducible curves with r A_μ-singularities where the T-property fails and

$$r\mu^2 = \sum_{i=1}^{r} \tau'(A_\mu)^2 \sim 9d^2, \tag{0.4}$$

which shows that we can expect T-property only if the sum of the squares of the Milnor number is bounded by a quadratic function in d.

For general surfaces Σ, basically only one sufficient condition for the T-property of $V_{|D|}(S_1, \ldots, S_r)$ at C is known:

$$\sum_{i=1}^{r} \tau'(S_i) - \varepsilon'(S_i) < -K_\Sigma \cdot C \tag{0.5}$$

where $\varepsilon'(S_i) \ge 0$ is the "isomorphism defect" (cf. [GK, GL]). Only for families of nodal curves there are improvements of (0.5). In the case of a $K3$-surface Σ the canonical divisor K_Σ is trivial, but A. Tannenbaum ([T]) shows that $V_{|D|}(n \cdot A_1)$ always has the T-property. For $\Sigma = \mathbb{P}^2_m$, the projective plane blown up in $m \ge 10$ generic points, we give in [GLS] an asymptotically improved sufficient condition for the T-property. Finally, L. Chiantini and E. Sernesi ([CS]) consider families of nodal curves on surfaces Σ with ample canonical divisor K_Σ. They study a rank 2 bundle on Σ associated to the nodes of C and use an inequality for the Bogomolov instability of this bundle in order to obtain a sufficient criterion for the T-property. Namely, if C is numerically equivalent to $p \cdot K_\Sigma$, $p \in \mathbb{Q}$, $p > 2$ then $V_{|C|}(n \cdot A_1)$ has the T-property provided

$$4n \ < \ p(p-2)K_\Sigma^2.$$

In the present article we generalize this approach in two directions: firstly, we allow arbitrary singularities and secondly, we weaken the assumption of K_Σ being ample, such that $\Sigma = \mathbb{P}^2$ is included. We obtain a numerical criterion for the T-property in Theorem 1, which in the case of plane curves improves conditions (0.2) and (0.3):

- $V_d(S_1,\dots,S_r)$ has the T-property if

$$\sum_{i=1}^{r}(\tau'(S_i)+1)^2 < d^2 \tag{0.6}$$

and

$$\sum_{i=1}^{r}\frac{\kappa(S_i)^2}{\kappa(S_i)-\tau'(S_i)} < d^2 \tag{0.7}$$

where κ denotes the intersection multiplicity of the curve germ with a generic polar.

- For curves with only simple singularities $S_1,\dots,S_r$ condition (0.7) is already implied by (0.6). In particular, for curves with n nodes and k cusps we obtain the sufficient condition

$$4n+9k \;<\; d^2.$$

In addition, in Section 3 we apply our criterion to families of curves on $K3$-surfaces and on smooth hypersurfaces in $\mathbb{P}^3$.

Finally, we use the method introduced in [Sh5] to construct a series of irreducible plane curves with precisely r D_μ-singularities, where the variety V is locally irreducible (even normal) of the expected dimension, but not smooth (in particular, there is no T-property), and again the analogue of (0.4) holds. This shows that in the case of plane curves with only simple singularities the new condition is optimal with respect to the occurring exponents of μ, d (cf. [Sh5], section 3.2).

Our methods of proof are based on the Chiantini-Sernesi idea, involving Bogomolov's theory of unstable vector bundles on surfaces, and on our previous local analysis of singularities and h^1-vanishing criteria (cf. [GLS1, Sh5]).

Notations and basic definitions

Let Σ be a smooth projective surface and $C \subset \Sigma$ be an irreducible curve with exactly r singularities of topological (resp. analytic) types $S_1,\dots,S_r$ (cf. [Sh4] for the definitions). All curves are assumed to be reduced.

Let $I^{ea}(C,z) = (f, f_x, f_y)\cdot \mathcal{O}_{\Sigma,z}$ (where f is a local equation for the germ (C,z)) denote the Tjurina ideal of (C,z) and

$$I^{es}(C,z) := \{g \in \mathcal{O}_{\Sigma,z} \mid f+\varepsilon g \text{ is equisingular over } \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)\}$$

denote the equisingularity ideal of (C,z) (cf. [Wa]). We introduce the schemes $X^{es}(C)$ (resp. $X^{ea}(C)$), concentrated at the singular points $z_1,\dots,z_r$ of C, and defined by the ideals $I^{es}(C, z_i)$ (resp. $I^{ea}(C, z_i)$). The corresponding ideal sheaves on Σ will be denoted by $\mathcal{J}_{X^{es}(C)}$ (resp. $\mathcal{J}_{X^{ea}(C)}$).

We shall often treat, simultaneously, the case of fixed topological types (the "es"-case) and the case of fixed analytic types ("ea"-case). Then we shall use the notation $X'(C)$ to denote $X^{es}(C)$ and $X^{ea}(C)$, respectively. Moreover, for a singularity type S_i we denote by $\tau'(S_i)$ the codimension of the corresponding equisingularity (resp. Tjurina) ideal, that is

- $\tau'(S_i) = \tau(S_i) - \mathrm{mod}(S_i)$ for a topological type S_i. (Here mod denotes the modality in the sense of [AGV].)
- $\tau'(S_i) = \tau(S_i)$, the Tjurina number, for an analytic type S_i.

Finally, by $\kappa(S_i)$ we denote the intersection multiplicity of a singularity of type S_i with a corresponding generic polar. Note that $\kappa(S_i) = \mu(S_i) + \mathrm{mt}(S_i) - 1$, where μ denotes the Milnor number and mt the multiplicity.

1 Curves on arbitrary surfaces

Let Σ be a smooth projective surface and $C \subset \Sigma$ be an irreducible curve with exactly r singularities of topological (resp. analytic) types $S_1, \ldots, S_r$. By [GK, GL], we know that the variety $V_{|C|}(S_1, \ldots, S_r)$ has the T-property at C if and only if

$$h^1\big(\Sigma, \mathcal{J}_{X^{es}(C)}(C)\big) \;=\; 0 \qquad (\text{resp. } h^1\big(\Sigma, \mathcal{J}_{X^{ea}(C)}(C)\big) \;=\; 0).$$

We introduce the number

$$A(\Sigma, C) := \frac{(CK_\Sigma)^2 - K_\Sigma^2 C^2}{4C^2}.$$

By the Hodge Index Theorem this is a non-negative number if C is ample.

Theorem 1 *Let Σ be a smooth projective surface and let $C \subset \Sigma$ be an irreducible curve with precisely r singularities at $z_1, \ldots, z_r$ of topological (resp. analytic) types $S_1, \ldots, S_r$, such that C, $C - K_\Sigma$ are ample and $C^2 \geq K_\Sigma^2$. If*

$$\sum_{i=1}^{r} \tau'(S_i) \;<\; \frac{(C - K_\Sigma)^2}{4}\,, \qquad \sum_{i=1}^{r} \kappa(S_i) \;<\; \frac{C(C - K_\Sigma)}{2} \tag{1.1}$$

and the following inequalities hold

$$\frac{\left(\sum_{i=1}^{r} (\tau'(S_i) + 1)\right)^2}{C^2} \;<\; \sum_{i=1}^{r} \left(1 - \frac{CK_\Sigma}{C^2}(\tau'(S_i) + 1)\right) - A(\Sigma, C) \tag{1.2}$$

$$\frac{\left(\sum_{i=1}^{r} \kappa(S_i)\right)^2}{C^2} \;<\; \sum_{i=1}^{r} \left(\kappa(S_i)\left(1 - \frac{CK_\Sigma}{C^2}\right) - \tau'(S_i)\right) - A(\Sigma, C), \tag{1.3}$$

then $V_{|C|}(S_1,\ldots,S_r)$ *has the T-property, that is, is a smooth variety of codimension* $\sum_{i=1}^r \tau'(S_i)$.

Remark. As will follow from the proof, in the conditions (1.1)–(1.3), we can replace $\tau'(S_i)$ by the maximum codimension of a complete intersection ideal containing $I^{es}(C,z_i)$ (respectively $I^{ea}(C,z_i)$).

Proof. Most parts of the proof are identical for topological types and analytic types. There we use the notation $X'(C)$ introduced above to denote $X^{es}(C)$ and $X^{ea}(C)$, respectively.

Assume that $V_{|C|}(S_1,\ldots,S_r)$ has no T-property at C, that is, $h^1\big(\Sigma,\mathcal{J}_{X'(C)}(C)\big)>0$. Let $\emptyset\neq X_0\subset X'(C)$ be minimal with non-vanishing $h^1\big(\Sigma,\mathcal{J}_{X_0}(C)\big)$ (that is, for each proper subscheme $\emptyset\neq \widetilde{X}_0\subsetneq X_0$ $h^1\big(\Sigma,\mathcal{J}_{\widetilde{X}_0}(C)\big)$ vanishes). Then, by Serre-Grothendieck duality $\mathrm{Ext}^1\big(\mathcal{J}_{X_0}(C),\mathcal{O}_\Sigma(K_\Sigma)\big)\neq 0$ and a generic element defines an extension

$$0\longrightarrow \mathcal{O}_\Sigma\longrightarrow E\longrightarrow \mathcal{J}_{X_0}(C-K_\Sigma)\longrightarrow 0, \tag{1.4}$$

where E is a rank 2 vector bundle on Σ with Chern classes

$$c_1(E)=C-K_\Sigma\,,\quad c_2(E)=\deg(X_0) \tag{1.5}$$

(cf., e.g., [L]). In particular, it follows that $X_0|_{z_i}$ is a complete intersection and we have

$$1\;\leq\;\deg(X_0)\;=:\;\sum_{i=1}^r\tau_i^0\;\leq\;\sum_{i=1}^r\big(\tau'(S_i)-\varepsilon(S_i)\big) \tag{1.6}$$

with $\varepsilon(S_i)\geq 0$ and $\varepsilon(S_i)=0$ implies that $X'(C)|_{z_i}$ is a complete intersection, too. By (1.1) we obtain

$$c_1(E)^2-4c_2(E)\;=\;(C-K_\Sigma)^2-4\deg(X_0)\;>\;0,$$

hence E is Bogomolov-unstable ([B, R, L]), that is, there is a finite subscheme $Z\subset\Sigma$ (possibly empty) plus a divisor M and an exact sequence

$$0\longrightarrow\mathcal{O}_\Sigma(M)\longrightarrow E\longrightarrow\mathcal{J}_Z(C-K_\Sigma-M)\longrightarrow 0 \tag{1.7}$$

such that $(2M-C+K_\Sigma)^2>0$ and for each ample divisor H on Σ we have

$$(2M-C+K_\Sigma)H\;>\;0.$$

Tensoring (1.7) by $\mathcal{O}_\Sigma(-M)$ we obtain the existence of a section $s\in H^0\big(\Sigma,E(-M)\big)$ and

$$\deg(X_0)-M(C-K_\Sigma-M)\;=\;c_2(E)+M^2-Mc_1(E)\;=\;c_2(E(-M))\;\geq\;0. \tag{1.8}$$

On the other hand, the complete linear system $|-M|$ is empty, since for each ample divisor H we have $(-2M)H<(K_\Sigma-C)H\leq 0$. The exact sequence (1.4) induces an injection

$$\emptyset\neq H^0\big(\Sigma,E(-M)\big)\hookrightarrow H^0\big(\Sigma,\mathcal{J}_{X_0}(C-K_\Sigma-M)\big),$$

that is, the existence of a curve $\Delta \in |C - K_\Sigma - M|$ vanishing at X_0. Δ cannot contain C as a component since for the ample divisor $C - K_\Sigma$ the above inequality gives

$$-2(K_\Sigma + M)(C - K_\Sigma) \;<\; -2K_\Sigma(C - K_\Sigma) - (C - K_\Sigma)^2 \;=\; K_\Sigma^2 - C^2 \;\leq\; 0.$$

Thus, applying Bézout's Theorem, we obtain

$$C(C - K_\Sigma - M) \;=\; C\Delta \;\geq\; \sum_{i=1}^{r} \dim \mathcal{O}_{\Sigma,z_i}/(f_i, g_i) \;=:\; \sum_{i=1}^{r} \kappa_i^0, \tag{1.9}$$

where f_i denotes a local equation of (C, z_i) and g_i denotes a generic element in the ideal of $X_0|_{z_i}$. Note that by Lemma 4.1 in [Sh5], $\kappa_i^0 \geq \tau_i^0 + 1$.

Since C is ample, we can apply the Hodge Index Theorem and obtain by (1.1) and (1.9)

$$\begin{aligned} C^2(2M - C + K_\Sigma)^2 \;&\leq\; \big(C(C - K_\Sigma) + 2C(M - C + K_\Sigma)\big)^2 \\ &\leq\; \Big(C(C - K_\Sigma) - 2\sum_{i=1}^{r} \kappa_i^0\Big)^2 . \end{aligned}$$

On the other hand, by (1.6) and (1.8), we have

$$(2M - C + K_\Sigma)^2 \;=\; 4\Big(M - \frac{C - K_\Sigma}{2}\Big)^2 \;\geq\; (C - K_\Sigma)^2 - 4\sum_{i=1}^{r} \tau_i^0 .$$

Finally,

$$\begin{aligned} &(C - K_\Sigma)^2 - 4\sum_{i=1}^{r} \tau_i^0 - \frac{\big(C(C - K_\Sigma) - 2\sum_{i=1}^{r} \kappa_i^0\big)^2}{C^2} = \\ &-\frac{(CK_\Sigma)^2 - K_\Sigma^2 C^2}{C^2} + 4\sum_{i=1}^{r}\Big(\kappa_i^0\Big(1 - \frac{CK_\Sigma}{C^2}\Big) - \tau_i^0\Big) - \frac{4\left(\sum_{i=1}^{r} \kappa_i^0\right)^2}{C^2} \end{aligned}$$

is a concave function in $\sum_{i=1}^{r} \kappa_i^0$ (for fixed $\sum_{i=1}^{r} \tau_i^0$), hence it takes its minimum either at $\sum_{i=1}^{r} \kappa_i^0 = \sum_{i=1}^{r} (\tau_i^0 + 1)$ or at $\sum_{i=1}^{r} \kappa_i^0 = \sum_{i=1}^{r} \kappa(S_i)$. In both cases the minimum is not smaller than the value for $\sum_{i=1}^{r} \tau_i^0 = \sum_{i=1}^{r} \tau'(S_i)$, which is positive by (1.2) and (1.3) in contradiction to the latter inequalities. □

2 Curves in $\mathbb{P}^2$

As an immediate consequence of Theorem 1 we obtain

Theorem 2 *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d with r singularities of topological (resp. analytic) types $S_1, \ldots, S_r$ as its only singularities. If*

$$\sum_{i=1}^{r} (\tau'(S_i)+1)^2 \;<\; d^2 \quad \textit{and} \quad \sum_{i=1}^{r} \frac{\kappa(S_i)^2}{\kappa(S_i)-\tau'(S_i)} \;<\; d^2, \tag{2.1}$$

then $V_d(S_1, \ldots, S_r)$ has the T-property.

Proof. First note that $A(\mathbb{P}^2, C) = 0$ and $-CK_{\mathbb{P}^2}/C^2 = 3/d$, hence condition (1.3) in Theorem 1 is satisfied, since applying the Cauchy inequality to the left-hand side of the second inequality in (2.1) gives

$$\frac{\left(\sum_{i=1}^{r} \kappa(S_i)\right)^2}{\sum_{i=1}^{r} (\kappa(S_i)-\tau'(S_i))} \;\leq\; \sum_{i=1}^{r} \frac{\kappa(S_i)^2}{\kappa(S_i)-\tau'(S_i)}.$$

The inequality (1.2) follows in an analogous way from the first inequality in (2.1), whereas the two inequalities in (1.1) are induced by (2.1), since we have $(\tau'(S_i)+1)^2 \geq 2\tau'(S_i)$ respectively $\tau'(S_i) \geq \kappa(S_i)/2$ by the following lemma. □

Lemma 2.2 *Let (C, z) be a plane curve singularity, then*

$$\kappa(C,z) \;\leq\; 2\big(\tau(C,z) - mod(C,z)\big).$$

Proof. As is well-known (cf. [DH]), $I^{es}(C,z)$ is contained in the equiclassical ideal $I^{ec}(C,z)$, which is contained in the equigeneric ideal $I^{eg}(C,z)$. Moreover,

$$\dim \mathbb{C}\{x,y\}/I^{ec}(C,z) \;=\; \delta(C,z) \quad \text{and} \quad \dim \mathbb{C}\{x,y\}/I^{eg}(C,z) \;=\; \kappa(C,z) - \delta(C,z).$$

Hence $\tau(C,z) - \mathrm{mod}(C,z) \geq \kappa(C,z) - \delta(C,z) \geq \kappa(C,z) - \big(\tau(C,z) - \mathrm{mod}(C,z)\big)$. □

Note that for an arbitrary reduced plane curve singularity (C,z) we have $\kappa(C,z) = \mu(C,z) + \mathrm{mt}(C,z) - 1$, where μ is the Milnor number and mt the multiplicity. Hence the contribution of an A_μ-singularity to the left-hand side of the conditions in (2.1) is just $(\mu+1)^2$ and the contribution of a D_μ- (resp. E_μ-) singularity is at most $(\mu+1)^2$. We obtain the following

Corollary 2.3 *Let $S_1, \ldots, S_r$ be simple singularities. Then $V_d(S_1, \ldots, S_r)$ has the T-property if*

$$\sum_{i=1}^{r} (\mu(S_i)+1)^2 \;<\; d^2.$$

In particular, the variety $V_d(n \cdot A_1, k \cdot A_2)$ of irreducible plane curves of degree d having n nodes and k cusps as its only singularities is a smooth variety of the expected dimension $d(d+3)/2 - n - 2k$ if $4n + 9k \;<\; d^2$.

3 Curves on special surfaces

In the following we consider further examples of surfaces Σ satisfying

$$(CK_\Sigma)^2 = C^2 K_\Sigma^2, \tag{3.1}$$

for each curve $C \subset \Sigma$, that is $A(\Sigma, C) = 0$ in the conditions of Theorem 1.

As a first example, we consider surfaces Σ with a trivial canonical divisor (e.g., K3-surfaces). In an analogous way to the case of plane curves, we obtain

Theorem 3 *Let Σ be a surface with $K_\Sigma = 0$ and let $C \subset \Sigma$ be an irreducible curve with precisely r singularities of topological (resp. analytic) types $S_1, \ldots, S_r$, such that C is ample. Then $V_{|C|}(S_1, \ldots, S_r)$ has the T-property, if*

$$\sum_{i=1}^{r} (\tau'(S_i) + 1)^2 \ < \ C^2 \quad \textit{and} \quad \sum_{i=1}^{r} \frac{\kappa(S_i)^2}{\kappa(S_i) - \tau'(S_i)} \ < \ C^2.$$

The condition (3.1) is also satisfied in case $C \equiv pK_\Sigma$, $p \in \mathbb{Q}$. As an example we mention the case of curves on a smooth hypersurface $\Sigma \subset \mathbb{P}^3$:

Theorem 4 *Let $\Sigma \subset \mathbb{P}^3$ be a smooth hypersurface of degree $d \geq 5$ and let $C \subset \Sigma$ be an irreducible curve with precisely r singularities of topological (resp. analytic) types $S_1, \ldots, S_r$, such that $C \equiv p(d-4)H$ (H being a hyperplane section), $p \in \mathbb{Q}$, $p > 1$. Suppose, moreover, that for each $1 \leq i \leq r$ there exists $N_i > 0$ with*

$$p \geq \textit{max} \left\{ \frac{\kappa(S_i)}{\kappa(S_i) - \tau'(S_i)} + N_i \, , \, \tau'(S_i) + 2 \right\}.$$

Then $V_{|C|}(S_1, \ldots, S_r)$ has the T-property, if

$$\sum_{i=1}^{r} \tau'(S_i) < \frac{(p-1)^2}{4}$$

and

$$\textit{max} \left\{ \sum_{i=1}^{r} \frac{(\kappa(S_i))^2 \left(\frac{1+N_i}{N_i} \kappa(S_i) - \tau'(S_i) \right)}{(\kappa(S_i) - \tau'(S_i))^2} \, , \, \sum_{i=1}^{r} (\tau'(S_i) + 1)^2 \right\} \ < \ p^2(d-4)^2. \tag{3.2}$$

Proof. Note that $K_\Sigma \equiv (d-4)H$. Since $p(d-4) > 0$ and $(p-1)(d-4) > 0$, then C, $C - K_\Sigma$ are ample and $p > 1$ implies $C^2 \geq K_\Sigma^2$.

Moreover, as in the case of plane curves, condition (1.1) is induced by (3.2). Finally, $CK_\Sigma/C^2 = 1/p$, whence

$$\begin{aligned}
\kappa(S_i)\left(1 - \frac{CK_\Sigma}{C^2}\right) - \tau'(S_i) &= (\kappa(S_i) - \tau'(S_i)) - \frac{\kappa(S_i)}{p} \\
&\geq (\kappa(S_i) - \tau'(S_i)) \cdot \left(1 - \frac{\kappa(S_i)}{(1+N_i)\kappa(S_i) - N_i\tau'(S_i)}\right) \\
&= \frac{(\kappa(S_i) - \tau'(S_i))^2}{\frac{1+N_i}{N_i}\kappa(S_i) - \tau'(S_i)}.
\end{aligned}$$

In addition, applying the Cauchy inequality to the left-hand side of (3.2) gives

$$\frac{\left(\sum_{i=1}^r \kappa(S_i)\right)^2}{\sum_{i=1}^r \left((1 - \frac{CK_\Sigma}{C^2})\kappa(S_i) - \tau'(S_i)\right)} \leq \sum_{i=1}^r \frac{(\kappa(S_i))^2\left(\frac{1+N_i}{N_i}\kappa(S_i) - \tau'(S_i)\right)}{(\kappa(S_i) - \tau'(S_i))^2}.$$

Condition (1.3) follows in the same manner, whence the statement of Theorem 4 holds. □

Corollary 3.3 *Let $\Sigma \subset \mathbb{P}^3$ be a smooth hypersurface of degree $d \geq 5$, C be an irreducible curve with precisely n nodes and k cusps, such that $C \equiv p(d-4)H$, $p > 5$. Then $V_{|C|}(n \cdot A_1, k \cdot A_2)$ has the T-property, if*

$$\frac{20}{3}n + \frac{45}{2}k < p^2(d-4)^2.$$

4 Non-smooth equisingular families

E. Shustin constructs in [Sh5] a series of irreducible plane curves C of degree $2pq$ ($p \geq 4$, $q \geq 3$) with precisely q^2 singularities of type A_{6p-1}, such that the variety $V_{2pq}(q^2 \cdot A_{6p-1})$ has no T-property at C. In this example,

$$\sum_{i=1}^r (\mu+1)^2 = 36p^2q^2 = 9d^2,$$

which implies that the condition in Corollary 2.3 is optimal with respect to the occurring exponents of d, μ. In the following theorem, we extend Shustin's construction to the case of D_μ-singularities to prove the optimality of our condition in the case of simple singularities. Moreover, we shall prove that, in both examples, the variety $V_{2pq}(q^2 \cdot A_{6p-1})$ (resp. $V_{2pq}(q^2 \cdot D_{6p+1})$) is locally irreducible at C (for $p \geq 9$).

Theorem 5 *Let $p \geq 4$ and $q \geq 3$, then there are irreducible curves C_{2pq} of degree $2pq$ having precisely q^2 A_{6p-1}-singularities (given by a local equation $y^2 - x^{6p}$), respectively q^2 D_{6p+1}-singularities (given by a local equation $x(y^2 - x^{6p-1})$), such that the variety $V = V_{2pq}(q^2 \cdot A_{6p-1})$ (resp. $V_{2pq}(q^2 \cdot D_{6p+1})$) has no T-property at C_{2pq}. Moreover, if $p \geq 5$ and $q \geq 3$, V has a component of the expected dimension but is singular at C_{2pq}.*

Theorem 6 *Using the notation of Theorem 5, let $p \geq 8$ (resp. $p \geq 9$) and $q \geq 3$. Then the variety V is locally a complete intersection, which is normal (hence analytically irreducible), but not smooth, at C_{2pq}.*

Proof of Theorem 5. The main idea of the proof is as follows. We are going to construct a curve $C_{2pq} \in V$ such that $h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(C_{2pq})}(2pq)\big) > 0$ and, in a neighbourhood of C_{2pq} there is a one-parameter family of curves $C \in V$ with $h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(C)}(2pq)\big) = 0$.

Firstly, this gives the failure of the T-property at $C_{2pq} \in V$. Secondly, this says that V has a singularity at C_{2pq}, because V has a component, which locally is a complete analytic intersection, generically smooth and transversal, but not transversal at C_{2pq}, which follows from the representation of V as an intersection of a (smooth) equisingular family of curves of large degree with the space corresponding to curves of degree $2pq$.

The case of A_μ-singularities is completely treated in [Sh5]. For D_μ-singularities in the first part of the proof, in particular the construction of the curve C_{2pq}, we proceed as in the case of A_μ-singularities. Hence we give only a sketch of it and refer to [Sh5], Section 4.5, for details.

For generic $\lambda \neq 0$, the projective closure C_{2p} of the curve

$$F(x,y) = x(y^3 + y - x^3)^2(1 + \lambda x^{2p-7}) - y^{2p} = 0$$

is an irreducible curve of degree $2p$ having exactly one singular point of type D_{6p+1} at the origin. Its polar curve

$$F_y(x,y) = 2x(y^3 + y - x^3)(3y^2 + 1)(1 + \lambda x^{2p-7}) - 2py^{2p-1} = 0$$

intersects the non-singular cubic C_3 given by $y^3 + y - x^3 = 0$ with multiplicity $6p - 3$ at the origin.

After choosing generic projective coordinates x_0, x_1, x_2 we consider the curves

$$C_{2pq}(x_0, x_1, x_2) = C_{2p}(x_0^q, x_1^q, x_2^q) \quad \text{and} \quad C_{3q}(x_0, x_1, x_2) = C_3(x_0^q, x_1^q, x_2^q).$$

C_{2pq} is an irreducible curve having q^2 singularities of type D_{6p+1} as its only singularities, whereas C_{3q} is non-singular. Since the polar F_x vanishes along C_3, the degree of $X^{es}(C_{2pq}, z) \cap C_{3q}$, $z \in \mathrm{Sing}(C_{2pq})$, can be calculated as the intersection multiplicity of C_3 with the polar F_y at the origin (which is $6p - 3$ by the above). We obtain

$$\mathcal{J}_{X^{es}(C_{2pq}) \cap C_{3q}/C_{3q}}(2pq) = \mathcal{O}_{C_{3q}}(D),$$

where D is a divisor of degree

$$2pq \cdot 3q - \deg(X^{es}(C_{2pq}) \cap C_{3q}) \;=\; 3q^2 \;<\; \frac{(3q-1)(3q-2)}{2} \;=\; g(C_{3q}),$$

if $q \geq 3$. Hence $h^1\big(C_{3q}, \mathcal{J}_{X^{es}(C_{2pq}) \cap C_{3q}/C_{3q}}(2pq)\big) \neq 0$, which implies the non-vanishing of $h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(C_{2pq})}(2pq)\big)$, that is, V has no T-property at C_{2pq}.

Now assume that $p \geq 5$, $q \geq 3$. As in the A_μ-case, we shall construct a one-parameter family of curves $C_{2pq}^{(\alpha)}$ in V with $C_{2pq} = C_{2pq}^{(0)}$, such that for a generic choice of α

$$h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(C_{2pq}^{(\alpha)})}(2pq)\big) = 0. \tag{4.1}$$

The following deformation of F

$$F^{(\alpha)}(x,y) = x(y^3+y-x^3)\big((y^3+y-x^3)(1+\lambda x^{2p-7}) + \alpha y^{p+1}\big) - y^{2p} = 0$$

is easily seen to preserve the D_{6p+1}-singularity at the origin. Moreover, for generic α, the polar curve

$$\begin{aligned} F_y^{(\alpha)}(x,y) &= x(y^3+y-x^3)\big((6y^2+2)(1+\lambda x^{2p-7}) + \alpha(p+1)y^p\big) \\ &\quad + \alpha x(3y^2+1)y^{p+1} - 2py^{2p-1} = 0 \end{aligned}$$

intersects C_3 with multiplicity $3p+4$ at the origin. As before, after choosing generic projective coordinates x_0, x_1, x_2, we obtain

$$C_{2pq}^{(\alpha)}(x_0,x_1,x_2) = C_{2p}^{(\alpha)}(x_0^q, x_1^q, x_2^q) \in V_{2pq}(q^2 \cdot D_{6k+1}).$$

Finally, we apply again the reduction procedure introduced in [Sh5] to deduce the vanishing of $h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(C_{2pq}^{(\alpha)})}(2pq)\big)$. But, different to the A_μ-case, we have to reduce three times, since we cannot assume $C_3^2 \supset X^{es}(C_{2pq}^{(\alpha)})$. By Noether's Criterion (cf. [W], p. 215), we have $C_3^3 \supset X^{es}(C_{2pq}^{(\alpha)})$, since the (smooth) local branches $\mathcal{P}_1, \mathcal{P}_2$ of $F_y^{(\alpha)}$ satisfy

$$(C_3^3, \mathcal{P}_1) = 3 = 1 + (F_x^{(\alpha)}, \mathcal{P}_1)\,, \;\; (C_3^3, \mathcal{P}_2) = 3(3p+3) \geq 6p = 1 + (F_x^{(\alpha)}, \mathcal{P}_2),$$

thus we can find A, B with $C_3^3 = A \cdot F_y^{(\alpha)} + B \cdot F_x^{(\alpha)}$.

Note that, in the same manner, it follows that $\mathfrak{m}_z \cdot C_{3q}^2$ is contained in $I^{es}(C_{2pq}^{(\alpha)}, z)$ for each $z \in \operatorname{Sing}(C_{2pq}^{(\alpha)})$, whence $\deg(X^{es}(C_{2pq}^{(\alpha)}) : C_{3q}^2) \leq q^2$. To save space, let $X^{es} := X^{es}(C_{2pq}^{(\alpha)})$. We have the exact sequences

$$\begin{aligned} &0 \to \mathcal{J}_{X^{es}:C_{3q}}(2pq-3q) \to \mathcal{J}_{X^{es}}(2pq) \to \mathcal{J}_{X^{es}\cap C_{3q}/C_{3q}}(2pq) \to 0, \\ &0 \to \mathcal{J}_{X^{es}:C_{3q}^2}(2pq-6q) \to \mathcal{J}_{X^{es}:C_{3q}}(2pq-3q) \to \mathcal{J}_{(X^{es}:C_{3q})\cap C_{3q}/C_{3q}}(2pq-3q) \to 0, \\ &0 \to \mathcal{O}_{\mathbb{P}^2}(2pq-9q) \to \mathcal{J}_{X^{es}:C_{3q}^2}(2pq-6q) \to \mathcal{J}_{X^{es}:C_{3q}^2/C_{3q}}(2pq-6q) \to 0. \end{aligned}$$

Since $2g(C_{3q}) - 2 = 9q^2 - 9q$, we obtain (4.1) from the following (obvious) inequalities

$$\begin{aligned} &6pq^2 - \deg(X^{es} \cap C_{3q}) \;=\; 6pq^2 - q^2(3p+4) \;\geq\; 11q^2, \\ &6pq^2 - 9q^2 - \deg((X^{es} : C_{3q}) \cap C_{3q}) \;\geq\; 6pq^2 - 9q^2 - q^2(3p-3) \;\geq\; 9q^2, \\ &6pq^2 - 18q^2 - \deg(X^{es} : C_{3q}^2) \geq 6pq^2 - 18q^2 - q^2 \geq 11q^2. \end{aligned}$$

The statement of the theorem follows from the considerations in [Sh5]. □

Proof of Theorem 6. Firstly, consider the case of A_μ-singularities (with $\mu = 6p-1$). Assume that V decomposes locally at C_{2pq} into (at least) two components V_1, V_2. Then we would have at C_{2pq}

$$\begin{aligned}\dim(\operatorname{Sing} V) &\ \geq\ \dim(V_1 \cap V_2)\ \geq\ \dim(V_1) + \dim(V_2) - \dim(V) \\ &\ \geq\ pq(2pq+3) - q^2\mu - h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(C_{2pq})}(2pq)\big), \end{aligned} \tag{4.2}$$

since V_1, V_2 have at least the expected dimension.

On the other hand, since $h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(C_{2pq})}(2pq)\big) > 0$, the proof of Theorem 1 shows the existence of a subscheme $X_0 \subset X^{es}(C_{2pq})$, such that $h^1\big(\mathbb{P}^2, \mathcal{J}_{X_0}(2pq)\big) > 0$, and of a curve Δ vanishing at X_0 of degree $\delta \leq (2pq+3)/2$ satisfying (cf. (1.8))

$$(2pq - \delta + 3)\cdot\delta\ \leq\ \deg(X_0)\ \leq\ q^2\mu,$$

that is,

$$\delta\ \leq\ pq + \tfrac{3}{2} - \sqrt{(pq)^2 + 3pq + \tfrac{9}{4} - q^2\mu}\ <\ pq + \tfrac{3}{2} - \sqrt{\big((p-4)q + \tfrac{3}{2}\big)^2}\ =\ 4q \tag{4.3}$$

as $p \geq 8$.

Moreover, in the construction of C_{2pq}, we have seen the existence of a smooth curve C_{3q} of degree $3q$ satisfying $C_{3q}^2 \supset X_0$. The exact sequences appearing in the reduction process imply that, due to the non-vanishing of $h^1\big(\mathbb{P}^2, \mathcal{J}_{X_0}(2pq)\big)$,

$$h^1\big(C_{3q}, \mathcal{J}_{X_0\cap C_{3q}/C_{3q}}(2pq)\big) > 0 \quad \text{or} \quad h^1\big(C_{3q}, \mathcal{J}_{X_0 : C_{3q}/C_{3q}}(2pq - 3q)\big) > 0.$$

But

$$\deg(X_0 : C_{3q})\ \leq\ \deg(X^{es}(C_{2pq}) : C_{3q})\ =\ q^2(\mu - 6p + 3)\ <\ 3q(2pq - 6q + 3),$$

whence (by [Sh5], Lemma 4.2) $h^1\big(C_{3q}, \mathcal{J}_{X_0 : C_{3q}/C_{3q}}(2pq - 3q)\big)$ vanishes. It follows that $h^1\big(C_{3q}, \mathcal{J}_{X_0\cap C_{3q}/C_{3q}}(2pq)\big)$ does not vanish, especially we have $C_{3q} \supset X_0$ (due to the minimality of X_0) and

$$\deg(X_0)\ \geq\ \deg(X_0 \cap C_{3q})\ \geq\ 3q(2pq - 3q + 3). \tag{4.4}$$

Now, let $\widetilde{C}$ be a curve in V close to C_{2pq} with

$$h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(\widetilde{C})}(2pq)\big)\ >\ 0.$$

Then we can assume that for the corresponding scheme $\widetilde{X}_0$ and curve $\widetilde{\Delta}$ of degree $\widetilde{\delta}$ we have

$$\widetilde{\delta}\ <\ 4q \quad \text{and} \quad \deg(\widetilde{X}_0)\ \geq\ \deg(X_0)$$

(due to the semicontinuity of cohomology).

Let us show that the curve $\widehat{\Delta}$ of the minimal degree containing $\widetilde{X}_0$ is (reduced) irreducible and its degree satisfies

$$3q \le \widehat{\delta} < 4q .$$

Note that for the scheme X_0 such a curve is just C_{3q}, because by Bézout's Theorem and the construction of C_{3q} we have for any curve C of degree $< 4q$, which does not contain C_{3q} as a component,

$$\begin{aligned} \deg\big(X^{es}(C_{2pq}) \cap C\big) \ &\le\ \frac{\mu}{6p-3}\, C \cdot C_{3q} \ <\ \frac{6p-1}{6p-3} \cdot 12q^2 \\ &<\ 3q(2pq - 3q + 3)\ \le\ \deg(X_0) \end{aligned} \tag{4.5}$$

as $p \ge 8$. It follows, that if $\widetilde{C}$ tends to C_{2pq} along V, then $\widehat{\Delta}$ tends to a curve (of degree $< 4q$) containing C_{3q} as a component. In particular, $\widehat{\Delta}$ has to contain a component $\widehat{\Delta}'$ of degree $\widehat{\delta}' \ge 3q$, and

$$\deg\big(X^{es}(\widetilde{C}) \cap \widehat{\Delta}'\big) \ >\ \deg(X_0) - \frac{\mu}{6p-3} \cdot 3q^2 \ \ge\ 3q(2pq - 5q + 3) \ \ge\ \frac{q^2\mu}{2} . \tag{4.6}$$

Hence $(\widehat{\Delta}')^2 \supset X^{es}(\widetilde{C})$, and by (4.6)

$$\begin{aligned} \deg(X^{es}(\widetilde{C}) : \widehat{\Delta}') \ &<\ q^2\mu - 3q(2pq - 5q + 3) \ =\ 14q^2 - 9q \ \le\ 3q(2pq - 6q + 3) \\ &\le\ \widehat{\delta}'(2pq - 2\widehat{\delta}' + 3), \end{aligned}$$

which as above means

$$h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(\widetilde{C})}(2pq)\big) > 0 \ \Longrightarrow\ h^1\big(\widehat{\Delta}', \mathcal{J}_{X^{es}(\widetilde{C}) \cap \widehat{\Delta}'/\widehat{\Delta}'}(2pq)\big) > 0 .$$

In other words, we can assume $\widehat{\Delta}' \supset \widetilde{X}_0$, hence $\widehat{\Delta} = \widehat{\Delta}'$. In addition, note that the non-vanishing of $h^1\big(\mathbb{P}^2, \mathcal{J}_{\widetilde{X}_0}(2pq)\big) = h^1\big(\widehat{\Delta}, \mathcal{J}_{\widetilde{X}_0}(2pq)\big)$ implies

$$\deg(\widetilde{X}_0) \ \ge\ \widehat{\delta}(2pq - \widehat{\delta} + 3) . \tag{4.7}$$

Moreover, since $\widehat{\Delta}$ tends to a curve containing C_{3q} as a component, the geometric genus $g(\widehat{\Delta})$ is greater or equal to

$$g(C_{3q}) = \frac{(3q-1)(3q-2)}{2} ,$$

which gives, in particular,

$$\begin{aligned} \sum_{z \in \widehat{\Delta}} (\mathrm{mt}(\widehat{\Delta}, z) - 1) \ &\le\ \sum_{z \in \widehat{\Delta}} \frac{\mathrm{mt}(\widehat{\Delta}, z)(\mathrm{mt}(\widehat{\Delta}, z) - 1)}{2} \ \le\ \frac{(\widehat{\delta} - 1)(\widehat{\delta} - 2) - (3q-1)(3q-2)}{2} \\ &=\ \frac{(\widehat{\delta} - 3q)(\widehat{\delta} + 3q - 3)}{2} , \end{aligned} \tag{4.8}$$

where mt is the multiplicity.

We can estimate the dimension of the space $\widetilde{V}$ of all such curves $\widetilde{C}$ near C_{2pq} by the degree of freedom N_1 in the choice of $\widetilde{X}_0$, plus the degree of freedom N_2 (for fixed $\widetilde{X}_0$) in the choice of $\mathrm{Sing}(\widetilde{C})$ and the singular infinitely near points, plus the dimension N_3 of the space of curves $\widetilde{C} \in V$ with fixed $\mathrm{Sing}(\widetilde{C})$ and fixed singular infinitely near points.

Given a point $z \in \widehat{\Delta}$, a subideal in $\mathcal{O}_{\mathbb{P}^2,z}$ containing $(\widehat{\Delta})$ and being generated by two non-singular germs, depends on no more than $\mathrm{mt}(\widehat{\Delta}, z) - 1$ parameters; hence by (4.8)

$$N_1 \;\le\; \frac{\widehat{\delta}(\widehat{\delta}+3)}{2} + q^2 + \sum_{z \in \widehat{\Delta}} (\mathrm{mt}(\widehat{\Delta}, z) - 1) \;\le\; \widehat{\delta}^2 - \frac{7q^2 - 9q}{2} \; .$$

Given a curve germ σ with singularity A_{6p-1}, there are exactly $3p$ singular infinitely near points in its resolution tree, and they all belong to $X^{es}(\sigma)$; hence by (4.7)

$$N_2 \;\le\; q^2\mu - \deg \widetilde{X}_0 \;\le\; (6p-1)q^2 - \widehat{\delta}(2pq - \widehat{\delta} + 3) \; .$$

Finally, given a curve $\widetilde{C} \in V$ and the scheme $Y(\widetilde{C})$ of all its double infinitely near points, then

$$\begin{aligned} N_3 \;&\le\; h^0\big(\mathbb{P}^2, \mathcal{J}_{Y(\widetilde{C})}(2pq)\big) - 1 \\ &=\; \frac{2pq(2pq+3)}{2} - \deg(Y(\widetilde{C})) + h^1\big(\mathbb{P}^2, \mathcal{J}_{Y(\widetilde{C})}(2pq)\big) \\ &\le\; \frac{2pq(2pq+3)}{2} - 9pq^2 + h^1\big(\mathbb{P}^2, \mathcal{J}_{Y(C_{2pq})}(2pq)\big) \; . \end{aligned}$$

Combining the estimates for N_1, N_2, N_3 with (4.2), one obtains

$$\begin{aligned} pq(2pq+3) - q^2(6p-1) - h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}(C_{2pq})}(2pq)\big) \;&\le\; N_1 + N_2 + N_3 \\ &\le\; \widehat{\delta}^2 - \frac{7q^2 - 9q}{2} + (6p-1)q^2 - \widehat{\delta}(2pq - \widehat{\delta} + 3) \\ &\quad + \frac{2pq(2pq+3)}{2} - 9pq^2 + h^1\big(\mathbb{P}^2, \mathcal{J}_{Y(C_{2pq})}(2pq)\big) \; . \end{aligned}$$

The right-hand side contains the expression $2\widehat{\delta}^2 - 2pq\widehat{\delta}$, which takes its maximum in the interval $[3q, 4q]$ at $\widehat{\delta} = 3q$, since $pq/2 \ge 4q$ as $p \ge 8$. Therefore, the latter inequality implies

$$h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}}(2pq)\big) + h^1\big(\mathbb{P}^2, \mathcal{J}_Y(2pq)\big) \;\ge\; \big(3p - \tfrac{25}{2}\big)\, q^2 + \tfrac{9}{2} q \; , \qquad (4.9)$$

with the shortened notation $X^{es} = X^{es}(C_{2pq})$, $Y = Y(C_{2pq})$.

On the other hand, we can again reduce by C_{3q} and consider the corresponding exact sequences. Since $C_{3q}^2 \supset Y \supset X^{es}$, we obtain

$$\begin{aligned} h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}}(2pq)\big) \;&\le\; h^1\big(C_{3q}, \mathcal{J}_{X^{es} \cap C_{3q}/C_{3q}}(2pq)\big) + h^1\big(C_{3q}, \mathcal{J}_{X^{es} : C_{3q}/C_{3q}}(2pq - 3q)\big), \\ h^1\big(\mathbb{P}^2, \mathcal{J}_{Y}(2pq)\big) \;&\le\; h^1\big(C_{3q}, \mathcal{J}_{Y \cap C_{3q}/C_{3q}}(2pq)\big) + h^1\big(C_{3q}, \mathcal{J}_{Y : C_{3q}/C_{3q}}(2pq - 3q)\big). \end{aligned}$$

Now, we know that

$$\mathcal{J}_{X^{es}\cap C_{3q}/C_{3q}}(2pq) = \mathcal{O}_{C_{3q}}(D)\,, \quad \mathcal{J}_{Y\cap C_{3q}/C_{3q}}(2pq) = \mathcal{O}_{C_{3q}}(\widetilde{D}),$$

for some effective divisors $D, \widetilde{D}$, and

$$\begin{aligned} \deg(X^{es}:C_{3q}) &= q^2(\mu - 6p + 3) \le 3q(2pq - 6q + 3), \\ \deg(Y:C_{3q}) &\le q^2(\mu + 1 + 3p - 6p + 3) \le 3q(2pq - 6q + 3), \end{aligned}$$

whence both $h^1\big(C_{3q}, \mathcal{J}_{X^{es}:C_{3q}/C_{3q}}(2pq - 3q)\big)$ and $h^1\big(C_{3q}, \mathcal{J}_{Y:C_{3q}/C_{3q}}(2pq - 3q)\big)$ vanish (cf. [Sh5], Lemma 4.2). Using the Riemann-Roch formula, we can estimate

$$\begin{aligned} h^1\big(\mathbb{P}^2, \mathcal{J}_{X^{es}}(2pq)\big) + h^1\big(\mathbb{P}^2, \mathcal{J}_Y(2pq)\big) &= h^1\big(C_{3q}, \mathcal{O}_{C_{3q}}(D)\big) + h^1\big(C_{3q}, \mathcal{O}_{C_{3q}}(\widetilde{D})\big) \\ &\le 2g(C_{3q}) = (3q-1)(3q-2), \end{aligned}$$

which contradicts the choice of p, q.

Moreover, we have at C_{2pq}

$$\dim(\mathrm{Sing}\,V) \le N_1 + N_2 + N_3 < \dim(V) - 2,$$

showing that V, being locally a complete intersection, is normal at C_{2pq}.

In the case of D_μ-singularities (with $\mu = 6p+1$) we proceed similarly. The inequality (4.2) remains the same. The inequality (4.3) holds as $p \ge 9$. Further, by construction of C_{2pq}, C_{3q}, we have $C_{3q}^3 \supset X_0$. Since

$$\begin{aligned} \deg(X_0 : C_{3q}) &\le \deg(X^{es}:C_{3q}) = q^2(6p + 1 - (6p-3)) < 3q(2pq - 6q + 3), \\ \deg(X_0 : C_{3q}^2) &\le q^2 < 3q(2pq - 9q + 3), \end{aligned}$$

as $p \ge 9$, then (4.4) holds true. Again, we deduce that for a generic curve $\widetilde{C} \in \mathrm{Sing}(V)$ in a neighbourhood of C_{2pq} there exists a reduced irreducible curve $\widehat{\Delta} \supset \breve{X}_0$ of degree $3q \le \widehat{\delta} < 4q$, that in a similar manner implies

$$N_1 \le \widehat{\delta}^2 - \frac{7q^2 - 9q}{2}\,, \quad N_2 \le (6p+1)q^2 - \widehat{\delta}(2pq - \widehat{\delta} + 3)$$

and

$$\begin{aligned} N_3 &\le \frac{2pq(2pq+3)}{2} - \deg(Y(\widetilde{C})) + h^1\big(\mathbb{P}^2, \mathcal{J}_{Y(\widetilde{C})}(2pq)\big) \\ &\le \frac{2pq(2pq+3)}{2} - (9p+2)q^2 + h^1\big(\mathbb{P}^2, \mathcal{J}_{Y(C_{2pq})}(2pq)\big)\,, \end{aligned}$$

Thus we come to the same inequality (4.9) and the same contradiction, saying that V is locally irreducible and normal at C_{2pq}. □

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