On analytic semigroups and cosine functions in Banach spaces

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Abstract

If A generates a bounded cosine function on a Banach space X then the negative square root B of A generates a holomorphic semigroup, and this semigroup is the conjugate potential transform of the cosine function. This connection is studied in detail, and it is used for a characterization of cosine function generators in terms of growth conditions on the semigroup generated by B. This characterization relies on new results on the inversion of the vector-valued conjugate potential transform.

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1. Introduction

In a Banach space X, consider a closed linear operator A which generates a cosine function $C(\cdot)$ (see e.g. Fattorini [6] or Goldstein [7] for more information about cosine operator functions). Then A generates a holomorphic semigroup $T(\cdot)$ of angle $\pi/2$. The semigroup and the cosine function are related by the abstract Weierstrass formula

$$T(t)x = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\tau^2/4t} C(\tau) x d\tau, \quad t > 0.$$

On the other hand, assume that A generates a C_0-semigroup $T(\cdot)$. If $T(\cdot)$ is uniformly bounded, then one can define the fractional powers $(-A)^\alpha$ of $-A$ for $0 < \alpha < 1$. We restrict ourselves to the case $\alpha = 1/2$. First define the operator J with domain $D(J) = D(A)$ by

$$Jx = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(\lambda - A)^{-1}(-A)x\,d\lambda, \quad x \in D(J).$$

Then J is closable and by definition, $(-A)^{1/2} := \overline{J}$ (see e.g. Yosida [16, p.260]).

The operator $B := -(-A)^{1/2}$ is the generator of a holomorphic semigroup $T_B(\cdot)$ which has an explicit representation (see [16, p.268]):

$$T_B(t)x = \frac{t}{2\sqrt{\pi}}\int_0^\infty e^{-t^2/4\tau}T(\tau)x\frac{d\tau}{\tau^{3/2}}, \quad x \in X,\ t > 0.$$

Combining the above facts, we see that whenever A generates a uniformly bounded cosine function $C(\cdot)$, the negative square root of A generates a bounded holomorphic semigroup of angle $\pi/2$ given by the formula

$$T_B(t)x = \frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2}C(\tau)x\,d\tau,\ x \in X, \quad t > 0. \tag{1}$$

It is our intention in this paper to study this connection in more details. In the first part, we introduce the general transformation: if $f : (0,\infty) \to X$ is measurable, and if the integral $\int_0^\infty \|f(\tau)\|/(t^2+\tau^2)\,d\tau$ converges for all $t \in (0,\infty)$ then we define

$$\mathcal{C}f(t) = \frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2}f(\tau)\,d\tau, \quad t \in (0,\infty),$$

and we call $\mathcal{C}f$ the conjugate potential transform of f. We provide a vector-valued inversion theory for the conjugate potential transform in the spirit of [14], using Widder's results on the inversion of convolution transforms [15].

In the second part we consider the relationship (1) and prove that $T_B(\cdot)$ satisfies the semigroup property iff $C(\cdot)$ satisfies the cosine functional equation. A similar relationship was studied by Dettman [4] in connection with the Cauchy problem. Our approach is operator theoretic.

A remarkable feature is the following: by using the sine function $S(\cdot)$ associated with the cosine function, one can recast formula (1) in the form

$$T_B(t) = \frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2}\,dS(\tau), \quad x \in X. \tag{2}$$

Now, if we do not assume that A generates a cosine function but rather that it generates a sine function which is Lipschitz-continuous in the strong operator topology, then we prove that the representation (2) implies that in fact A generates a strongly continuous cosine function. This is to be likened to Arendt [1] where a similar phenomenon occurs in the relationship between resolvents and integrated semigroup. More precisely, the fact that Widder's theorem holds for general Banach spaces only in an *integrated form* while it holds in all Banach spaces in the usual form for resolvents of densely defined linear operators.

The results of the first section can then be used to recover $C(\cdot)$ from $T_B(\cdot)$ in the representation (1). We provide an explicit representation to that effect. Another interesting fact is that since the transform of Section 2 was studied for general vector-valued functions, it can be used, along with the inversion formula to relate the solution of the second order Cauchy problem associated with A to that of the first order Cauchy problem associated with the negative square root of A.

1. Inversion of the conjugate potential transform

If $f : (0,\infty) \to X$ is measurable with $\int_0^\infty \|f(t)\| / (s^2+t^2)\, dt < \infty$ for all $s \in (0,\infty)$ then we define

$$\mathcal{C}f(s) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2} f(t)\, dt, \quad s \in (0,\infty).$$

In this section we give an inversion formula which recovers any bounded continuous function f from the transformed function $\mathcal{C}f$, and we characterize those functions $F : (0,\infty) \to X$ which can be represented as

$$F(s) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\, d\phi(t), \quad s \in (0,\infty),$$

where $\phi : (0,\infty) \to X$ is Lipschitz-continuous.

Before we state the inversion formula we introduce some notations. For $\Omega \subseteq \mathbf{R}$ open and $f : \Omega \to X$ differentiable, we set

$$Df(s) = f'(s) \quad \text{and} \quad \Delta f(s) = sf'(s), \quad s \in \Omega.$$

For $n \in \mathbf{N}$, we denote by E_n the polynomial

$$E_n(s) = \prod_{k=0}^{n-1}\left(1 - \frac{s^2}{(2k+1)^2}\right),$$

and we put $E_0(s) = 1$. If $f \in C^{2n}$ then we put

$$E_n^D[f] = E_n(D)f \quad \text{and} \quad E_n^\Delta[f] = E_n(\Delta)f.$$

With these notations the proposed inversion formula takes the form:

THEOREM 1 *If $f : (0,\infty) \to X$ is bounded and continuous then, for all $s \in (0,\infty)$,*

$$\lim_{n\to\infty} E_n^\Delta[\mathcal{C}f](s) = f(s).$$

This theorem will be proven using Widder's results on the inversion of convolution transforms (see [15] and Theorem 2). This is possible because the operator $\mathcal{C}$ can be "translated" into a convolution transform in the following way:

If $f : (0,\infty) \to X$ is any function then, for $u \in \mathbf{R}$, put $\Gamma f(u) = f(e^u)$. If $f \in L_\infty((0,\infty), X)$ then

$$\begin{aligned} \Gamma(\mathcal{C}f)(s) &= \frac{2}{\pi}\int_0^\infty \frac{e^s}{e^{2s}+t^2} f(t)\,dt \\ &= \frac{2}{\pi}\int_{-\infty}^\infty \frac{e^{s-u}}{e^{2(s-u)}+1}\Gamma f(u)\,du \\ &= K * \Gamma f(s), \end{aligned}$$

where the convolutional kernel $K \in L_1(\mathbf{R})$ is given by

$$K(u) = \frac{2}{\pi}\frac{e^u}{e^{2u}+1}.$$

The convolution transform $g \mapsto K * g$ can be inverted by using the following theorem, which is a special case of [15, Chapter 7, Theorem 7].

THEOREM 2 *Let $K : \mathbf{R} \to \mathbf{R}$ be a measurable function with the following properties:*

(i) The bilateral Laplace transform of K converges in a strip symmetric about the imaginary axis.

(ii) $F(s) = \int_{-\infty}^{\infty} e^{-su} K(u)\, du$ *has no zeros in a strip* $|\Re(s)| < \sigma$, *and* $E(s) = F(s)^{-1}$ *can be written as*

$$E(s) = \prod_{k=0}^{\infty} \left(1 - \frac{s}{a_k}\right),$$

where the numbers $a_k \in \mathbf{R} \setminus \{0\}$ *are such that* $\lim_{n\to\infty} \sum_{k=0}^{n} 1/a_k = 0$ *and* $\sum_{k=0}^{\infty} 1/a_k^2 < \infty$.

If $g : \mathbf{R} \to \mathbf{R}$ *is bounded and continuous then* $K * g \in C^{\infty}(\mathbf{R})$, *and, for all* $s \in \mathbf{R}$,

$$\lim_{n\to\infty} \prod_{k=0}^{n} \left(1 - \frac{D}{a_k}\right) [K * g](s) = g(s).$$

We show next that the kernel $K(u) = 2\pi^{-1} e^{u} (e^{2u} + 1)^{-1}$ fulfills the assumptions of the foregoing theorem. The bilateral Laplace transform

$$F(s) = \int_{-\infty}^{\infty} e^{-su} K(u)\, du = \frac{2}{\pi} \int_{-\infty}^{\infty} e^{-su} \frac{e^{u}}{e^{2u} + 1}\, du$$

of K exists in the strip $|\Re(s)| < 1$, and, by substitution,

$$F(s) = \frac{2}{\pi} \int_{0}^{\infty} \frac{t^s}{1 + t^2}\, dt = \frac{1}{\cos(s\pi/2)}. \tag{3}$$

Hence F has no zeros in the strip $|\Re(s)| < 1$. Moreover, by [8, p.484], $E(s) = F(s)^{-1}$ can be written as

$$E(s) = \cos(s\pi/2) = \prod_{k=0}^{\infty} \left(1 - \frac{s^2}{(2k+1)^2}\right) = \prod_{k=0}^{\infty} \left(1 - \frac{s}{a_k}\right),$$

where $a_k = k + 1$ if k is even, and $a_k = -k$ if k is odd. Moreover,

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{a_k} = 0 \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{1}{a_k^2} < \infty.$$

Hence K fulfills the assumptions of Theorem 2. Since $E(s) = \lim_{n\to\infty} E_n(s)$ we can use Theorem 2 for the proof of the following proposition.

Proposition 3 *Let $g : \mathbf{R} \to X$ be bounded and continuous. Then $K * g \in C^\infty(\mathbf{R}, X)$ and, for all $s \in \mathbf{R}$,*

$$\lim_{n\to\infty} E_n^D[K * g](s) = g(s).$$

Proof. We consider first a real-valued bounded and continuous function $g : \mathbf{R} \to \mathbf{R}$. Since K satisfies the assumptions of Theorem 2 it follows that for all $s \in \mathbf{R}$,

$$\lim_{n\to\infty} E_n^D[K * g](s) = g(s). \tag{4}$$

In order to prove the conclusion for X-valued functions we make the following observations:

(a) Let $K_n = E_n^D[K]$, for $n = 0, 1, 2, \ldots$. By induction it can be proven easily that

$$K_n(u) = c_n \frac{e^{(2n+1)u}}{(e^{2u} + 1)^{2n+1}},$$

where c_n is a positive constant depending only on n. In particular, K_n is positive for all n.

(b) Let $\hat{K}_n$ denote the Fourier transform of K_n. Then, by (3),

$$\hat{K}_n(\omega) = \widehat{E_n^D[K]}(\omega) = E_n(i\omega)\hat{K}(\omega) = \frac{E_n(i\omega)}{\cos(i\omega\pi/2)}.$$

Consequently, $\int_{-\infty}^{\infty} K_n(t)\, dt = \hat{K}(0) = 1$. Since, by (a), K_n is positive we have $\|K_n\|_{L_1} = 1$.

(c) Since K_n belongs to $L_1(\mathbf{R})$ for all $n \in \mathbf{N}$ it follows that

$$E_n^D[K * g] = E_n^D[K] * g = K_n * g.$$

If $g : \mathbf{R} \to X$ is bounded and continuous then $K * g$ belongs to $C^\infty(\mathbf{R}, X)$. For $u, s \in \mathbf{R}$ define $\tau_s(u) = \|g(s) - g(s + u)\|$. Then $\tau_s : \mathbf{R} \to \mathbf{R}$ is bounded and continuous. So we may conclude from (a) - (c) together with (4) that

$$\begin{aligned}
&\limsup_{n\to\infty} \left\| g(s) - E_n^D[K * g](s) \right\| \\
= \;& \limsup_{n\to\infty} \left\| \int_{-\infty}^{\infty} K_n(u)(g(s) - g(s - u))\, du \right\| \\
\leq \;& \lim_{n\to\infty} \int_{-\infty}^{\infty} K_n(u)\tau_s(-u)\, dt \\
= \;& \lim_{n\to\infty} K_n * \tau_s(0) = \tau(0) = 0,
\end{aligned}$$

and the proof is complete. $\equiv$

In order to deduce Theorem 1 from Proposition 3 we note that $\Gamma(\Delta F) = D(\Gamma F)$ if $F \in C^1((0,\infty),X)$, and

$$\Gamma\left(E_n^\Delta[F]\right) = E_n^D[\Gamma F] \tag{5}$$

for $f \in C^{2n}((0,\infty),X)$.

Proof. (*of Theorem 1*) Let $f:(0,\infty)\to X$ be bounded and continuous. Then $F = \mathcal{C}f$ belongs to $C^\infty((0,\infty),X)$, and by (5),

$$\Gamma\left(E_n^\Delta[F]\right) = E_n^D[\Gamma F] = E_n^D[K * \Gamma f].$$

Since $\Gamma f : \mathbf{R} \to X$ is bounded and continuous we can apply Proposition 3 to Γf. Hence

$$\begin{aligned}
\lim_{n\to\infty} E_n^\Delta[F](s) &= \lim_{n\to\infty} \Gamma\left(E_n^\Delta[F]\right)(\log s) \\
&= \lim_{n\to\infty} E_n^D[K * \Gamma f](\log s) \\
&= \Gamma f(\log s) \\
&= f(s),
\end{aligned}$$

for all $s \in (0,\infty)$. $\equiv$

In the following section we need the injectivity of $\mathcal{C}$ on $L_\infty([0,\infty),X)$. Therefore, we prove the following corollary to Proposition 3.

Corollary 4 *Let $f \in L_\infty([0,\infty),X)$. If $\mathcal{C}f = 0$ then $f = 0$.*

Proof. Since $\Gamma : L_\infty([0,\infty),X) \to L_\infty(\mathbf{R},X)$ is an isometric isomorphism, and $\Gamma(\mathcal{C}f) = K * \Gamma f$ for $f \in L_\infty([0,\infty),X)$, it is sufficient to prove that $K * g = 0$ implies $g = 0$ for $g \in L_\infty(\mathbf{R},X)$. If $K * g = 0$ then, for all $h \in L_1(\mathbf{R})$,

$$0 = (K * g) * h = K * (g * h).$$

Since $g * h : \mathbf{R} \to X$ is bounded and continuous Proposition 3 implies that

$$0 = g * h(0) = \int_{-\infty}^{\infty} g(t)h(-t)\,dt, \quad \text{for all } h \in L_1(\mathbf{R}).$$

Consequently, $g = 0$. $\equiv$

The inversion formula in Theorem 1 is the key for a characterization of those function $F : (0,\infty) \to X$ which have a representation

$$F(s) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\,d\phi(t), \quad s \in (0,\infty),$$

where $\phi : [0,\infty) \to X$ is Lipschitz-continuous. Our next task is to state and prove such a characterization. To this end, we need some more notations, and we recall some facts about vector-valued Lipschitz-continuous functions, which may be found in [14, Chapter 1, Section 3].

For Lipschitz-continuous functions $\phi : [0,\infty) \to X$ we introduce the Lipschitz norm

$$\|\phi\|_{Lip} = \sup\Big\{\frac{\|\phi(s)-\phi(t)\|}{s-t} : 0 \le s < t < \infty\Big\}. \tag{6}$$

By $Lip([0,\infty),X)$ we denote the space of all Lipschitz-continuous functions $\phi : [0,\infty) \to X$ with $\phi(0) = 0$. The space $Lip([0,\infty),X)$ supplied with the norm defined in (6) is a Banach space. Moreover, we have the following proposition (see e.g. [14, Proposition 1.3.5]).

PROPOSITION 5 *The mapping which assigns to $\phi \in Lip([0,\infty),X)$ the operator $T_\phi : L_1([0,\infty)) \to X$ defined by*

$$T_\phi h = \int_0^\infty h(t)\,d\phi(t)$$

is an isometric isomorphism.

If $\psi : \Omega \to X$, $\Omega \subseteq \mathbf{R}$, is any function, and if $x^* \in X^*$, then $x^* \circ \psi$ stands for the scalar-valued function given by $x^* \circ \psi(t) = x^*(\psi(t))$, $t \in \Omega$.

THEOREM 6 *Let $F : (0,\infty) \to X$ be any function, and let M be a positive real number. Then the following two assertions are equivalent:*

(i) There exists $\phi \in Lip([0,\infty),X)$, with $\|\phi\|_{Lip} \le M$, such that, for all $s > 0$,

$$F(s) = \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\,d\phi(t). \tag{7}$$

(ii) $F \in C^\infty((0,\infty),X)$ *and*

$$\sup_{n\in\mathbf{N}\cup\{0\}} \left\| E_n^\Delta[F] \right\|_\infty \le M. \tag{8}$$

Proof. (i)⇒(ii) Let $\phi \in Lip([0,\infty),X)$ have Lipschitz norm equal to M. Then F defined by (7) belongs to $C^\infty((0,\infty),X)$. In order to prove (8) it is sufficient to show $\sup_{n\in\mathbf{N}} \left\| E_n^\Delta[x^* \circ F] \right\|_\infty \le M$ for all $x^* \in X^*$ with $\|x^*\| \le 1$. If $x^* \in X^*$ has norm less than or equal to one then $x^* \circ \phi$ is a scalar-valued Lipschitz-continuous function with $\|x^* \circ \phi\|_{Lip} \le M$. Hence, $x^* \circ \phi$ has a Radon-Nikodym derivative f_{x^*} with $\|f_{x^*}\|_\infty \le M$. Moreover, for $s \in (0,\infty)$,

$$\begin{aligned} x^* \circ F(s) &= \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2}\, d(x^*\circ\phi)(t) \\ &= \frac{2}{\pi}\int_0^\infty \frac{s}{s^2+t^2} f_{x^*}(t)\, dt \\ &= \mathcal{C} f_{x^*}(s). \end{aligned}$$

Therefore, we have to show $\left\| E_n^\Delta[\mathcal{C} f_{x^*}] \right\|_\infty \le M$. But by (5), this estimate is an immediate consequence of

$$\begin{aligned} \left\| E_n^D[\Gamma(\mathcal{C} f_{x^*})] \right\|_\infty &= \|K_n * \Gamma f_{x^*}\|_\infty \\ &\le \|K_n\|_{L_1} \|\Gamma f_{x^*}\|_\infty \\ &\le M. \end{aligned}$$

(ii)⇒(i) Let $F \in C^\infty((0,\infty),X)$ fulfill (8). Then for $n \in \mathbf{N}\cup\{0\}$, the operators $T_n : L_1([0,\infty)) \to X$ defined by

$$T_n h = \int_0^\infty h(t) E_n^\Delta[F](t)\, dt$$

have norm less than or equal to M. We claim that the family (T_n) converges pointwise to an operator $T : L_1([0,\infty)) \to X$ with $\|T\| \le M$. To see this we rewrite $T_n h$ in the following way:

$$\begin{aligned} T_n h &= \int_0^\infty h(t) E_n^\Delta[F](t)\, dt \\ &= \int_{-\infty}^\infty e^u h(e^u) E_n^\Delta[F](e^u)\, du \\ &= \int_{-\infty}^\infty \Gamma_1 h(u) E_n^D[\Gamma F](u)\, du, \end{aligned}$$

where $\Gamma_1 : L_1([0,\infty)) \to L_1(\mathbf{R})$ is given by $\Gamma_1 h(u) = e^u h(e^u)$. Since Γ_1 is an isometric isomorphism it is enough to show that the operators $S_n : L_1(\mathbf{R}) \to X$ given by $S_n g = \int_{-\infty}^{\infty} g(u) E_n^D[\Gamma F](u)\,du$ converge towards an operator $S : L_1(\mathbf{R}) \to X$. To see this, take $s \in \mathbf{R}$ and consider $K_s(u) = K(s-u)$. Then, by Proposition 3,

$$\begin{aligned}
\lim_{n\to\infty} S_n K_s &= \lim_{n\to\infty} \int_{-\infty}^{\infty} K(s-u) E_n^D[\Gamma F](u)\,du \\
&= \lim_{n\to\infty} K * E_n^D[\Gamma F](s) \\
&= \lim_{n\to\infty} K_n * \Gamma F(s) = \Gamma F(s).
\end{aligned}$$

Hence, $S_n g$ converges for all g in the subset $\kappa = \{K_s : s \in \mathbf{R}\} \subseteq L_1(\mathbf{R})$. We know from (3) that the Fourier transform of K has no zeros. Hence, by Wiener's Tauberian theorem [16, Theorem XI.16.3] it follows that κ is total in $L_1(\mathbf{R})$. In addition, the family (S_n) is bounded, since $\|S_n\| = \|T_n\| \leq M$. Hence, by the Banach-Steinhaus theorem, (S_n) converges pointwise to an operator $S : L_1(\mathbf{R}) \to X$. In particular, $SK_s = \Gamma F(s)$. Consequently, (T_n) converges pointwise to an operator $T : L_1([0,\infty)) \to X$ with $\|T\| \leq M$, and S and T are related by $Th = S(\Gamma_1 h)$.

Now, by Proposition 5, there exists $\phi \in Lip([0,\infty), X)$ with $\|\phi\|_{Lip} \leq \|T\| \leq M$, such that T has a representation

$$Th = \int_0^{\infty} h(t)\,d\phi(t), \quad h \in L_1([0,\infty)).$$

Let $k_s(t) = \dfrac{2}{\pi}\dfrac{s}{s^2+t^2}$. Then $\Gamma_1 k_s(u) = K_{\log s}(u)$. Consequently,

$$F(s) = \Gamma F(\log s) = SK_{\log s} = S(\Gamma_1 k_s) = Tk_s = \frac{2}{\pi}\int_0^{\infty} \frac{s}{s^2+t^2}\,d\phi(t),$$

and the proof is complete. $\equiv$

2. A characterization of uniformly bounded cosine functions

Let us first recall the following definitions: A mapping $T(\cdot) : (0,\infty) \to \mathbf{L}(X)$ has the semigroup property if

$$T(t+u) = T(t)T(u), \quad t, u > 0,$$

and $T(\cdot)$ is a C_0-semigroup if, in addition, $T(\cdot)$ is strongly continuous in $[0,\infty)$ and $T(0) = \mathrm{Id}$. A mapping $C(\cdot) : \mathbf{R} \to \mathbf{L}(X)$ satisfies the cosine functional equation if

$$C(t)C(u) = \frac{1}{2}\left[C(t+u) + C(t-u)\right], \quad t,u \in \mathbf{R}, \tag{9}$$

and $S(\cdot) : \mathbf{R} \to \mathbf{L}(X)$ fulfills the sine functional equation if S is strongly measurable with

$$S(t)S(u) = \frac{1}{2}\int_0^u \left[S(t+\sigma) + S(t-\sigma)\right] d\sigma, \quad t,u \in \mathbf{R}. \tag{10}$$

If, in addition to (9), $C(\cdot)$ is strongly continuous with $C(0) = \mathrm{Id}$ then $C(\cdot)$ is a cosine function. $S(\cdot)$ is a sine function if, in addition to (10), $S(\cdot)$ is non-degenerate, that is $S(t)x = 0$ for all $t \in \mathbf{R}$ implies $x = 0$.

If $C(\cdot)$ is a cosine function, then the generator A of $C(\cdot)$ is defined by

$$D(A) = \{x \in X : \; C(\cdot)x \in C^2(\mathbf{R}, X)\} \quad \text{and} \quad Ax = C''(0)x \text{ for } x \text{ in } D(A).$$

The generator A of a sine function $S(\cdot)$ is given by: x belongs to $D(A)$ if and only if there exists $y \in X$ such that, for all $\tau \in \mathbf{R}$,

$$S(\tau)x = \tau x + \int_0^\tau (\tau - \sigma)S(\sigma)y \, d\sigma. \tag{11}$$

In this case $Ax = y$. Note that y is uniquely determined by (11) since $S(\cdot)$ is non degenerate. In case we assume that the sine function is exponentially bounded, with densely defined generator, one can provide equivalent definitions using the Laplace transform (see [9], and [13]).

If $C : \mathbf{R} \to \mathbf{L}(X)$ is strongly continuous and even, and if $S : \mathbf{R} \to \mathbf{L}(X)$ is defined by

$$S(t) = \int_0^t C(\tau)\, d\tau, \quad t \in \mathbf{R},$$

then it follows by straightforward calculations that $C(\cdot)$ fulfills the cosine functional equation if and only if $S(\cdot)$ fulfills the sine functional equation. Consequently, $C(\cdot)$ is a cosine function if and only if $S(\cdot)$ is a sine function. In this case the generators of $C(\cdot)$ and $S(\cdot)$ are the same.

Let A be the generator of a bounded C_0-semigroup $T(\cdot)$. Then, by [16, Chapter IX.11] (see also the introduction), we can define $B = -(-A)^{1/2}$, and

B is the generator of a bounded C_0-semigroup $T_B(\cdot)$. If A generates a cosine function $C(\cdot)$ then we have the fundamental connection (see Introduction)

$$T_B(t) = \frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2} C(\tau)\,d\tau, \tag{12}$$

and if $S(\cdot)$ is the sine function generated by A then

$$T_B(t) = \frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2}\,dS(\tau). \tag{13}$$

Unless otherwise stated, integrals involving operator-valued functions will be understood in the strong operator topology henceforth. Our main goal in this section is to show that the converse of the above assertion holds; more precisely,

Theorem 7 *Let A be the generator of a bounded C_0-semigroup and let $T_B(\cdot)$ be the C_0-semigroup generated by $B = -(-A)^{1/2}$. Then A generates a bounded cosine function if and only if there exists a strongly Lipschitz-continuous function $S(\cdot) : [0,\infty) \to \mathbf{L}(X)$ such that*

$$T_B(t) = \frac{2}{\pi}\int_0^\infty \frac{t}{t^2+\tau^2}\,dS(\tau), \quad t > 0. \tag{14}$$

Before we prove Theorem 7 we need a couple of lemmas and propositions, and we make a few remarks.

Remark 8 (i) If $F : \mathbf{R} \to \mathbf{L}(X)$ is a strongly Lipschitz continuous function then, as a consequence of the uniform boundedness principle, F is Lipschitz continuous with respect to the operator norm. Therefore, it is enough to prove Theorem 7 for Lipschitz continuous sine functions.

(ii) If the densely defined operator A generates a Lipschitz-continuous sine function $S(\cdot)$ then A generates a bounded strongly continuous analytic semigroup $T(\cdot)$ given by

$$T(t)x = \frac{1}{2\sqrt{\pi}t^{3/2}}\int_0^\infty e^{-\tau^2/4t}\tau S(\tau)x d\tau \tag{15}$$

(see Arendt-Kellermann [3]). If we proceed as in the introduction, we find that the semigroup $T_B(\cdot)$ generated by the negative square root B of A has the representation:

$$T_B(t) = \frac{4}{\pi}\int_0^\infty \frac{\tau t}{(t^2+\tau^2)^2}\, S(\tau)d\tau. \tag{16}$$

It is well-known that there are operators that generate sine functions but do not generate cosine functions (see [3], [9] and [5]). Proposition 10 below states that the semigroup property corresponds to the cosine functional equation via (12) and to the sine functional equation via (13).

(iii) In the case where X has the Radon-Nikodym property (see [14] or [1]), the assumption on $S(\cdot)$ implies the existence of a derivative $S'(\cdot) = C(\cdot)$ which is bounded. The cosine functional equation for $C(\cdot)$ combined with strong measurability imply that $C(\cdot)$ is strongly continuous (see [6, Theorem 1.1, p. 24] or [12]; these results extend the corresponding facts for the semigroup functional equation [10]).

For our further investigations it is useful to introduce the Poisson kernels

$$P_s(\sigma) = \frac{1}{\pi}\frac{s}{s^2+\sigma^2}, \quad s > 0,\ \sigma \in \mathbf{R}.$$

We note that the family (P_s) has the following semigroup property

$$P_s * P_t = P_{s+t}, \quad s,t > 0. \tag{17}$$

If f is bounded and measurable on $\mathbf{R}$ then we let

$$\mathcal{P}f(t) = \int_{-\infty}^\infty P_t(\tau)f(\tau)\,d\tau, \quad t \in \mathbf{R}.$$

We note that $\mathcal{P}f = 0$ implies $f = 0$ if $f \in L_\infty(\mathbf{R}, X)$ is even. This follows from Corollary 4 since for even functions $f \in L_\infty(\mathbf{R}, X)$

$$\mathcal{P}f(t) = 2\int_0^\infty P_t(\tau)f(\tau)\,d\tau = (\mathcal{C}f_{|[0,\infty)})(t).$$

In the sequel we write $Q_t = -P_t'$.

Lemma 9 *If $f : \mathbf{R} \to X$ is odd and Lipschitz-continuous, and if*

$$\int_{-\infty}^\infty Q_t(\tau)f(\tau)\,d\tau = 0, \quad \textit{for all } t > 0 \tag{18}$$

then $f(\tau) = 0$ for all $\tau \in \mathbf{R}$.

Proof. It is enough to prove the lemma for scalar-valued functions. Then the vector-valued case follows by applying the Hahn-Banach theorem. Let f be an odd, scalar-valued Lipschitz-continuous function with (18). Then f has an even, bounded Radon Nikodym derivative f'. By partial integration it follows that

$$0 = \int_{-\infty}^{\infty} Q_t(\tau) f(\tau)\, d\tau = \int_{-\infty}^{\infty} P_t(\tau) f'(\tau)\, d\tau.$$

Since the operator $\mathcal{P}$ is injective on even functions we conclude that $f' = 0$. Consequently, f is constant. But a constant function which is odd must be 0. $\equiv$

PROPOSITION 10 *Let $T(\cdot) : [0, \infty) \to \mathbf{L}(X)$ be bounded and strongly continuous.*

(i) If $C(\cdot) : \mathbf{R} \to \mathbf{L}(X)$ is bounded, strongly continuous and even, and if $C(\cdot)$ and $T(\cdot)$ are related by

$$T(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + \tau^2} C(\tau)\, d\tau, \quad t > 0,$$

then $T(\cdot)$ has the semigroup property if and only if $C(\cdot)$ fulfills the cosine functional equation. Moreover, $T(0) = C(0)$.

(ii) If $S(\cdot) : \mathbf{R} \to \mathbf{L}(X)$ is strongly Lipschitz-continuous and odd, and if $S(\cdot)$ and $T(\cdot)$ are connected by

$$T(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + \tau^2}\, dS(\tau), \quad t > 0,$$

then $T(\cdot)$ has the semigroup property if and only if $S(\cdot)$ fulfills the sine functional equation.

Proof. We first prove (ii).

(ii) By partial integration it follows that

$$T(t) = \int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau) = \int_{-\infty}^{\infty} Q_t(\tau) S(\tau)\, d\tau.$$

Consequently,

$$T(s)T(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_s(\sigma)Q_t(\tau)\, S(\sigma)S(\tau)\, d\tau d\sigma$$

The semigroup property of the Poisson kernels gives

$$Q_{s+t}(\tau) = -\frac{d}{d\tau}P_{s+t}(\tau) = -\frac{d}{d\tau}(P_s * P_t)(\tau) = Q_s * P_t(\tau).$$

Since S and Q_s are odd it follows that

$$\begin{aligned} T(s+t) &= \int_{-\infty}^{\infty} Q_{s+t}(\rho)S(\rho)\, d\rho \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_s(\rho-\tau)P_t(\tau)\, d\tau\, S(\rho)\, d\rho \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_s(\sigma)P_t(\tau)S(\sigma+\tau)\, d\tau d\sigma \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_s(\sigma)P_t(\tau)\,\frac{1}{2}\,[S(\sigma+\tau)+S(\sigma-\tau)]\; d\tau d\sigma. \end{aligned}$$

Integrating the right hand side of the above equation by parts gives

$$T(s+t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_s(\sigma)Q_t(\tau)\left(\frac{1}{2}\int_0^{\tau}[S(\sigma+\rho)+S(\sigma-\rho)]\, d\rho\right)\, d\tau d\sigma.$$

If $S(\cdot)$ fulfills the sine functional equation then it follows directly that $T(\cdot)$ has the semigroup property in $(0,\infty)$. That $T(\cdot)$ has the semigroup property in the closed interval $[0,\infty)$ follows from the strong continuity of $T(\cdot)$.

Conversely, if $T(\cdot)$ has the semigroup property then we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_s(\sigma)Q_t(\tau)\, S(\sigma)S(\tau)\, d\tau d\sigma \\ =\; &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_s(\sigma)Q_t(\tau)\left(\frac{1}{2}\int_0^{\tau}[S(\sigma+\rho)+S(\sigma-\rho)]d\rho\right)\, d\tau d\sigma, \end{aligned}$$

for all $s,t \geq 0$. Since the functions

$$(\sigma,\tau) \mapsto \int_0^{\tau}[S(\sigma+\rho)+S(\sigma-\rho)]d\rho \quad \text{and} \quad (\sigma,\tau)\mapsto S(\sigma)S(\tau)$$

are odd in σ for τ fixed, and in τ for σ fixed, it follows from Lemma 9 that

$$S(\sigma)S(\tau) = \frac{1}{2}\int_0^{\tau}[S(\sigma+\rho)+S(\sigma-\rho)]\, d\rho,$$

whence $S(\cdot)$ fulfills the sine functional equation.

(i) Define $S(\cdot) : \mathbf{R} \to \mathbf{L}(X)$ by

$$S(t) = \int_0^t C(\tau)\, d\tau.$$

Since $C(\cdot)$ is even it follows that $S(\cdot)$ fulfills the sine functional equation if and only $C(\cdot)$ fulfills the cosine functional equation. Moreover,

$$T(t) = \int_{-\infty}^{\infty} P_t(\tau)C(\tau)\, d\tau = \int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau).$$

Hence it follows from (ii) that $C(\cdot)$ fulfills the cosine functional equation if and only if $T(\cdot)$ has the semigroup property.

Moreover, since the family of Poisson kernels (P_t) is an approximate identity it follows that

$$T(0) = \lim_{t\to 0^+} T(t) = \lim_{t\to 0^+} \int_{-\infty}^{\infty} P_t(\tau)C(\tau)\, d\tau = C(0).$$

$\equiv$

If A generates an integrated semigroup $U(\cdot)$ then, for all $x \in X$ and $\tau > 0$,

$$\int_0^\tau U(\sigma)x\, d\sigma \in D(A), \quad \text{and} \quad U(\tau)x = \tau x + A\int_0^\tau U(\sigma)x\, d\sigma$$

(see Arendt [1, Proposition 3.3]). If we consider sine functions instead of integrated semigroups then, by Arendt [2], we obtain the following result.

Lemma 11 *Let $S(\cdot)$ be a sine function with generator A. Then, for all $x \in X$ and $\tau \in \mathbf{R}$,*

$$\int_0^\tau (\tau-\sigma)S(\sigma)x\, d\sigma \in D(A), \ \text{and} \ S(\tau)x = \tau x + A\int_0^\tau (\tau-\sigma)S(\sigma)x\, d\sigma. \quad (19)$$

Proof. Let $\tau \in \mathbf{R}$, $x \in X$ and set $x_\tau = \int_0^\tau (\tau - \sigma)S(\sigma)x\, d\sigma$. Then

$$\begin{aligned} S(t)x_\tau &= S(t)\int_0^\tau (\tau-\sigma)S(\sigma)x\, d\sigma \\ &= \frac{1}{2}\int_0^\tau (\tau-\sigma)\int_0^\sigma [S(t+\rho)+S(t-\rho)]x\, d\rho d\sigma \\ &= \frac{1}{2}\int_0^\tau (\tau-\sigma)\left[\int_t^{t+\sigma} S(\rho)x\, d\rho - \int_t^{t-\sigma} S(\rho)x\, d\rho\right] d\sigma \\ &= \frac{1}{2}\int_0^\tau (\tau-\sigma)\int_{t-\sigma}^{t+\sigma} S(\rho)x\, d\rho d\sigma. \end{aligned}$$

It follows that

$$\frac{d}{dt}S(t)x_\tau = \frac{1}{2}\int_0^\tau (\tau-\sigma)[S(t+\sigma)-S(t-\sigma)]x\,d\sigma. \tag{20}$$

In particular, $S'(0)x_\tau = x_\tau$. From (20) we infer

$$\frac{d}{dt}S(t)x_\tau = \frac{1}{2}\int_t^{t+\tau}(\tau+t-\sigma)S(\sigma)x\,d\sigma + \frac{1}{2}\int_t^{t-\tau}(\tau-t+\sigma)S(\sigma)x\,d\sigma,$$

whence

$$\begin{aligned}\frac{d^2}{dt^2}S(t)x_\tau &= \frac{1}{2}\left[\int_t^{t+\tau}S(\sigma)x\,d\sigma - \tau S(t)x - \int_t^{t-\tau}S(\sigma)\,d\sigma - \tau S(t)x\right] \\ &= \frac{1}{2}\int_{t-\tau}^{t+\tau}S(\sigma)x\,d\sigma - \tau S(t)x = [S(\tau)-\tau]S(t)x.\end{aligned}$$

Therefore,

$$\begin{aligned}S(t)x_\tau &= \int_0^t (t-\sigma)S''(\sigma)x_\tau\,d\sigma + tS'(0)x_\tau + S(0)x_\tau \\ &= tx_\tau + \int_0^t (t-\sigma)S(\sigma)[S(\tau)-\tau]x\,d\sigma.\end{aligned}$$

Consequently, $x_\tau \in D(A)$ and $Ax_\tau = S(\tau)x - \tau x$. $\equiv$

PROPOSITION 12 *Let B generate a C_0-semigroup $T_B(\cdot)$ on X and let A be the generator of a strongly Lipschitz-continuous sine function $S(\cdot)$. If*

$$T_B(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{\tau^2+t^2}\,dS(\tau), \quad t>0, \tag{21}$$

then $B^2 = -A$.

Proof. T_B is infinitely often differentiable in $t>0$; this follows easily from the representation (21) (actually, $T_B(\cdot)$ is analytic). Hence $T_B(t)x$ belongs to $D(B^n)$ for all $t>0$, $x\in X$, $n\in\mathbf{N}$, and

$$B^nT_B(t)x = \frac{d^n}{dt^n}T_B(t)x.$$

In order to prove that $B^2 = -A$, we use integration by parts combined with the estimates $\|S(\tau)x\| \leq M\tau \,\|x\|$ and $\|\int_0^\tau S(\rho)x\, d\rho\| \leq M\tau^2 \,\|x\|$ for some number $M > 0$, and the fundamental formula (Lemma 11, 19) for sine function generators:

$$\begin{aligned}
B^2 T_B(t)x &= \frac{d^2}{dt^2} T_B(t)x = \int_{-\infty}^{\infty} \frac{d^2}{dt^2} P_t(\tau)\, dS(\tau)x \\
&= \int_{-\infty}^{\infty} -\frac{d^2}{d\tau^2} P_t(\tau)\, d\left(\tau x + A\int_0^\tau (\tau-\sigma)S(\sigma)x\, d\sigma\right) \\
&= -A\int_{-\infty}^{\infty} \frac{d^2}{d\tau^2} P_t(\tau) \left(\int_0^\tau S(\sigma)x\, d\sigma\right) d\tau \\
&= A\int_{-\infty}^{\infty} \frac{d}{d\tau} P_t(\tau) S(\tau)x\, d\tau \\
&= -A\int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau)x \\
&= -AT_B(t)x.
\end{aligned}$$

Let $x \in D(B^2)$. Then

$$\lim_{t\to 0^+} -AT_B(t)x = \lim_{t\to 0^+} B^2 T_B(t)x = \lim_{t\to 0^+} T_B(t)B^2 x = B^2 x.$$

Since A is closed and $\lim_{t\to 0^+} T_B(t)x = x$ it follows that $x \in D(A)$ and $-Ax = B^2 x$. Conversely, if $x \in D(A)$ then

$$\begin{aligned}
\lim_{t\to 0^+} B^2 T_B(t)x &= -\lim_{t\to 0^+} AT_B(t)x = -\lim_{t\to 0^+} A\int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau)x \\
&= -\lim_{t\to 0^+} \int_{-\infty}^{\infty} P_t(\tau)\, dS(\tau)Ax = -\lim_{t\to 0^+} T_B(t)Ax = -Ax.
\end{aligned}$$

Consequently, by the closedness of B^2 implies that $x \in D(B^2)$ and $B^2 x = -Ax$. $\equiv$

Now we are in the position to prove the main theorem (Theorem 7).

Proof. (*of Theorem 7*) Assume first that A generates a bounded cosine function $C(\cdot)$. Then A is the generator of a sine function $S(\cdot)$ which is given by

$$S(t) = \int_0^t C(\tau)\, d\tau.$$

Hence, since $C(\cdot)$ is bounded $S(\cdot)$ is Lipschitz-continuous, and (14) follows from (12).

Conversely, assume that there exists a Lipschitz-continuous function $S(\cdot) : [0,\infty) \to \mathbf{L}(X)$ such that $T_B(\cdot)$ and $S(\cdot)$ are connected by (14). We may assume without loss of generality that $S(0) = 0$. Then $S(\cdot)$ can be extended to an odd, strongly Lipschitz-continuous function $S : \mathbf{R} \to \mathbf{L}(X)$ by putting $S(t) = -S(-t)$ for $t < 0$. Then

$$T_B(t) = \frac{2}{\pi}\int_0^\infty P_t(\tau)\,dS(\tau) = \frac{1}{\pi}\int_{-\infty}^\infty P_t(\tau)\,dS(\tau).$$

Therefore, Proposition 10 implies that $S(\cdot)$ fulfills the sine functional equation. Moreover, if $S(t)x = 0$ for all $t \in \mathbf{R}$ then it follows from (14) that $T(t)x = 0$ for all $t > 0$, whence $x = 0$. Consequently, $S(\cdot)$ is a sine function, which, by Proposition 12, is generated by $-B^2 = A$.

It remains to show that $S(\cdot)$ has a strong derivative $C(\cdot)$. Let $x \in D(A)$. Then

$$S(t)x = tx + \int_0^t (t-\tau)S(\tau)Ax\,d\tau.$$

Hence $S(t)x$ is continuously differentiable and we can define

$$\Phi(x)(t) = S'(t)x = x + \int_0^t S(\tau)Ax\,d\tau, \quad t \in \mathbf{R}.$$

Since $S(\cdot)$ is Lipschitz-continuous we have

$$\|\Phi(x)\|_\infty = \|S(\cdot)x\|_{Lip} \le \|S(\cdot)\|_{Lip}\,\|x\|\,. \tag{22}$$

Hence $\Phi : D(A) \to C_b(\mathbf{R}, X)$ is a bounded linear operator. Consequently, Φ has a unique bounded linear extension to $\overline{D(A)} = X$. Define $C(t)x = \Phi(x)(t)$. Then, for every $t \in \mathbf{R}$,

$$\sup_{\|x\|\le 1} \|C(t)x\| \le \|S(\cdot)\|_{Lip}\,.$$

Hence, $C(t) \in \mathbf{L}(X)$ for each $t \in \mathbf{R}$, and $C(\cdot) : \mathbf{R} \to \mathbf{L}(X)$ is bounded and strongly continuous. Moreover, $C(\cdot)$ is a cosine function, since $S(\cdot)$ is a sine function, and $C(\cdot)$ is generated by A since $S(\cdot)$ is generated by A. $\equiv$

Combining Theorem 1, Theorem 6 and Theorem 7 we obtain the following

Corollary 13 *Let A be the generator of a bounded C_0-semigroup, and let $B = -(-A)^{1/2}$ generate the semigroup $T_B(\cdot)$. Then A generates a bounded cosine function if and only if there exists $M > 0$ such that*

$$\left\| E_n^{\Delta}[T_B](t) \right\| \leq M \quad \textit{for all } n = 0, 1, 2, \ldots \textit{ and } t > 0.$$

In this case, the cosine function $C(\cdot)$ generated by A is given by

$$C(t)x = C(-t)x = \lim_{n\to\infty} E_n^{\Delta}[T_B](t)x, \quad t \geq 0,\ x \in X.$$

We now provide an explicit description of $E_n^{\Delta}[T_B](t)$. We claim first that $E_n^{\Delta}[T_B](t) = p_n(tB)T_B(t)$, where p_n is a polynomial of degree $2n$. This statement is certainly true for $n = 0$, with $p_0(t) = 1$. For any polynomial p let us define $(\Phi p)(t) = t[p(t) + p'(t)]$. If the statement holds for $n > 0$ then

$$\begin{aligned} \Delta E_n^{\Delta}[T_B](t) &= \Delta p_n(tB)T_B(T) \\ &= t[Bp_n'(tB)T_B(t) + p_n(tB)BT_B(t)] = (\Phi p_n)(tB)T_B(t). \end{aligned}$$

Consequently, $E_{n+1}^{\Delta}[T_B](t) = p_{n+1}(tB)T_B(t)$, where

$$p_{n+1} = \left(1 - \frac{\Phi^2}{(2n+1)^2}\right) p_n = \prod_{k=0}^{n} \left(1 - \frac{\Phi^2}{(2k+1)^2}\right) p_0 = E_n^{\Phi}[p_0]$$

is a polynomial of degree $2n + 2 = 2(n+1)$.

Secondly, we describe the p_n's explicitly. Let $p_n(t) = a_{2n}t^{2n} + a_{2n-1}t^{2n-1} + \ldots + a_1 t + a_0$. The polynomial p_n is uniquely determined by the equation

$$E_n^{\Delta}[e^t] = p_n(t)e^t = \sum_{j=0}^{2n} a_j t^j \cdot \sum_{l=0}^{\infty} \frac{t^l}{l!} = \sum_{l=0}^{\infty} b_l t^l, \tag{23}$$

where $b_l = \sum_{j=0}^{\min(l,2n)} a_j/(l-j)!$. On the other hand, since $\Delta(t^l) = lt^l$ we have

$$E_n^{\Delta}[e^t] = \sum_{l=0}^{\infty} \frac{E_n^{\Delta}[t^l]}{l!} = \sum_{l=0}^{\infty} \frac{t^l}{l!} \prod_{k=0}^{n-1} \left(1 - \frac{l^2}{(2k+1)^2}\right) = \sum_{l=0}^{\infty} c_l t^l, \tag{24}$$

where $c_l = E_n(l)/l!$. Combining (23) and (24) we have

$$\sum_{j=0}^{l} \frac{a_j}{(l-j)!} = c_l, \quad l = 0, 1, \ldots, 2n. \tag{25}$$

Let $\alpha = (a_0, \ldots, a_{2n})$, $\gamma = (c_0, \ldots, c_{2n})$. Then (25) may be written as $\mathcal{A}\alpha = \gamma$, where $\mathcal{A}$ is the matrix given by

$$\mathcal{A} = \left(\begin{array}{cccccccc} 1 & 0 & \cdots & \cdot & \cdot & \cdots & \cdot & 0 \\ 1 & 1 & 0 & \cdot & \cdot & \cdots & \cdot & 0 \\ 1/2 & 1 & 1 & 0 & \cdot & \cdots & \cdot & 0 \\ 1/6 & 1/2 & 1 & 1 & 0 & \cdots & \cdot & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & 1 & 1 & 0 & \cdot \\ \cdot & & & & & 1 & 1 & 0 \\ 1/(2n)! & \cdot & \cdot & \cdot & \cdot & \cdots & 1 & 1 \end{array} \right).$$

Consequently, $\alpha = \mathcal{A}^{-1}\gamma$, where

$$\mathcal{A}^{-1} = \left(\begin{array}{cccccccc} 1 & 0 & \cdots & \cdot & \cdot & \cdots & \cdot & 0 \\ -1 & 1 & 0 & \cdot & \cdot & \cdots & \cdot & 0 \\ 1/2 & -1 & 1 & 0 & \cdot & \cdots & \cdot & 0 \\ -1/6 & 1/2 & -1 & 1 & 0 & \cdots & \cdot & 0 \\ \cdot & & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & -1 & 1 & 0 & \cdot \\ \cdot & & & & & -1 & 1 & 0 \\ 1/(2n)! & \cdot & \cdot & \cdot & \cdot & \cdots & -1 & 1 \end{array} \right).$$

Since $c_1 = c_3 = \ldots = c_{2n-1} = 0$ we obtain the following representation of $E_n^{\Delta}[T_B](t)$:

Proposition 14 *If $T_B(\cdot)$ is a differentiable semigroup which is generated by B, then*

$$E_n^{\Delta}[T_B](t) = \left[a_{2n}(tB)^{2n} + \ldots + a_1(tB) + a_0\right] T_B(t),$$

where

$$a_k = \frac{1}{k!} \sum_{l=0}^{[k/2]} \left[(-1)^k \binom{k}{2l} \prod_{j=0}^{n-1} \left(1 - \frac{(2l)^2}{(2j+1)^2} \right) \right], \quad k = 0, 1, \ldots, 2n,$$

and $[k/2]$ denotes the greatest nonnegative integer not exceeding $k/2$.

Finally, if we consider the Laplace operator on one of the spaces $L^p(\mathbf{R})$ $1 \leq p < \infty$, $C_0(\mathbf{R})$ or $BUC(\mathbf{R})$ (with maximal distributional domain for $L^p(\mathbf{R})$, $1 \leq p < \infty$), then the semigroup $T_B(\cdot)$ corresponds to the classical Poisson transform for which an inversion theory has been carried out in [14].

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