Characterization of operators of positive scalar type

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Abstract

Let X be a Banach lattice. Necessary and sufficient conditions for a linear operator $A : D(A) \to X$, $D(A) \subseteq X$, to be of positive C^0-scalar type are given. In addition, the question is discussed which conditions on the Banach lattice imply that every operator of positive C^0-scalar type is necessarily of positive scalar type.

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0 Introduction

A linear operator A on a Banach space X is a scalar-type-operator (or equivalently of scalar-type) on $[0, \infty)$ if $\sigma(A) \subseteq [0, \infty)$ and if there exists a spectral measure E on the Borel subsets of $[0, \infty)$ such that

$$D(A) = \{x \in X : \lim_{n\to\infty} \int_0^n \lambda\, E(d\lambda)x \text{ exists}\}$$

and

$$Ax = \lim_{n\to\infty} \int_0^n \lambda\, E(d\lambda), \quad x \in D(A).$$

If X is Banach lattice then we say that A is of positive scalar type on $[0, \infty)$ if A is of scalar type on $[0, \infty)$ with positive spectral measure E, i.e. $E(M)$ is a positive projection for all Borel measurable subsets $M \subseteq [0, \infty)$.

We refer the reader to [4] for a brief introduction into the history and the importance of scalar-type operators. Operators of positive scalar type where studied e.g. in [1]. In addition, it was shown in [2], if A is an operator of scalar type on a cyclic Banach space X then there exists an ordering and an equivalent norm on X, such that X becomes a Banach lattice, and such that X is of positive scalar type on X.

By $C_0[0, \infty)$ we denote the space of complex-valued functions on $[0, \infty)$ vanishing at infinity, and $\mathbf{L}(X)$ denotes the space of linear bounded operators on X. If A is of scalar type on $[0, \infty)$ with spectral measure E then there exists a bounded algebra homomorphism $T\Phi : C_0[0, \infty) \to \mathbf{L}(X)$ given by

$$\Phi(f)x = \int_0^\infty \lambda \, E(d\lambda)x.$$

If we denote by ρ_s the function $\rho_s(t) = 1/(s + t)$ then $\Phi(\rho_s) = (s + A)^{-1}$ for every $s > 0$. Operators for which such an algebra homomorphism exists are called C^0-scalar-type operators on $[0, \infty)$. If X is a Banach lattice and if Φ is a positive algebra homomorphism then A is said to be of positive C^0-scalar type.

We note that every operator of (positive) scalar type is of (positive) C^0-scalar type where the algebra homomorphism Φ is given by

$$\Phi(f) = \int_0^\infty f(t) \, E(dt), \quad f \in C_0[0, \infty).$$

Conversely, if a scalar-type-operator A is of positive C^0-scalar type then A automatically is of positive scalar type.

The notion of scalar type and C^0-scalar type operators lead to the following problems:

(I) Find conditions on A (or on the resolvent of A or the semigroup generated by A) which are necessary and sufficient for A being of C^0-scalar type.

(II) In which Banach spaces are all C^0-scalar type operators of scalar type?

These two problems were discussed in detail in [6]. In this note we focus on the characterization of operators of positive scalar type.

1 Operators of positive scalar type

We study now operators acting on a Banach lattice X. For elementary properties of positive operators in Banach lattices we refer the reader to [5]. Recall that the linear operator A on X is of positive scalar type on $[0,\infty)$ if A is of scalar type on $[0,\infty)$ with positive spectral measure E, and that A is of positive C^0-scalar type if A is of C^0-scalar type with positive algebra homomorphism.

We pose the following two problems:

(I+) Find conditions on A (or on the resolvent of A or the semigroup generated by A) which are necessary and sufficient for A being of positive C^0-scalar type.

(II+) In which Banach lattices are all operators of positive C^0-scalar type operator of positive scalar type?

The answer to problem (I+) is the following

THEOREM 1. *Let A be a densely defined operator on X with $\sigma(A) \subseteq [0,\infty)$. Then the following assertions are equivalent:*

(a) *A is of positive C^0-scalar type on $[0,\infty)$.*

(b) *$t(t+A)^{-1}$ is uniformly bounded, $(t+A)^{-1} \geq 0$ and $A^k(t+A)^{-2k} \geq 0$ for $k = 1,2,\ldots$ and $t > 0$.*

(c) *$A^k(1+A)^{-(k+n)} \geq 0$ for $k,n = 0,1,2,\ldots$.*

(d) $-A$ *generates a* C_0*-semigroup* $(U(t))_{t\geq 0}$ *such that* $U(t)X \subseteq D(A)$ *for every* $t > 0$*, and* $A^k U(t) \geq 0$ *for* $k = 0, 1, 2, \ldots$ *and* $t > 0$*.*

The proof of Theorem 1 is an easy combination of the following ingredients:

(a) The proof of the characterization of C^0-scalar type-operators given in [6], Theorem 6.

(b) A characterization of Stieltjes transforms of positive measures [7], Theorem VIII.17c., and its application to the resolvent $(\lambda + A)^{-1}$, $\lambda > 0$..

(c) A description of completely monotonic sequences [7], Theorem III.4a., and its application to the sequence $((1 + A)^{-n})_{n=0,1,2,\ldots}$.

(d) Bernstein's theorem on the characterization of Laplace transforms of positive measures, and its application to the semigroup generated by $-A$.

A partly answer to the second problem (II+) will follow from the solution of problem (II) given in [6], which we recall now. Assume A to be of scalar type on $[0, \infty)$ with spectral measure E. Then A is of C^0-scalar type on $[0, \infty)$ with corresponding algebra homomorphism Φ given by

$$\Phi f(x) = \int_0^\infty f(t)\, E(dt)x.$$

If we denote by $\Phi[x] : C_0[0, \infty) \to X$ the operator $\Phi[x]f = \Phi f(x)$, and if we define $E[x]$ to be the vector measure defined by $E[x](E) = \mu(E)x$, then the" components" $\Phi[x]$ of Φ can be represented by the countably additive vector measure $E[x]$, i.e.

$$\Phi[x]f = \int_0^\infty f(t) E[x](dt).$$

In [6] it is shown that the converse is also true, i.e. if A is a linear operator on X with $(-\infty, 0) \subseteq \rho(A)$, then A is of scalar type on $[0, \infty)$ if and only if A is of C^0-scalar type on $[0, \infty)$ and the components $\Phi[x]$ of the corresponding algebra homomorphism Φ can be represented by a countably additive vector measure for all $x \in X$. The following theorem is an immediate consequence of the foregoing considerations.

THEOREM 2 *The operator A on X is of positive scalar type on $[0, \infty)$ if and only if A is of positive C^0-scalar type on $[0, \infty)$ with corresponding algebra homomorphism Φ and, for all $x \in X_+$, the, necessarily positive, components $\Phi[x]$ of Φ can be represented by a, necessarily positive, countably additive vector measure.*

It is well known that every operator $T : C_0[0, \infty) \to X$ has a representation by a countably additive vector measure if and only if X does not contain an isomorphic copy of the space of complex valued null sequences c_0 [6]. Hence, by the statement preceding Theorem 2, if X does not contain an isomorphic copy of c_0 then every C^0-scalar type-operator on X is of scalar type on X. Moreover, Doust [3] showed that if c_0 is contained in X then there exists an operator of C^0-scalar type on X which is not of scalar type.

Since we are interested in a characterization of those Banach lattices X in which every operator of positive C^0-scalar type is of positive scalar type the following two open problems should be solved:

(III+) Give a characterization of those Banach lattices X such that every positive operator $T : C_0[0, \infty) \to X$ can be represented by a, necessarily positive, countably additive vector measure.

(IV+) If X is a Banach lattice with the property that not every positive operator $T : C_0[0, \infty) \to X$ can be represented by a countably additive vector measure is it then possible to construct an operator A on X which is of positive C^0-scalar type on $[0, \infty)$, but which is not of positive scalar type on $[0, \infty)$?

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