

TOYING WITH JORDAN MATRICES

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It is shown that an important resolvent estimate is instable under small perturbations.

In many applications operators T are of interest which fulfill the condition: the resolvent set $\rho(T)$ contains the negative reals, and there is a constant $\gamma > 0$, such that for all positive reals s the "resolvent estimate" holds

$$(res) \quad \|(sI + T)^{-1}\| \leq \frac{\gamma}{s}.$$

In the theory of semigroups (see e.g. J.A. GOLDSTEIN [1, p. 20]) or in the theory of evolutionary integral equations (J. PRUESS [4, p. 69]) this condition is well known. In the investigation of regularization methods for ill-posed equations in Banach spaces the condition (res) is necessary for the convergence of the so-called Lavrentiev regularization (see R. PLATO [3], E. SCHOCK/V. PHONG [6], SPECKERT [7]).

The following simple example will show that the condition (res) is unstable under arbitrary small perturbations. We will construct operators J built up with Jordan matrices which act on block subspaces of ℓ_p .

By $J_n(\lambda)$ we denote an $(n \times n)$ -Jordan matrix with the eigenvalue λ and with 1's in the upper diagonal. The following Lemma is well known.

Lemma 1. *Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then*

$$J_n(\lambda)^{-1} = \begin{pmatrix} \lambda^{-1} & -\lambda^{-2} & \dots & (-1)^{n-1} \lambda^{-n} & \\ & \lambda^{-1} & \dots & (-1)^{n-2} \lambda^{-n+1} & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \lambda^{-1} \end{pmatrix}$$

Lemma 2. *In $X = \ell_p^n = (\mathbb{C}^n, \|\cdot\|_p)$ for $0 < \lambda \leq 1$ holds*

$$\|J_n(\lambda)^{-1}\| \geq \lambda^{-n}.$$

Proof. Let e_n be the n^{th} unit vector, then

$$\|J_n(\lambda)^{-1}e_n\|_p^p = \sum_{k=1}^n \lambda^{-pk} \geq \lambda^{-pn}.$$

□

Let $D_n(\lambda)$ be the $(n \times n)$ -diagonal matrix with the eigenvalue λ , then for $\varepsilon > 0$

$$D_n(\lambda) + \varepsilon J_n(0) = \varepsilon J_n\left(\frac{\lambda}{\varepsilon}\right).$$

Proposition 3. Let (λ_k) be a sequence of non-negative reals converging to zero, $\varepsilon > 0$, $n \in \mathbb{N}$ and

$$J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n\left(\frac{\lambda_k}{\varepsilon}\right).$$

Then for all s with $0 < s \leq \varepsilon$

$$\|(sI + J_\varepsilon)^{-1}\| \geq \frac{\varepsilon^{n-1}}{s^n}.$$

Proof. We have

$$sI + J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n\left(\frac{\lambda_k + s}{\varepsilon}\right).$$

By Lemma 2 for $x = e_{nm}$

$$\|(sI + J_\varepsilon)^{-1}x\|_p^p = \sum_{k=1}^n \varepsilon^{-p} \frac{\varepsilon^{pk}}{(\lambda_m + s)^{pk}} \geq \left(\frac{\varepsilon^{n-1}}{(\lambda_m + s)^n}\right)^p,$$

thus

$$\|(sI + J_\varepsilon)^{-1}\| \geq \sup_m \frac{\varepsilon^{n-1}}{(\lambda_m + s)^n} = \frac{\varepsilon^{n-1}}{s^n}.$$

□

Let $D = \bigoplus_{k=1}^{\infty} D_n(\lambda_k)$, then

$$\|D - J_\varepsilon\| = \varepsilon$$

because $D - J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n(0)$, and $J_n(0)$ acts as a shift operator.

Corollary 4. The operator D fulfills (res), but for every $\varepsilon > 0$ the operator J_ε , which is an ε -perturbation of D , violates this condition.

In [4] it is shown, that the Lavrentiev regularization diverges, if (res) is violated. On the other hand, in the treatment of ill-posed problems the discussion of perturbations is indispensable, hence the Lavrentiev method is useless for ill-posed equations in Banach spaces.

The smallest positive integer r , such that for all positive integers k

$$\text{Ker}(\lambda I - T)^r = \text{Ker}(\lambda I - T)^{r+k}$$

and

$$\text{Range}(\lambda I - T)^r = \text{Range}(\lambda I - T)^{r+k}$$

is called the Riesz number of the compact operator T . (see e.g. R. KRESS [2, p. 27]).

For compact selfadjoint operators in Hilbert spaces the Riesz number of each eigenvalue $\lambda \neq 0$ is equal to unity. An operator T in a Hilbert space X is said to be *highly non-selfadjoint*, if for any positive integer n there is a finite dimensional T -invariant subspace $E \subset X$, such that the restriction of T onto E has an eigenvalue λ with Riesz number $> n$.

The following construction of a highly non-selfadjoint operator is a refinement of the construction above.

Let (λ_k) be a sequence of non-negative reals, tending to zero. Then

$$T = \bigoplus_{k=1}^{\infty} J_k(\lambda_k)$$

is highly non-selfadjoint, since λ_k is an eigenvalue of T with Riesz-number k . The spectrum of T is the unit disk Δ and the sequence (λ_k) .

$$sI - T = \bigoplus_{k=1}^{\infty} J_k(s - \lambda_k).$$

The entry with number $(1, k)$ of the matrix $(sI - T)^{-1}$ has the modulus $|s - \lambda_k|^{-k}$ hence $\sup_k |s - \lambda_k|^{-k} < \infty$, if $s \notin \{\lambda_k, k \in \mathbb{N}\}$ or if $|s| > 1$ (since $\lim \lambda_k = 0$). Of course, T is not compact.

Although each matrix $J_k(0)$ is nilpotent, $N = \bigoplus_{k=1}^{\infty} J_k(0)$ is not quasinilpotent, since the spectrum of N is the unit disk Δ .

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