TOYING WITH JORDAN MATRICES

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It is shown that an important resolvent estimate is instable under small perturbations.

In many applications operators T are of interest which fulfill the condition: the resolvent set $\rho(T)$ contains the negative reals, and there is a constant $\gamma > 0$, such that for all positive reals s the "resolvent estimate" holds

$$\left\|(sI+T)^{-1}\right\| \le \tfrac{\gamma}{s}\,. \tag{res}$$

In the theory of semigroups (see e.g. J.A. GOLDSTEIN [1, p. 20]) or in the theory of evolutionary integral equations (J. PRUESS [4, p. 69]) this condition is well known. In the investigation of regularization methods for ill-posed equations in Banach spaces the condition (res) is necessary for the convergence of the so-called Lavrentiev regularization (see R. PLATO [3], E. SCHOCK/V. PHONG [6], SPECKERT [7]).
The following simple example will show that the condition (res) is unstable under arbitrary small perturbations. We will construct operators J built up with Jordan matrices which act on block subspaces of ℓ_p.
By $J_n(\lambda)$ we denote an $(n \times n)$-Jordan matrix with the eigenvalue λ and with 1's in the upper diagonal. The following Lemma is well known.

Lemma 1. *Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then*

$$J_n(\lambda)^{-1} = \begin{pmatrix} \lambda^{-1} & -\lambda^{-2} & \cdots & (-1)^{n-1} & \lambda^{-n} \\ & \lambda^{-1} & \cdots & (-1)^{n-2} & \lambda^{-n+1} \\ & & \ddots & & \\ & O & & \ddots & \\ & & & & \lambda^{-1} \end{pmatrix}$$

Lemma 2. *In $X = \ell_p^n = \left(\mathbb{C}^n, \|\ \|_p\right)$ for $0 < \lambda \le 1$ holds*

$$\left\|J_n(\lambda)^{-1}\right\| \ge \lambda^{-n}.$$

Proof. Let e_n be the nth unit vector, then

$$\|J_n(\lambda)^{-1}e_n\|_p^p = \sum_{k=1}^{n} \lambda^{-pk} \geq \lambda^{-pn}.$$

$\square$

Let $D_n(\lambda)$ be the $(n \times n)$-diagonal matrix with the eigenvalue λ, then for $\varepsilon > 0$

$$D_n(\lambda) + \varepsilon J_n(0) = \varepsilon J_n\left(\tfrac{\lambda}{\varepsilon}\right).$$

Proposition 3. *Let (λ_k) be a sequence of non-negative reals converging to zero, $\varepsilon > 0$, $n \in \mathbb{N}$ and*

$$J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n\left(\tfrac{\lambda_k}{\varepsilon}\right).$$

Then for all s with $0 < s \leq \varepsilon$

$$\left\|(sI + J_\varepsilon)^{-1}\right\| \geq \tfrac{\varepsilon^{n-1}}{s^n}.$$

Proof. We have

$$sI + J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n\left(\tfrac{\lambda_k + s}{\varepsilon}\right).$$

By Lemma 2 for $x = e_{nm}$

$$\left\|(sI + J_\varepsilon)^{-1} x\right\|_p^p = \sum_{k=1}^{n} \varepsilon^{-p} \tfrac{\varepsilon^{pk}}{(\lambda_m + s)^{pk}} \geq \left(\tfrac{\varepsilon^{n-1}}{(\lambda_m + s)^n}\right)^p,$$

thus

$$\left\|(sI + J_\varepsilon)^{-1}\right\| \geq \sup_m \tfrac{\varepsilon^{n-1}}{(\lambda_m + s)^n} = \tfrac{\varepsilon^{n-1}}{s^n}.$$

$\square$

Let $D = \bigoplus_{k=1}^{\infty} D_n(\lambda_k)$, then

$$\|D - J_\varepsilon\| = \varepsilon$$

because $D - J_\varepsilon = \bigoplus_{k=1}^{\infty} \varepsilon J_n(0)$, and $J_n(0)$ acts as a shift operator.

Corollary 4. *The operator D fulfills (res), but for every $\varepsilon > 0$ the operator J_ε, which is an ε-perturbation of D, violates this condition.*

In [4] it is shown, that the Lavrentiev regularization diverges, if (res) is violated. On the other hand, in the treatment of ill-posed problems the discussion of perturbations is indispensable, hence the Lavrentiev method is useless for ill-posed equations in Banach spaces.

The smallest positive integer r, such that for all positive integers k

$$Ker\,(\lambda I - T)^r = Ker(\lambda I - T)^{r+k}$$

and

$$Range(\lambda I - T)^r = Range(\lambda I - T)^{r+k}$$

is called the Riesz number of the compact operator T. (see e.g. R. KRESS [2, p. 27]).

For compact selfadjoint operators in Hilbert spaces the Riesz number of each eigenvalue $\lambda \neq 0$ is equal to unity. An operator T in a Hilbert space X is said to be *highly non-selfadjoint*, if for any positive integer n there is a finite dimensional T-invariant subspace $E \subset X$, such that the restriction of T onto E has an eigenvalue λ with Riesz number $> n$.

The following construction of a highly non-selfadjoint operator is a refinement of the construction above.
Let (λ_k) be a sequence of non-negative reals, tending to zero. Then

$$T = \bigoplus_{k=1}^{\infty} J_k\,(\lambda_k)$$

is highly non-selfadjoint, since λ_k is an eigenvalue of T with Riesz-number k. The spectrum of T is the unit disk Δ and the sequence (λ_k).

$$sI - T = \bigoplus_{k=1}^{\infty} J_k\,(s - \lambda_k)\,.$$

The entry with number $(1, k)$ of the matrix $(sI - T)^{-1}$ has the modulus $|s - \lambda_k|^{-k}$ hence $\sup_k |s - \lambda_k|^{-k} < \infty$, if $s \notin \{\lambda_k, k \in \mathbb{N}\}$ or if $|s| > 1$ (since $\lim \lambda_k = 0$). Of course, T is not compact.
Although each matrix $J_k(0)$ is nilpotent, $N = \bigoplus_{k=1}^{\infty} J_k\,(0)$ is not quasinilpotent, since the spectrum of N is the unit disk Δ.

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