Iterative Methods with Perturbations for Ill-Posed Problems

Schock, E., Dr., Professor
Kaiserslautern University, Germany
Veretennikov, A., Dr., Professor
Institute for Information Transmission Problems, Moscow *

Abstract

We consider regularizing iterative procedures for ill-posed problems with random and nonrandom additive errors. The rate of square–mean convergence for iterative procedures with random errors is studied. The comparison theorem is established for the convergence of procedures with and without additive errors.

1 Introduction

Many problems arising in mathematical applications may be represented in the form

$$Tx = y \tag{1}$$

where T is a certain operator. In particular, one may regard at the components of this equation as follows: x represents the unknown data or input, T – the known transformation of these data, and y – the result or output. In slightly different language one may regard at x as at input signal, T being a certain transformation and y – the signal output. The problem is to reconstruct the input signal given y and T. The problem may be more difficult if this reconstruction problem is ill-posed and both y and T are known with some errors. In this paper certain new results are established both for deterministic and random additive errors. For random errors we, in some sense, complete the theory of mean-square optimal order rate procedures.

Assume that T is a bounded linear operator in Hilbert space H, $\|T\| \leq 1$, equation (1) has a unique solution $\hat{x}$. One of important regularizing methods has the form

*this investigation for the second author was partially supported by Department of Mathematics of Kaiserslautern University

$$x_n = x_{n-1} + s(T^*T)T^*(Tx_{n-1} - y), \qquad n \geq 1,$$

where $s(\lambda)$ is some real function of the variable λ. Algorithms of this type in ill-posed problems were considered in [5], [3] et al. We consider the case when the operator T and the right-hand side y are not known exactly,

$$\|T_\eta - T\| \leq \eta, \qquad \|y_\delta - y\| \leq \delta,$$

and hence the algorithm has the form

(2) $$\tilde{x}_n = \tilde{x}_{n-1} + s(T_\eta^* T_\eta)T_\eta^*(T_\eta \tilde{x}_{n-1} - y_\delta), \qquad n \geq 1.$$

Below procedures with random additive errors will be considered. Let us denote $r(\lambda) = 1 + \lambda s(\lambda)$. Then algorithm (4) may be represented in the form (with another s)

(3) $$\tilde{x}_n = r(T_\eta^* T_\eta)\tilde{x}_{n-1} + s(T_\eta^* T_\eta)T_\eta^* y_\delta.$$

Assume that function s from (3) is measurable,

(4) $$0 < s(\lambda) \leq 1, \qquad \sup_{c \leq \lambda \leq 1} \|1 - \lambda s(\lambda)\| < 1 \qquad \forall c > 0.$$

and

(5) $$\|r(\lambda) - r(\lambda')\| + \|s(\lambda) - s(\lambda')\| \leq C\|\lambda - \lambda'\|,$$

Then the following assertion holds true (see [4]): for any $p > 0$

(6) $$\sup_{n \geq 0, \lambda \in [0,1]} \|\lambda^p r(\lambda)^n\| n^p \leq C_p < \infty.$$

The regularizing parameter in algorithms (1) or (2) is often a discrepancy stop-rule of the type

$$\tau := \inf(n \geq 0 : \|T_\eta \tilde{x}_n - y_\delta\| \leq \mu),$$

$$\mu = b_1\delta + \hat{b}\eta \qquad (b_1 > 2, \quad \hat{b} > 2\|\hat{x}\|).$$

Comment. The algorithm may be improved in a standard way so that it will use the values $\|\tilde{x}_n\|$ instead of $\|\hat{x}\|$, see for ex. [5]. Besides, inequalities $b_1 > 2, \quad \hat{b} > 2\|\hat{x}\|$ may be improved to $b_1 > 1, \quad \hat{b} > \|\hat{x}\|$. It requires standard modifications in the proof.

2 Main results

At first we will consider general iterative methods with "exact" T and deterministic perturbations which make sense for random setting as well. Let

$$\tilde{x}_{n+1} - \tilde{x}_n = s_n(T^*T)T^*(T\tilde{x}_n - y_n) - v_{n+1}, \tag{7}$$

where (y_n) is a sequence with $\lim y_n = y$ and (v_n) is an arbitrary sequence. It was shown in [2] that the sufficient general implicit iterative method is given by

$$(I + \gamma_n T^*T)\tilde{x}_{n+1} = (I - \beta_n T^*T)\tilde{x}_n + (\beta_n + \gamma_n)T^*y_n + w_{n+1}. \tag{8}$$

We compare the sequence $(\tilde{x}_n)$ with the "exact" sequence (x_n), defined by

$$(I + \gamma_n T^*T)x_{n+1} = (I - \beta_n T^*T)x_n + (\beta_n + \gamma_n)T^*y.$$

Then we have the following result – "comparison" theorem.

Theorem 1 *Let $y \in R(T)$, $y_n \in H$, such that $\lim y_n = y$. Let (w_n) be a sequence in H. Let β_n, γ_n be positive reals with $0 \leq \beta_n \leq \gamma_n + 2$ and $\sum(\beta_n + \gamma_n) = \infty$. If*

$$\sum_{n=1}^{\infty}{}^{*} \frac{\beta_n + \gamma_n}{\sqrt{\gamma_n}} \|y - y_n\| < \infty$$

and

$$\sum \|w_n\| < \infty$$

then

$$\lim \tilde{x}_n = \hat{x}.$$

($\sum^*$ indicates

$$\sum_{n=1}^{\infty}{}^{*} \frac{\beta_n + \gamma_n}{\sqrt{\gamma_n}} \|y - y_n\| = \sum_{\gamma_n \neq 0} \frac{\beta_n + \gamma_n}{\sqrt{\gamma_n}} \|y - y_n\| + \sum_{\gamma_n = 0} \beta_n \|y - y_n\| \quad)$$

Corollary 1 *Let, additionally to Theorem 1, $T_n : H \to H$ be a sequence with*

$$\sum (1 + \gamma_n)||T - T_n|| < \infty.$$

Let (z_n) be given by

$$(I + \gamma_n T_n^*T_n)z_{n+1} = (I - \beta_n T_n^*T_n)z_n + (\beta_n + \gamma_n)T_n^*y_n + w_n \tag{9}$$

then

$$\lim z_n = \hat{x}.$$

In the sequel the algorithm has a form

$$\tilde{x}_n - \tilde{x}_{n-1} = s(T_\eta^* T_\eta) T_\eta^* (T_\eta \tilde{x}_{n-1} - y_\delta) + w_n, \tag{10}$$

where (w_n) is a sequance of random values. Assume that (w_n) are independent and identically distributed random values (i.i.d.r.v.) in H and

$$Ew_n = 0, \qquad E\|w_n\|^2 \leq \varepsilon^2. \tag{11}$$

An important case in the deterministic theory is provided by the assumption

$$\tilde{x}_0 - \hat{x} = |T|^p v, \quad p > 0, \quad ||v|| \leq r \quad (r > 0). \tag{12}$$

One has for the algorithm (10) the following result in this case.

Theorem 2 *Let*

$$\varepsilon = O(\delta + \eta)$$

and

$$\mu^2 \geq (1 + 2\kappa)(3(\delta + \eta\|\hat{x}\|)^2 + \varepsilon^2), \qquad \kappa > 0.$$

Then under assumptions of (4)-(5) and (12)

$$(\delta + \eta)^2 E\tau \to 0,$$

$$E|\tilde{x}_\tau - \hat{x}|^2 \to 0.$$

By virtue of theorem 2 we may expect (similar to the nonrandom case) that it is not necessary to repeat steps of our procedure longer than $a\mu^{-2}$ with any constant $a > 0$. Moreover, by virtue of some technical reasons in random case it is necessary to stop our procedure in such a way if we would like to obtain the optimal rate of convergence (cf. [4]). So let us fix any $a > 0$ and consider a new stop-rule

$$\sigma := \min(\inf(n \geq 0 : \|T_\eta x_n - y_\delta\| \leq \mu), a\mu^{-2}). \tag{13}$$

Theorem 3 *Let*

$$\varepsilon = O(\delta + \eta), \qquad a > 0.$$

Then under assumptions of theorem 2

$$(\delta + \eta)^2 E\sigma \to 0,$$

$$E|\tilde{x}_\sigma - \hat{x}|^2 \to 0.$$

The following theorem gives one the optimal convergence rate (compare with deterministic theory) under a certain restriction on random errors similar to that on the initial value $\tilde{x}_0$. Note that similar results may be obtained in general case as well (cf. with [6] for the rate of convergence in the mean) under more restrictive assumptions on the value ε (at any rate, in general case $\varepsilon = o(\delta+\eta)^2$).

Theorem 4 *Let assumptions of theorem 2 be satisfied,* $\varepsilon = O(\delta+\eta)^2$,

$$w_n = |T|^p v_n, \tag{14}$$

for any $n \geq 0$, *where* (v_n) *are also i.i.d.r.v. and*

$$Ev_n = 0, \qquad E\|v_n\|^q \leq C_q \varepsilon^q \qquad \forall q.$$

Then for the stop-rule (13)

$$E\sigma \leq C_{p,r,t}(\delta+\eta)^{-2/(p+1)},$$

$$E\sigma^2 \leq C_{p,r,t}(\delta+\eta)^{-4/(p+1)},$$

$$E\|\tilde{x}_\sigma - \hat{x}\|^2 \leq C_{p,r,t}(\delta+\eta)^{2p/(p+1)}.$$

A P P E N D I X

Proof of theorem 1. Let $\tilde{x}_n - x_n = d_n$, then

$$d_{n+1} = R_n d_n + Q_n(y_n - y) + P_n w_n$$

with

$$R_n = (I + \gamma_n T^*T)^{-1}(I - \beta_n T^*T),$$

$$Q_n = (\beta_n + \gamma_n)(I + \gamma_n T^*T)^{-1}T^*,$$

$$P_n = (I + \gamma_n T^*T)^{-1}.$$

Then we see

$$\|R_n\| \leq 1, \qquad \|P_n\| \leq 1,$$

$$\|Q_n\| \leq \tfrac{\beta_n+\gamma_n}{\sqrt{\gamma_n}} \quad \text{resp } \|Q_n\| \leq \beta_n, \quad \text{if } \gamma_n = 0,$$

By induction we obtain

$$d_{n+1} = \prod_{j=k}^{n} R_j d_k + \sum_{j=k}^{n} \prod_{i=j+1}^{n} R_i Q_j (y_j - y) + \sum_{j=k}^{n} \prod_{i=j+1}^{n} R_i P_j w_{j+1}.$$

The condition $\sum(\beta_j + \gamma_j) = \infty$ implies the strong convergence of $\prod R_j$ (see [2]), then

$$\|d_n\| \leq \| \prod_{j=k}^{n} R_j d_k\| + \sum_{j=k}^{n}{}^{*} \frac{\beta_n + \gamma_n}{\sqrt{\gamma_n}} \|y - y_j\| + \sum_{j=k}^{n} \|w_j\|$$

and for all $\varepsilon > 0$ there exist k_0 such that for all $n \geq k \geq k_0$ we have

$$\|d_{n+1}\| \leq \varepsilon.$$

Theorem 1 is proved.

Proof of corollary 1. We will show $\lim (z_n - x_n) = 0$, where (x_n) is the ,,exact" sequence of Theorem 1. Let

$$r_n = (I + \gamma_n T^*T) z_{n+1} - (I - \beta_n T^*T) z_n - (\beta_n + \gamma_n) T^* y \,, \tag{15}$$

then

$$(I + \gamma_n T^*T)(x_{n+1} - z_{n+1}) = (I - \beta_n T^*T)(x_n - z_n) - r_n.$$

Because of Theorem 1 we have to show

$$\sum ||r_n|| < \infty$$

We have, combining (9) and (15),

$$r_n = \gamma_n (T^*T - T_n^*T_n) z_{n+1} + \beta_n (T^*T - T_n^*T_n) z_n$$

$$+(\beta_n + \gamma_n)(T^*y - T_n^* y_n) + w_n$$

$$\|r_n\| \leq (\gamma_n + \beta_n)(\|T^*T - T_n^*T_n\| + \|T^*y - T_n^*y_n\|) \sup \|z_n\| + \|w_n\|$$

hence $\sum ||r_n|| < \infty$ and by Theorem 1 convergence holds true. Corollary 1 is proved.

Example. To show that the conditions of Theorem 1 are sharp, we consider the real equation

$$x = 0$$

i.e. $T = Id : IR \to IR\,, \quad y = 0\,,$ and solve it using the Landweber iteration

$$x_{n+1} = (1-\beta_n)x_n + w_n$$

with the special choice $\beta_n = w_n = \frac{1}{n}$. Then we obtain

$$x_n = \prod_{j=k}^{n-1}(1-\beta_j)x_k + \sum_{j=k}^{n-1} w_j \prod_{i=j+1}^{n-1}(1-\beta_i)$$

The first term tends to zero for $n \to \infty$.
Computation of the second term:

$$\log \prod_{i=j+1}^{n-1}(1-\frac{1}{i}) = \sum_{i=j+1}^{n-1} \log(1-\frac{1}{i}) \approx -\sum \frac{1}{i}$$

$$\approx -\int_{j+1}^{n} \frac{1}{x}\,dx = \log \frac{j+1}{n}$$

$$\sum_{j=k}^{n} \frac{1}{j}\,\frac{j+1}{n} \approx \frac{n-k}{n} = 1 - \frac{k}{n}$$

i.e. for all k and all $n \geq 2k$ we have $\frac{n-k}{n} \geq \frac{1}{2}$.

Thus the second term of the equation does not tend to zero.
Otherwise, if $\sum |w_n| < \infty$, then

$$|\sum_{j=k}^{n} w_j \prod_{i=j+1}^{n}(1-\frac{1}{i})| \leq \sum_{j=k}^{n} |w_j| \leq \varepsilon\,,$$

if k is sufficiently large and $n \geq k$. □

Comment. Usually it is presumed that T^*y_δ is computed exactly. But, due to the ill-posed nature of the operator T it may be very likely, that one actually deals with a perturebed quantily $T^*y_\delta + w_n$, so that w does not belong to the range of T^*. Such situation in the finite dimensional framework has been considered in [1].

Proof of theorem 2. Denote

$$d := (T_\eta - T)\hat{x} + (y_\delta - y), \;\; q_n = \tilde{x}_n - \hat{x}, \;\; B_\eta = I - s(T_\eta^* T_\eta)T_\eta^* T_\eta.$$

Let us consider the conditional expectation

$$E(\|q_{n+1}\|^2|q_n) = \|B_\eta q_n\|^2 - 2\langle B_\eta q_n, s(T_\eta^* T_\eta)T_\eta^* d\rangle \tag{16}$$

$$+\|s(T_\eta^* T_\eta)T_\eta^* d\|^2 + E\|w_{n+1}\|^2.$$

Let us use the bounds (11),

$$2|\langle B_\eta q_n, d\rangle| \le c^2\|B_\eta q_n\|^2 + c^{-2}\|d\|^2 \qquad (c>0) \tag{17}$$

$$\le c^2\mu^2 + c^{-2}\|d\|^2 \qquad (n<\sigma),$$

and

$$\|B_\eta q_n\|^2 = \|q_n\|^2 - 2\langle q_n, (I-B_\eta)q_n\rangle + \|(I-B_\eta)q_n\|^2 \tag{18}$$

$$= \|q_n\|^2 - 2\langle q_n, s(T_\eta^* T_\eta)T_\eta^* T_\eta q_n\rangle + \|s(T_\eta^* T_\eta)T_\eta^* T_\eta q_n\|^2$$

$$= \|q_n\|^2 - 2\|s(T_\eta^* T_\eta)^{1/2}(T_\eta^* T_\eta)^{1/2} q_n\|^2$$

$$+\|s(T_\eta^* T_\eta)T_\eta^* T_\eta q_n\|^2$$

$$\le \|q_n\|^2 - \|(T_\eta^* T_\eta)^{1/2} q_n\|^2$$

$$\le \|q_n\|^2 - \mu^2 \qquad (n<\sigma).$$

Now, (16)–(18) with $c = 2^{-1/2}$ and $n<\sigma$ imply

$$E(\|q_{n+1}\|^2|q_n) \le \|q_n\|^2 - 2^{-1}(\mu^2 - 3d^2 - \varepsilon^2)$$

$$\le \|q_n\|^2 - \kappa d^2.$$

Thus, for any n

$$EI(n+1<\sigma)\|q_{n+1}\|^2 \le EI(n<\sigma)\|q_{n+1}\|^2$$

$$\le EI(n<\sigma)\|q_n\|^2 - \kappa d^2 EI(n<\sigma).$$

If we take the sum from any $m \ge 0$ to n $(n \ge m)$ we obtain

$$EI(n+1<\sigma)\|q_{n+1}\|^2$$

$$\le EI(m<\sigma)\|q_m\|^2 - \kappa d^2 \textstyle\sum_{k=m}^n EI(k<\sigma).$$

As $n \to \infty$, we have

$$\kappa d^2 \sum_{k=m}^{\infty} EI(k < \sigma) \leq EI(m < \sigma)\|q_m\|^2. \tag{19}$$

In particular,

$$\kappa d^2 \sum_{k=0}^{\infty} EI(k < \sigma) = \kappa d^2 E\sigma \leq \|q_0\|^2.$$

Moreover, (19) implies for any m

$$\kappa d^2 E\sigma \leq \kappa d^2 m + E\|q_m\|^2. \tag{20}$$

If we show that $E\|q_m\|^2 \to 0, \delta + \eta \to 0$, then it will follow from (20) the assertion $d^2 E\sigma \to 0$.

Denote $u := T_\eta^*((T_\eta - T)\hat{x} - (y_\delta - y))$. Then we have the identity

$$q_n = B_\eta^n q_0 + \sum_{k=0}^{n-1} B_\eta^k s(T_\eta^* T_\eta)(u + w_{n-k}). \tag{21}$$

Then

$$E\|q_m\|^2 \leq 2E\|B_\eta^n q_0\|^2 + 2m^2\|u\|^2 + m\varepsilon^2.$$

So it is sufficient to prove the assertion

$$\lim_m \limsup_{\eta \to 0} E\|B_\eta^m q_0\|^2 = 0,$$

or, equivalently,

$$\lim_m \|B^m q_0\|^2 = 0 \quad (B := B_\eta|_{\eta=0}).$$

But this equality is well-known (see [4]). Theorem 2 is proved.

Proof of theorem 3 follows from the same considerations as in the proof of theorem 2.

Proof of theorem 4. From (21) one gets

$$T_\eta q_n = T_\eta B_\eta^n q_0 + \sum_{k=0}^{n-1} T_\eta B_\eta^k s(T_\eta^* T_\eta) u + \sum_{k=0}^{n-1} T_\eta B_\eta^k s(T_\eta^* T_\eta) w_{n-k}.$$

By virtue of lemma 3.2.2 [4] or lemma 4.1.2 [5] one finds

$$\begin{aligned} \|T_\eta B_\eta^n q_0\| &\leq \|(T_\eta^* T_\eta)^{1/2} B_\eta^n (T_\eta^* T_\eta)^{p/2} v\| + \\ &+ \|(T_\eta^* T_\eta)^{1/2} B_\eta^n ((T_\eta^* T_\eta)^{p/2} - (T^* T)^{p/2}) v\| \\ &\leq Crn^{-(p+1)/2} + C\eta\varepsilon_{r-1,p}, \end{aligned}$$

where $\varepsilon_{n,p} \to 0,\ n \to \infty$. Hence, for $n = \sigma - 1$ one has

$$\mu \le Cr(\sigma-1)^{-(p+1)/2} + C(\sigma-1)^{-1/2}\eta\varepsilon_{\sigma-1,p} +$$

$$+C\| \textstyle\sum_{k=0}^{\sigma-2} T_\eta B_\eta^k w_{\sigma-1-k}\|$$

or

$$\mu \le Cr(\sigma-1)^{-(p+1)/2} + C(\sigma-1)^{-1/2}\eta\varepsilon_{\sigma-1,p} +$$

$$+C\| \textstyle\sum_{k=0}^{\sigma-2} T_\eta B_\eta^k |T|^p v_{\sigma-1-k}\|.$$

Further, since

$$\| \textstyle\sum_{k=0}^{\sigma-2} T_\eta B_\eta^k |T|^p v_{\sigma-1-k}\| \le$$

$$\le \textstyle\sum_{k=0}^{\sigma-2} \|T_\eta B_\eta^k |T_\eta|^p v_{\sigma-1-k}\| + \sum_{k=0}^{\sigma-2} \|T_\eta B_\eta^k |T_\eta|^p - |T|^p v_{\sigma-1-k}\|,$$

then again due to lemma 3.3.2 [4] or lemma 4.1.2 [5] one gets

$$\| \textstyle\sum_{k=0}^{\sigma-2} T_\eta B_\eta^k |T|^p v_{\sigma-1-k}\| \le$$

$$\le C \textstyle\sum_{k=0}^{\sigma-2} k^{-(p+1)/2} \|v_{\sigma-1-k}\| + \sum_{k=0}^{\sigma-2} \eta\varepsilon_{k,p} \|v_{\sigma-1-k}\|.$$

Because

$$E\sum_{k=0}^{\sigma-2} \eta\varepsilon_{k,p}\|v_{\sigma-1-k}\| \le \eta\varepsilon o(E(\sigma-1)^2) = o(\mu),$$

one obtains

$$\mu \le C_r E(\sigma-1)^{-(p+1)/p} + (E\sum_{k=0}^{\sigma-2}\|v_{\sigma-1-k}\|^2)^{1/2}(E\sum_{k=0}^{\sigma-2} k^{-1-p})^{1/2}.$$

(We consider the main case with $\sigma \to \infty$ in probability; otherwise, the estimate is trivial.)

Since

$$E\sum_{k=0}^{\sigma-2}\|v_{\sigma-1-k}\|^2 \le C\varepsilon^2 E(\sigma-1) = o(\varepsilon^2\mu^{-2}),$$

it follows

$$\mu \le CE(\sigma-1)^{-(p+1)/2}.$$

Since x^{-1} is a convex function, we get by virtue of Jensen's inequality

$$E(\sigma-1) \le (E(\sigma-1)^{-(p+1)/2})^{-2/(p+1)} \le C\mu^{-2/(p+1)}.$$

Similarly

$$E(\sigma-1)^2 \le (E(\sigma-1)^{-(p+1)/2})^{-4/(p+1)} \le C\mu^{-4/(p+1)}.$$

(Though we do not need it in this paper, note that similarly

$$E(\sigma-1)^q \le (E(\sigma-1)^{-(p+1)/2})^{-2q/(p+1)} \le C\mu^{-2q/(p+1)}$$

for any $q > 0$.) This consideration corrects the calculations in [6] which were incorrect for $p < 1$.

Let us prove the second inequality. Similarly to [6] we have

$$E\|q_\sigma\|^2 \le 3(E\|B_\eta^\sigma q_0\|^2 + E\|\sum_{k=0}^{\sigma-1} B_\eta^k w_{\sigma-k-1}\|^2 + E\|\sum_{k=0}^{\sigma-1} B_\eta^k D_\eta \nu\|^2),$$

where

$$\nu = -(T_\eta - T)\hat{x} + (y_\delta - y), D_\eta = s(T_\eta^* T_\eta)T_\eta^*.$$

We have

$$E\|B_\eta^\sigma q_0\|^2 = E\|B_\eta^\sigma (T^*T)^{p/2} v\|^2$$

$$\le 2E\|B_\eta^\sigma((T_\eta^*T_\eta)^{p/2} - (T^*T)^{p/2})v\|^2 + 2E\|B_\eta^\sigma(T_\eta^*T_\eta)^{p/2}v\|^2$$

$$\le 2E\|B_\eta^\sigma(T_\eta^*T_\eta)^{p/2}v\|^2 + C\eta E\varepsilon_{\sigma-1,p}.$$

The value $\|B_\eta^\sigma(T_\eta^*T_\eta)^{p/2}v\|^2$ may be estimated similarly to [5] and [6] with the help of moments' inequality:

$$\|B_\eta^\sigma(T_\eta^*T_\eta)^{p/2}v\|^2 \le C_{p,r}\mu^{2p/(p+1)}$$

(see the proof of theorem 2 in [6]). Next,

$$E\|\sum_{k=0}^{\sigma-1} B_\eta^k D_\eta \nu\|^2 = E\|(I - B_\eta^\sigma)D_\eta \nu\|^2 \le \|\nu\| \le \mu^{2p/(p+1)}.$$

Let us estimate the second term:

$$E\|\textstyle\sum_{k=0}^{\sigma-1} B_\eta^k w_{\sigma-k-1}\|^2 = E\|\sum_{k=0}^{\sigma-1} B_\eta^k |T|^p v_{\sigma-k-1}\|^2$$

$$\le 2E\|\textstyle\sum_{k=0}^{\sigma-1} B_\eta^k |T_\eta|^p v_{\sigma-k-1}\|^2 + 2E\|\sum_{k=0}^{\sigma-1} B_\eta^k(|T_\eta|^p - |T|^p)v_{\sigma-k-1}\|^2.$$

Now,

$$E\| \sum_{k=0}^{\sigma-1} B_\eta^k |T_\eta|^p v_{\sigma-k-1}\|^2 \leq (E\sum_{k=0}^{\sigma-1} \|v_{\sigma-k-1}\|^2)^{1/2}(E\sum_{k=0}^{\sigma-1} k^{-p-1})^{1/2}$$

$$\leq C\varepsilon(E\sigma)^{1/2} \leq C\varepsilon\mu^{-2/(p+1)} = o(\mu^{2p/(p+1)}).$$

Finally, by virtue of the same lemma from [5]

$$E\| \sum_{k=0}^{\sigma-1} B_\eta^k(|T_\eta|^p - |T|^p)v_{\sigma-1-k}\|^2$$

$$\leq E(C\eta^{\min(1,p)} \sum_{k=0}^{\sigma-1} \|v_{\sigma-1-k}\|)^2.$$

We have

$$E(\sum_{k=0}^{\sigma-1} \|v_{\sigma-1-k}\|)^2$$

$$\leq 2E(\sum_{k=0}^{\sigma-1}(\|v_{\sigma-1-k}\| - E\|v_{\sigma-1-k}\|))^2 + 2E(\sum_{k=0}^{\sigma-1} E\|v_{\sigma-1-k}\|)^2$$

$$\leq C(\varepsilon^2 E\sigma + \varepsilon^2 E\sigma^2)$$

$$\leq C(\mu^{4-2p/(p+1)} + \mu^{4-4p/(p+1)}).$$

Thus,

$$\eta^{\min(1,p)} E(\sum_{k=0}^{\sigma-1} \|v_{\sigma-1-k}\|)^2 = o(\mu^{2p/(p+1)})$$

both for $p \geq 1$ and $p < 1$. Theorem 4 is proved.

References

[1] Schock, E. (1986) What are the proper condition numbers of discretized ill-posed problems? Linear Algebra and Appl., vol. 81, pp. 129-136.

[2] Schock, E. (1987) Implicit Iterative Methods for the Approximate Solution of Ill–Posed Problems. Boll. U.M.I. (7) 1-B, pp. 1171-1184.

[3] Schock, E. (1988) Pointwise Rational Approximation and Iterative Methods for Ill-Posed Problems. Numer. Mathem., vol. 54, pp. 91-103.

[4] Vainikko, G. (1982) Methods for solving of linear ill-posed problems in Hilbert spaces. Tartu State Univ., Tartu (in Russian)

[5] Vainikko, G.M. and Veretennikov, A.Yu. (1986) Iterative procedures in ill–posed problems. Moscow, Nauka (in Russian)

[6] Veretennikov, A.Yu. (1988) Rate of convergence of stochastic iteration procedures in ill-posed problems. Aut. and Remote Control, vol.49, no.1, pp. 56-60.

[7] Veretennikov, A.Yu. (1989) On square-mean convergence of stochastic iterative procedures in ill-posed problems. USSR Comp. Math. Maths. Phys., vol. 28, no. 3, pp. 125-129.