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LETTERS

Optimal dynamic reinsurance with worst-case default of the reinsurer

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Abstract
We consider the optimization problem of a large insurance company that wants to maximize the expected utility of its surplus through the optimal control of the proportional reinsurance. In addition, the insurer is exposed to the risk of default of its reinsurer at the worst possible time, a setting that is closely related to a scenario of the Swiss Solvency Test.

Keywords Dynamic proportional reinsurance · Reinsurer default · Stress scenario · Swiss Solvency Test · Worst-case scenario approach

1 Introduction

We model the surplus process of an insurance company with proportional reinsurance, using a diffusion model as in [13, 14, 17]. In contrast to these works, where the primary goal is to minimize the ruin probability, we consider an expected utility maximization problem. This is also done in [2, 12], among others. Furthermore, our optimization problem includes the default of the reinsurer. In [1, 5], the authors incorporate the reinsurer counterparty default risk in a one-period model. We are adapting the worst-case scenario approach as introduced to portfolio optimization by Korn and Willmott [11]. Further publications, based on this portfolio optimization approach, are [6, 9, 10, 15], among others. In [7], the investment problem of an insurance company with crash-risk is solved and in [8] the authors consider the control of a surplus process with a worst-case claim development.

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To model the default of a reinsurer, we look at a translated quotation from the technical description of the scenarios in the Swiss Solvency Test [16]:

> This scenario deals with the risk of a reinsurer default. It assumes a situation where the insurer is faced with a large insurance loss. In addition, the economic environment for reinsurers is adverse, leading to their ratings being downgraded. A number of reinsurers default, so that they can no longer (fully) meet their obligations. The primary insurer suffers a loss as a result, which is composed as follows
>
> - The reinsurers can no longer cover the reinsured portion of the occurred major loss.
> - Since a number of reinsurers have defaulted, the primary insurer has to buy new cover and pay another premium for it.
> - Reinsurers can only partially settle the primary insurer's outstanding debts from old claims.

A detailed motivation for the use of a worst-case approach, based on Knightian uncertainty, for the modeling of stress scenarios is given in [10].

2 Mathematical setting and model description

We consider a complete probability space $(\Omega, \mathcal{F}, P)$, equipped with a filtration $\mathbb{F} = \left(\mathcal{F}_t\right)_{t\in[0,T]}$, for a finite time horizon $T > 0$. The filtration $\mathbb{F}$ is extended to $[0,T] \cup \{\infty\}$ by letting $\mathcal{F}_\infty := \mathcal{F}_T$ and Θ is the corresponding set of all $[0,T] \cup \{\infty\}$ -valued stopping times.

To model the surplus process of the primary insurer we choose a diffusion model and a proportional reinsurance. We include the possibility of a default of the reinsurer at time $\tau \in \Theta$, where $\{\tau = \infty\}$ denotes the no-default scenario.

The set $\mathbb{A}$ of admissible risk exposures of the insurer contains all predictable processes a defined on [0, T] with values in [0, 1]. For $a \in \mathbb{A}$ we denote by a_1 the risk exposure up to the default and by a_0 the corresponding risk exposure after the default. In addition, we require that a_1 has right-continuous paths.

To model the default we use the above quoted technical description of the Swiss Solvency Test. We note that the primary insurer's loss can be divided into two main components. First, there is a loss C that is independent of the risk exposure and results from the major damage, which is the trigger for the reinsurer's default. In addition, the insurer suffers a loss $(1 - a_1(\tau))F$ that depends on the risk exposure due to the new covers to be purchased and the only partially settled former claims.

By $\mu \in \mathbb{R}_{>0}$ and $\sigma \in \mathbb{R}_{>0}$ we denote the drift and volatility of the surplus process without reinsurance. Further, λ, C, $F \in \mathbb{R}_{>0}$ are used to include premiums to the reinsurer and the above decomposition of the loss. For $a \in \mathbb{A}$, $\tau \in \Theta$ and a one-dimensional Brownian motion $(W(t))_{t\in[0,T]}$ with respect to $\mathbb{F}$, the (approximated) dynamics for the surplus process $R^{a,\tau}$ are defined as

$$R^{a,\tau}(0) = x, \ x \in \mathbb{R},$$
$$dR^{a,\tau}(t) = (\mu - (1 - a(t))\lambda)dt + a(t)\sigma dW(t) - L^{a,\tau} dN(t),$$

for $t \in [0, T]$, with $N(t) := \mathbb{1}_{\{\tau \leq t\}}$ and $L^{a,\tau} := \mathbb{1}_{\{\tau < \infty\}}\big(C + (1 - a_1(\tau))F\big)$.

3 Worst-case optimization problem

Using an exponential utility function with constant absolute risk aversion $\gamma > 0$, we define the **optimal dynamic reinsurance problem with worst-case default of the reinsurer** as

$$\sup_{a \in \mathbb{A}} \inf_{\tau \in \Theta} \mathbb{E}(-\exp(-\gamma R^{a,\tau}(T))). \tag{1}$$

Standard results from portfolio optimization and stochastic control, as e.g. [3], imply that the optimal risk exposure $\bar{a}_0$ after the default is

$$\bar{a}_0 = \min\left\{\frac{\lambda}{\gamma\sigma^2}, 1\right\}. \tag{2}$$

Let v_0 denote the value function after the default has occurred. Then we obtain for $\tau \in \Theta$ that we have

$$\begin{aligned} v_0(\tau, R^{a,\tau}(\tau)) &:= \sup_{a \in \mathbb{A}} \mathbb{E}\big(-\exp(-\gamma R^{a,\tau}(T))|\mathcal{F}_\tau\big) \\ &= \sup_{\substack{a \in \mathbb{A} \\ a_0 = \bar{a}_0}} \mathbb{E}\big(-\exp(-\gamma R^{a,\tau}(T))|\mathcal{F}_\tau\big) \\ &= -\exp\Big(-\gamma R^{a,\tau}(\tau) - \gamma\Big(\mu - (1 - \bar{a}_0)\lambda - \frac{1}{2}\gamma\bar{a}_0^2\sigma^2\Big)(T - \tau)\Big), \end{aligned}$$

on $\{\omega \in \Omega | \tau(\omega) < \infty\}$. Thus, we reduce the set $\mathbb{A}$ to all risk exposures a with corresponding $a_0 = \bar{a}_0$. Further, we note that for every risk exposure $a \in \mathbb{A}$, due to the constant loss $C > 0$ and the non-negativity of $(1 - a_1(T))F$, a default at time T is worse than no default. Therefore we also reduce the set Θ to all $[0, T]$-valued stopping times and obtain the following reformulation of our optimization problem (1):

$$\sup_{a\in\mathbb{A}}\inf_{\tau\in\Theta}\mathbb{E}(-\exp(-\gamma R^{a,\tau}(T))) = \sup_{a\in\mathbb{A}}\inf_{\tau\in\Theta}\mathbb{E}\big(\mathbb{E}\big(-\exp(-\gamma R^{a,\tau}(T))|\mathcal{F}_\tau\big)\big)$$
$$= \sup_{a\in\mathbb{A}}\inf_{\tau\in\Theta}\mathbb{E}\big(v_0(\tau, R^{a,\tau}(\tau))\big) = \sup_{a\in\mathbb{A}}\inf_{\tau\in\Theta}\mathbb{E}\big(v_0(\tau, R^{a,\tau}(\tau-) - L^{a,\tau})\big).$$

4 Main result

To determine the optimal risk exposure $\bar{a}_1$, used up to the time of default τ, we first look at the indifference optimality principle, which is well-known in the worst-case scenario literature. Further, we consider the special case of a default at terminal time T.

Lemma 1 (Indifference optimality principle) *Let $\bar{a} \in \mathbb{A}$ be a risk exposure such that*

$$\mathbb{E}\big(v_0\big(\tau, R^{\bar{a},\tau}(\tau-) - L^{\bar{a},\tau}\big)\big) = \mathbb{E}\big(v_0\big(\xi, R^{\bar{a},\xi}(\xi-) - L^{\bar{a},\xi}\big)\big),$$

for all $\tau, \xi \in \Theta$. If for every $a \in \mathbb{A}$ there exists a $\bar{\tau} \in \Theta$ with

$$\mathbb{E}\big(v_0\big(\bar{\tau}, R^{\bar{a},\bar{\tau}}(\bar{\tau}-) - L^{\bar{a},\bar{\tau}}\big)\big) \geq \mathbb{E}\big(v_0\big(\bar{\tau}, R^{a,\bar{\tau}}(\bar{\tau}-) - L^{a,\bar{\tau}}\big)\big),$$

then $\bar{a}$ is optimal for the optimization problem (1).

Proof Analogous to the proof of Proposition 4.1 in [15].□

Lemma 2 (*Default at terminal time*) *Let $\bar{a} \in \mathbb{A}$ be a risk exposure with $\bar{a}(T) = 1$. Then, at the terminal time T we have*

$$\mathbb{E}\big(v_0\big(T, R^{\bar{a},T}(T-) - L^{\bar{a},T}\big)\big) = \mathbb{E}\left(-\exp\left(-\gamma(x - C) - \gamma\int_0^T \phi(\bar{a}_1(t))dt\right)\right),$$

for $\phi : [0,1] \mapsto \mathbb{R}$ defined by $\phi(a) := \mu - (1-a)\lambda - \frac{1}{2}\gamma a^2\sigma^2$.

Proof Note that $L^{\bar{a},T} = C$. Then the proof follows from the definition of the surplus process and $\mathbb{A}$.□

We can use these results to compute the optimal pre-default risk exposure $\bar{a}_1$.

Theorem 1 (Optimal risk exposure) *Let $\bar{a} \in \mathbb{A}$ with $\bar{a}_0$ defined in* (2).

1. *If $\lambda < \gamma\sigma^2$, we define $\bar{a}_1(t)$, for $t \in [0,T]$, by*

$$\bar{a}_1(t) = \bar{a}_0 + \frac{2F}{\gamma\sigma^2(T-t) + (2F/(1-\bar{a}_0))}.$$

2. *If $\lambda \geq \gamma\sigma^2$, we set $\bar{a}_1 = 1$.*

Then $\bar{a}$ solves the portfolio optimization problem (1).

Proof

1. We give the proof in three steps. But first we show that $\bar{a}$ is well-defined, i.e. that $\bar{a}_1(t) \in [0,1]$ for all $t \in [0,T]$. For this note that $\bar{a}_1 \geq \bar{a}_0 \geq 0$. Further, $\bar{a}_1(t)$ is increasing in t with $\bar{a}_1(T) = 1$ and thus $\bar{a}_1 \leq 1$.

   (a) Applying Itô's formula and using the definition of $\bar{a}_1$ leads to

   $$dv_0\left(t, R^{\bar{a},t}(t-) - L^{\bar{a},t}\right) = -v_0\left(t, R^{\bar{a},t}(t-) - L^{\bar{a},t}\right)\gamma \bar{a}_1(t)\sigma dW(t),$$

   for $t \in [0,T]$. Thus, $v_0\left(t, R^{\bar{a},t}(t-) - L^{\bar{a},t}\right)$ is a martingale on [0, T]. Due to Doob's optional sampling theorem we can apply Lemma 1 from now on.

   (b) Now let $a \in \mathbb{A}$ and define τ_a as

   $$\tau_a = \inf\{t \in [0,T] | a_1(t) < \bar{a}_1(t)\}, \text{with } \inf \emptyset := \infty.$$

   We construct $\tilde{a} \in \mathbb{A}$ such that $\tilde{a}_1(t) = a_1(t)$ on $[0, \tau_a)$ and $\tilde{a}_1(t) = \bar{a}_1(t)$ else. Due to the right-continuity of a_1 we have $L^{\tilde{a},\tau_a} = L^{\bar{a},\tau_a} \leq L^{a,\tau_a}$ if $\tau_a < \infty$. Together with the above martingale property we obtain

   $$\inf_{\tau \in \Theta} \mathbb{E}\left(v_0\left(\tau, R^{\bar{a},\tau}(\tau-) - L^{\bar{a},\tau}\right)\right) = \inf_{\substack{\tau \in \Theta \\ \tau \leq \tau_a}} \mathbb{E}\left(v_0\left(\tau, R^{\tilde{a},\tau}(\tau-) - L^{\tilde{a},\tau}\right)\right)$$

   $$\geq \inf_{\substack{\tau \in \Theta \\ \tau \leq \tau_a}} \mathbb{E}\left(v_0(\tau, R^{a,\tau}(\tau-) - L^{a,\tau})\right) \geq \inf_{\tau \in \Theta} \mathbb{E}\left(v_0(\tau, R^{a,\tau}(\tau-) - L^{a,\tau})\right).$$

   Thus, we can reduce the search for the optimal risk exposure to all $a \in \mathbb{A}$ with $a_1(t) \geq \bar{a}_1(t)$ for all $t \in [0,T]$.

   (c) Next we fix such an arbitrary $a \in \mathbb{A}$. In particular, we have that $a_1(t) \geq \bar{a}_1(t) \geq \bar{a}_0$ with $a(T) = \bar{a}(T) = 1$. Now we look at the quadratic, strictly concave function ϕ as defined in Lemma 2. This function is maximized on [0, 1] by $\bar{a}_0$, and we thus obtain for $t \in [0,T]$ that

   $$\max\{\phi(a_1(t)), \phi(\bar{a}_1(t))\} = \phi(\bar{a}_1(t)).$$

   Now we apply Lemma 2 and obtain at terminal time T:

   $$\mathbb{E}\left(v_0\left(T, R^{\bar{a},T}(T-) - L^{\bar{a},T}\right)\right) \geq \mathbb{E}\left(v_0\left(T, R^{a,T}(T-) - L^{a,T}\right)\right).$$

   Then the claim follows with Lemma 1.

2. As in this case we have $\bar{a}_1 = \bar{a}_0 = 1$, the assumptions of Lemmas 1 and 2 are valid and the proof follows as in part 1. □

Future Research. Maximizing expected utility is a classic objective in portfolio optimization, but less common in the control of a surplus process, where the usual goal is to minimize the probability of ruin. Browne [3] shows an interesting connection between minimizing the risk of ruin and maximizing the utility of terminal wealth, which serves as a motivation for considering the probability of ruin in our model with worst-case default of the reinsurer. Moreover, it is of interest to look at a much wider class of reinsurance treaties, as motivated by [18], respectively reinsurance designs based on risk measures (see [4] for a review).

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