Hyperbolic planes

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Introduction

A *plane* is a two-dimensional *right* vector space V over a division algebra D, which we assume in this paper has an involution; a hermitian form $h : V \times V \longrightarrow D$ is *hyperbolic*, if, with respect to a properly choosen basis, it is given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; the pair (V, h) is called a *hyperbolic plane*. If D is central simple over a number field K, a hyperbolic plane (V, h) gives rise to a reductive K-group $G_D = U(V, h)$ and a simple K-group $SG_D = SU(V, h)$. In this paper we study the case where the real Lie group $G(\mathbb{R})$ is of *hermitian type*, in other words that the symmetric space of maximal compact subgroups of $G(\mathbb{R})$ is a hermitian symmetric space. Let $d = \deg D$ (that is, $\dim_K D = d^2$); we consider the following cases:

1) $d = 1$: k is a totally real number field of degree f over $\mathbb{Q}$, $K|k$ is an imaginary quadratic extension, and $D := K$, $V := K^2$. The hyperbolic plane is denoted (K^2, h).

2) $d = 2$: k as in 1), D a totally indefinite quaternion division algebra, central simple over k, $V := D^2$. The hyperbolic plane is denoted (D^2, h).

3) $d \geq 3$: k as above, $K|k$ an imaginary quadratic extension, D a central simple division algebra over K with a $K|k$-involution of the second kind, $V := D^2$; this is again denoted (D^2, h).

Of course, 1) is the same case as 3), if we view K as central simple over itself; the $K|k$ involution is just the Galois action of K over k. Our first observation (Proposition 2.3) is that the case $d = 1$ is none other than the case familiar from the study of Hilbert modular varieties. If $K = k(\sqrt{-\eta})$, with $\eta \in k^+$, then the isomorphism between $SL_2(k)$ and $SU(K^2, h)$[1] is simply

$$\begin{array}{ccc} SL_2(k) & \xrightarrow{\sim} & SU(K^2, h) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} d & 2c/\sqrt{-\eta} \\ b\sqrt{-\eta}/2 & a \end{pmatrix}. \end{array}$$

The following properties of Hilbert modular surfaces are well-known.

i) the domain $\mathcal{D} \cong \mathfrak{H} \times \cdots \times \mathfrak{H}$ is a product of f copies of the upper half plane, and the rational boundary components are zero-dimensional, i.e., reduce to points.

ii) For the maximal order $\mathcal{O}_k \subset k$, the arithmetic subgroup $\Gamma := SL_2(\mathcal{O}_k)$ acts on $\mathcal{D}$, the quotient $X_\Gamma = \Gamma\backslash\mathcal{D}$ being non-compact, with a natural singular compactification $(\Gamma\backslash\mathcal{D})^*$ obtained by adding a point at each boundary component (the cusps). Furthermore, X_Γ has exactly $h(k)$ cusps, where $h(k)$ denotes the class number of k.

[1] In this paper we will reserve the notation $SL_2(k)$ for that group, and otherwise will use notations like $SL(n, K)$, etc.

iii) There are modular subvarieties, known in the case of Hilbert modular surfaces as Hirzebruch-Zagier cycles.

These are some of the properties it would be reasonable to consider for hyperbolic planes over higher-dimensional D. The main results of this paper contribute towards that end. First, in analogy to i), we find that for a hyperbolic plane (D^2, h), the domain $\mathcal{D}$ is a product of f copies of an irreducible domain $\mathcal{D}_1$, and $\mathcal{D}_1$ is a tube domain[2]. It is the disc for $d = 1$, Siegel space of degree two (type $\mathbf{III_2}$) for $d = 2$, and of type $\mathbf{I_{d,d}}$ for $d \geq 3$, respectively. The rational boundary components for the groups G_D are points. It already follows from this that the Baily-Borel compactification X_Γ^* of the arithmetic quotient $X_\Gamma = \Gamma\backslash\mathcal{D}$, $\Gamma \subset G_D$ arithmetic, is obtained by adding finitely many isolated cusps to X_Γ, much as in the situation for Hilbert modular surfaces. We remark that although we will not study the smooth compactifications of $\overline{X}_\Gamma$, Satake in [S1] has considered these, and in particular has given a formula for a "cusp defect".

To introduce arithmetic subgroups one fixes a maximal order $\Delta \subset D$, and considers $\Gamma_\Delta = M_2(\Delta) \cap G_D$ and $S\Gamma_\Delta = M_2(\Delta) \cap SG_D$. $S\Gamma_\Delta$ is the generalisation of the Hilbert modular group ($d = 1$) to the higher d case, and Γ_Δ is the generalisation of the extended Hilbert modular group. The arithmetic quotients we study are the $\Gamma_\Delta\backslash\mathcal{D} = X_{\Gamma_\Delta}$ and $S\Gamma_\Delta\backslash\mathcal{D} = X_{S\Gamma_\Delta}$. The first surprise is that one can easily calculate the class number, that is, the (finite) number of cusps on X_{Γ_Δ}. We prove (Corollary 4.11)

Theorem 0.1 *The number of cusps of Γ_Δ is the class number of K for $d \geq 3$ and the class number of k for $d = 1, 2$.*

This result is just a straightforward generalisation of the known $d = 1$ result, as mentioned in ii). We then turn to a generalisation of iii) above, and describe a rough analogue of the Hirzebruch-Zagier cycle F_1 to the case of higher d. This results from the enrichment of the algebraic theory, which arises from *subfields* of the division algebra D. Let us briefly explain this in the simplest case, $d \geq 3$. Then D is a cyclic algebra $D = (L/K, \sigma, \gamma)$, $\gamma \in K^*$, and the subfield $L \subset D$ is of degree d over K; it is a splitting field for D. These matters are recalled in the first paragraph for the convenience of the reader, as they are an essential part of the remainder of the paper. The subfield $L \subset D$ defines a subspace $L^2 \subset D^2$, and $h_{|L^2} =: h_L$ is again hyperbolic. In other words, in a hyperbolic plane over a degree d cyclic division algebra, we have a subhyperbolic plane (L^2, h); this is of the type $d = 1$, but with $f = d$. This results in corresponding subgroups $G_L \subset G_D$, subdomains $\mathcal{D}_L \subset \mathcal{D}$ and arithmetic subgroups $\Gamma_{\mathcal{O}_L} \subset \Gamma_\Delta$; finally these yield modular subvarieties $M_L \subset X_{\Gamma_\Delta}$. By the result mentioned above, this means: *on an arithmetic quotient X_{Γ_Δ} for $d \geq 3$, there are Hilbert modular subvarieties of dimension d.* For the general statement, see Theorem 4.14 below. An additional analysis leads to the following strengthening (Theorem 4.16):

Theorem 0.2 *Given a cusp $p \in X_{\Gamma_\Delta}^*\backslash X_{\Gamma_\Delta}$ there is a modular subvariety $M_{L,p} \subset X_{\Gamma_\Delta}$, such that $p \in M_{L,p}^*$, that is, p is also a cusp for $M_{L,p}$.*

It is now quite straightforward to apply Shimura's theory to give the moduli interpretation of the arithmetic quotients X_{Γ_Δ}. A second application of this theory gives the moduli interpretation of the modular subvarieties M_L. This is summarised as follows:

1) $d = 1$: abelian varieties of dimension $2f$ with complex multiplication by K, of signature $(1, 1)$, OR: abelian varieties of dimension f with real multiplication by k.

2) $d = 2$: abelian varieties of dimension $4f$, with multiplication by D.

[2]this is certainly well-known to experts

3) $d = 3$: abelian varieties of dimension $2d^2 f$, with multiplication by D.

We note that the two possibilities for $d = 1$ are related by the fact that, given A^f, an abelian variety of dimension f with real multiplication by k, we get one of dimension $2f$ with complex multiplication as follows: if the field $K = k(\sqrt{-\eta})$, let $K' = \mathbb{Q}(\sqrt{-\eta})$. Then if $A^f \cong \mathbb{C}^f/\Lambda$, we have a lattice in $\mathbb{C}^{2f}$ given by $\Lambda' := \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{K'}$. The abelian variety $A^{2f} := \mathbb{C}^{2f}/\Lambda'$ is the desired abelian variety with complex multiplication. Conversely, if A^{2f} is given, it is easily seen that the lattice Λ' has a sublattice Λ such that $\Lambda' \cong \Lambda \otimes_{\mathbb{Z}} \mathcal{O}_{K'}$, and $\mathbb{C}^f/\Lambda$ is an abelian variety of half the dimension with real multiplication.

The modular subvarieties, for $d \geq 2$, parameterise:

2) $d = 2$: the abelian variety A^{4f} with multiplication by D splits, A^{4f} is isogenous to a product $A^{2f} \times A^{2f}$, where A^{2f} has complex multiplication by an imaginary quadratic extension field $L \subset D$ of the center of D, k.

3) $d \geq 3$: the abelian variety $A^{2d^2 f}$ with multiplication by D splits, $A^{2d^2 f}$ is isogenous to a product $\underbrace{A^{2df} \times \cdots \times A^{2df}}_{d \text{ factors}}$, where A^{2df} has complex multiplication by the splitting field $L \subset D$.

In the fifth paragraph we discuss in more detail a central simple division algebra D of degree three over an imaginary quadratic extension $K|\mathbb{Q}$, which defines unitary groups which it appears may be related to Mumford's fake projective plane.

1 Cyclic algebras with involution

We will be considering matrix algebras $M_n(D)$, where D is a simple division algebra with an involution. There are two kinds of involution to be considered. Fix, for the rest of the paper, a totally real number field k of finite degree f over $\mathbb{Q}$, and assume D is a division algebra over k with involution. The two cases are:

(i) D is central simple over k, and k is a maximal field which is symmetric with respect to the involution on D (involution of the first kind).

(ii) D is central simple over an imaginary quadratic extension $K|k$, and k is the maximal subfield of the center which is symmetric with respect to the involution on D; the involution restricts on K to the $K|k$ involution (involution of the second kind).

If one sets $K := k$ in case (i), then in both cases one speaks of a $K|k$-involution. With these notations, let d be the degree of D over K (i.e., $\dim_K D = d^2$). Specifically, we will be considering the following cases:

1) $d = 1$, $D = K$ with an involution of the second kind, namely the $K|k$ involution.

2) $d = 2$, D a totally indefinite quaternion division algebra over k with the canonical involution (involution of the first kind). (1)

3) $d \geq 3$, D a central simple division algebra of degree d over K with a $K|k$-involution (involution of the second kind).

Next recall that any division algebra over K is a *cyclic algebra* ([A], Thm. 9.21, 9.22). These algebras are constructed as follows. Let L be a cyclic extension of degree d over K and let σ denote a generator of the Galois group $Gal_{L/K}$. For any $\gamma \in K^*$, one forms the algebra generated by L and an element e,

$$\begin{aligned} A = (L/K, \sigma, \gamma) &:= L \oplus eL \oplus \cdots \oplus e^{d-1}L, \\ e^d = \gamma, \qquad & e \cdot z = z^\sigma \cdot e, \quad \forall_{z \in L}. \end{aligned} \tag{2}$$

This algebra can be constructed as a subalgebra of $M_d(L)$, by setting:

$$e = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 1 \\ \gamma & 0 & \cdots & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} z & \cdots & \cdots & 0 \\ & z^{\sigma} & & \\ & & \ddots & \\ 0 & \cdots & \cdots & z^{\sigma^{d-1}} \end{pmatrix}. \tag{3}$$

One verifies easily that these matrices fulfill the relations (2), and letting B be the algebra in $M_d(L)$ generated by e and $z \in L$, we have $(L/K, \sigma, \gamma) \cong B$, and clearly $B \otimes_K L \cong M_n(L)$, giving the explicit splitting. It is known that the cyclic algebra $A = (L/K, \sigma, \gamma)$ is split if and only if $\gamma \in N_{L|K}(L^*)$ ([A], Thm. 5.14). We will assume that A is a division algebra; we will denote it in the sequel by D.

Now assuming $d \geq 2$, we consider the question of involutions on D. It is well-known that for $d = 2$, an involution of the second kind is the base change of an involution of the first kind ([A], Thm. 10.21), so the assumption above that for $d = 2$ the involution is of the first kind is no real restriction[3]. Furthermore, if $d \geq 3$, there are no involutions of the first kind, since an involution of the first kind gives an isomorphism $A \cong A^{op}$ (A^{op} the opposite algebra), implying that the class $[A]$ in the Brauer group Br_K is of exponent one or two. Hence, for $d \geq 3$ the restriction that the involution is of the second kind is no restriction at all. Finally in case $d = 1$, an involution of the first kind on D is trivial, so this case is not hermitian, and we will not consider it.

Let us now consider the quaternion algebras D. Recall that D is said to be *totally definite* (respectively *totally indefinite*), if at all real primes ν, the local algebra D_ν is the skew field over $\mathbb{R}$ of the Hamiltonian quaternions $\mathbb{H}$ (respectively, if for all real primes ν, the local algebra D_ν is split). Recall also that D *ramifies* at a finite prime $\mathfrak{p}$ if $D_{\mathfrak{p}}$ is a division algebra; the isomorphism class of D is determined by the (finite) set of primes $\mathfrak{p}$ at which it ramifies and its isomorphism class at those primes ([A], Thm. 9.34). As a special case of cyclic algebras, quaternion algebras can be displayed as algebras in $M_2(\ell)$, where ℓ/k is a real quadratic extension, $\ell = k(\sqrt{a})$. Then there is some $b \in k^*$ such that

$$e = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} z & 0 \\ 0 & z^{\sigma} \end{pmatrix}, \quad z \in \ell,$$

where $z^{\sigma} = (z_1 + \sqrt{a} z_2)^{\sigma} = z_1 - \sqrt{a} z_2$. In other words, we can write for any $\alpha \in D$,

$$\alpha = \begin{pmatrix} a_0 + a_1\sqrt{a} & a_2 + a_3\sqrt{a} \\ b(a_2 - a_3\sqrt{a}) & a_0 - a_1\sqrt{a} \end{pmatrix}. \tag{4}$$

Then the canonical involution is given by the involution

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{5}$$

on $M_2(\ell)$, and the norm and trace are just the determinant and trace of the matrix (4). We will use the notation (a, b) to denote this algebra $(\ell/k, \sigma, b)$, $\ell = k(\sqrt{a})$, if no confusion can arise from this. We remark also that for quaternion algebras we have

$$Tr_{D|k}(x) = x + \overline{x}, \quad N_{D|k}(x) = x\overline{x},$$

where the trace and norm are the reduced traces and norms. All traces and norms occuring in this paper are the reduced ones unless stated otherwise.

[3]perhaps somewhat more pragmatic is the remark that because of the exceptional isomorphism between domains of type $\mathbf{I_{2,2}}$ and $\mathbf{IV_4}$, the case $d = 2$ and involution of the second kind can be translated into bilinear forms in eight variables, hence does not fit in well in the framework of hyperbolic planes.

In the case of involutions of the second kind, note first that the $K|k$-conjugation extends to the splitting field L; its invariant subfield ℓ is then a totally real extension of k, also cyclic with Galois group generated by σ. We have the following diagram:

$$\begin{tikzcd} & L \arrow[dl, dash] \arrow[dr, dash] & \\ \ell \arrow[dr, dash] & & K \arrow[dl, dash] \\ & k & \end{tikzcd} \tag{6}$$

and the conjugations on L and K give the action of the Galois group on the extensions L/ℓ and K/k; these are ordinary imaginary quadratic extensions. There are precise relations known under which D admits a $K|k$-involution of the second kind.

Theorem 1.1 ([A], Thm. 10.18) *A cyclic algebra $D = (L/K, \sigma, \gamma)$ has an involution of the second kind $\iff$ there is an element $\omega \in \ell$ such that*

$$\gamma\overline{\gamma} = N_{K|k}(\gamma) = N_{\ell|k}(\omega) = \omega \cdot \omega^{\sigma} \cdots \omega^{\sigma^{d-1}}.$$

If this condition holds, then an involution is given explicitly by setting:

$$(e^k)^J = \omega \cdots \omega^{\sigma^{k-1}} (e^k)^{-1}, \left(\sum e^i z_i\right)^J = \sum \overline{z}_i (e^i)^J, \tag{7}$$

where $x \mapsto \overline{x}$ denotes the L/ℓ-involution. In particular for $x \in L$ we have

$$x^J = \overline{x}, \text{ and } x = x^J \iff x \in \ell.$$

Later it will be convenient to have a description for when $x + x^J = 0$. This results from the following.

Theorem 1.2 ([A], Thm. 10.10) *Given an involution J of the second kind on an algebra A, central simple of degree d over K, there are elements $u_1, \ldots, u_d$, with $u_i = u_i^J$, such that A is generated over K by $u_1, \ldots, u_d$. Furthermore, there is an element $q \in A$, $q^J = -q$, $q^2 \in k$, such that, as a k-vector space,*

$$A = A^+ + qA^+,$$

where $A^+ = \{x \in A | x = x^J\}$.

If $x \in A$ is arbitrary, then $\frac{1}{2}(x + x^J) \in A^+$, while $\frac{1}{2}(x - x^J) \in A^-$. For example, we have $e^i + (e^i)^J =: E^i \in A^+$, and then we have an isomorphism

$$A^+ \cong \ell \oplus E\ell \oplus \cdots \oplus E^{d-1}\ell. \tag{8}$$

If, as above, $K = k(\sqrt{-\eta})$, $L = \ell(\sqrt{-\eta})$, then we may take $\sqrt{-\eta} = q$ in the theorem above, and for elements in

$$qA^+ \cong \sqrt{-\eta}\ell \oplus E\sqrt{-\eta}\ell \oplus \cdots \oplus E^{d-1}\sqrt{-\eta}\ell, \tag{9}$$

we have $y = -y^J$. In particular, the dimension of qA^+ is d^2 as a k-vector space, and the dimension of A is $2d^2$.

2 Algebraic groups

Let D be a central simple K-division algebra with a $K|k$-involution $x \mapsto \overline{x}$ as discussed above (denoted $x \mapsto x^J$ above, but unless it is a cause of confusion we fix the notation $x \mapsto \overline{x}$ for the remainder of the paper), with $K|k$ imaginary quadratic for $d = 1, \geq 3$ and $K = k$ for $d = 2$. Consider the simple algebra $M_2(D)$; it is endowed with an involution, $M \mapsto {}^t\overline{M} =: M^*$. The group of units of $M_2(D)$ is denoted $GL(2, D)$, its derived group is denoted by $SL(2, D)$. Consider a non-degenerate hermitian form on D^2 given by

$$h(\mathrm{x}, \mathrm{y}) = x_1\overline{y}_2 + x_2\overline{y}_1, \tag{10}$$

where $\mathrm{x} = (x_1, x_2), \mathrm{y} = (y_1, y_2) \in D^2$. This means the hermitian form is given by the matrix $H := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The unitary and special unitary groups for h are

$$U(D^2, h) = \left\{ g \in GL(2, D) \middle| gHg^* = H \right\}, \qquad SU(D^2, h) = U(D^2, h) \cap SL(2, D). \tag{11}$$

The equations defining $U(D^2, h)$ are then

$$U(D^2, h) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a\overline{d} + b\overline{c} = 1, a\overline{b} + b\overline{a} = c\overline{d} + d\overline{c} = 0 \right\}. \tag{12}$$

The additional equation defining $SU(D^2, h)$ can be written in terms of determinants, using Dieudonné's theory of determinants over skew fields, (see [Ar], p. 157)

$$\det(g) = N_{D|k}(ad - aca^{-1}b) = 1. \tag{13}$$

The center of $U(D^2, h)$, which we will denote by $\mathcal{C}$, is given by

$$\mathcal{C} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in K, a\overline{a} = 1 \right\} \cong U(1) \cap K,$$

where we view $K \subset \mathbb{C}$ as a subfield of the complex numbers. In particular:

Lemma 2.1 *For $d = 2$, $K = k$ a real field, we have $\mathcal{C} = \{\pm 1\}$.*

In other words, for the case $d = 2$, there is no essential difference between the unitary and special unitary groups. Otherwise, $U(D^2, h)$ is a reductive K-group, and $SU(D^2, h)$ is the corresponding simple group.

Proposition 2.2 *Let SG_D denote $SU(D^2, h)$, the simple algebraic K-group defined by the relations (12) and (13). Then SG_D is simple with the following index (cf. [T])*

1) $d = 1$, ${}^2A^1_{1,1}$.

2) $d = 2$, $C^2_{2,1}$.

3) $d \geq 3$, ${}^2A^d_{2d-1,1}$.

Proof: This is immediate from the description in [T] of these indices. □

The unitary group for $d = 1$ is familiar in different terms. If we set $A = \begin{pmatrix} -1/2 & -1 \\ -1/2 & 1 \end{pmatrix}$, then

$$A \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^tA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In particular, these two hermitian forms are equivalent. For the form $H_1 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, the corresponding unitary group is customarily denoted by $U(1,1)$. Hence for $K \subset \mathbb{C}$ imaginary quadratic over a real field, we have an isomorphism

$$\begin{array}{rcl} U(K^2,h) & \xrightarrow{\sim} & U(1,1;K) \\ g & \mapsto & AgA^{-1}; \end{array} \tag{14}$$

the group $U(1,1;K)$ is given by the familiar conditions

$$U(1,1;K) = \left\{ \left(\begin{array}{cc} \alpha c & \beta c \\ \overline{\beta c} & \overline{\alpha c} \end{array} \right) \Big| \alpha, \beta \in K, \alpha \neq 0, c\overline{c} = 1, \alpha\overline{\alpha} - \beta\overline{\beta} = 1 \right\},$$

and the special unitary group is given by

$$SU(1,1;K) = \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{array} \right) \in GL(2,K) \Big| \alpha\overline{\alpha} - \beta\overline{\beta} = 1 \right\}.$$

Recall that this latter group is isomorphic to a subgroup of $GL(2,\mathbb{R})$, in fact we have the well-known isomorphism

$$\begin{array}{rcl} SU(1,1;\mathbb{C}) & \xrightarrow{\sim} & SL(2,\mathbb{R}) \\ g & \mapsto & BgB^{-1}, \end{array} \tag{15}$$

where $B = \left(\begin{smallmatrix} -i & i \\ 1 & 1 \end{smallmatrix}\right)$ is the matrix of a fractional linear transformation mapping the disk to the upper half plane. If, instead of B, we use

$$B_\eta = \left(\begin{array}{cc} -i & i \\ \sqrt{\eta} & \sqrt{\eta} \end{array} \right), \qquad K = k(\sqrt{-\eta}), \tag{16}$$

then we have in fact

Proposition 2.3 *The matrix B_η gives an isomorphism*

$$\begin{array}{rcl} SU(1,1;K) & \xrightarrow{\sim} & SL_2(k) \\ & \mapsto & B_\eta g B_\eta^{-1}. \end{array}$$

In particular, we have an isomorphism

$$\begin{array}{rcl} SU(K^2,h) & \xrightarrow{\sim} & SL_2(k) \\ g & \mapsto & B_\eta AgA^{-1}B_\eta^{-1}. \end{array}$$

Proof: We only have to check that $B_\eta g B_\eta^{-1}$ is real, as it is clearly a matrix in $SL(2,K)$, and $SL(2,K) \cap SL(2,\mathbb{R}) = SL_2(k)$. If $g = \left(\begin{smallmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$ then

$$B_\eta g B_\eta^{-1} = \frac{1}{-2\sqrt{-\eta}} \left(\begin{array}{cc} \sqrt{-\eta}(-\alpha - \overline{\alpha} + \beta + \overline{\beta}) & -\alpha - \beta + \overline{\alpha} + \overline{\beta} \\ \eta(\alpha - \beta + \overline{\beta} - \overline{\alpha}) & -\sqrt{-\eta}(\alpha + \beta + \overline{\alpha} + \overline{\beta}) \end{array} \right) =$$

$$\left(\begin{array}{cc} Re(\alpha) - Re(\beta) & Im(\alpha) + Im(\beta) \\ \eta(-Im(\alpha) + Im(\beta)) & Re(\alpha) + Re(\beta) \end{array} \right),$$

which is clearly real. □

Note that the inverse $SL_2(k) \longrightarrow SU(K^2,h)$ is given by conjugating with $C = (B_\eta A)^{-1} = \begin{pmatrix} 0 & -2i \\ \sqrt{\eta} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{\eta}} \\ \frac{i}{2} & 0 \end{pmatrix}$, so we may describe the latter group as:

$$SU(K^2,h) = \left\{ \begin{pmatrix} \delta & 2\gamma/\sqrt{-\eta} \\ \beta\sqrt{-\eta}/2 & \alpha \end{pmatrix} \Big| \alpha,\beta,\gamma,\delta \in k, \alpha\delta - \gamma\beta = 1 \right\}. \tag{17}$$

We might remark at this point that it is a bit messy to describe the unitary groups in the same fashion, as they are *not* isomorphic to subgroups of $GL(2,\mathbb{R})$. We have the following extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & SU(1,1;K) & \longrightarrow & U(1,1;K) & \longrightarrow & U(1)\cap K & \longrightarrow 1 \\ & & \wr\| & & & & & \\ 1 & \longrightarrow & SL_2(k) & \longrightarrow & GL(2,k) & \longrightarrow & k^* & \longrightarrow 1, \end{array} \tag{18}$$

the first being an extension by a compact torus, the second by a split torus. That is why the map of Proposition 2.3, if extended to $U(1,1;K)$, does not land in $GL(2,\mathbb{R})$. It is also appropriate to remark here that, for $d=2$, the *reductive* group associated with the problem is the group of symplectic similtudes, i.e., $g \in GL(2,D)$ such that $gHg^* = H\lambda$ for some non-zero $\lambda \in D$. This gives an extension of $SU(D^2,h)$ similar to the second sequence above.

Next we describe some subgroups of $G_D = U(D^2,h)$ and SG_D. For this, recall that we are dealing with right vector spaces, and matrix multiplication is done from the right, i.e.,

$$(x_1,x_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (x_1a + x_2c, x_1b + x_2d).$$

First, the vectors $(1,0)$ and $(0,1)$ are isotropic; the stabilisers of the lines they span in D^2 are maximal (and minimal) parabolics. For example, the stabiliser of $(0,1)D$ is

$$P = \left\{ g \in G_D \Big| (0,1)g = (0,1)\lambda, \lambda \in D^* \right\} = \left\{ g = \begin{pmatrix} a & b \\ 0 & \overline{a}^{-1} \end{pmatrix} \Big| a \in D^*,\ b + \overline{b} = 0 \right\}. \tag{19}$$

Indeed, it is immediate that for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P$, we must have $c = 0$, while the first relation of (12) reduces to $a\overline{d} = 1$, or $d = \overline{a}^{-1}$. Since a parabolic is a semidirect product of a Levi factor and the unipotent radical, to calculate the latter we may assume $a = 1$. Then the second relation of (12) is $a\overline{b} + b\overline{a} = \overline{b} + b = 0$. The corresponding parabolic in SG_D takes the form

$$SP = \left\{ g \in SG_D \Big| g = \begin{pmatrix} a & b \\ 0 & \overline{a}^{-1} \end{pmatrix},\ a \in D^*,\ b + \overline{b} = 0 \right\}. \tag{20}$$

Note that g in (20) has K-determinant

$$\det_K(g) = N_{D|K}(a\overline{a}^{-1}) = N_{D|K}(a)N_{D|K}(\overline{a})^{-1},$$

so the requirement on g is that a must satisfy $N_{K|k}(N_{D|K}(a)/N_{D|K}(\overline{a})) = 1$. This is satisfied for all $a \in K^*$, where a is viewed as an element of D^*; if $a \notin K$, then the condition becomes $N_{D|K}(a) = \overline{N_{D|K}(a)}$, i.e., $N_{D|K}(a) \in k$. It follows that the Levi component is a product

$$L = \left\{ g = \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \Big| a \in D^* \right\} \cong (U(1)\cap K)\cdot \left\{ g = \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \Big| a \in D^*,\ N_{D|K}(a) \in k \right\}.$$

Now recall from (9) that for $d \geq 3$, the set of elements fulfilling $b + \overline{b} = 0$ in D is $\sqrt{-\eta}D^+ \cong \sqrt{-\eta}\ell \oplus E\sqrt{-\eta}\ell \oplus \cdots \oplus E^{d-1}\sqrt{-\eta}\ell$, where $E^i = e^i + \overline{e^i}$. In particular, the unipotent radical of SP, generated by $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b + \overline{b} = 0$, has dimension d^2 over k.

We also have non-isotropic vectors $(1,1)$ and $(1,-1)$. Any element $g \in G_D$ preserving one preserves also the other (as they are orthogonal). Hence the stabiliser of $(1,1)$ is

$$\mathcal{K} = \left\{ g = \begin{pmatrix} a & c \\ c & a \end{pmatrix} \middle| a\overline{a} + c\overline{c} = 1,\ a\overline{c} + c\overline{a} = 0 \right\}. \tag{21}$$

For $d = 1$, this can be more precisely described as

$$\mathcal{K} = \left\{ g = \begin{pmatrix} a & \frac{c}{\sqrt{-\eta}} \\ -\frac{c\sqrt{-\eta}}{\eta} & a \end{pmatrix},\ a, c \in k, a^2 - \frac{c^2}{\eta} = 1 \right\}.$$

For $d = 2$, the relations give, viewed as equations over k, four relations, so the group $\mathcal{K}$ is four-dimensional. Finally, for $d \geq 3$, we get (over k) d relations (as elements $a\overline{a}$ are in ℓ, so diagonal) from the first condition and $d^2 - d + 1$ conditions from the second, leaving $d^2 + d + 2$ parameters for the group $\mathcal{K}$.

If D has an involution of the second kind, we have the splitting field L ($= K$ for $d = 1$), which is an imaginary quadratic extension of the totally real field ℓ ($= k$ for $d = 1$), see the diagram (6). Suppose then $d = 2$, D quaternionic, say $D = (\ell/k, \sigma, b)$, where $\ell = k(\sqrt{a})$, $e^2 = b \in k^*$; in this case ℓ is a splitting field. Recall the conjugation σ is given by

$$(z_1 + \sqrt{a}z_2)^\sigma = z_1 - \sqrt{a}z_2,$$

so the element $c = diag(\sqrt{a}, -\sqrt{a})$ representing $\sqrt{a} \in \ell$ satisfies $c^\sigma = -c$, while $e^\sigma = e$. Consequently, the relation (2) for c is $ec = c^\sigma e = -ce$, and

$$(ec)^2 = (ec)(-ce) = -ec^2e = -ab, \tag{22}$$

so $k(ec) \cong k(\sqrt{-ab})$. If we assume, as we may, that $a > 0, b > 0$ (otherwise replace $-ab$ in what follows by the negative one of a, b), then $L := \sqrt{-ab}$ is an imaginary quadratic extension of k which is a subfield of D. So in all cases ($d = 1, 2, \geq 3$) we have an imaginary quadratic extension of ℓ ($d = 1, \geq 3$) or k ($d = 2$), L, which is a subfield of D. We now claim that $U(L^2, h)$ is a natural subgroup of $U(D^2, h)$. In all cases $U(L^2, h)$ and $U(D^2, h)$ are defined by the same relations (12), so we have

$$U(L^2, h) = U(D^2, h) \cap M_2(L), \tag{23}$$

which verifies the claim. In the case $d = 1$, where we took $L = K$, this is in fact the whole group. But even in this case we can get subgroups in this manner. Consider taking, instead of $L = K$, $L = \mathbb{Q}(\sqrt{-\eta})$ for $K = k(\sqrt{-\eta})$. More generally, for any $\mathbb{Q} \subseteq k' \subseteq k$, we have the subfield $k'(\sqrt{-\eta}) \subset K$, and we may consider the corresponding subgroup:

$$U(k'(\sqrt{-\eta})^2, h) = U(K^2, h) \cap M_2(k'(\sqrt{-\eta})).$$

So, setting $L = k'(\sqrt{-\eta})$ in (23), these subgroups are defined by the same relations (12). Now recall the description (17) for groups of the type (23).

Proposition 2.4 *We have the following subgroups of $U(D^2, h)$.*

1) $d = 1$, for any $\mathbb{Q} \subseteq k' \subseteq k$ we have the subgroup

$$SL_2(k') \cong \left\{ \begin{pmatrix} \alpha & 2\beta/\sqrt{-\eta} \\ \gamma\sqrt{-\eta}/2 & \delta \end{pmatrix} \middle| \alpha, \beta, \gamma, \delta \in k', \alpha\delta - \gamma\beta = 1 \right\} \subset U(K^2, h).$$

2) $d = 2$, for the field $L = k(\sqrt{-ab})$ above, we have the subgroup

$$SL_2(k) \cong \left\{ \begin{pmatrix} \alpha & 2\beta/\sqrt{-ab} \\ \gamma\sqrt{-ab}/2 & \delta \end{pmatrix} \Big| \alpha, \beta, \gamma, \delta \in k, \alpha\delta - \gamma\beta = 1 \right\} \subset U(D^2, h).$$

3) $d \geq 3$, for the degree d cyclic extension L/K we have the following subgroup

$$SL_2(\ell) \cong \left\{ \begin{pmatrix} \alpha & 2\beta/\sqrt{-\eta} \\ \gamma\sqrt{-\eta}/2 & \delta \end{pmatrix} \Big| \alpha, \beta, \gamma, \delta \in \ell, \alpha\delta - \gamma\beta = 1 \right\} \subset U(D^2, h).$$

These are interesting subgroups, whose existence is a natural part of the description of D as a cyclic algebra. In all cases we may view these subgroups as stabilisers. Let L denote one of the fields $k'(\sqrt{-\eta}), k(\sqrt{-ab}), \ell(\sqrt{-\eta})$ as in Proposition 2.4; let $L^2 \subset D^2$ denote the k-vector subspace of D^2, viewing D as a k-vector space, and consider the stabiliser in G_D. Noting that $L^2 \subset D^2$ may be spanned over L by two non-isotropic vectors $(1,1)$ and $(1,-1)$, it is clear that $g \in U(L^2, h)$ preserves the L span of these two vectors, that is, L^2, hence $U(L^2, h)$ is contained in the stabiliser. The converse is also true: if an endomorphism of D^2 preserves the L span, then its matrix representation is in $M_2(L)$, so the stabiliser is contained in the intersection $U(D^2, h) \cap M_2(L) = U(L^2, h)$. Let us record this as

Observation 2.5 *The subgroups of Proposition 2.4 are stabilisers (with determinant 1) of k-vector subspaces of the k-vector space D^2.*

3 Domains

Recall that given an almost simple algebraic group G over k, by taking the $\mathbb{R}$ points one gets a semisimple real Lie group $G(\mathbb{R})$. For the three kinds of groups above we determine these now.

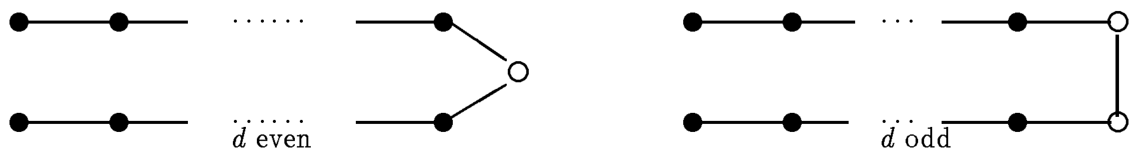
Proposition 3.1 *The real groups $G_D(\mathbb{R})$ are the following.*

1) $d = 1$, $G_D(\mathbb{R}) = (U(1,1))^f, SG_D(\mathbb{R}) \cong (SL(2,\mathbb{R}))^f$.

2) $d = 2$, $G_D(\mathbb{R}) = SG_D(\mathbb{R}) = (Sp(4,\mathbb{R}))^f$.

3) $d \geq 3$, $G_D(\mathbb{R}) = (U(d,d))^f$, $SG_D(\mathbb{R}) = (SU(d,d))^f$.

Proof: The $d = 1$ case is obvious. For $d = 2$, note that since D is totally indefinite, $D_\nu \cong M_2(\mathbb{R})$ for all real primes, and consequently $(G_D)_\nu$ is a simple group of type Sp for each ν, not compact as G_D is isotropic, of rank 2, hence $Sp(4,\mathbb{R})$. Suppose now that $d \geq 3$. By Proposition 2.2 we know the index of G_D is ${}^2A^d_{2d-1,1}$, and the index, as displayed in [T], is



(24)

where the number of black vertices is $2d-2$, $d-1$ on each branch. This shows in particular that the *isotropic* root is the one farthest to the right. If a root is isotropic over $\mathbb{Q}$, then all the more over $\mathbb{R}$. Hence, considering the Satake diagram of the real groups of type ${}^2A$, we see that the only possibility is $SU(d,d)$, as this is the only real group of this type for which the right most vertex is $\mathbb{R}$-isotropic. Actually, this only proves that *at least one* factor $(G_D)_\nu$ is $SU(d,d)$; but our hyperbolic form has signature (d,d) at *any* real prime, so it holds for *all* (non-compact) factors. That there are no compact factors follows from the fact that G_D is isotropic. In all cases there are f factors, as this is the degree of $k|\mathbb{Q}$, so the $\mathbb{Q}$-group, $Res_{k|\mathbb{Q}}G_D$, has f factors over $\mathbb{R}$. This verifies the proposition in all cases. $\square$

It is a consequence of this proposition that the symmetric spaces associated to the real groups $G_D(\mathbb{R})$ are *hermitian symmetric*; indeed, this is why we have choosen those D to deal with. More precisely, one has the following statement.

Theorem 3.2 *The hermitian symmetric domains defined by the $G_D(\mathbb{R})$ as in Proposition 3.1*

- *are tube domains, products of irreducible components of the types* $\mathbf{I_{1,1}}$, $\mathbf{III_2}$, $\mathbf{I_{d,d}}$ *in the cases* $d=1$, $d=2$ *and* $d\geq 3$, *respectively.*
- *For any rational parabolic $P\subset G_D$, the corresponding boundary component F such that $P(\mathbb{R})=N(F)$ is a point.*

Proof: The first statement follows immediately from Proposition 3.1. The second is well known in the case $d=1$ from the study of Hilbert modular varieties. For $d=2$ the statement clearly holds for $k=\mathbb{Q}$ (just look at the index, from which one sees the boundary components are points, not one-dimensional), and the general statement follows from this: all boundary components are of the type $(c_1,\ldots,c_f)\in\mathcal{D}_1^*\times\cdots\times\mathcal{D}_f^*$, where $c_i\in\mathcal{D}_i^*$ is a point, which is a general fact about $\mathbb{Q}$-parabolics in a *simple* $\mathbb{Q}$-group (this applies directly to $Res_{k|\mathbb{Q}}SG_D$, but G_D and SG_D define the same domain and boundary components). Finally, for $d\geq 3$, we point out that in the diagram (24), the isotropic vertex (farthest right) corresponds to a maximal parabolic which stabilises a point, establishing the statement for $k=\mathbb{Q}$. Then the general statement follows as above from this. $\square$

We now consider the structure of the real parabolics for $d\geq 2$ in more detail. For this we refer to Satake's book, §III.4.

- $d=2$: Here we have the case denote $b=n$ in Satake's book, and the parabolic in $Sp(4,\mathbb{R})$ is

$$P(\mathbb{R})=(G^{(1)}\cdot G^{(2)})\rtimes U\cdot V=1\cdot GL(2,\mathbb{R})\rtimes Sym_2(\mathbb{R}),$$

where $Sym_2(\mathbb{R})$ denotes the set of symmetric real 2×2 matrices. Refering to (19), we may identify the Levi component of P with

$$L\cong\left\{\left(\begin{array}{cc} a & 0\\ 0 & \overline{a}^{-1}\end{array}\right)\Big| a\in D^*\right\}\cong D^*,$$

and, as D is indefinite, $D^*(\mathbb{R})\cong GL(2,\mathbb{R})$; this is the factor $GL(2,\mathbb{R})$. The unipotent radical is

$$U\cong\left\{\left(\begin{array}{cc} 1 & b\\ 0 & 1\end{array}\right)\Big|\, Tr(b)=0\right\}\cong D^0 \text{ (=totally imaginary elements)},$$

which is three-dimensional, and an isomorphism $U(\mathbb{R})\cong Sym_2(\mathbb{R})$ is given by

$$\left(\begin{array}{cc} a_1\sqrt{a} & a_2+a_3\sqrt{a}\\ b(a_2-a_3\sqrt{a}) & -a_1\sqrt{a}\end{array}\right)\mapsto\left(\begin{array}{cc} a_1 & a_2\\ a_2 & a_3\end{array}\right).$$

This describes the real parabolic completely in this case.

- $d\geq 3$: Here we have $b=d$ in Satake's notation, $d=p=q$. The real parabolic is

$$P(\mathbb{R})=(G^{(1)}\cdot G^{(2)})\rtimes U\cdot V=(U(1)\cdot GL^0(d,\mathbb{C}))\rtimes\mathcal{H}_d(\mathbb{C}),$$

where $GL^0(d,\mathbb{C}) = \{g \in GL(d,\mathbb{C}) | \det(g) \in \mathbb{R}^*\}$ and $\mathcal{H}_d(\mathbb{C})$ denotes the space of hermitian $d \times d$ matrices. We must now consider the parabolic SP as in (20). The Levi component is again

$$L \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a}^{-1} \end{pmatrix} \Big| a \in D^*,\ N_{D|k}(a\overline{a}^{-1}) = 1 \right\},$$

the second condition because of determinant 1. From the multiplicativity of the norm, the second relation can be written $N_{D|K}(a) = N_{D|K}(\overline{a})$. Since the norm is given by the determinant, this means, in $D^*(\mathbb{R})$,

$$L(\mathbb{R}) \cong \left\{ a \in D^*(\mathbb{R}) \Big| a \notin K \Rightarrow \det(a) \in \mathbb{R}^* \right\} \cong U(1) \cdot GL^0(d,\mathbb{C}),$$

where the $U(1)$ factor arises from the elements in $U(1) \cap K$. Similarly, the unipotent radical takes the form

$$U \cong \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Big| b \in D,\ b + \overline{b} = 0 \right\} \cong \sqrt{-\eta} D^+,$$

which is d^2-dimensional. An explicit isomorphism $U(\mathbb{R}) \cong \mathcal{H}_d(\mathbb{C})$ is given by the identity on $D^+(\mathbb{R})$, since $x \in D^+ \Rightarrow x = \overline{x}$, hence as a real matrix, this element is hermitian.

The maximal compact subgroups of $G_D(\mathbb{R})$ are the groups $\mathcal{K}(\mathbb{R})$, where $\mathcal{K}$ is the group of (21). Indeed, it is easy to see that the stabiliser of the non-isotropic vector $(1,1)$ is the stabiliser, in the domain $\mathcal{D}$, of the base point, which is by definition the maximal compact subgroup.

Now consider the subgroups $U(L^2, h) \subset G_D$; let us introduce the notation G_L for these subgroups. Then, since G_L is of the type $d = 1$ in Proposition 3.1, it follows that Theorem 3.2 applies to $G_L(\mathbb{R})$ as well as to $G_D(\mathbb{R})$.

Proposition 3.3 *We have a commutative diagram*

$$\begin{array}{ccc} G_L(\mathbb{R}) & \hookrightarrow & G_D(\mathbb{R}) \\ \downarrow & & \downarrow \\ \mathcal{D}_L & \hookrightarrow & \mathcal{D}_D, \end{array}$$

where $\mathcal{D}_L$ and $\mathcal{D}_D$ are hermitian symmetric domains, and the maps $G(\mathbb{R}) \longrightarrow \mathcal{D}$ are the natural projections. Moreover, the subdomains $\mathcal{D}_L$ are as follows:

1) *$d = 1$; We now assume that the extension k/k' is Galois. Then, if $\deg_{\mathbb{Q}} k' = f'$, $f/f' = m$, we have $\mathcal{D}_L \cong (\mathfrak{H})^{f'}$ and $\mathcal{D}_D \cong ((\mathfrak{H})^m)^{f'}$ and the embedding $\mathcal{D}_L \subset \mathcal{D}_D$ is given by $\mathfrak{H} \hookrightarrow (\mathfrak{H})^m$ diagonally, and the product of this f' times.*

2) *$d = 2$; $\mathcal{D}_L \cong \begin{pmatrix} \tau_1 & 0 \\ 0 & b^{\zeta_1}\tau_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} \tau_1 & 0 \\ 0 & b^{\zeta_f}\tau_1 \end{pmatrix}$, where $\zeta_i : k \longrightarrow \mathbb{R}$ denote the distinct real embeddings of k.*

3) *$d \geq 3$; $\mathcal{D}_L \cong \begin{pmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_d \end{pmatrix}^f$.*

Proof: Considering the tube realisation of the domain $\mathcal{D}_D$, we have the base point

$$o = (\mathrm{diag}(i, i, \ldots, i))^f,$$

and we just calculate the orbit of the $\mathbb{R}$-group $G_L(\mathbb{R})$ of this point; this gives the corresponding subdomain. The commutativity of the diagram follows from this.

1) $d = 1$: Since k/k' is Galois, we have that $G_L \subset G_D$ is the subgroup fixed under the natural $Gal_{k|k'}$ action; hence lifting the $\mathbb{Q}$-groups to $\mathbb{R}$, we get the identification

$$(Res_{k'|\mathbb{Q}} G_L)_{\mathbb{R}} \cong (Res_{k|\mathbb{Q}} G_D)_{\mathbb{R}}^{Gal_{k|k'}}.$$

In terms of the domains, if $\mathcal{D}_D = (\mathfrak{H})^f$, then $Gal_{k|k'}$ acts by permuting m copies at a time, and the invariant part is the diagonally embedded $\mathfrak{H} \hookrightarrow (\mathfrak{H})^m,\ z \mapsto (z, \ldots, z)$. Since $\mathcal{D}_L \cong (\mathfrak{H})^{f'}$, the result follows in this case.

2) $d = 2$: First we remark that $G_{\mathbb{R}}$ is conjugate to the *standard* $Sp(4, \mathbb{R})$ (by which we mean the symplectic group with respect to the symplectic form $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$) under the element

$$q = \left(\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & -1 & 0 \end{array} \right),$$

that is $qG_{\mathbb{R}}q^{-1} = Sp(4, \mathbb{R})$. To see this, we recall how the (D-valued) hyperbolic (hermitian) form is related to alternating forms. Since $D_{\mathbb{R}} \cong M_2(\mathbb{R})$, our vector space D^2 over $\mathbb{R}$ is $V_{\mathbb{R}} = (M_2(\mathbb{R}))^2$; the hyperbolic form gives a $M_2(\mathbb{R})$-valued form on $V_{\mathbb{R}}$. We have matrix units $e_{ij},\ i, j = 1, 2$ in $M_2(\mathbb{R})$, and setting $W_1 = e_{11}V,\ W_2 = e_{22}V$, the spaces W_i are two-dimensional vector spaces over $\mathbb{R}$. The hermitian form $h(\mathbf{x}, \mathbf{y})$, restricted to W_1, turns out to have values in $\mathbb{R}e_{12}$: if

$$\mathbf{x} = \left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \begin{pmatrix} x'_{11} & x'_{12} \\ x'_{21} & x'_{22} \end{pmatrix} \right), \quad \mathbf{y} = \left(\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \begin{pmatrix} y'_{11} & y'_{12} \\ y'_{21} & y'_{22} \end{pmatrix} \right),$$

then $\mathbf{x}, \mathbf{y} \in W_1 \iff x_{2i} = x'_{2i} = y_{2i} = y'_{2i} = 0$. Then we have[4]

$$h(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} x_{11} & x_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -y'_{12} \\ 0 & y'_{11} \end{pmatrix} + \begin{pmatrix} x'_{11} & x'_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -y_{12} \\ 0 & y_{11} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -x_{11}y'_{12} + x_{12}y'_{11} - x'_{11}y_{12} + x'_{12}y_{11} \\ 0 & 0 \end{pmatrix}.$$

Then the alternating form $\langle \mathbf{x}, \mathbf{y} \rangle$ is defined by

$$h(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle \cdot e_{12},\ \mathbf{x}, \mathbf{y} \in W_1.$$

In other words, viewing the symplectic group as a subgroup of $GL(2, M_2(\mathbb{R}))$, the symplectic form will be given by $J_1 = \left(\begin{smallmatrix} 0 & J_2 \\ J_3 & 0 \end{smallmatrix}\right)$, where J_2 and J_3 are symplectic forms on $\mathbb{R}^2$. It is clear that $q\left(\begin{smallmatrix} 0 & 1_2 \\ 1_2 & 0 \end{smallmatrix}\right)^t q = J_1$ is of this form, and this shows $qG_{\mathbb{R}}q^{-1} = Sp(4, \mathbb{R})$.

Next, recalling that we view D as a cyclic algebra of 2×2 matrices, an element $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in M_2(D)$ is actually a 4×4 matrix; each of $\alpha, \beta, \gamma, \delta$ are of the form (4), and this 4×4 matrix must be conjugated by q, then the element acts in the well known manner on the domain $\mathcal{D}_D$:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{R}), \quad \tau = \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix} \in \mathcal{D}_D, \quad M \cdot \tau = (A\tau + B)(C\tau + D)^{-1}.$$

[4] the involution on $M_2(k)$ is as in (5).

Finally we note that the subgroup $U(L^2, h) \subset U(D^2, h)$ consists of the elements of the form as in Proposition 2.4 2):

$$x = \begin{pmatrix} \alpha & 2\beta/\sqrt{-ab} \\ \gamma\sqrt{-ab}/2 & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in k,$$

and the element $\sqrt{-ab} = ec$ as in (22), so that $(\sqrt{-ab})^{-1} = (ec)^{-1}$. In other words, as a 4×4 matrix, the element x above is

$$x = \left(\begin{array}{cc|cc} \alpha & 0 & 0 & \frac{2\beta}{b\sqrt{a}} \\ 0 & \alpha & \frac{2\beta}{-\sqrt{a}} & 0 \\ \hline 0 & \frac{-\gamma\sqrt{a}}{2} & \delta & 0 \\ \frac{b\gamma\sqrt{a}}{2} & 0 & 0 & \delta \end{array}\right), \tag{25}$$

and after conjugating with q this becomes

$$qxq^{-1} = \left(\begin{array}{cc|cc} \alpha & 0 & \frac{2\beta}{b\sqrt{a}} & 0 \\ 0 & \alpha & 0 & \frac{2\beta}{\sqrt{a}} \\ \hline \frac{b\gamma\sqrt{a}}{2} & 0 & \delta & 0 \\ 0 & \frac{\gamma\sqrt{a}}{2} & 0 & \delta \end{array}\right). \tag{26}$$

From the form of the matrix (26) we see that the corresponding subdomain is contained in the set of diagonal matrices $\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)^f$. Now calculating, setting $g = qxq^{-1}$, we have

$$g\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}^f = \left(\begin{pmatrix} \left(\frac{\alpha b\tau_1 + \frac{2\beta\sqrt{a}}{a}}{\frac{\gamma\sqrt{a}}{2}b\tau_1 + \delta}\right)\frac{1}{b} & 0 \\ 0 & \frac{\alpha\tau_2 + \frac{2\beta\sqrt{a}}{a}}{\frac{\gamma\sqrt{a}}{2}\tau_2 + \delta} \end{pmatrix}^{\zeta_1}, \ldots, \begin{pmatrix} \left(\frac{\alpha b\tau_1 + \frac{2\beta\sqrt{a}}{a}}{\frac{\gamma\sqrt{a}}{2}b\tau_1 + \delta}\right)\frac{1}{b} & 0 \\ 0 & \frac{\alpha\tau_2 + \frac{2\beta\sqrt{a}}{a}}{\frac{\gamma\sqrt{a}}{2}\tau_2 + \delta} \end{pmatrix}^{\zeta_f}\right), \tag{27}$$

So in particular, G_L maps the set of matrices

$$\mathcal{D}_L = \begin{pmatrix} \tau_1 & 0 \\ 0 & b^{\zeta_1}\tau_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} \tau_1 & 0 \\ 0 & b^{\zeta_f}\tau_1 \end{pmatrix} \tag{28}$$

into itself. (In the expression (27), the bracket $(\ldots)^{\zeta_i}$ means ζ_i is applied to all matrix elements). This proves the theorem in the case $d = 2$.

Remark 3.4 The description of the subdomain $\mathcal{D}$ in terms of b, depends on the way we described D as a cyclic algebra. We took $\ell = k(\sqrt{a})$, $D = (\ell/k, \sigma, b)$. But we could also consider $\ell' = k(\sqrt{b})$, $D' = (\ell'/k, \sigma', a)$; these two cyclic algebras are isomorphic, an isomorphism being given by extending the identity on the center k by

$$\begin{array}{ccc} D & \xrightarrow{\sim} & D' \\ e & \mapsto & c' \\ c & \mapsto & e', \end{array}$$

where

$$e = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \qquad e' = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \quad c' = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & -\sqrt{b} \end{pmatrix}.$$

Indeed, as one sees immediately, $(ec)^2 = -ab = (e'c')^2$, so the map above is a morphism, which is clearly bijective. Consequently, it would be better to denote (28) $\mathcal{D}_{L,b}$, since we *also* have a subdomain

$$\mathcal{D}_{L,a} = \begin{pmatrix} \tau_1 & 0 \\ 0 & a^{\zeta_1}\tau_1 \end{pmatrix} \times \cdots \times \begin{pmatrix} \tau_1 & 0 \\ 0 & a^{\zeta_f}\tau_1 \end{pmatrix},$$

neither of which is a priori privledged.

3) $d \geq 3$: Here the situation is simpler; the matrix H is the matrix of the hermitian form whose symmetry group acts on the usual unbounded realisation of the domain of type $\mathbf{I_{d,d}}$, hence no conjugation is necessary. Let G_L be the subgroup we wish to consider, and recall from Proposition 2.4 that it consists of matrices

$$x = \begin{pmatrix} \alpha & 2\beta/\sqrt{-\eta} \\ \gamma\sqrt{-\eta}/2 & \delta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta \in k, \ \alpha\delta - \gamma\beta = 1.$$

Thinking of D itself as $d \times d$ matrices, this element is

$$x = \left(\begin{array}{cccc|cccc} \alpha & & & 0 & \frac{2\beta}{\sqrt{-\eta}} & & & 0 \\ & \alpha^{\sigma} & & & & \frac{2\beta^{\sigma}}{\sqrt{-\eta}} & & \\ & & \ddots & & & & \ddots & \\ 0 & & & \alpha^{\sigma^{d-1}} & 0 & & & \frac{2\beta^{\sigma^{d-1}}}{\sqrt{-\eta}} \\ \hline \frac{\gamma\sqrt{-\eta}}{2} & & & 0 & \delta & & & 0 \\ & \frac{\gamma^{\sigma}\sqrt{-\eta}}{2} & & & & \delta^{\sigma} & & \\ & & \ddots & & & & \ddots & \\ 0 & & & \frac{\gamma^{\sigma^{d-1}}\sqrt{-\eta}}{2} & 0 & & & \delta^{\sigma^{d-1}} \end{array}\right), \tag{29}$$

and the orbit of the base point $\mathrm{diag}(i, \ldots, i)^f$ under elements as x is clearly the diagonal subdomain $(\mathrm{diag}(\tau_1, \ldots, \tau_d))^f$, and the action is, for $k = \mathbb{Q}$,

$$\mathrm{diag}(\tau_1, \ldots, \tau_d) \mapsto \left(\frac{\alpha\tau_1 + \frac{2\beta}{\sqrt{-\eta}}}{\frac{\gamma\sqrt{-\eta}}{2}\tau_1 + \delta}, \ldots, \frac{\alpha^{\sigma^{d-1}}\tau_d + \frac{2\beta^{\sigma^{d-1}}}{\sqrt{-\eta}}}{\frac{\gamma^{\sigma^{d-1}}\sqrt{-\eta}}{2}\tau_d + \delta^{\sigma^{d-1}}} \right) =: \begin{pmatrix} \alpha & \frac{2\beta}{\sqrt{-\eta}} \\ \frac{\gamma\sqrt{-\eta}}{2} & \delta \end{pmatrix} (\tau_1, \ldots, \tau_d), \tag{30}$$

and for general k, letting as above $\zeta_1, \ldots, \zeta_f$ denote the f embeddings of k,

$$(\mathrm{diag}(\tau_1, \ldots, \tau_d))^f \mapsto \left(\begin{pmatrix} \alpha & \frac{2\beta}{\sqrt{-\eta}} \\ \frac{\gamma\sqrt{-\eta}}{2} & \delta \end{pmatrix}^{\zeta_1} (\tau_1, \ldots, \tau_d), \ldots, \begin{pmatrix} \alpha & \frac{2\beta}{\sqrt{-\eta}} \\ \frac{\gamma\sqrt{-\eta}}{2} & \delta \end{pmatrix}^{\zeta_f} (\tau_1, \ldots, \tau_d) \right). \tag{31}$$

This establishes all statements of the theorem. □

Finally, let F be the rational boundary component $\in \mathcal{D}_D^*$ corresponding to the isotropic vector $(0, 1)$, of which the parabolic P of (19) is the normaliser, $P(\mathbb{R}) = N(F)$. Note that by construction, $(0, 1)$ is also isotropic for G_L; let F_L be the rational boundary component $\in \mathcal{D}_L^*$ corresponding to it, so the parabolic in G_L, P_L of (19) is the normaliser, $P_L(\mathbb{R}) = N(F_L)$. Then clearly

$$P_L = G_L \cap P. \tag{32}$$

Also we have

Proposition 3.5 *Let $\mathcal{D}_L \subset \mathcal{D}_D$, $F_L \in \mathcal{D}_L^*$, $F \in \mathcal{D}_D^*$ be as above, and let $i : \mathcal{D}_L^* \hookrightarrow \mathcal{D}_D^*$ be the induced inclusion of Satake compactifications of the domains. Then $i(F_L) = F$.*

We will refer to this as "the subdomain $\mathcal{D}_L$ contains F as a rational boundary component." The proposition itself is, after unraveling the definitions, nothing by (32).

4 Arithmetic groups

It is well-known how to construct arithmetic groups $\Gamma \subset G_D(K)$; we recall this briefly. View D^2 as a D-vector space, and let $\Delta \subset D$ be a maximal order, that is an $\mathcal{O}_K$-module (where D is central simple over K, $K = k$ if $d = 2$), which spans D as a K-vector space, is a sub*ring* of D, and is maximal with these properties. Then set

$$\Gamma_\Delta := G_D(K) \cap M_2(\Delta). \tag{33}$$

If we view $\Delta^2 \subset D^2$ as a lattice, then this is also described as

$$\Gamma_\Delta = \left\{ g \in G_D(K) \middle| g(\Delta^2) \subset \Delta^2 \right\}.$$

From the second description it is clear that Γ_Δ is an *arithmetic* subgroup. We will denote the quotient of the domain $\mathcal{D}_D$ by this arithmetic subgroup by

$$X_{\Gamma_\Delta} = X_\Delta = \Gamma_\Delta \backslash \mathcal{D}_D. \tag{34}$$

If Γ_Δ were fixed-point free on $\mathcal{D}_D$ this would be a non-compact complex manifold; if Γ_Δ has fixed points (as it usually does), these yield singularities of the space X_Δ, which is still an analytic variety. It is a V-variety in the language of Satake. In fact it has a *global* finite smooth cover: let $\Gamma \subset \Gamma_\Delta$ be a normal subgroup of finite index which does act freely (such exist by a theorem of Selberg). Then we have a Galois cover

$$X_\Gamma \longrightarrow X_{\Gamma_\Delta},$$

and X_Γ is smooth. Hence the singularities of X_{Γ_Δ} are encoded in the action of the finite Galois group of the cover.

The Baily-Borel compactification of X_Γ ($\Gamma \subset \Gamma_\Delta$) is an embedding

$$X_\Gamma \subset X_\Gamma^* \subset \mathbb{P}^N,$$

where X_Γ^* is a normal algebraic variety. Since the boundary components of $\mathcal{D}_D$ are points (Theorem 3.2), it follows that the singularities of X_Γ^* contained in the boundary $X_\Gamma^* - X_\Gamma$ are also *isolated points.* In particular, if $\Gamma \subset \Gamma_\Delta$ has no elements of finite order, then X_Γ^* is smooth outside a finite set of isolated points, all contained in the compactification locus.

The singularities of X_Γ^* may be resolved by means of toroidal compactifications; let us denote such a compactification by $\overline{X}_\Gamma$, for which we make the assumptions:

i) $\overline{X}_\Gamma - X_\Gamma$ is a normal crossings divisor,

ii) $\overline{X}_\Gamma \longrightarrow X_\Gamma^*$ is a resolution of singularities and $\overline{X}_\Gamma$ is projective algebraic.

Such a compactification $\overline{X}_\Gamma$ depends on a set of polyhedral cones, and one can make choices such that i) and ii) are satisfied, by the results of [SC]. We will not need these in detail; the mere existence will be sufficient.

Recall that a cusp of an arithmetic group Γ is a maximal flag of boundary components of Γ; in the case at hand this is just a point in $X_\Gamma^* - X_\Gamma$. The *number of cusps* is the number of points in $X_\Gamma^* - X_\Gamma$, and may be defined in the following ways:

a) It is the number of Γ-equivalence classes of parabolic subgroups of G_D or SG_D; here Γ acts on the parabolic subgroups by conjugation.

b) It is the number of Γ-equivalence classes of isotropic vectors of the hermitian plane (D^2, h); here Γ is acting as a subgroup of G_D on the vector space.

In the $d = 1$ case we are talking about the number of cusps of a Hilbert modular variety. Let us recall how these numbers are determined. For this we consider first the group $SL_2(k)$ (instead of $SU(K^2, h)$), which acts on a product of upper half planes, $\mathcal{D}_D = \mathfrak{H}^f$. The boundary is the product $(\mathbb{P}^1(\mathbb{R}))^f$, and the rational boundary components are the points of the image $\mathbb{P}^1(k) \hookrightarrow (\mathbb{P}^1(\mathbb{R}))^f$ given by $x \mapsto (x^{\zeta_1}, \ldots, x^{\zeta_f})$. Hence we may denote the boundary components by $\xi = (\xi_1 : \xi_2)$, $\xi_i \in \mathcal{O}_k$. Let $\mathfrak{a}_\xi$ denote the ideal generated by ξ_1, ξ_2. Then one has

Proposition 4.1 *Two boundary components $\xi = (\xi_1 : \xi_2)$ and $\eta = (\eta_1 : \eta_2)$ are equivalent under $SL_2(\mathcal{O}_k) \iff$ the ideals $\mathfrak{a}_\xi$ and $\mathfrak{a}_\eta$ are in the same class. In particular, the number of cusps is the class number of k.*

Proof: One direction is trivial: if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \eta_1 \\ \eta_2 \end{smallmatrix}\right)$, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathcal{O}_k)$, then the ideals $\mathfrak{a}_\xi$ and $\mathfrak{a}_\eta$ are the *same*. This is because g is unimodular, so affects just a change of base in the ideal $\mathfrak{a}_\xi$. Conversely, suppose ξ and η are given, and suppose that $\mathfrak{a}_\xi$ and $\mathfrak{a}_\eta$ have the same class. After multiplication by an element $c \in k^*$, we may assume $\mathfrak{a}_\xi = \mathfrak{a}_\eta = \mathfrak{a}$. Furthermore, writing $\mathcal{O}_k = \mathfrak{a}\mathfrak{a}^{-1} = \xi_1\mathfrak{a}^{-1} + \xi_2\mathfrak{a}^{-1}$, we see that $1 \in \mathcal{O}_k$ can be written

$$1 = \xi_1\xi_2' - \xi_2\xi_1' = \eta_1\eta_2' - \eta_2\eta_1', \quad \xi_i', \eta_i' \in \mathfrak{a}^{-1}. \tag{35}$$

But this means the matrices (acting from the left)

$$M_\xi = \begin{pmatrix} \xi_1 & \xi_1' \\ \xi_2 & \xi_2' \end{pmatrix}, \qquad M_\eta = \begin{pmatrix} \eta_1 & \eta_1' \\ \eta_2 & \eta_2' \end{pmatrix}, \tag{36}$$

are in $SL_2(k)$, and

$$M_\xi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad M_\eta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{37}$$

Of course M_ξ, M_η are not in $SL_2(\mathcal{O}_k)$, but from the fact that $\xi_i', \eta_i' \in \mathfrak{a}^{-1}$ it follows that $M_\xi M_\eta^{-1} \in SL_2(\mathcal{O}_k)$. Hence

$$(M_\xi M_\eta^{-1})\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

and the cusps are conjugate under $SL_2(\mathcal{O}_k)$. □

We now translate this into the corresponding statement for the hyperbolic plane (K^2, h), where $K|k$ is an imaginary quadratic extension. First note that if $\left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right) \in K^2$ is *isotropic*, then

$$Tr_{K|k}(\xi_1\overline{\xi}_2) = \xi_1\overline{\xi}_2 + \xi_2\overline{\xi}_1 = 0.$$

In fact, the converse also holds,

Lemma 4.2 *A vector $\left(\begin{smallmatrix} \xi_1 \\ \xi_2 \end{smallmatrix}\right) \in K^2$ is isotropic with respect to $h \iff Tr(\xi_1\overline{\xi}_2) = Tr(\overline{\xi}_1\xi_2) = 0 \iff \xi_2 = 0$ or $Tr(\xi_1\xi_2^{-1}) = 0$.*

Proof: By definition, $\binom{\xi_1}{\xi_2}$ is isotropic $\iff$ $\xi_1\overline{\xi}_2 + \xi_2\overline{\xi}_1 = 0$, but this is $Tr(\xi_1\overline{\xi}_2) = 0$. Noting that $\xi_2^{-1} = \frac{1}{N(\xi_2)}\overline{\xi}_2$, if $\xi_2 \neq 0$, we get the second equivalence, from the linearity of the trace, $Tr(\xi_1\xi_2^{-1}) = Tr(\xi_1\frac{1}{N(\xi_2)}\overline{\xi}_2) = \frac{1}{N(\xi_2)}Tr(\xi_1\overline{\xi}_2)$. If $\xi_2 = 0$, then $\binom{\xi_1}{\xi_2} = \binom{\xi_1}{0}$ is isotropic anyway, establishing both equivalences as stated. $\square$

Let K^0 denote the purely imaginary (traceless) elements of K, $K^0 \cong \sqrt{-\eta}k$ for $K = k(\sqrt{-\eta})$. It follows from Lemma 4.2 that any $x \in K^0$ of the form $x = \xi_1\xi_2^{-1}$ or $x = \xi_1\overline{\xi}_2$ determines an isotropic vector $\binom{\xi_1}{\xi_2}$ of K^2. Consider in particular the case that $\binom{\xi_1}{\xi_2} \in \mathcal{O}_K^2$ is integral. Recall that (non-zero) ξ_1 and ξ_2 are relatively prime $\iff$ there are $x, y \in \mathcal{O}_K$ such that $x\xi_1 + y\xi_2 = 1$. Define the set of relatively prime integral isotropic vectors,

$$\mathfrak{I} = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathcal{O}_K^2 - (0,0) \middle| Tr(\xi_1\overline{\xi}_2) = 0; \text{ and if } \xi_i \neq 0, i = 1,2, \text{ then } \exists_{x,y\in\mathcal{O}_K}\ x\xi_1 + y\xi_2 = 1 \right\}. \quad (38)$$

Following the proof of Proposition 4.1, we consider when two isotropic integral vectors are conjugate under $S\Gamma_\Delta = SU(\mathcal{O}_K^2, h)$. Let now $\mathfrak{a}_\xi$ be the ideal generated by ξ_1, ξ_2 in $\mathcal{O}_K$.

Proposition 4.3 *Two isotropic vectors $\xi = \binom{\xi_1}{\xi_2}$ and $\xi' = \binom{\xi_1'}{\xi_2'} \in \mathfrak{I}$ are conjugate under $SU(\mathcal{O}_K^2, h)$ $\iff$ the ideals $\mathfrak{a}_\xi$ and $\mathfrak{a}_{\xi'}$ are equivalent in K $\iff$ the ideals $N(\mathfrak{a}_\xi)$ and $N(\mathfrak{a}_{\xi'})$ are equivalent in k.*

Proof: If $g\binom{\xi_1}{\xi_2} = \binom{\xi_1'}{\xi_2'}$, $g \in SU(\mathcal{O}_K^2, h)$, then since g is unimodular, the ideal classes coincide. Conversely, suppose $\binom{\xi_1}{\xi_2}$ and $\binom{\xi_1'}{\xi_2'}$ are equivalent; again we may assume $\mathfrak{a}_\xi = \mathfrak{a}_{\xi'} = \mathfrak{a}$. Then, as above, there are $\rho,\ \rho' \in (\mathfrak{a}^{-1})^2$ such that (35) holds. Consequently we have M_ξ, $M_{\xi'}$, but now in $SU(K^2, h)$, such that (36) and (37) hold. It follows that $M_\xi M_{\xi'}^{-1} \in SU(\mathcal{O}_K^2, h)$ maps ξ' to ξ. This completes verification of the first $\iff$. The second equivalence then follows from Proposition 2.3 and Proposition 4.1. $\square$

We now derive an analogue of the above for general D as in (1). We first note that Lemma 4.2 is true here also.

Lemma 4.4 *Let $\xi = (\xi_1, \xi_2) \in D^2$ be given. Then ξ is isotropic $\iff$ $\xi_1\overline{\xi}_2 + \xi_2\overline{\xi}_1 = 0$.*

Proof: We have $h(\xi, \xi) = \xi_1\overline{\xi}_2 + \xi_2\overline{\xi}_1 = 0$, so that $h(\xi, \xi) = 0$ if and only if $\xi_1\overline{\xi}_2 + \xi_2\overline{\xi}_1 = 0$. $\square$

Next we note that any isotropic vector is conjugate in G_D to the standard isotropic vector $(0, 1)$.

Lemma 4.5 *Let $\xi = (\xi_1, \xi_2) \in D^2$ be isotropic. Then there is a matrix $M_\xi \in G_D$ such that $(0,1)M_\xi = \xi$.*

Proof: If $M_\xi = \begin{pmatrix} \xi_1' & \xi_2' \\ \xi_1 & \xi_2 \end{pmatrix}$, then $(0,1)M_\xi = \xi$. So we must show the existence of such an $M_\xi \in G_D$. The equations to be solved (for ξ_i') are

$$\begin{aligned} &1)\ \xi_1'\overline{\xi}_2 + \xi_2'\overline{\xi}_1 = 1. \\ &2)\ \xi_1'\overline{\xi}_2' + \xi_2'\overline{\xi}_1' = 0. \\ &3)\ \xi_1\overline{\xi}_2 + \xi_2\overline{\xi}_1 = 0. \end{aligned} \quad (39)$$

Since ξ is isotropic, 3) is fulfilled by Lemma 4.4. If $\xi_1 \neq 0$, then setting $\xi_1' = 0, \xi_2' = \overline{\xi}_1^{-1}$, ξ' fulfills both 1) and 2). If $\xi_1 = 0$, then $\xi_2 \neq 0$ and we set similarly $\xi_1' = \overline{\xi}_2^{-1}$ and $\xi_2' = 0$. $\square$

Now we consider as above integral isotropic vectors. Let $\Delta \subset D$ be a maximal order, and set

$$\mathfrak{I}_\Delta = \left\{ (\xi_1, \xi_2) \in \Delta^2 - \{(0,0)\} \middle| \xi_1\overline{\xi}_2 + \xi_2\overline{\xi}_1 = 0, \xi_1, \xi_2 \neq 0 \Rightarrow \exists_{x,y\in\Delta}\ \xi_1 x + \xi_2 y = 1 \right\}.$$

Lemma 4.6 *Let* $(\xi_1, \xi_2) \in \mathfrak{I}_\Delta$. *Then there exists a matrix*

$$M_\xi = \begin{pmatrix} \xi_1'' & \xi_2'' \\ \xi_1 & \xi_2 \end{pmatrix} \in \Gamma_\Delta.$$

Proof: If $\xi_2 = 0$, then $\xi_2'' = 1$, any ξ_1'' with $\xi_1'' + \overline{\xi}_1'' = 0$ gives such a matrix, similarly, if $\xi_1 = 0$, then $\xi_1'' = 1$ gives a solution. So we assume $\xi_i \neq 0$; then by assumption we have $(x, y) \in \Delta^2$ such that $\xi_1 x + \xi_2 y = 1$, hence $\overline{x}\overline{\xi}_1 + \overline{y}\overline{\xi}_2 = 1$. So setting $\xi_2' = \overline{x}, \xi_1' = \overline{y}$, the first relation of (39) is satisfied. Again 3) is satisfied by assumption, so we must verify 2). It turns out that we may have to alter ξ_i' to achieve this. The relation 1) can be expressed as $h(\xi', \xi) = 1$, where $\xi = (\xi_1, \xi_2)$, $\xi' = (\xi_1', \xi_2')$. Since ξ is isotropic by Lemma 4.4, $h(\xi, \xi) = 0$, and with respect to the base $< \xi, \xi' >$ of D^2, h is given by a matrix $H_{\xi,\xi'} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & \epsilon \end{smallmatrix}\right)$, where $\epsilon = h(\xi', \xi') = \delta + \overline{\delta}$, $\delta = \xi_1'\overline{\xi}_2'$ (if $\epsilon \neq 0$; otherwise we are done), in particular $\delta \in \Delta$. Now setting

$$\xi'' = (\xi_1'', \xi_2'') = (-\xi_1\overline{\delta} + \xi_1', -\xi_2\overline{\delta} + \xi_2') \in \Delta^2,$$

we can easily verify $h(\xi'', \xi'') = 0, \quad h(\xi, \xi'') = h(\xi'', \xi) = 1$, so this vector ξ'' gives a matrix $M_\xi \in \Gamma_\Delta$, as was to be shown. $\square$

We require the following refinement of Lemma 4.5. For this, given $(\xi_1, \xi_2) \in \Delta^2$, let $\mathfrak{a}_\xi$ denote the left ideal in Δ generated by the elements ξ_1, ξ_2.

Lemma 4.7 *Given* $\xi = (\xi_1, \xi_2) \in \Delta^2$ *isotropic, the entries* ξ_i' *of the matrix* M_ξ *of Lemma 4.5 are in* $\mathfrak{a}_\xi^{-1}$.

Proof: This follows from the proof of 4.5, as we took just inverses of elements ξ_i. $\square$

We can now consider when two integral isotropic vectors are equivalent under Γ_Δ.

Proposition 4.8 *Let* $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in \Delta^2$ *be two integral isotropic vectors,* $\mathfrak{a}_\xi$, $\mathfrak{a}_\eta$ *the left ideals generated by the components of* ξ *and* η*, respectively. Then* ξ *and* η *are equivalent under an element of* $\Gamma_\Delta \iff \mathfrak{a}_\eta \cong \mathfrak{a}_\xi$ *as* Δ*-modules.*

Proof: If there is an element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\Delta$ with $(\xi_1, \xi_2)g = (\eta_1, \eta_2)$, then $\eta_1 = \xi_1 a + \xi_2 c$, $\eta_2 = \xi_1 b + \xi_2 d$, hence $\mathfrak{a}_\eta \subset \mathfrak{a}_\xi$ and the map

$$\begin{array}{rcl} \phi : \mathfrak{a}_\eta & \longrightarrow & \mathfrak{a}_\xi \\ \eta_1\lambda_1 + \eta_2\lambda_2 & \mapsto & \xi_1(a\lambda_1 + b\lambda_2) + \xi_2(c\lambda_1 + d\lambda_2) \end{array}$$

is an isomorphism of Δ-modules. The inverse is given is a similar manner by g^{-1}. Conversely, if $\mathfrak{a}_\eta \cong \mathfrak{a}_\xi$ we may assume $\mathfrak{a}_\xi = \mathfrak{a}_\eta$, hence also $\mathfrak{a}_\xi^{-1} = \mathfrak{a}_\eta^{-1}$. Applying Lemma 4.7 and making use of $\mathfrak{a}_\xi\mathfrak{a}_\xi^{-1} = \Delta$, which holds in central simple division algebras over number fields, we find for the matrices M_ξ, M_η associated by Lemma 4.5 to ξ and η, respectively, that $M_\xi M_\eta^{-1} \in \Gamma_\Delta$. For example, if $\xi_1 \neq 0$, $\eta_1 \neq 0$, then

$$M_\xi M_\eta^{-1} = \begin{pmatrix} 0 & \overline{\xi}_1^{-1} \\ \xi_1 & \xi_2 \end{pmatrix} \begin{pmatrix} \overline{\eta}_2 & \eta_1^{-1} \\ \overline{\eta}_1 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\xi}_1^{-1}\overline{\eta}_1 & 0 \\ \xi_1\overline{\eta}_2 + \xi_2\overline{\eta}_1 & \xi_1\eta_1^{-1} \end{pmatrix}.$$

This verifies "$\Leftarrow$" and completes the proof. $\square$

Finally, as above we can now express this in terms of ideal classes. The ideals $\mathfrak{a}_\xi$ and $\mathfrak{a}_\eta$ of Proposition 4.8 are equivalent, according to the usual definition. For general division algebras, one requires a weaker notion of equivalence than isomorphism as Δ-modules, the notion of *stable equivalence*. However it is standard that for central simple division algebras over number fields, excluding definite quaternion algebras, the stronger notion coincides with the weaker one (see [R], 35.13). This is: Δ-modules M and N are stably equivalent if

$$M + \Delta^r \cong N + \Delta^r,$$

for some $r \geq 0$, "$\cong$" being isomorphism of Δ-modules.

At any rate, one defines the *class ray group* (depending on D) as follows. Let $S \subset \Sigma_\infty$ denote the set of infinite primes which ramify in D, and set

$$\mathcal{C}\ell_D(\mathcal{O}_K) := \{ideals\} / \{\alpha\mathcal{O}_K,\ \alpha \in K^*, \alpha_\nu > 0 \text{ for all } \nu \in S\}.$$

Furthermore, let $\mathcal{C}\ell(\Delta)$ denote the set of equivalence classes of left ideals in Δ. The two are related by

$$\begin{array}{ccc} \mathcal{C}\ell(\Delta) & \xrightarrow{\sim} & \mathcal{C}\ell_D(\mathcal{O}_K) \\ {[M]} & \mapsto & [N_{D|K}(M)]. \end{array}$$

For details on these matters, see [R].

Now notice that for the division algebras of (1), for $d \geq 2$, we have $\mathcal{C}\ell_D(\mathcal{O}_K) = \mathcal{C}\ell(\mathcal{O}_K)$ (where again $K = k$ if $d = 2$). Hence

Proposition 4.9 *For the division algebras D we are considering, if $d \geq 2$ and $\Delta \subset D$ is a maximal order, then*

$$\mathcal{C}\ell(\Delta) \xrightarrow{\sim} \mathcal{C}\ell(\mathcal{O}_K).$$

We can now proceed to carry out the program sketched above for the division algebras D.

Theorem 4.10 *Let D be a central simple division algebra over K as in (1), D^2 the right vector space with the hyperbolic form (10). Let $\xi = (\xi_1, \xi_2)$ and $\epsilon = (\eta_1, \eta_2)$ be two isotropic vectors in $\mathfrak{I}_\Delta$. Then*

- *(i) If $d = 2$, then ξ and η are equivalent under $\Gamma_\Delta \iff$ the ideals $N_{D|k}(\mathfrak{a}_\xi)$ and $N_{D|k}(\mathfrak{a}_\eta)$ are equivalent in $\mathcal{O}_k$.*
- *(ii) If $d \geq 3$, then ξ and η are equivalent under $\Gamma_\Delta \iff$ the ideals $N_{D|K}(\mathfrak{a}_\xi)$ and $N_{D|K}(\mathfrak{a}_\eta)$ are equivalent in $\mathcal{O}_K$.*

Proof: By Proposition 4.8, ξ and η are equivalent under $\Gamma_\Delta \iff$ the ideals $\mathfrak{a}_\xi$ and $\mathfrak{a}_\eta$ are equivalent. By Proposition 4.9 $\mathfrak{a}_\xi \cong \mathfrak{a}_\eta \iff N_{D|K}(\mathfrak{a}_\xi) \cong N_{D|K}(\mathfrak{a}_\eta)$ (where $K = k$ for $d = 2$). This is the statement of the theorem. $\square$

As a corollary of this

Corollary 4.11 *The number of cusps of Γ_Δ is the class number of K ($d \geq 3$) or the class number of k ($d = 1, 2$).*

Proof: It follows from Theorem 4.10 that the number of cusps is *at most* the class number in question; it remains to verify that it is at least the class number. If $\mathfrak{a}_\xi$ denotes the ideal in Δ, and $N_{D|K}(\mathfrak{a}_\xi)$ denotes the corresponding ideal in $\mathcal{O}_K$, then ξ_1, ξ_2 form a basis of $\mathfrak{a}_\xi$, while $N_{D|K}(\xi_1)$, $N_{D|K}(\xi_2)$ form an $\mathcal{O}_K$-basis of the norm ideal. So the question here is, given an ideal $(a_1, a_2) \subset \mathcal{O}_K$ generated by two elements, is there an ideal (b_1, b_2) in the same ideal class of (a_1, a_2), such that $b_i = N_{D|K}(a_i')$ for some $a_i' \in \Delta$? But this is just the statement of Proposition 4.9: given any ideal $\mathfrak{a}$, there is an ideal

$\mathfrak{b} \subset [\mathfrak{a}]$, and an ideal $\mathfrak{b}' \subset \Delta$ such that $\mathfrak{b} = N_{D|K}(\mathfrak{b}')$. We further require that $\mathfrak{b}'$ defines an isotropic vector in Δ^2, in the above sense. There should be $b_i \in \mathfrak{b}'$ which generate $\mathfrak{b}'$, such that the vector $(b_1, b_2) \in \Delta^2$ is isotropic with respect to h. This in turn is a question about what happens to the relation $\xi_1\overline{\xi}_2 + \xi_2\overline{\xi}_1 = 0$ under the norm map. This relation turns into

$$N(\xi_1)\overline{N(\xi_2)} + N(\xi_2)\overline{N(\xi_1)} = 0,$$

or setting $b_1 = N(\xi_1)$, $b_2 = N(\xi_2)$, $b_1\overline{b}_2 + b_2\overline{b}_1 = 0$. This just says that (b_1, b_2) is isotropic in the hyperbolic plane K^2, hence (b_1, b_2) is isotropic in the hyperbolic plane D^2. In sum, for any isotropic vector $v \in K^2$, the vector is also an isotropic vector $v \in D^2$. Any isotropic vector of D^2 yields by the norm map an isotropic vector of K^2. It follows from Proposition 4.9 that this gives an isomorphism on equivalence classes, or in other words, *the isomorphism of Proposition 4.9 can be represented geometrically by isotropic vectors.* □

Remark 4.12 We have not been very precise about the groups, but the number of cusps of Γ_Δ and $S\Gamma_\Delta$ are the same, as is quite clear.

Now assume $d \geq 2$ and recall the subfield $L \subset D$ and subgroups $G_L \subset G_D$ of Proposition 2.4. By Proposition 3.3 these give rise to subdomains of the domain $\mathcal{D}_D$. Consider, in G_L, the discrete group $\Gamma_{\mathcal{O}_L} \subset G_L(K)$.

Lemma 4.13 *We have for any maximal order $\Delta \subset D$,*

$$\Gamma_\Delta \cap G_L = \Gamma_{\mathcal{O}_L}.$$

Proof: This follows from the definitions and the fact that $\Delta \cap L = \mathcal{O}_L$. □

Let M_L denote the arithmetic quotient

$$M_L = \Gamma_{\mathcal{O}_L} \backslash \mathcal{D}_L. \tag{40}$$

It follows from Proposition 3.3 that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}_L & \hookrightarrow & \mathcal{D}_D \\ \downarrow & & \downarrow \\ M_L & \hookrightarrow & X_{\Gamma_\Delta}. \end{array} \tag{41}$$

Each M_L has its own Baily-Borel compactification M_L^*, which is also affected by adding isolated points to M_L. Moreover, from Proposition 3.5, we see that the embedding $M_L \hookrightarrow X_{\Gamma_\Delta}$ can be extended to the cusp denoted $F_L \in \mathcal{D}_L^*$ respectively $F \in \mathcal{D}_D^*$ there. By Theorem 3 of [S2], we actually get embeddings of the Baily-Borel embeddings of M_L and X_{Γ_Δ}, respectively.

Theorem 4.14 *Let $M_L^* \subset \mathbb{P}^N$, $X_{\Gamma_\Delta}^* \subset \mathbb{P}^{N'}$ be Baily-Borel embeddings. Then there is a linear injection $\mathbb{P}^N \hookrightarrow \mathbb{P}^{N'}$ making the diagram*

$$\begin{array}{ccc} M_L^* & \hookrightarrow & \mathbb{P}^N \\ \cap & & \cap \\ X_{\Gamma_\Delta}^* & \hookrightarrow & \mathbb{P}^{N'} \end{array}$$

commute and making $M_L^* \in X_{\Gamma_\Delta}^*$ an algebraic subvariety.

Proof: We have an injective holomorphic embedding $\mathcal{D}_L \hookrightarrow \mathcal{D}_D$ which comes from a $\mathbb{Q}$-morphism $\rho : (G_L)_\mathbb{C} \hookrightarrow (G_D)_\mathbb{C}$ (for this one takes the restriction of scalars from k to $\mathbb{Q}$ yielding an injection $Res_{k|\mathbb{Q}}G_L \hookrightarrow Res_{k|\mathbb{Q}}G_D$, then lifts this to $\mathbb{C}$) such that $\rho(\Gamma_{\mathcal{O}_L}) \subset \Gamma_\Delta$. Hence we map apply [S2], Thm. 3, and the theorem follows. □

Corollary 4.15 *If $d = 2$, then there are modular curves M_L^* on the algebraic threefold $X_{\Gamma_\Delta}^*$ such that the cusps of M_L^* are cusps of $X_{\Gamma_\Delta}^*$. If $d \geq 3$, then we have Hilbert modular varieties of dimension d, $M_L^* \subset X_{\Gamma_\Delta}^*$ in the d^2-dimensional algebraic variety $X_{\Gamma_\Delta}^*$, such that the cusps of M_L^* are cusps of $X_{\Gamma_\Delta}^*$.*

The previous Theorem 4.14 applies to the cusp of $X_{\Gamma_\Delta}^*$ which represents the equivalence class of the isotropic vector $(0, 1)$. We now consider the others. Given (ξ_1, ξ_2), an isotropic vector representing a class of cusps, let M_ξ be the matrix in G_D of Lemma 4.5 which maps it to $(0, 1)$. Then

$$\Gamma_{\Delta,\xi} := M_\xi \cdot \Gamma_\Delta \cdot M_\xi^{-1} \subset U(D^2, h)$$

is a discrete subgroup of G_D, and the cusp (ξ_1, ξ_2) of Γ_Δ is equivalent to the cusp $(0, 1)$ of $\Gamma_{\Delta,\xi}$. Letting, as above, $\Gamma_{\mathcal{O}_L} \subset \Gamma_\Delta$ denote the discrete subgroup defining the modular subvariety M_L above, we have

$$\Gamma_{\mathcal{O}_L,\xi} := M_\xi \cdot \Gamma_{\mathcal{O}_L} \cdot M_\xi^{-1} \subset \Gamma_{\Delta,\xi}, \tag{42}$$

and without difficulty this gives a subdomain

$$\mathcal{D}_{L,\xi} \subset \mathcal{D}_D, \tag{43}$$

such that, if F_ξ denotes the boundary component corresponding to ξ, then $F_\xi \in \mathcal{D}_{L,\xi}^*$. More precisely, $G_{L,\xi} := M_\xi \cdot G_L \cdot M_\xi^{-1}$ is the k-group whose $\mathbb{R}$-points $G_{L,\xi}(\mathbb{R})$ define $\mathcal{D}_{L,\xi}$. Then the parabolic subgroup of $\Gamma_{\Delta,\xi}$ is $\Gamma_{\Delta,\xi} \cap P$ (P as in (19) the normaliser of $(0, 1)$), and similarly for $\Gamma_{\mathcal{O}_L,\xi}$. The corresponding modular subvariety

$$M_{L,\xi} = \Gamma_{\mathcal{O}_L,\xi} \backslash \mathcal{D}_{L,\xi}, \tag{44}$$

is, as is easily checked, a Hilbert modular variety for the group $SL_2(\mathcal{O}_\ell, \mathfrak{b}^2)$, where for any ideal $\mathfrak{c}$ one defines

$$SL_2(\mathcal{O}_\ell, \mathfrak{c}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad - cb = 1,\ a, d \in \mathcal{O}_\ell, b \in \mathfrak{c}^{-1},\ c \in \mathfrak{c} \right\}, \tag{45}$$

and where $\mathfrak{b}$ is the intersection of the ideal $\mathfrak{a}_\xi$ with $\mathcal{O}_\ell$. The same arguments as above then yield

Theorem 4.16 *Given any cusp $p \in X_{\Gamma_\Delta}^* \backslash X_{\Gamma_\Delta}$, there is a modular subvariety $M_{L,p} \subset X_{\Gamma_\Delta}$ such that $p \in M_{L,p}^*$.*

Proof: If the cusp p is represented by the isotropic vector $\xi = (\xi_1, \xi_2)$, then the modular subvariety is $M_{L,\xi}$ as in (44). Just as in the proof of Theorem 4.14, we get an embedding of the Baily-Borel embeddings, and as mentioned above, $F_\xi \in \mathcal{D}_{L,\xi}^*$, and p is the image of F_ξ under the natural projection $\pi : \mathcal{D}_{L,\xi}^* \longrightarrow M_{L,\xi}^* = M_{L,p}^*$. □

Corollary 4.17 *If $d = 2$, there are modular curves $M_{L,p}^* \subset X_{\Gamma_\Delta}^*$ passing through each cusp of $X_{\Gamma_\Delta}^*$. If $d \geq 3$, there are Hilbert modular varieties (to groups as in (45)) of dimension d passing through any cusp of $X_{\Gamma_\Delta}^*$.*

5 An example

The theory of quaternion algebras is quite well established, and the corresponding arithmetic quotients have already been studied in [Ara] and [H]. However the $d \geq 3$ case seems not to have drawn much attention as of yet, so we will give an example to illustrate the theory. Curiously enough, I ran across this example in the construction of Mumford's fake projective plane. Recall that this is an algebraic surface S of general type with $c_1^2 = 3c_2$, $c_1^2 = 9$, $c_2 = 3$, just as for the projective plane. It then follows from Yau's theorem that S is the quotient of the two-ball $\mathbb{B}_2$ by a discrete subgroup,

$$S = \Gamma\backslash\mathbb{B}_2, \tag{46}$$

where Γ is cocompact and fixed point free. Mumford's construction involves lifting a quotient surface from a 2-adic field to the complex numbers, and while the 2-adic group is clearly not arithmetic (as a subgroup of $SL(3, \mathbb{Q}_2)$, its elements are unbounded in the 2-adic valuation), it is not clear from the construction whether Γ is arithmetic. Now the *arithmetic* cocompact groups are known: these derive either from anisotropic groups (over $\mathbb{Q}$) which are of the form $U(1, D)$ or $SU(1, D)$, where D is a central simple division algebra over an imaginary quadratic extension K of a totally real field k, such that if the corresponding hermitian symmetric domain is irreducible, then $k = \mathbb{Q}$, and such that D has a $K|\mathbb{Q}$-involution, or from unitary groups over field extensions $k|\mathbb{Q}$ of degree $d \geq 2$, such that, for all but one infinite prime, the real groups G_ν are compact. The strange thing is that in Mumford's construction such a D comes up (implicitly) naturally, and it is this D we will introduce. It is a fascinating question whether the Γ of (46) occurs as a discrete subgroup of $U(1, D)$, a question which will be left unanswered here.

This is a cyclic algebra, central simple over K, with splitting field L. The fields involved are

$$L = \mathbb{Q}(\zeta), \;\; \zeta = \exp(\frac{2\pi i}{7}); \quad K = \mathbb{Q}(\sqrt{-7}). \tag{47}$$

Lemma 5.1 *L is a cyclic extension of K, of degree three. The Galois group is generated by the transformation*

$$\sigma(\zeta) = \zeta^2, \; \sigma(\zeta^2) = \zeta^4, \; \sigma(\zeta^4) = \zeta,$$

where $1, \zeta, \zeta^2$ generate L over K.

Proof: Let $\gamma = \frac{-1+\sqrt{-7}}{2} \in K$, so $\mathcal{O}_K = \mathbb{Z} \oplus \gamma\mathbb{Z}$. The ring of integers of L is generated by the powers of ζ,

$$\mathcal{O}_L = \mathbb{Z} \oplus \zeta\mathbb{Z} \oplus \cdots \oplus \zeta^5\mathbb{Z},$$

(remember that $\zeta^6 = -1 - \zeta - \cdots - \zeta^5$), with the inclusion $\mathbb{Z} \subset \mathcal{O}_K, \mathbb{Z} \subset \mathcal{O}_L$ on the first factors. The key identity is the following

$$\gamma = \zeta + \zeta^2 + \zeta^4, \qquad \overline{\gamma} = \zeta^3 + \zeta^5 + \zeta^6. \tag{48}$$

From this relation we see that $\mathcal{O}_L$ is generated over $\mathcal{O}_K$ by ζ and ζ^2,

$$\mathcal{O}_L = \mathcal{O}_K \oplus \zeta\mathcal{O}_K \oplus \zeta^2\mathcal{O}_K.$$

The conjugation on K extends to (complex) conjugation on L, which we will denote by $x \mapsto \overline{x}$. Its action on $\mathcal{O}_K$ is clear, $\gamma \mapsto \overline{\gamma}$, and on $\mathcal{O}_L$ it affects

$$\overline{\zeta} = \zeta^6, \quad \overline{\zeta}^2 = \zeta^5, \quad \overline{\zeta}^3 = \zeta^4.$$

It is clear that $L|K$ is cyclic, with a generator of the Galois group

$$1 \mapsto 1, \quad \zeta \mapsto \zeta^2, \quad \zeta^2 \mapsto \gamma - \zeta - \zeta^2,$$

and by (48), this is the statement of the Lemma. $\square$

Now consider the fixed field in L under conjugation, $\ell \subset L$.

Lemma 5.2 *ℓ is a degree three cyclic extension of $\mathbb{Q}$, with a generator of the Galois group being*

$$\sigma: \quad \zeta+\zeta^6 \mapsto \zeta^2+\zeta^5 \mapsto \zeta^3+\zeta^4 \mapsto \zeta+\zeta^6.$$

Proof: It is clear that $\eta_1=\zeta+\zeta^6, \eta_2=\zeta^2+\zeta^5,\ \eta_3=\zeta^3+\zeta^4$ are in ℓ (they are $\eta_1=Tr_{L|\ell}(\zeta),\ \eta_2=Tr_{L|\ell}(\zeta^2),\ \eta_3=Tr_{L|\ell}(\zeta^3)$). Moreover they generate the integral closure of $\mathbb{Z}$ in ℓ, hence

$$\mathcal{O}_\ell=\mathbb{Z}\oplus\eta_1\mathbb{Z}\oplus\eta_2\mathbb{Z},$$

and there is a relation $\eta_3=-\eta_1-\eta_2-1$. Now observe that σ of Lemma 5.1 acts as follows

$$\sigma(\eta_1)=\sigma(\zeta+\zeta^6)=\sigma(\zeta)+\sigma(\zeta^6)=\zeta^2+\zeta^5=\eta_2, \quad \sigma(\eta_2)=\eta_3, \quad \sigma(\eta_3)=\eta_1,$$

and this is the map specified in the lemma. □

Next consider the cyclic algebra $D=(L/K,\sigma,\gamma)$ for the element γ above. Then D will be split $\iff$ γ is the L/K norm of some element. However, it seems difficult to verify this condition explicitly, so we show directly that D is a division algebra.

Proposition 5.3 *$D=(L/K,\sigma,\gamma)$, with $\gamma=\frac{-1+\sqrt{-7}}{2}$, is a division algebra, central simple over K.*

Proof: Note first that since the degree of D over K is three, a prime, D is a division algebra $\iff$ D is not split. To show that D is not split, it suffices to find a prime $\mathfrak{p}\in\mathcal{O}_K$ for which the local algebra $D_\mathfrak{p}$ is not split. This, it turns out, is easy to find. Quite generally we know that $D_\mathfrak{p}$ is split for almost all $\mathfrak{p}$, and non-split at the divisors of the discriminant. Note that

$$N_{K|\mathbb{Q}}(\gamma)=\gamma\overline{\gamma}=2,$$

while K ramifies over $\mathbb{Q}$ only at the prime $\sqrt{-7}$. These are the two primes where $D_\mathfrak{p}$ might ramify. To see that for $\mathfrak{p}=(\gamma)$ $D_\mathfrak{p}$ actually does ramify, we determine the Hasse invariant $inv_\mathfrak{p}(D)\in\mathbb{Q}/\mathbb{Z}$ of $D_\mathfrak{p}$.

To do this we must first understand the action of the Frobenius acting as generator for the maximal unramified extension $L_\mathfrak{p}/K_\mathfrak{p}$. But Frobenius in characteristic two is just a squaring map, so is clearly the mod($\mathfrak{p}$) reduction of the σ of Lemma 5.1. We denote this by $\sigma_\mathfrak{p}$. Finally, since we are at the prime $\mathfrak{p}=(\gamma)$, we may take the image of γ as local uniformising element $\pi_\mathfrak{p}$. Then

$$D_\mathfrak{p}=(L_\mathfrak{p}/K_\mathfrak{p},\sigma_\mathfrak{p},\pi_\mathfrak{p}),$$

which is the cyclic algebra with $inv_\mathfrak{p}(D)=\frac{1}{3}$. From this it follows that $D_\mathfrak{p}$ is not split. That D is central simple over K is clear from construction. □

Finally we require a $K|\mathbb{Q}$-involution on D. It is necessary and sufficient for the existence of such an involution J that there exists an element $g\in\ell$, such that

$$N_{K|\mathbb{Q}}(\gamma)=\gamma\overline{\gamma}=N_{\ell|\mathbb{Q}}(g)=gg^\sigma g^{\sigma^2}. \tag{49}$$

Given such an element g, the involution is given by (see (7))

$$e^J=ge^{-1}, \ \ (e^2)^J=gg^\sigma(e^2)^{-1}, \quad (\sum_0^2 e^iz_i)^J:=\sum_0^2\overline{z}_i(e^i)^J.$$

An alternative to finding an explicit g is to use a theorem of Landherr ([Sch], Thm. 10.2.4): D admits a $K|\mathbb{Q}$-involution $\Longleftrightarrow$

$$\begin{array}{l} \bullet\ inv_{\mathfrak{p}}(D) = 0, \quad \forall_{\mathfrak{p}=\overline{\mathfrak{p}}}, \text{ and} \\ \bullet\ inv_{\mathfrak{p}}(D) + inv_{\overline{\mathfrak{p}}}(D) = 0, \quad \forall_{\mathfrak{p}\neq\overline{\mathfrak{p}}}. \end{array} \tag{50}$$

This condition is satisfied for all $\mathfrak{p}$ for which $D_{\mathfrak{p}}$ splits, so must be verified only for those primes which divide the discriminant. For us, this means at most $\pm\sqrt{-7}, \gamma,\ \overline{\gamma}$, which we will denote by $\pm\mathfrak{q}, \mathfrak{p}, \overline{\mathfrak{p}}$, respectively. We showed above that $inv_{\mathfrak{p}}(D) = \frac{1}{3}$. The same argument shows that $inv_{\overline{\mathfrak{p}}}(D) = -\frac{1}{3}$. The negative sign occurs because at the prime $\overline{\mathfrak{p}}$, $\overline{\gamma}$ is a local uniformising element $\pi_{\overline{\mathfrak{p}}}$, so localising γ at $\overline{\mathfrak{p}}$ gives $\pi_{\overline{\mathfrak{p}}}^{-1}$. This verifies (50) for the primes $\mathfrak{p} \neq \overline{\mathfrak{p}}$ lying over 2. Consider the prime $\pm\mathfrak{q}$. Here it is easy to see that the image of γ is actually *invertible*, and hence, that $D_{\mathfrak{q}}$ splits, so $inv_{\mathfrak{q}}(D) = 0$, as was to be shown. □

This completes the proof of the following.

Theorem 5.4 *The cyclic algebra $D = (L/K, \sigma, \gamma)$ constructed above is a central simple division algebra over K with a $K|\mathbb{Q}$-involution. It ramifies at the two primes γ and $\overline{\gamma}$, and is split at all others.*

This algebra gives rise to an anisotropic group $GL(1, D) \cong D^*$, and $D^*(\mathbb{R})$ is a twisted real form of GL_3; since it cannot be compact, it must be $U(2,1)$. A maximal order $\Delta \subset D$ gives rise to an arithmetic subgroup of $D^*(\mathbb{R})$, and the quotient is then a compact ball quotient. As remarked above, this may be related with Mumford's fake projective plane.

We can consider the hyperbolic space (D^2, h), and the corresponding $\mathbb{Q}$-groups (here $k = \mathbb{Q}$) G_D and SG_D as in (11) and (13), respectively, as well as the arithmetic subgroups Γ_Δ and $S\Gamma_\Delta$. By Corollary 4.11, we see that Γ_Δ has $h(K)$ cusps, where $h(K)$ is the class number of K; this is known to be 1. So the arithmetic quotient has only 1 cusp. We also have the subgroups $\Gamma_{\mathcal{O}_L}$ as in Lemma 4.13, as well as the modular subvarieties M_L^* of $X_{\Gamma_\Delta}^*$. We note that these are Hilbert modular threefolds coming from the cyclic cubic extension $\ell/\mathbb{Q}$. Such threefolds have been considered in [TV].

6 Moduli interpretation

The moduli interpretation of the arithmetic quotients X_{Γ_Δ}, as well as of the modular subvarieties M_L, by which we mean the description of these spaces as moduli spaces, is a straightforward application of Shimura's theory. Fix a hyperbolic plane (D^2, h), and consider the endomorphism algebra $M_2(D)$ of D^2, endowed with the involution $x \mapsto {}^t\overline{x}$, $x \in M_2(D)$, where the bar denotes the involution in D. Now view D^2 as a $\mathbb{Q}$-vector space, of dimension $4f$, $8f$ and $4d^2f$, in the cases $d = 1$, $d = 2$ and $d \geq 3$, respectively. The data determining one of Shimura's moduli spaces (with no level structure) is $(D, \Phi, *)$, $(T, \mathcal{M})$, where D is a central simple division algebra over K, Φ is a representation of D in $\mathfrak{gl}(n, \mathbb{C})$, $*$ is an involution on D, T is a $*$-skew hermitian form (matrix) on a right D vector space V of dimension m, with $\mathfrak{gl}(n, \mathbb{C}) \cong End(V, V)_{\mathbb{R}}$, and finally, $\mathcal{M} \subset D^m$ is a $\mathbb{Z}$-lattice. Then for suitable $x_i \in V$,

$$\Lambda = \left\{ \sum_{i=1}^{m} \Phi(a_i)x_i \middle| a_i \in \mathcal{M} \right\} \subset \mathbb{C}^n \tag{51}$$

is a lattice and $\mathbb{C}^n/\Lambda$ is abelian variety with multiplication by D. The data $(D, \Phi, *)$ will be given in our cases as follows. D is our central simple division algebra over K with a $K|k$-involution, and the representation $\Phi : D \longrightarrow M_N(\mathbb{C})$ is obtained by base change from the natural operation of D on D^2 by right multiplication. Explicitly,

$$\Phi : D \longrightarrow End_D(D^2, D^2) \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(D) \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(D \otimes_{\mathbb{Q}} \mathbb{R}) \cong M_2(\mathbb{R}^N) \cong M_N(\mathbb{C}), \tag{52}$$

where $N = \dim_{\mathbb{Q}} D = 2f,\ 4f,\ 2d^2 f$ in the cases $d = 1,\ d = 2$ and $d \geq 3$, respectively. The involution $*$ on D will be our involution, which we still denote by $x \mapsto \overline{x}$. Then a $^-$-skew hermitian matrix $T \in M_2(D)$ will be one such that $T = -T^*$, where $(t_{ij})^* = (\overline{t}_{ji})$, the canonical involution on $M_2(D)$ induced by the involution on D. Note that for any $c \in D^*$ such that $c = -\overline{c}$, the matrix $T = cH$ (H our hyperbolic matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$) has this property. To be more specific, then, we set

$$\begin{aligned}
&1)\ d = 1\colon T = \sqrt{-\eta} H = \begin{pmatrix} 0 & \sqrt{-\eta} \\ \sqrt{-\eta} & 0 \end{pmatrix}. \\
&2)\ d = 2\colon T_a = \sqrt{a} H = \begin{pmatrix} 0 & \sqrt{a} \\ \sqrt{a} & 0 \end{pmatrix} \text{ or } T_b = eH = \left(\begin{smallmatrix} 0 & e \\ e & 0 \end{smallmatrix}\right), \text{ where } e = \left(\begin{smallmatrix} 0 & 1 \\ b & 0 \end{smallmatrix}\right). \\
&3)\ d \geq 3\colon T = \sqrt{-\eta} H = \begin{pmatrix} 0 & \sqrt{-\eta} \\ \sqrt{-\eta} & 0 \end{pmatrix}.
\end{aligned} \tag{53}$$

Since two such forms T are equivalent when they are scalar multiples of one another, assuming T of the form in (53) is no real restriction. Finally the lattice $\mathcal{M}$ will be $\Delta^2 \subset D^2$. Then $D^2 \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^N$, N as above, and for "suitable" vectors $x_1, x_2 \in D^2$, the lattice

$$\Lambda_x = \{\Phi(a_1)x_1 + \Phi(a_2)x_2 | (a_1, a_2) \in \Delta^2\} \tag{54}$$

gives rise to an abelian variety $A_x = \mathbb{C}^N / \Lambda_x$. Shimura has determined exactly what "suitable" means; the conditions determine certain unbounded realisations of hermitian symmetric spaces, in our cases just the domains $\mathcal{D}_D$. The Riemann form on A_x is given by the alternating form $E(x, y)$ on $\mathbb{C}^N$ defined by:

$$E(\sum_1^2 \Phi(\alpha_i)x_i, \sum_1^2 \Phi(\beta_j)x_j) = Tr_{D|\mathbb{Q}}(\sum_{i,j=1}^2 \alpha_i t_{ij} \overline{\beta}_j), \tag{55}$$

for $\alpha_i, \beta_j \in D_{\mathbb{R}}$, and $(t_{ij}) = T$ is the matrix above. In particular, in all cases dealt with here the abelian varieties are *principally polarised.*

Shimura shows that for each $z \in \mathcal{D}_D$, vectors $x_1, x_2 \in D^2$ are uniquely determined, hence by (54) a lattice, denoted $\Lambda(z, T, \mathcal{M})$. The data determine an arithmetic group, which for our cases is just Γ_Δ defined above (not $S\Gamma_\Delta$), cf. [Sh] (38). The basic result, applied to our concrete situation, is

Theorem 6.1 ([Sh], Thm. 2) *The arithmetic quotient X_{Γ_Δ} is the moduli space of isomorphism classes of abelian varieties determined by the data:*

$$(D, \Phi, ^-), \quad (T, \Delta^2),$$

where Φ is given in (52), T in (53).

The corresponding classes of abelian varieties can be described as follows:

1) $d = 1$. Here we have *two* families, relating from the isomorphism of Proposition 2.3. The first, for $D = k,\ D^2 = k^2$ yields $D_{\mathbb{R}}^2 \cong \mathbb{C}^f$, and we have abelian varieties of dimension f with real multiplication by k. Secondly, for $D = K,\ D^2 = K^2$, we have abelian varieties of dimension

$2f$ with complex mulitplication by K, with signature $(1,1)$, that is, for each eigenvalue χ of the differential of the action, also $\overline{\chi}$ occurs. If $K = k(\sqrt{-\eta})$, then setting $K' = \mathbb{Q}(\sqrt{-\eta})$ we have $k \otimes_{\mathbb{Q}} K' \cong K$, hence $k^2 \otimes_{\mathbb{Q}} K' \cong K^2$ and $k^2_{\mathbb{R}} \otimes K'_{\mathbb{R}} \cong K^2_{\mathbb{R}}$, giving the relation between the $\mathbb{Q}$-vector spaces and their real points. Moreover, $\mathcal{O}_k \otimes_{\mathbb{Z}} \mathcal{O}_{K'} \cong \mathcal{O}_K$, and if

$$\Lambda_{x,k} = \{\sum_1^2 \Phi(a_i)x_i \Big| (a_1, a_2) \in \mathcal{O}_k^2\} \tag{56}$$

is a lattice giving an abelian variety with mulitiplication by k,

$$A_x := \mathbb{C}^f / \Lambda_{x,k},$$

then $\Lambda_{x,k} \otimes_{\mathbb{Z}} \mathcal{O}_{K'} = \Lambda_{x,K}$ is a lattice in $\mathbb{C}^{2f}$, and determines an abelian variety

$$A'_x := \mathbb{C}^{2f} / \Lambda_{x,K}. \tag{57}$$

This abelian variety determines a point x' in its moduli space, and the mapping $x \mapsto x'$ gives the isomorphism

$$\Gamma_{\mathcal{O}_k} \backslash \mathfrak{H}^f \xrightarrow{\sim} \Gamma_{\mathcal{O}_K} \backslash \mathfrak{H}^f. \tag{58}$$

Remark 6.2 It turns out that this case is one of the exceptions of Theorem 5 in [Sh], denoted case d) there. The actual endomorphism ring of the generic member of the family is larger than K:

Theorem 6.3 ([Sh], Prop. 18) *The endomorphism ring E of the generic element of the family (58) is a totally indefinite quaternion algebra over k, having K as a quadratic subfield.*

In our situation, the totally indefinite quaternion algebra E over k is constructed as the cyclic algebra $E = (K/k, \sigma, \lambda)$, where $\lambda = -u^{-1}v$, if the matrix T of (53) is diagonalized $T = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right)$. So in our case we have $\lambda = 1$ and hence the algebra E is split; the corresponding abelian variety is isogenous to a product of two copies of a simple abelian variety B with real multiplication by k, as has been described already above. The conclusion follows from our choice of T, i.e., of hyperbolic form. It would seem one gets more interesting quaternion algebras by choosing different hermitian forms (which, by the way, will also lead to other polarisations).

2) $d = 2$. $D = (\ell/k, \sigma, b) = (a, b)$ is a totally indefinite quaternion algebra, cental simple over k, with canonical involution. We have $D_\nu \cong M_2(\mathbb{R})$, and $D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^4$, while $M_2(D) \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R}^4) \cong M_4(\mathbb{C})$. Let $\Delta \subset D$ be a maximal order, $\Gamma_\Delta \subset G_D$ the corresponding arithmetic group. Two vectors $x_1, x_2 \in D^2$ arising from a point in the domain $\mathbb{S}_2$ (Siegel space of degree 2) determine a lattice Λ_x as in (56), with $(a_1, a_2) \in \Delta^2$, and $A_x = \mathbb{C}^{4f}/\Lambda_x$ is the corresponding abelian variety.

3) $d \geq 3$. In this case D is the cyclic algebra of degree d over K, and the abelian varieties are of dimension $2d^2 f$.

Now we come to the most interesting point of the whole story – the moduli interpretation of the modular subvarieties $M_L \subset X_{\Gamma_\Delta}$ of (41). The moduli interpretation of the M_L is as stated in Theorem 6.1, $d = 1$ case. Disregarding the case $d = 1$, the subvarieties have the following interpretations.

2) $d=2$: As M_L arises from the group $U(L^2,h)$, where $L=k(\sqrt{-ab})$ in our notations above, this is the moduli space of abelian varieties of dimension $2f$ with complex multiplication by L. On the other hand, the space X_{Γ_Δ} parameterizes abelian varieties of dimension $4f$ with multiplication by D. The relation is given as follows. By definition we have $D=\ell\oplus e\ell$, which in terms of matrices, is

$$D\cong\left\{\left(\begin{array}{cc} a_0+a_1\sqrt{a} & 0\\ 0 & a_0-a_1\sqrt{a}\end{array}\right)\oplus\left(\begin{array}{cc} 0 & a_2+a_3\sqrt{a}\\ b(a_2-a_3\sqrt{a}) & 0\end{array}\right)\right\}.$$

Now our subfield L is the set of matrices of the form

$$L\cong\left\{\left(\begin{array}{cc} a_0 & a_3\sqrt{a}\\ -ba_3\sqrt{a} & a_0\end{array}\right)\right\}.$$

If we let $c=\mathrm{diag}(\sqrt{a},-\sqrt{a})$ be the element representing $\sqrt{a}$ and $e=\left(\begin{smallmatrix}0&1\\b&0\end{smallmatrix}\right)$, then $c(ec)=c(-ce)=-c^2e=-ae$, so we can generate D over $\mathbb{Q}$ by c and (ec). Recall that $L\cong k(ec)$, hence

$$D\cong L\oplus cL. \tag{59}$$

Now consider the representation Φ; we have $\Phi(D)=\Phi(L\oplus cL)=\Phi_{|L}(L)\oplus\Phi_{|cL}(cL)$. The lattice $\Delta^2\subset D^2$ gives rise to a lattice Λ_x as in (54), and we would like to determine when the splitting (59) gives rise to a splitting of the lattice Λ_x, hence of the abelian variety A_x. Consider the order

$$\Delta':=\mathcal{O}_L\oplus c\mathcal{O}_L\subset\Delta; \tag{60}$$

Δ' is in general not a maximal order, but it is of finite index in Δ. Note that Δ' and a point x (consisting of two vectors $x_1,x_2\in D^2$) determine a lattice

$$\Lambda'_x=\{\sum\Phi(a_i)x_i|(a_1,a_2)\in\Delta'\},$$

which is also of finite index in Λ_x. Therefore $A'_x=\mathbb{C}^4/\Lambda'_x$ and A_x are isogenous.

We now assume $x_i\in L^2$. From (60) we can write $a_i=a_i^1+ca_i^2$ for $a_i\in\Delta'$, hence $\Phi(a_i)=\Phi(a_i^1+ca_i^2)=\Phi_{|\mathcal{O}_L}(a_i^1)+c\Phi_{|\mathcal{O}_L}(a_i^2)$. Then we have

$$\begin{aligned}\Lambda'_x=\left\{\sum\Phi(a_i)x_i\Big|(a_1,a_2)\in\Delta'^2\right\} &= \left\{\sum\left(\Phi_{|L}(a_i^1)+c\Phi_{|L}(a_i^2)\right)x_i\Big|(a_1,a_2)\in\Delta'^2\right\} \\ &= \left\{\Phi_{|L}(a_1^1)x_1+\Phi_{|L}(a_2^1)x_2+c\left(\Phi_{|L}(a_1^2)x_1+\Phi_{|L}(a_2^2)x_2\right)\right\} \\ &= \Lambda^1\oplus c\Lambda^2,\end{aligned} \tag{61}$$

and each of Λ^i has complex multiplication by L. It follows from this that

$$A'_x\cong A'^1_x\times A'^2_x,\quad x=(x_1,x_2)\in L^2,$$

and each abelian variety A'^i_x, of dimension $2f$, has complex multiplication by L. Since $A'_x\longrightarrow A_x$ is an isogeny, we have

Proposition 6.4 *In case $d=2$, the abelian varieties parameterised by the modular subvariety M_L are isogenous to products of two abelian varieties of dimensions $2f$ with complex multiplication by L.*

3) $d \geq 3$: Again the situation is somewhat simpler here. As above, the subvariety M_L parameterises abelian varieties of dimension $2df$ with complex multiplication by L. Again we consider the order

$$\Delta' := \mathcal{O}_L \oplus e\mathcal{O}_L \oplus \cdots \oplus e^{d-1}\mathcal{O}_L \subset \Delta,$$

where we assume e is as in (3) and $\gamma \in \mathcal{O}_k$; then as above we can write the lattice Λ'_x for $x_1, x_2 \in L^2$, in terms of $\Phi_{|L}$. If we write $a_i = a_i^1 + ea_i^2 + \cdots + e^{d-1}a_i^d$ with $a_i^j \in \mathcal{O}_L$, then $\Phi(a_i) = \Phi_{|L}(a_i^1) + e\Phi_{|L}(a_i^2) + \cdots + e^{d-1}\Phi_{|L}(a_i^d)$, and consequently

$$\begin{aligned} \Phi(a_1)x_1 + \Phi(a_2)x_2 &= \left(\Phi_{|L}(a_1^1)x_1 + \Phi_{|L}(a_2^1)x_2\right) + e\left(\Phi_{|L}(a_1^2)x_1 + \Phi_{|L}(a_2^2)x_2\right) + \\ &\quad + \ldots + e^{d-1}\left(\Phi_{|L}(a_1^d)x_1 + \Phi_{|L}(a_2^d)x_2\right), \\ \left\{\sum \Phi(a_i)x_i \middle| (a_1, a_2) \in \Delta'^2\right\} &= \Lambda^1 \oplus e\Lambda^2 \oplus \cdots \oplus e^{d-1}\Lambda^d, \end{aligned}$$

and each sublattice Λ^i has complex multiplication by L. Again, the abelian variety A'_x so determined is isogenous to A_x, and hence

Proposition 6.5 *In case $d \geq 3$, the abelian varieties parameterised by the modular subvariety M_L are isogenous to the product of d abelian varieties of dimension $2df$ with complex multiplication by the field L.*

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