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M. A. Borges-Trenard and H. Pérez-Rosés

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email: `greuel@mathematik.uni-kl.de`
URL: `http://www.mathematik.uni-kl.de/~zca/`

COMPLETE PRESENTATIONS OF DIRECT PRODUCTS OF GROUPS

Miguel A. Borges-Trenard
Hebert Pérez-Rosés

Department of Mathematics, Faculty of Sciences,
University of Oriente, Santiago de Cuba 90500, Cuba;
{ mborges, hebert }@csd.uo.edu.cu

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Abstract

Complete presentations provide a natural solution to the word problem in monoids and groups. Here we give a simple way to construct complete presentations for the direct product of groups, when such presentations are available for the factors. Actually, the construction we are referring to is just the classical construction for direct products of groups, which has been known for a long time, but whose completeness-preserving properties had not been detected. Using this result and some known facts about Coxeter groups, we sketch an algorithm to obtain the complete presentation of any finite Coxeter group. A similar application to Abelian and Hamiltonian groups is mentioned.

Keywords. Word problem, complete presentations, direct product, Coxeter groups, Abelian groups, Hamiltonian groups.

Introduction.

From the computational viewpoint it is very useful to have group presentations that make possible the solution of the word problem, and the obtainment of such presentations has received increasing attention in recent years. In a pioneering article, Gilman gave a computational procedure to convert (when possible) an arbitrary group presentation into a complete one, which is a presentation that makes it possible to solve the word problem, and thereby give answer to several basic questions in regard to the corresponding group, such as its finiteness [8].

Later on, Le Chenadec used an implementation of the Knuth-Bendix procedure [14] to obtain complete presentations for some important groups, including a large subfamily of Coxeter groups [15, 16, 17].

Following the trend left by Le Chenadec, researchers at the University of Oriente have found complete presentations for dicyclic and dihedral groups [3], symmetric groups [2, 19], Dyck groups [20, 12], fractional linear groups [24], alternating groups [9], and Coxeter groups with three generators [25]. To obtain these results, an implementation of Mora's procedure for the computation of Gröbner bases in noncommutative algebras has been used [21, 23]. In fact, when it is applied to monoid and group presentations, Mora's procedure is equivalent to the aforementioned Knuth-Bendix procedure.

It is well known that the Knuth-Bendix procedure may not stop because there is no finite complete presentation based on a specific set of generators, whereas there may be a finite complete presentation on a *different* set of generators. In order to overcome the computing difficulties, it may be necessary to change the generating set, or to change the ordering used for the computations, or both; for examples of these manipulations see the work of Hermiller [10]. A common feature of all the presentations obtained at the University of Oriente is that they keep the original "standard" generators. Keeping the standard generators could be advantageous in many cases, for these generators usually have a straightforward geometric interpretation, and several associated results; this is the case of Coxeter groups, for example (see section 3).

In 1995, Hermiller and Meier showed that the graph product of finitely many groups that admit finite complete presentations, also admits a finite complete presentation [11] (the graph product is a group operation that involves direct and free products). The construction by Hermiller and Meier introduces a new set of generators, that are tuples of the original ones. In general, the new presentation thus defined may be substantially larger than the union of the input presentations.

In section 2 of this paper we re-address the issue of computing complete presentations for direct products, and give a simple (though useful) result for that special case. Basically, we show that the classical straightforward construction of direct product presentations preserves completeness. The classical procedure to construct the presentation of a direct product of groups amounts to a little more than the mere union of the input presentations, and thus, it is simpler than the more general construction of Hermiller's and Meier's, and the resulting presentations are smaller.

In section 3 we show how the latter result can be used to construct the complete presentation of any finite Coxeter group. In order to accomplish this, it is first necessary to have complete presentations for the *irreducible* Coxeter groups, *on the standard generators*, because the standard generators are needed to decompose an arbitrary finite Coxeter group in irreducible components of the same type. Similar applications to Abelian and Hamiltonian groups are detailed in section 4.

1 Basic Definitions.

Let T be a finite set of symbols; we shall denote by $\langle T\rangle$ the free monoid generated by T, i.e. the set of words on the alphabet T together with the operation of concatenation of words, where the identity element is the empty word, and will be denoted as 1. If we have a binary relation R on $\langle T\rangle$, then $\langle R\rangle$ will stand for the congruence generated by R, i.e. the reflexive-symmetric-transitive closure of R, that is compatible with the concatenation operation. To make it simpler, we shall write the pairs of words $(\alpha, \beta) \in R$, as rewriting rules of the form $\alpha \to \beta$, or equalities of the form $\alpha = \beta$, and $\overline{\alpha}$ will be the equivalence class of α modulo $\langle R\rangle$. The rules $\alpha \to \beta$ could also be written as binomials $\alpha - \beta$, thus enabling the use of the concepts and tools of non commutative polynomial ring theory, such as Gröbner bases [5, 18].
Let T^{-1} be the set of inverse symbols of T:

$$T^{-1} = \{x^{-1} \mid x \in T\}, \quad \text{and} \quad I = \{xx^{-1} = x^{-1}x = 1 \mid x \in T\}.$$

The set I is a binary relation on $\langle T \cup T^{-1} \rangle$, and the quotient monoid

$$F = \frac{\langle T \cup T^{-1} \rangle}{\langle I \rangle}$$

is the free group generated by T. Now, if R is any binary relation on $\langle T \cup T^{-1} \rangle$, then it is said that $\langle T; R \rangle$ is a presentation of the group

$$G = \frac{\langle T \cup T^{-1} \rangle}{\langle R \cup I \rangle},$$

or any group canonically isomorphic to G.
Clearly, if we let $T = T \cup T^{-1}$, and $R = R \cup I$, then we have a monoid presentation, so we can restrict ourselves to presentations of this kind.
The theory of rewriting systems and its applications to group theory is extensively treated by Book and Otto [4], and also surveyed by Cohen [6]. The monographs by Le Chenadec [16] and Sims [26] contain more specific, as well as introductory material on groups. In the remaining part of this section we summarize some basic notions and results of this theory that will be used throughout the paper.
If there is (α, β) in R, and v, w are arbitrary words of $\langle T \rangle$, then we say that $v\alpha w$ **reduces** or **rewrites** to $v\beta w$ (with respect to R), and we also write $v\alpha w \to v\beta w$. In this case, the word $v\alpha w$ is called **reducible**; otherwise, it is said to be **irreducible** (notice that $\to$ now stands for a new binary relation R' on $\langle T \rangle$, such that $(v\alpha w, v\beta w) \in R'$ iff $(\alpha, \beta) \in R$). From the computational viewpoint, it would be desirable that there be exactly one irreducible word in each equivalence class modulo $\langle R \rangle$ and that this irreducible word can be computed by repeated application of a finite number of reduction steps $\alpha \to \beta$.
We write $\xrightarrow{*}$ for the reflexive-transitive closure of R', and we say that the words α, β are **joinable** (denoted $\alpha \downarrow \beta$) if there exists a word γ such that $\alpha \xrightarrow{*} \gamma$ and $\beta \xrightarrow{*} \gamma$. R is called **terminating**, **well-founded** or **noetherian** if there is no infinite sequence $\alpha_1 \to \alpha_2 \to \ldots \to \alpha_n \to \ldots$; R is called **confluent** if $\alpha \xrightarrow{*} \beta$ and $\alpha \xrightarrow{*} \gamma$ implies $\beta \downarrow \gamma$, and it is called **locally confluent** if $\alpha \to \beta$ and $\alpha \to \gamma$ implies $\beta \downarrow \gamma$. If R is terminating then local confluence amounts to confluence (Diamond Lemma or Newman's Lemma).
R is said to have the **Church-Rosser property** if $\alpha \equiv \beta \ (\mathrm{mod}\langle R \rangle)$ implies $\alpha \downarrow \beta$. If R is both terminating and Church-Rosser, it is said to be **complete**, **canonical** or **convergent**; completeness guarantees the existence of

a unique irreducible word in each equivalence class modulo $\langle R \rangle$. Since the Church-Rosser property is equivalent to confluence, in order to check for completeness, it suffices to check first for termination, and then for local confluence.
The termination property is undecidable (as well as confluence), but we could prove it if we could find a well-founded strict partial ordering $<$, such that $<$ is **admissible** (i.e. $u < v$ implies $xuy < xvy$, for all $u, v, x, y \in \langle T \rangle$), and $<$ is **compatible** with R (i.e. $\beta < \alpha$ for all rules $\alpha \to \beta$ in R). In fact, termination of R is equivalent to the existence of such an ordering. Admissible well-founded orderings are also called **term orderings**.
Once R is proved terminating, completeness can be checked with the aid of the Knuth-Bendix criterion, [14], given below. Let R be terminating, and let $LRED(\alpha)$ denote the irreducible word of $\overline{\alpha}$, obtained from α by repeated reduction of the leftmost possible substring. An **overlap ambiguity** is a triple of non-empty strings, α, μ, β such that there are rules $\alpha\mu \to \gamma_1$ and $\mu\beta \to \gamma_2$ in R; the pair of strings $\gamma_1\beta$ and $\alpha\gamma_2$ is then called a **critical pair**, since $\alpha\mu\beta$ reduces to them both.
The triple α, μ, β is called an **inclusion ambiguity** if there are rules $\mu \to \gamma_1$ and $\alpha\mu\beta \to \gamma_2$ in R, where $\gamma_1 \neq \gamma_2$ if α and β are both empty. The pair of words $\alpha\gamma_1\beta$ and γ_2 is the corresponding critical pair.
The terminating set of rules R is locally confluent (and hence complete) if, and only if, for any critical pair δ_1, δ_2 we have $LRED(\delta_1) = LRED(\delta_2)$ [14]; clearly, it suffices to check whether $\delta_1 \downarrow \delta_2$. In this case we say that the critical pair **resolves**. In fact, the equality between the corresponding irreducible words of $\overline{\delta_1}$ and $\overline{\delta_2}$ should be true no matter the reduction strategy one uses to obtain them.
The Knuth-Bendix procedure receives a terminating set of rules R as input, and produces a complete one R' (if it ever stops) by adding the critical pairs that do not resolve. The procedure makes use of the underlying ordering $<$ in an essential way: pairs are added so that the new set of rules remains compatible with $<$, in order to guarantee its termination. A group given by a complete presentation $\langle T; R \rangle$ has a solvable word problem. Since R is terminating and finite, we can always compute $LRED(\alpha)$ for every word α, and $LRED(\alpha)$ is the only irreducible word in the class $\overline{\alpha}$.
R is called **normalized** if for every rule $\alpha \to \beta$ of R, $\beta \in Irr(R)$, and $\alpha \in Irr(R \setminus \{\alpha \to \beta\})$; normalized presentations do not contain relations that are obviously redundant.

2 Complete Presentations of Direct Products.

Let G_1 and G_2 be groups with finite disjoint generating sets T_1 and T_2, and finite sets of defining relations R_1 and R_2 on T_1 and T_2 respectively. Let us define the set B of **bridge relations** between T_1 and T_2 as follows:

$$B = \{yx \to xy \mid x \in T_1, y \in T_2\}.$$

It is well known that the set $T_1 \cup T_2$ generates the directproduct G_1 x G_2, and $R_1 \cup R_2 \cup B$ is a set of defining relations for G_1 x G_2 ([7], p. 3). Moreover, we are going to prove the following, stronger result:

Theorem 1: Let G_1 and G_2 be groups, let T_1, T_2, R_1, R_2 and B be defined as above, and let us assume that $\langle T_1; R_1 \rangle$ and $\langle T_2; R_2 \rangle$ are complete presentations of G_1 and G_2 respectively. Then $\langle T_1 \cup T_2; R_1 \cup R_2 \cup B \rangle$ is a complete presentation of the direct product G_1 x G_2. Moreover, if R_1 and R_2 are normalized, so is $R_1 \cup R_2 \cup B$

Proof: First we have to show that $R_1 \cup R_2 \cup B$ is terminating, which is equivalent to finding an admissible well-founded strict partial ordering $<_T$ on $\langle T_1 \cup T_2 \rangle$, compatible with $R_1 \cup R_2 \cup B$. As R_1 and R_2 are terminating, there exist admissible well-founded orderings $<_1$ and $<_2$, such that $<_1$ is compatible with R_1 and $<_2$ is compatible with R_2. Let us construct $<_T$ from $<_1$ and $<_2$ as follows.
Every word α in $\langle T_1 \cup T_2 \rangle$ can be transformed in a finite number of steps, with the aid of the bridge rules only, into a word of the form $\alpha_1\alpha_2$ where $\alpha_1 \in \langle T_1 \rangle$ and $\alpha_2 \in \langle T_2 \rangle$. In order to transform α into this "canonical form", we may apply the bridge rules in different ways, but the number of steps needed is constant. Let $\alpha, \beta \in \langle T_1 \cup T_2 \rangle$ and let $\alpha_1\alpha_2$ and $\beta_1\beta_2$ be their corresponding "canonical forms"; moreover, let us suppose that we needed k steps to obtain $\alpha_1\alpha_2$ from α, and m for $\beta_1\beta_2$. If $\alpha_1 \leq_1 \beta_1$ and $\alpha_2 <_2 \beta_2$, or if $\alpha_1 <_1 \beta_1$ and $\alpha_2 \leq_2 \beta_2$, then we take $\alpha <_T \beta$. On the other hand, if $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$, and $k < m$, then we take $\alpha <_T \beta$ too (notice that $<_T$ need not be a total ordering). It is easy to see that $<_T$ coincides with $<_1$ on $\langle T_1 \rangle$, and with $<_2$ on $\langle T_2 \rangle$. Moreover, it is easy to verify that $<_T$ is admissible, well founded and compatible with $R_1 \cup R_2 \cup B$, thus proving that this set of rules is terminating.

Now, we only have to prove that for every critical pair v, w that arises from $R_1 \cup R_2 \cup B$, $v \downarrow w$.
First, every critical pair arising from R_1 is obviously joinable, since R_1 is complete, and the same happens if the critical pair arises from two relations in R_2. On the other hand, no unresolved ambiguities can arise from $R_1 \cup R_2$, so we only have to examine possible overlap ambiguities resulting from a bridge relation together with a relation in R_1 or R_2.
Let $yx \to xy$ be a bridge relation, and $x\alpha \to \beta$ a relation in R_1, where $\alpha, \beta \in \langle T_1 \rangle$. This pair of relations gives rise to the critical pair $xy\alpha, y\beta$. Now, $y \in T_2$ and $\alpha, \beta \in \langle T_1 \rangle$, hence $xy\alpha \xrightarrow{*} x\alpha y \to \beta y$ and $y\beta \xrightarrow{*} \beta y$, hence $xy\alpha \downarrow y\beta$.
Similarly, if $\alpha y \to \beta$ is a relation in R_2 then we have the critical pair $\alpha xy, \beta x$, where $x \in T_1$ and $\alpha, \beta \in \langle T_2 \rangle$, so $\alpha xy \xrightarrow{*} x\alpha y \to x\beta$ and $\beta x \xrightarrow{*} x\beta$, hence $\alpha xy \downarrow \beta x$.
That $R_1 \cup R_2 \cup B$ is normalized, if R_1 and R_2 are, is a triviality. $\square$

Obviously, we obtain the same result if we define the set of bridge relations as $B = \{xy \to yx \mid x \in T_1, y \in T_2\}$. Notice also that, as explained in the proof, we can work with two complete presentations that have been obtained with the aid of different orderings: $<_1$ and $<_2$, to construct the complete presentation of the direct product.

3 Application to Coxeter Groups.

A **Coxeter group** on the n generators $x_1, x_2, \ldots, x_n$ is defined by a presentation of the form

$$x_i^2 = 1, \ 1 \le i \le n,$$
$$(x_i x_j)^{m_{ij}} = 1, \ 1 \le i < j \le n,$$

where the m_{ij} are integers that are either zero or greater than one. If $m_{ij} = 0$ then we have the trivial relation $1 = 1$. The generators $x_1, x_2, \ldots, x_n$ can be seen as reflections of a real Euclidean vector space V, with respect to n different hyperplanes of V. Coxeter and Moser devote one chapter of their book [7] to the study of these groups; and the monograph by Humphreys [13] is also a good source on the subject.
Given a Coxeter group G as defined above, the **Coxeter graph** H of G is constructed as follows: the vertex set of H is $\{x_1, x_2, \ldots, x_n\}$, and there is

an edge joining x_i and x_j iff $m_{ij} \neq 2$; that edge (x_i, x_j) is then labeled with m_{ij}; labels equal to 3 are conventionally not written.
If H is connected then the corresponding group G is called **irreducible** or **indecomposable**, otherwise it is said to be **reducible** or **decomposable**. A reducible Coxeter group is the direct product of smaller irreducible groups of the same family, corresponding to the connected components of its graph H. Figure 1 shows the finite irreducible Coxeter groups.

Every finite Coxeter group is either irreducible or it is the direct product of two or more irreducible ones.
If we had a complete presentation for each one of these irreducible groups, then, with the aid of Theorem 1 and the results mentioned above, we could easily construct a complete presentation for any finite Coxeter group.
Le Chenadec's general result covers the Coxeter groups, for which all the exponents m_{ij} are different from 2, and he treated the finite irreducible Coxeter groups as special cases that do not fit in this general setting [15, 17]. Our approach differs essentially from Le Chenadec's, and it looks more promising for obtaining complete presentations in the cases where they are still missing.
The groups in the family $\mathbf{A_n}$ are precisely the symmetric groups; i.e. $\mathbf{A_n} = S_{n+1}$, where S_{n+1} is the symmetric group of degree $n+1$ and order $(n+1)!$, taking as generators the transpositions $x_i = (i \quad i+1)$. A complete normalized presentation for these groups, based on the degree lexicographic ordering (deglex), is given by Le Chenadec [17]:

$$\begin{array}{ll} x_i^2 = 1 & (1 \leq i \leq n), \\ x_i x_j = x_j x_i & (3 \leq i \leq n; 1 \leq j \leq i-2), \\ x_i x_{i-1} \dots x_{i-j} x_i = x_{i-1} x_i x_{i-1} \dots x_{i-j} & (2 \leq i \leq n; 1 \leq j \leq i-1). \end{array}$$

Curiously, this same presentation has been obtained by H. Martínez-Silva [19] using another term ordering, proposed by Mora [22], which is defined as follows: Let us assume that there is a linear ordering defined on the set T, and let $MS(w)$ denote the maximal symbol that occurs in the word $w \in \langle T \rangle$. Given w_1 and w_2 with $w_1 \neq w_2$, if $MS(w_1) < MS(w_2)$, then $w_1 < w_2$, otherwise if $MS(w_1) = MS(w_2) = x$, then we can write:
$w_1 = u_1 x u_2 x \dots u_{r-1} x u_r$, and $w_2 = v_1 x v_2 x \dots v_{p-1} x v_p$, where the u_i and v_j do not contain any occurrences of x; then if $r < p$ we take $w_1 < w_2$. Finally, if $r = p$ then we set $q = \max\{i / u_i \neq v_i\}$, and if $u_q < v_q$ then $w_1 < w_2$.
This ordering has been shown to have useful properties [5] and it has been used to calculate a number of complete presentations, mentioned earlier, in

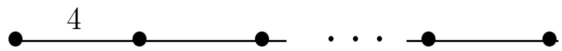
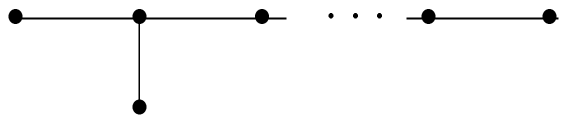
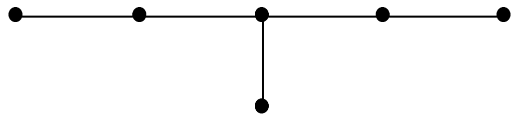
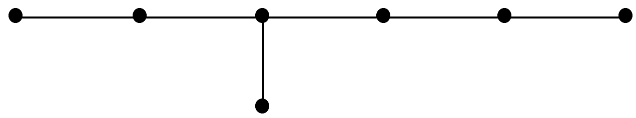
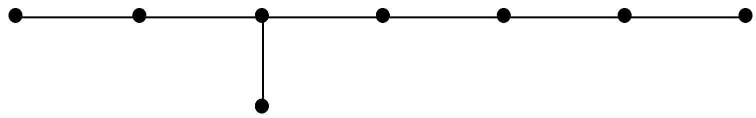
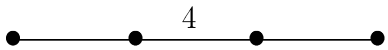
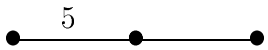
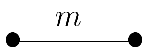
Name	**Order**	**Coxeter graph**
$A_n \quad (n \geq 1)$	$(n+1)!$	
$B_n \quad (n \geq 2)$	$2^n n!$	
$D_n \quad (n \geq 4)$	$2^{n-1} n!$	
E_6	51840	
E_7	2903040	
E_8	696729600	
F_4	1152	
H_3	120	
H_4	14400	
$I_2(m)$	$2m$	

Figure 1: The irreducible finite Coxeter groups

the introduction of this paper [2, 3, 12, 20, 25, 19, 24, 9]. Moreover, unless otherwise stated, this is the ordering that we have used in all the computational results that we are going to present from now on. (Also, all presentations shown in this paper are normalized).
Le Chenadec also gives complete presentations for the families $\mathbf{B_n}$ and $\mathbf{I_2(m)}$, as well as for the individual group $\mathbf{H_3}$ (based on the deglex ordering), which are shown below. In turn, we have computed complete presentations for $\mathbf{H_4}$ and $\mathbf{F_4}$, which are given in the appendix.

$\mathbf{B_n}$:

$$\begin{array}{ll}
x_i^2 = 1 & (1 \leq i \leq n), \\
x_i x_j = x_j x_i & (3 \leq i \leq n; 1 \leq j \leq i-2), \\
x_i x_{i-1} \ldots x_{i-j} x_i = x_{i-1} x_i x_{i-1} \ldots x_{i-j} & (2 \leq i \leq n-1; 1 \leq j \leq i-1), \\
(x_n x_{n-1} \ldots x_{n-j})^2 = & \\
\quad = x_{n-1} x_n x_{n-1} \ldots x_{n-j} x_n x_{n-1} \ldots x_{n-j+1} & (1 \leq j \leq n-1).
\end{array}$$

$\mathbf{I_2(m)}$:

$$a^2 = b^2 = 1,$$
$$baba \ldots (m \text{ entries}) = abab \ldots (m \text{ entries}).$$

$\mathbf{H_3}$:

$$a^2 = b^2 = c^2 = 1,$$
$$babab = ababa, \quad ca = ac, \quad cbc = bcb,$$
$$cbac = bcba, \quad cbabcb = bcbabc, \quad cbabacbaba = bcbabacbab.$$

For the family $\mathbf{D_n}$ and the individual groups $\mathbf{E_6}$, $\mathbf{E_7}$ and $\mathbf{E_8}$, Le Chenadec reported not being able to obtain complete presentations. We have managed to compute a complete presentation for $\mathbf{E_6}$ (see appendix), but we too have failed so far with $\mathbf{D_n}$, $\mathbf{E_7}$ and $\mathbf{E_8}$.

4 Abelian and Hamiltonian Groups.

It is well known that every finitely generated Abelian group G is isomorphic to a direct product of cyclic groups of the form

$$Z_{p_1^{r_1}} \times Z_{p_2^{r_2}} \times \cdots \times Z_{p_n^{r_n}} \times Z \times Z \times \cdots \times Z \qquad (4.1)$$

where the p_i are prime and not necessarily distinct; or G is also isomorphic to a direct product of the form

$$Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r} \times Z \times Z \times \cdots \times Z \qquad (4.2)$$

where m_i divides m_{i+1}, for $i = 1, 2, \ldots, r-1$. These decompositions are unique except for rearrangements of the factors.

Now, if a is a generator of the cyclic group Z_m , then $a^m = 1$ is a complete presentation of Z_m , and Z is a free group generated by one element. Hence, if we have the canonical decomposition (4.1) or (4.2) of an Abelian group G, then, with the aid of Theorem 1 we can readily construct a complete presentation for G. For example, a complete presentation of $Z_2 \times Z_6 \times Z$ would be

$$\langle a, b, c, d; \quad a^2 = b^6 = cd = dc = 1, \;\; ba = ab,$$
$$ca = ac, \;\; cb = bc, \;\; da = ad, \;\; db = bd\rangle$$

These complete presentations for Abelian groups are, of course, well known; our intention here is to exhibit them as a straightforward consequence of our main result.

Conversely, if we have any complete presentation of an Abelian group G, then it is also possible to obtain its canonical decomposition (4.1) (see, for example, [16] and [26]).

These facts can also be used to obtain complete presentations of the Hamiltonian groups, i.e. the non-Abelian groups, all of whose subgroups are normal. The finite Hamiltonian groups have the form $Q \times A \times B$, where Q is the quaternion group, A is an Abelian group of odd order and B is an Abelian group of type $Z_2 \times Z_2 \times \cdots \times Z_2$ (m times, $m \geq 0$) ([7], p. 8).

For the quaternion group Q, Coxeter and Moser give two presentations: $\langle a, b; \;\; a^2 = b^2 = (ab)^2\rangle$ and $\langle a, b, c; \;\; a^2 = b^2 = c^2 = abc\rangle$, from which we have computed the complete presentations:

$$\langle a, b; \quad a^4 = 1, \;\; ba = a^3b, \;\; b^2 = a^2, \rangle$$

and

$$\langle a, b, c; \quad a^4 = 1, \;\; ba = a^3b, \;\; b^2 = a^2, \;\; c = ab\rangle,$$

respectively. Note that in the second presentation we are rewarded with the fact that generator c is superfluous. For these computations we have also used Mora's order.

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Appendix: Complete presentation of some irreducible finite Coxeter groups.

$\mathbf{H_4}$:

$$a^2 = b^2 = c^2 = d^2 = 1,$$

$$babab = ababa, ca = ac, da = ad, db = bd, cbc = bcb, dcd = cdc,$$

$$cbac = bcba, dcbd = cdcb, dcbad = cdcba, cbabcb = bcbabc,$$

$$dcbabd = cdcbab, dcbabad = cdcbaba, dcbabcdc = cdcbabcd,$$

$$dcbabacdc = cdcbabacd, cbabacbaba = bcbabacbab,$$

$$dcbabacbdcb = cdcbabacbdc, dcbabacbadcbaba = cdcbabacbadcbab,$$

$$dcbabacbabdcbabacbab = cdcbabacbabdcbabacba,$$

$$dcbabacbabcdcbabacbabdcbabacba = cdcbabacbabcdcbabacbabdcbabacb,$$

$$dcbabacbabcdcbabacbabcdcbabacbadcbab = \ldots$$

$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbadcba,$$

$$dcbabacbabcdcbabacbabcdcbabacbabdcbabc = \ldots$$
$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbabdcbab,$$
$$dcbabacbabcdcbabacbabcdcbabacbabdcbabac = \ldots$$
$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbabdcbaba,$$
$$dcbabacbabcdcbabacbabcdcbabacbabcdcbabcd = \ldots$$
$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbabcdcbabc,$$
$$dcbabacbabcdcbabacbabcdcbabacbabcdcbabacd = \ldots$$
$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbabcdcbabac,$$
$$dcbabacbabcdcbabacbabcdcbabacbabcdcbabacbd = \ldots$$
$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbabcdcbabacb,$$
$$dcbabacbabcdcbabacbabcdcbabacbabcdcbabacbad = \ldots$$
$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbabcdcbabacba,$$
$$dcbabacbabcdcbabacbabcdcbabacbabcdcbabacbabd = \ldots$$
$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbabcdcbabacbab,$$
$$dcbabacbabcdcbabacbabcdcbabacbabcdcbabacbabcdc = \ldots$$
$$\ldots = cdcbabacbabcdcbabacbabcdcbabacbabcdcbabacbabcd.$$

$\mathbf{F_4}$:

$$a^2 = b^2 = c^2 = d^2 = 1,$$
$$ca = ac, da = ad, db = bd, bab = aba, dcd = cdc,$$
$$cbcb = bcbc, dcbd = cdcb, dcbad = cdcba, cbacba = bcbacb,$$
$$dcbcdc = cdcbcd, dcbacdc = cdcbacd, dcbacbdcbc = cdcbacbdcb,$$
$$dcbacbdcbac = cdcbacbdcba, dcbacbcdcbcd = cdcbacbcdcbc,$$
$$dcbacbcdcbacd = cdcbacbcdcbac,$$
$$dcbacbcdcbacbd = cdcbacbcdcbacb,$$
$$dcbacbcdcbacbcdc = cdcbacbcdcbacbcd.$$

$\mathbf{E_6}$:

$$a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = 1,$$
$$cb = bc, db = bd, dc = cd, ea = ae, eb = be, ec = ce,$$
$$fa = af, fb = bf, fd = df, fe = ef, bab = aba, cac = aca,$$
$$dad = ada, ede = ded, fcd = dfc, fce = efc, fcf = cfc,$$
$$caba = bcab, cabc = acab, daba = bdab, dabd = adab,$$
$$daca = cdac, dacd = adac, edae = deda, fcae = efca,$$
$$fcaf = cfca, dabcd = adabc, edabe = dedab, edace = dedac,$$
$$fcabe = efcab, fcabf = cfcab, fcada = dfcad, fcadf = cfcad,$$
$$dabcab = cdabca, edabce = dedabc, fcabdf = cfcabd,$$
$$fcaded = efcade, fcadef = cfcade, dabcada = adabcad,$$
$$edabcae = dedabca, fcabdab = dfcabda, fcabdaf = cfcabda,$$
$$fcabded = efcabde, fcabdef = cfcabde, fcabdaef = cfcabdae,$$
$$edabcaded = dedabcade, fcabdacfc = cfcabdacf,$$
$$fcabdaeda = efcabdaed, fcabdaedf = cfcabdaed,$$
$$fcabdacefc = cfcabdacef, fcabdacedac = efcabdaceda,$$
$$fcabdacedfc = cfcabdacedf, fcabdacedabc = efcabdacedab,$$
$$fcabdacedafca = cfcabdacedafc, fcabdacedabfcab = cfcabdacedabfca.$$

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