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Cycle decompositions of pathwidth-6 graphs

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Abstract

Hajós' conjecture asserts that a simple Eulerian graph on n vertices can be decomposed into at most $\lfloor (n-1)/2 \rfloor$ cycles. The conjecture is only proved for graph classes in which every element contains vertices of degree 2 or 4. We develop new techniques to construct cycle decompositions. They work on the common neighborhood of two degree-6 vertices. With these techniques, we find structures that cannot occur in a minimal counterexample to Hajós' conjecture and verify the conjecture for Eulerian graphs of pathwidth at most 6. This implies that these graphs satisfy the *small cycle double cover conjecture*.

KEYWORDS

circuit, cycle decomposition, decomposition, Eulerian graphs, Hajós conjecture

1 | INTRODUCTION

It is well-known that the edge set of an Eulerian graph can be decomposed into cycles. In this context, a natural question arises: How many cycles are needed to decompose the edge set of an Eulerian graph? Clearly, a graph G with a vertex of degree $|V(G)| - 1$ cannot be decomposed into less than $\lfloor (|V(G)| - 1)/2 \rfloor$ many cycles. Thus, for a general graph G, we cannot expect to find a cycle decomposition with less than $\lfloor (|V(G)| - 1)/2 \rfloor$ many cycles. Hajós' conjectured that this number of cycles will always suffice. (Originally, Hajós' conjectured a bound of $\lfloor |V(G)|/2 \rfloor$. Dean [4] showed that Hajós' conjecture is equivalent to the conjecture with bound $\lfloor (|V(G)| - 1)/2 \rfloor$).

Conjecture 1 (Hajós' conjecture (see [11])). *Every simple Eulerian graph G has a cycle decomposition with at most $(|V(G)| - 1)/2$ many cycles.*

Consider a sequence $T_1, \ldots, T_k$ of triangles such that $|V(T_i) \cap V(T_{i+1})| = 1$ for $i \in \{1, \ldots, k-1\}$ and $V(T_i) \cap V(T_j) = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ with $|i-j| > 1$. The graph $G_k = T_1 \cup \cdots \cup T_k$ has a unique cycle decomposition into the k triangles $T_1, \ldots, T_k$ and $V(G) = 2k+1$. This shows that the bound $\lfloor (|V(G)| - 1)/2 \rfloor$ is best possible.

More generally, Granville and Moisiadis [7] showed that for every $n \geq 3$ and every $i \in \{1, \ldots, \lfloor (|V(G)| - 1)/2 \rfloor\}$, there exists a connected graph with n vertices and maximum degree at most 4 whose minimal cycle decomposition consists of exactly i cycles.

A simple lower bound on the minimal number of necessary cycles is the maximum degree divided by 2. This bound is achieved by the complete bipartite graph $K_{2k,2k}$ that can be decomposed into k Hamiltonian cycles (see [10]). In general, all graphs with a Hamilton decomposition (eg, complete graphs K_{2k+1} [1]) trivially satisfy Hajós' conjecture.

Hajós' conjecture remains wide open for most classes. Heinrich, Natale, and Streicher [9] verified Hajós' conjecture for small graphs by exploiting Lemma 6, 8, 10, and 11 of this paper as well as random heuristics and integer programming techniques:

Theorem 2 (Heinrich, Natale, and Streicher [9]). *Every simple Eulerian graph with at most* 12 *vertices satisfies Hajós' Conjecture.*

Apart from Hamilton decomposable (and small) graphs, the conjecture has (to our knowledge) only been shown for graph classes in which every element contains vertices of degree at most 4. Granville and Moisiadis [7] showed that Hajós' conjecture is satisfied for all Eulerian graphs with maximum degree at most 4. Fan and Xu [6] showed that all Eulerian graphs that are embeddable in the projective plane or do not contain the minor K_6^- satisfy Hajós' conjecture. To show this, they provided four operations involving vertices of degree less than 6 that transform an Eulerian graph not satisfying Hajós' conjecture into another Eulerian graph not satisfying the conjecture that contains at most one vertex of degree less than 6. This statement generalizes the work of Granville and Moisiadis [7]. As all four operations preserve planarity, the statement further implies that planar graphs satisfy Hajós' conjecture. This was shown by Seyffarth [12] before. The conjecture is still open for toroidal graphs. Xu and Wang [13] showed that the edge set of each Eulerian graph that can be embedded on the torus can be decomposed into at most $\lfloor (|V(G)| + 3)/2 \rfloor$ cycles. Heinrich and Krumke [8] introduced a linear time procedure that computes minimum cycle decompositions in treewidth-2 graphs of maximum degree 4.

We contribute to the sparse list of graph classes satisfying Hajós' conjecture. Our class contains graphs without any vertex of degree 2 or 4—in contrast to the above mentioned graph classes.

Theorem 3. *Every Eulerian graph G of pathwidth at most* 6 *satisfies Hajós' conjecture.*

As graphs of pathwidth at most 5 contain two vertices of degree less than 6, it suffices to concentrate on graphs of pathwidth exactly 6. All such graphs with at most one vertex of degree 2 or 4 contain two degree-6 vertices that are either nonadjacent with the same neighborhood or adjacent with four or five common neighbors. We use these structures to construct cycle decompositions.

With similar ideas, it is possible attack graphs of treewidth 6. As more substructures may occur, we restrict ourselves to graphs of pathwidth 6.

A *cycle double cover* of a graph G is a collection C of cycles of G such that each edge of G is contained in exactly two elements of C. The popular *cycle double cover conjecture* asserts that every 2-edge connected graph admits a cycle double cover. This conjecture is trivially satisfied for Eulerian graphs. Hajós' conjecture implies a conjecture of Bondy regarding the Cycle double cover conjecture.

Conjecture 4 (Small Cycle Double Cover Conjecture (Bondy [3])). *Every simple 2-edge connected graph G admits a cycle double cover of at most $|V(G)| - 1$ many cycles.*

As a cycle double cover may contain a cycle twice, we can conclude the following directly from Theorem 3.

Corollary 5. *Every Eulerian graph G of pathwidth at most 6 satisfies the small cycle double cover conjecture.*

2 | REDUCIBLE STRUCTURES

All graphs considered in this paper are finite, simple and Eulerian. We use standard graph theory notation as can be found in the book of Diestel [5].

To prove our main theorem, we consider a cycle decomposition of a graph G as a coloring of the edges of G where each color class is a cycle. We define a *legal coloring* c of a graph G as a map

$$c : E(G) \mapsto \{1, \ldots, \lfloor (|V(G)| - 1)/2 \rfloor\}$$

where each color class $c^{-1}(i)$ for $i \in \{1, \ldots, \lfloor (|V(G)| - 1)/2 \rfloor\}$ is the edge set of a cycle of G. A legal coloring is thus associated to a cycle decomposition of G that satisfies Hajós' conjecture.

Using recoloring techniques we show the following lemmas for two degree-6 vertices with common neighborhood N of size 4, 5, or 6. All proofs can be found in Section 4.

Lemma 6. *Let G be an Eulerian graph with two degree-6 vertices u, v with*

$$N(u) = N \cup \{v\} \quad N(v) = N \cup \{u\}.$$

Let all Eulerian graphs obtained from $G - \{u, v\}$ by addition or deletion of edges with both end vertices in N have a legal coloring.

If $G[N]$ contains at least one edge, or if $G - \{u, v\}$ contains a vertex that is adjacent to at least three vertices of N, then G also has a legal coloring.

Lemma 7. *Let G be an Eulerian graph with two degree-6 vertices u, v with*

$$N(u) = N \cup \{v, x_v\} \quad N(v) = N \cup \{u, x_u\}.$$

Let P be an x_u–x_v-path in $G - \{u, v\} - N$. Further, let all Eulerian graphs obtained from $G - \{u, v\}$ by addition and deletion of edges with both end vertices in $N \cup \{x_u, x_v\}$ and by optional deletion of $E(P)$ have a legal coloring.

If $G[N \cup \{x_u, x_v\}]$ *contains at least one edge not equal to* $x_u x_v$*, or if* $G - \{u, v\}$ *contains a vertex that is adjacent to at least three vertices of* N*, then* G *also has a legal coloring.*

Lemma 8. *Let* G *be an Eulerian graph with two degree-*6 *vertices* u, v *with*

$$N(u) = N(v) = N.$$

Let all Eulerian graphs obtained from $G - \{u, v\}$ *by addition or deletion of edges with both end vertices in* N *have a legal coloring.*

If $G[N]$ *contains at least one edge, or if* $G - \{u, v\}$ *contains a vertex that is adjacent to at least three vertices of* N*, then* G *also has a legal coloring.*

The next two results are not necessary for the proof of Theorem 3. We nevertheless state them here.

The first lemma is useful for graphs with an odd number of vertices.

Lemma 9. *Let* G *be an Eulerian graph on an odd number* n *of vertices that contains a vertex* u *of degree* 2 *or* 4 *with neighborhood* N*. If all Eulerian graphs that can be obtained from* $G - \{u\}$ *by addition or deletion of arbitrary edges in* $G[N]$ *have a legal coloring, then* G *has a legal coloring.*

If a graph G contains a degree-2 vertex v with independent neighbors x_1, x_2, then it is clear that a legal coloring of $G - v + x_1x_2$ can be transformed into a legal coloring of G. Granville and Moisiadis [7] observed a similar relation for a degree-4 vertex.

Lemma 10 (Granville and Moisiadis [7]. *Let* G *be an Eulerian graph containing a vertex* v *with neighborhood* $N = \{x_1, \ldots, x_4\}$ *such that* $G[N]$ *contains the edge* x_1x_2 *but not the edge* x_3x_4*. If* $G - \{vx_3, vx_4\} + \{x_3x_4\}$ *has a legal coloring, then* G *also has a legal coloring.*

Generalizing this idea, we analyze the neighborhood of a degree-6 vertex.

Lemma 11. *Let* G *be an Eulerian graph that contains a degree-*6 *vertex* u *with neighborhood* $N_G(u) = \{x_1, \ldots, x_6\}$ *such that* $\{x_1, x_2, x_3, x_4\}$ *is a clique and* $x_5x_6 \notin E(G)$*. If* $G' = G - \{x_5u, ux_6\} + \{x_5x_6\}$ *has a legal coloring, then* G *has a legal coloring.*

3 | RECOLORING TECHNIQUES

In this section, we provide recoloring techniques that are necessary to prove Lemma 6, 7, and 8. For a path P or a cycle C we write $c(P) = i$ or $c(C) = i$ to express that all edges of P respectively C are colored with color i. We start with a statement about monochromatic triangles.

Lemma 12. *Let* H *be a graph with legal coloring* c *that contains a clique* $\{x_1, x_2, x_3, y\}$*. Then there is a legal coloring* c' *of* H *in which the cycle* $x_1x_2x_3x_1$ *is not monochromatic.*

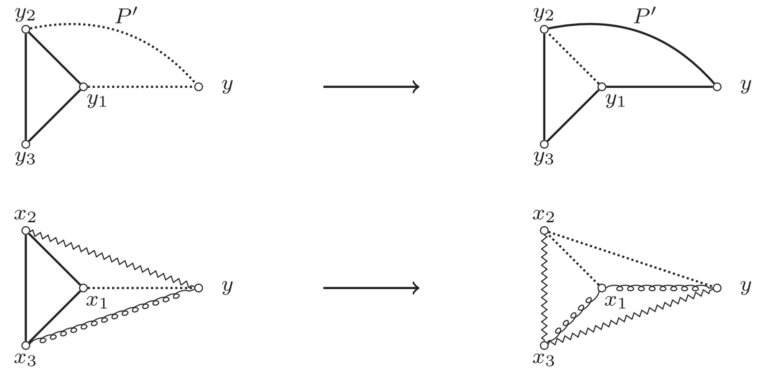

FIGURE 1 The two possible cases in Lemma 12 to obtain a coloring in which a fixed triangle is not monochromatic; the different styles of the edges represent the colors

Proof. Figure 1 illustrates the recolorings described in this proof. Assume that $x_1x_2x_3x_1$ is monochromatic of color i in c. First assume that

$$\text{an edge of colour } j := c(y_1y) \text{ is adjacent to } y_2 \tag{1}$$

for two distinct vertices y_1, y_2 in $\{x_1, x_2, x_3\}$. Without loss of generality, the path P' of color j between y and y_2 along the path $c^{-1}(j) - \{yy_1\}$ does not contain the vertex y_3 (where $\{y_3\} = \{x_1, x_2, x_3\} - \{y_1, y_2\}$). Flip the colors of the monochromatic paths y_1y_2 and $y_1yP'y_2$, ie, set $c'(y_1y_2) = j$, $c'(y_1yP'y_2) = c(y_1y_2)$ and $c'(e) = c(e)$ for all other edges $e \in E(H)$. The obtained coloring is legal: By construction, all color classes are cycles and at most $\lfloor(|V(H)| - 1)/2\rfloor$ many colors are used. Further, the cycle $x_1x_2x_3x_1$ is not monochromatic.

If (1) does not hold, we can get rid of one color. Set $c'(x_1x_2y) = c(x_1y)$, $c'(x_2x_3y) = c(x_2y)$, $c'(x_3x_1y) = c(x_3y)$, and $c'(e) = c(e)$ for all other edges $e \in E(H)$. By construction, all color classes are cycles and $x_1x_2x_3x_1$ is not monochromatic. □

Figure 2 illustrates the following simple observation.

Observation 13. Let P_1 be an x_1-y_1-path that is vertex-disjoint from an x_2-y_2-path P_2. Then there are three possibilities to connect $\{x_1, y_1\}$ and $\{x_2, y_2\}$ by two vertex-disjoint paths that do not intersect $V(P_i) - \{x_i, y_i\}$ for $i = 1, 2$. Two of the possibilities yield a cycle—the third way leads to two cycles.

Lemma 14, 15, and 16 are all based on the same elementary fact: Let G and G' be graphs with $|V(G)| = |V(G')| + 2$. If G' allows for a cycle decomposition with at most $\lfloor(|V(G')| - 1)/2\rfloor$ cycles, then any cycle decomposition of G that uses at most one cycle more than the cycle decomposition of G' shows that G is not a counterexample to Hajós' conjecture.

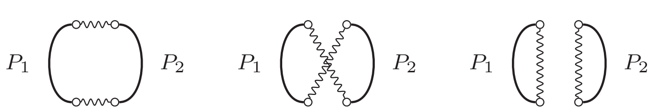

FIGURE 2 The three possible ways to connect the end vertices of two paths P_1 and P_2; the connection between the end vertices is drawn with jagged lines

This fact leads us to the following inductive approach: Given a graph G with two vertices u and v of degree 6, we remove u and v from G and might remove or add edges to obtain a graph G'. If G' has a cycle decomposition with at most $\lfloor (|V(G')| - 1)/2 \rfloor$ cycles we construct a cycle decomposition of G from it. We reroute some of the cycles in an appropriate way such that u and v are each touched by two cycles. Now, there remain some edges in G that are not covered. If those edges form a cycle, we have found a cycle decomposition of G. If a cycle is not rerouted to u or v twice, the cycle decomposition of G satisfies Hajós' conjecture.

To describe this inductive approach in a coherent way, we regard the cycle decomposition of G' as a legal coloring. Then we regard the above reroutings as recolorings where we have to make sure that no color appears twice at u or v. If the edges that have not yet received a color form a cycle, we associate the new color $\lfloor (|V(G)| - 1)/2 \rfloor$ to this cycle. The obtained coloring of the edges then uses at most $\lfloor (|V(G)| - 1)/2 \rfloor$ many colors and each color class is a cycle. Thus, we have constructed a legal coloring.

Lemma 14. *Let G be an Eulerian graph without legal coloring that contains two adjacent vertices u and v of degree 6 with common neighborhood $N = \{x_1, \ldots, x_5\}$. Define $G' = G - \{u, v\}$ and let c' be a legal coloring of G'.*

(i) If $G[N]$ contains a path $P' = y_1y_2y_3y_4$ of length 3 then P' is monochromatic in c'.
(ii) Let $G[N]$ contain an independent set $S = \{y_1, y_2, y_3\}$ of size 3. If N is not an independent set or if there is a vertex in G' that is adjacent to y_1, y_2, and y_3, then $G'' = G' + \{y_1y_2, y_2y_3, y_3y_1\}$ does not have a legal coloring.
(iii) If $G[N]$ contains an induced path $y_1y_2y_3y_4$ of length 3 then $G'' = G' - \{y_2y_3\} + \{y_2y_4, y_4y_1, y_1y_3\}$ does not have a legal coloring.
(iv) If $G[N]$ contains a triangle $y_1y_2y_3y_1$, a vertex y_4 that is not adjacent to y_1 and y_3 and a vertex $y_5 \in N - \{y_1, y_2, y_3, y_4\}$ adjacent to y_4 then $G'' = G' - \{y_1y_3\} + \{y_1y_4, y_3y_4\}$ does not have a legal coloring.

Proof of (i). If y_3y_4 has a color different from y_1y_2 and y_2y_3, then set

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_2vy_3) = c'(y_2y_3) \quad c(y_3uvy_4) = c'(y_3y_4).$$

If y_2y_3 has a color different from y_1y_2 and y_3y_4, then set

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_2vuy_3) = c'(y_2y_3) \quad c(y_3vy_4) = c'(y_3y_4).$$

The case distinction makes sure that the modified color classes remain cycles. By further setting $c(y_1y_2y_3y_4uy_5vy_1) = \lfloor (|V(G)| - 1)/2 \rfloor$ and $c(e) = c'(e)$ for all other edges e we have constructed a legal coloring c of G. □

Proof of (ii). Set $\{y_4, y_5\} = N - \{y_1, y_2, y_3\}$ and let c'' be a legal coloring of G''.

First assume that $c''(y_1y_2) \notin \{c''(y_2y_3), c''(y_3y_1)\}$. Then one can easily check that the following is a legal coloring of G.

$$c(y_2uvy_1) = c''(y_2y_1) \quad c(y_2vy_3) = c''(y_2y_3) \quad c(y_3uy_1) = c''(y_3y_1)$$
$$c(y_4uy_5vy_4) = \lfloor(|V(G)| - 1)/2\rfloor \qquad (2)$$
$$c(e) = c''(e) \quad \text{for all other edges } e$$

By symmetry, we are done unless the triangle $y_1y_2y_3y_1$ is monochromatic in c''. By Lemma 12, we can suppose that there is no vertex y in G'' that is adjacent to y_1, y_2 and y_3. Suppose that N is not independent. Without loss of generality, we can assume that $G[N]$ contains an edge, say y_4y_1 incident to one of the vertices of the independent 3-set. (Otherwise, we can choose another suitable independent 3-set in $G[N]$). Then by construction the following is a legal coloring of G.

$$c(y_1uvy_4) = c''(y_1y_4) \quad c(y_2uy_3) = c''(y_2y_3) \quad c(y_2vy_3) = c''(y_2y_1y_3)$$
$$c(y_1y_4uy_5vy_1) = \lfloor(|V(G)| - 1)/2\rfloor$$
$$c(e) = c''(e) \quad \text{for all other edges } e \qquad (3)$$

□

Proof of (iii). Let G'' have a legal coloring c'' and let y_5 be the unique vertex in $N - \{y_1, y_2, y_3, y_4\}$.

If $c''(y_2y_4) = c''(y_4y_1) = c''(y_1y_3)$, set

$$c(y_1uy_2) = c''(y_1y_2) \quad c(y_2vuy_3) = c''(y_2y_4y_1y_3) \quad c(y_3vy_4) = c''(y_3y_4)$$
$$c(uy_5vy_1y_2y_3y_4u) = \lfloor(|V(G)| - 1)/2\rfloor.$$

If $c''(y_1y_3)$ is different from $c''(y_2y_4)$ and $c''(y_4y_1)$, set

$$c(y_1uvy_3) = c''(y_1y_3) \quad c(y_4vy_1) = c''(y_4y_1) \quad c(y_2uy_4) = c''(y_2y_4)$$
$$c(uy_5vy_2y_3u) = \lfloor(|V(G)| - 1)/2\rfloor.$$

If $c''(y_2y_4)$ is different from $c''(y_1y_3)$ and $c''(y_1y_4)$, the coloring is defined similarly by relabeling the vertices $y_1, ..., y_5$.

If $c''(y_4y_1)$ is different from $c''(y_1y_3)$ and $c''(y_2y_4)$, set

$$c(y_1vy_3) = c''(y_1y_3) \quad c(y_4vuy_1) = c''(y_4y_1) \quad c(y_2uy_4) = c''(y_2y_4)$$
$$c(uy_5vy_2y_3u) = \lfloor(|V(G)| - 1)/2\rfloor.$$

Further set $c(e) = c''(e)$ for all other edges e in all cases. Again, the case distinction makes sure that all color classes are cycles and we have constructed a legal coloring. □

Proof of (iv). Let c'' be a legal coloring of G''. First assume that $c''(y_2y_3) \notin \{c''(y_3y_4), c''(y_1y_4)\}$. Then set

$$c(y_2vuy_3) = c''(y_2y_3) \quad c(y_1uy_4) = c''(y_1y_4) \quad c(y_3vy_4) = c''(y_3y_4)$$
$$c(uy_5vy_1y_3y_2u) = \lfloor(|V(G)| - 1)/2\rfloor.$$

If $c''(y_1y_2) \notin \{c''(y_3y_4), c''(y_1y_4)\}$, the coloring is defined as above by interchanging the roles of y_1 and y_3.

Now assume that $c''(y_2y_3), c''(y_1y_2) \in \{c''(y_3y_4), c''(y_1y_4)\}$. If $c''(y_3y_4) = c''(y_1y_4)$, then the cycle $y_1y_2y_3y_4y_1$ is monochromatic. Set

$$c(y_4uvy_5) = c''(y_4y_5)$$
$$c(y_1vy_3y_2uy_1) = c''(y_1y_2y_3y_4y_1) \quad c(y_1y_3uy_5y_4vy_2y_1) = \lfloor(|V(G)| - 1)/2\rfloor.$$

If $c''(y_3y_4) \neq c''(y_1y_4)$, then either $c''(y_2y_3) = c''(y_3y_4)$ or $c''(y_2y_3) = c''(y_1y_4)$. If $c''(y_2y_3) = c''(y_3y_4)$, set

$$c(y_2uy_3vy_4) = c''(y_2y_3y_4) \quad c(y_1vuy_4) = c''(y_1y_4)$$
$$c(y_1uy_5vy_2y_3y_1) = \lfloor(|V(G)| - 1)/2\rfloor.$$

If $c''(y_2y_3) = c''(y_1y_4)$, set

$$c(y_2uy_3) = c''(y_2y_3) \quad c(y_1vy_4) = c''(y_1y_4) \quad c(y_3vuy_4) = c''(y_3y_4)$$
$$c(y_1uy_5vy_2y_3y_1) = \lfloor(|V(G)| - 1)/2\rfloor.$$

By setting $c(e) = c''(e)$ for all other edges e we have constructed a legal coloring for G in all cases. □

If u and v are adjacent degree-6 vertices that have a common neighborhood N of size 4, we call the two vertices that are adjacent with exactly one of u, v the *private neighbors* of u and v. Here, we denote them by x_u and x_v. If there is a x_u–x_v-path P in $G - \{u, v\} - N$, it is possible to translate all techniques of Lemma 16. It suffices to delete u, v and $E(P)$ to obtain another Eulerian graph: In all recolorings of Lemma 16, the edges uy, vy for one vertex $y \in N$ are contained in the new color class $\lfloor(|V(G)| - 1)/2\rfloor$. If we have two private neighbors x_u and x_v it suffices to replace the path uyv by the path ux_uPx_vv in this color class. This means, we can regard x_uPx_v as a single vertex y.

Lemma 15. *Let G be an Eulerian graph without legal coloring that contains two adjacent vertices u and v of degree 6 with common neighborhood* $N = \{x_1, \ldots, x_4\}$ *and* $N_G(u) = N \cup \{x_u, v\}$ *as well as* $N_G(v) = N \cup \{x_v, u\}$. *Let P be an* x_u–x_v*-path in* $G - \{u, v\} - N$. *Define* $G' = G - \{u, v\} - E(P)$ *and let c′ be a legal coloring of G′.*

(i) If $G[N \cup \{x_u, x_v\}]$ *contains a path* $P' = y_1y_2y_3y_4$ *with* $y_2, y_3, y_4 \in N$ *of length* 3 *then P′ is monochromatic in c′.*
(ii) Let $G[N]$ *contain an independent set* $S = \{y_1, y_2, y_3\}$ *of size* 3. *If* $G[N \cup \{x_u, x_v\}]$ *contains an edge* $x_ix_j \neq x_ux_v$ *or if there is a vertex in G′ that is adjacent to* y_1, y_2 *and* y_3 *then* $G'' = G' + \{y_1y_2, y_2y_3, y_3y_1\}$ *does not have a legal coloring.*

(iii) *If* $G[N \cup \{x_u, x_v\}]$ *does not contain the edges* $x_u y_1, y_1 y_2, y_2 x_v$ *for two vertices* $y_1, y_2 \in N$ *but contains an edge with end vertex* y_1 *or* y_2 *then* $G'' = G - \{u, v\} + \{x_u y_1, y_1 y_2, y_2 x_v\}$ *does not have a legal coloring.*

(iv) *If* G *contains the edges* $y_1 y_2, y_3 y_4, y_1 y_5$ *with* $y_1, y_2, y_3, y_4 \in N$ *and* $y_5 \in \{x_u, x_v\}$ *but not the edges* $y_1 y_3, y_2 y_3$ *then* $G'' = G' - \{y_1 y_2\} + \{y_1 y_3, y_3 y_2\}$ *does not have a legal coloring.*

(v) *If* $G[N \cup \{x_u, x_v\}]$ *contains a triangle* $y_1 y_2 y_3 y_1$ *with* $y_1, y_2, y_3 \in N$*, a vertex* $y_4 \in N - \{y_1, y_2, y_3\}$ *that is not adjacent to* y_1 *and* y_3 *and a vertex* $y_5 \in \{x_u, x_v\}$ *adjacent to* y_4 *then* $G'' = G' - \{y_1 y_3\} + \{y_1 y_4, y_3 y_4\}$ *does not have a legal coloring.*

Proof of (i). The proof is very similar to the proof of Lemma 14.(i) if we regard $x_u P x_v$ as one single vertex. We will nevertheless give a detailed proof. By symmetry of u and v (and thus of x_u and x_v), we can assume that y_1 is either contained in N or is equal to x_u. Suppose that P is not monochromatic.

If $c'(y_1 y_2) \notin \{c'(y_2 y_3), c'(y_3 y_4)\}$, then set

$$c(y_1 u v y_2) = c'(y_1 y_2) \quad c(y_2 u y_3) = c'(y_2 y_3) \quad c(y_3 v y_4) = c'(y_3 y_4).$$

If $c'(y_2 y_3) \notin \{c'(y_1 y_2), c'(y_3 y_4)\}$, then set

$$c(y_1 u y_2) = c'(y_1 y_2) \quad c(y_2 v u y_3) = c'(y_2 y_3) \quad c(y_3 v y_4) = c'(y_3 y_4).$$

If $c'(y_3 y_4) \notin \{c'(y_1 y_2), c'(y_2 y_3)\}$, then set

$$c(y_1 u y_2) = c'(y_1 y_2) \quad c(y_2 v y_3) = c'(y_2 y_3) \quad c(y_3 u v y_4) = c'(y_3 y_4).$$

If $y_1 \in N$ the following completes by construction a legal coloring c of G:

$$\begin{aligned} c(y_1 y_2 y_3 y_4 u x_u P x_v v y_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \quad \text{for all other edges } e \end{aligned}$$

Now suppose that $y_1 = x_u$ and that x_4, x_v are not contained in the path $y_1 y_2 y_3 y_4$. Then, the following completes by construction a legal coloring c of G:

$$\begin{aligned} c(y_1 y_2 y_3 y_4 u x_4 v x_v P y_1) &= \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) &= c'(e) \quad \text{for all other edges } e \end{aligned}$$

□

Proof of (ii). The proof is very similar to the proof of Lemma 14.(ii) if we regard $x_u P x_v$ as one single vertex. □

Proof of (iii). Assume that c'' is a legal coloring of G'' and let $\{y_3, y_4\} = N - \{y_1, y_2\}$. By symmetry of u and v (and thus of y_1 and y_2) we can suppose that $y_1 y_4 \in E(G)$.

If $y_1 x_u$ has a color different from the color of $y_1 y_2$ and $y_2 x_v$, set

$$c(x_u uvy_1) = c''(x_u y_1) \quad c(y_1 uy_2) = c''(y_1 y_2) \quad c(y_2 vx_v) = c''(y_2 x_v)$$
$$c(y_3 uy_4 vy_3) = \lfloor (|V(G)| - 1)/2 \rfloor.$$

An analogous coloring can be defined if $x_v y_2$ has a color different from the color of $y_1 y_2$ and $x_u y_1$.

If $y_1 y_2$ has a color different from the color of $y_1 x_u$ and $y_2 x_v$, then set

$$c(x_u uy_1) = c''(x_u y_1) \quad c(y_1 vuy_2) = c''(y_1 y_2) \quad c(y_2 vx_v) = c''(y_2 x_v)$$
$$c(y_3 uy_4 vy_3) = \lfloor (|V(G)| - 1)/2 \rfloor.$$

Now suppose that all three edges $x_u y_1, y_1 y_2, y_2 x_v$ have the same color. Then, $y_1 y_4$ has a different color. Set

$$c(x_u uy_2) = c''(x_u y_1 y_2) \quad c(y_1 uvy_4) = c''(y_1 y_4) \quad c(y_2 vx_v) = c''(y_2 x_v)$$
$$c(uy_3 vy_1 y_4 u) = \lfloor (|V(G)| - 1)/2 \rfloor.$$

In all cases, set $c(e) = c''(e)$ for all other edges e. The case distinction now makes sure that we constructed a legal coloring for G . □

Proof of (iv). Assume that c'' is a legal coloring of G''. Without loss of generality let $y_5 = x_u$.

First suppose that all three edges $x_u y_1, y_1 y_3, y_3 y_2$ have the same color. Then, $y_3 y_4$ has a different color and the following gives by construction a legal coloring for G:

$$c(x_u uy_1) = c''(x_u y_1) \quad c(y_1 vy_2) = c''(y_1 y_3 y_2) \quad c(y_3 uvy_4) = c''(y_3 y_4)$$
$$c(uy_2 y_1 x_u P x_v vy_3 y_4 u) = \lfloor (|V(G)| - 1)/2 \rfloor$$
$$c(e) = c'(e) \quad \text{for all other edges } e$$

Now suppose that $x_u y_1 y_3 y_2$ is not monochromatic.

If $x_u y_1$ has a color different from the colors of $y_1 y_3$ and $y_3 y_2$, set

$$c(x_u uvy_1) = c''(x_u y_1) \quad c(y_1 uy_3) = c''(y_1 y_3) \quad c(y_3 vy_2) = c''(y_3 y_2).$$

If $y_1 y_3$ has a color different from the colors of $x_u y_1$ and $y_3 y_2$, set

$$c(x_u uy_1) = c''(x_u y_1) \quad c(y_1 vuy_3) = c''(y_1 y_3) \quad c(y_3 vy_2) = c''(y_3 y_2).$$

If $y_3 y_2$ has a color different from the colors of $x_u y_1$ and $y_1 y_3$, set

$$c(x_u uy_1) = c''(x_u y_1) \quad c(y_1 vy_3) = c''(y_1 y_3) \quad c(y_3 uvy_2) = c''(y_3 y_2).$$

By setting $c(uy_2 y_1 x_u P x_v vy_4 u) = \lfloor (|V(G)| - 1)/2 \rfloor$ and $c(e) = c''(e)$ for all other edges e, we obtain by construction in all cases a legal coloring for G. □

Proof of (v). The proof is very similar to the proof of Lemma 14.(iv) if we regard $x_u P x_v$ as one single vertex. □

In our last recoloring lemma, we consider two degree-6 vertices that are not adjacent but have six common neighbors $x_1, ..., x_6$. Some of the recoloring techniques of this lemma need a somewhat deeper look into the cycle decomposition. They rely on a generalization of the recolorings used in Lemma 14 and 12. We introduce two pieces of notation. For two distinct vertices $x_i, x_j \in N = \{x_1, ..., x_6\}$, a path $P_{x_i x_j}$ always denotes an x_i–x_j-path that is not intersecting with $N - \{x_i, x_j\}$.

For a cycle C and two distinct vertices $x_i, x_j \in N = \{x_1, ..., x_6\} \cap V(C)$ there are two x_i–x_j-paths along C. If there is a unique path that is not intersecting with $N - \{x_i, x_j\}$, we denote this path by $C_{x_i x_j}$.

Lemma 16. *Let G be an Eulerian graph without legal coloring and let G contain two degree-*6 *vertices u and v with common neighborhood* $N = \{x_1, ..., x_6\}$. *Define* $G' = G - \{u, v\}$ *and let c′ be a legal coloring of G′.*

(i) If G′ contains two vertex-disjoint paths $P_{y_1y_2}P_{y_2y_3}$ *and* $P_{y'_1y'_2}P_{y'_2y'_3}$ *with* $\{y_1, y_2, y_3, y'_1, y'_2, y'_3\} = N$ *where the four paths* $P_{y_1y_2}, P_{y_2y_3}, P_{y'_1y'_2}, P_{y'_2y'_3}$ *are monochromatic in c′, then at least three of the four paths have the same color in c′.*

(ii) Let G′ contain a path $P' = P_{y_1y_2}P_{y_2y_3}P_{y_3y_4}P_{y_4y_5}$ *with* $\{y_1, ..., y_5\} \subset N$ *where* $P_{y_iy_{i+1}}$ *is monochromatic in c′ for each* $i \in \{1, 2, 3, 4\}$. *Then* $c'(P_{y_1y_2}) = c'(P_{y_3y_4})$ *or* $c'(P_{y_2y_3}) = c'(P_{y_4y_5})$.

(iii) If $G[N]$ *contains an independent set* $S = \{y_1, y_2, y_3\}$ *of size* 3 *and if* $G[N]$ *contains at least one edge or there is a vertex in G′ that is adjacent to* y_1, y_2 *and* y_3 *then* $G'' = G' + \{y_1y_2, y_2y_3, y_3y_1\}$ *does not have a legal coloring.*

(iv) If $G[N]$ *contains a path* $P' = y_1y_2y_3y_4$ *of length* 3, *then P′ is monochromatic in c′.*

Proof of (i). Suppose that less than three of the paths have the same color. Then, without loss of generality $c'(P_{y'_1y'_2}) \neq c'(P_{y_1y_2})$ and $c'(P_{y'_2y'_3}) \neq c'(P_{y_2y_3})$ and the following is by construction a legal coloring of G:

$$\begin{gathered} c(y_1uy_2) = c'(P_{y_1y_2}) \quad c(y_2vy_3) = c'(P_{y_2y_3}) \\ c(y'_1uy'_2) = c'(P_{y'_1y'_2}) \quad c(y'_2vy'_3) = c'(P_{y'_2y'_3}) \\ c(y_1P_{y_1y_2}P_{y_2y_3}y_3uy'_3P_{y'_3y'_2}P_{y'_2y'_1}y'_1vy_1) = \lfloor (|V(G)| - 1)/2 \rfloor \\ c(e) = c'(e) \quad \text{for all other edges } e \end{gathered}$$

□

Proof of (ii). Suppose that $c'(P_{y_1y_2}) \neq c'(P_{y_3y_4})$ and $c'(P_{y_2y_3}) \neq c'(P_{y_4y_5})$ and let y_6 be the vertex of N not contained in P'. Then, the following is by construction a legal coloring of G:

$$c(y_1uy_2) = c'(P_{y_1y_2}) \quad c(y_2vy_3) = c'(P_{y_2y_3})$$
$$c(y_3uy_4) = c'(P_{y_3y_4}) \quad c(y_4vy_5) = c'(P_{y_4y_5})$$
$$c(y_1P_{y_1y_2}P_{y_2y_3}P_{y_3y_4}P_{y_4y_5}y_5uy_6vy_1) = \lfloor(|V(G)| - 1)/2\rfloor$$
$$c(e) = c'(e) \quad \text{for all other edges} \quad e$$

□

Proof of (iii). The proof uses ideas of the proof of Lemma 14.(ii).

Let c'' be a legal coloring of G''. First suppose that $i := c''(y_1y_2) \notin \{c''(y_2y_3), c''(y_3y_1)\}$ and let $C = c^{-1}(i)$ be the monochromatic cycle in G'' with color i.

If there is a vertex $y_6 \in N - \{y_1, y_2, y_3\}$ that is not contained in C set $\{y_4, y_5\} = N - \{y_1, y_2, y_3, y_6\}$ and use the recoloring (2) where the edge uv is replaced by the path uy_6v.

Otherwise, $\{y_4, y_5, y_6\} := N - \{y_1, y_2, y_3\}$ is a subset of $V(C)$. Without loss of generality, we may assume that $C_{y_6y_5}$ and $C_{y_6y_1}$ exist. (The cycle C is either of the form $C = y_1y_2Cy_2y_iC_{y_iy_j}C_{y_jy_k}C_{y_ky_1}$ with $\{y_i, y_j, y_k\} = \{y_4, y_5, y_6\}$ or $C = y_1y_2Cy_2y_iC_{y_iy_j}C_{y_jy_k}C_{y_ky_l}C_{y_ly_1}$ with $\{y_i, y_j, y_k, y_l\} = \{y_3, y_4, y_5, y_6\}$. We may assume by the symmetry of the elements in $\{y_4, y_5, y_6\}$ and those in $\{y_1, y_2\}$ that $C_{y_6y_1}$ and $C_{y_6y_5}$ exist). By construction, the following is a legal coloring of G:

$$c(y_2uy_3) = c''(y_2y_3) \quad c(y_1vy_3) = c''(y_1y_3) \quad c(y_2vy_6C_{y_6y_1}y_1uy_5) = c''(y_1y_2),$$
$$c(y_5vy_4uy_6C_{y_5y_6}) = \lfloor(|V(G)| - 1)/2\rfloor \quad \text{and}$$
$$c(e) = c''(e) \quad \text{for all other edges} \quad e.$$

Assume that the triangle $y_1y_2y_3y_1$ is monochromatic in c''. By Lemma 12, there is no vertex y in G' that is adjacent to y_1, y_2 and y_3. Suppose that N is not independent. Without loss of generality $G[N]$ contains the edge y_4y_1. Set $\{y_5, y_6\} = N - \{y_1, y_2, y_3, y_4\}$.

If there is a vertex in $\{y_5, y_6\}$, say y_6, that is not contained in the cycle $C = c^{-1}(j)$ of color $j := c'(y_1y_4)$, use the recoloring (3) where the edge uv is replaced by the path uy_6v.

If y_5 and y_6 are both contained in C, let S be the segment of $C - \{y_1y_4\}$ that connects y_4 with y_5. By symmetry of y_5 and y_6, we can suppose that $y_6 \notin S$. By construction, the following is a legal coloring of G:

$$c(y_1vy_4) = c''(y_1y_4) \quad c(y_5uy_4) = c''(S)$$
$$c(y_2uy_3vy_2) = c(y_1y_2y_3y_1) \quad c(y_1y_4Sy_5vy_6uy_1) = \lfloor(|V(G)| - 1)/2\rfloor$$
$$c(e) = c''(e) \quad \text{for all other edges} \quad e$$

□

Proof of (iv). Suppose that P is not monochromatic in c' and set $\{y_5, y_6\} = N - \{y_1, y_2, y_3, y_4\}$.

First assume that $c'(y_3y_4) \notin \{c'(y_1y_2), c'(y_2y_3)\}$. Let C be the cycle of color $c'(y_3y_4)$ in G'.

If there is a vertex in $\{y_5, y_6\}$, say y_5, that is not in C, then by construction the following is a legal coloring of G:

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_2vy_3) = c'(y_2y_3) \quad c(y_3uy_5vy_4) = c'(y_3y_4)$$
$$c(y_1y_2y_3y_4uy_6vy_1) = \lfloor(|V(G)| - 1)/2\rfloor$$
$$c(e) = c'(e) \quad \text{for all other edges} \quad e$$

Now assume that y_5 and y_6 are contained in C. If $C_{y_5y_1}$, $C_{y_6y_1}$, $C_{y_5y_4}$ or $C_{y_6y_4}$ exists then we can apply (ii). Thus, $C_{y_5y_6}$ must exist and by symmetry $C_{y_5y_2}$ and $C_{y_6y_3}$ exist. We can apply (ii) to $y_1y_2, y_2y_3, C_{y_3y_6}$ and $C_{y_5y_6}$.

Thus, for the rest of the proof we can assume that

$$c'(y_2y_3) =: i \notin \{c'(y_1y_2), c'(y_3y_4)\}.$$

Let $C' = c^{-1}(i)$ be the cycle of color i in G'. If there is a vertex in $\{y_5, y_6\}$, say y_5, that is not in C', then by construction the following is a legal coloring of G:

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_3vy_4) = c'(y_3y_4) \quad c(y_2vy_5uy_3) = c'(y_2y_3)$$
$$c(y_1y_2y_3y_4uy_6vy_1) = \lfloor(|V(G)| - 1)/2\rfloor$$
$$c(e) = c'(e) \quad \text{for all other edges} \quad e$$

Thus, we can assume that

$$y_5 \quad \text{and} \quad y_6 \quad \text{are contained in} \quad C'.$$

Now, there are three cases up to symmetry: y_1 and y_4 both are not contained in C', y_1 is contained in C' but y_4 is not, and y_1 and y_4 are both contained in C'.

First assume that y_1 and y_4 are not contained in C'. Then, by symmetry, C' is the cycle consisting of $y_2y_3, C'_{y_3y_6}, C'_{y_6y_5}, C'_{y_5y_2}$. We are done by applying (i) to the vertex-disjoint paths $y_1y_2, C'_{y_2y_5}$ and $y_4y_3, C'_{y_3y_6}$.

Next assume that y_1 is contained in C' and y_4 is not contained in C'. First suppose that $C'_{y_6y_3}$ exists. As $C'_{y_5y_1}$ or $C'_{y_5y_2}$ must exist, we are done with (i).

By symmetry, we can now suppose that neither $C'_{y_6y_3}$ nor $C'_{y_5y_3}$ exists. Then $C'_{y_3y_1}$ exists. We can suppose without loss of generality that C' is the cycle consisting of $C'_{y_1y_3}, y_3y_2, C'_{y_2y_6}, C'_{y_6y_5}, C'_{y_5y_1}$ and by construction the following is a legal coloring of G:

$$c(y_1uy_2) = c'(y_1y_2) \quad c(y_3vy_4) = c'(y_3y_4)$$
$$c(y_3y_2vy_1C'_{y_1y_5}C'_{y_5y_6}y_6uy_4y_3) = i$$
$$c(y_3uy_5vy_6C'_{y_6y_2}y_2y_1C'_{y_1y_3}y_3) = \lfloor(|V(G)| - 1)/2\rfloor$$
$$c(e) = c'(e) \quad \text{for all other edges} \quad e$$

Last, assume that y_1 and y_4 are both contained in C'. First, suppose that $C'_{y_5y_6}$ does not exist. Without loss of generality, we can suppose that $C'_{y_5y_1}$ exists. Now neither $C'_{y_6y_3}$ nor $C'_{y_6y_4}$ exists; otherwise, we are done with (i). Thus, $C'_{y_6y_1}$ and $C'_{y_6y_2}$ must exist. Thus, $C'_{y_5y_4}$ exists and we are done with (i).

Now suppose that $C'_{y_5y_6}$ exists. First, suppose that $C'_{y_5y_2}$ exists. Then, we are done with (i) if $C'_{y_6y_3}$ or $C'_{y_6y_4}$ exists. As C' is a cycle, $C'_{y_6y_1}$ and thus also $C'_{y_4y_1}$ and $C'_{y_4y_3}$ exist. The following is by construction a legal coloring of G:

$$c(y_3uy_4) = c'(y_3y_4) \quad c(y_1vy_2) = c'(y_1y_2)$$
$$c(y_5vy_3C'_{y_3y_4}C'_{y_4y_1}y_1y_2uy_6C'_{y_6y_5}y_5) = i$$
$$c(y_2y_3y_4vy_6C'_{y_6y_1}y_1uy_5C'_{y_5y_2}y_2) = \lfloor(|V(G)| - 1)/2\rfloor$$
$$c(e) = c'(e) \quad \text{for all other edges} \quad e$$

Thus, we can suppose that none of $C'_{y_5y_2}, C'_{y_5y_3}, C'_{y_6y_2}, C'_{y_6y_3}$ exists. Without loss of generality, $C'_{y_5y_1}$ exists. As $C'_{y_6y_4}$ must exist we are done with (i). □

4 | PROOFS FOR THE REDUCIBLE STRUCTURES

In this section, we prove Lemma 6, 7, 8, 9, and 11. In the first three proofs, we use the following observation:

Observation 17. Let x be a vertex of degree at least 3 in a graph H with a legal coloring. Then the neighborhood $N_H(x)$ of x contains an independent set of size 3 or $G[\{x\} \cup N_H(x)]$ contains a path of length 3 that is not monochromatic.

Proof of Lemma 6. If $G[N]$ contains a vertex of degree at least 3 we are done by applying Observation 17 as well as Lemma 14.(i) and (ii).

Now, suppose that $G[N]$ contains a vertex, say x_1 of degree 0. As we have seen, $G[N]$ contains no vertex of degree 3 or 4. Thus, $G[N] - \{x_1\}$ contains two nonadjacent vertices, say x_2 and x_3. Then, $\{x_1, x_2, x_3\}$ is an independent set and we are done by Lemma 14.(ii).

We can conclude that all vertices in $G[N]$ have degree 1 or 2. Consequently, the graph is isomorphic to C_5, $K_3 \dot{\cup} P_2$, $P_3 \dot{\cup} P_2$ or P_5. The 5-cycle C_5 contains an induced P_4, the graph $K_3 \dot{\cup} P_2$ contains a triangle and a vertex that is not adjacent to two of the triangle vertices, the latter two graphs contain an independent set of size 3. Thus, we are done by (iii), (iv), and (ii) of Lemma 14. □

Proof of Lemma 7. If $G[N]$ contains a vertex of degree at least 3 we are done by applying Observation 17 as well as Lemma 15.(i) and (ii). Thus, $G[N]$ must be isomorphic to one of the graphs that we will treat now.

First, suppose that $G[N]$ is isomorphic to $\overline{K_4}$, $P_2 \cup \overline{K_2}$ or $P_3 \cup K_1$. Then, $G[N]$ contains an independent 3-set and, hence, G has a legal coloring by Lemma 15.(ii).

Next, suppose that the edge set of $G[N]$ is equal to $\{x_1x_2, x_3x_4\}$ or to $\{x_1x_2, x_1x_3, x_3x_4\}$. If x_u is adjacent to x_2, apply Lemma 15.(iv) to get a legal coloring: the edges x_ux_2, x_2x_1, x_3x_4 exist while x_4 is neither adjacent to x_1 nor to x_2. Similarly, we can apply Lemma 15.(iv) if $x_3x_v \in E(G)$. Thus, we can suppose that neither x_2x_u nor x_3x_v exists in G and we are done with Lemma 15.(iii): the edges x_ux_2, x_2x_3, x_3x_v do not exist while $x_2x_1 \in E(G)$.

Now, suppose that the edge set of $G[N]$ consists of x_1x_2, x_2x_3, x_3x_1. If x_u is adjacent to x_1, not all paths of length 3 can be monochromatic and we can apply Lemma 15.(i). Thus we can suppose that $x_u x_1 \notin E(G)$. If $x_4 x_v \notin E(G)$ then we can apply Lemma 15.(iii) to $x_u x_1, x_1 x_4, x_4 x_v \notin E(G)$ and $x_1 x_3 \in E(G)$ to obtain a legal coloring of G. If $x_4 x_v \in E(G)$ we are done by Lemma 15.(v).

Last, suppose that the edge set of $G[N]$ consists of $x_1x_2, x_2x_3, x_3x_4, x_4x_1$. If the 4-cycle is not monochromatic, the cycle contains a P_4 that is not monochromatic and we are done by Lemma 15.(i). Suppose that $x_1 x_u$ is an edge of G. Then, $x_u x_1 x_2 x_3$ is a P_4 that is not monochromatic. By symmetry, we get that neither x_u nor x_v is adjacent to a vertex of N. But then apply Lemma 15.(iii) to $x_u x_1, x_1 x_3, x_3 x_v \notin E(G)$ and $x_1 x_2 \in E(G)$ to obtain a legal coloring of G. □

Proof of Lemma 8. If $G[N]$ contains a vertex of degree at least 3 we are done by applying Observation 17 as well as Lemma 16.(iii) and 16.(iv).

Now, suppose that $G[N]$ contains a vertex, say x_1 of degree 0. As we have seen, $G[N]$ contains no vertex of degree at least 3. Thus, $G[N] - \{x_1\}$ contains two nonadjacent vertices, say x_2 and x_3. Then, $\{x_1, x_2, x_3\}$ is an independent set and we are done by Lemma 16.(iii).

We can conclude that all vertices in $G[N]$ have degree 1 or 2. Thus, $G[N]$ is isomorphic to one of the following graphs: $C_3 \dot\cup C_3$, C_6, $C_4 \dot\cup P_2$, $C_3 \dot\cup P_3$, $P_3 \dot\cup P_3$, $P_4 \dot\cup P_2$, $P_2 \dot\cup P_2 \dot\cup P_2$.

If $G[N]$ is isomorphic to $C_3 \dot\cup C_3$, we can apply Lemma 16.(i). It is not possible that all pairs of 3-paths have three edges of the same color. In all other cases, we can apply Lemma 16.(iii). □

Proof of Lemma 9. The proof is based on the following observation: a legal coloring c' of G' consists of at most $\lfloor (|V(G)| - 2)/2 \rfloor = \lfloor (|V(G)| - 3)/2 \rfloor$ colors while a legal coloring of G can consist of $\lfloor (|V(G)| - 3)/2 \rfloor + 1 = \lfloor (|V(G)| - 1)/2 \rfloor$ many colors. We will now consider the neighborhood of u in G.

If u has exactly two neighbors x_1 and x_2 that are nonadjacent, set $G' = G - \{u\} + \{x_1x_2\}$ and set $c(x_1 u x_2) = c'(x_1 x_2)$.

If u has exactly two neighbors x_1 and x_2 that are adjacent, set $G' = G - \{u\} - \{x_1x_2\}$ and set $c(x_1 u x_2 x_1) = \lfloor (|V(G)| - 1)/2 \rfloor$. Further, set $c(e) = c'(e)$ for all other edges in both cases to obtain a legal coloring.

If u has exactly four neighbors $x_1, \ldots, x_4$ such that $x_1x_2, x_3x_4 \notin E(G)$ set $G' = G - \{u\} + \{x_1x_2, x_3x_4\}$ and set $c(x_1 u x_2) = c'(x_1x_2)$ and $c(x_3 u x_4) = c'(x_3x_4)$. If $c'(x_1x_2) \neq c'(x_3x_4)$, setting $c(e) = c'(e)$ for all other edges gives a legal coloring. If $c'(x_1x_2) = c'(x_3x_4)$, we again set $c(e) = c'(e)$ for all other edges. Now, c is a coloring of G where one color class consists of two cycles intersecting only at u. We can split up this color class into two cycles to obtain a legal coloring of G.

By Lemma 10, we are done unless u has four neighbors x_1, x_2, x_3, x_4 that form a clique. In that case, set $G' = G - \{u\} - \{x_1x_3, x_2x_4\}$ and set $c(x_1 u x_2) = c'(x_1x_2)$, $c(x_1 x_2 x_4 u x_3 x_1) = \lfloor (|V(G)| - 1)/2 \rfloor$ and $c(e) = c'(e)$ for all other edges. □

Proof of Lemma 11. We transform the legal coloring c' of G' into a legal coloring c of G. For this, we first note that u has degree 4 in G', that is, $\{x_1, x_2, x_3, x_4\}$ splits up into two pairs $\{a, a'\}$ and $\{b, b'\}$ with $c'(ua) = c'(ua')$ and $c'(ub) = c'(ub')$ and $c'(ua) \neq c'(ub)$.

If the color $c'(x_5x_6)$ is not incident with u in G', set $c(x_5 u x_6) = c'(x_5x_6)$ and leave all other edge colors untouched to get a legal coloring.

Now suppose that $c'(x_5x_6)$ is incident with u (say $c'(aua') = c'(x_5x_6)$), but the set $\{c'(ua), c'(aa'), c'(ub), c'(bb')\}$ consists of at least three different colors. Then, there are two possible configurations. First, let $c'(aa') \neq c'(x_5x_6)$ and $c'(aa') \neq c'(bub')$. Then, set $c(x_5ux_6) = c'(x_5x_6)$, flip the colors of the edges aua' and aa' and leave all other edge colors untouched to get a legal coloring.

If $c'(aa') = c'(bub')$ and $c'(bb') \neq c'(x_5x_6)$, set $c(x_5ux_6) = c'(x_5x_6)$, flip the colors of the edges aua' and aa' and the colors of the edges bub' and bb', and leave all other edge colors untouched to get a legal coloring.

Thus, without loss of generality $c'(x_5x_6) = c'(aua') = c'(bb')$ and $c'(aa') = c'(bub')$. That is, among the considered edges there are only two colors. We may assume that $c'(a'b) \neq c'(x_5x_6)$ and $c'(a'b') \neq c'(x_5x_6)$. Because, if eg, then $c'(ab) \neq c'(x_5x_6)$ and $c'(ab') \neq c'(x_5x_6)$. This is symmetric to the assumption.

If $c'(a'b) = c'(bub')$ and $c'(a'b') \neq c'(bub')$ the following is a legal coloring for G:

$$c(aa') = c'(aua') \quad c(a'ub') = c'(a'b')$$
$$c(auba'b') = c'(aa'bub') \quad c(x_5ux_6) = c'(x_5x_6)$$
$$c(e) = c'(e) \quad \text{for all other edges} \quad e$$

If $c'(a'b') = c'(bub')$, $\quad c'(a'b) \neq c'(bub')$, the following coloring for G is legal:

$$c(aa') = c'(aua') \quad c(a'ub) = c'(a'b)$$
$$c(aub'a'b) = c'(aa'b'ub) \quad c(x_5ux_6) = c'(x_5x_6)$$
$$c(e) = c'(e) \quad \text{for all other edges} \quad e$$

Otherwise by Observation 13, one of the following is a legal coloring for G:

$$c_1(aub') = c'(aa') \quad c_1(a'b) = c'(aa')$$
$$c_1(a'ub) = c'(a'b) \quad c_1(aa') = c'(aua')$$
$$c_1(x_5ux_6) = c'(x_5x_6)$$
$$c_1(e) = c'(e) \quad \text{for all other edges} \quad e$$

or

$$c_2(aub) = c'(aa') \quad c_2(a'b') = c'(aa')$$
$$c_2(a'ub') = c'(a'b') \quad c_2(aa') = c'(aua')$$
$$c_2(x_5ux_6) = c'(x_5x_6)$$
$$c_2(e) = c'(e) \quad \text{for all other edges} \quad e$$

□

5 | PATH-DECOMPOSITIONS

For a graph G a *path-decomposition* $(\mathcal{P}, \mathcal{B})$ consists of a path $\mathcal{P}$ and a collection $\mathcal{B} = \{B_t : t \in V(\mathcal{P})\}$ of *bags* $B_t \subset V(G)$ such that

- $V(G) = \bigcup_{t \in V(\mathcal{P})} B_t$,
- for each edge $vw \in E(G)$ there exists a vertex $t \in V(\mathcal{P})$ such that $v, w \in B_t$, and
- if $v \in B_s \cap B_t$, then $v \in B_r$ for each vertex r on the path connecting s and t in $\mathcal{P}$.

A path-decomposition $(\mathcal{P}, \mathcal{B})$ has *width* k if each bag has a size of at most $k + 1$. The *pathwidth* of G is the smallest integer k for which there is a width k path-decomposition of G.

In this paper, all paths $\mathcal{P}$ have vertex set $\{1, \ldots, n'\}$ and edge set $\{i(i + 1) : i \in \{1, \ldots, n' - 1\}\}$. We denote with $|\mathcal{P}|$ the *length* of $\mathcal{P}$, that is, $|\mathcal{P}| = n' - 1$. A path-decomposition $(P, \mathcal{B})$ of width k is *smooth* if

- $|B_i| = k + 1$ for all $i \in \{1, \ldots, n'\}$ and
- $|B_i \cap B_{i+1}| = k$ for all $i \in \{1, \ldots, n' - 1\}$.

A graph of pathwidth at most k always has a smooth path-decomposition of width k; see Bodlaender [2]. Note that this path-decomposition has exactly $n' = |V(G)| - k$ many bags.

If $(\mathcal{P}, \mathcal{B})$ is a path-decomposition of the graph G, then for any connected vertex set W of G we denote by $\mathcal{P}(W)$ the subpath of $\mathcal{P}$ that consists of those bags that contain a vertex of W. Further, if $\mathcal{P}(W)$ is the path on vertex set $\{s, s + 1, \ldots, t - 1, t\}$ with $s \leq t$ we denote s by $s(W)$ and t by $t(W)$. For $W = \{v\}$, we abuse notation and denote $\mathcal{P}(W)$, $s(W)$ and $t(W)$ by $\mathcal{P}(v)$, $s(v)$ and $t(v)$.

We note: in a smooth path-decomposition, for an edge $st \in E(\mathcal{P})$, there is exactly one vertex $v \in V(G)$ with $v \in B_s$ and $v \notin B_t$. We call this vertex $v(s, t)$. Thus for any vertex v of G, the number of vertices in the union of all bags containing v is at most $|\mathcal{P}(v)| + k$ and

$$\deg(v) \leq |\mathcal{P}(v)| + k - 1. \tag{4}$$

The vertex set of $P(v(i, i + 1))$ is contained in $\{1, 2, \ldots, i\}$ and the vertices of $P(v(n' + 1 - i, n' - i))$ are a subset of $\{n', n' - 1, \ldots, n' - i + 1\}$. With (4) we obtain:

$$\begin{gathered} \deg(v(i, i + 1)), \deg(v(n' + 1 - i, n' - i)) \leq k + i - 1 \\ \text{for every} \quad i \in \{1, \ldots, n' - 1\}. \end{gathered} \tag{5}$$

Based on (5) and on Lemma 6, 7, and 8 we finally show that Hajós' conjecture is satisfied for all Eulerian graphs of pathwidth 6.

Theorem 18. *Every Eulerian graph G of pathwidth at most 6 satisfies Hajós' conjecture.*

Proof. To prove a slightly more general statement, we introduce the following classes:

$$\begin{aligned} \mathcal{G}_6 &:= \{G \quad \text{is a simple graph of pathwidth at most} \quad 6\}, \\ \mathcal{G}_7 &:= \{G \quad \text{is a simple graph of pathwidth} \quad 7\}, \\ \mathcal{G}_7^- &:= \{G \in \mathcal{G}_7 \colon \exists\, p \in V(G) \colon \deg_G(p) = 2, \operatorname{pw}(G - p) = 6, p \quad \text{lies on a triangle}\}. \end{aligned}$$

The class $\mathcal{G}_7^-$ is the natural extension of $\mathcal{G}_6$ in the context of minimal counterexamples to Hajós' Conjecture. A detailed explanation of this is given in the Appendix. We prove the following statement: *The class* $\mathcal{G}_6 \cup \mathcal{G}_7^-$ *satisfies Hajós' conjecture*. Suppose toward a contradiction that there exists a counterexample to Hajós' Conjecture in $\mathcal{G}_6 \cup \mathcal{G}_7^-$. Let $G \in \mathcal{G}_6 \cup \mathcal{G}_7^-$ be a counterexample of minimum order. By Theorem 2, G has at least 13 vertices. By Lemma 22 we may assume that

$$G \text{ contains at most one vertex of degree 2 or 4.} \tag{6}$$

In the rest of the proof, we show that a certain vertex has at most five possible neighbors. Since G is even, we conclude that the degree of this vertex is either 2 or 4. We then exploit (6) and conclude that this vertex is the unique vertex with a degree in $\{2, 4\}$.

Case A: $\mathbf{G} \in \mathcal{G}_6$: By (6) the three vertices $v(i, i+1)$ with $i = 1, 2, 3$ or the three vertices $v(i, i-1)$ with $i = n', n'-1, n'-2$ all have degree at least 6. (Observe that $|V(G)| \geq 13$ implies that these vertices are distinct). Without loss of generality,

$$\deg(u), \deg(v), \deg(w) \geq 6 \quad \text{for} \quad u := v(1,2), v := v(2,3), w := v(3,4). \tag{7}$$

As u and v are both of degree 6 and $N_G(u) \subseteq B_1, N_G(v) \subseteq B_1 \cup B_2$, there are three possibilities, cf. Figure 3.

(I) u and v have common neighborhood $N = \{x_1, ..., x_6\}$, or
(II) u and v are adjacent with common neighborhood $N = \{x_1, ..., x_5\}$, or
(III) u and v are adjacent with common neighborhood $N = \{x_1, ..., x_4\}$ and private neighbors x_u and x_v.

We will now always delete u *and* v *and optionally some edges. Further, we optionally add some edges in the neighborhood of the two vertices. The obtained graph is still of pathwidth at most 6 since all elements of* N *(respectively* $N \cup \{x_u, x_v\}$*) are contained in the bag* B_2 *and consequently, it has a legal coloring.*

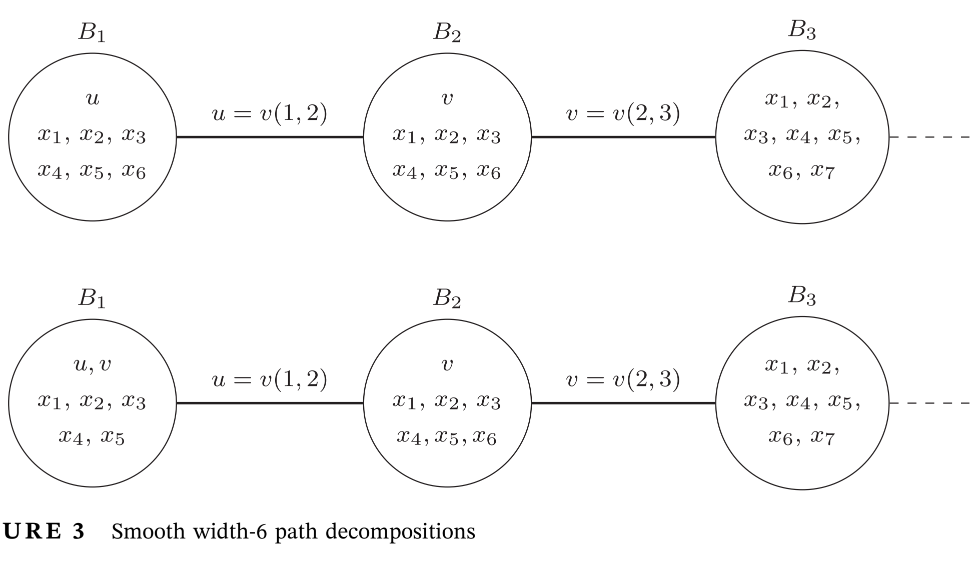

FIGURE 3 Smooth width-6 path decompositions

First assume (I) or (II). By Lemma 8 and Lemma 6, N is an independent set and there is no vertex in $G - \{u, v\}$ that has at least three neighbors in N. This is not possible as w must have at least six neighbors in $B_1 \cup B_2 \cup B_3$ by (7) and, hence, either $w \in N$ and w has a neighbor in N, or, $w \notin N$ and w has at least three neighbors in N.

Last assume (III) and define $u' = v(n', n' - 1), v' = v(n' - 1, n' - 2)$ and $w' = v(n' - 2, n' - 3)$. Observe that

$$B_1 = \{u, v, x_1, x_2, x_3, x_4, x_u\} \quad \text{and} \\ B_2 = \{v, x_1, x_2, x_3, x_4, x_u, x_v\} \tag{8}$$

as depicted in the upper drawing of Figure 4.

If $\deg_G(u'), \deg_G(v'), \deg_G(w') \geq 6$ and the two vertices u' and v' are twins or u' and v' are adjacent with five common neighbors, then with the same reasoning as above for u and v, we obtain a contradiction. Consequently,

a) u' and v' are two adjacent degree-6 vertices with common neighborhood $N' = \{x'_1, ..., x'_4\}$ and private neighbors x'_u and x'_v and $\deg(w') \geq 6$, or
b) there is a vertex y of degree less than 6 among u', v', w'.

Our aim is to find a path between x_u and x_v in $G - R$ with $R = N \cup \{u, v\}$ (respectively a path between $x_{u'}$ and $x_{v'}$ in $G - R'$ with $R' = N' \cup \{u', v'\}$). The existence of this path implies that N is an independent set and there is no vertex in $G - \{u, v\}$ that has at least three neighbors in N by Lemma 7. This is not possible as w must have at least six neighbors by (7) and $N_G(w) \subseteq B_1 \cup B_2 \cup B_3$ by the choice of w. Consequently, either $w \in N$ and w has at least one neighbor in N, or, $w \in \{x_u, x_v\}$ has exactly one neighbor in $\{u, v\}$ and at least three neighbors in N, or, $w \notin R \cup \{x_u, x_v\}$, that is, $w = v(3, 4) = v(3, 2)$. Thus $N(w) = B_3 \backslash \{w\}$.

Suppose that there is no path between x_u and x_v in $G - R$ and denote the set of vertices in the component of x_u in $G - R$ by V_u. Similarly, we define V_v. The vertex z of V_u

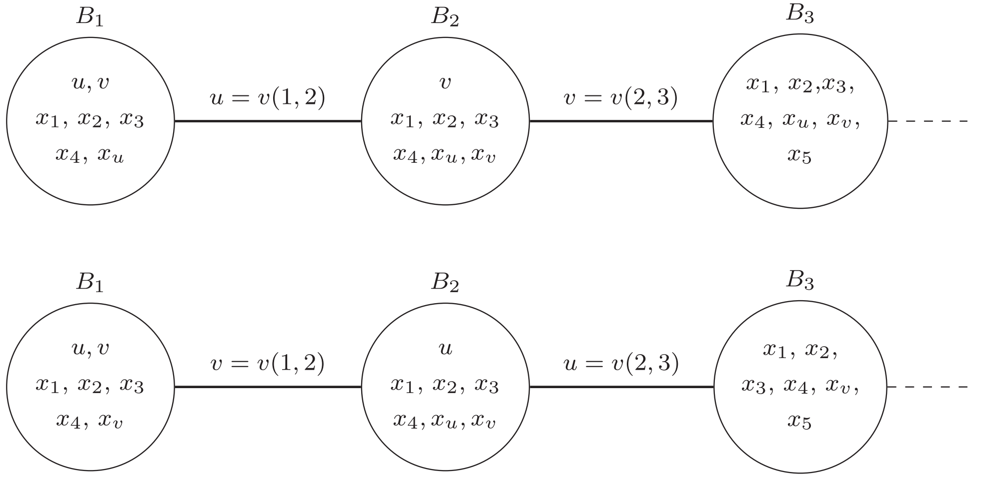

FIGURE 4 Two smooth path-decompositions for the same graph in the case: $\deg(u) = \deg(v) = 6$ and $|N_G(u) \cap N_G(v)| = 4$

(respectively V_v) that maximizes $s(z)$ is denoted by z_u (respectively z_v). Note that the neighborhood of z_a (for $a = u$ and $a = v$) satisfies

$$N(z_a) \subseteq B_{s(z_a)} \tag{9}$$

since every neighbor of z_a is contained in a common bag with z_a, and, by the choice of z_a no neighbor of z_a appears first in a bag of higher index than $s(z_a)$. By (8) it holds that

$$s(V_u) = 1 \quad \text{and} \quad s(V_v) = 2. \tag{10}$$

If $t(V_u) = t(V_v)$, then $t(V_u) = t(V_v) = n'$ since $|B_{i+1} \backslash B_i| = 1$ for $i \in \{1, \ldots n' - 1\}$ by the smoothness of $(\mathcal{P}, \mathcal{B})$. (In particular, at most one of the two components may have its last vertex in B_i). Together with (10) it follows that every bag B_i with $i \in \{2, \ldots, n'\}$ contains a vertex of V_u and a vertex of V_v. By (9), the neighbors of z_u and z_v are contained in the sets $B_{s(z_u)}$ and $B_{s(z_v)}$. Now $\deg(z_v) \leq 4$ since $B_{s(z_v)}$ contains a vertex from V_u. Analogously, if $s(z_u) \geq 2$, we obtain that $\deg(z_u) \leq 4$. Observe that $s(z_u) = 1$ implies $z_u = x_u$. Since x_u is u's private neighbor, the vertex $v \in B_{s(x_u)}$ is not adjacent to x_u and we obtain from (9) that $\deg(x_u) \leq 4$. This contradicts (6).

It remains to consider the situation

$$t(V_u) < t(V_v) \leq n'. \tag{11}$$

(The case $t(V_v) < t(V_u)$ is analogous to (11) by interchanging the roles of u and v, see Figure 4).

Now, z_v might have degree 6, but

$$z_u \text{ has degree less than } 6,$$

since the bag $B_{s(z_u)}$ contains a non-neighbor of z_u as in the case $t(V_u) = t(V_v)$. We split up the proof.

First assume (a). We apply the previous part of the proof and find a vertex $z'_{u'}$ of degree 2 or 4 in the component $V'_{u'} \neq V'_{v'}$ in $G - R'$ (with $R' = \{u', v'\} \cup N'$). It holds that $z_u = z'_{u'}$ by (6). Since $z_u \in V_u \cap V'_{u'}$, there is an x_u–z_u-path P_{x_u,z_u} in $G - R$ and an $x'_u - z_u$-path $P_{x'_u,z_u}$ in $G - R'$. There is no x_u–$x'_{u'}$-path $P_{x_u,x'_{u'}}$ in $G - R - R'$ by (11). Hence, P_{x_u,z_u} contains a vertex r' of $R' \subseteq B_{n'}$ or $P_{x'_u,z_u}$ contains a vertex of $R \subseteq B_1$ which contradicts (11).

Now assume that (b) holds. We obtain from (6) that $y = z_u$.

If $z_u = y = u'$, then $t(V_u) = n'$ which contradicts (11). If $z_u = y = v'$, then $t(V_u) = n' - 1$ and $t(V_v) = n'$ by (11). Hence, $z_v = u'$ since $\emptyset \neq B_{n'} \cap V_v \subseteq N_{G-R}(u')$ by (6). In particular, $N(z_u) \subseteq B_{n'-1} \backslash \{z_u\} = B_{n'} \backslash \{z_v\} = N(z_v)$. There exists an x_u–z_u-path P_u in V_u and an x_v–z_v-path P_v in V_v. The neighbor of z_u in P_u is not contained in R since P_u is a subgraph of V_u and it is a neighbor of z_v. Thus, the paths P_u and P_v can be combined to an x_u–x_v-path in $G - R$ (possibly using edges from $G[B_{n'-1} \cup B_{n'}]$). This contradicts the assumption that no such path exists.

It remains the case $z_u = y = w'$. We obtain $\deg_G(u') = \deg_G(v') = 6$ by (b). If u' and v' are adjacent with exactly four common neighbors, then we apply (a). If u' and v' are adjacent with five common neighbors, then $\deg_G(v(n'-3, n'-4)) \geq 6$ by (6) and $N_G(v(n'-3, n'-4)) \subseteq B_{n'-3} \cup \cdots \cup B_{n'}$. The set on the right contains exactly ten elements of which eight are contained in $B_{n'-1} \cup B_{n'}$ and, hence, $v(n'-3, n'-4)$ is either contained in $N_G(u') \cap N_G(v')$ and has a neighbor in $N_G(u') \cap N_G(v')$ or it is not contained in $N_G(u') \cap N_G(v')$ and has three neighbors in $N_G(u') \cap N_G(v')$. Thus we can apply Lemma 6 to get a legal coloring of G.

The remaining case is that u' and v' are degree-6 twins with common neighborhood N'. In particular, $B_{n'} = \{u'\} \cup N'$ and $B_{n'-1} = \{v'\} \cup N'$. By (11), we have that $B_{n'-1}$ contains a vertex of V_v. Consequently $v' \in V_v$ since $v' \notin R$ and $N_G(v') = B_{n'-1} \backslash \{v'\}$. Hence, there is an x_u–w'-path Q_u in V_u and an x_v–v'-path Q_v in V_v. If $w' \in N'$, then w a neighbor of v'. If, otherwise $w' \notin N'$, then $B_{n'-2} \backslash \{w'\} = N'$ and $N_G(w') \subseteq N'$. We obtain that the unique neighbor of w' in Q_u is a neighbor of v'. In both cases, the paths Q_u and Q_v can be combined to an x_u–x_v-path in $G - R$ which is a contradiction.

Case B: **G** $\in \mathcal{G}_7^-$: By (6), G does not contain a degree-4 vertex and G contains exactly one degree-2 vertex p. The vertex p lies on a triangle pq_1q_2p and $\mathrm{pw}(G - p) = 6$. By Corollary 24, there is a path-decomposition $(\mathcal{P}, \mathcal{B})$ of G with:

(i) There exists exactly one bag B_{i^*} containing p,
(ii) $i^* \notin \{1, 2, n'-1, n'\}$,
(iii) for all $i \in \{1, ..., n'-1\} \backslash \{i^*-1, i^*\}$ it holds that $|B_i \cap B_{i+1}| = 6$,
(iv) $B_{i^*+1} = B_{i^*-1} = B_{i^*} \backslash \{p\}$,
(v) only the bags B_{i^*-1}, B_{i^*} and B_{i^*+1} contain both vertices q_1 and q_2,
(vi) $|B_{i^*}| = 8$ and $|B_i| = 7$ for all $i \in \{1, ... n'\}$ with $i \neq i^*$,
(vii) $\{q_1, q_2\} \subseteq B_i$ holds only for the bags B_{i^*-1}, B_{i^*} and B_{i^*+1}.

For each $i \in \{1, ..., n'-1\} \backslash \{i^*-1\}$, let $v(i, i+1)$ be the unique vertex in $B_i \backslash B_{i+1}$. For each $j \in \{2, ... n'\} \backslash \{i^*+1\}$, let $v(i, i-1)$ be the unique vertex in $B_i \backslash B_{i-1}$. From the structure of the decomposition $(\mathcal{P}, \mathcal{B})$, we obtain that $n' \geq 8$ (B_1 contains seven vertices and every bag except B_{i^*+1} contributes exactly one new vertex). Without loss of generality $i^* \notin \{3, 4\}$. (If $i^* \in \{3, 4\}$, we change the labels of the bag such that they appear in reversed order). Since $i^* \notin \{1, 2, 3, 4, n'-1, n'\}$, we obtain that the vertices

$$\begin{aligned} u &:= v(1, 2), \\ v &:= v(2, 3), \\ w &:= v(3, 4) \text{ and} \\ u' &:= v(n', n'-1) \end{aligned}$$

exist and

$$\begin{aligned} &\deg_G(u) = \deg_G(v) = \deg_G(u') = 6 \quad \text{and} \\ &\deg_G(w) \geq 6. \end{aligned}$$

As in Case A, there are three possibilities:

(I) u and v have common neighborhood $N = \{x_1, ..., x_6\}$, or
(II) u and v are adjacent with common neighborhood $N = \{x_1, ..., x_5\}$, or

(III) u and v are adjacent with common neighborhood $N = \{x_1, ..., x_4\}$ and private neighbors x_u and x_v.

If (I) or (II) holds true, then we obtain a contradiction along the same lines as in Case A. Now assume (III). The bags B_1 and B_2 appear as described in (8), and, by symmetry of the two sides of the path $\mathcal{P}$ of G's path-decomposition, we can suppose that

a) u' and v' are two adjacent degree-6 vertices with common neighborhood $N' = \{x'_1, ..., x'_4\}$ and private neighbors x'_u and x'_v, or,
b) $i^* = n' - 2$, or,
c) $i^* = n' - 3$.

Our aim is now to find a path between x_u and x_v in $G - R$ with $R = N \cup \{u, v\}$ (respectively a path between $x_{u'}$ and $x_{v'}$ in $G - R'$ with $R' = N' \cup \{u', v'\}$). This settles the claim as described in Case A. Suppose toward a contradiction that there is no path between x_u and x_v in $G - R$ with $R = N \cup \{u, v\}$. Define V_u, V_v, z_u and z_v as in Case A. Again, the neighborhood of z_a (for $a \in \{u, v\}$) satisfies

$$N(z_a) \subseteq B_{s(z_a)}. \tag{12}$$

By (8) it holds that

$$s(V_u) = 1 \quad \text{and} \quad s(V_v) = 2. \tag{13}$$

In analogy to Case A, we have that $t(V_u) = t(V_v)$ implies $t(V_u) = t(V_v) = n'$ and, hence, every bag B_i with $i \in \{2, ..., n'\}$ contains a vertex of V_u and a vertex of V_v. In the following, we lead this to a contradiction to (6) (as in Case A) by showing that both vertices z_u and z_v are of degree less than six:

If $s(z_a) = i^*$ for $a \in \{u, v\}$, then $z_a = p$ and, hence, $\deg_G(z_a) = 2$.

By (12), the neighbors of z_u and z_v are contained in the sets $B_{s(z_u)}$ and $B_{s(z_v)}$. If $s(z_v) \neq i^*$, then $\deg(z_v) \leq 4$ since $B_{s(z_v)}$ contains a vertex from V_u and $s(z_v) \neq i^*$ yields that there are only five potential neighbors of z_v in $B_{s(z_v)}$.

Analogously, if $s(z_u) \geq 2$ and $s(z_u) \neq i^*$, we obtain that $\deg(z_u) \leq 4$.

If, otherwise, $s(z_u) = 1$ then $z_u = x_u$. By assumption v is not adjacent to x_u and we obtain from (12) that $\deg(x_u) \leq 4$. This contradicts (6). Thus we can assume that

$$t(V_u) < t(V_v) (\leq n'). \tag{14}$$

(The case $t(V_v) < t(V_u)$ is analogous by interchanging the roles of u and v as in Case A, cf. Figure 4). The vertex z_v might have degree 6, but z_u has degree less than 6 since the bag $B_{s(z_u)}$ contains a non-neighbor of z_u if $s(z_u) \neq i^*$. In particular,

$$p = z_u. \tag{15}$$

As in Case A, we split up the proof.

First, assume that (a) holds. We obtain a contradiction in analogy to Case A.

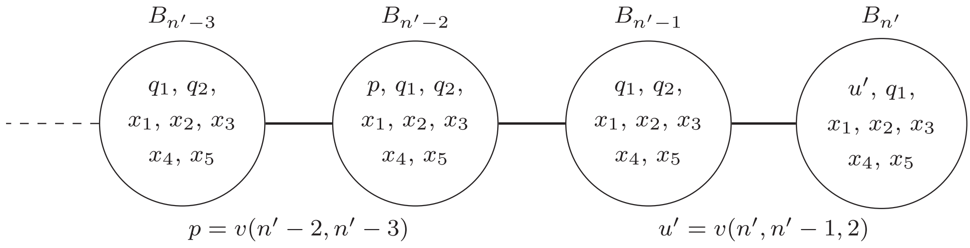

FIGURE 5 The bags $B_{n'-3}, B_{n'-2}, B_{n'-1}$ and $B_{n'}$ of a smooth path-decomposition of a graph $G \in \mathcal{G}_7^-$ with $i^* = n' - 2$ are illustrated

Now assume that (b) holds. This situation is illustrated in Figure 5. We may assume that $q_1 \in B_{n'}$ without loss of generality.

Observe that $s(z_v) > s(z_u)$ (Otherwise, $B_{s(z_v)}$ contains a nonneighbour of z_v which yields $\deg_G(z_v) < 6$. This contradicts (6)). This implies that $z_v = u'$ and, hence, there exists an x_v–u'-path $P_{x_v,u'}$ in $G - R$. There exists a path P_{p,x_u} in $G - R$. At least one of the two neighbors q_1, q_2 of p is contained in P_{p,x_u}. Since $t(V_u) < n'$ and $p_1 \in B_{n'}$, we have that $q_2 \in V(P_{p,x_u})$ and $q_1 \notin V(P_{p,x_u})$.

If $q_1 \notin R$, then $P_{x_v,u}uq_1pP_{p,x_u}$ is an x_u–x_v-path in $G - R$ which is a contradiction. If, otherwise, $q_1 \in R$, then q_1 is contained in every bag of the path decomposition. In particular, $q_2 \notin B_{n'-4}$ since $B_{n'-3}, B_{n'-2}, B_{n'-1}$ are the only bags containing both neighbors of p. But now, the following is a width-6 path decomposition of G: Set

$$\begin{aligned} X &:= B_{n'-3} \backslash \{q_1, q_2\}, \\ \tilde{B}_{n'-3} &:= \{q_1, u'\} \cup X, \\ \tilde{B}_{n'-2} &:= \{q_1, q_2\} \cup X \quad \text{and} \\ \tilde{B}_{n'-1} &:= \{q_1, q_2, p\}. \end{aligned}$$

Remove the vertex n' from $\mathcal{P}$ and replace B_i with $\tilde{B}_i$ for $i \in \{n'-1, n'-2, n'-3\}$ to obtain a path-decomposition of G of width 6. This contradicts $G \in \mathcal{G}_7^-$.

Last, assume that (c) holds, that is $i^* = n' - 3$. The vertex $v' := v(n'-1, n'-2)$ exists and has degree 6. If u' and v' have four common neighbors and are adjacent, then (a) holds and we are done. If u' and v' have five common neighbors and are adjacent, or, u' and v' have six common neighbors, then by Lemma 8 and Lemma 6, N' is an independent set and there is no vertex in $G - \{u', v'\}$ that has at least three neighbors in N. This is impossible since the vertex $w' := v(n'-4, n'-5)$ has degree not less than 6 and satisfies

$$N_G(w') \subseteq B_{n'-4} \cup B_{n'-3} \cup B_{n'-2} \cup B_{n'-1} \cup B_{n'} = B_{n'-3} \cup B_{n'-1} \cup B_{n'},$$

since $n' - 3 = i^*$ and B_{i^*} contains its neighboring bags. The set on the right side contains ten elements of which at least five are in N'. If $w' \in N'$, then w has at least one neighbor in N'. If, otherwise, $w' \notin N'$, then w' has at least four neighbors in N', since w' is neither adjacent to u' nor to v'. □

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APPENDIX

Fan and Xu [6] considered a generalized version of Hajós' conjecture that includes loopless graphs with parallel edges.

Conjecture 19 (Generalized Hajós' conjecture). *If G is a loopless Eulerian graph, then G allows for a cycle decomposition with not more than*

$$\left\lfloor \frac{V(G) + m(G) - 1}{2} \right\rfloor$$

cycles, where $m(G)$ is the minimum number of edges in G that need to be removed to obtain a simple graph.

Observation 20. Let G be obtained from a graph G' by subdividing an edge $e \in E(G')$ which is parallel to some other edge of G' with a new vertex u. A minimum cycle decomposition of G clearly corresponds to a minimum cycle decomposition of G'.

Furthermore, it holds that $m(G) + |V(G)| = m(G') + |V(G')|$. As a consequence G satisfies the generalized Hajós' conjecture if and only if G' satisfies Hajós' conjecture. In particular: Every counterexample to the generalized Hajós' conjecture can be transformed into a simple counterexample to Hajós' conjecture by subdivision of parallel edges.

The following notion is introduced in [6]: Let G be a graph. A *reduction* of G is a graph obtained by recursively applying one of the following operations:

(i) Remove the edges of a cycle.
(ii) Delete an isolated vertex.
(iii) Remove a vertex u of degree 2 and add an edge joining its two neighbors.
(iv) Let u be a degree-4 vertex with distinct neighbors x, y, z, w such that $xy \in E(G)$ and $zw \notin E(G)$. Delete u and add two new edges – one joining x and y and the other one joining z and w.

Now we are ready to state Fan and Xu's Theorem:

Theorem 21 (Fan and Xu [6]). *If G is an Eulerian graph that does not satisfy the generalized Hajós' conjecture, then there exists a reduction H of G that does not satisfy the generalized Hajós' conjecture and the number of vertices of degree less than six in H plus $m(H)$ is at most one.*

It is observed in [6] that none of the above reduction operations changes the minor-free property if the minor is a simple graph. In particular,

$$\text{If } H \text{ is a reduction of } G\text{, then } \mathrm{pw}(H) \leq \mathrm{pw}(G). \tag{A1}$$

Our aim is to transfer these reductions to *simple graphs* to show that a minimum counterexample to Hajós' conjecture for graphs of pathwidth at most six contains at most one vertex of degree at most 2. Observation 20 leads to a straight-forward approach: Subdivide parallel edges that may arise by operation (iii) or (iv). The resulting graph is simple and a counterexample to Hajós' conjecture. However, as demonstrated in Figure A1, this is not pathwidth-preserving.

The removal of the subdivision vertex always leads to a graph of pathwidth at most 6. We exploit this operation and enlarge the considered class to save the subdivision approach.

Let $\mathcal{G}_6$ be the class of all simple graphs of pathwidth at most 6. Further, let $\mathcal{G}_7^-$ be the class of all simple graphs G of pathwidth 7 with the following property: There exists a degree-2 vertex $p \in V(G)$ such that the two neighbors of p are adjacent and $\mathrm{pw}(G - p) = 6$.

Lemma 22. *Let $G \in \mathcal{G}_6 \cup \mathcal{G}_7^-$ be a graph that does not satisfy Hajós' conjecture. There exists a reduction H' of G that does not satisfy the generalized Hajós' conjecture and the number of vertices of degree less than six in H' plus $m(H')$ is at most one. The graph*

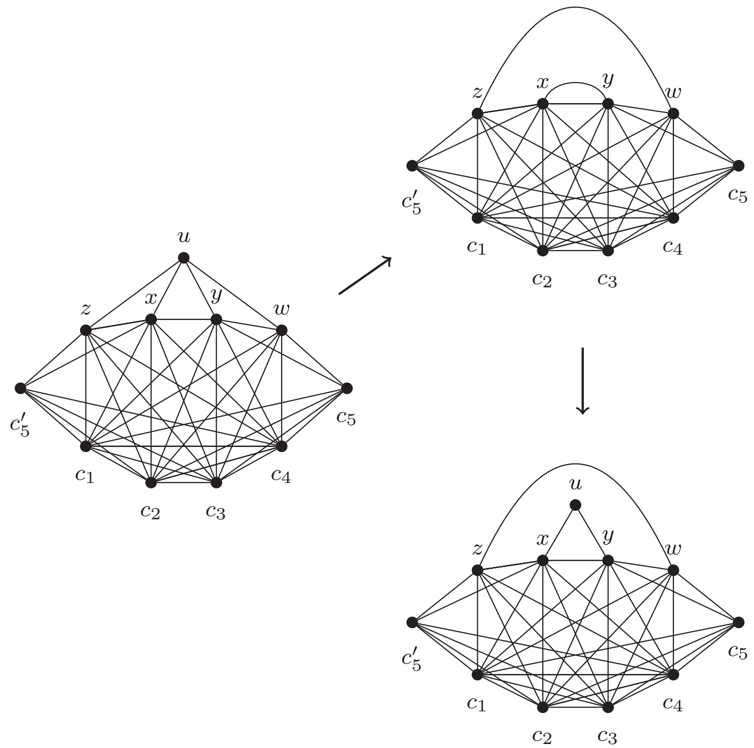

FIGURE A1 Applying first operation (iv) and then subdividing the new parallel edge can increase the pathwidth. The left and the upper graph are of pathwidth 6, the lower graph is of pathwidth 7

$$H := \begin{cases} H' & \text{if} \quad m(H') = 0, \\ H' - e + q_1 p + q_2 p & \text{if} \quad m(q_1 q_2) = 1 \text{ and } e \text{ is an edge joining } q_1 \text{ and } q_2, \end{cases}$$

has the following properties:

(i) H is a counterexample to Hajós' conjecture,
(ii) H contains at most one vertex of degree less than six,
(iii) $H \in \mathcal{G}_6 \cup \mathcal{G}_7^-$, and,
(iv) $|V(H)| \leq |V(G)|$ and $|E(H)| \leq |E(G)|$.

Proof. Set

$$G' := \begin{cases} G & \text{if} \quad G \in \mathcal{G}_6 \quad \text{and} \\ G - p' + e_{q'_1 q'_2} & \text{if} \quad G \in \mathcal{G}_7^-, \end{cases}$$

where p' is a degree-2 vertex lying on a triangle $p'q'_1q'_2p'$ in G with $\text{pw}(G - p') = 6$, and, $e_{q'_1 q'_2}$ denotes a new parallel edge joining q'_1 and q'_2. Observe that G' is a reduction of G, G' is a counterexample to the generalized Hajós' conjecture by Observation 20 and $\text{pw}(G') = 6$. We apply Theorem 21 to obtain a reduction H' of G' which satisfies:

$m(H') \in \{0, 1\}$, at most one vertex in H' is of degree less than 6, and, if $m(H') = 1$, then H' does not contain a vertex of degree less than 6.

It holds that H is simple. Applying Observation 20, we obtain that (i) is satisfied.

If H' is simple, then $H = H'$ and (ii) is clearly satisfied. If, otherwise, $m(H') = 1$, then H' does not contain any vertex of degree less than 6 and, hence, H's only vertex of degree less than 6 is the subdivision vertex p, that is, (ii) holds.

By (A1), we have that $\mathrm{pw}(H') \le \mathrm{pw}(G') \le \mathrm{pw}(G)$. It follows from the construction of H that (iii) holds.

It remains to prove (iv). This is clear if $H = H'$. Otherwise, $H = H' - e + q_1p + q_2p$ and $m(H') = 1$. Since G is a simple graph and H' is a reduction of G, we conclude that at least one of the reductions (iii) and (iv) is applied to G to obtain H' (the other two reduction operations do not generate parallel edges). Both operations (iii) and (iv) have the property that they strictly decrease the order of the graph and the number of edges. Altogether, we obtain that (iv) holds. This settles the claim. □

Lemma 23. *Let G be an Eulerian pathwidth-7 graph that contains a degree-2 vertex p lying in a triangle pq_1q_2p. If $\mathrm{pw}(G - p) = 6$ and $(\mathcal{P}', \mathcal{B}')$ is a smooth width-6 path decomposition of $G - p$, then*

(i) there is exactly one bag $B_{i'} \in \mathcal{B}'$ containing both vertices q_1 and q_2, and,
(ii) i' is not a leaf of $\mathcal{P}'$.

Proof. Suppose that $G - p$ allows for a smooth path-decomposition $(\mathcal{P}', \mathcal{B}')$ such that there are two distinct bags $B_j{}', B_k{}' \in \mathcal{B}'$ containing both vertices q_1 and q_2. This implies that each bag on the subpath of $\mathcal{P}'$ with ends j and k every bag contains q_1 and q_2. In particular, a neighboring bag of $B_j{}'$, say B'_{j+1}, contains q_1 and q_2. The decomposition $(\mathcal{P}', \mathcal{B}')$ can be extended to a width-6-decomposition of G: Add the bag $B'_{j+\frac{1}{2}} \coloneqq B'_j \cap B'_{j+1} \cup \{p\}$ and replace the edge $j(j+1)$ in $\mathcal{P}'$ with the length-2 path $j\left(j + \frac{1}{2}\right)(j+1)$. This is a contradiction to $\mathrm{pw}(G) = 7$.

We may now assume that $(\mathcal{P}', \mathcal{B}')$ is a smooth path decomposition of $G - p$ such that exactly one bag $B_{i'} \in \mathcal{B}'$ contains q_1 and q_2. Suppose for a contradiction that i' is a leaf of $\mathcal{P}'$. Let $u \in B_{i'} \backslash \{q_1, q_2\}$. Set $B_0 \coloneqq (B_{i'} \backslash \{u\}) \cup \{p\}$. We obtain a smooth path-decomposition $(\mathcal{P}, \mathcal{B})$ of G, where $\mathcal{B} \coloneqq \mathcal{B}' \cup \{B_0\}$ and $\mathcal{P}$ is obtained by adding the vertex 0 and the edge $0i'$ to $\mathcal{P}'$. This is a contradiction since $(\mathcal{P}, \mathcal{B})$ is a width-6 decomposition. □

Corollary 24. *Let G be an Eulerian pathwidth-7 graph that contains a degree-2 vertex p lying in a triangle pq_1q_2p. If $\mathrm{pw}(G - p) = 6$, then G allows for a path decomposition $(\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a path on the vertex set $\{1, \dots, n'\}$ and edge set $\{i(i+1): 1 \le i \le n' - 1\}$, with the following properties:*

(i) There exists exactly one bag B_{i^*} containing p, both neighbors of p are also contained in B_{i^*}, and,
(ii) $i^* \notin \{1, 2, n' - 1, n'\}$,
(iii) for all $i \in \{1, \dots, n' - 1\} \backslash \{i^* - 1, i^*\}$ it holds that $|B_i \cap B_{i+1}| = 6$,
(iv) $B_{i^*+1} = B_{i^*-1} = B_{i^*} \backslash \{p\}$,
(v) $|B_{i^*}| = 8$ and $|B_i| = 7$ for all $i \in \{1, \dots n'\}$ with $i \ne i^*$, and,

(vi) $\{q_1, q_2\} \subseteq B_i$ *holds only for the bags* B_{i^*-1}, B_{i^*} *and* B_{i^*+1}.

Proof. Let $(\mathcal{P}', \mathcal{B}')$ be a smooth path decomposition of $G - p$ of width 6. According to Lemma 23, there is exactly one bag $\tilde{B}_{\tilde{i}} \in \mathcal{B}'$ with $\{q_1, q_2\} \in \tilde{B}$. Set

$$B_{i^*} := \tilde{B}_{\tilde{i}} \cup \{p\} \quad \text{and}$$
$$B_{i^*-1} := B_{i^*+1} := \tilde{B}_{\tilde{i}}.$$

We obtain a path $\mathcal{P}$ by replacing the subpath $(\tilde{i} - 1)\tilde{i}(\tilde{i} + 1)$ of $\mathcal{P}'$ with the new subpath $(\tilde{i} - 1)(i^* - 1)i^*(i^* + 1)(\tilde{i} + 1)$. Set

$$\mathcal{B} := (\mathcal{B}' \backslash \tilde{B}_{\tilde{i}}) \cup \{B_{i^*-1}, B_{i^*}, B_{i^*+1}\}.$$

Now, $(\mathcal{P}, \mathcal{B})$ is a path-decomposition of G and we may assume that it is labeled as in the claim. Properties (i)– (vi) are satisfied by construction. □