

# On Moment-Dissipative Stochastic Dynamical Systems

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**ABSTRACT:** Nonlinear dissipativity, asymptotical stability, and contractivity of (ordinary) stochastic differential equations (SDEs) with some dissipative structure and their discretizations are studied in terms of their moments in the spirit of Pliss (1977). For this purpose, we introduce the notions and discuss related concepts of dissipativity, growth-bounded and monotone coefficient systems, asymptotical stability and contractivity in wide and narrow sense, nonlinear A-stability, AN-stability, B-stability and BN-stability for stochastic dynamical systems – more or less as stochastic counterparts to deterministic concepts. The test class of in a broad sense interpreted dissipative SDEs as natural analogon to dissipative deterministic differential systems is suggested for stochastic-numerical methods. Then, in particular, a kind of mean square calculus is developed, although most of ideas and analysis can be carried over to general ‘stochastic  $L^p$ -case’ ( $p \geq 1$ ). For example, we prove mean square dissipativity, asymptotical mean square stability in wide and narrow sense, and mean square contractivity for fully drift-implicit Euler method for any choice of ‘admissible’ step sizes  $\Delta_k > 0$ . (A sequence of step sizes is called ‘admissible’ if  $0 < \Delta_k \leq \sup_{n \in \mathbb{N}} \Delta_n < +\infty$  and  $\sum_{n=0}^{+\infty} \Delta_n = +\infty$ .) By this natural restriction, the new stochastic concepts are theoretically meaningful, as in deterministic analysis. Since the choice of step sizes then plays no essential role in related proofs, we even obtain nonlinear A-stability, AN-stability, B-stability and BN-stability in the mean square sense for this implicit method with respect to appropriate test classes of moment-dissipative SDEs.

**AMS (MOS) subject classification.** 60H10, 65C05, 65C20, 65U05

## 1. INTRODUCTION

Numerous applications in Natural Sciences, Engineering, Environmental Sciences and Econometrics lead to models governed by nonlinear, but dissipative differential and difference systems perturbed by additive or parametric random noise. The occurring systems are often considered as systems of nonlinear (stochastic) differential and difference equations in  $\mathbb{R}^d$ . Therein and here, the concept of dissipativity can be interpreted in a fairly wide sense. Roughly speaking, a dissipative system in a Banach space  $E$  is a dynamical system (e.g. an evolution equation) which possesses a bounded set of  $E$  into which every orbit eventually enters and remains.

Here we want to study dissipative stochastic dynamical systems where the dissipativity can be observed in terms of their absolute moments. A general classification with respect to dissipativity, stability and contractivity and a presentation of an unified approach to qualitative analysis of (nonlinear) stochastic dynamical systems with different time scales is the main objective of this paper. For this purpose, the related notions and concepts need to be introduced and justified. In particular, we will examine classes of (ordinary) stochastic differential equations and related (adequate) discretizations.

The paper is organized as follows. Section 2 commences with statement of very general definitions of ( $p$ -th mean and uniform) dissipativity for stochastic dynamical systems on time scales. In section 3 we discuss the related concepts. The  $p$ -th mean dissipativity of absolute norm as well as of initial perturbations are proven for stochastic differential equations (SDEs) driven by standard Brownian motions and for their discretizations by implicit Euler methods. For this purpose we will state nonautonomous linear variation of constants inequalities (see section 7) as natural generalizations of well-known Bellman-Gronwall inequality. In section 4 we continue with some refinements of presented concepts of dissipativity. So further classifying notions are introduced: uniform boundedness and monotonicity of coefficient systems, asymptotical stability and contractivity in wide and narrow sense, i.e. notions related to qualitative behaviour of absolute norm of state and initial perturbations of dynamical systems as integration time  $t$  goes to infinity, respectively. Section 5 is devoted to property of asymptotical  $p$ -th mean stability for nonlinear stochastic dynamical systems. Sufficient conditions for mean square exponential stability in wide and narrow sense are stated for continuous time stochastic systems. Additionally we shall prove a new theorem concerning asymptotic behaviour of absolute norm of state process for their discretizations (3.6). Since the obtained result will be independent of ‘admissible step sizes’ and valid in an uniform sense with respect to appropriate test class of nonlinear SDEs in case of fully drift-implicit Euler method (i.e. the implicit Euler method with implicitness  $\alpha = 1$  taking into account only implicitness in the drift part), one receives the properties of nonlinear (mean square) A- and AN-stability of this method. Thereafter, in section 6, the stochastic dynamical systems are investigated in view of initial perturbations and its propagation. For this purpose we introduce the notions of contractivity of initial perturbations in wide and narrow sense for stochastic dynamical systems. Mean square contractivity of continuous time SDEs in wide and narrow sense is proven at first. Then the same property can be verified for fully drift-implicit Euler method. Since this fact is valid for all ‘admissible step sizes’ and in an uniform sense with respect to appropriate test class of nonlinear SDEs, the fairly general notions of mean square B- and BN-stability are introduced for parameterized stochastic dynamical systems in view of deterministic counterparts (cf. Burrage and Butcher [4], [5], and related discussions of Dahlquist [7], [8], Hairer and Wanner [13] Hundsdorfer and Spijker [16] Van Veldhuizen [42] Verwer [43]). Thus mean square B- and BN-stability of fully drift implicit Euler method can be noticed. For convenience, section 7 supplements the presented analysis with some linear variation of constants inequalities. A brief summary and remarks finish this contribution to nonlinear stochastic dynamical systems by section 8.

## 2. CONCEPTS OF DISSIPATIVITY ON TIME SCALES

To unify first considerations and reduce the space taken by necessary new definitions, let  $\mathbb{T}$  be a (completely ordered) time scale with minimum element  $t_{min} > -\infty$ , with no maximum element, but supremum  $+\infty$ . For deterministic, theoretical background, see e.g. [15]. For simplicity of stochastic analysis (e.g. difficulties may arise when considering anticipating random or stopping times), we exclusively refer to the case of deterministic time scales (or time scales whose elements are monotone random variables being independent of naturally underlying filtration). The most popular examples of time scales are given by  $\mathbb{T} = [0, +\infty)$  or  $\mathbb{T} = \{t_i \in \mathbb{R} : t_i < t_{i+1}, i \in \mathbb{N}\}$  which correspond to classical time horizons of continuous time or discrete time dynamical systems, respectively, as occurring in numerical integration of continuous time differential systems. Furthermore, let  $X_{t_0, X_0}(t) (= X_t)$  denote the value of stochastic real-valued dynamical system at time  $t \in \mathbb{T}$ , started at value  $X_0$  at time  $t_0 \in \mathbb{T}$ .  $\mathbb{D}$  is supposed to be a sufficiently large domain of  $\mathbb{R}^d$  where the given dynamical system is defined and which is left invariant by its dynamic mapping (a.s.). For simplicity, we will only consider dynamical systems forward in time, i.e. we impose  $t \geq t_0$  (recall the existence of a minimum element of that time scale). Let  $\|\cdot\|$  be the Euclidean vector norm for the sake of simplicity. (In principle other norms can be treated, but statements and above all the conditions below will slightly change while referring to other norms.) Throughout this paper we suppose that corresponding stochastic basis  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{T}}, \mathbb{P})$  is given with natural filtration  $\mathcal{F}_t$ .

Then the following definition – in analogy to deterministic counterpart (i.e. in the sense of Pliss [25] and Hale [14] – can be introduced, provided that related moments globally exist on that time scale  $\mathbb{T} \subset \mathbb{R}^1$ .

**Definition 2.1** *A stochastic dynamical system  $\{X_t : t \in \mathbb{T}\} \subseteq \mathbb{D} \subseteq \mathbb{R}^d$  is called **p–th mean dissipative** ( $p \in \mathbb{R}^1 \setminus \{0\}$  fixed) iff there exist finite positive real numbers  $r_p, R_p$  such that*

$$\limsup_{t \rightarrow +\infty} \mathbb{E} \|X_{t_0, X_0}(t)\|^p < R_p \quad (2.1)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^p < r_p.$$

In addition, it is said to be **uniformly p–th mean dissipative** ( $p \in \mathbb{R}^1 \setminus \{0\}$  fixed) iff there exist finite positive real numbers  $\bar{r}_p, \bar{R}_p$  such that

$$\lim_{t \rightarrow +\infty} \mathbb{E} \sup_{t_0 \leq s \leq t} \|X_{t_0, X_0}(s)\|^p < \bar{R}_p \quad (2.2)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^p < \bar{r}_p.$$

**Remark.** The concept of uniform p–th mean dissipativity (2.2) seems to be a very strong requirement in general (caused by nature of unbounded random noise). In

fact, to our knowledge it makes only sense when the stochastic dynamical system is uniformly bounded (a.s.). Because of this restriction in mind we do not pursue the uniform analysis with (2.2) here. For simple examples to manifest our opinion one may consider stochastic differential equations with purely additive noise or innovation diffusions as in Schurz [31]. Rather we want to devote our studies to systems with property of  $p$ -th mean dissipativity (2.1), in particular when  $p = 2$ , here. In case of  $p = 2$  we also term related property as **mean square dissipativity**. As the (stochastic) dynamical system itself, we call *the generating mapping, flow, equation, method, scheme, solution as dissipative* ones in the sense as above. That this definition is meaningful in stochastic analysis too is an objective of this paper.

So far we have considered the absolute norm of dynamical systems and concept of dissipativity. One can also introduce a similar characterization of temporal propagation of their initial perturbations. In doing so we meet the following convention.

**Definition 2.2** *A stochastic dynamical system  $\{X_t : t \in \mathbb{T}\} \subseteq \mathbb{D} \subseteq \mathbb{R}^d$  has  **$p$ -th mean dissipative initial perturbations** ( $p \in \mathbb{R}^1 \setminus \{0\}$  fixed) iff there exist finite positive real numbers  $r_p, R_p$  such that*

$$\limsup_{t \rightarrow +\infty} \mathbb{E} \|X_{t_0, X_0}(t) - X_{t_0, Y_0}(t)\|^p < R_p \quad (2.3)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0, Y_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^p < +\infty, \quad \mathbb{E} \|Y_0\|^p < +\infty, \quad \mathbb{E} \|X_0 - Y_0\|^p < r_p.$$

In addition, their **initial perturbations** are said to be **uniformly  $p$ -th mean dissipative** ( $p \in \mathbb{R}^1 \setminus \{0\}$  fixed) iff there exist finite positive real numbers  $\bar{r}_p, \bar{R}_p$  such that

$$\lim_{t \rightarrow +\infty} \mathbb{E} \sup_{t_0 \leq s \leq t} \|X_{t_0, X_0}(s) - X_{t_0, Y_0}(s)\|^p < \bar{R}_p \quad (2.4)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0, Y_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^p < +\infty, \quad \mathbb{E} \|Y_0\|^p < +\infty, \quad \mathbb{E} \|X_0 - Y_0\|^p < \bar{r}_p.$$

**Remarks.** Note that concepts of dissipativity arising by both definitions 2.1 and 2.2 do not coincide in general! However,  $p$ -th mean dissipativity of norm of state process implies  $p$ -th mean dissipative initial perturbations. This fact can be seen by application of triangular or Minkowski's inequality. Besides,  $(p+\varepsilon)$ -th mean dissipativity guarantees  $p$ -th mean dissipativity for  $p \geq 1, \varepsilon \geq 0$  – a fact which is verified by use of Lyapunov's inequality. For systems satisfying the requirement of dissipativity on the propagation of their initial perturbations, see below. Systems with property of dissipative initial perturbations can not possess exploding perturbations in  $p$ -th moment sense on that given time scale  $\mathbb{T}$ . Thus one may also speak of **systems with  $p$ -th mean controlled propagation of perturbations**. The concept of controllable perturbations has large importance in qualitative analysis of numerical methods for continuous time SDEs (error propagation control).

As already said, we are particularly interested in examinations with respect to dissipativity in terms of given squared norm. We have seen that there are two basic

directions where investigations will go to, when following the definitions from above. First, dissipativity of state evolution of dynamical systems leading to assertions on global boundedness and asymptotical stability – in conformance with definition from above. Second, dissipativity of initial perturbations yielding some control on ‘initial error propagation’ of those stochastic dynamical systems to be discussed.

### 3. MOMENT-DISSIPATIVITY OF STOCHASTIC SYSTEMS

At first, let us turn to continuous time dynamical systems governed by stochastic differential equations (SDEs) with traditional time scale  $\mathbb{T} = [t_0, +\infty)$  on deterministic domains  $\mathbb{D} \subseteq \mathbb{R}^d$ . In statements below, if not additionally stated, let indices  $j$  run in  $\{1, 2, \dots, m\}$ .

#### 3.1 Dissipativity of continuous time SDEs

Given (ordinary) stochastic differential equations (SDEs)

$$dX_t = a(t, X_t) dt + \sum_{j=1}^m b^j(t, X_t) dW_t^j \quad (3.1)$$

where  $a, b^j$  are  $d$ -dimensional, real-valued vectors, and  $W_t^j$  represent real-valued, scalar, independent, identically distributed, continuous time martingales with bounded quadratic variation. Then one may consider  $W_t^j$  as standard Wiener processes, thanks to the well-known and fundamental work of P. Levy. Let us do so. (In general, one might even think about carrying over the approach to be presented here to semimartingales  $Z_t^j$  with some appropriately bounded variation instead of  $W_t^j$ .) Assume that solution of (3.1) exists in an appropriate sense, i.e. at least in mean square ( $p$ -th mean) sense on any finite time-interval  $[t_0, T]$ , and solutions are measurable with respect to natural filtration

$$\mathcal{F}_t = \sigma\{W_s^j : t_0 \leq s \leq t, j = 1, 2, \dots, m\}.$$

Without loss of generality, we may suppose that system (3.1) is given in Itô interpretation (Otherwise one transforms given stochastic calculus to Itô one.). For theory on SDEs, see Arnold [1], Khas'minskij [18], Gard [12], Karatzas and Shreve [17], Protter [26] or Rogers and Williams [27].

**Theorem 3.1** *Assume that  $\mathbb{T} = [t_0, +\infty)$  and coefficients  $a, b^j$  of SDE (3.1) are measurable with respect to time  $t \in \mathbb{T}$  and continuous in  $x \in \mathbb{D} \subseteq \mathbb{R}^d$  where domain  $\mathbb{D}$  is (a.s.) left invariant by this SDE.*

*Then SDE of type (3.1) is mean square dissipative (i.e.  $p = 2$ ) if there exist locally  $L^1$ -integrable functions  $K_1(t), K_2(t)$  with  $t \in \mathbb{T}$  with respect to Lebesgue measure on*

$\mathbb{T}$  such that, for all  $x \in \mathbb{D}$ , it holds

$$2 \langle x, a(t, x) \rangle + \sum_{j=1}^m \|b^j(t, x)\|^2 \leq K_1(t) + K_2(t) \cdot \|x\|^2 \quad (3.2)$$

where

$$\sup_{t,s \in \mathbb{T}: t \geq s} \int_s^t K_2(u) du + \sup_{t,s \in \mathbb{T}: t \geq s} \left( \int_s^t K_1(u) \exp \left( \int_u^t K_2(v) dv \right) du \right) < +\infty. \quad (3.3)$$

**Proof.** Apply well-known Dynkin's formula (see Dynkin [10] or Itô formula) to functional  $f(x) = \|x\|^2$ ,  $x \in \mathbb{R}^d$ . Define  $v(t) := \mathbb{E} \|X_{s, X_s}(t)\|^2$ . Then one obtains

$$v(t) = v(s) + \int_s^t \mathbb{E} \left[ 2 \langle X_{s, X_s}(u), a(u, X_{s, X_s}(u)) \rangle + \sum_{j=1}^m \|b^j(u, X_{s, X_s}(u))\|^2 \right] du,$$

thanks to assumptions of Theorem 3.1, for all  $t \geq s$  where  $t, s \in \mathbb{T}$ . Under condition (3.2) it follows

$$v(t) \leq v(s) + \int_s^t K_1(u) du + \int_s^t [K_2(u) \cdot \mathbb{E} \|X_{s, X_s}(u)\|^2] du.$$

Using generalized Bellman–Gronwall–type estimates (as in Mao [21] or variation of constants inequalities as in section 7) one receives

$$v(t) \leq \left( \mathbb{E} \|X_s\|^2 + \int_s^t K_1(u) \exp \left( - \int_s^u K_2(v) dv \right) du \right) \cdot \exp \left( \int_s^t K_2(u) du \right)$$

for all  $t \geq s$ , all  $s \in \mathbb{T}$ . Then requirement (3.3) yields mean square dissipativity of related SDE with dissipativity constant

$$R_2 \leq \sup_{t,s \in \mathbb{T}: t \geq s} \left( r_2 + \int_s^t K_1(u) \exp \left( - \int_s^u K_2(v) dv \right) du \right) \cdot \exp \left( \int_s^t K_2(u) du \right),$$

hence  $R_2 < +\infty$ . This proves Theorem 3.1.  $\diamond$

**Remarks.** One notices that the choice of vector norm  $\|\cdot\|$  (here Euclidean norm) is essential for the verification of (mean square) dissipativity of absolute norm and initial perturbations of SDEs (for latter property, see below). This fact results from possible changings in application of Dynkin's formula (or Itô formula). Other norms, in general, can lead to different inequalities in Theorem 3.1, and hence to changings in functions (or constants)  $K_1$  and  $K_2$ , or difficulties to find efficient estimations. Thus, for verification of dissipativity, it arises the problem to choose an efficient vector norm under application of Dynkin's formula (or Itô formula). This is where 'real art' of presented qualitative analysis begins. Note that similar effects have been observed in deterministic analysis.

**Theorem 3.2** Assume that  $\mathbb{T} = [t_0, +\infty)$  and coefficients  $a, b^j$  of SDE (3.1) are measurable with respect to time  $t \in \mathbb{T}$  and continuous in  $x \in \mathbb{D} \subseteq \mathbb{R}^d$  where domain  $\mathbb{D}$  is (a.s.) left invariant by this SDE.

Then SDE of type (3.1) has mean square dissipative (i.e.  $p = 2$ ) initial perturbations if there exist locally  $L^1$ -integrable functions  $K_3(t), K_4(t)$  with  $t \in \mathbb{T}$  with respect to Lebesgue measure on  $\mathbb{T}$  such that, for all  $t \in \mathbb{T}$ , for all  $x, y \in \mathbb{D}$ , it holds

$$2 \langle x - y, a(t, x) - a(t, y) \rangle + \sum_{j=1}^m \|b^j(t, x) - b^j(t, y)\|^2 \leq K_3(t) + K_4(t) \cdot \|x - y\|^2 \quad (3.4)$$

where

$$\sup_{t, s \in \mathbb{T}: t \geq s} \int_s^t K_4(u) du + \sup_{t, s \in \mathbb{T}: t \geq s} \left( \int_s^t K_3(u) \exp \left( \int_u^t K_4(v) dv \right) du \right) < +\infty. \quad (3.5)$$

**Proof.** Introduce new stochastic process  $(X_{t_0, X_0}(t), X_{t_0, Y_0}(t))$  on  $\mathbb{R}^{2d}$  where single components satisfy SDE (3.1) with start-values  $X_0 \in \mathbb{R}^d$  and  $Y_0 \in \mathbb{R}^d$ , respectively. Apply once again Dynkin's formula (or Itô formula) to functional

$$f(x, y) = \|x - y\|^2, x, y \in \mathbb{R}^d$$

in order to obtain

$$\begin{aligned} \mathbb{E} \|X_{s, X_s}(t) - X_{s, Y_s}(t)\|^2 &= \mathbb{E} \|X_s - Y_s\|^2 + \\ &+ \int_s^t \mathbb{E} \left[ 2 \langle X_{s, X_s}(u) - X_{s, Y_s}(u), a(u, X_{s, X_s}(u)) - a(u, X_{s, Y_s}(u)) \rangle + \right. \\ &\quad \left. + \sum_{j=1}^m \|b^j(u, X_{s, X_s}(u)) - b^j(u, X_{s, Y_s}(u))\|^2 \right] du, \end{aligned}$$

thanks to assumptions of Theorem 3.2, for all  $t \geq s$  where  $t, s \in \mathbb{T}$ . Now, define the expression  $v(t) := \mathbb{E} \|X_{s, X_s}(t) - X_{s, Y_s}(t)\|^2$ . Under condition (3.4) it follows

$$v(t) \leq v(s) + \int_s^t K_3(u) du + \int_s^t \left[ K_4(u) \cdot \mathbb{E} \|X_{s, X_s}(u) - X_{s, Y_s}(u)\|^2 \right] du.$$

Using generalized Bellman–Gronwall–type estimates (as in Mao [21]) and done in proof before, cf. also section 7 with variation of constants formula) one receives

$$v(t) \leq \left( v(s) + \int_s^t K_3(u) \exp \left( - \int_s^u K_4(v) dv \right) du \right) \cdot \exp \left( \int_s^t K_4(u) du \right)$$

for all  $t \geq s$ , all  $s \in \mathbb{T}$ . Then requirement (3.5) yields mean square dissipative initial perturbations of related SDE with dissipativity constant

$$\hat{R}_2 \leq \sup_{t, s \in \mathbb{T}: t \geq s} \left( \hat{r}_2 + \int_s^t K_3(u) \exp \left( - \int_s^u K_4(v) dv \right) du \right) \cdot \exp \left( \int_s^t K_4(u) du \right),$$

hence  $\hat{R}_2 < +\infty$ . This proves Theorem 3.2.  $\diamond$

**Remark.** The conditions (3.3) and (3.5) of Theorems 3.1 and 3.2 are trivially satisfied if indefinite integrals over positive parts of that characteristics  $K_1, K_2$  and  $K_3, K_4$  of dissipativity exist, respectively.

Let us turn our attention to frequently discussed discrete time stochastic dynamical systems. In the following subsection we only refer to classical (deterministic) time scales of the form  $\mathbb{T} = \{t_i \in \mathbb{R} : t_i < t_{i+1}, i \in \mathbb{N}\}$  satisfying  $\sup_{i \in \mathbb{N}} |t_{i+1} - t_i| < +\infty$  and related discrete time systems which leave some deterministic domain  $\mathbb{D} \subseteq \mathbb{R}^d$  invariant.

### 3.2 Dissipativity of some discrete time systems

The previously discussed continuous time stochastic systems have a large variety of counterparts under discretization. For examples, see [12], [19], [22] - [24], [28], [30] - [35] and [39]. We are especially interested in qualitative behaviour of the simplest and most used numerical methods as integration time tends to infinity. The simplest numerical methods are performed by the **family of implicit Euler methods**. Their schemes applied to system (3.1) with current deterministic step size  $\Delta_n = t_{n+1} - t_n > 0$  and (bounded) deterministic, real implicitness  $\alpha_n$  have the implicit form

$$X_{n+1} = X_n + [\alpha_n a(t_{n+1}, X_{n+1}) + (1 - \alpha_n) a(t_n, X_n)]\Delta_n + \sqrt{\Delta_n} \sum_{j=1}^m b^j(t_n, X_n) \xi_n^j \quad (3.6)$$

where independent scalar random variables  $\xi_n^j$ , for example

$$\xi_n^j = (W^j(t_{n+1}) - W^j(t_n))/\sqrt{\Delta_n}, \quad (3.7)$$

can be considered as centered, identically distributed with normalized second moments, along given time-discretization  $-\infty < t_0 < t_1 < \dots < t_n < t_{n+1} < \dots$  (i.e. on related discrete time scale). Well-known members of this family are (**explicit Euler** (i.e.  $\alpha_n \equiv 0$ ), **implicit trapezoidal** (i.e.  $\alpha_n \equiv 0.5$ , sometimes called *improved Euler method* which is also identical with *midpoint method* in linear autonomous case) and **implicit Euler method** (i.e.  $\alpha_n \equiv 1$ ). For sake of identification, we also term the latter method as **fully drift-implicit Euler method**. All these methods belong to the more general classes of **stochastic  $\theta$ -methods** (for introduction, see [28]) and **stochastic Runge-Kutta methods** (see [19] or [39]). Note that most of the schemes of these classes only take into account an incorporation of implicitness in those terms carried by drift part of underlying continuous time dynamics. For our purposes, such type of implicitness shall be sufficient. An appropriate incorporation of stochastic-implicit terms is fairly complicated without changing stochastic calculus in the presence of multiplicative noise. For such a treatment, see [22]. Besides 'stochastic implicitness' shall not be necessary for the purpose of our considerations. This fact also follows from papers [30], [32]. Thereby we do not need to consider such representatives here.

The class of implicit Euler methods (3.6) while using random variables (3.7) provides numerically mean square converging solutions to SDEs (3.1) with convergence order (rate)  $\gamma = 1.0$ , under natural assumptions of growth-boundedness and Lipschitz continuity of  $a^j \equiv b^j$  in space coordinate. For the class of in the wide sense mean square dissipative and contractive stochastic systems of Itô SDEs, one receives the same result for the fully drift-implicit Euler methods at least. For showing this fact one has to verify that related discretization is dissipative, nonlinearly stable and contractive in wide sense too.

In this subsection, for sake of simplicity, we commence with analysis of *implicit Euler method* with  $\alpha_n \equiv 1$  (i.e. the fully drift-implicit Euler method) with respect to dissipativity. Other members of family of implicit Euler methods could be treated in a similar way. One arrives at the following result concerning mean square dissipativity



of the fully drift-implicit Euler method. Set  $\sum_{k=n+1}^n c_k = 0$ , no matter what  $c_k$  is to be.

**Theorem 3.3** *Assume that  $\mathbb{T} = [t_0, +\infty)$  and coefficients  $a, b^j$  of SDE (3.1) are measurable with respect to time  $t \in \mathbb{T}$  and continuous in  $x \in \mathbb{D} \subseteq \mathbb{R}^d$ . Furthermore, let domain  $\mathbb{D}$  be left invariant (a.s.) by implicit Euler method (3.6) with  $\alpha_n \equiv 1$ .*

*Then the implicit Euler method (3.6) with  $\alpha_n \equiv 1$  is mean square dissipative (i.e.  $p = 2$ ) if there exist locally  $L^1$ -integrable functions  $K_1^a(t), K_2^a(t), K_1^b(t), K_2^b(t)$  with  $t \in \mathbb{T}$  with respect to Lebesgue measure on  $\mathbb{T}$  such that, for all  $x \in \mathbb{D}$ , it holds*

$$\langle x, a(t, x) \rangle \leq K_1^a(t) + K_2^a(t) \cdot \|x\|^2 \quad \text{and} \quad (3.8)$$

$$\sum_{j=1}^m \|b^j(t, x)\|^2 \leq K_1^b(t) + K_2^b(t) \cdot \|x\|^2 \quad (3.9)$$

where related coefficients

$$C_1(t_n, t_{n+1}) := \frac{[2K_1^a(t_{n+1}) + K_1^b(t_n)]}{[1 - 2K_2^a(t_{n+1})(t_{n+1} - t_n)]} (t_{n+1} - t_n) \quad \text{and}$$

$$C_2(t_n, t_{n+1}) := \frac{[2K_2^a(t_{n+1}) + K_2^b(t_n)]}{[1 - 2K_2^a(t_{n+1})(t_{n+1} - t_n)]} (t_{n+1} - t_n)$$

satisfy  $1 - 2K_2^a(t_{n+1})(t_{n+1} - t_n) > 0$  for all  $n \in \mathbb{N}$  and

$$\sup_{n, k \in \mathbb{N}: n \geq k} \sum_{l=k}^n C_2(t_l, t_{l+1}) < +\infty, \quad (3.10)$$

$$\sup_{n, k \in \mathbb{N}: n \geq k} \sum_{l=k}^n C_1(t_l, t_{l+1}) \cdot \exp\left(\sum_{r=l+1}^n C_2(t_r, t_{r+1})\right) < +\infty.$$

These conditions on coefficients of fully drift-implicit Euler method yield the estimate

$$\begin{aligned} \limsup_{n \in \mathbb{N}} \mathbb{E} \|X_{n+1}\|^2 &\leq R_2 \leq r_2 \cdot \exp\left(\sup_{n, k \in \mathbb{N}: n \geq k} \sum_{l=k}^n C_2(t_l, t_{l+1})\right) + \\ &+ \sup_{n, k \in \mathbb{N}: n \geq k} \sum_{l=k}^n C_1(t_l, t_{l+1}) \cdot \exp\left(\sum_{r=l+1}^n C_2(t_r, t_{r+1})\right) < +\infty. \end{aligned}$$

**Proof.** First, rearrange the scheme (3.6) of implicit Euler methods to

$$X_{n+1} - \alpha_n \Delta_n a(t_{n+1}, X_{n+1}) = X_n + (1 - \alpha_n) a(t_n, X_n) \Delta_n + \sqrt{\Delta_n} \sum_{j=1}^m b^j(t_n, X_n) \xi_n^j.$$

Square the rearranged scheme with respect to Euclidean vector norm. After taking expectation on both sides one gets

$$\mathbb{E} \|X_{n+1}\|^2 - 2\alpha_n \Delta_n \mathbb{E} \langle X_{n+1}, a(t_{n+1}, X_{n+1}) \rangle + \alpha_n^2 \Delta_n^2 \mathbb{E} \|a(t_{n+1}, X_{n+1})\|^2$$

$$\begin{aligned}
&= \mathbb{E} \|X_n\|^2 + 2(1 - \alpha_n)\Delta_n \mathbb{E} \langle X_n, a(t_n, X_n) \rangle + (1 - \alpha_n)^2 \Delta_n^2 \mathbb{E} \|a(t_n, X_n)\|^2 + \\
&\quad + \Delta_n \sum_{j=1}^m \mathbb{E} \|b^j(t_n, X_n)\|^2.
\end{aligned}$$

Now, take  $\alpha_n \equiv 1$  and estimate the right hand side by using requirements (3.8) and (3.9). By this procedure one receives

$$\begin{aligned}
v_{n+1} &:= \mathbb{E} \|X_{n+1}\|^2 \\
&\leq \mathbb{E} \|X_n\|^2 + \Delta_n \mathbb{E} \left[ 2 \langle X_{n+1}, a(t_{n+1}, X_{n+1}) \rangle + \sum_{j=1}^m \|b^j(t_n, X_n)\|^2 \right] \\
&\leq \frac{[1 + K_2^b(t_n)\Delta_n]}{[1 - 2K_2^a(t_{n+1})\Delta_n]} v_n + \frac{[2K_1^a(t_{n+1}) + K_1^b(t_n)]}{[1 - 2K_2^a(t_{n+1})\Delta_n]} \Delta_n \text{ if } 1 - 2K_2^a(t_{n+1})\Delta_n > 0.
\end{aligned}$$

Now, apply auxiliary Lemma 7.1 (i.e. discrete version of variation of constants formula from section 7) and Corollary 7.13 in order to obtain claimed assertions. This proves Theorem 3.3.  $\diamond$

**Remarks.** If positive parts of occurring coefficients  $C_1$  and  $C_2$  are zero then one even receives mean square dissipativity of the state process with

$$\limsup_{n \in \mathbb{N}} v_n \leq r_2 < +\infty.$$

Analogous results one can find for **autonomously drift-implicit Euler methods** with scheme

$$X_{n+1} = X_n + [\alpha_n a(t_n, X_{n+1}) + (1 - \alpha_n) a(t_n, X_n)]\Delta_n + \sqrt{\Delta_n} \sum_{j=1}^m b^j(t_n, X_n) \xi_n^j \quad (3.11)$$

which only differs from method (3.6) in nonautonomous situation. In special case  $\alpha_n \equiv 1$  we also term this method as **autonomously fully drift-implicit Euler method**. For analysis of this method (as in Theorem 3.3), one has to substitute coefficients  $C_1, C_2$  in Theorem 3.3 by

$$\begin{aligned}
\hat{C}_1(t_n, t_{n+1}) &:= \frac{[2K_1^a(t_n) + K_1^b(t_n)]}{[1 - 2K_2^a(t_n)(t_{n+1} - t_n)]} (t_{n+1} - t_n) \quad \text{and} \\
\hat{C}_2(t_n, t_{n+1}) &:= \frac{[2K_2^a(t_n) + K_2^b(t_n)]}{[1 - 2K_2^a(t_n)(t_{n+1} - t_n)]} (t_{n+1} - t_n)
\end{aligned}$$

while requiring  $1 - 2K_2^a(t_n)\Delta_n > 0$  for all  $n \in \mathbb{N}$ . A similar procedure can be carried out for (nonautonomously) **drift-implicit midpoint method** with scheme

$$X_{n+1} = X_n + [a(\frac{t_{n+1} + t_n}{2}, \frac{X_{n+1} + X_n}{2})]\Delta_n + \sqrt{\Delta_n} \sum_{j=1}^m b^j(t_n, X_n) \xi_n^j, \quad (3.12)$$

for (nonautonomously) **fully Itô-implicit midpoint method** with scheme

$$X_{n+1} = X_n + [a(\frac{t_{n+1} + t_n}{2}, \frac{X_{n+1} + X_n}{2})]\Delta_n + \sqrt{\Delta_n} \sum_{j=1}^m b^j(\frac{t_{n+1} + t_n}{2}, X_n) \xi_n^j \quad (3.13)$$

or for semi-implicit, linear-implicit methods and their ‘autonomously implicit derivatives’, etc., as well as for other sequences of implicitness  $(\alpha_n)_{n \in \mathbb{N}}$ . These refinements are left to the interest of readership. It is worth noting that numerical methods (3.12) and (3.13) coincide with implicit trapezoidal method in linear case with time-independent coefficient  $a = a(x)$ , e.g.  $a(x) = a^0 + Ax$  with constant vector  $a^0$  and matrix  $A$  of underlying SDE.

To complete dissipativity analysis of fully drift-implicit Euler methods we continue with investigation with respect to propagation of their initial perturbations. In stating assertions for discrete time dynamical systems,  $X_n, Y_n \in \mathbb{D}$  denote the values of related dynamical system at discrete time  $t_n \in \mathbb{T}$ , started at values  $X_0, Y_0$ , respectively, at initial time  $t_0 \in \mathbb{T}$ .

**Theorem 3.4** *Assume that  $\mathbb{T} = [t_0, +\infty)$  and coefficients  $a, b^j$  of SDE (3.1) are measurable with respect to time  $t \in \mathbb{T}$  and continuous in  $x \in \mathbb{D} \subseteq \mathbb{R}^d$ . Furthermore, let domain  $\mathbb{D}$  be left invariant (a.s.) by implicit Euler method (3.6) with  $\alpha_n \equiv 1$ .*

*Then the implicit Euler method (3.6) with  $\alpha_n \equiv 1$  has mean square dissipative initial perturbations (i.e.  $p = 2$ ) if there exist locally  $L^1$ -integrable functions  $K_3^a(t), K_4^a(t), K_3^b(t), K_4^b(t)$  with  $t \in \mathbb{T}$  with respect to Lebesgue measure on  $\mathbb{T}$  such that, for all  $x, y \in \mathbb{D}$ , it holds*

$$\langle x - y, a(t, x) - a(t, y) \rangle \leq K_3^a(t) + K_4^a(t) \cdot \|x - y\|^2 \quad \text{and} \quad (3.14)$$

$$\sum_{j=1}^m \|b^j(t, x) - b^j(t, y)\|^2 \leq K_3^b(t) + K_4^b(t) \cdot \|x - y\|^2 \quad (3.15)$$

where related coefficients

$$C_3(t_n, t_{n+1}) := \frac{[2K_3^a(t_{n+1}) + K_3^b(t_n)]}{[1 - 2K_4^a(t_{n+1})(t_{n+1} - t_n)]} (t_{n+1} - t_n) \quad \text{and}$$

$$C_4(t_n, t_{n+1}) := \frac{[2K_4^a(t_{n+1}) + K_4^b(t_n)]}{[1 - 2K_4^a(t_{n+1})(t_{n+1} - t_n)]} (t_{n+1} - t_n)$$

satisfy  $1 - 2K_4^a(t_{n+1})(t_{n+1} - t_n) > 0$  for all  $n \in \mathbb{N}$  and

$$\sup_{n, k \in \mathbb{N}: n \geq k} \sum_{l=k}^n C_4(t_l, t_{l+1}) < +\infty, \quad (3.16)$$

$$\sup_{n, k \in \mathbb{N}: n \geq k} \sum_{l=k}^n C_3(t_l, t_{l+1}) \cdot \exp\left(\sum_{r=l+1}^n C_4(t_r, t_{r+1})\right) < +\infty.$$

These conditions on coefficients of fully drift-implicit Euler method yield the estimate

$$\begin{aligned} \limsup_{n \in \mathbb{N}} \mathbb{E} \|X_{n+1} - Y_{n+1}\|^2 &\leq \hat{R}_2 \leq \hat{r}_2 \cdot \exp\left(\sup_{n, k \in \mathbb{N}: n \geq k} \sum_{l=k}^n C_4(t_l, t_{l+1})\right) + \\ &+ \sup_{n, k \in \mathbb{N}: n \geq k} \sum_{l=k}^n C_3(t_l, t_{l+1}) \cdot \exp\left(\sum_{r=l+1}^n C_4(t_r, t_{r+1})\right), \end{aligned}$$

hence  $\hat{R}_2 < +\infty$ .

**Proof.** In view of a fair comparison with respect to dissipativity of initial perturbations, let numerical schemes of  $(X_n)_{n \in \mathbb{N}}$  and  $(Y_n)_{n \in \mathbb{N}}$  be constructed on a common probability space. First, rearrange the scheme (3.6) of implicit Euler methods to

$$X_{n+1} - \alpha_n \Delta_n a(t_{n+1}, X_{n+1}) = X_n + (1 - \alpha_n) a(t_n, X_n) \Delta_n + \sqrt{\Delta_n} \sum_{j=1}^m b^j(t_n, X_n) \xi_n^j.$$

Consider this scheme for different start-values  $X_0, Y_0$ , respectively. Square the difference of related schemes for  $X_n, Y_n$  with respect to Euclidean vector norm while using the same sequences of step sizes  $\Delta_n$  and implicitnesses  $\alpha_n$ . After taking expectation on both sides one gets

$$\begin{aligned} & \mathbb{E} \|X_{n+1} - Y_{n+1}\|^2 - 2\alpha_n \Delta_n \mathbb{E} \langle X_{n+1} - Y_{n+1}, a(t_{n+1}, X_{n+1}) - a(t_{n+1}, Y_{n+1}) \rangle + \\ & \quad + \alpha_n^2 \Delta_n^2 \mathbb{E} \|a(t_{n+1}, X_{n+1}) - a(t_{n+1}, Y_{n+1})\|^2 \\ = & \mathbb{E} \|X_n - Y_n\|^2 + 2(1 - \alpha_n) \Delta_n \mathbb{E} \langle X_n - Y_n, a(t_n, X_n) - a(t_n, Y_n) \rangle + \\ & \quad + (1 - \alpha_n)^2 \Delta_n^2 \mathbb{E} \|a(t_n, X_n) - a(t_n, Y_n)\|^2 + \Delta_n \sum_{j=1}^m \mathbb{E} \|b^j(t_n, X_n) - b^j(t_n, Y_n)\|^2. \end{aligned}$$

Now, take  $\alpha_n \equiv 1$  and estimate the right hand side by using requirements (3.14) and (3.15). By this procedure one receives

$$\begin{aligned} \hat{v}_{n+1} & := \mathbb{E} \|X_{n+1} - Y_{n+1}\|^2 \\ & \leq \mathbb{E} \|X_n - Y_n\|^2 + \Delta_n \mathbb{E} \left[ 2 \langle X_{n+1} - Y_{n+1}, a(t_{n+1}, X_{n+1}) - a(t_{n+1}, Y_{n+1}) \rangle \right. \\ & \quad \left. + \sum_{j=1}^m \|b^j(t_n, X_n) - b^j(t_n, Y_n)\|^2 \right] \\ & \leq \frac{[1 + K_4^b(t_n) \Delta_n]}{[1 - 2K_4^a(t_{n+1}) \Delta_n]} v_n + \frac{[2K_3^a(t_{n+1}) + K_3^b(t_n)]}{[1 - 2K_4^a(t_{n+1}) \Delta_n]} \Delta_n \text{ if } 1 - 2K_4^a(t_{n+1}) \Delta_n > 0. \end{aligned}$$

Now apply auxiliary Lemma 7.1 (i.e. discrete version of variation of constants formula from section 7) and Corollary 7.13 in order to obtain claimed assertions. This proves Theorem 3.4.  $\diamond$

**Remarks.** If positive parts of occurring coefficients  $C_3$  and  $C_4$  are zero then one even receives dissipativity of initial perturbations with

$$\limsup_{n \in \mathbb{N}} \hat{v}_n \leq \hat{r}_2 < +\infty$$

for fully drift-implicit Euler method. Analogous results one can find for autonomously fully drift-implicit Euler method with scheme (3.11). For this purpose one has to substitute coefficients  $C_3, C_4$  in Theorem 3.4 by

$$\begin{aligned} \hat{C}_3(t_n, t_{n+1}) & := \frac{[2K_3^a(t_n) + K_3^b(t_n)]}{[1 - 2K_4^a(t_n)(t_{n+1} - t_n)]} (t_{n+1} - t_n) \quad \text{and} \\ \hat{C}_4(t_n, t_{n+1}) & := \frac{[2K_4^a(t_n) + K_4^b(t_n)]}{[1 - 2K_4^a(t_n)(t_{n+1} - t_n)]} (t_{n+1} - t_n) \end{aligned}$$

while requiring  $1 - 2K_4^a(t_n)\Delta_n > 0$  for all  $n \in \mathbb{N}$ . A similar procedure can be carried out for other (nonautonomous) implicit methods (like midpoint method) or for other sequences of implicitness  $(\alpha_n)_{n \in \mathbb{N}}$  (but this is left to the reader).

#### 4. FURTHER CLASSIFICATION OF DISSIPATIVE SYSTEMS

The main attention in the further exposition is drawn to the case of time-independent characteristics  $K_i$  and the following classes of SDEs. However, we emphasize that this restriction is made only for convenience and clearness of main ideas.

##### 4.1 Linearly growth-bounded coefficient systems

**Definition 4.3** *A stochastic process governed by SDE (3.1) as well as its SDE is said to have an (uniformly) mean square linearly growth-bounded coefficient system  $(a, b^j)$  on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  on any finite time-interval iff there exist finite, time-independent real constants  $K_1^a, K_2^a, K_1^b, K_2^b$  such that*

$$\langle x, a(t, x) \rangle \leq K_1^a + K_2^a \cdot \|x\|^2 \quad (4.1)$$

$$\sum_{j=1}^m \|b^j(t, x)\|^2 \leq K_1^b + K_2^b \cdot \|x\|^2 \quad (4.2)$$

for all  $t \in \mathbb{R}$ , for all  $x \in \mathbb{D} \subseteq \mathbb{R}^d$ .

Additionally, if

$$2K_1^a + K_1^b \leq 0 \quad \text{and} \quad 2K_2^a + K_2^b \leq 0$$

the mean square linearly growth-bounded coefficient system  $(a, b^j)$  is called mean square dissipative in narrow sense, otherwise mean square dissipative in wide sense. An in the narrow sense mean square dissipative coefficient system  $(a, b^j)$  is said to be strictly (uniformly) mean square dissipative if  $2K_2^a + K_2^b < 0$ .

**Remarks** (Existence of solutions, dissipativity, stationarity). Such class of processes guarantees global existence of mean square solutions for SDEs. In case of additive noise (state-independent diffusion parts  $b^j$ ), it additionally implies global existence of strong solutions for SDEs in general (cf. [35]). The growth-boundedness (more general for p-th mean) in the sense of this definition allows to find a weighted norm of the form

$$\|X\|_{\mathbb{T}, \varepsilon} = \sup_{t \in \mathbb{T}} \left[ \exp \left( - (2|K_2^a| + K_2^b + \varepsilon) \cdot (t - t_0) \right) \cdot \mathbb{E} \|X_{t_0, X_0}(t)\|^2 \right]$$

such that the original dynamical system gets the property of (p-th mean) dissipativity with respect to this new norm for all  $\varepsilon > 0$ . Besides, if  $\varepsilon = 0$  and  $2K_1^a + K_1^b \leq 0$  then one receives at least dissipativity constant  $R_2 \leq r_2$ . In case of time-independent

coefficient system  $(a, b^j)$  with  $K_2^a < 0$  one may apply Khas'minskij's stationarity criterion which also gives the existence of a stationary Markovian solution (see [18]). If  $2K_2^a + K_2^b < 0$  then one can establish the property of **global dissipativity**.

**Corollary 4.1** *Let the assumptions of Theorem 3.1 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is mean square linearly growth-bounded on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_1^a, K_2^a, K_1^b, K_2^b$  satisfying  $2K_2^a + K_2^b < 0$ .*

*Then the stochastic process  $X_{t_0, X_0}(t)$  governed by SDE (3.1) is (global) mean square dissipative on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with*

$$R_2 \leq -\frac{[2K_1^a + K_1^b]_+}{2K_2^a + K_2^b} \quad (4.3)$$

where  $[\cdot]_+$  represents the positive part of inscribed expression.

For the one-dimensional Ornstein–Uhlenbeck process  $X_t \in \mathbb{R}^1$  (see [1]) satisfying SDE

$$dX_t = -cX_t dt + \sigma dW_t$$

where parameter  $c > 0$ , and  $W_t$  ( $t \in \mathbb{R}^1$ ) is a standard one-dimensional Wiener process, this estimate (4.3) turns out to be sharp with  $R_2 = \sigma^2/2c$ .

**Proof.** Plug in time-independent values  $K_1 = [2K_1^a + K_1^b]_+$  and  $K_2 = 2K_2^a + K_2^b$  into conclusion of Theorem 3.1. Then one receives the mean square dissipativity with claimed (4.3). For Ornstein–Uhlenbeck Process  $X_t$ , take  $K_1^a = 0, K_2^a = -c, K_1^b = \sigma^2, K_2^b = 0$ .  $\diamond$

**Remark.** For linear, homogeneous, one-dimensional SDE driven by multiplicative White noise with  $\sigma^2 - 2c < 0$ , (global) mean square dissipativity reduces to (global) asymptotical mean square stability (i.e.  $R_2 = 0$ ).

## 4.2 A- and AN-dissipativity

Let us classify dissipative numerical methods in view of dissipative SDEs to be solved. For this purpose we consider the generated sequence of discrete time values  $X_n$  as stochastic dynamical system on countable, discrete time scale as discretization of underlying continuous time scale. Recall that one speaks of an **autonomous SDE** if related coefficient system  $(a(x), b^j(x))$  does not depend on time  $t \in \mathbb{T}$ , otherwise **nonautonomous SDE**.

**Definition 4.4** *A numerical method  $(X_n)_{n \in \mathbb{N}}$  applied to SDEs (3.1) with step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  is called **mean square AN-dissipative** iff it holds*

$$\forall n \in \mathbb{N} : \mathbb{E} \|X_{n+1}\|^2 \leq \mathbb{E} \|X_n\|^2 \quad (4.4)$$

*for all nonautonomous SDEs (3.1) with in the narrow sense mean square dissipative coefficient systems  $(a(t, x), b^j(t, x))$  and for any step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  with*

$\sum_{k=0}^{+\infty} \Delta_k = +\infty$ . A numerical method  $(X_n)_{n \in \mathbb{N}}$  applied to SDEs (3.1) with step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  is said to be **mean square A-dissipative** iff relation (4.4) holds for all autonomous SDEs (3.1) with in the narrow sense mean square dissipative coefficient systems  $(a(x), b^j(x))$  and for any step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  with  $\sum_{k=0}^{+\infty} \Delta_k = +\infty$ .

**Remark.** In an analogous way one might think of the notion of  $p$ -th mean  $A(N)$ -dissipativity of numerical methods or of parametrized dynamical systems.

**Theorem 4.5** *The autonomously fully drift-implicit Euler method with (3.11) is mean square AN-dissipative. Fully drift-implicit Euler methods are mean square A-dissipative.*

**Proof.** Consider autonomously drift-implicit Euler method (3.11) with  $\alpha_n \equiv 1$  and return to estimation in proof of Theorem 3.3 with coefficients  $\hat{C}_1 \leq 0, \hat{C}_2 \leq 0$  corresponding to autonomously fully drift-implicit Euler method. For in the narrow sense dissipative coefficient systems, estimation in that proof reduces here to relation (4.4). As this relation holds for all sequences of step sizes the first assertion is proven. The second assertion is verified in a similar way. This completes the proof of Theorem 4.5.  $\diamond$

**Remark.** Other methods could be examined with respect to  $A(N)$ -dissipativity in an analogous way. However, this examination is omitted here. It is clear that there exist simple examples of numerical methods which are not  $A(N)$ -dissipative (Consider explicit methods with step size large enough and linear systems, cf. Schurz [30]).

### 4.3 Monotone coefficient systems

To examine qualitative behaviour of stochastic dynamics in view of initial perturbations and asymptotic stability, we introduce a further definition motivated by the notion of *monotone functions* in mathematical treatment of partial differential equations (PDEs).

**Definition 4.5** *A stochastic process governed by SDE (3.1) as well as its SDE have an (uniformly) mean square monotone coefficient system  $(a, b^j)$  in the wide sense on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  on any finite time-interval iff there exist finite, real constants  $K_3^a, K_4^a, K_3^b, K_4^b$  such that*

$$\langle x - y, a(t, x) - a(t, y) \rangle \leq K_3^a + K_4^a \cdot \|x - y\|^2 \quad (4.5)$$

$$\sum_{j=1}^m \|b^j(t, x) - b^j(t, y)\|^2 \leq K_3^b + K_4^b \cdot \|x - y\|^2 \quad (4.6)$$

for all  $t \in \mathbb{R}$ , for all  $x, y \in \mathbb{D} \subseteq \mathbb{R}^d$ .

Additionally, the monotone coefficient system  $(a, b^j)$  is called (uniformly) mean square nonexploding if  $2K_4^a + K_4^b \leq 0$ , (uniformly) mean square monotone nonincreasing if  $2K_3^a + K_3^b \leq 0, 2K_4^a + K_4^b \leq 0$  and strictly (uniformly) mean square monotone decreasing if  $2K_4^a + K_4^b < 0$  and  $2K_3^a + K_3^b \leq 0$ .

**Remarks** (Perturbation analysis). The first property with  $2K_3^a + K_3^b \leq 0$  yields mean square uniqueness of SDEs in general, and strong uniqueness of SDEs in case of additive noise on any finite time-interval (i.e. *global uniqueness*). The latter two properties even imply some uniform control on mean square initial perturbations, i.e. that they are not ‘mean square expansive’ or even ‘mean square contractive’ on any finite time-interval.

**Corollary 4.2** *Let the assumptions of Theorem 3.2 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is mean square monotone in the wide sense with real constants  $K_3^a, K_4^a, K_3^b, K_4^b$  and  $2K_4^a + K_4^b < 0$ .*

*Then the stochastic process  $X_{t_0, X_0}(t)$  governed by SDE (3.1) has (global) mean square dissipative initial perturbations with*

$$\hat{R}_2 \leq -\frac{[2K_3^a + K_3^b]_+}{2K_4^a + K_4^b} \quad (4.7)$$

where  $[\cdot]_+$  represents the positive part of inscribed expression.

For the one-dimensional Ornstein–Uhlenbeck process  $X_t \in \mathbb{R}^1$  (see [1]) satisfying SDE

$$dX_t = -cX_t dt + \sigma dW_t$$

where parameter  $c > 0$ , and  $W_t$  ( $t \in \mathbb{R}^1$ ) is a standard one-dimensional Wiener process, this estimate (4.7) turns out to be sharp with  $\hat{R}_2 = 0$ .

**Proof.** Plug in time-independent values  $K_3 = [2K_3^a + K_3^b]_+$  and  $K_4 = 2K_4^a + K_4^b$  into conclusion of Theorem 3.2. Then one receives the mean square dissipativity with claimed (4.7). For Ornstein–Uhlenbeck Process  $X_t$ , take  $K_3^a = 0, K_4^a = -c, K_3^b = 0, K_4^b = 0$ .  $\diamond$

**Remark.** For linear, homogeneous, autonomous, one-dimensional SDE driven by multiplicative White noise with  $\sigma^2 - 2c \leq 0$ , mean square dissipativity of initial perturbations follows from mean square stability of trivial solution (i.e.  $\hat{R}_2 = 0$  in case of asymptotical stability). Hence, we have seen that, for bilinear SDEs with multiplicative noise, the concepts of mean square dissipativity can coincide. However, for example, these concepts differ when linear dynamical systems with additive noise are considered. An illustrative example is given by  $d$ -dimensional, real-valued SDEs

$$dX_t = \sum_{j=1}^m b^j(t) dW_t^j \quad \text{with} \quad \sum_{j=1}^m \int_{t_0}^{+\infty} \|b^j(u)\|^2 du = +\infty \quad (4.8)$$

where  $W_t^j$  are independent, standard Wiener processes, and  $t \geq t_0$ . These stochastic processes  $X_t$  possess an exploding mean square evolution. In contrast to that, one notices a constant propagation of initial perturbations with

$$0 \leq \mathbb{E} \|X_{t_0, X_0}(t) - X_{t_0, Y_0}(t)\|^2 = \mathbb{E} \|X_0 - Y_0\|^2 < \hat{r}_2 =: \hat{R}_2$$

as time  $t$  advances. In that case (asymptotical) mean square stability does not hold, but mean square dissipativity of initial perturbations does. This observation can also be found for systems with inhomogeneities in principle (thus also for ODEs).



#### 4.4 B- and BN-dissipativity

Numerical methods applied to SDEs with monotone coefficient systems  $(a, b^j)$  can be classified further. In stating definition and assertions below, we suppose that  $X_n$  and  $Y_n$  denote the value of one and the same numerical method with start-values  $X_0$  and  $Y_0$ , respectively. We also assume that their related second moments globally exist.

**Definition 4.6** *A numerical method  $(X_n)_{n \in \mathbb{N}}$  applied to SDEs (3.1) with the step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  is called **mean square BN-dissipative** iff it holds*

$$\forall n \in \mathbb{N} : \mathbb{E} \|X_{n+1} - Y_{n+1}\|^2 \leq \mathbb{E} \|X_n - Y_n\|^2 \quad (4.9)$$

for all nonautonomous SDEs (3.1) with mean square monotone nonincreasing coefficient systems  $(a(t, x), b^j(t, x))$  and for any sequences  $(\Delta_n)_{n \in \mathbb{N}}$  with  $\sum_{k=0}^{+\infty} \Delta_k = +\infty$ . A numerical method  $(X_n)_{n \in \mathbb{N}}$  applied to SDEs (3.1) with the step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  is said to be **mean square B-dissipative** iff relation (4.9) holds for all autonomous SDEs (3.1) with mean square monotone nonincreasing coefficient systems  $(a(x), b^j(x))$  and for any sequences  $(\Delta_n)_{n \in \mathbb{N}}$  with  $\sum_{k=0}^{+\infty} \Delta_k = +\infty$ .

**Remark.** In an analogous way one might introduce the notion of *p-th mean B(N)-dissipativity* of numerical methods or more general of parametrized dynamical systems.

**Theorem 4.6** *The autonomously fully drift-implicit Euler method with (3.11) is mean square BN-dissipative. Fully drift-implicit Euler methods are mean square B-dissipative.*

**Proof.** Consider autonomously drift-implicit Euler method (3.11) with  $\alpha_n \equiv 1$  and return to estimation in proof of Theorem 3.4 with coefficients  $\hat{C}_3, \hat{C}_4$  corresponding to autonomously fully drift-implicit Euler method. For monotone nonincreasing coefficient systems, estimation in that proof reduces here to relation (4.9). As this relation holds for all sequences of step sizes the first assertion is proven. The second assertion is verified in a similar way. This completes the proof of Theorem 4.6.  $\diamond$

**Remark.** Other methods could be examined with respect to B(N)-dissipativity in an analogous way. However, this examination is omitted here. It is clear that there exist simple examples of numerical methods which are not B(N)-dissipative (Consider explicit methods with step size large enough and linear systems).

Let us reconsider dissipative stochastic dynamical systems with uniform mean square growth-bounded and monotone coefficient systems, respectively.

#### 4.5 Estimates for growth-bounded coefficient systems

Consider stochastic dynamical systems with mean square growth-bounded coefficient systems  $(a, b^j)$  on deterministic, traditional time scales  $\mathbb{T} = [t_{min}, +\infty)$  or

its discretization  $\mathbb{T} = \{t_n : t_n < t_{n+1}, t_n \in [t_{min}, +\infty)\}$  where  $t_{min} > -\infty$ . Then the following two basic assertions concerning the temporal mean square evolution in nonlinear dynamical systems come up (in p–th mean calculus analogous assertions). These assertions are natural implications of Theorems 3.1 and 3.3, respectively, under additional assumptions of mean square growth–boundedness on related coefficient systems  $(a, b^j)$ . Therefore there is no extra need to prove these statements.

**Corollary 4.3** *Let the assumptions of Theorem 3.1 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is (uniformly) mean square linearly growth–bounded on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_1^a, K_2^a, K_1^b, K_2^b$ .*

*Then the stochastic process  $X_{s, X_s}(t)$  governed by SDE (3.1) and started at  $\mathcal{F}_s$ –measurable  $X_s \in \mathbb{D}$ , satisfies*

$$\mathbb{E} \|X_{s, X_s}(t)\|^2 \leq \mathbb{E} \|X_s\|^2 \cdot \exp\left(C_2 \cdot (t - s)\right) + C_1 \cdot \frac{\exp\left(C_2 \cdot (t - s)\right) - 1}{C_2} \quad (4.10)$$

for all  $t, s \in \mathbb{T}$  with  $t \geq s$ , where

$$C_1 = 2K_1^a + K_1^b, \quad C_2 = 2K_2^a + K_2^b.$$

**Remarks.** In case of  $C_2 = 0$  we understand the occurring expressions in (4.10) as limit taken  $C_2$  towards zero. The obtained estimate is asymptotically sharp (i.e. as time  $t$  tends to infinity). For verification of latter fact, consider bilinear, one–dimensional SDEs.

Define  $\sum_{k=n+1}^n c_k := 0$ , no matter what  $c_k$  is to be.

**Corollary 4.4** *Let the assumptions of Theorem 3.3 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is (uniformly) mean square linearly growth–bounded on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_1^a, K_2^a, K_1^b, K_2^b$ . Let domain  $\mathbb{D}$  be left invariant by fully drift–implicit Euler method (3.6) and autonomously fully drift–implicit Euler method (3.11), respectively.*

*Then the fully drift–implicit Euler method (3.6) and autonomously fully drift–implicit Euler method (3.11) applied to SDE (3.1) and started at  $\mathcal{F}_{t_n}$ –measurable  $X_n \in \mathbb{D}$ , satisfy*

$$\mathbb{E} \|X_{n+l}\|^2 \leq \mathbb{E} \|X_n\|^2 \cdot \exp\left(\sum_{k=n}^{n+l-1} c_2(k)\right) + \sum_{k=n}^{n+l-1} c_1(k) \cdot \exp\left(\sum_{r=k+1}^{n+l-1} c_2(r)\right) \quad (4.11)$$

for all  $n, l \in \mathbb{N}$  with  $l \geq 1$ , where  $c_1(\cdot)$  and  $c_2(\cdot)$  are coefficient sequences with

$$c_1(k) := C_1(t_k, t_{k+1}) = \frac{[2K_1^a + K_1^b]}{[1 - 2K_2^a \Delta_k]} \Delta_k,$$

$$c_2(k) := C_2(t_k, t_{k+1}) = \frac{[2K_2^a + K_2^b]}{[1 - 2K_2^a \Delta_k]} \Delta_k,$$

$\Delta_k = t_{k+1} - t_k, t_k \in \mathbb{T}$  and  $1 - 2K_2^a \Delta_k > 0$  for all  $k \in \mathbb{N}$ .

**Remark.** The coefficient sequences  $C_1(\cdot, \cdot)$  and  $\hat{C}_1(\cdot, \cdot)$  as well as  $C_2(\cdot, \cdot)$  and  $\hat{C}_2(\cdot, \cdot)$  for fully drift-implicit methods (3.6) and (3.11), respectively, coincide in case of uniformly mean square growth-bounded coefficient systems  $(a, b^j)$ . Thus, both numerical methods have same mean square estimations of their state process.

## 4.6 Propagation of initial perturbations under monotonicity

Consider stochastic dynamical systems with in the wide sense mean square monotone coefficient systems  $(a, b^j)$  on deterministic, traditional time scales  $\mathbb{T} = [t_{min}, +\infty)$  or its discretization  $\mathbb{T} = \{t_n : t_n < t_{n+1}, t_n \in [t_{min}, +\infty)\}$  where  $t_{min} > -\infty$ . Then the following two basic assertions concerning the propagation of initial mean square perturbations / errors in dynamical systems come up within the framework of mean square calculus (in p-th mean calculus analogously). These assertions are natural implications of Theorems 3.2 and 3.4, respectively, under additional assumptions of mean square monotonicity on related coefficient systems  $(a, b^j)$ . Therefore there is no need to prove it extra.

**Corollary 4.5** *Let the assumptions of Theorem 3.2 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is (uniformly) mean square monotone in the wide sense on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_3^a, K_4^a, K_3^b, K_4^b$ .*

*Then the stochastic process  $X_{s, X_s}(t)$  governed by SDE (3.1) and started at  $\mathcal{F}_s$ -measurable  $X_s, Y_s \in \mathbb{D}$ , respectively, satisfies*

$$v(t) \leq v(s) \cdot \exp\left(C_4 \cdot (t - s)\right) + C_3 \cdot \frac{\exp\left(C_4 \cdot (t - s)\right) - 1}{C_4} \quad (4.12)$$

where  $v(t) = \mathbb{E} \|X_{s, X_s}(t) - X_{s, Y_s}(t)\|^2$  for all  $t, s \in \mathbb{T}$  with  $t \geq s$ , and where

$$C_3 = 2K_3^a + K_3^b, \quad C_4 = 2K_4^a + K_4^b.$$

**Remarks.** In case of  $C_4 = 0$  we understand the occurring expressions in (4.12) as limit taken  $C_4$  towards zero. The obtained estimate is asymptotically sharp (i.e. as time  $t$  tends to infinity). For verification of latter fact, consider bilinear, one-dimensional SDEs.

Set again  $\sum_{k=n+1}^n c_k = 0$ , no matter what  $c_k$  is to be.

**Corollary 4.6** *Let the assumptions of Theorem 3.4 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is (uniformly) mean square monotone in the wide sense on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_3^a, K_4^a, K_3^b, K_4^b$ . Let domain  $\mathbb{D}$  be left invariant by fully drift-implicit Euler method (3.6) and autonomously fully drift-implicit Euler method (3.11), respectively.*

*Then the fully drift-implicit Euler method (3.6) and autonomously fully drift-implicit Euler method (3.11) applied to SDE (3.1) and started at  $\mathcal{F}_{t_n}$ -measurable*

$X_n, Y_n \in \mathbb{D}$ , respectively, satisfy

$$\mathbb{E} \|X_{n+l} - Y_{n+l}\|^2 \leq \mathbb{E} \|X_n - Y_n\|^2 \cdot \exp\left(\sum_{k=n}^{n+l-1} c_4(k)\right) + \sum_{k=n}^{n+l-1} c_3(k) \cdot \exp\left(\sum_{r=k+1}^{n+l-1} c_4(r)\right) \quad (4.13)$$

for all  $n, l \in \mathbb{N}$  with  $l \geq 1$ , where  $c_3(\cdot)$  and  $c_4(\cdot)$  are coefficient sequences with

$$c_3(k) := C_3(t_k, t_{k+1}) = \frac{[2K_3^a + K_3^b]}{[1 - 2K_4^a \Delta_k]} \Delta_k,$$

$$c_4(k) := C_4(t_k, t_{k+1}) = \frac{[2K_4^a + K_4^b]}{[1 - 2K_4^a \Delta_k]} \Delta_k,$$

$\Delta_k = t_{k+1} - t_k$ ,  $t_k \in \mathbb{T}$  and  $1 - 2K_4^a \Delta_k > 0$  for all  $k \in \mathbb{N}$ .

**Remark.** The coefficient sequences  $C_3(\cdot, \cdot)$  and  $\hat{C}_3(\cdot, \cdot)$  as well as  $C_4(\cdot, \cdot)$  and  $\hat{C}_4(\cdot, \cdot)$  for fully drift-implicit methods (3.6) and (3.11), respectively, coincide in case of uniformly mean square monotone coefficient systems  $(a, b^j)$ . Thus, both numerical methods have same mean square estimations concerning propagation of initial perturbations (errors).

**Further Remark.** One could continue with reformulation of latter estimates in case of equidistant (i.e. constant step size) or other numerical methods. Then interesting relations involving discrete time series and number theory come up. However, this detailed analysis is omitted here.

## 5. P-TH MEAN STABILITY IN WIDE AND NARROW SENSE

The traditional concepts of asymptotical stability usually refer to an equilibrium (steady state, trivial solution, etc.) of dynamical system to be examined. These concepts are termed as **stability concepts in narrow sense** in this paper. It is of more general nature to describe the asymptotical behaviour of dynamical systems without referring to an equilibrium solution. In concern with asymptotical stability, we are just requiring that the related dynamics should asymptotically be declining to zero. This fairly general concept we are calling **stability concept in wide sense**. The same distinguishing by-names are used for the concept of exponential stability. We will mostly devote the following examinations to the more general approach of stability in the wide sense. Thus, for convenience, we may drop the additional words ‘of trivial solution’, ‘of equilibrium’, of ‘steady state’, etc. as well as ‘in the wide sense’ in general in this section.

In stochastic analysis one has already discussed a plenty of stability concepts. We follow the ‘p-th mean approach’ here, and, in particular, the analysis related to second moment behaviour (i.e. mean square stability with  $p = 2$ ). For this purpose p-th moments of related dynamical system are assumed to exist globally on that

time scale  $\mathbb{T} \subset \mathbb{R}^1$  (i.e. p–th mean dissipativity of state process should be clarified before). In stating definitions below, let  $\mathbb{D} \subseteq \mathbb{R}^d$  be a domain which contains a open centered ball with positive (finite or infinite) radius. We also assume that this domain  $\mathbb{D}$  almost surely contains all image values of considered dynamical systems, provided that it has started at domain  $\mathbb{D}$ .

## 5.1 Asymptotical p–th mean stability on time scales

The following definitions – in analogy to deterministic counterparts – can be introduced.

**Definition 5.7** *A stochastic dynamical system  $\{X_t : t \in \mathbb{T}\} \subseteq \mathbb{D} \subseteq \mathbb{R}^d$  is called **asymptotically p–th mean stable in wide sense** ( $p \in \mathbb{R}^1 \setminus \{0\}$  fixed) iff there exists a finite positive real number  $\varepsilon_p$  such that*

$$\lim_{t \rightarrow +\infty} \mathbb{E} \|X_{t_0, X_0}(t)\|^p = 0 \quad (5.1)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^p < \varepsilon_p.$$

In addition, if constant  $\varepsilon_p$  can be taken as  $\varepsilon_p = +\infty$  an in the wide sense asymptotically p–th mean stable dynamical system is said to be **globally asymptotically p–th mean stable in wide sense**.

A stochastic dynamical system  $\{X_t : t \in \mathbb{T}\} \subseteq \mathbb{D} \subseteq \mathbb{R}^d$  is called **(globally) exponentially p–th mean stable in wide sense** ( $p \in \mathbb{R}^1 \setminus \{0\}$  fixed) iff there exist finite, positive real numbers  $C_1(p), C_2(p) > 0$  such that

$$\mathbb{E} \|X_{t_0, X_0}(t)\|^p \leq C_1(p) \cdot \mathbb{E} \|X_0\|^p \cdot \exp\left(-C_2(p)(t - t_0)\right) \quad (5.2)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^p < +\infty.$$

**Remarks.** Since the introduction of concept of uniform p–th mean stability seems to be a very strong requirement (i.e. contrary to nature of unbounded random noise), we do not pursue the related approach and analysis here. For illustrative effects in this respect, however, consider bilinear stochastic differential equations with time-independent, purely additive White noise. We rather want to study stochastic systems with respect to property of p–th mean stability (5.1), in particular when  $p = 2$ , here. In case of  $p = 2$  we also term related property as **asymptotical mean square stability**. As the (stochastic) dynamical system itself, we call *the generating mapping, flow, equation, method, scheme, solution as asymptotically mean square stable* (in the wide sense as above) ones. Of course, the requirement of exponential p–th mean stability is much stronger than that of asymptotical p–th mean stability as in deterministic analysis.

## 5.2 Exponential stability for nonlinear SDEs in wide sense

Confine to continuous time SDEs for simplicity of first considerations.

**Corollary 5.7** *Let the assumptions of Theorem 3.1 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is strictly (uniformly) mean square dissipative on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_1^a, K_2^a, K_1^b, K_2^b$ .*

*Then the stochastic process  $X_{t_0, X_0}(t)$  governed by SDE (3.1) is (globally) exponentially mean square stable with*

$$C_1(2) = 1.0 \quad \text{and} \quad C_2(2) \leq -[2K_2^a + K_2^b]. \quad (5.3)$$

**Proof.** Plug in time-independent values  $K_1 = 2K_1^a + K_1^b$  and  $K_2 = 2K_2^a + K_2^b$  into conclusion of Theorem 3.1 (or use Corollary 4.3). Then one receives the exponential mean square stability with claimed constants (5.3).  $\diamond$

**Remark.** Consider systems (3.1) with trivial solution  $X \equiv x_*$ . In traditional stability theory it is common to require that  $a(t, x_*) \equiv 0, b^j(t, x_*) \equiv 0$  for all  $t \geq t_0$ , for all  $j = 0, 1, 2, \dots, m$ . Under these additional presumptions Corollary 5.7 also tells us about sufficient conditions for exponential stability in narrow sense, i.e. exponential stability of trivial solution of related dynamical system.

## 5.3 Exponential stability for nonlinear numerical methods

Consider families of drift-implicit Euler methods governed by (3.6) or (3.11).

**Corollary 5.8** *Let the assumptions of Theorem 3.3 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is strictly (uniformly) mean square dissipative on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_1^a, K_2^a, K_1^b, K_2^b$ , and it holds*

$$0 < \Delta_k < \sup_{n \in \mathbb{N}} \Delta_n < +\infty, \quad \sum_{k=0}^{+\infty} \Delta_k = +\infty. \quad (5.4)$$

*Then the fully drift-implicit Euler methods  $(X_n)_{n \in \mathbb{N}}$  following schemes (3.6) or (3.11) while applying to SDE (3.1) with step size sequence  $(\Delta_n)_{n \in \mathbb{N}}$  are (globally) exponentially mean square stable with*

$$C_1(2) = 1.0 \quad \text{and} \quad C_2(2) \leq -\frac{2K_2^a + K_2^b}{1 - 2K_2^a \sup_{i \in \mathbb{N}} \Delta_i}, \quad (5.5)$$

and thus they satisfy

$$\mathbb{E} \|X_{n+1}\|^2 \leq C_1(2) \cdot \mathbb{E} \|X_0\|^2 \cdot \exp\left(-C_2(2) \cdot (t - t_0)\right) \quad (5.6)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^2 < +\infty.$$

**Proof.** Plug in time-independent values  $K_1^a = K_1^b = 0$ ,  $K_2^a, K_2^b$  with the relation  $2K_2^a + K_2^b < 0$  into conclusion of Theorem 3.3 (or use Corollary 4.4). Then one receives the exponential mean square stability with claimed constants (5.5).  $\diamond$

**Remark.** Other numerical methods can be considered as well. This is left to the interest of readership. Since the observed fact is fairly independent of ‘admissible sequences’ of step sizes, we obtain the following property.

## 5.4 Nonlinear A– and AN–stability

The properties of A– and AN–stability have attracted many researchers so far. It is natural to prefer such numerical method which can be used without any restriction on step size or their parameters while keeping the property of asymptotical stability, and hence control on error propagation. The key idea originates from outstanding work of Dahlquist. For some of his highlights, see [7], [8].

**Definition 5.8** *A numerical method  $(X_n)_{n \in \mathbb{N}}$  applied to SDEs (3.1) with the step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  is called **mean square AN–stable** iff it holds*

$$\lim_{n \rightarrow +\infty} \mathbb{E} \|X_{n+1}\|^2 = 0 \quad (5.7)$$

*for all nonautonomous SDEs (3.1) with strictly (uniformly) mean square dissipative coefficient systems  $(a(t, x), b^j(t, x))$  and for any admissible sequences  $(\Delta_n)_{n \in \mathbb{N}}$  satisfying (5.4).*

*A numerical method  $(X_n)_{n \in \mathbb{N}}$  applied to SDEs (3.1) with the step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  is said to be **mean square A–stable** iff relation (5.7) holds for all autonomous SDEs (3.1) with strictly (uniformly) mean square dissipative coefficient systems  $(a(x), b^j(x))$  and for any admissible sequences  $(\Delta_n)_{n \in \mathbb{N}}$  satisfying (5.4).*

**Remark.** In an analogous way one might introduce and think of the notion of *p–th mean A(N)–stability* of numerical methods or more general of parametrized stochastic dynamical systems.

Consider once again families of drift–implicit Euler methods governed by schemes (3.6) and (3.11).

**Corollary 5.9** *The fully drift–implicit Euler methods  $(X_n)_{n \in \mathbb{N}}$  following schemes (3.6) or (3.11) with admissible step size sequences  $(\Delta_n)_{n \in \mathbb{N}}$  are mean square A– and AN–stable.*

**Proof.** Plug in time-independent values  $K_1^a = K_1^b = 0$ ,  $K_2^a, K_2^b$  under the condition  $2K_2^a + K_2^b < 0$  into conclusion of Theorem 3.3. (Or, alternatively, take estimation (5.6) from exponential stability). Then one trivially receives the property of nonlinear mean square AN– and A–stability as claimed in Corollary 5.9.  $\diamond$

**Remark.** Other numerical methods can be considered as well. However, it seems to be extremely difficult to find nonlinear A(N)–stable stochastic methods. The

requirement of admissible sequences of step sizes turns out to be essential for efficient estimation concerning A- and AN-stability while permitting variable step sizes.

## 6. MOMENT-CONTRACTIVITY IN WIDE AND NARROW SENSE

This section exhibits some results on contractivity of initial perturbations of stochastic dynamical systems as time  $t$  advances. We follow the ‘p-th mean approach’ here, and, in particular, the analysis related to asymptotical propagation of second moments of initial perturbations (i.e. mean square contractivity with  $p = 2$ ). For this purpose p-th moments of related dynamical system are assumed to exist globally on that time scale  $\mathbb{T} \subset \mathbb{R}^1$  (i.e. p-th mean dissipativity with respect to initial perturbations should be clarified before). In stating definitions below, let  $\mathbb{D} \subseteq \mathbb{R}^d$  be a (deterministic) domain which contains a open centered ball with positive radius. We also assume that this domain  $\mathbb{D}$  almost surely contains all image values of considered dynamical systems, provided that it has started at domain  $\mathbb{D}$ .

### 6.1 Contractivity of initial perturbations on time scales

The following definitions – in analogy to deterministic counterparts – can be introduced.

**Definition 6.9** *A stochastic dynamical system  $\{X_t : t \in \mathbb{T}\} \subseteq \mathbb{D} \subseteq \mathbb{R}^d$  is called **asymptotically p-th mean contractive in wide sense** ( $p \in \mathbb{R}^1 \setminus \{0\}$  fixed) iff there exists a finite positive real number  $c_p$  such that*

$$\lim_{t \rightarrow +\infty} \mathbb{E} \|X_{t_0, X_0}(t) - X_{t_0, Y_0}(t)\|^p = 0 \quad (6.1)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0, Y_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^p + \mathbb{E} \|Y_0\|^p < +\infty, \quad \mathbb{E} \|X_0 - Y_0\|^p < c_p.$$

In addition, if constant  $c_p$  can be taken as  $c_p = +\infty$  in the wide sense asymptotically p-th mean contractive dynamical system is said to be **globally asymptotically p-th mean contractive in wide sense**.

A stochastic dynamical system  $\{X_t : t \in \mathbb{T}\} \subseteq \mathbb{D} \subseteq \mathbb{R}^d$  is called **(globally) exponentially p-th mean contractive** ( $p \in \mathbb{R}^1 \setminus \{0\}$  fixed) iff there exist finite, positive real numbers  $C_1(p), C_2(p) > 0$  such that

$$\mathbb{E} \|X_{t_0, X_0}(t) - X_{t_0, Y_0}(t)\|^p \leq C_1(p) \cdot \mathbb{E} \|X_0 - Y_0\|^p \cdot \exp\left(-C_2(p)(t - t_0)\right) \quad (6.2)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0, Y_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^p + \mathbb{E} \|Y_0\|^p < +\infty.$$



**Remarks.** Since the introduction of concept of uniform  $p$ -th mean contractivity seems to be a very strong requirement (i.e. contrary to nature of unbounded random noise), we do not pursue the related approach and analysis here. For illustrative effects in this respect, however, consider bilinear stochastic differential equations driven by multiplicative White noise. We rather want to study stochastic systems with respect to property of  $p$ -th mean contractivity (6.1), in particular when  $p = 2$ , here. In case of  $p = 2$  we also term related property as **asymptotical mean square contractivity**. As the (stochastic) dynamical system itself, we call *the generating mapping, flow, equation, method, scheme, solution as asymptotically mean square contractive* (in the wide sense as above) ones. One may also consider the concept of exponential contractivity as the one of **asymptotical contractivity in the narrow sense**.

## 6.2 Contractivity of initial perturbations of SDEs

Let us turn our attention to continuous time systems governed by continuous time SDEs.

**Corollary 6.10** *Let the assumptions of Theorem 3.2 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is strictly (uniformly) mean square monotone decreasing on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_3^a, K_4^a, K_3^b, K_4^b$ .*

*Then the stochastic process  $X_{t_0, X_0}(t)$  governed by SDE (3.1) is (globally) exponentially mean square contractive with*

$$C_3(2) = 1.0 \quad \text{and} \quad C_4(2) \leq -[2K_4^a + K_4^b]. \quad (6.3)$$

**Proof.** Plug in time-independent values  $K_3 = 0$  and  $K_4 = 2K_4^a + K_4^b$  into conclusion of Theorem 3.2 (or use Corollary 4.5). Then one receives (global) exponential mean square contractivity with claimed constants (6.3). $\diamond$

**Remarks.** Consider systems (3.1) with trivial solution  $X \equiv x_*$ . In traditional stability theory, as already noted, it is common to require that  $a(t, x_*) \equiv 0, b^j(t, x_*) \equiv 0$  for all  $t \geq t_0$ , for all  $j = 0, 1, 2, \dots, m$ . Under these additional presumptions Corollary 6.10 also tells us about sufficient conditions for exponential stability in narrow sense, i.e. exponential stability of trivial solution of related dynamical system.

**Examples.** Many examples can be found for illustration of asymptotically contractive SDEs. Most simple ones are provided with bilinear SDEs when the eigenvalues of Jacobian of related diffusion matrices  $B^j(t)$  are ‘dominated’ by eigenvalues of Jacobian drift matrix  $A(t)$  in appropriate manner (i.e. in terms of corresponding real parts) for all  $t \in [t_0, +\infty)$ . An interesting example for exponentially mean square contractive, nonlinear SDEs driven by standard Wiener processes  $W_t^j$  is given by one-dimensional dynamics of

$$dX_t = -[a_0 \sqrt{|X_t|} \cdot \text{sign}(X_t) + a_1 X_t + \gamma \cdot (X_t)^{2n+1}] dt + \sum_{j=1}^m [\sigma_j^0 + \sigma_j^1 X_t] dW_t^j \quad (6.4)$$

where  $2a_1 - \sum_{j=1}^m (\sigma_j^1)^2 > 0$ ,  $a_0, \gamma \geq 0$ ,  $n \in \mathbb{N}$ , and  $\text{sign}(x)$  represents the signum of inscribed real expression. In passing we note that it has exponentially mean square stable trivial solution  $X \equiv 0$  provided that additionally  $\sigma_j^0 \equiv 0$  (for all  $j$ ).

### 6.3 Contractivity of some numerical methods

As before, we want to investigate nonlinear, fully drift-implicit Euler methods (3.6) or (3.11) in concern with stochastic asymptotical contractivity.

**Corollary 6.11** *Let the assumptions of Theorem 3.4 for SDE (3.1) be satisfied. Furthermore, assume that coefficient system  $(a, b^j)$  related to SDE (3.1) is strictly (uniformly) mean square monotone decreasing on domain  $\mathbb{D} \subseteq \mathbb{R}^d$  with real constants  $K_3^a, K_4^a, K_3^b, K_4^b$ , and it holds (5.4)*

*Then the fully drift-implicit Euler methods  $(X_n)_{n \in \mathbb{N}}$  following schemes (3.6) or (3.11) while applying to SDE (3.1) with step size sequence  $(\Delta_n)_{n \in \mathbb{N}}$  are (globally) exponentially mean square contractive with*

$$C_3(2) = 1.0 \quad \text{and} \quad C_4(2) \leq -\frac{2K_4^a + K_4^b}{1 - 2K_4^a \sup_{i \in \mathbb{N}} \Delta_i} \quad (6.5)$$

and thus they satisfy

$$\mathbb{E} \|X_{n+1} - Y_{n+1}\|^2 \leq C_3(2) \cdot \mathbb{E} \|X_0 - Y_0\|^2 \cdot \exp\left(-C_4(2) \cdot (t - t_0)\right) \quad (6.6)$$

for all  $t_0 \in \mathbb{T}$ , all  $X_0, Y_0 \in \mathbb{D}$  which are measurable with respect to  $\mathcal{F}_{t_0}$  with

$$\mathbb{E} \|X_0\|^2 + \mathbb{E} \|Y_0\|^2 < +\infty.$$

**Proof.** Plug in time-independent values  $K_3^a = K_3^b = 0$ ,  $K_4^a, K_4^b$  with  $2K_4^a + K_4^b < 0$  into conclusion of Theorem 3.4 (or use Corollary 4.6). Then one receives (global) exponential mean square contractivity with claimed constants (6.5).  $\diamond$

Since the result of our corollary is fairly independent of ‘admissible’ sequences of step sizes  $\Delta_n$ , the concepts of stochastic B- and BN-stability can be introduced and related properties can be verified for some numerical methods.

### 6.4 Nonlinear B- and BN-stability

The following notions and concepts are motivated by deterministic counterparts, see e.g. Burrage and Butcher [3] - [5] to a certain extent. For this purpose, assume that involved numerical methods are defined on corresponding domains.

**Definition 6.10** *A numerical method  $(X_n)_{n \in \mathbb{N}}$  applied to SDEs (3.1) and started at  $\mathcal{F}_0$ -measurable initial values  $X_0, Y_0 \in \mathbb{D}$  with step sizes  $(\Delta_n)_{n \in \mathbb{N}}$  is called **mean square BN-stable** iff it is mean square BN-dissipative with*

$$\lim_{n \rightarrow +\infty} \mathbb{E} \|X_{n+1} - Y_{n+1}\|^2 = 0 \quad (6.7)$$

for all nonautonomous SDEs (3.1) with strictly (uniformly) mean square monotone decreasing coefficient systems  $(a(t, x), b^j(t, x))$  and for any admissible sequences of step sizes  $(\Delta_n)_{n \in \mathbb{N}}$  satisfying (5.4).

A numerical method  $(X_n)_{n \in \mathbb{N}}$  applied to SDEs (3.1) with step sizes  $(\Delta_n)_{n \in \mathbb{N}}$  is said to be **mean square B-stable** iff it is mean square B-dissipative and relation (6.7) holds for all autonomous SDEs (3.1) with strictly (uniformly) mean square monotone decreasing coefficient systems  $(a(x), b^j(x))$  and for any admissible sequences of step sizes  $(\Delta_n)_{n \in \mathbb{N}}$  satisfying (5.4).

**Remark.** In an analogous way one might introduce and think of the notion of  $p$ -th mean  $B(N)$ -stability of numerical methods or more general of parametrized stochastic dynamical systems.

Consider once again families of drift-implicit Euler methods governed by schemes (3.6) and (3.11).

**Corollary 6.12** *The fully drift-implicit Euler methods  $(X_n)_{n \in \mathbb{N}}$  following schemes (3.6) or (3.11) with admissible step sizes  $(\Delta_n)_{n \in \mathbb{N}}$  are mean square B- and BN-stable.*

**Proof.** From Theorem 4.6, we know about mean square BN- and B-dissipativity of fully drift-implicit Euler methods, respectively. Now, plug in time-independent values  $K_3^a = K_3^b = 0$ ,  $K_4^a, K_4^b$  with  $2K_4^a + K_4^b < 0$  into conclusion of Theorem 3.4. (Or, alternatively, take estimation (6.6) from exponential contractivity). Then one receives the properties of nonlinear mean square BN- and B-stability, respectively, as claimed in Corollary 6.12.  $\diamond$

**Remark.** Other numerical methods can be considered as well. However, it seems to be extremely difficult to find nonlinear B(N)-stable stochastic methods. Besides, as in the analysis concerning AN- and A-stability, the restriction to admissible sequences of step sizes turns out to be essential in analysis with respect to BN- and B-stability while permitting variable step sizes.

## 7. LINEAR VARIATION-OF-CONSTANTS INEQUALITIES

The following lemmas have often been used in proofs before. They represent natural generalization of the well-known Bellman-Gronwall Lemma. Define  $\sum_{k=n+1}^n c_k := 0$ , no matter what  $c_k$  is to be. For certain relations to discrete time estimations which, for example, might occur in numerical analysis, let  $\mathbb{T} = \{t_n \in \mathbb{R} : t_n < t_{n+1}, n \in \mathbb{N}\}$  with finite minimum element  $t_0$ , with no maximum element, but with the supremum  $t_{+\infty} \leq +\infty$ .

**Remark.** The following estimate can be called **discrete time version of variation of constants formula for linear integral inequalities**, cf. Coddington and

Levinson [6] for some motivation arising from deterministic, continuous time differential systems. We clearly recognize the additively separated contributions which come from the homogeneous part and which come from the variation over the inhomogeneous part of considered inequality - in analogy to corresponding continuous time situation.

**Lemma 7.1** *Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence of nonnegative, finite real numbers related to given time scale  $\mathbb{T}$  and satisfying the estimate*

$$v_{n+1} \leq v_n \cdot \left(1 + C_1(t_n, t_{n+1})\right) + C_0(t_n, t_{n+1}) \quad (7.1)$$

with finite values  $C_0(t_n, t_{n+1}), C_1(t_n, t_{n+1})$  for all  $n \in \mathbb{N}$  along that time scale  $\mathbb{T}$ .

Then it holds the **discrete variation of constants inequality**, i.e.

$$v_{n+1} \leq v_0 \cdot \exp\left(\sum_{k=0}^n C_1(t_k, t_{k+1})\right) + \sum_{k=0}^n C_0(t_k, t_{k+1}) \cdot \exp\left(\sum_{l=k+1}^n C_1(t_l, t_{l+1})\right). \quad (7.2)$$

**Proof.** Via induction on  $n \in \mathbb{N}$ . For  $n = 0$ , the assertion is trivially satisfied. Now assume that the assertion of Lemma 7.1 has already been proven for  $n-1, n \in \mathbb{N} \setminus \{0\}$ . Then, from (7.1), it follows

$$v_{n+1} \leq v_n \cdot \exp\left(C_1(t_n, t_{n+1})\right) + C_0(t_n, t_{n+1}) \quad (7.3)$$

for all  $n \in \mathbb{N}$  along that time scale  $\mathbb{T}$ . Using assumption of induction and relation (7.3) yields

$$\begin{aligned} v_{n+1} &\leq \left[ v_0 \cdot \exp\left(\sum_{k=0}^{n-1} C_1(t_k, t_{k+1})\right) + \right. \\ &\quad \left. + \sum_{k=0}^{n-1} C_0(t_k, t_{k+1}) \cdot \exp\left(\sum_{l=k+1}^{n-1} C_1(t_l, t_{l+1})\right) \right] \cdot \exp\left(C_1(t_n, t_{n+1})\right) + C_0(t_n, t_{n+1}) \\ &\leq v_0 \cdot \exp\left(\sum_{k=0}^n C_1(t_k, t_{k+1})\right) + \sum_{k=0}^{n-1} C_0(t_k, t_{k+1}) \cdot \exp\left(\sum_{l=k+1}^n C_1(t_l, t_{l+1})\right) + C_0(t_n, t_{n+1}) \\ &= v_0 \cdot \exp\left(\sum_{k=0}^n C_1(t_k, t_{k+1})\right) + \sum_{k=0}^n C_0(t_k, t_{k+1}) \cdot \exp\left(\sum_{l=k+1}^n C_1(t_l, t_{l+1})\right) \end{aligned}$$

Consequently, the validity of Lemma 7.1 has been proven.  $\diamond$

**Example.** The most popular example arising in numerical analysis as application of this type of inequalities is given by the case when  $C_0$  and  $C_1$  can be chosen with

$$C_0(s, t) = \hat{C}_0(s) \cdot (t - s), \quad C_1(s, t) = \hat{C}_1(s) \cdot (t - s); \quad t \geq s; \quad t, s \in \mathbb{R}^1$$

where  $\hat{C}_0(s), \hat{C}_1(s), s \in \mathbb{R}^1$  are absolutely integrable functions (with respect to Lebesgue measure on  $\mathbb{R}^1$ ). This could also be seen in previous sections to a large extent.

**Corollary 7.13** *Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence as assumed in Lemma 7.1. Furthermore, assume that*

$$\forall t_k, t_{k+1} \in \mathbb{T} : C_0(t_k, t_{k+1}) \leq K \text{ and } \limsup_{n \rightarrow +\infty} \sum_{k=0}^n \exp \left( \sum_{l=k+1}^n C_1(t_l, t_{l+1}) \right) < +\infty.$$

where  $K \geq 0$  is an appropriate real constant.

Then it holds

$$\limsup_{n \rightarrow +\infty} v_n \leq v_0 \cdot \exp \left( \sum_{k=0}^{+\infty} C_1(t_k, t_{k+1}) \right) + K \cdot \limsup_{n \rightarrow +\infty} \sum_{k=0}^n \exp \left( \sum_{l=k+1}^n C_1(t_l, t_{l+1}) \right),$$

i.e. ‘dissipativity’ of sequence  $(v_n)_{n \in \mathbb{N}}$ . Moreover, if constant  $K < 0$ , then

$$\limsup_{n \rightarrow +\infty} v_n \leq v_0 \cdot \exp \left( \sum_{k=0}^{+\infty} C_1(t_k, t_{k+1}) \right) + K \cdot \liminf_{n \rightarrow +\infty} \sum_{k=0}^n \exp \left( \sum_{l=k+1}^n C_1(t_l, t_{l+1}) \right),$$

**Remarks.** The assertion of Corollary 7.13 immediately follows from lemma above, hence there is no need to prove it here. Further refinements concerning the size or estimates of limits as  $n$  goes to infinity are possible, but left to the interest of readership. We have essentially made use of Lemma 7.1 in the course of previous sections in order to examine nonautonomous systems in view of their asymptotic qualitative behaviour.

The continuous time version of Lemma 7.1 can be obtained through taking the limit as step sizes go to zero in variation of constants formula of that Lemma. Thus, proof is either not needed to state additionally it. It can be called **continuous time, version of the variation of constants formula for linear integral inequalities**.

**Lemma 7.2** *Let  $v(t), t \geq t_0$  be a nonnegative real-valued function which is absolutely integrable on  $[t_0, +\infty)$ . Assume that  $K_1(t), K_2(t)$  are absolutely integrable with*

$$\int_s^t K_1(u) \cdot \exp \left( \int_u^t K_2(v) dv \right) du < +\infty$$

for all  $t, s$  with  $t \geq s \geq t_0$ . Furthermore,  $v(t)$  satisfies

$$v(t) \leq v(s) + \int_s^t K_1(u) du + \int_s^t K_2(u) \cdot v(u) du \tag{7.4}$$

for all  $t, s$  with  $t \geq s \geq t_0$ . Then it holds the **continuous variation of constants inequality**, i.e.

$$v(t) \leq v(s) \cdot \exp \left( \int_s^t K_2(u) du \right) + \left( \int_s^t K_1(u) \cdot \exp \left( - \int_s^u K_2(v) dv \right) du \right) \cdot \exp \left( \int_s^t K_2(u) du \right) \tag{7.5}$$

for all  $t, s$  with  $t \geq s \geq t_0$ .

**Remark.** The uniform requirement ‘for all  $t, s$  with  $t \geq s \geq t_0$ ’ is essential for validity of this assertion in certain cases, cf. also Mao [21]. This uniform condition is not necessary in Bellman–Gronwall Lemma with nonnegative kernel  $K_2(t)$ . For some linear systems, the estimates of Lemmas 7.1 and 7.2 are asymptotically sharp (i.e. as time  $t$  goes to  $+\infty$ ). See examples in section 4.

## 8. CONCLUSIONS AND REMARKS

This paper represents a trial to unified approach to the qualitative analysis of moment-dissipative stochastic dynamical systems such as continuous time stochastic differential equations (SDEs) and their discretizations. The basic notions of dissipativity, stability, contractivity and their refinements have been introduced for nonlinear, nonautonomous stochastic systems. We have essentially studied the related concepts by the use of monotonicity arguments, as already noted by Krylov [20]. Moment-dissipative SDEs perform a reasonable test class for the qualitative behaviour of numerical methods. In particular, we have seen the ‘theoretical goodness’ of fully drift-implicit Euler method in numerical treatment of nonlinear, nonautonomous SDEs. This method is mean square A-, AN-dissipative, B-, BN-dissipative, A-, AN-stable, contractive, B- and BN-stable for natural test classes. Eventually, one gets a justification for applying these numerical methods to study the (global) asymptotical behaviour and characteristics of nonlinear SDEs through controlled numerical dynamics.

Further detailed studies could now follow concerning qualitative (asymptotical) analysis of nonlinear and nonautonomous stochastic systems using the same and similar ideas as here. For example, estimation of asymptotical characteristics from below instead of the presented estimation from above are of special interest too. This can be carried out by similar mathematical techniques as those used in this paper. Furthermore, a computation or estimation of the exponential growth rates of moment evolutions is a desirable task to be solved, both within the concept of stability and contractivity without using the classical idea of linearization (e.g. by monotonicity argumentation, see also the forthcoming paper of the author [34] in this respect).

Many interesting questions are left open. Among them: Can one find other A- and AN-dissipative numerical methods? Which role do linear-implicit or semi-implicit methods play for the asymptotically adequate integration of dissipative systems? Is the midpoint method p-th mean BN-stable? Does there other stochastic nonlinear AN-stable numerical methods exist (e.g. stochastic Runge–Kutta methods)? What can one say about convergence of nonlinear stochastic numerical methods for dissipative continuous time systems? Which rules have to be paid attention during choice of variable step size algorithms for numerical solution of dissipative stochastic differential equations? When do the global (or local) attractors of continuous time and related discrete time dynamical systems coincide? Which numerical methods are appropriate to approximate stochastic attractors fairly accurate? How efficient is the presented analysis in numerical integration of infinite-dimensional stochastic dynamical systems or singularly perturbed stochastic systems?

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