MOMENT CONTRACTIVITY AND STABILITY EXPONENTS OF NONLINEAR STOCHASTIC DYNAMICAL SYSTEMS

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Abstract. Nonlinear stochastic dynamical systems as ordinary stochastic differential equations and stochastic difference methods are in the center of this presentation in view of the asymptotical behaviour of their moments. We study the exponential p-th mean growth behaviour of their solutions as integration time tends to infinity. For this purpose, the concepts of nonlinear contractivity and stability exponents for moments are introduced as generalizations of well-known moment Lyapunov exponents of linear systems. Under appropriate monotonicity assumptions we gain uniform estimates of these exponents from above and below. Eventually, these concepts are generalized to describe the exponential growth behaviour along certain Lyapunov-type functionals.

1. INTRODUCTION

The analysis of stochastic dynamical systems with respect to their asymptotical behaviour has attracted many researchers, see e.g. Arnold [1] - [7], Baxendale [8] - [12], Freidlin and Wentzell [15], Imkeller [19] - [20], Khas'minskij [25], Kifer [26], Mao [31] or, for a recent and comprehensive treatment, see Arnold [7]. Among them systems which have a 'finite asymptotical structure' when integration time tends to infinity are given by the class of dissipative ones (see Hale [18]). Roughly speaking, these systems have some compact attracting sets such that their trajectories at least in the vicinity of these sets will approach to these sets and stay there afterwards. Under randomness we can observe a similar behaviour, mainly classified by the behaviour of moments and by the pathwise behaviour of trajectories.

We will examine the case of 'moment dissipativity' here, since this approach permits to carry over the deterministic analysis to stochastic one in a fairly straight forward manner. We are also inspired by the results from Hale [18], Krylov [29] and Schurz [35], [36] (Note: The terminology 'moment dissipativity' becomes clear from latter two works.). Especially, the interest of this contribution lies in the estimation of maximum and minimum exponential growth rates of 'moment dissipative stochastic systems'. We will distinguish between the exponents characterizing the exponential growth behaviour of the absolute norm of state process (a property also called stability) and those exponents characterizing the exponential growth behaviour of initial perturbations (a property here called contractivity). In the

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linear situation these exponents coincide with the generally common moment Lyapunov exponents (see [3], [4]). It is worth noting that there is a well–known relation between moment and sample Lyapunov exponents (for a formula, see [2]). The existence of sample Lyapunov exponents can be justified by the fundamental multiplicative ergodic theorem of Oseledec [33] which induces a decomposition of the original domain of definition into random Oseledec spaces. We will not go into details of the structure of these spaces. For this and related problems, see Imkeller [19], [20]. In the nonlinear situation sample Lyapunov exponents replicate the behaviour of the corresponding linearized dynamics, hence in general the local behaviour in a small neighborhood of equilibria is determined in general. We are rather interested in the global exponential growth behaviour of moments characterized by some significant deterministic numbers without using the idea of linearization and without using anticipative calculus. The presented concepts turn out to be very appropriate for the study of asymptotical moment behaviour of stochastic dynamical systems both in continuous and discrete time as integration time tends to infinity.

The paper is organized as follows. In section 2 we introduce the definition of moment stability exponents and give first uniform estimations. Section 3 presents the concept of moment contractivity exponents. Some uniform estimates can be derived for contractivity exponents as well. Both section 2 and 3 provide estimates from below and above for stochastic differential equations (SDEs) and the simplest, discrete time, iterative, random mappings (such as numerical methods for SDEs using monotonicity conditions as in Krylov [29] or Schurz [35], [36]. The paper continues with section 4 containing examples of linear systems illustrating that some of the obtained estimates are sharp and illustrating some relations between the concepts of stability and contractivity. Eventually, we end up with a generalization to $V-$ exponents and some illustrations of their usefulness in case of nonlinear systems in section 5. Some final and summarizing remarks finish this paper by section 6.

2. Moment stability exponents

Throughout this paper let $< .,. >$ denote the Euclidean scalar product defined by $< x, y >= \sum_{i=1}^{d} x_i y_i$ for vectors x, y in $\mathbb{R}^d$, $d \in \mathbb{N}_+ \setminus \{0\}$, and $\|.\|$ the Euclidean vector norm in $\mathbb{R}^d$. Furthermore, $(.)_+$ or $[.]_+$ will denote the positive part of inscribed expression, $(.)_-$ or $[.]_-$ the negative part. Fix a $p \in \mathbb{R}_+, p > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and suppose $\mathcal{T}$ to be a discrete or continuous, deterministic time scale, respectively. Consider a stochastic process $(X(t))_{(t\in\mathcal{T})}$ defined for all $t \in \mathcal{T} \subset [t_0, +\infty]$ on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in domain $\mathbb{D} \subseteq \mathbb{R}^d$ (a.s.) and with finite p-th absolute moments for all finite times $t \in \mathcal{T}$.

Definition 1. The *upper (forward p–th moment) stability exponent* of the given stochastic process $(X(t))_{(t\in\mathcal{T})}$ in domain $\mathbb{D}$ is defined to be

$$\overline{\lambda}_p \;:=\; \limsup_{t\to+\infty} \frac{1}{t}\, ln\, \mathbb{E}\, \|X(t)\|^p \tag{2.1}$$

for $X(t_0) \in \mathbb{D}$ (a.s.), provided that this limit exists. The *lower (forward p–th moment) stability exponent* of the given stochastic process $(X(t))_{(t\in\mathcal{T})}$ in domain $\mathbb{D}$ is defined to be

$$\underline{\lambda}_p \;:=\; \liminf_{t\to+\infty} \frac{1}{t}\, ln\, \mathbb{E}\, \|X(t)\|^p \tag{2.2}$$

for $X(t_0) \in \mathbb{D}$ (a.s.), provided that this limit exists.

Remark. A similar definition one could introduce for stochastic fields where the time scale is a partially ordered set with maximum element '$+\infty$'. However, that definition can also be understood only in one specific direction of given multivariate time scale.

2.1. **Stability exponents of SDEs.** Let us look closer at uniform estimates of those stability exponents in case of a class of nonlinear stochastic differential equations (SDEs) with monotone coefficients. For general theory of SDEs, see Arnold [1, 7], Dynkin [14], Friedman [16], Freidlin and Wentzell [15], Gard [17], Khas'minskij [25], Krylov [30] or Protter [34]. Throughout this paper we consider W_t^j as scalar, independent Wiener processes. However, the presented results can be carried over to other stochastic processes than W_t^j (with some necessary corrections).

Theorem 2.1. *Let process $(X(t))_{(t\geq t_0)}$ satisfy the Itô SDE*

$$dX(t) \ = \ a(t,X(t))\,dt + \sum_{j=1}^{m} b^j(t,X(t))\,dW_t^j \tag{2.3}$$

with values in deterministic domain $I\!D \subseteq I\!R^d$, where deterministic coefficients a, b^j are such that strong solution of this SDE with finite p–th absolute moments exists. Assume that

$$< a(t,x),x > + \frac{1}{2}\sum_{j=1}^{m}\|b^j(t,x)\|^2 + \frac{(p-2)}{2}\sum_{j=1}^{m}\frac{< b^j(t,x),x >^2}{\|x\|^2}$$

$$\leq \ \overline{K}_p(t)\|x\|^2 \tag{2.4}$$

for all $t \in [t_0,+\infty)$, for all $x \in I\!D$, where deterministic function $\overline{K}_p(t)$ is L^1–integrable on $[t_0,+\infty)$ with respect to the Lebesque measure, and

$$I\!P\{\omega\in\Omega : X(t)(\omega) \in I\!D, \forall t\in[t_0,+\infty)\} \ = \ 1\,. \tag{2.5}$$

Then it holds

$$\overline{\lambda}_p \ \leq \ p\limsup_{t\to+\infty}\frac{\int_{t_0}^{t}\overline{K}_p(s)ds}{t}\,. \tag{2.6}$$

Furthermore, if

$$< a(t,x),x > + \frac{1}{2}\sum_{j=1}^{m}\|b^j(t,x)\|^2 + \frac{(p-2)}{2}\sum_{j=1}^{m}\frac{< b^j(t,x),x >^2}{\|x\|^2}$$

$$\geq \ \underline{K}_p(t)\|x\|^2 \tag{2.7}$$

for all $t \in [t_0,+\infty)$, for all $x \in I\!D$, where deterministic function $\underline{K}_p(t)$ is L^1–integrable on $[t_0,+\infty)$ with respect to the Lebesque measure, then this implies

$$\underline{\lambda}_p \ \geq \ p\liminf_{t\to+\infty}\frac{\int_{t_0}^{t}\underline{K}_p(s)ds}{t}\,. \tag{2.8}$$

Remark. This theorem provides an uniform estimate of the 'spectrum' of (forward) p–th moment stability exponents for the class of SDEs satisfying monotonicity conditions (2.4) and (2.7). Of course, these estimates are 'worst case estimates' (but also sharp estimates, see linear systems as in subsection 4.1). While requiring L^1-integrability here and later throughout this paper, we only refer to the classical spaces $L^1_{loc}([t_0,+\infty),\mathcal{B}([t_0,+\infty)),\mu)$ with respect to the Lebesque measure μ and equipped with the Borel σ-algebra $\mathcal{B}([t_o,+\infty))$ of interval $[t_0,+\infty)$.

Proof. The main idea is to apply Dynkin's formula [14], to evaluate the arising linear partial differential operators under the required monotonicity conditions and finally a generalized Gronwall–Bellman Lemma (see [31] and [35]). The required L^1–integrability ensures us that the expressions in the definition of stability exponents exist, and together with the existence of finite p–th absolute moments, that we can apply (the unstopped form of) Dynkin's formula at any time $t \in [t_0,+\infty)$. The application of linear partial differential operator

$$\mathcal{L} = \frac{\partial}{\partial t} + < a(t,x), \nabla_x > + \frac{1}{2}\sum_{j=1}^{m}\sum_{k,l=1}^{d} b_k^j(t,x)b_l^j(t,x)\frac{\partial^2}{\partial x_k \partial x_l}$$

to $\|x\|^p, p>0$ for corresponding diffusion process $X(t)$ exactly gives $\mathcal{L}\|x\|^p =$

$$p\left(< a(t,x),x > + \frac{1}{2}\sum_{j=1}^{m}\|b^j(t,x)\|^2 + \frac{(p-2)}{2}\sum_{j=1}^{m}\frac{< b^j(t,x),x >^2}{\|x\|^2}\right)\|x\|^{p-2}$$

after some laborious calculations, hence

$$\mathcal{L}\|x\|^p \;\le\; p\overline{K}_p(t)\|x\|^p \quad \text{and} \quad \mathcal{L}\|x\|^p \;\ge\; p\underline{K}_p(t)\|x\|^p\,,$$

respectively, presuming the validity of inequalities (2.4) and (2.7). Note that

$$\nabla_x \;=\; \left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_d}\right)^T$$

represents the d-dimensional gradient vector in $x=(x_1,...,x_d)^T$–direction. By the formula of Dynkin we know that

$$\mathbb{E}\,\|X(t)\|^p \;=\; \mathbb{E}\,\|X(s)\|^p + \mathbb{E}\int_s^t \mathcal{L}\|X(u)\|^p du$$

for all s,t with $t \ge s; s,t \in [t_0,+\infty)$. Under the monotonicity assumptions (2.4) and (2.7), this implies

$$\overline{v}(t) := \mathbb{E}\,\|X(t)\|^p \;\le\; \mathbb{E}\,\|X(s)\|^p + p\int_s^t \overline{K}_p(u)\mathbb{E}\,\|X(u)\|^p du$$

and

$$\underline{v}(t) := \mathbb{E}\,\|X(t)\|^p \;\ge\; \mathbb{E}\,\|X(s)\|^p + p\int_s^t \underline{K}_p(u)\mathbb{E}\,\|X(u)\|^p du\,,$$

respectively. Now one applies the generalized Gronwall–Bellman Lemma (see [31], [35]) to $\overline{v}(t)$ and $\underline{v}(t)$, respectively, takes the limit as time t tends to $+\infty$ and encounters with desired result which completes the proof. $\diamond$

2.2. **Stability exponents of stochastic iterative mappings.** Let us now look at uniform estimates of moment–stability exponents in case of a class of nonlinear stochastic difference equations with monotone coefficients. Such difference equations arise in numerical methods for SDEs when one discretizes these continuous time systems, or when one examines time series. For a general theory of stochastic-numerical methods, see Kloeden, Platen and Schurz [28], Mil'shtein [32], Schurz

[35] and Talay [38], among many others. Now consider d-dimensional iterations

$$X_{n+1} = X_n + \Phi_0^I(X_i : i \leq n+1)\Delta_n + \Phi_0^E(X_i : i \leq n)\Delta_n + \sum_{j=1}^{m} \Phi_j(X_i : i \leq n)\xi_n^j \sqrt{\Delta_n} \tag{2.9}$$

where $\Delta_n := t_{n+1} - t_n$ can be interpreted as a sequence of step sizes with monotonically increasing time-instants $(t_i)_{i \in \mathbb{N}}$ and $\lim_{i \to +\infty} t_i = +\infty$; $\Phi_0^I, \Phi_0^E, \Phi_j$ where $j = 1, 2, ..., m$ represent deterministic mappings from all currently generated values into $\mathbb{R}^d$ (They admit past–path–dependence in general!), and ξ_n^j are real-valued, independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with moments

$$\mathbb{E}\, \xi_n^j = 0 \quad \text{and} \quad \mathbb{E}\, |\xi_n^j|^2 = (\sigma_n^j)^2 < +\infty\,.$$

Theorem 2.2. *Let process $(X_n)_{n \in \mathbb{N}}$ satisfy the stochastic difference equation (2.9) under the above mentioned conditions for all $n \in \mathbb{N}$, whereas all ξ_n^j are independent of X_0 as well. Assume that $\forall n \in \mathbb{N}\, \forall x^{(l)} \in \mathbb{R}^d (l = 0, 1, ..., n+1)\ \forall j = 1, 2, ..., m$:*

$$\begin{aligned}
< \Phi_0^I(x^{(l)} : l \leq n+1), x^{(n+1)} > \ &\leq \ \overline{k}_I(n)\|x^{(n+1)}\|^2 \\
< \Phi_0^E(x^{(l)} : l \leq n), x^{(n)} > \ &\leq \ \overline{k}_E(n)\|x^{(n)}\|^2 \\
\|\Phi_0^I(x^{(l)} : l \leq n+1)\|^2 \ &\geq \ \overline{k}_0^I(n)\|x^{(n+1)}\|^2 \\
\|\Phi_0^E(x^{(l)} : l \leq n)\|^2 \ &\leq \ \overline{k}_0^E(n)\|x^{(n)}\|^2 \\
\|\Phi_j(x^{(l)} : l \leq n)\|^2 \ &\leq \ \overline{k}_j(n)\|x^{(n)}\|^2 \\
2\overline{k}_I(n)\Delta_n \ &< \ 1 + \overline{k}_0^I(i)\Delta_n^2
\end{aligned} \tag{2.10}$$

where $\overline{k}_I, \overline{k}_E(n), \overline{k}_0^I(n), \overline{k}_0^E(n), \overline{k}_j(n)$ are finite, deterministic, real numbers. Then

$$\overline{\lambda}_2 \leq \limsup_{n \to +\infty} \frac{\displaystyle\sum_{i=0}^{n} \Big(\frac{2\overline{k}_E(i) + 2\overline{k}_I(i) + (\overline{k}_0^E(i) - \overline{k}_0^I(i))\Delta_i + \sum_{j=1}^{m} (\sigma_i^j)^2 \overline{k}_j(i)}{1 - 2\overline{k}_I(i)\Delta_i + \overline{k}_0^I(i)\Delta_i^2} \Big) \Delta_i}{t_{n+1}}\,. \tag{2.11}$$

Furthermore, if $\forall n \in \mathbb{N}\, \forall x^{(l)} \in \mathbb{R}^d (l = 0, 1, ..., n+1)\ \forall j = 1, 2, ..., m$:

$$\begin{aligned}
< \Phi_0^I(x^{(l)} : l \leq n+1), x^{(n+1)} > \ &\geq \ \underline{k}_I(n)\|x^{(n+1)}\|^2 \\
< \Phi_0^E(x^{(l)} : l \leq n), x^{(n)} > \ &\geq \ \underline{k}_E(n)\|x^{(n)}\|^2 \\
\|\Phi_0^I(x^{(l)} : l \leq n+1)\|^2 \ &\leq \ \underline{k}_0^I(n)\|x^{(n+1)}\|^2 \\
\|\Phi_0^E(x^{(l)} : l \leq n)\|^2 \ &\geq \ \underline{k}_0^E(n)\|x^{(n)}\|^2 \\
\|\Phi_j(x^{(l)} : l \leq n)\|^2 \ &\geq \ \underline{k}_j(n)\|x^{(n)}\|^2 \\
-2\underline{k}_E(n)\Delta_n \ &< \ 1 + \underline{k}_0^E(n)\Delta_n^2 + \sum_{j=1}^{m} (\sigma_n^j)^2 \underline{k}_j(n)\Delta_n
\end{aligned} \tag{2.12}$$

where $\underline{k}_I, \underline{k}_E(n), \underline{k}_0^I(n), \underline{k}_0^E(n), \underline{k}_j(n)$ are finite, deterministic, real numbers, then

(2.13)

$$\underline{\lambda}_2 \;\geq\; \liminf_{n\to+\infty} \frac{\displaystyle\sum_{i=0}^{n} \Big(\frac{2\underline{k}_E(i) + 2\underline{k}_I(i) + (\underline{k}_0^E(i) - \underline{k}_0^I(i))\Delta_i + \sum_{j=1}^{m} (\sigma_i^j)^2 \underline{k}_j(i)}{1 + 2\underline{k}_E(i)\Delta_i + \underline{k}_0^E(i)\Delta_i^2 + \sum_{j=1}^{m}(\sigma_i^j)^2 \underline{k}_j(i)\Delta_i} \Big) \Delta_i}{t_{n+1}} .$$

Remark. This theorem provides an uniform estimate of the 'spectrum' of (forward) mean square stability exponents for the class of stochastic difference equations satisfying (2.10) and (2.12). Of course these estimates are 'worst case estimates', but sharp ones (see linear systems as in subsection 4.1). Since the analysis of nonlinear, nonautonomous, discrete time stochastic mappings turns out to be very difficult, we restrict our attention only to the feasible case of mean square calculus.

Proof. Define $v_n := \mathbb{E}\, \|X_n\|^2$. Taking the square of the Euclidean vector norm of random vector $X_{n+1} - \Phi_0^I(X_i : i \leq n+1)\Delta_n$ and its expectation value afterwards one receives

$$\begin{aligned}
& v_{n+1} \left(1 - 2\Delta_n \overline{k}_I(n) + \Delta_n^2 \overline{k}_0^I(n)\right) \\
&\leq\; v_{n+1} - 2\Delta_n \mathbb{E}\, < \Phi_0^I(X_i : i \leq n+1), X_{n+1} > + \Delta_n^2 \mathbb{E}\, \|\Phi_0^I(X_i : i \leq n+1)\|^2 \\
&=\; v_n + 2\Delta_n \mathbb{E}\, < \Phi_0^E(X_i : i \leq n), X_n > + \Delta_n^2 \mathbb{E}\, \|\Phi_0^E(X_i : i \leq n)\|^2 \\
&\qquad + \Delta_n \sum_{j=1}^{m} (\sigma_n^j)^2 \mathbb{E}\, \|\Phi_j(X_i : i \leq n)\|^2 \\
&\leq\; v_n \left(1 + 2\Delta_n \overline{k}_E(n) + \Delta_n^2 \overline{k}_0^E(n) + \Delta_n \sum_{j=1}^{m} (\sigma_n^j)^2 \overline{k}_j(n) \right)
\end{aligned}$$

using the mutual independence of random variables ξ_n^j, that $\mathbb{E}\, (\xi_n^j)^2 = (\sigma_n^j)^2$ and conditions (2.10). A similar estimate is derived under conditions (2.12). In the other direction we obtain

$$\begin{aligned}
& v_n \left(1 + 2\Delta_n \underline{k}_E(n) + \Delta_n^2 \underline{k}_0^E(n) + \Delta_n \sum_{j=1}^{m} (\sigma_n^j)^2 \underline{k}_j(n) \right) \\
&\leq\; v_n + 2\Delta_n \mathbb{E}\, < \Phi_0^E(X_i : i \leq n), X_n > + \Delta_n^2 \mathbb{E}\, \|\Phi_0^E(X_i : i \leq n)\|^2 \\
&\qquad + \Delta_n \sum_{j=1}^{m} (\sigma_n^j)^2 \mathbb{E}\, \|\Phi_j(X_i : i \leq n)\|^2 \\
&=\; v_{n+1} - 2\Delta_n \mathbb{E}\, < \Phi_0^I(X_i : i \leq n+1), X_{n+1} > + \Delta_n^2 \mathbb{E}\, \|\Phi_0^I(X_i : i \leq n+1)\|^2 \\
&\leq\; v_{n+1} \left(1 - 2\Delta_n \underline{k}_I(n) + \Delta_n^2 \underline{k}_0^I(n)\right)
\end{aligned}$$

Obviously, under conditions (2.10) and (2.12), one gains the estimates

$$
\begin{aligned}
v_{n+1} &\leq v_n \cdot \left(\frac{1 + 2\overline{k}_E(n)\Delta_n + \overline{k}_0^E(n)\Delta_n^2 + \sum_{j=1}^m (\sigma_n^j)^2 \overline{k}_j(n)\Delta_n}{1 - 2\overline{k}_I(n)\Delta_n + \overline{k}_0^I(n)\Delta_n^2} \right) \\
&\leq v_0 \cdot \prod_{i=0}^n \left(\frac{1 + \overline{k}_E(i)\Delta_i + \overline{k}_0^E(i)\Delta_i^2 + \sum_{j=1}^m (\sigma_i^j)^2 \overline{k}_j(i)\Delta_i}{1 - 2\overline{k}_I(i)\Delta_i + \overline{k}_0^I(i)\Delta_i^2} \right)_+ \\
&\leq v_0 \cdot \exp \left(\sum_{i=0}^n \Big(\frac{2\overline{k}_E(i) + 2\overline{k}_I(i) + (\overline{k}_0^E(i) - \overline{k}_0^I(i))\Delta_i + \sum_{j=1}^m (\sigma_i^j)^2 \overline{k}_j(i)}{1 - 2\overline{k}_I(i)\Delta_i + \overline{k}_0^I(i)\Delta_i^2} \Big) \Delta_i \right)
\end{aligned}
$$

and in the other direction

$$
\begin{aligned}
v_n &\leq v_{n+1} \cdot \left(\frac{1 - 2\underline{k}_I(n)\Delta_n + \underline{k}_0^I(n)\Delta_n^2}{1 + 2\underline{k}_E(n)\Delta_n + \underline{k}_0^E(n)\Delta_n^2 + \sum_{j=1}^m (\sigma_n^j)^2 \underline{k}_j(n)\Delta_n} \right) \\
&\leq v_{n+1} \cdot \exp \left(\frac{-2\underline{k}_E(n) - 2\underline{k}_I(n) - (\underline{k}_0^E(n) - \underline{k}_0^I(n))\Delta_n - \sum_{j=1}^m (\sigma_n^j)^2 \underline{k}_j(n)}{1 + 2\underline{k}_E(n)\Delta_n + \underline{k}_0^E(n)\Delta_n^2 + \sum_{j=1}^m (\sigma_n^j)^2 \underline{k}_j(n)\Delta_n} \Delta_n \right),
\end{aligned}
$$

and therefore

$$
v_{n+1} \geq v_0 \cdot \exp \left(\sum_{i=0}^n \frac{2\underline{k}_E(i) + 2\underline{k}_I(i) + (\underline{k}_0^E(i) - \underline{k}_0^I(i))\Delta_i + \sum_{j=1}^m (\sigma_i^j)^2 \underline{k}_j(i)}{1 + 2\underline{k}_E(i)\Delta_i + \underline{k}_0^E(i)\Delta_i^2 + \sum_{j=1}^m (\sigma_i^j)^2 \underline{k}_j(i)\Delta_i} \Delta_i \right),
$$

respectively, using twice the elementary inequality

$$
\frac{1+y}{1+x} \leq \exp\left(\frac{y-x}{1+x}\right).
$$

Taking the logarithm and the limit as n tends to $+\infty$ yield the desired estimates. Consequently, the proof is completed. ◇

Remark. An useful information which we also obtained by Theorem 2.2 is provided for the problem how to split drift parts of SDEs into an explicit and implicit treatment such that some control on the asymptotical stability behaviour of numerical methods can be achieved. The splitting is represented by the choice of implicit part $\Phi_0^I(.)$ and explicit part $\Phi_0^E(.)$. The remaining task consists of finding such an explicit–implicit splitting such that the conditions (2.10) and / or (2.12) are met. One can justify the construction of linear- or partial-implicit methods (cf. [37])

and their use for the long-time integration of dissipative stochastic systems by this theorem. A more detailed application is left to future research (for an indication, see also subsection 2.3).

2.3. **A nonlinear example.** An illustrative example for mean square stability exponents, for what we achieved by our results and its applications is given by the discretization of the nonlinear one-dimensional Itô SDE

$$dX(t) \;=\; [(\alpha X(t) - \beta X^3(t)]dt + \sigma X(t)dW_t$$

where α, β, σ^2 are nonnegative real constants - an equation which is met in physical field theory (there a Stratonovich interpretation is studied, but this is included in our model class by application of transformation rules between calculi, see [1]). For this equation one can show that

$$\overline{\lambda}_p(x) \leq p\left(\alpha + \frac{p-1}{2}\sigma^2\right)$$

and

$$\overline{\lambda}_2 = \underline{\lambda}_2 = 2\alpha + \sigma^2 \,.$$

Now, the problem is how to simulate and discretize this SDE in a nonanticipative and efficient way such that a control on the stability behaviour of used discretization method can be achieved under the presence of nonlinearities (i.e. $\beta > 0$). Applying our result, we suggest to take the following explicit-implicit method

$$\begin{aligned} Y_{n+1} \;&=\; Y_n + (\alpha)_+ Y_n - (\alpha)_- Y_{n+1} - \beta Y_n^2 Y_{n+1})\Delta_n + \sigma Y_n \Delta W_n \\ &\quad + |c|(Y_n - Y_{n+1})|\Delta W_n| \\ &=\; Y_n \left(\frac{1 + (\alpha)_+\Delta_n + \sigma\Delta W_n + |c\Delta W_n|}{1 + (\alpha)_-\Delta_n + \beta Y_n^2\Delta_n + |c\Delta W_n|}\right) \end{aligned}$$

where c is a further control parameter. Obviously, it can be made explicitly by simple algebraic rearrangements – an advantage from practical implementation point of view. This scheme also establishes a numerical method with the same numerical L^2-convergence order towards the exact solution as the well-known and most-used Euler method does. However, our explicit-implicit method is able to achieve a complete control on the asymptotical stability behaviour in contrast to that of the explicit Euler method, where (random) step size restrictions are needed. The assumptions of Theorem 2.2 are fulfilled by taking

$$\overline{k}_I(n) = -(\alpha)_-, \overline{k}_0^I(n) = 0, \overline{k}_E(n) = (\alpha)_+, \overline{k}_0^E(n) = ((\alpha)_+)^2, \overline{k}_j(n) = \sigma^2 \,.$$

Thus, the upper stability exponent λ_2 of considered numerical method is under control for all initial values (for simplicity, consider the cases $\alpha < 0$ and $\alpha > 0$ separated). For example, when $2\alpha + \sigma^2 < 0$ one knows about the asymptotical mean square stability of trivial solution of that SDE. The same is true for the suggested discretization method using any step size. Namely, for equidistant step size Δ and $c = 0$, one estimates

$$\overline{\lambda}_2 \;\leq\; \frac{2\alpha + \sigma^2}{1 + 2(\alpha)_-\Delta} + ((\alpha)_+)^2\Delta \,.$$

Thus, the asymptotical stability behaviour of underlying SDE is replicated by our explicit-implicit discretization, and it can be used to approximate the upper stability exponents. It is not hard to find a constellation where the explicit Euler method fails in this respect (e.g. take a sufficiently large equidistant step size and a bilinear equation). Moreover, by the choice of an appropriate parameter c, one even gains control on the boundedness of sample paths (a.s.). Note that the chosen splitting into explicit and implicit treatment is very important for an asymptotically adequate numerical integration. Of course, a control on λ_2 could also be obtained by the use of fully drift-implicit Euler method (for the definition, see [28]) under the condition $2\alpha + \sigma^2 < 0$. However, the fully drift-implicit Euler method requires the local algebraic resolution of implicit equations at each integration step, hence more computational effort and additional local errors. Consequently, we may prefer our explicit-implicit technique in view of adequate stability control with discretization of SDEs.

3. Moment contractivity exponents

Fix a $p \in \mathbb{R}_+, p > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{T}$ a discrete or continuous, deterministic time scale, respectively. Consider a stochastic process $(X(t,z))_{(t\in\mathcal{T})}$ defined for all $t \in \mathcal{T} \subset [t_0, +\infty]$ on $(\Omega, \mathcal{F}, \mathbb{P})$, started at values z in domain $\mathbb{D}$ at time $t_0 \in \mathcal{T}$, with values in domain $\mathbb{D} \subset \mathbb{R}^d$ for all times $t \in \mathcal{T}$ (a.s.) and with finite p-th absolute moments for all finite times $t \in \mathcal{T}$.

Definition 2. The *upper (forward p–th moment) contractivity exponent* of the given stochastic process $(X(t,z))_{(t\in\mathcal{T})}$ in domain $\mathbb{D}$ is defined to be

$$\overline{\kappa}_p \ := \ \limsup_{t\to+\infty} \frac{1}{t}\, ln\ (\mathbb{E}\ \|X(t,x) - X(t,y)\|^p) \tag{3.1}$$

for $X(t_0,x) = x \in \mathbb{D}$ (a.s.), $X(t_0,y) = y \in \mathbb{D}$ (a.s.), provided that this limit exists. The *lower (forward p–th moment) contractivity exponent* of the given stochastic process $(X(t,z))_{(t\in\mathcal{T})}$ in domain $\mathbb{D}$ is defined to be

$$\underline{\kappa}_p \ := \ \liminf_{t\to+\infty} \frac{1}{t}\, ln\ (\mathbb{E}\ \|X(t,x) - X(t,y)\|^p) \tag{3.2}$$

for $X(t_0,x) = x \in \mathbb{D}$ (a.s.), $X(t_0,y) = y \in \mathbb{D}$ (a.s.), provided that this limit exists.

Remark. A similar definition one could introduce for stochastic fields where the time scale is a partially ordered set with maximum element '$+\infty$'. However, that definition can also be understood only in one specific direction of their multivariate time scales (a comment as in the case of stability exponents). It is worth noting that contractivity exponents could theoretically depend on initial values x, y, and hence on the considered domain $\mathbb{D}$ as well. However, for example, there is not a such dependence on initial values in the linear situation (cf. subsection 4.3).

3.1. **Contractivity exponents of SDEs.**

Let us look closer at uniform estimates of those contractivity exponents in case of a class of nonlinear stochastic differential equations (SDEs) with monotone coefficients.

Theorem 3.1. *Let process $(X(t,z))_{t\geq t_0}$ satisfy the Itô SDE*

$$dX(t,z) \ = \ a(t, X(t,z))\, dt \ + \ \sum_{j=1}^{m} b^j(t, X(t,z))\, dW_t^j \tag{3.3}$$

with values in deterministic domain $\mathbb{D} \subseteq \mathbb{R}^d$ (a.s.), where deterministic coefficients a, b^j are such that strong solution of this SDE with finite p–th absolute moments *exists. Assume that*

$$< a(t,x) - a(t,y), x-y > + \frac{1}{2}\sum_{j=1}^{m} ||b^j(t,x) - b^j(t,y)||^2$$

$$+\frac{(p-2)}{2}\sum_{j=1}^{m}\frac{< b^j(t,x) - b^j(t,y), x-y >^2}{||x-y||^2} \quad \leq \quad \overline{C}_p(t)||x-y||^2 \tag{3.4}$$

for all $t \in [t_0, +\infty)$, *for all* $x, y \in I\!D$, *where deterministic function* $\overline{C}_p(t)$ *is* L^1*–integrable on* $[t_0, +\infty)$ *with respect to the Lebesque measure, and*

$$I\!P\{\omega \in \Omega : X(t,z)(\omega) \in I\!D, \forall t \in [t_0, +\infty), \forall z \in I\!D\} \ = \ 1\,. \tag{3.5}$$

Then it holds

$$\overline{\kappa}_p \ \leq \ p \limsup_{t\to+\infty} \frac{\int_{t_0}^t \overline{C}_p(s)ds}{t}\,. \tag{3.6}$$

Furthermore, if

$$< a(t,x) - a(t,y), x-y > + \frac{1}{2}\sum_{j=1}^{m} ||b^j(t,x) - b^j(t,y)||^2$$

$$+\frac{(p-2)}{2}\sum_{j=1}^{m}\frac{< b^j(t,x) - b^j(t,y), x-y >^2}{||x-y||^2} \quad \geq \quad \underline{C}_p(t)||x-y||^2 \tag{3.7}$$

for all $t \in [t_0, +\infty)$, *for all* $x, y \in I\!D$, *where deterministic function* $\underline{C}_p(t)$ *is* L^1*–integrable on* $[t_0, +\infty)$ *with respect to the Lebesque measure, then this implies*

$$\underline{\kappa}_p \ \geq \ p \liminf_{t\to+\infty} \frac{\int_{t_0}^t \underline{C}_p(s)ds}{t}\,. \tag{3.8}$$

Remark. This theorem provides an uniform estimate of the 'spectrum' of (forward) p–th moment contractivity exponents for the class of SDEs satisfying monotonicity conditions (3.4) and (3.7). Of course these estimates are 'worst case estimates', but sharp ones (cf. case of linear SDEs). In passing we note that assertions on contractivity exponents can be used to estimate the stability exponents for nonlinear SDEs, and vice versa. For example, when drift $a(t,x)$ and all diffusion parts $b^j(t,x)$ possess one and the same trivial equilibrium $x_* = 0$ and the investigated system has finite contractivity exponent one observes this fact. However, the property of contractivity is more comprehensive and more appropriate for control on error propagation in stochastic numerical analysis than that of stability. See also section 4 for more details.

Proof. The main idea is to enlarge the dynamical system on $\mathbb{D} \times \mathbb{D} \subseteq \mathbb{R}^{2d}$, apply Dynkin's formula [14] to this new system, to evaluate the arising linear partial differential operators under the required monotonicity conditions and finally to apply a generalized Gronwall–Bellman Lemma (see [31] and [35]). The required L^1–integrability ensures us that the expressions in the definition of contractivity exponents exist, and together with the existence of finite p–th absolute moments, that we can apply (the unstopped form of) Dynkin's formula at any time $t \in$

$[t_0, +\infty)$. Now, apply the extended linear partial differential operator

$$\hat{\mathcal{L}} = \frac{\partial}{\partial t} + < a(t,x), \nabla_x > + < a(t,y), \nabla_y > + \frac{1}{2}\sum_{j=1}^{m}\sum_{k,l=1}^{2d} b_k^j(t,x,y) b_l^j(t,x,y) \frac{\partial^2}{\partial x_k \partial x_l}$$

to $||x-y||^p, p > 0$, where

$$b_k^j(t,x,y) := b_k^j(t,x) \quad \text{and} \quad b_{k+d}^j(t,x,y) := b_k^j(t,y), k = 1,2,...,d$$

for corresponding $2d$-dimensional diffusion process $\hat{X}(t,z)$ on $\mathbb{D} \times \mathbb{D}$. After some tedious calculations, this procedure exactly gives

$$\begin{aligned} \hat{\mathcal{L}}||x-y||^p &= p\Bigg(< a(t,x) - a(t,y), x-y > + \frac{1}{2}\sum_{j=1}^{m} ||b^j(t,x) - b^j(t,y)||^2 \\ &\quad + \frac{(p-2)}{2}\sum_{j=1}^{m} \frac{< b^j(t,x) - b^j(t,y), x-y >^2}{||x-y||^2} \Bigg) ||x-y||^{p-2} \end{aligned}$$

for all $x, y \in \mathbb{D}$ with $||x-y|| > 0$, hence

$$\hat{\mathcal{L}}||x-y||^p \leq p\overline{C}_p(t)||x-y||^p \quad \text{and} \quad \hat{\mathcal{L}}||x-y||^p \geq p\underline{C}_p(t)||x-y||^p,$$

respectively, presuming the validity of inequalities (3.4) and (3.7). Note that

$$\nabla_y = \Big(\frac{\partial}{\partial x_{d+1}}, ..., \frac{\partial}{\partial x_{2d}}\Big)^T$$

represents the d-dimensional gradient vector in $y = (x_{d+1}, ..., x_{2d})^T$–direction, and ∇_x the gradient in x-direction as in previous section. By the formula of Dynkin we know that

$$\mathbb{E}\, ||X(t,x) - X(t,y)||^p = \mathbb{E}\, ||X(s,x) - X(s,y)||^p + \mathbb{E} \int_s^t \hat{\mathcal{L}}||X(u,x) - X(u,y)||^p du$$

for all s, t with $t \geq s; s, t \in [t_0, +\infty)$. Define

$$v(t) := \mathbb{E}\, ||X(t,x) - X(t,y)||^p$$

for all $x, y \in \mathbb{D}$. Now, fix initial values x, y. Under the monotonicity assumptions (3.4) and (3.7), this implies

$$v(t) \leq v(s) + p\int_s^t \overline{C}_p(u) v(u) du$$

and

$$v(t) \geq v(s) + p\int_s^t \underline{C}_p(u) v(u) du,$$

respectively. Now one applies the generalized Gronwall–Bellman lemma (see [31], [35]) to both estimates of $v(t)$, respectively, takes the limit as time t tends to $+\infty$ and encounters with desired result which completes the proof. ⋄

3.2. **Contractivity exponents of stochastic iterative mappings.** Let us now look at uniform estimates of those contractivity exponents in case of a class of nonlinear stochastic difference equations with monotone coefficients. Fix a deterministic domain $\mathbb{D} \subset \mathbb{R}^d$. Now consider again the d-dimensional iterative mappings

$$\begin{aligned} X_{n+1}(z) \;&=\; X_n(z) + \Phi_0^I(X_i(z) : i \leq n+1)\Delta_n + \Phi_0^E(X_i(z) : i \leq n)\Delta_n \\ &\quad + \sum_{j=1}^{m} \Phi_j(X_i(z) : i \leq n)\xi_n^j\sqrt{\Delta_n} \end{aligned} \tag{3.9}$$

on deterministic domain $\mathbb{D}$ (a.s.), started at any $z \in \mathbb{D}$, where $\Delta_n := t_{n+1} - t_n$ is a sequence of step sizes with monotonically increasing time-instants $(t_i)_{i\in\mathbb{N}}$ and $\lim_{i\to+\infty} t_i = +\infty$, and ξ_n^j are real-valued, independent random variables on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with moments

$$\mathbb{E}\, \xi_n^j = 0 \quad \text{and} \quad \mathbb{E}\, |\xi_n^j|^2 = (\sigma_n^j)^2 < +\infty\,.$$

For convenience of statement, define $\delta_n(x,y) := x^{(n)} - y^{(n)}$.

Theorem 3.2. *Let process $(X_n(z))_{n\in\mathbb{N}}$ satisfy the stochastic difference equation (3.9) started at value $z \in \mathbb{D}$ under the above mentioned conditions for all $n \in \mathbb{N}$, whereas all ξ_n^j are independent of $X_0(z)$ as well. Assume that $\forall n \in \mathbb{N}\, \forall x^{(l)}, y^{(l)} \in \mathbb{R}^d (l = 0, 1, ..., n+1)\; \forall j = 1, 2, ..., m$:*

$$\begin{aligned} < \Phi_0^I(x^{(l)} : l \leq n+1) - \Phi_0^I(y^{(l)} : l \leq n+1), \delta_{n+1}(x,y) > \;&\leq\; \overline{c}_I(n)\|\delta_{n+1}(x,y)\|^2 \\ < \Phi_0^E(x^{(l)} : l \leq n) - \Phi_0^E(y^{(l)} : l \leq n), \delta_n(x,y) > \;&\leq\; \overline{c}_E(n)\|\delta_n(x,y)\|^2 \\ \|\Phi_0^I(x^{(l)} : l \leq n+1) - \Phi_0^I(y^{(l)} : l \leq n+1)\|^2 \;&\geq\; \overline{c}_0^I(n)\|\delta_{n+1}(x,y)\|^2 \\ \|\Phi_0^E(x^{(l)} : l \leq n) - \Phi_0^E(y^{(l)} : l \leq n)\|^2 \;&\leq\; \overline{c}_0^E(n)\|\delta_n(x,y)\|^2 \\ \|\Phi_j(x^{(l)} : l \leq n) - \Phi_j(y^{(l)} : l \leq n)\|^2 \;&\leq\; \overline{c}_j(n)\|\delta_n(x,y)\|^2 \\ 2\overline{c}_I(n)\Delta_n \;&<\; 1 + \overline{c}_0^I(i)\Delta_n^2 \end{aligned} \tag{3.10}$$

where $\overline{c}_I, \overline{c}_E(n), \overline{c}_0^I(n), \overline{c}_0^E(n), \overline{c}_j(n)$ are finite, deterministic, real numbers. Then

$$\overline{\kappa}_2 \;\leq\; \limsup_{n\to+\infty} \frac{\sum_{i=0}^{n} \Big(\dfrac{2\overline{c}_E(i) + 2\overline{c}_I(i) + (\overline{c}_0^E(i) - \overline{c}_0^I(i))\Delta_i + \sum_{j=1}^{m}(\sigma_i^j)^2\overline{c}_j(i)}{1 - 2\overline{c}_I(i)\Delta_i + \overline{c}_0^I(i)\Delta_i^2} \Big)\Delta_i}{t_{n+1}}\,. \tag{3.11}$$

Furthermore, if $\forall n \in \mathbb{N}\, \forall x^{(l)}, y^{(l)} \in \mathbb{R}^d (l = 0, 1, ..., n+1)\; \forall j = 1, 2, ..., m$:

$$\begin{aligned} < \Phi_0^I(x^{(l)} : l \leq n+1) - \Phi_0^I(y^{(l)} : l \leq n+1), \delta_{n+1}(x,y) > \;&\geq\; \underline{c}_I(n)\|\delta_{n+1}(x,y)\|^2 \\ < \Phi_0^E(x^{(l)} : l \leq n) - \Phi_0^E(y^{(l)} : l \leq n), \delta_n(x,y) > \;&\geq\; \underline{c}_E(n)\|\delta_n(x,y)\|^2 \\ \|\Phi_0^I(x^{(l)} : l \leq n+1) - \Phi_0^I(y^{(l)} : l \leq n+1)\|^2 \;&\leq\; \underline{c}_0^I(n)\|\delta_{n+1}(x,y)\|^2 \\ \|\Phi_0^E(x^{(l)} : l \leq n) - \Phi_0^E(y^{(l)} : l \leq n)\|^2 \;&\geq\; \underline{c}_0^E(n)\|\delta_n(x,y)\|^2 \\ \|\Phi_j(x^{(l)} : l \leq n) - \Phi_j(y^{(l)} : l \leq n)\|^2 \;&\geq\; \underline{c}_j(n)\|\delta_n(x,y)\|^2 \\ 1 + \underline{c}_0^E(n)\Delta_n^2 + \sum_{j=1}^{m}(\sigma_n^j)^2\underline{c}_j(n)\Delta_n \;&>\; -2\underline{c}_E(n)\Delta_n \end{aligned} \tag{3.12}$$

where $\underline{c}_I, \underline{c}_E(n), \underline{c}_0^I(n), \underline{c}_0^E(n), \underline{c}_j(n)$ *are finite, deterministic, real numbers, then*

(3.13)

$$\underline{\kappa}_2 \geq \liminf_{n\to+\infty} \frac{\sum_{i=0}^{n} \Big(\frac{2\underline{c}_E(i) + 2\underline{c}_I(i) + (\underline{c}_0^E(i) - \underline{c}_0^I(i))\Delta_i + \sum_{j=1}^{m} (\sigma_i^j)^2 \underline{c}_j(i)}{1 + 2\underline{c}_E(i)\Delta_i + \underline{c}_0^E(i)\Delta_i^2 + \sum_{j=1}^{m} (\sigma_i^j)^2 \underline{c}_j(i)\Delta_i} \Big) \Delta_i}{t_{n+1}}.$$

Remark. This theorem provides an uniform estimate of the 'spectrum' of (forward) p–th moment contractivity exponents for the class of stochastic difference equations satisfying monotonicity conditions (3.10) and (3.12). Of course these estimates are 'worst case estimates' (but sharp ones) again. Since the analysis of nonlinear, nonautonomous, discrete time stochastic mappings turns out to be very difficult, we restrict our attention only to the case of mean square calculus as before. The obtained result is useful to control the propagation of initial errors by explicit–implicit numerical methods (cf. [37] and future publications). The latter property can be used to prove convergence of nonlinear numerical methods as in deterministic analysis.

Proof. Fix any $x, y \in \mathbb{D}$. Define $v_n := \mathbb{E}\, \|X_n(x) - X_n(y)\|^2$. Taking the square of Euclidean vector norm of

$$X_{n+1}(x) - \Phi_0^I(X_l(x) : l \leq n+1)\Delta_n - X_{n+1}(y) + \Phi_0^I(X_l(x) : l \leq n+1)\Delta_n$$

and its expectation value afterwards one receives

$$\begin{aligned} & v_{n+1}\left(1 - 2\Delta_n \overline{c}_I(n) + \Delta_n^2 \overline{c}_0^I(n)\right) \\ \leq\ & v_n + 2\Delta_n \mathbb{E} < \Phi_0^E(X_i : i \leq n) - \Phi_0^E(X_i(y) : i \leq n), X_n(x) - X_n(y) > \\ & +\Delta_n^2 \mathbb{E}\, \|\Phi_0^E(X_i(x) : i \leq n) - \Phi_0^E(X_i(y) : i \leq n)\|^2 \\ & +\Delta_n \sum_{j=1}^{m} (\sigma_n^j)^2 \mathbb{E}\, \|\Phi_j(X_i(x) : i \leq n) - \Phi_j(X_i(y) : i \leq n)\|^2 \\ \leq\ & v_n \left(1 + 2\Delta_n \overline{c}_E(n) + \Delta_n^2 \overline{c}_0^E(n) + \Delta_n \sum_{j=1}^{m} (\sigma_n^j)^2 \overline{c}_j(n) \right) \end{aligned}$$

using the independence of ξ_n^j, that $\mathbb{E}\, (\xi_n^j)^2 = (\sigma_n^j)^2$ and conditions (3.10). A similar estimate is derived under conditions (3.12). In the other direction we obtain

$$\begin{aligned} & v_n \left(1 + 2\Delta_n \underline{c}_E(n) + \Delta_n^2 \underline{c}_0^E(n) + \Delta_n \sum_{j=1}^{m} (\sigma_n^j)^2 \underline{c}_j(n) \right) \\ \leq\ & v_n + 2\Delta_n \mathbb{E} < \Phi_0^E(X_i(x) : i \leq n) - \Phi_0^E(X_i(y) : i \leq n), X_n(x) - X_n(y) > \\ & +\Delta_n^2 \mathbb{E}\, \|\Phi_0^E(X_i(x) : i \leq n) - \Phi_0^E(X_i(y) : i \leq n)\|^2 \\ & +\Delta_n \sum_{j=1}^{m} (\sigma_n^j)^2 \mathbb{E}\, \|\Phi_j(X_i : i \leq n) - \Phi_j(X_i(y) : i \leq n)\|^2 \\ \leq\ & v_{n+1}\left(1 - 2\Delta_n \underline{c}_I(n) + \Delta_n^2 \underline{c}_0^I(n)\right) \end{aligned}$$

Obviously, under (3.10) and (3.12), one gains the estimates

$$v_{n+1} \leq v_n \cdot \left(\frac{1 + 2\overline{c}_E(n)\Delta_n + \overline{c}_0^E(n)\Delta_n^2 + \sum_{j=1}^{m} (\sigma_n^j)^2 \overline{c}_j(n)\Delta_n}{1 - 2\overline{c}_I(n)\Delta_n + \overline{c}_0^I(n)\Delta_n^2} \right)$$

$$\leq v_0 \cdot \prod_{i=0}^{n} \left(\frac{1 + \overline{c}_E(i)\Delta_i + \overline{c}_0^E(i)\Delta_i^2 + \sum_{j=1}^{m} (\sigma_i^j)^2 \overline{c}_j(i)\Delta_i}{1 - 2\overline{c}_I(i)\Delta_i + \overline{c}_0^I(i)\Delta_i^2} \right)_+$$

$$\leq v_0 \cdot \exp \left(\sum_{i=0}^{n} \Big(\frac{2\overline{c}_E(i) + 2\overline{c}_I(i) + (\overline{c}_0^E(i) - \overline{c}_0^I(i))\Delta_i + \sum_{j=1}^{m} (\sigma_i^j)^2 \overline{c}_j(i)}{1 - 2\overline{c}_I(i)\Delta_i + \overline{c}_0^I(i)\Delta_i^2} \Big) \Delta_i \right)$$

and in the other direction

$$v_n \leq v_{n+1} \cdot \left(\frac{1 - 2\underline{c}_I(n)\Delta_n + \underline{c}_0^I(n)\Delta_n^2}{1 + 2\underline{c}_E(n)\Delta_n + \underline{c}_0^E(n)\Delta_n^2 + \sum_{j=1}^{m} (\sigma_n^j)^2 \underline{c}_j(n)\Delta_n} \right)$$

$$\leq v_{n+1} \cdot \exp \left(\frac{-2\underline{c}_E(n) - 2\underline{c}_I(n) - (\underline{c}_0^E(n) - \underline{c}_0^I(n))\Delta_n - \sum_{j=1}^{m} (\sigma_n^j)^2 \underline{c}_j(n)}{1 + 2\underline{c}_E(n)\Delta_n + \underline{c}_0^E(n)\Delta_n^2 + \sum_{j=1}^{m} (\sigma_n^j)^2 \underline{c}_j(n)\Delta_n} \Delta_n \right),$$

and therefore

$$v_{n+1} \geq v_0 \cdot \exp \left(\sum_{i=0}^{n} \frac{2\underline{c}_E(i) + 2\underline{c}_I(i) + (\underline{c}_0^E(i) - \underline{c}_0^I(i))\Delta_i + \sum_{j=1}^{m} (\sigma_i^j)^2 \underline{c}_j(i)}{1 + 2\underline{c}_E(i)\Delta_i + \underline{c}_0^E(i)\Delta_i^2 + \sum_{j=1}^{m} (\sigma_i^j)^2 \underline{c}_j(i)\Delta_i} \Delta_i \right),$$

respectively, using twice the elementary inequality

$$\frac{1+y}{1+x} \leq \exp\left(\frac{y-x}{1+x}\right).$$

Taking the logarithm and the limit as n tends to $+\infty$ yield the desired estimates. Consequently, the proof is completed. ◇

4. The example of linear systems and correlations

Consider nonautonomous linear stochastic systems

$$dX(t) = A(t)\,X(t)\,dt + \sum_{j=1}^{m} (B^j(t)\,X(t) + b^j(t))\,dW^j(t) \tag{4.1}$$

where $X(t)$ denotes their d–dimensional solution; $A, B^j (j = 1, 2, ..., m)$ deterministic, real–valued matrices, b^j deterministic, real–valued vectors and W^j are uncorrelated, one-dimensional standard Wiener processes. Take $\mathbb{D} = \mathbb{R}^d$.

4.1. **A corollary for linear systems.** As a consequence of presented analysis, we get the following corollary. The proof can be carried out by the application of continuous variation-of-constants inequalities (see Lemma 8.10.2 in [35]) as a generalization of the well-known Bellman-Gronwall Lemma. The detailed proof is omitted here.

Corollary 4.1. *Consider $X(t)$ satisfying Itô SDE (4.1) for $t \in [t_0, +\infty)$. Assume that $\mathbb{E}\,\|X(t_0\|^2 < +\infty$ and*

$$\forall x \in \mathbb{R}^d, \forall t \geq t_0 \;:\; 2 < x, A(t)x > + \sum_{j=1}^{m} \|B^j(t)x + b^j(t)\|^2 \;\leq\; \overline{K}_1(t) + \overline{K}_2(t)\|x\|^2$$

where the deterministic functions $\overline{K}_1, \overline{K}_2$ are L^1-integrable with respect to Lebesque measure, and assume

$$\forall t > t_0 \;:\; \mathbb{E}\,\|X(t_0)\|^2 + \int_{t_0}^{t} \overline{K}_1(s) \exp(-\int_{t_0}^{s} \overline{K}_2(z)dz)\, ds > 0\,.$$

Then, it holds

$$\overline{\lambda_2} \;\leq\; \limsup_{t \to +\infty} \frac{ln\left(\mathbb{E}\,\|X(t_0)\|^2 + \int_{t_0}^{t} \overline{K}_1(s) \exp(-\int_{t_0}^{s} \overline{K}_2(z)dz\, ds)\right) + \int_{t_0}^{t} \overline{K}_2(s)ds}{t}\,.$$

Furthermore, assume that

$$\forall x \in \mathbb{R}^d, \forall t \geq t_0 \;:\; 2 < x, A(t)x > + \sum_{j=1}^{m} \|B^j(t)x + b^j(t)\|^2 \;\geq\; \underline{K}_1(t) + \underline{K}_2(t)\|x\|^2$$

where the deterministic functions $\underline{K}_1, \underline{K}_2$ are L^1-integrable with respect to Lebesque measure, and assume

$$\forall t > t_0 \;:\; \mathbb{E}\,\|X(t_0)\|^2 + \int_{t_0}^{t} \underline{K}_1(s) \exp(-\int_{t_0}^{s} \underline{K}_2(z)dz)\, ds > 0\,.$$

Then

$$\underline{\lambda_2} \;\geq\; \liminf_{t \to +\infty} \frac{ln\left(\mathbb{E}\,\|X(t_0)\|^2 + \int_{t_0}^{t} \underline{K}_1(s) \exp(-\int_{t_0}^{s} \underline{K}_2(z)dz)\, ds\right) + \int_{t_0}^{t} \underline{K}_2(s)ds}{t}\,.$$

Remark. The limits do not depend on initial values really. The above assumptions on $\overline{K}_i, \underline{K}_i$ are only made to exclude nonmeaningful extreme cases. If $b^j \equiv 0$ (i.e. the case of fundamental solution), matrices A and all B^j have complete basis systems of eigenvectors, then $\overline{\lambda}_2$ and $\underline{\lambda}_2$ are exclusively controlled by the interaction of eigenvalues of $A, B^1, ..., B^m$. For example, if the matrices A, B^j are time-independent, then

$$\overline{\lambda}_2 \;\leq\; 2\, \max_i\{\mu_i\} + \sum_{j=1}^{m} \max_i\{\rho_{i,j}^2\}$$

and

$$\underline{\lambda}_2 \;\geq\; 2\, \min_i\{\mu_i\} + \sum_{j=1}^{m} \min_i\{\rho_{i,j}^2\}$$

where μ_i are the eigenvalues of matrix A, $\rho_{i,j}^2$ the eigenvalues of positive semi-definite $B^j(B^j)^T$. Note, in the linear autonomous case with pure multiplicative noise, the concept of moment stability exponents coincides with that of moment Lyapunov exponents. In the one-dimensional case we can obtain the equality in these estimates, hence sharp estimates have been found by Corollary 4.1.

A similar corollary holds for the contractivity exponents $\underline{\kappa}, \overline{\kappa}$ of the linear equation as above. For example, under

$$\underline{C}_2(t)\|x-y\|^2 \;\leq\; 2 < x-y, A(t)(x-y) > + \sum_{j=1}^{m} \|B^j(t)(x-y)\|^2 \;\leq\; \overline{C}_2(t)\|x-y\|^2$$

for all $x, y \in \mathbb{R}^d$ and all $t \geq t_0$, where the deterministic functions $\overline{C}_2, \underline{C}_2$ are L^1-integrable with respect to Lebesque measure, one can show that

$$\liminf_{t\to+\infty} \frac{\int_{t_0}^t \underline{C}_2(s)ds}{t} \;\leq\; \underline{\kappa_2} \;\leq\; \overline{\kappa_2} \;\leq\; \limsup_{t\to+\infty} \frac{\int_{t_0}^t \overline{C}_2(s)ds}{t}\,.$$

It is worth noting that the behaviour of inhomogeneouity $b^j = b^j(t)$ does not play any role within the concept of asymptotical moment contractivity, in contrast to the concept of asymptotical moment stability.

4.2. **Contractivity and stability are not always the same!** The following illustrative example of a onedimensional, linear, but nonautonomous SDE shows that the concepts of contractivity and stability exponents do not coincide in general! Moreover, these concepts will provide assertions of totally different qualitative behaviour, and the concept of contractivity is more appropriate in case of additive noise. Consider SDE

$$dX(t) \;=\; \mu\, X(t)\, dt \;+\; \sigma \exp(\alpha t)\, dW_t \tag{4.2}$$

where μ, σ, α are deterministic, real parameters, and W_t represents the standard Wiener process.

Lemma 4.2. *Assume $p \geq 1$. The class of stochastic processes $(X(t))_{(t\geq t_0)}$ governed by SDE (4.2) possesses characteristic exponents with*

$$\overline{\lambda}_p \geq \underline{\lambda}_p \geq p\mu$$

and

$$\overline{\kappa}_p = \underline{\kappa}_p = p\mu\,.$$

Moreover, assume $\sigma^2 > 0$. Then the class of stochastic processes $(X(t))_{(t\geq t_0)}$ governed by SDE (4.2) satisfies the estimate

$$\underline{\lambda}_p \leq \overline{\lambda}_p \leq p\max(\mu, \alpha)$$

where $p \geq 2$. If $p = 2$, it holds the equality

$$\overline{\lambda}_2 = \underline{\lambda}_2 = 2\max(\mu, \alpha)\,.$$

Remark. If $p = 2$ and $\alpha > 0, \mu < 0$ then, loosely speaking, one has an asymptotically contractive SDE, but not an asymptotically stable one. In another words, one can always find a SDE (bilinear with nonautonomous diffusion part) with negative contractivity exponent, but with positive stability exponent.

Proof. Assume $p \geq 1$. Without loss of generality, we may set set $t_0 = 0$. First, consider an estimation of the stability exponents. Define $v(t) := \mathbb{E}\;|X(t)|^p$. A calculation gives

$$\begin{aligned} v(t) &= v(0) + p\int_0^t \left(\sigma^2 \frac{p-1}{2}\exp(2\alpha s)(\mathbb{E}\,[|X(s)|^{p-2}]) + \mu v(s)\right) ds \\ &\geq v(0) + p\mu \int_0^t v(s)ds \end{aligned}$$

by the use of Dynkin's formula and simple monotonicity arguments. Consequently, $\overline{\lambda}_p \geq \underline{\lambda}_p \geq p\mu$ by the application of Bellman-Gronwall inequality with constant kernel. If $p \geq 2$, the evolution of $v(t)$ satisfies

$$\begin{aligned} v(t) &\leq v(0) + \int_0^t \left(\mathbb{E}\,[p(p-1)\frac{\sigma^2}{2}\exp(2\alpha s)|X(s)|^{p-2}] + p\mu v(s)\right) ds \\ &\leq v(0) + \int_0^t \left(\varepsilon\sigma^2 \left[\frac{p-1}{\varepsilon}\right]^{p/2} \exp(p\alpha s) + (p\mu + \varepsilon(p-2)\frac{\sigma^2}{2})v(s)\right) ds \end{aligned}$$

for all $\varepsilon > 0$, by the use of Young's inequality applied to

$$c(s)|y(s)|^{p-2} \;\leq\; \frac{2}{p}|c(s)|^{p/2} + \frac{p-2}{p}|y(s)|^p$$

where

$$c(s) = \frac{p-1}{\varepsilon}\exp(2\alpha s),\; y(s) = |X(s)|\,.$$

Now, apply the continuous variation-of-constants inequality from [35] (see Lemma 8.10.2) to obtained inequality for $v(t)$. Thereby one arrives at the estimate

$$v(t) \leq \left(v(0) + \int_0^t K_1(u)\exp\left(-\int_0^u K_2(z)dz\right)du\right)\cdot \exp\left(\int_0^t K_2(u)du\right)$$

where

$$K_1(u) = \sigma^2\left[\frac{p-1}{\varepsilon}\right]^{p/2} \exp(p\alpha u),\; K_2(u) = p\mu + \varepsilon(p-2)\frac{\sigma^2}{2}\,.$$

By taking the logarithm and the limit as time t tends to $+\infty$ one arrives at

$$\overline{\lambda}_p \leq p\max(\mu + \varepsilon\frac{\sigma^2(p-2)}{2}, \alpha)\,.$$

This observation holds for all $\varepsilon > 0$, hence the claimed result for stability exponents

$$\overline{\lambda}_p \leq p\max(\mu,\alpha)$$

can be established. In case of $p = 2$, one may explicitly calculate the exact value of the stability exponent. It is clear that upper and lower exponents must coincide in this case. One encounters with

$$v(t) \;=\; v(0) + \int_0^t [\sigma^2\exp(2\alpha s) + 2\mu v(s)]ds\,.$$

This equation can be solved by the method of variation of constants. It yields

$$\begin{aligned} \frac{ln[v(t)]}{t} &= 2\mu + \frac{ln\left[v(0) + \sigma^2\int_0^t \exp(2(\alpha-\mu)s)ds\right]}{t} \\ &= 2\mu + \frac{ln\left[v(0) + \sigma^2\frac{\exp(2(\alpha-\mu)t) - 1}{2(\alpha-\mu)}\right]}{t}\,. \end{aligned}$$

Therefore, by taking the limit $t \to +\infty$, one arrives at

$$\overline{\lambda}_2 = \underline{\lambda}_2 = 2\mu + 2(\alpha-\mu)_+ = 2\max(\alpha,\mu)$$

provided that $\sigma^2 > 0$. The argumentation for the contractivity exponents is much easier, since

$$\begin{aligned} u(t,x,y) &= \mathbb{E}\,|X(t,x) - X(t,y)|^p = u(0,x,y) + p\mu \int_0^t u(s,x,y)ds \\ &= u(0,x,y)\exp\Big(p\mu t\Big) \end{aligned}$$

for any fixed, finite $x, y \in \mathbb{R}^1$. After the application of Bellman-Gronwall Lemma, one finds the predicted estimate. Thus, the proof can be completed. $\diamond$

Remark. The proof can also be carried out by using the pathwise expression of exact solution

$$X(t) = \exp(\mu t)\left(X(0) + \sigma \int_0^t \exp((\alpha - \mu)s)dW_s\right)$$

and well-known moment martingale inequalities. However, this idea is restricted to that specific one-dimensional equation by the apriori knowledge of its exact solution. In contrast to that, our suggested inequality approach combining variation-of-constants inequalities and monotonicity arguments can be applied to much more general cases of SDEs.

4.3. **When contractivity and stability coincide.** There is a certain specific situation when the estimates for moment contractivity and stability coincide. This will be the case for bilinear systems with multiplicative noise or for nonlinear systems where $x_* = 0$ is a trivial equilibrium. For simplicity, consider $\mathbb{D} = \mathbb{R}^d$.

Proposition 4.3. *Let process $X(t,z)_{(t \geq t_0)}$ be a stochastic dynamical system with*

$$\exists x_* \in \mathbb{D}\ \forall t \geq t_0 \ :\ X(t,x_*) = 0 \ (a.s.)\,.$$

Then, moment contractivity and moment stability exponents coincide with respect to equilibrium x_*, i.e.

$$\forall z \in \mathbb{D}\ :\ \overline{\lambda}_p(z) = \overline{\kappa}_p(z,x_*), \underline{\lambda}_p(z) = \underline{\kappa}_p(z,x_*)\,.$$

Moreover, in the situation of Itô SDEs with infinitesimal generator $\mathcal{L}$ when there are nonnegative real constants $K_{1,p}, K_{2,p}$ such that for all $x \in \mathbb{D}$

$$\mathcal{L}||x||^p \ \leq\ K_{1,p} + K_{2,p}||x||^p$$

and initial values which are independent of all σ–fields $\mathcal{F}^j_\infty = \sigma\{W^j_t : t \geq t_0\}$, then one finds uniform estimates of upper exponents which do not depend on the choice of initial value z. Additionally, for Itô SDEs with infinitesimal generator $\mathcal{L}$ and nonnegative real constants $C_{1,p}, C_{2,p}$ such that for all $x \in \mathbb{D}$

$$\mathcal{L}||x||^p \ \geq\ C_{1,p} - C_{2,p}||x||^p$$

and initial values which are independent of all σ–fields $\mathcal{F}^j_\infty = \sigma\{W^j_t : t \geq t_0\}$, there are uniform estimates of lower exponents which do not depend on the choice of initial value z.

Proof. For the first assertion, one only has to note

$$\mathbb{E}\,||X(t,x) - X(t,x_*)||^p \ =\ \mathbb{E}\,||X(t,x)||^p\,.$$

The second assertion for Itô SDEs becomes clear from the following. From the inequality of Minkowski one knows

$$(\mathbb{E}\,\|X(t,x)-X(t,y)\|^p)^{1/p} \le (\mathbb{E}\,\|X(t,x)\|^p)^{1/p} + (\mathbb{E}\,\|X(t,y)\|^p)^{1/p}\,.$$

After taking this inequality to the power p one recognizes that the stability evolution can be used to dominate the evolution of initial perturbations (hence that of contractivity, see also next subsection). Under the condition above, by Dynkin's formula, we also receive

$$\mathbb{E}\,\|X(t,z)\|^p \ \le\ (\|z\|^p + K_{1,p}(t-t_0))\exp(K_{2,p}(t-t_0))\,,$$

hence $\overline{\lambda}_p \le K_{2,p}$ and $\overline{\kappa}_p \le K_{2,p}$. The third assertion can be seen after application of the inverse triangular inequality for vector norms, namely

$$|(\mathbb{E}\,\|X(t,x)\|^p)^{1/p} - (\mathbb{E}\,\|X(t,y)\|^p)^{1/p}| \ \le\ (\mathbb{E}\,\|X(t,x)-X(t,y)\|^p)^{1/p}\,.$$

Analogously to steps for second assertion, one confirms the third assertion and obtains $\min(\underline{\lambda}_p, \underline{\kappa}_p) \ge -C_{2,p}\chi_{\{C_{1,p}=0\}}$ where $\chi_{\{.\}}$ is the characteristic function of inscribed set. Thus, proof is complete. ⋄

Remark. The coincidence of stability and contractivity concept for fixed $x_* = 0$ can also be motivated from the simple fact

$$< f(t,x_*) - f(t,y), x_* - y > = < f(t,y), y >$$

provided that $x_* = 0, f(t,0) = 0$. The operator $\mathcal{L}$ applied to $\|x\|^p$ can always be bounded from below and above if all coefficients a, b^j of the considered SDE are globally Lipschitz continuous (uniformly with respect to time t). Thus, for the class of SDEs with that property, we receive an estimate for the spectrum of stability and contractivity exponents which does not depend on the initial values - as a consequence of our proposition (spectrum of stability and contractivity exponents on deterministic, open domain $\mathbb{D} = \mathbb{R}^d$ means here the distance

$$\sup_{x\in\mathbb{D}} \overline{\lambda}_p(x) - \inf_{x\in\mathbb{D}} \underline{\lambda}_p(x) + \sup_{x,y\in\mathbb{D}} \overline{\kappa}_p(x,y) - \inf_{x,y\in\mathbb{D}} \underline{\kappa}_p(x,y)\,.$$

4.4. **Upper stability exponents dominate contractivity exponents.** As already noted in proof of Proposition 4.3, the stability concept is dominating that of contractivity. This fact can be manifested by the following assertion.

Proposition 4.4. *Let process $X(t,z)_{(t\ge t_0)}$ be a stochastic dynamical system with finite p-th moment ($p \ge 1$) for all finite times t and with finite upper stability exponents $\overline{\lambda}_p(x)$ and $\overline{\lambda}_p(y)$ where $x, y \in \mathbb{D}$. Then, it holds*

$$\overline{\kappa}_p(x,y) \ \le\ \max(\overline{\lambda}_p(x), \overline{\lambda}_p(y))\,.$$

Proof. By application of the inequalities of Minkowski and Hoelder one knows

$$u(t,x,y) := \mathbb{E}\,\|X(t,x)-X(t,y)\|^p \le 2^{p-1}\,(\mathbb{E}\,\|X(t,x)\|^p + \mathbb{E}\,\|X(t,y)\|^p)\ .$$

This implies

$$\begin{aligned} ln[u(t,x,y)] \ &\le\ ln[2^{p-1}] + \max(\overline{\lambda}_p(x), \overline{\lambda}_p(y))t \\ &\quad + ln[\exp\left(-\max(\overline{\lambda}_p(x), \overline{\lambda}_p(y))t\right)(\mathbb{E}\,\|X(t,x)\|^p + \mathbb{E}\,\|X(t,y)\|^p)]. \end{aligned}$$

The latter logarithm must converge to a finite number as time t tends to infinity because of the existence of upper stability exponents. Now, one takes the limit with respect to time and finds

$$\limsup_{t \to +\infty} \frac{ln[u(t,x,y)]}{t} \leq \max(\overline{\lambda}_p(x), \overline{\lambda}_p(y)).$$

This completes the proof. $\diamond$

5. Generalized V-exponents and examples

More general than in previous sections, the following definiton provides with concepts for the qualitative description of asymptotical growth behaviour along certain functionals of underlying stochastic process. This definition is particularly introduced to describe and investigate the dissipativity, stability and contractivity properties of nonlinear stochastic dynamical systems. In passing, the concepts of V–dissipativity (as well as V–stability and V–contractivity) have been introduced in [35]. Here we only continue with a refinement concerning those concepts (i.e. in order to find appropriate *speed measures of nonlinear exponential growth*).

5.1. **Definition of V-exponents.** Our aim is to incorporate a larger class of nonlinear stochastic systems than before in order to derive estimates of their asymptotical growth behaviour. Thus, we introduce the following definition.

Definition 3. The *upper (forward moment) V–exponent* of a given stochastic process $(X_t)_{t \in \mathcal{T}}$ on domain $\mathbb{D}$ is defined to be

$$\overline{\lambda}_V := \limsup_{t \to +\infty} ln \left(\mathbb{E}\, V(t, X(t)) \right) \tag{5.1}$$

for a fixed deterministic function $V(t,x) : [t_0, +\infty) \times \mathbb{D} \longrightarrow \mathbb{R}_+$ (or positive functional), provided that this expression exists. The *lower (forward moment) V–exponent* of a given stochastic process $(X_t)_{t \in \mathcal{T}}$ on domain $\mathbb{D}$ is defined to be

$$\underline{\lambda}_V := \liminf_{t \to +\infty} ln \left(\mathbb{E}\, V(t, X(t)) \right) \tag{5.2}$$

for a fixed deterministic function $V(t,x) : [t_0, +\infty) \times \mathbb{D} \longrightarrow \mathbb{R}_+$ (or positive functional), provided that this expression exists.

Obviously, the art consists in finding appropriate functionals V. We will demonstrate with two illustrative, nonlinear examples how such functions could look like.

5.2. **Asymptotics of a nonlinear stochastic oscillator.** In Mechanical and Electronical Engineering one encounters with nonlinear oscillators perturbed by noise and excited by periodic forces (e.g. in modelling of stabilizing electric circuits or of mechanical structures). For simplicity, consider the one-degree of freedom oscillator

$$\ddot{x} + \alpha^2 x + \frac{\beta^2 r^2}{\dot{x}} - \gamma^2 (r^2 - \alpha^2 x^2 - (\dot{x})^2)\dot{x} = \sigma \xi_t \tag{5.3}$$

where $r, \alpha, \beta, \gamma, \sigma \in \mathbb{R}$ and ξ_t is white or coloured noise. $x = x(t)$ represents the displacement of oscillations from the rest-point, whereas $\dot{x} = \dot{x}(t)$ the velocity of the oscillations. Let us restrict to the white noise case only. Here $r > 0$ is a real parameter controlling the asymptotical behaviour. We will see that $|r|$ and $|\alpha|$ determine the shape of a limit ellipsoid where all the ellipsoid–interior trajectories converge in probability to. Additionally, we are interested in a measure for the convergence speed of the interior trajectories, without considering the possibility of 'crossing' trajectories or leaves after first entrance. Define the open deterministic domain

$$\mathbb{D} := \left\{ (x,y) \in \mathbb{R}^2 \setminus (\mathbb{R}^1 \times \{0\}) : r^2 - \alpha^2 x^2 - y^2 > 0 \right\} .$$

Lemma 5.1. *Consider the stochastic process $(x(t), \dot{x}(t))$, $t \in \mathbb{R}_+$, governed by SDE (5.3), started at any values $(x(0), \dot{x}(0)) \in \mathbb{D}$. Assume $2\beta^2 r^2 - \sigma^2 < 0$. Then*

$$\forall \alpha, \gamma \in \mathbb{R}^1 \; : \; V(x(t), \dot{x}(t)) := (r^2 - \alpha^2 x^2(t) - \dot{x}^2(t))_+ \quad \underset{t \to +\infty}{\overset{\mathbb{P}}{\longrightarrow}} \quad 0 \,,$$

i.e. convergence in probability. In general its V-exponents satisfy

$$\overline{\lambda}_V(r,\beta,\gamma,\sigma) \left\{ \begin{array}{lll} = & -\infty & \text{if } \; 2\beta^2 r^2 - \sigma^2 < 0 \\ \leq & 0 & \text{if } \; 2\beta^2 r^2 - \sigma^2 \geq 0 \end{array} \right.$$

and

$$\underline{\lambda}_V(r,\beta,\gamma,\sigma) \left\{ \begin{array}{lll} \geq & -2r^2\gamma^2 & \text{if } \; 2\beta^2 r^2 - \sigma^2 \geq 0 \\ = & -\infty & \text{if } \; 2\beta^2 r^2 - \sigma^2 < 0 \end{array} \right. .$$

Remark. The received relation exhibits an interesting relation between noise intensity σ, nonlinearity parameter β and limit radius $|r|$. For example, if the noise intensity is large enough (i.e. $\sigma^2 > 2\beta^2 r^2$) then all ellipsoid–interior trajectories are attracted with an infinite exponential speed, i.e. they approach to the limit ellipsoid at finite times. Obviously, the situation $2\beta^2 r^2 = \sigma^2$ represents a **bifurcation** point of corresponding dynamical system in the sense of a qualitative change of convergence speed of interior trajectories towards the limit ellipsoid with radius $|r| > 0$ and scale parameter $|\alpha|$. In the situation $\sigma^2 \leq 2\beta^2 r^2$ the ellipsoid–interior trajectories may take an infinite time in the mean sense to approach to the limit cycle. To be honest and complete, we consider by this Lyapunov functional V only the stochastic process stopped at the first hitting of the ellipsoid with radius $|r| > 0$ and scale parameter $|\alpha|$, and leaving all its future values at that point where it has hitted this ellipsoid at first. With this procedure we avoid considering the contribution of 'crossing trajectories' for the sake of simplicity of our considerations here. Besides, of course, the assumption $r^2 > 0$ is essential for the meaningful construction of considered Lyapunov functional. It is interesting to note that growing noise intensity leads to stabilizing effects on the convergence towards the limit ellipsoid on the boundary of domain $\mathbb{D}$.

Proof. Consider Lyapunov functional

$$V(x,y) := (r^2 - \alpha^2 x^2 - y^2)_+$$

for fixed parameters $r, \alpha, \beta, \gamma, \sigma$. Then, by Dynkin's formula (stopped at first hitting the boundary of deterministic domain $\mathbb{D}$) one arrives at

$$\begin{aligned} & \Big(v(0) + (2\beta^2 r^2 - \sigma^2)t \Big)_+ \; \geq \\ v(t) \;\; := \;\; & \mathbb{E}\, V(x(t), \dot{x}(t)) \; = \; \mathbb{E}\, V(x(0), \dot{x}(0)) + \mathbb{E} \int_0^t \mathcal{L}V(x(s), \dot{x}(s)) ds \\ \geq \;\; & \Big(\mathbb{E}\, V(x(0), \dot{x}(0)) + (2\beta^2 r^2 - \sigma^2)t - 2\gamma^2 r^2 \int_0^t \mathbb{E}\, V(x(s), \dot{x}(s)) ds \Big)_+ \\ \geq \;\; & \Big(\mathbb{E}\, V(x(0), \dot{x}(0)) + (2\beta^2 r^2 - \sigma^2)t \Big)_+ \exp(-2\gamma^2 r^2 t) \end{aligned}$$

where

$$\begin{aligned}\mathcal{L}V(x,y) &= \left(y\frac{\partial}{\partial x} + [\gamma^2(r^2-\alpha^2x^2-y^2)y - \frac{r^2\beta^2}{y} - \alpha^2 x]\frac{\partial}{\partial y} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}\right)V(x,y)\\ &= (-\alpha^2 2xy + 2\beta^2r^2 + \alpha^2 2xy - 2\gamma^2y^2V(x,y) - \sigma^2)\chi_{\{r^2-\alpha^2x^2-y^2>0\}}(x,y)\\ &= (2\beta^2r^2 - \sigma^2 - 2\gamma^2y^2V(x,y) - \sigma^2)\chi_{\{r^2-\alpha^2x^2-y^2>0\}}(x,y)\,.\end{aligned}$$

If $2\beta^2r^2 < \sigma^2$ then there is a finite critical time $\hat{t} > 0$ such that for all $t > \hat{t}$ one finds $v(t) = 0$, hence $\overline{\lambda}_V = \underline{\lambda}_V = -\infty$. Now, assume $2\beta^2r^2 \geq \sigma^2$. Then we can immediately conclude from the above application of Dynkin's formula that

$$-2\gamma^2r^2 \;\leq\; \underline{\lambda}_V \;\leq\; \overline{\lambda}_V \;\leq 0\,.$$

Now assume $2\beta^2r^2 < \sigma^2$. Then the convergence of $V(x(t),\dot{x}(t))$ towards 0 in probability can be established by application of well-known Chebyshev-inequality, i.e. this fact follows from

$$\forall\varepsilon > 0 : \mathbb{P}\{V(x(t),\dot{x}(t)) \geq \varepsilon\} \;\leq\; \frac{\mathbb{E}\,V(x(t),\dot{x}(t))}{\varepsilon}$$

and the polynomially fast decline of $\mathbb{E}\,V(x(t),\dot{x}(t))$ towards 0 at finite time, hence the proof is complete. ⋄

5.3. **How noise may stabilize nonlinear oscillations.** Let us slightly change our model from previous subsection in order to observe another interesting effect. Consider Itô SDE

$$\ddot{x} + \alpha^2 x - \gamma^2(r^2-\alpha^2x^2-(\dot{x})^2)\dot{x} \;=\; \sigma\sqrt{|r^2-\alpha^2x^2-(\dot{x})^2|}\,\xi_t \tag{5.4}$$

where $r,\alpha,\gamma,\sigma \in \mathbb{R}$ and ξ_t is white noise. Again $x = x(t)$ represents the displacement of oscillations from the rest-point, whereas $\dot{x} = \dot{x}(t)$ the velocity of the oscillations. Define the open deterministic domain

$$\mathbb{D} := \left\{(x,y) \in \mathbb{R}^2 : r^2 - \alpha^2x^2 - y^2 > 0\right\}$$

as interior of an ellipsoid with radius $|r|$ and scale parameter $|\alpha|$.

Lemma 5.2. *Assume that the stochastic process $(x(t),\dot{x}(t))_{(t\in\mathbb{R}_+)}$ is generated by Itô SDE (5.4), started at any values $(x(0),\dot{x}(0)) \in \mathbb{D}$ with parameters $r^2,\sigma^2 > 0$. Then*

$$\forall\alpha,\gamma \in \mathbb{R}^1 \;:\; V(x(t),\dot{x}(t)) := (r^2 - \alpha^2x^2(t) - \dot{x}^2(t))_+ \;\underset{t\to+\infty}{\overset{\mathbb{P}}{\longrightarrow}}\; 0\,,$$

i.e. convergence in probability. In general this process possesses V-exponents

$$-2\gamma^2r^2 - \sigma^2 \;\leq\; \underline{\lambda}_V(r,\gamma,\sigma) \;\leq\; \overline{\lambda}_V(r,\gamma,\sigma) \;\leq\; -\sigma^2\,.$$

Remark. The process $(x(t),\dot{x}(t))$, $t \in \mathbb{R}_+$ satisfying (5.4) needs an infinite time (in the mean sense) to reach the boundary of domain $\mathbb{D}$ (i.e. when $r^2 > 0$), hence to the limit ellipsoid. However, on the infinite time-horizon it approaches to that limit set with exponential speed stabilized by increasing noise intensities $|\sigma|$.

Proof. Consider Lyapunov functional

$$V(x,y) := (r^2 - \alpha^2x^2 - y^2)_+$$

for fixed parameters $r, \alpha, \gamma, \sigma$. Define $v(t) := \mathbb{E}\, V(x(t), \dot{x}(t))$. Then, by Dynkin's formula (stopped at first hitting the boundary of deterministic domain $\mathbb{D}$) one arrives at

$$\Big(v(0) - \sigma^2 \int_0^t v(s)ds\Big)_+ \geq v(t) = v(0) + \mathbb{E} \int_0^t \mathcal{L}V(x(s), \dot{x}(s))ds$$

$$\geq \Big(\mathbb{E}\, V(x(0), \dot{x}(0)) - (2\gamma^2 r^2 + \sigma^2) \int_0^t \mathbb{E}\, V(x(s), \dot{x}(s))ds\Big)_+$$

where

$$\begin{aligned}\mathcal{L}V(x,y) &= \Big(y\frac{\partial}{\partial x} + [\gamma^2 V(x,y)y - \alpha^2 x]\frac{\partial}{\partial y} + \frac{\sigma^2}{2}V(x,y)\frac{\partial^2}{\partial y^2}\Big) V(x,y)\\ &= (-2\alpha^2 xy + 2\alpha^2 xy - (2\gamma^2 y^2 + \sigma^2)V(x,y))\chi_{\{r^2-\alpha^2x^2-y^2>0\}}(x,y)\\ &= -(2\gamma^2 y^2 + \sigma^2)V(x,y)\chi_{\{r^2-\alpha^2x^2-y^2>0\}}(x,y)\,,\end{aligned}$$

where $\chi_{\{.\}}$ represents the characteristic function of inscribed set. After application of the Bellman-Gronwall inequality with nonpositive kernels, taking the logarithm and the limit as time t tends to infinity, one arrives at the claimed estimation of V-exponents. Now assume $r^2, \sigma^2 > 0$. Then the convergence of $V(x(t), \dot{x}(t))$ towards 0 in probability can be established again by application of Chebyshev-inequality, i.e. this fact follows from

$$\forall \varepsilon > 0 : \mathbb{P}\{V(x(t), \dot{x}(t)) \geq \varepsilon\} \leq \frac{\mathbb{E}\, V(x(t), \dot{x}(t))}{\varepsilon}$$

and the exponentially fast decline of $\mathbb{E}\, V(x(t), \dot{x}(t))$ towards 0 as integration time advances, hence the proof is complete. ⋄

5.4. **Asymptotics of a randomly excited generalized Duffing-type oscillator.** Models with Duffing–type oscillations (i.e. with cubic type of dissipative nonlinearities) are found in Mechanical Engineering fairly often in order to describe the qualitative behaviour of structures under external and random loads. There, in particular, the problem of reliability of mechanical structures arises. For simplicity, let us here consider the one-degree of freedom, randomly perturbed *generalized Duffing-type oscillator* (thought as one component of a multi-degree of freedom system with independent noise sources)

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 f(x) = \sigma_1\, \xi_t^1 + \sigma_2 \sqrt{h^2(\dot{x})}\, \xi_t^2 \tag{5.5}$$

where $\sigma_1, \sigma_2 \in \mathbb{R}$ and ξ_t^1, ξ_t^2 represent independent white or coloured noises. Thus, the classical model of *Duffing's oscillator* is included by the case $f(x) = x + \gamma x^3$ with real parameter $\gamma > 0$. Also the *Ueda's oscillator* is contained by the choice $f(x) = -x + \gamma x^3$ where $\gamma > 0$, and the damped harmonic oscillator as well. $x = x(t)$ again represents the displacement of oscillations from the rest-point, whereas $\dot{x} = \dot{x}(t)$ the velocity of the oscillations. The parameter $\zeta \in \mathbb{R}_+$ controls the intensity of damping part, and parameter $\omega \in \mathbb{R}_+$ is the eigenfrequency and determines the stiffness of the system. $h(.)$ is a further control function on noise influence. Let us restrict to the white noise case only. Here $\gamma, \zeta, \omega, \sigma_2$ essentially control the asymptotical behaviour. We will see that the interplay of ζ, ω, σ_2 establishes the exponential growth behaviour of trajectories in a decisive manner. It is clear that the requirements f is integrable and $\int^x f(z)dz \geq k_1 - k_2\|x\|^2$ with constants $k_1, k_2 \geq 0$ (or $\gamma \geq 0$) together with that of the finiteness of certain initial moments are essential for existence of nonexploding solutions at all time-instants t. Take $\mathbb{D} = \mathbb{R}^2$.

Lemma 5.3. *Consider the stochastic process* $(x(t), \dot{x}(t))_{t \in \mathbb{R}_+}$, *governed by Itô SDE (5.5), started at any values* $(x(0), \dot{x}(0))$ *such that*

$$\mathbb{E}\, V(x(0), \dot{x}(0)) \ < \ +\infty$$

with the Lyapunov functional

$$V(x, \dot{x}) \ = \ \frac{\dot{x}^2}{2} + \omega^2 \int^x f(z)\, dz\, .$$

Assume that there are real, deterministic constants c_1, c_2 *such that for all* $(x,y) \in \mathbb{D}$

$$\int^x f(z)dz \geq 0, \ h^2(y) \leq c_1^2 + c_2^2 y^2\, .$$

Then, if $c_2^2\sigma_2^2 < 4\zeta\omega$, *it holds*

$$\forall \gamma \geq 0, \sigma_1 \ : \ \Big(\dot{x}(t)\Big)^2 \ \underset{t \to +\infty}{\overset{\mathbb{P}}{\longrightarrow}} \ 0$$

i.e. convergence in probability. The $(\dot{x})^2$*-exponents satisfy*

$$\underline{\lambda}_{(\dot{x})^2}(c_2, \zeta, \omega, \sigma_2) \ \leq \ \overline{\lambda}_{(\dot{x})^2}(c_2, \zeta, \omega, \sigma_2) \ \leq \ c_2^2\sigma_2^2 - 4\zeta\omega\, .$$

Moreover, the stochastic process $(x(t), \dot{x}(t))$ *has* V*-exponents*

$$\overline{\lambda}_V(c_2, \zeta, \omega, \sigma_2) \left\{ \begin{array}{lll} \leq & \dfrac{(c_2^2\sigma_2^2 - 4\zeta\omega)_+}{2} & \textit{else} \\ = & 0 & \textit{if} \quad c_2^2\sigma_2^2 = \zeta\omega = 0 \end{array} \right.$$

and

$$\underline{\lambda}_V(c_2, \zeta, \omega, \sigma_2) \left\{ \begin{array}{lll} \geq & -2(\zeta\omega)_+ & \textit{else} \\ = & 0 & \textit{if} \quad c_2^2\sigma_2^2 = \zeta\omega = 0 \end{array} \right. .$$

Remark. The parameters of function f (like γ) do not explicitly occur in the estimates for the exponents, however f is very important for the construction of the Lyapunov functional $V = V(f)$ and in obtaining of these estimates in case of presence of nonlinearities in f. The randomly perturbed generalized Duffing-type oscillator (5.5) is moment dissipative with respect to above given Lyapunov functional V when $\sigma_1^2 + c_1^2 = 0$ and $c_2^2\sigma_2^2 - 4\zeta\omega \leq 0$ (for exact definition of moment V-dissipativity see [14]). In contrast to that, if $c_2^2\sigma_2^2 - 4\zeta\omega \geq 0$ and $\sigma_1^2 + h^2(y) \geq \varepsilon > 0 (\forall y)$ this property cannot be established. Anyway, the time-weighted Lyapunov functional

$$\hat{V}(t,x,y) = \exp\Big(-2|c_2^2\sigma_2^2 - 4\zeta\omega| t\Big)\Big(\frac{\sigma_1^2 + c_1^2\sigma_2^2}{2|c_2^2\sigma_2^2 - 4\zeta\omega|} + V(x,y)\Big)$$

would render the randomly perturbed generalized Duffing oscillator (5.5) to a moment $\hat{V}$–dissipative stochastic system whenever $c_2^2\sigma_2^2 - 4\zeta\omega > 0$. The same would happen with

$$\hat{V}(t,x,y) \ = \ \exp\Big(- (c_1^2\sigma_2^2 + \sigma_1^2)t\Big)\Big(\frac{1}{2} + V(x,y)\Big)$$

provided that $4\zeta\omega = c_2^2\sigma_2^2$.

Proof. Consider Lyapunov functional

$$V(x,y) \ := \ \frac{y^2}{2} + \omega^2 \int^x f(z)dz$$

where parameter ω is fixed. Define $v(t) := \mathbb{E}\, V(x(t), \dot{x}(t))$. Then, by Dynkin's formula one arrives at

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left(\dot{x}(t)\right)^2 \le v(t) &= v(0) + \mathbb{E}\int_0^t \mathcal{L}V(x(s), \dot{x}(s))ds \\
&= v(0) + \frac{\sigma_1^2}{2}t + \mathbb{E}\int_0^t \left(\frac{\sigma_2^2}{2}h^2(\dot{x}(s)) - 2\zeta\omega(\dot{x}(s))^2\right)ds \\
&\le v(0) + \frac{c_1^2\sigma_2^2 + \sigma_1^2}{2}t + \int_0^t \left(\frac{c_2^2\sigma_2^2}{2} - 2\zeta\omega\right)(\dot{x}(s))^2\, ds \\
&\le v(0) + \frac{c_1^2\sigma_2^2 + \sigma_1^2}{2}t + \int_0^t \left(\frac{c_2^2\sigma_2^2}{2} - 2\zeta\omega\right)_+ v(s)\, ds\,.
\end{aligned}$$

After taking the logarithm and limit as t tends to $+\infty$, this immediately implies

$$-2(\zeta\omega)_+ \le \underline{\lambda}_V \le \overline{\lambda}_V \le \frac{(c_2^2\sigma_2^2 - 4\zeta\omega)_+}{2}\,.$$

If $c_2^2\sigma_2^2 \le 4\zeta\omega$ then one finds $\overline{\lambda}_V \le 0$, as well as, if $\zeta = 0$ or $\omega = 0$ then $\underline{\lambda}_V \ge 0$. The equality $\overline{\lambda}_V = \underline{\lambda}_V = 0$ if $c_2^2\sigma_2^2 = \zeta\omega = 0$ is obvious after a careful view on the application of Dynkin's formula (see above). In a similar way we can conclude estimates for $(\dot{x})^2$-exponents from above. Now, assume $c_2^2\sigma_2^2 < 4\zeta\omega$. Then the convergence of $(\dot{x}(t))^2$ towards 0 in probability can be established by application of well-known Chebyshev-inequality, i.e. again this fact follows from

$$\forall \varepsilon > 0 : \mathbb{P}\{(\dot{x}(t))^2 \ge \varepsilon\} \le \frac{\mathbb{E}\,(\dot{x}(t))^2}{\varepsilon}$$

and the exponentially fast decline of $\mathbb{E}\,(\dot{x}(t))^2$ towards zero, hence the proof is complete. ⋄

Remark. One has to be very careful with the construction of appropriate Lyapunov functionals V. An 'overscaling' could lead to inefficient conclusions about exponential growth speed of trajectories of stochastic processes. So far there is no systematic way for appropriate constructions of such functionals for general nonlinear SDEs. However, it is clear one has to heavily take into account the specific form of nonlinearities.

5.5. **On general V-asymptotics of iterative random mappings.** In analogy to deterministic analysis, we receive the following discrete inequality. Let $(t_n)_{n\in\mathbb{N}}$ be a monotonically nondecreasing sequence of deterministic time-instants with t_n diverging to $+\infty$ as n tends to $+\infty$, and define

$$\Delta\mathbb{E}\, V_n := \mathbb{E}\, V(n+1, X_{n+1}) - \mathbb{E}\, V(n, X_n)$$

for a discrete time $\mathbb{D}$-valued stochastic process $X = (X_n)_{n\in\mathbb{N}}$ on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\mathbb{N}}, \mathbb{P})$.

Lemma 5.4. *Assume that $\mathbb{E}\, V(0, X_0) < +\infty$ for a function $V : \mathbb{N} \times \mathbb{D} \longrightarrow \mathbb{R}^1_+$ with*

$$\underline{k}_n \mathbb{E}\, V(n, X_n) \le \Delta\mathbb{E}\, V_n \le \overline{k}_n \mathbb{E}\, V(n, X_n)$$

for all $n \in \mathbb{N}$, where $\underline{k}_i, \overline{k}_i$ are deterministic, real constants along the dynamics of process $(X_n)_{n\in\mathbb{N}}$, and for all $n \in \mathbb{N}$

$$1 + \underline{k}_n > 0\,.$$

Then, for all $n \in I\!N$, it holds

$$\exp\left(\sum_{i=0}^{n} \frac{\underline{k}_i}{1+\underline{k}_i}\right) I\!E\, V(0,X_0) \le I\!E\, V(n+1,X_{n+1}) \le \exp\left(\sum_{i=0}^{n} \overline{k}_i\right) I\!E\, V(0,X_0)$$

and, if the limits exist, then

$$\liminf_{n\to+\infty} \frac{\sum_{i=0}^{n-1} \frac{\underline{k}_i}{1+\underline{k}_i}}{t_n} \le \underline{\lambda}_V \le \overline{\lambda}_V \le \limsup_{n\to+\infty} \frac{\sum_{i=0}^{n-1} \overline{k}_i}{t_n}.$$

Proof. First, assume $\Delta \mathbb{E}\ V_n \le \overline{k}_n \mathbb{E}\ V(n,X_n)$ (for all $n \in \mathbb{N}$). Making use of elementary splitting

$$z(n+1) = z(n) + z(n+1) - z(n)$$

with $z(n+1) := \mathbb{E}\ V(n+1,X_{n+1})$, one concludes

$$z(n+1) \le z(n)(1+\overline{k}_n) \le z(0)\prod_{i=0}^{n}(1+\overline{k}_i)_+ \le z(0)\exp\left(\sum_{i=0}^{n}\overline{k}_i\right).$$

On the other hand, when $\Delta \mathbb{E}\ V_n \le \underline{k}_n \mathbb{E}\ V(n,X_n)$ and $1+\underline{k}_n > 0$ (for all $n \in \mathbb{N}$), one recognizes the validity of

$$z(n) \le \frac{z(n+1)}{1+\underline{k}_n} \le z(n+1)\exp\left(\frac{-\underline{k}_n}{1+\underline{k}_n}\right)$$

which implies

$$z(n+1) \ge z(n)\exp\left(\frac{\underline{k}_n}{1+\underline{k}_n}\right) \ge z(0)\exp\left(\sum_{i=0}^{n}\frac{\underline{k}_i}{1+\underline{k}_i}\right),$$

using elementary inequality

$$\frac{1}{1+x} \le \exp\left(-\frac{x}{1+x}\right).$$

Now one arrives at the second result by taking the exponential logarithm and limit when integration time t_n advances. Therefore, the proof is complete. ⋄

Remark. Exactly, we used this lemma to prove previous results in the discrete time case during the estimation of moment stability and contractivity exponents with $V(x) = \|x\|^2$ where $\|.\|$ is the Euclidean vector norm. Under the existence of Riemann-integrals $\int_{t_0}^{t} K(s)ds$ with $k_i = K(t_i)\Delta_i$ one can also derive corresponding continuous time versions by taking the limit of arising Riemann sums in corresponding discrete time inequalities. It is always possible to find $\underline{k}_i$ such that $1+\underline{k}_i \ge 0$. If only one $\underline{k}_{i*}$ with $\underline{k}_{i*} = -1$ exists, then our estimate of sequence $z(n)$ from below reduces to the trivial one, i.e. $z(n) \ge 0$ at least for all $n \ge i*$. Thus, this latter case would not be very meaningful in the estimation process anyway.

5.6. **An example for V-asymptotics in discrete time.** To our suprise, even in deterministic numerical analysis there is not too much written in standard textbooks about the asymptotical behaviour of variable step size algorithms (i.e. on asymptotics of nonautonomous discrete dynamical systems). For the sake of simplicity and illustration, we shall consider the stochastic oscillator with multiplicative white noise

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x \;=\; \sigma\dot{x}\,\xi_t \tag{5.6}$$

where $\zeta, \omega > 0$ and the stochastic integration is understood in the sense of Itô. Then the corresponding deterministic equation has an asymptotically stable zero solution if $0 < \zeta < 1$, and does not exponentially grow if $0 \leq \zeta \leq 1$. Thanks to Lemma 5.3, we know about the stochastic version that the upper V-exponent with $V(x,y) = y^2 + \omega^2 x^2$ is not larger than zero if $0 \leq \sigma^2 \leq 4\zeta\omega$. Let us now look at the discretization of such a equation by numerical methods. Define

$$V(n+1,x,y) := \omega^2 x^2 + (1 + 2\zeta\omega\Delta_n)y^2$$

where $\Delta_n = t_{n+1} - t_n$ is current step size, and $v_{n+1} := \mathbb{E}\, V(n+1, X_{n+1}, Y_{n+1})$.

Lemma 5.5. *Assume that the stochastic oscillator (5.6) is discretized by the fully drift-implicit Euler method given by*

$$\begin{aligned} X_{n+1} &= X_n + Y_{n+1}\Delta_n \\ Y_{n+1} &= Y_n - (2\zeta\omega Y_{n+1} + \omega^2 X_{n+1})\Delta_n + \sigma Y_n \Delta W_n \end{aligned} \tag{5.7}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$ *along a time-discretization* $(t_n)_{n\in\mathbb{N}}$, *and*

$$\mathbb{E}\,[X_0^2 + Y_0^2] < +\infty.$$

Then, for all $n \in \mathbb{N}$, *all* $l \in \mathbb{N}$ *with* $1 \leq l < n$, *it holds*

$$v_l \exp\left(\sum_{i=l}^{n} \frac{\underline{k}_i}{1+\underline{k}_i}\right) \;\leq\; v_{n+1} = \mathbb{E}\, V(n+1, X_{n+1}, Y_{n+1}) \;\leq\; v_l \exp\left(\sum_{i=l}^{n} \overline{k}_i\right)$$

where

$$\overline{k}_i = \frac{-\omega^2\Delta_i^2(1+2\zeta\omega\Delta_{i-1}) + [(\sigma^2 - 2\zeta\omega)\Delta_i - 2\zeta\omega\Delta_{i-1}(1+2\zeta\omega\Delta_i)]_+}{(1+2\zeta\omega\Delta_{i-1})(1+2\zeta\omega\Delta_i + \omega^2\Delta_i^2)}$$

and

$$\underline{k}_i = \frac{-\omega^2\Delta_i^2(1+2\zeta\omega\Delta_{i-1}) - [(\sigma^2 - 2\zeta\omega)\Delta_i - 2\zeta\omega\Delta_{i-1}(1+2\zeta\omega\Delta_i)]_-}{(1+2\zeta\omega\Delta_{i-1})(1+2\zeta\omega\Delta_i + \omega^2\Delta_i^2)}.$$

Furthermore, if $(\Delta_n)_{n\in\mathbb{N}}$ *is a deterministic sequence then the* V*-exponents can be estimated by*

$$\liminf_{n\to+\infty} \frac{1}{t_n}\sum_{i=1}^{n-1} \frac{\underline{k}_i}{1+\underline{k}_i} \;\leq\; \underline{\lambda}_V \;\leq\; \overline{\lambda}_V \;\leq\; \limsup_{n\to+\infty} \frac{1}{t_n}\sum_{i=1}^{n-1} \overline{k}_i\,.$$

Additionally, in the following assume that

$$\exists \Delta_a, \Delta_b \in \mathbb{R}_+ \;:\; \forall n \in \mathbb{N} \quad 0 < \Delta_b \leq \Delta_n \leq \Delta_a < +\infty\,. \tag{5.8}$$

If

$$(\sigma^2 - 2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n) \;\leq\; 0 \tag{5.9}$$

for all $n \in \mathbb{N}$ *then*

$$\overline{\lambda}_V \leq -\frac{\omega^2\Delta_b}{1+2\zeta\omega\Delta_a + \omega^2\Delta_a^2}\,.$$

If

$$(\sigma^2 - 2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n) \;\geq\; 0 \tag{5.10}$$

for all $n \in \mathbb{N}$ *then*

$$\underline{\lambda}_V \geq -\frac{\omega^2\Delta_a}{1+2\zeta\omega\Delta_b}\,.$$

Proof. First, we equivalently rearrange the scheme (5.7) to an explicit one. Thus, one arrives at

$$
\begin{aligned}
(5.11)\qquad X_{n+1} &= \frac{1+2\zeta\omega\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n + \frac{(1+\Delta W_n)\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n \\
Y_{n+1} &= -\frac{\omega^2\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n + \frac{(1+\Delta W_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n\,.
\end{aligned}
$$

After some calculations this relation implies

$$
v_{n+1} = \omega^2 \mathbb{E}\left[\frac{1+2\zeta\omega\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}X_n^2\right] + \mathbb{E}\left[\frac{1+\sigma^2\Delta_n}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right],
$$

hence

$$
-\frac{\omega^2\Delta_a\Delta_n}{1+2\zeta\omega\Delta_b+\omega^2\Delta_b^2}v_n - \mathbb{E}\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right]_- \le
$$

$$
\begin{aligned}
\Delta \mathbb{E}\, V_n &= -\mathbb{E}\left[\frac{\omega^2\Delta_n^2}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}\omega^2 X_n^2\right] \\
&\quad + \mathbb{E}\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n - \omega^2\Delta_n^2 - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right] \\
&= -\frac{\omega^2\Delta_n^2}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}v_n + \mathbb{E}\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right] \\
&\le -\frac{\omega^2\Delta_b\Delta_n}{1+2\zeta\omega\Delta_a+\omega^2\Delta_a^2}v_n + \mathbb{E}\left[\frac{(\sigma^2-2\zeta\omega)\Delta_n - 2\zeta\omega\Delta_{n-1}(1+2\zeta\omega\Delta_n)}{1+2\zeta\omega\Delta_n+\omega^2\Delta_n^2}Y_n^2\right]_+
\end{aligned}
$$

Now, we may choose $\overline{k}_n, \underline{k}_n$ as indicated above, and apply the Lemma 5.4 to our constellation with that $\overline{k}_n, \underline{k}_n$. Thus, the assertions of Lemma 5.5 follow straight forward by elementary analysis of received exponential expressions, and the proof can be completed. ⋄

Remark. Most of the clever variable step size algorithms have implemented conditions on the step size selection like that of (5.8). We can conclude from our assertion that the fully drift-implicit Euler method (5.7) applied to stochastic oscillator (5.6) produces overdamped approximations compared to the asymptotics of exact solution. This can be seen particularly in the critical case (the energy-conservative case) when $\sigma^2 = 4\zeta\omega$ under the condition (5.8). However, the oberved effect of numerical stabilization also explains that the requirement (5.8) is meaningful in variable step size algorithms in order to achieve asymptotically stable approximations (i.e. with 'sure side argumentation'). Asymptotically considered, when maximum step size Δ_a tends to zero, the V-exponents of the continuous time dynamics are correctly replicated by the discretization method (5.7) what we would naturally expect, and with a convergence order Δ_a.

6. Conclusions and remarks

This paper is a continuation of our works [35] and [36]. Here some asymptotical moment characterization by deterministic numbers could be made for nonlinear stochastic dynamical systems. The sign of these characteristics decides about a possible existence of finite 'asymptotical structures'. Only if these numbers are zero one can expect the existence of nontrivial limit laws of corresponding moments. In this latter case one needs further investigations to refine the asymptotical behaviour of examined systems, for example with different rescaled functionals in the definition (i.e. in order to distinguish between V–dissipative and (polynomially) exploding solutions). However, meeting an 'optimal' scaling turns out to be very difficult and very case-sensitive. All estimates here are based on the generalized Gronwall-Bellman Lemma with nonautonomous kernels suggested by [31], [35] in the context of stochastic systems (We also sometimes called its generalization as variation-of-constants inequality in this contribution.). Other integral inequalities (like that of Bihari's or Wendroff's Lemma) can be used to find more delicate estimates of characteristic exponents. The introduced exponents fully determine the global exponential growth of solutions of stochastic differential equations or stochastic difference equations in forward direction. A similar analysis can be carried out in backward direction (i.e. when time t tends to $-\infty$). Also, the convergence proofs of numerical methods should be revised under the knowledge on finiteness of characteristic exponents of underlying continuous time differential systems. A careful look at our uniform estimates of both contractivity and stability exponents yields the conclusion that the drift part $a(t,x)$ in the continuous time case and $\Phi_0^I(x^{(i)} : i \leq n)$ in the discrete time case are sometimes able to compensate the diffusion behaviour given by $b^j(t,x)$ and $\Phi_j(x^{(i)} : i \leq n)$ through appropriate terms, respectively. In contrast to that fact, diffusion terms always lead to an increase of moment contractivity and stability exponents within the Itô calculus (provided that $p \geq 2$). The latter fact can dramatically change under other stochastic calculi like that of Stratonovich due to different correction terms in their stochastic chain rules. Besides, it would be interesting to develope a similar concept in the 'almost sure sense' instead of exclusive consideration of moments (i.e. under assumptions like 'almost sure monotonicity' of involved stochastic terms). In the linear framework with multiplicative noise the presented concepts of moment contractivity and stability exponents coincide with the well-known concept of moment Lyapunov exponents (connected by Arnold's formula [2] with sample Lyapunov exponents arising from the multiplicative ergodic theorem of Oseledec [33]). Thus, the main purpose of presented paper is aiming at an analysis of nonlinear and nonautonomous stochastic systems where nontrivial equilibria may occur. Moreover, in any framework where the zero solution is an equilibrium position (a.s.), the concepts of contractivity and stability are identical. These concepts may differ when stochasticity (or inhomogeneouities) comes into play as it can be seen by the trivial example of Ornstein-Uhlenbeck processes with exponentially exploding nonautonomous diffusion part. The concepts also differ in the deterministic situation. A simple example to manifest this assertion is given by the one-dimensional equation

$$\dot{x} \; = \; -\alpha^2 x + \beta^2$$

where parameters $\alpha^2 > 0$ and $\beta^2 > 0$. This equation satisfies

$$\underline{\kappa}_p = \overline{\kappa}_p = -p\alpha^2 < 0 = \underline{\lambda}_p = \overline{\lambda}_p$$

for all start values $x(0)$, and trivially for all exponents $p > 0$. As seen by the latter example, it may be noted that the concept of contractivity replicates more the asymptotical-qualitative behaviour of fundamental solutions in comparison to that of stability where inhomogeneous parts have to take into account in addition. The fairly new concept of stochastic moment contractivity permits to treat stochastic systems with both additive and multiplicative noise by an unified approach. This is a fact in which the major advantage of contractivity analysis can be seen. It is particularly appropriate to describe the propagation of initial perturbations generated by nonlinear stochastic dynamical systems. All in all we do have the hope that the concept of contractivity sheds some new light on the qualitative theory of dynamical systems.

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