On the distribution of the maximum of sums of dependent random variables

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1 On the distribution of maximum of the consecutive sums of m-dependent random variables

The study of queuing theory brings us to the problems of finding to find the limit distribution of the maximal sum of a sequence of random variables and of estimating how close this distribution is to the distribution of the sum.

In this chapter this problem is solved for m-dependent stationary random variables.

Let $X_1, X_2, \ldots$ be a sequence of m-dependent random variables which is stationary in the strict sense. Let $a > 0$ and

$$S_k = \sum_{j=1}^{k} X_j, \quad \overline{S}_n = \max_{1 \le k \le n} S_k, \quad G(x) = \sqrt{\frac{2}{\pi}} \int_0^{x^+} e^{-y^2/2} dy,$$

$$Y_k = X_k I_{(|X_k| \le a)}, \quad Z_k = Y_k - \mathbb{E}Y_k,$$

$$\sigma_n = \sum_{j=1}^{n} Z_j, \quad \sigma^2 = \mathbb{E}Z_1^2 + 2 \sum_{k=2}^{m+1} \mathbb{E}Z_1 Z_k,$$

$$A_n = n|MY_1|, \ \Gamma_n = nP(|X_1| > a),$$

$$B_n^2 = n\ \mathbb{E}Z_1^2 + 2 \sum_{k=1}^{m+1} (n-k+1)\mathbb{E}(Z_1 Z_2).$$

With this notation we prove the following theorem:

Theorem 1 *Let a and b be any positive numbers. Then the following estimate holds:*

$$\sup_x |P(\overline{S}_n < bx) - G(x)| \le \Gamma_n + \frac{2}{\sqrt{2\pi}} \frac{2a(2m + n_0 + 1) + \delta}{B_n} +$$

$$+2\Delta_n + \sqrt{\frac{2}{\pi}} \frac{A_n}{B_n} + \sqrt{\frac{2}{\pi e}} \{\max(\frac{b}{B_n}, \frac{B_n}{b}) - 1\}$$

where

$$\Delta_n = \frac{1}{\sqrt{n}} \left(\frac{7(m+1)^2 \mathbb{E}|Z_1|^3}{(s^2 - \frac{m(m+1)}{n}\mathbb{E}Z_1^2)^{3/2}} + \frac{44\sqrt{m}(m+1)\mathbb{E}^{2/3}|Z_1|^3}{\pi(s^2 - \frac{m(m+1)}{n}\mathbb{E}Z_1^2)} + \right.$$

$$\left. + \frac{8}{\sqrt{2\pi} T} + \frac{\ln n}{n} \left(\frac{64\sqrt{2}m(m+1)\mathbb{E}^{2/3}|Z_1|^3(1 + \ln T)}{s^2 - \frac{m(m+1)}{n}\mathbb{E}Z_1^2} \right) \right),$$

$$T = \min\left(\frac{\sqrt{s^2 - \frac{m(m+1)}{n}\mathbb{E}Z_1^2}}{2\sqrt{2}(m+1)\mathbb{E}^{1/3}|Z_1|^3}, \frac{(s^2 - \frac{m(m+1)}{n}\mathbb{E}Z_1^2)^{3/2}}{28(m+1)^2\mathbb{E}|Z_1|^3} \right),$$

$$n_0 = [\max(n_1, 2\pi n_1 \Delta_{n_1}^2)] + 1, \;\; n_1 = [\frac{2m(m+1)}{\sigma^2}(2a)^2],$$

$$\textit{and } \delta = 4\sqrt{3}a\sqrt{n_0}\ln(\frac{\sqrt{n_0}}{\sqrt{n_0} - \sqrt{2\pi}\Delta}) \textit{ where } \Delta = \sqrt{n_1}\,\Delta_{n_1}.$$

Proof: To prove this theorem we use the proof of Theorem 1 from [1] and the estimate of the speed of convergence in the CLT for m-dependent random variables from [2]. The latter gives:

$$P\left(\sigma_{n+n_0+m} - \sigma_{k+m} < x \sqrt{\operatorname{var}\left(\sum_{j=k+m+1}^{n+n_0+m} Z_j \right)} \right) \le \Phi(x) + \Delta_{n+n_0-k}.$$

Denote

$$D = \operatorname{var}(\sigma_{n+n_0+m} - \sigma_{m+k}).$$

Then

$$P(\sigma_{n+n_0+m} - \sigma_{k+m} \ge t) \le \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{tD^{-\frac{1}{2}}} e^{-y^2/2} dy + \Delta_{n+n_0-k}. \qquad (1)$$

We shall find t such that for all m, n, k the inequality

$$P(\sigma_{n+n_0+m} - \sigma_{k+m} \ge t) \le \frac{1}{2} \qquad (2)$$

holds. Since $(y\ e^{-y^2/2})' = e^{-y^2/2} - y^2 e^{-y^2/2}$, we have

$$\int_0^x (y\, e^{-y^2/2})' dy = \int_0^x (e^{-y^2/2} - y^2 e^{-y^2/2}) dy.$$

Therefore

$$\int_0^x \ e^{-y^2/2} dy = x\ e^{-\frac{x^2}{2}} + \int_0^x \ y^2 e^{-y^2/2} dy.$$

From the inequality

$$\int_0^x \ y^2 e^{-y^2/2} dy - \int_0^x (2y-1)\, e^{-y^2/2} dy = \int_0^x \ e^{-y^2/2} (y-1)^2 dy \geq 0$$

we have

$$\int_0^x \ y^2 e^{-y^2/2} dy \geq \int_0^x (2y-1)\, e^{-y^2/2} dy.$$

Then $\int_0^x \ e^{-y^2/2} dy \geq \frac{x}{2} e^{-x^2/2} + \int_0^x y\ e^{-y^2/2} dy = -\int_0^x \ e^{-y^2/2} d(-\frac{y^2}{2}) + \frac{x}{2}\, e^{-x^2/2} = 1 - e^{-x^2/2} + \frac{x}{2}\, e^{-x^2/2}$, i.e.

$$\int_0^x \ e^{-y^2/2} dy \geq \frac{x}{2}\, e^{-x^2/2} + 1 - e^{-x^2/2}.$$

It follows from (1) that for the inequality (2) ot is sufficient to prove the inequality

$$\frac{1}{\sqrt{2\pi}} \int_0^{tD^{-1/2}} \ e^{-y^2/2} dy > \Delta_{n+n_0-k}.$$

Substituting the previous inequality and taking into account that $\frac{\Delta}{\sqrt{n+n_0-k}} \geq \Delta_{n+n_0-k}$, we need only

$$\begin{aligned} \left(-\frac{1}{\sqrt{2\pi}} + \frac{tD^{-1/2}}{2\sqrt{2\pi}}\right) e^{-\frac{t^2 D^{-1}}{2}} \quad &\geq \quad \frac{\Delta}{\sqrt{n+n_0-k}} - \frac{1}{\sqrt{2\pi}}, \\ \text{i.e.} \ \left(\frac{tD^{-1/2}}{2} - 1\right) e^{-t^2 D^{-1} \frac{1}{2}} \quad &\geq \quad \frac{\Delta\sqrt{2\pi}}{\sqrt{n+n_0-k}} - 1. \end{aligned} \qquad (3)$$

Now $\frac{tD^{-1/2}}{2} - 1 \geq -e^{-\frac{tD^{-1/2}}{2}}$ since $e^{-z} \geq 1 - z$ for $z \geq 0$. Therefore (3) is implied by

$$-e^{-\frac{tD^{-\frac{1}{2}}}{2}} \cdot \ e^{-\frac{t^2 D^1}{2}} \geq \frac{\Delta\sqrt{2\pi}}{\sqrt{n+n_0-k}} - 1,$$

or

$$\frac{tD^{-1/2}}{2} + \frac{t^2 D^{-1}}{2} \geq \ln\left(\frac{1}{1 - \frac{\sqrt{2\pi}\Delta}{\sqrt{n+n_0-k}}}\right).$$

Since $\frac{t^2D^{-1}}{2} \geq 0$, it suffices to choose t according to the condition $t \geq -2D^{\frac{1}{2}} \ln(1-\frac{\sqrt{2\pi}\Delta}{\sqrt{n+n_0-k}})$. Using properties of the variance we have $D \leq 3(n+n_0-k)(2a)^2$. Thus t may be choosen according to

$$t \geq -4\sqrt{3}a\sqrt{2\pi}\Delta \ln(1-\frac{\sqrt{2\pi}\Delta}{\sqrt{n+n_0-k}})^{\frac{\sqrt{n+n_0-k}}{\sqrt{2\pi}\Delta}}. \tag{4}$$

We know that the sequence $(1-\frac{1}{n})^n$ increases. We substitute the right-hand side of (4) by bigger value and get:

$$t \geq 4\sqrt{3}a\sqrt{n_0}\ln\left(\frac{\sqrt{n_0}}{\sqrt{n_0}-\sqrt{2\pi}\Delta}\right) = \delta,$$

i.e. for $t \geq \delta$ (2) holds.

Decomposing the events $\overline{\sigma}_n \geq x$, using (2) and using the m-dependence of our random variables we have:

$$P(\overline{\sigma}_n \geq x) = \sum_{k=1}^{n} P(\sigma_1 < x, \ldots, \sigma_{k-1} < x, \sigma_k \geq x) \geq$$

$$\geq \; 2\sum_{k=1}^{n} P(\sigma_1 < x, \ldots, \sigma_{k-1} < x, \sigma_k \geq x)\; P(\sigma_{n+n_0+m} - \sigma_{k+m} \geq \delta) =$$

$$= \; 2\sum_{k=1}^{n} P(\sigma_1 < x, \ldots, \sigma_{k-1} < x, \sigma_k \geq x, \sigma_{n+n_0+m} - \sigma_{k+m} \geq \delta).$$

On the other hand $(\sigma_j - \sigma_l) \leq 2s|l-j|$ for all j, l by definition of the σ_n and Z_i. Thus

$$\begin{aligned} P(\overline{\sigma}_n \geq x) &\geq \{\sigma_{k+m} \leq \sigma_{k-1} + 2a(m+1)\} \geq \\ &\geq 2\sum_{k=1}^{n} P(\sigma_1 < x, \ldots, \sigma_{k-1} < x, \sigma_k \geq x, \sigma_n \geq x + \\ &\quad +2a(2m+n_0+1)+\delta) = 2P(\sigma_n \geq x + 2a(2m+n_0+1)+\delta). \end{aligned}$$

One can prove analogously that

$$P(\sigma_{n+n_0+m} - \sigma_{k+m} \leq t) \leq \frac{1}{2}, \quad \text{for } t \leq -\delta,$$

and for all x $P(\overline{\sigma}_n \geq x) \leq 2P(\sigma_n \geq x - 2a(m+n_0) - \delta)$.
Using the previous inequalities we have

$$1-2P(\sigma_n \geq B_n x - 2a(2m+n_0)-\delta) \leq P(\overline{\sigma}_n < B_n x) \leq 1-2P(\sigma_n \geq B_n x + 2a(2m+n_0+1)+\delta.$$

Now for $x \geq 0 : 2\Phi(x) - 1 = G(x)$, and for $x < 0 \quad 2\Phi(x) - 1 = G(-x)$. Therefore for $x \geq 0$,
$P(\overline{\sigma}_n < B_n x) - G(x) \leq 2[\Phi(x + \frac{2a(2m+n_0+1)+\delta}{B_n}) - \Phi(x)] + 2\Delta_n$
$P(\overline{\sigma}_n < B_n x) - G(x) \geq 2[\Phi(x - \frac{2a(2m+n_0)+\delta}{B_n}) - \Phi(x)] - 2\Delta_n$,
and for $x < 0$:
$0 \leq P(\overline{\sigma}_n < B_n x) - G(x) \leq 2\Phi(\frac{2a(2m+n_0+1)+\delta}{B_n}) - 1 + 2\Delta_n$.

Moreover the following inequalities hold: $|\Phi(x+q) - \Phi(x)| \leq \frac{1}{\sqrt{2\pi}}|q|$, and $|\Phi(px) - \Phi(x)| \leq \frac{1}{\sqrt{2e\pi}}\{\max(p, \frac{1}{p}) - 1\}$. Hence

$$\sup_x |P(\overline{\sigma}_n < B_n x) - G(x)| \leq \frac{2}{\sqrt{2\pi}} \frac{2a(2m+n_0+1)+\delta}{B_n} + 2\Delta_n. \qquad (5)$$

Using the inclusions
$\{\max(Y_1, Y_1+Y_2, \ldots, Y_1+Y_2+\ldots+Y_n) < x\} \subset \{\max(Z_1, Z_1+Z_2, \ldots, Z_1+\ldots+Z_n) < x + A\}$ and
$\{\max(Z_1, Z_1+Z_2, \ldots, Z_1+\ldots+Z_n) < x - A_n\} \subset \{\max(Y_1, Y_1+Y_2, \ldots, Y_2+\ldots+Y_n) < x\}$
we have

$$P(\overline{\sigma}_n < bx - A_n) \leq P(\max_{i \leq k \leq n} \sum_{j=1}^{k} Y_j < bx) \leq P(\overline{\sigma}_n < bx + A_n).$$

Taking into account (5) we get

$$P(\max_{1 \leq k \leq n} \sum_{j=1}^{k} Y_j < bx) - G(x) \leq P(\overline{\sigma}_n \leq bx + A_n) - G(\frac{bx+A_n}{B_n}) + G(\frac{bx+A_n}{B_n}) - G(x) \leq$$

$$\leq \frac{2}{\sqrt{2\pi}} \frac{2a(2m+n_0+1)+\delta}{B_n} + 2\Delta_n + \sqrt{\frac{2}{\pi}} \frac{A_n}{B_n} + \sqrt{\frac{2}{e\pi}}\{\max(\frac{b}{B_n}, \frac{B_n}{b}) - 1\}$$

and analogously

$$P(\max_{1 \leq k \leq n} \sum_{j=1}^{k} Y_j < bx) - G(x) \quad \geq \quad -\frac{2}{\sqrt{2\pi}} \frac{2a(2m+n_0+1)+\delta}{B_n} - 2\Delta_n - \sqrt{\frac{2}{\pi}} \frac{A_n}{B_n} - \sqrt{\frac{2}{e\pi}}\{\max(\frac{b}{B_n}, \frac{B_n}{b}) - 1\}.$$

Furthermore

$$\sup_x |P(\max_{1 \leq k \leq n} \sum_{j=1}^{k} Y_j < x) - P(\max_{1 \leq k \leq n} \sum_{j=1}^{k} X_j < x)| \leq n \; P(|X_1| > a) = \Gamma_n.$$

Therefore

$$\sup_x |P(\overline{S}_n < bx) - G(x)| \le \sup_x |P(\overline{S}_n < bx) - P(\max_{1\le k\le n} \sum_{j=1}^{k} Y_j < bx)|$$

$$+ \sup_x |P(\max_{1\le k\le n} \sum_{j=1}^{k} Y_j < bx) - G(x)| \le$$

$$\le \Gamma_n + \frac{2}{\sqrt{2\pi}} \frac{2a(2m+n_0+1)+\delta}{B_n} + 2\Delta_n + \sqrt{\frac{2}{\pi}} \frac{A_n}{B_n} + \sqrt{\frac{2}{e\pi}} \{\max(\frac{b}{B_n}, \frac{B_n}{b}) - 1\}.$$

The theorem is proved. ∎

Corollary 1 *Let* $\mathbb{E}X_k = 0,\ k = \overline{1,n}$ *and assume that there exist some constants* $c > 0,\ c_0 > 0,\ \delta > 0,$ *s.t.*

$$\sup_{k\le n} \mathbb{E}|X_k|^{2+\delta} \le c, \quad B_n^2 \ge c_0 n.$$

Then the estimate in theorem 1 looks like

$$\sup_x |P(\overline{S}_n < B_n x) - G(x)| \le B_n^{-\gamma_1} \tag{6}$$

where $\gamma_1 = \frac{\delta}{4+2\delta}$ *and the bounded random variable* B *depends only on* c, c_0, m.

The proof of the corollary follows from inequalities

$$\Gamma_n = n\ P(|X_1| > a_n) \le n\ \frac{\mathbb{E}X_1^{2+\delta}}{a_n^{2+\delta}} \le \frac{nc}{a_n^{2+\delta}},$$

$$A_n = n|MX_1 I_{(|X_1|\le a)}| = n|\mathbb{E}X_1 I_{(|X_1|>a)}| \le \frac{cn}{a_n^{1+\delta}}.$$

Choose $a = a_n = n^{\alpha},\ \alpha > 0$ and take the minimum with respect to α of the expression in theorem 1. Having done this we get the estimate (6).

2 On the distribution of maximum of consecutive sums of random variables with strong mixing

In this chapter we treat the analogous problem for stationary sequences. Let $X_1, X_2, \ldots$ be a stationary sequence of random variables. Let $\mathcal{R}_a^b$ denote the

σ-algebra generated by the random variables X_j,
$j \in [a,b]$ Let

$$\alpha(\tau) = \{\sup |P(AB) - P(A)P(B)| : t \in \mathbb{N},\ \ A \in \mathcal{R}_0^t,\ \ B \in \mathcal{R}_{t+\tau}^{\infty}\}.$$

As before we use the following notations

$$\begin{aligned} S_k &= \sum_{j=1}^{k} X_j,\ \overline{S}_n = \max_{1\le k\le n} S_k,\ \ Y_k = X_k I_{(|X_k|\le a)}, \\ Z_k &= Y_k - \mathbb{E}Y_k,\ s^2 = \mathbb{E}Z_1^2 + 2\sum_{k=2}^{\infty} \mathbb{E}Z_1 Z_k, \quad A_n = n|\mathbb{E}Y_1|, \\ \Gamma_n &= n\ P(|X_1| > a),\ B_n^2 = \mathbb{E}(\sum_{j=1}^{n} Z_j)^2,\ \sigma_n = \sum_{j=1}^{n} Z_j \end{aligned}$$

$$\Delta_n = \sup_z |P(\frac{\sigma_n}{B_n} \le Z) - \frac{1}{\sqrt{2\pi}} \int_0^Z e^{-x^2/2} dx|,$$

$$G(x) = \begin{cases} 0, & ?\ \ x \le 0 \\ \sqrt{\frac{2}{\pi}} \int_0^x e^{-y^2/2} dy, & ?\ \ x < 0. \end{cases}$$

In this notation the following theorem holds:

Theorem 2 *Let a and b be any positive numbers and let m be any number then the following estimate holds:*

$$\sup_x |P(\overline{S}_n < bx) - G(x)| \le \Gamma_n + \frac{2}{\sqrt{2\pi}} \frac{2a(2m + n_0 + 1) + \delta(n)}{B_n} +$$
$$+ \ \ 2\Delta_n + \sqrt{\frac{2}{\pi}} \frac{A_n}{B_n} + \sqrt{\frac{2}{\pi e}} \{\max(\frac{b}{B_n}, \frac{B_n}{b}) - 1\} + 2n\alpha(m) \tag{7}$$

were the constant n_0 is choosen according to $\Delta_{n_0} < \frac{1}{\sqrt{2\pi}}$ and

$$\delta(n) = -\frac{2\sigma}{\Delta_{n_0}} \ln(1 - \sqrt{2\pi}\Delta_{n_0}) \max_{0\le k\le n} \{\sqrt{k + n_0}\, \Delta_{k+n_0}\}.$$

To prove this theorem we use the proof of theorem 1 from [1] and the estimate of the speed of convergence in the CLT for strongly mixing random variables. In our notations we have:

$$P\left(\sigma_{n+n_0+m} - \sigma_{k+m} < x\ D^{1/2} \left(\sum_{j=k+m+1}^{n+n_0+m} Z_j \right)\right) \le \Phi(x) - \Delta_{n+n_0-k}.$$

Denote $D = D(\sigma_{n+n_0+m} - \sigma_{k+m})$. Then

$$P(\sigma_{n+n_0+m} - \sigma_{k+m} \ge t) \le \frac{1}{2} - \int_0^{tD^{-1/2}} e^{-y^2/2} dy + \Delta_{n+n_0-k}. \tag{8}$$

We shall find t, s.t. for all m, n, k the inequality holds

$$P(\sigma_{n+n_0+m} - \sigma_{k+m} \geq t) \leq \frac{1}{2}. \tag{9}$$

In the chapter 1 we have shown that

$$\int_0^x e^{-y^2/2} dy \geq \frac{x}{2} e^{-x^2/2} + 1 - e^{-x^2/2}.$$

It follows from (8) that the inequality (9) is implied by the inequality

$$\frac{1}{\sqrt{2\pi}} \int_0^{tD^{-1/2}} e^{-y^2/2} dy > \Delta_{n+n_0-k}.$$

Analogously, as before, we can choose t according to

$$t \geq -2D^{1/2} \ln(1 - \sqrt{2\pi}\, \Delta_{n+n_0-k}).$$

Using the properties of variance we have $D \leq (n + n_0 - k)\sigma^2$. Therefore we can choose t according to

$$t \geq -2\sigma\sqrt{2\pi}\, \Delta_{n+n_0-k} \sqrt{n+n_0-k} \ln(1 - \sqrt{2\pi}\, \Delta_{n+n_0-k})^{\frac{1}{\sqrt{2\pi}}\Delta_{n+n_0-k}}. \tag{10}$$

The sequence $(1-x)^{\frac{1}{x}}$ increases as x decreases.

Substituting the right-hand side of (10) by a bigger value we have:

$$t \geq -\frac{2\sigma}{\Delta_{n_0}} \ln(1 - \sqrt{2\pi}\, \Delta_{n_0}) \max_{0\leq k\leq n} \{\sqrt{k+n_0}\, \Delta_{k+n_0}\} = \delta(n)$$

i.e. for $t \geq \delta$ (9) holds.

Decomposing the events $(\overline{\sigma}_n \geq x)$, using (9) and using the α-dependence of our random variables, we have

$$P(\overline{\sigma}_n \geq x) = \sum_{k=1}^{n} P(\sigma_1 < x, \ldots, \sigma_{k-1} < x, \sigma_k \geq x) \geq$$

$$\geq \ 2\sum_{k=1}^{n} P(\sigma_1 < x, \ldots, \sigma_{k-1} < x, \sigma_k \geq x)\ P(\sigma_{n+n_0+m} - \sigma_{k+m} \geq \delta(n)) \geq$$

$$\geq \ 2\sum_{k=1}^{n} P\left(\sigma_1 < x, \ldots, \sigma_{k-1} < x, \sigma_k \geq x, \sigma_{n+n_0+m} - \sigma_{k+m} \geq \delta(n)\right) - 2n\alpha(m).$$

On the other hand $\sigma_{k+m} \leq \sigma_{k-1} + 2a(m+1)$ and hence $P(\overline{\sigma}_n \geq x)$ is bounded below by $2\sum_{k=1}^{n} P(\sigma_1 < x, \ldots, \sigma_{k-1} < x, \sigma_k \geq x, \sigma_n \geq x + 2a(2m + n_0 + 1) + \delta(n)) - 2n\alpha(n) = 2P\left(\sigma_n \geq x + 2a(2m + n_0 + 1) + \delta(n)\right) - 2n\alpha(m)$.

One can prove analogously that

$$P(\sigma_{n+n_0+m} - \sigma_{k+m} \le t) \le \frac{1}{2}, \quad \text{for } t \le -\delta(n).$$

Then for all x

$$P(\overline{\sigma}_n \ge x) \le 2P\left(\sigma_n \ge x - 2a(2m+n_0) - \delta(n)\right) + 2n\alpha(m).$$

From the previous inequalities we have
$1 - 2P(\sigma_n \ge B_n x - 2a(2m+n_0) - \delta(n)) - 2n\alpha(m) \le P(\overline{\sigma}_n < B_n x) \le$
$\le 1 - 2P(\sigma_n \ge B_n x + 2a(2m+n_0+1) + \delta(n)) + 2n\alpha(m).$

As before, we then get for $x \ge 0$

$$P(\overline{\sigma}_n < B_n x) - G(x) \le 2[\mathcal{P}(x + \tfrac{2a(2m+n_0+1)+\delta(n)}{B_n}) - \mathcal{P}(x)] + 2\Delta_n + 2n\alpha(m)$$
$$P(\overline{\sigma}_n < B_n x) - G(x) \ge 2[\mathcal{P}(x - \tfrac{2a(2m+n_0)+\delta(n)}{B_n}) - \mathcal{P}(x)] - 2\Delta_n - 2n\alpha(m)$$

and for $x < 0$

$$0 \le P(\overline{\sigma}_n < B_n x) - G(x) \le 2\mathcal{P}(\frac{2a(2m+n_0+1)+\delta(n)}{B_n}) - 1 + 2\Delta_n + 2n\alpha(m).$$

Because of the inequalities $|\mathcal{P}(x+q) - \mathcal{P}(x)| \le \frac{1}{\sqrt{2\pi}}|q|$, $|\mathcal{P}(px) - \mathcal{P}(x)| \le \frac{1}{\sqrt{2e\pi}}\{\max(p, \frac{1}{p}) - 1\}$ we have

$$\sup_x |P(\overline{\sigma}_n < B_n x) - G(x)| \le \frac{2}{\sqrt{2\pi}} \frac{2a(2m+n_0+1)+\delta(n)}{B_n} + 2\Delta_n + 2n\alpha(m). \tag{11}$$

Using the inclusions

$\{\max(Y_1, Y_1+Y_2, \ldots, Y_1+\ldots+Y_n) < x\} \subset \{\max(Z_1, Z_1+Z_2, \ldots, Z_1+\ldots+Z_n) < x + A_n\}$
and
$\{\max(Z_1, Z_1+Z_2, \ldots, Z_1+\ldots+Z_n) < x - A_n\} \subset \{\max(Y_1, Y_1+Y_2, \ldots, Y_1+\ldots+Y_n) < x\}$,
we have

$$P(\overline{\sigma}_n < bx - A_n) \le P(\max_{1\le k\le n} \sum_{j=1}^{k} Y_j < bx) \le P(\overline{\sigma}_n bx + A_n).$$

Taking into account (11) we have

$$P(\max_{1\le k\le n}\sum_{j=1}^{k} Y_j < bx) - G(x) \le P(\overline{\sigma}_n \le bx + A_n) - G(\frac{bx+A_n}{B_n}) +$$

$$+ \quad G(\frac{bx+A_n}{B_n}) - G(x) \le \frac{2}{\sqrt{2\pi}}\frac{2a(2m+n_0+1)+\delta(n)}{B_n} + 2\Delta_n +$$

$$+ \quad \sqrt{\frac{2}{\pi}}\frac{A_n}{B_n} + \sqrt{\frac{2}{e\pi}}\{\max(\frac{b}{B_n},\frac{B_n}{b}) - 1\} + 2n\alpha(m),$$

and analogously

$$P(\max_{1\le k\le n}\sum_{j=1}^{k} Y_j < bx) - G(x) \ge -\frac{2}{\sqrt{2\pi}}\frac{2a(2m+n_0+1)+\delta(n)}{B_n}$$

$$-2\Delta_n - \sqrt{\frac{2}{\pi}}\frac{A_n}{B_n} - \sqrt{\frac{2}{e\pi}}\{\max(\frac{b}{B_n},\frac{B_n}{b}) - 1\} - 2n\alpha(m).$$

Furthermore

$$\sup_x |P(\max_{1\le k\le n}\sum_{j=1}^{k} Y_j < x) - P(\max_{1\le k\le n}\sum_{j=1}^{k} X_j < x)| \le P(|X_1| > a) = \Gamma_n.$$

Therefore $\sup_x |P(\overline{S}_n < bx) - G(x)| \le \sup_x |P(\overline{S}_n < bx) - P(\max_{1\le k\le n}\sum_{j=1}^{k} Y_j < bx)| +$

$$+\sup_x |P(\max_{1\le k\le n}\sum_{j=1}^{k} Y_j < bx) - G(x)| \le \Gamma_n + \frac{2}{\sqrt{2\pi}}\frac{2a(2m+n_0+1)+\delta(n)}{B_n} +$$

$$+2\Delta_n + \sqrt{\frac{2}{\pi}}\frac{A_n}{B_n} + \sqrt{\frac{2}{e\pi}}\{\max(\frac{b}{B_n},\frac{B_n}{B}) - 1\} + 2n\alpha(m)$$

The theorem is proved.

Corollary 2 *Let* $MX_k = 0,\ k = \overline{1,n}$ *and assume that there exists such constants* $c > 0,\ c_0 > 0,\ K > 0,\ 0 < \delta \le 1,\ \beta \ge 2$ *that*

$$\sup_{k\le n} M|X_k|^{2+\delta} \le c,\ \ B_n^2 \ge c_0 n,\ \ \alpha(\tau) = K\tau^{-\frac{2+\delta}{\delta}\beta},$$

then the estimate in (7) looks like

$$\sup_x |P(\overline{S}_n < B_n x) - G(x)| \le B\, n^{-\gamma} \tag{12}$$

where

$$\gamma = (1+\delta)\,\frac{\beta(2+\delta)-\delta/2}{\delta(1+\delta)+\beta(2+\delta)^2} - \frac{1}{2}$$

and the bounded random variable B depends only on c, c_0, k. Using [3] we have:

$$\Delta_n \leq B\, n^{-\frac{\delta}{2}}(\frac{\beta-1}{\beta+2\delta})$$

The proof is analogous to the proof of corollary 1.

References

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Postscriptum. This paper is a translation of the author's graduation thesis, presented at the university of Minsk. The author, tragically, died in Kaiserslautern in January 1999 while preparing an extension of these results. He will stay in our memory.