Received: 16 June 2021 | Accepted: 23 September 2021

DOI: 10.1002/pamm.202100085

Boundary Conditions in a Lattice Boltzmann Method For Plane Strain Problems

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The Lattice Boltzmann Method (LBM), e.g. in [1] and [2], can be interpreted as an alternative method for the numerical solution of certain partial differential equations that is not restricted to its origin in computational fluid mechanics. The interpretation of the LBM as a general numerical tool allows to extend the LBM to solid mechanics as well, see e.g. [3], which is concerned with the simulation of elastic solids under simplified deformation assumptions, and [4] as well as [5] which propose LBMs for the general plane strain case. In previous works on a LBM for plain strain such as [5], the treatment of practically relevant boundary conditions like Neumann and Dirichlet type boundary conditions is not the main focus and thus periodic conditions or absorbing layers are specified to simulate numerical examples. In this work, we show how Neumann and Dirichlet type boundary conditions are implemented in our LBM for plane strain from [4].

1 Representing the Plane Strain Problem by Wave Equations

For a linear elastic body with density ρ and Lamé parameters λ and μ under plane strain assumption, the volume dilatation $\boldsymbol{\nabla} \cdot \boldsymbol{u} = \phi$ as well as the only non-zero component of the rotation vector $(\boldsymbol{\nabla} \times \boldsymbol{u})_z = \psi$ are governed by the separate wave equations

$$c_d^2 \Delta\phi = \frac{\partial^2 \phi}{\partial t^2} \quad \text{and} \quad c_s^2 \Delta\psi = \frac{\partial^2 \psi}{\partial t^2}, \quad \text{where } c_d = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \; c_s = \sqrt{\frac{\mu}{\rho}}. \tag{1}$$

Herein, $\boldsymbol{u}$ denotes the displacement field. The dilatation and the rotation vector are coupled via the Navier equation

$$c_d^2 \nabla (\nabla \cdot \boldsymbol{u}) - c_s^2 \nabla \times (\nabla \times \boldsymbol{u}) = \ddot{\boldsymbol{u}}, \tag{2}$$

which results from the balance of linear momentum, Hooke's law for isotropic linear elastic material and the definition of the linearized strain tensor.

2 Boundary Conditions in a Lattice Boltzmann Method for Plane Strain Problems

In the LBM, information is represented by distribution functions that are defined on a discretized lattice. The lattice consists of spatially discrete lattice points, which are connected via lattice links denoted by α. The lattice links are associated with a lattice speed $\boldsymbol{c}^\alpha$, that determines to which neighbor information may travel in one time step Δt. The distribution functions are updated via the explicit rule

$$f^\alpha (\boldsymbol{x} + \boldsymbol{c}^\alpha \Delta t, t + \Delta t) = f^\alpha (\boldsymbol{x}, t) - \frac{1}{\tau} \left[f^\alpha(\boldsymbol{x}, t) - f^\alpha_{\text{eq}}(\boldsymbol{x}, t) \right], \tag{3}$$

where τ is a relaxation time. The distribution functions must be interpreted in relation to the macroscopic fields. We consider two sets of distribution functions for the dilatation ϕ and the non-zero displacement component ψ of the rotation vector. These sets of distribution functions are interpreted according to

$$\sum_\alpha f^\alpha_\psi = \psi, \quad \text{and} \quad \sum_\alpha f^\alpha_\phi = \phi. \tag{4}$$

The particular mesoscopic evolution law is then given by (3), where the equilibrium distribution functions $f^\alpha_{\psi,\text{eq}}$, $f^\alpha_{\psi,\text{eq}}$ and the relaxation times τ_ψ and τ_ϕ have to be chosen such that (3) together with the interpretation (4) yields the desired macroscopic behavior (1). In order to accomplish this, we choose the LBM for the wave equation, proposed by [1]. This model is two-dimensional and has five lattice velocities at each lattice point, i.e. it is referred to as a D2Q5-model. The same lattice and time step Δt is used for both sets of distribution functions. This is possible due to the fact, that the model [1] allows to adjust the macroscopic wave speed independently of the lattice spacing and the time step by adapting the f_{eq}.

The overall algorithm computes the accelerations $\ddot{\boldsymbol{u}}^n$ at a particular time step t_n from the Navier equation (2) and finite difference approximations for $\nabla\phi^n$ as well as $\nabla \times \psi^n$. The acceleration $\ddot{\boldsymbol{u}}^n$ at boundary lattice points needs to be determined

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from the boundary conditions. For Neumann boundary conditions, this involves the evaluation of a local balance of momentum for cells that are generated around each boundary lattice point, as is shown in Fig. 1 a). Assuming, that ρ and $\ddot{\boldsymbol{u}}^n$ are constant in each cell, the acceleration at lattice point k with neighbors r can be approximated as

$$\ddot{\boldsymbol{u}}_k^n \approx \frac{1}{V_C \rho} \left(\sum_{r \in \text{Neighbors}} \boldsymbol{\sigma}_{kr}(\boldsymbol{u}^n) \boldsymbol{n}_{kr} l_{kr} + \int_{\partial C_{\text{ext}}} \boldsymbol{t}^{*n} \mathrm{d}a \right), \quad \boldsymbol{\sigma}_{kr} \approx \frac{1}{2}\boldsymbol{\sigma}_k + \frac{1}{2}\boldsymbol{\sigma}_r, \tag{5}$$

where $\boldsymbol{\sigma}_{k/r}$ refers to the Cauchy stress tensor at the respective lattice points, $\boldsymbol{n}_{kr}$ is the outward normal at the shared internal boundary between the cells of k and the neighbor r, and l_{kr} is the length of this internal boundary. If Dirichlet boundary conditions $\boldsymbol{u}^*$ are specified at a lattice point k, the acceleration is computed by substituting $\boldsymbol{u}_k^{n+1} = \boldsymbol{u}^*$ into the integration scheme, e.g. an explicit Newmark scheme. This relation can subsequently be solved for the acceleration, e.g.

$$\ddot{\boldsymbol{u}}_k^n = \frac{2}{\Delta t^2} \left(\boldsymbol{u}^* - \boldsymbol{u}_k^n \right) - \frac{2}{\Delta t} \dot{\boldsymbol{u}}_k^n. \tag{6}$$

Once $\ddot{\boldsymbol{u}}^n$ is known at each lattice point, the displacement $\boldsymbol{u}^{n+1}$ is computed by integration with the explicit Newmark scheme. Subsequently, the rotation and dilatation is updated at the boundary lattice points by means of a finite difference approximation of $\psi_k^{n+1} = (\nabla \times \boldsymbol{u}_k^{n+1})_z$ and $\phi_k^{n+1} = \nabla \cdot \boldsymbol{u}_k^{n+1}$. These updated fields are used to set the distribution functions at the boundary lattice points such that they are consistent with

$$\sum_\alpha f_\psi^\alpha(\boldsymbol{x}_k, t_n) = \psi_k^{n+1}, \quad \sum_\alpha f_\phi^\alpha(\boldsymbol{x}_k, t_n) = \phi_k^{n+1}. \tag{7}$$

Finally, the distribution functions at t^{n+1} at internal lattice points as well as ψ^{n+1} and ϕ^{n+1} are determined by (3) and (4), respectively. At the boundary lattice points the distribution functions are not changed again, i.e. $f_{\psi/\phi}^\alpha(\boldsymbol{x}_k, t_{n+1}) = f_{\psi/\phi}^\alpha(\boldsymbol{x}_k, t_n)$. This algorithm allows to simulate problems with boundary conditions that are relevant in computational solid mechanics such as a plate with a hole that is subject to tensile Neumann boundary conditions at the top and at the bottom edge, as is shown in Fig. 1 b).

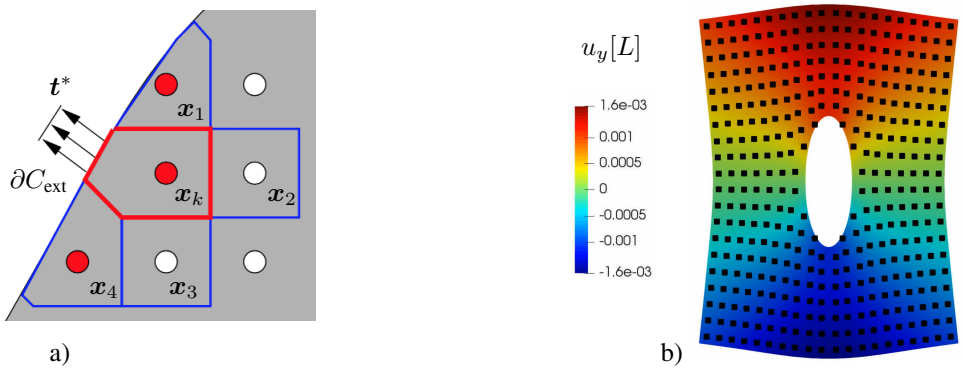

Fig. 1: a) The implementation of Neumann boundary conditions. b) Comparison of the LBM and the FEM for a tension loaded quadratic domain ($L \times L$) with a circular hole (radius $0.266L$). The plot shows the scaled deformed configuration for the LBM (dots represent the lattice points) as well as for a benchmark FEM simulation (contour plot of the vertical displacement component u_y in the background).

Acknowledgements The authors gratefully acknowledge the funding by the German Research Foundation (DFG) within the project 423809639. Open access funding enabled and organized by Projekt DEAL.

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