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Hypersurface singularities with monomial Jacobian ideal

Raul Epure | **Mathias Schulze**

Department of Mathematics, TU Kaiserslautern, Kaiserslautern, Germany

Correspondence
Mathias Schulze, Department of Mathematics, TU Kaiserslautern, Kaiserslautern, Germany.
Email: mschulze@mathematik.uni-kl.de

Abstract

We show that every convergent power series with monomial extended Jacobian ideal is right equivalent to a Thom–Sebastiani polynomial. This solves a problem posed by Hauser and Schicho. On the combinatorial side, we introduce a notion of Jacobian semigroup ideal involving a transversal matroid. For any such ideal, we construct a defining Thom–Sebastiani polynomial. On the analytic side, we show that power series with a quasihomogeneous extended Jacobian ideal are strongly Euler homogeneous. Due to a Mather–Yau-type theorem, such power series are determined by their Jacobian ideal up to right equivalence.

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1 | INTRODUCTION

Let $f : U \to \mathbb{C}$ be a holomorphic function, defined in some open neighborhood $U \subseteq \mathbb{C}^n$ of $\mathbf{0} \in \mathbb{C}^n$. By considering arbitrarily small U, f can be considered as a convergent power series $f \in \mathbb{C}\{\mathbf{x}\}$ in variables $\mathbf{x} = x_1, \dots, x_n$, which are local coordinates on $\mathbb{C}^n$ at $\mathbf{0}$. Similarly, a local biholomorphic map of $\mathbb{C}^n$ at the origin can be seen as a $\mathbb{C}$-algebra automorphism $\varphi \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}\{\mathbf{x}\}$, or a local coordinate change of $\mathbf{x}$.

The *Jacobian ideal* and *extended Jacobian ideal* of f are defined by

$$J_f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle \trianglelefteq \mathbb{C}\{\mathbf{x}\}, \quad \tilde{J}_f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, f \right\rangle \trianglelefteq \mathbb{C}\{\mathbf{x}\},$$

respectively.

Remark 1.1. The ideals $\mathcal{J}_f$ and $\tilde{\mathcal{J}}_f$ are analytic invariants of $f \in \mathbb{C}\{\mathbf{x}\}$ and of the ideal $\langle f \rangle \trianglelefteq \mathbb{C}\{\mathbf{x}\}$, respectively. This means that if $\varphi \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}\{\mathbf{x}\}$ is a $\mathbb{C}$-algebra automorphism and $u \in \mathbb{C}\{\mathbf{x}\}^*$ a unit power series, then

$$\varphi(\mathcal{J}_f) = \mathcal{J}_{\varphi(f)}, \quad \tilde{\mathcal{J}}_{u\cdot f} = \tilde{\mathcal{J}}_f.$$

In particular, φ induces $\mathbb{C}$-algebra isomorphisms

$$\mathbb{C}\{\mathbf{x}\}/\mathcal{J}_f \to \mathbb{C}\{\mathbf{x}\}/\mathcal{J}_{\varphi(f)}, \quad \mathbb{C}\{\mathbf{x}\}/\tilde{\mathcal{J}}_f \to \mathbb{C}\{\mathbf{x}\}/\tilde{\mathcal{J}}_{\varphi(f)}.$$

However, $\mathcal{J}_{u\cdot f} \neq \mathcal{J}_f$, for instance, for $f = x^5 + x^2y^2 + x^5$ and $u = 1 + x$.

In the language of analytic geometry, the arbitrarily small neighborhoods $U \subseteq \mathbb{C}^n$ of $\mathbf{0} \in \mathbb{C}^n$ with functions $f \in \mathbb{C}\{\mathbf{x}\}$ on it form a *smooth space germ*

$$Y = (\mathbb{C}^n, \mathbf{0}), \quad \mathcal{O}_Y = \mathbb{C}\{\mathbf{x}\} \rhd \langle \mathbf{x} \rangle = \mathfrak{m}_Y.$$

A *subspace germ* $X = V(\mathcal{I}_{X/Y}) \subseteq Y$ is the zero locus of an ideal $\mathcal{I}_{X/Y} \trianglelefteq \mathcal{O}_Y$, equipped with the $\mathbb{C}$-algebra $\mathcal{O}_X = \mathcal{O}_Y/\mathcal{I}_{X/Y}$. If $\mathcal{I}_{X/Y}$ is a radical ideal, that is, $\mathcal{O}_X$ and X are *reduced*, this algebraic structure is redundant. If $\mathcal{I}_{X/Y} = \langle f \rangle$ is generated by a single power series $f \in \mathbb{C}\{\mathbf{x}\}$, then

$$\emptyset \neq V(f) \iff \mathbf{0} \in V(f) \iff f \in \mathfrak{m}_Y, \quad V(f) \subsetneq Y \iff f \neq 0.$$

If both latter conditions hold true, then the subspace germ

$$\mathbf{0} \in X = V(f) \subsetneq Y, \quad \mathcal{O}_X = \mathcal{O}_Y/\langle f \rangle \rhd \mathfrak{m}_Y/\langle f \rangle = \mathfrak{m}_X,$$

is called a *hypersurface singularity*. This means that X is of (pure) codimension 1 in the smooth space germ Y. The points in Y where all partial derivatives of f vanish are called *critical points* of f, those in X, *singular points* of X. The corresponding subspace germ is the *singular locus* of X,

$$\mathrm{Sing}\, X = V(\tilde{\mathcal{J}}_f) \subseteq X, \quad \mathcal{O}_{\mathrm{Sing}\, X} = \mathcal{O}_Y/\tilde{\mathcal{J}}_f = \mathcal{O}_X/\mathcal{O}_X\mathcal{J}_f.$$

If $\mathrm{Sing}\, X = \emptyset$ is empty, then f is a coordinate and $X \cong (\mathbb{C}^{n-1}, \mathbf{0})$ is smooth. Otherwise,

$$\mathbf{0} \in \mathrm{Sing}\, X \iff 0 \neq f \in \mathfrak{m}_Y^2.$$

If $\mathrm{Sing}\, X = \{\mathbf{0}\}$ is a point, then f has an *isolated critical point* $\mathbf{0}$, and $X = V(f)$ is called an *isolated hypersurface singularity*.

Remark 1.2. In more intrinsic terms, n equals the *embedding dimension*

$$\mathrm{edim}\, X := \dim_{\mathbb{C}}(\mathfrak{m}_X/\mathfrak{m}_X^2)$$

of the hypersurface singularity $X = V(f)$, unless $\mathrm{Sing}\, X = \emptyset$ where $\mathrm{edim}\, X = n - 1$. Furthermore, the *Jacobian ideal* $\mathcal{O}_X\mathcal{J}_f \trianglelefteq \mathcal{O}_X$ of X defining $\mathrm{Sing}\, X$ is the Fitting ideal of order $\dim X$ of the $\mathcal{O}_X$-module Ω^1_X of differential 1-forms on X, and $\tilde{\mathcal{J}}_f$ is its contraction to $\mathcal{O}_Y$.

Any isomorphism $X \cong X' = V(f') \subseteq Y$ lifts to an automorphism of Y, defined by some $\varphi \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}\{\mathbf{x}\}$ with $\varphi(f) = u \cdot f'$ for some unit $u \in \mathbb{C}\{\mathbf{x}\}^*$.

Definition 1.3. Two power series $f, f' \in \mathbb{C}\{\mathbf{x}\}$ are called *contact equivalent* if $\varphi(f) = u \cdot f'$ for some $\mathbb{C}$-algebra automorphism $\varphi \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}\{\mathbf{x}\}$ and some unit power series $u \in \mathbb{C}\{\mathbf{x}\}^*$. They are called *right equivalent* if $u = 1$.

Due to Remark 1.1, $X \cong X'$ then implies that $\mathrm{Sing}\, X \cong \mathrm{Sing}\, X'$. The converse implication is a celebrated theorem of Mather and Yau in the case of isolated hypersurface singularities (see [13]), and of Gaffney and Hauser for general hypersurface singularities (see [5, Part I] and [9, Theorem 2]) under the following mild restriction (see [9, §3, Definition]).

Definition 1.4. The *isosingular locus* of the germ $X = (\mathcal{X}, x)$ of a space $\mathcal{X}$ at x is the subspace germ

$$\mathrm{Iso}\, X = (\{x' \in \mathcal{X} \mid (\mathcal{X}, x') \cong X\}, x) \subseteq X.$$

Note that $\mathrm{Iso}\, X \subseteq \mathrm{Iso}\, \mathrm{Sing}\, X$ if $\mathrm{Sing}\, X \neq \emptyset$. If $\mathrm{Iso}\, X \supseteq \mathrm{Iso}\, \mathrm{Sing}\, X$, then X is called *harmonic*, and *dissonant* otherwise.

Due to Ephraim (see [2, Theorem 0.2]),

$$\mathrm{Iso}\, X \cong (\mathbb{C}^k, \mathbf{0}), \quad X \cong X' \times \mathrm{Iso}\, X, \quad \mathrm{Sing}\, X \cong \mathrm{Sing}\, X' \times \mathrm{Iso}\, X. \tag{1.1}$$

If $k = 0$, then we say that $\mathrm{Iso}\, X$ is *trivial*.

Theorem 1.5 (Mather–Yau, Gaffney–Hauser). *If X and X' are harmonic hypersurface singularities, then $X \cong X'$ if and only if* $\dim X = \dim X'$ *and* $\mathrm{Sing}\, X \cong \mathrm{Sing}\, X'$.

In other words, the extended Jacobian ideal $\tilde{J}_f$ determines the geometry of the hypersurface singularity $X = V(f)$. This stunning fact has not been exploited systematically so far. It is natural to study hypersurface singularities X where $\mathrm{Sing}\, X$ is particularly simple. In this spirit, Hauser and Schicho (see [11, Problem 2^*]) formulated the following

Problem 1.6 (Hauser–Schicho). Describe all power series $f \in \mathbb{C}\{\mathbf{x}\}$ for which the extended Jacobian ideal $\tilde{J}_f$ is a *monomial ideal*, that is, generated by monomials in terms of some local coordinates $\mathbf{x}$ on $\mathbb{C}^n$ at $\mathbf{0}$.

Obviously this happens if f is of the following type.

Definition 1.7. We call a nonzero sum of non-constant monomials in disjoint sets of variables a *Thom–Sebastiani polynomial*. The particular case of a nonzero sum of positive powers of all variables is called a *Brieskorn–Pham polynomial*.

Example 1.8. The *Whitney umbrella* $X = V(f)$ defined by the Thom–Sebastiani polynomial $f = x^2 + y^2 z$ has a monomial extended Jacobian ideal $\tilde{J}_f = \langle x, y^2, yz \rangle$.

We record some obvious properties of Thom–Sebastiani polynomials.

Remark 1.9. Let $f \in \mathbb{C}\{\mathbf{x}\}$ be a Thom–Sebastiani polynomial.

(a) Then f is quasihomogeneous (see Remark 4.3).
(b) Unless f is a monomial, it is squarefree. Indeed, any multiple factor g of f divides a monomial $\frac{\partial f}{\partial x_i}$, which forces g and hence f to be a monomial.
(c) For any unit $u \in \mathbb{C}\{\mathbf{x}\}^*$, $u \cdot f$ is right equivalent to a Thom–Sebastiani polynomial. In fact, for each monomial $\mathbf{x}^\alpha$ of f and a choice of i with $\alpha_i \neq 0$, considering $\sqrt[\alpha_i]{u} \cdot x_i$ as a new variable eliminates u.
(d) By right equivalence, all degree two monomials of f can be turned into squares. Indeed, a linear coordinate change replaces $x_i x_j$ by $x_i^2 + x_j^2$.

Our main result solves Problem 1.6 of Hauser and Schicho.

Theorem 1.10. *The extended Jacobian ideal $\tilde{\mathcal{J}}_f$ of a power series $0 \neq f \in \langle \mathbf{x} \rangle \lhd \mathbb{C}\{\mathbf{x}\}$ is monomial if and only if f is right equivalent to a Thom–Sebastiani polynomial.*

Its proof in § 5 relies on a combinatorial study of monomial Jacobian ideals in § 2: By passing to exponents of f, we introduce a notion of *Jacobian semigroup ideal* which implements the underlying linear algebra in terms of a transversal matroid (see Definition 2.1 and Remark 2.2). In Proposition 2.4, we show that every such semigroup ideal arises from the exponents of a Thom–Sebastiani polynomial f'. The claimed right equivalence of f and f' then follows from a Mather–Yau-type Theorem 3.4 for strongly Euler homogeneous power series (see Definition 3.1). The homogeneity hypothesis is satisfied if $\tilde{\mathcal{J}}_f$ is monomial due to Theorem 4.6, which generalizes a result of Xu and Yau in the isolated singularity case (see [20, Theorem 1.2]).

We collect some consequences of Theorem 1.10. In the isolated singularity case, a result of K. Saito yields

Corollary 1.11. *If $f \in \mathbb{C}\{\mathbf{x}\}$ has an isolated critical point and $\tilde{\mathcal{J}}_f$ is monomial, then f is right equivalent to a Brieskorn–Pham polynomial.*

Proof. By Theorem 1.10, we may assume that $0 \neq f \in \mathfrak{m}_Y^2$ is a Thom–Sebastiani polynomial with isolated critical point. In particular, f is quasihomogeneous by Remark 1.9.(a). Then there must be, for each $i \in \{1, \dots, n\}$, a monomial x_i^m or $x_i^m x_j$ in f where $m \geqslant 1$ (see [16, Korollar 1.6]). In the second case, switching i and j forces $m = 1$ and Remark 1.9.(d) applies. □

In geometric terms, $\tilde{\mathcal{J}}_f$ monomial means that $\operatorname{Sing} X$ is normal crossing (see Definition 5.1). Using Remarks 1.9.(a) and (b) and Proposition 6.4, we obtain

Corollary 1.12. *Any hypersurface singularity $X = V(f)$ with normal crossing singular locus $\operatorname{Sing} X$ is quasihomogeneous, holonomic and either reduced, or a (possibly non-reduced) normal crossing divisor.* □

Due to the Aleksandrov–Terao Theorem, the notion of *Saito-free divisor* generalizes by requiring Cohen–Macaulayness of a generalized Jacobian ideal (see [8, Definition 5.1], [18, Definition 5.5],

[15, Definition 4.3]). Results of Epure and Pol (see [15, Corollary 5.5] and [1, Theorem 2]) yield the final conclusion in

Corollary 1.13. *A reduced normal crossing singular locus of a hypersurface singularity is a Cartesian product of equidimensional unions of coordinate subspaces, and hence, a free singularity.* □

We conclude with an application to E. Faber's conjecture, which aims for characterizing non-smooth normal crossing divisors as hypersurface singularities $X = V(f)$ with $\mathcal{J}_f$ radical and equidimensional of height 2 (see [4, Conjecture 2]). From $\mathcal{J}_f$ radical, it follows that f is Euler homogeneous (see [4, Lemma 1]). In particular, $\mathcal{J}_f = \tilde{\mathcal{J}}_f$ depends only on X.

Corollary 1.14. *A non-smooth hypersurface singularity $X = V(f)$ is a reduced normal crossing divisor if and only if $\mathcal{J}_f$ is monomial and radical of height 2.*

Proof. Suppose that $\mathcal{J}_f$ is monomial and radical of height 2. By Theorem 1.10, f is then a Thom–Sebastiani polynomial of squarefree monomials in terms of variables $\mathbf{x} = x_1, \dots, x_n$. Each monomial contributes 1 to the height of $\mathcal{J}_f$ if quadratic, and 2 otherwise. Then either $f = x_1^2 + x_2^2$, or $f \in \langle \mathbf{x} \rangle^3$ is a monomial. In the first case, $f = x_1 x_2$ after a linear coordinate change. Thus, X is a reduced normal crossing divisor in both cases. □

2 | JACOBIAN SEMIGROUP IDEALS

In this section, we describe the combinatorics underlying our problem by combining the structures of semigroup ideals and transversal matroids.

Fix $n \in \mathbb{N}$ and set $[n] := \{1, \dots, n\}$. Recall that a *matroid* M on $[n]$ axiomatizes the notion of linear dependence of families of n vectors in a vector space (see [14]). Among other options, it can be defined by the data of *independent sets*, or that of *bases*, that is, maximal independent sets. Both are distinguished subsets of the ground set $[n]$, subject to corresponding matroid axioms. These latter implement standard theorems of linear algebra such as the Steinitz exchange lemma and the basis extension theorem. The *rank* $\mathrm{rk}(S)$ of a subset $S \subseteq [n]$ is the maximal cardinality of an independent subset of S. The rank $\mathrm{rk}\,\mathsf{M} := \mathrm{rk}([n])$ of the matroid M equals the cardinality of any basis.

Consider the commutative monoid $M := (\mathbb{N}^n, +)$. The *support* of an element $\alpha \in M$ is the set

$$[\alpha] := \{i \in [n] \mid \alpha_i \neq 0\}, \tag{2.1}$$

its *degree* is defined by

$$|\alpha| := \alpha_1 + \cdots + \alpha_n.$$

For any subset $F \subseteq M$, we consider the union of supports of all its elements,

$$[F] := \bigcup_{\alpha \in F} [\alpha]. \tag{2.2}$$

For any semigroup ideal $I \trianglelefteq M$, the set of minima $\mathrm{Min}(I)$ with respect to the partial ordering is the (unique) minimal set of generators. By Dickson's Lemma, it has finite cardinality

$$\mu(I) := |\mathrm{Min}(I)| < \infty.$$

For $i \in [n]$, denote by $\mathbf{e}_i := (\delta_{i,j})_j \in M$ the ith unit vector and define *partial differentiation operators*

$$\delta_i : M \to M \cup \{\infty\}, \quad \delta_i(\alpha) := \begin{cases} \infty & \text{if } \alpha_i = 0, \\ \alpha - \mathbf{e}_i & \text{otherwise.} \end{cases}$$

For $F \subseteq M \ni \alpha$, we write

$$\delta(F) := \bigcup_{i\in[n]} \delta_i(F) \setminus \{\infty\} \subseteq M, \quad \delta(\alpha) := \delta(\{\alpha\}).$$

Definition 2.1. Let $F \subseteq M$ be a subset and consider the semigroup ideal

$$J_F := \langle \delta(F) \rangle \trianglelefteq M.$$

We call the *transversal matroid* M_F associated with the covering of $\mathrm{Min}(J_F)$

$$\{\mathrm{Min}(J_F) \cap \delta_i(F) \mid i \in [n]\} \subseteq 2^{\mathrm{Min}(J_F)}$$

the *Jacobian matroid* of F (see [14, §1.6]). Its independent sets are the *partial transversals* of the covering, that is, injective maps

$$[n] \supseteq I \overset{\psi}{\hookrightarrow} \mathrm{Min}(J_F), \quad \text{where } \psi(i) \in \delta_i(F) \text{ for all } i \in I.$$

Note that the *rank* of ψ equals $\mathrm{rk}(\psi) = |I|$. If $\mathrm{rk}\,\mathsf{M}_F = \mu(J_F)$, then refer to J_F as the *Jacobian semigroup ideal* of F.

Note that it is a strong requirement on F to have a Jacobian semigroup ideal J_F. In fact, typically $|\mu(J_F)| > n$, whereas $\mathrm{rk}\,\mathsf{M}_F \leqslant n$.

Our terminology is motivated by the following

Remark 2.2. Consider the *support* of a power series in variables $\mathbf{x} = x_1, \dots, x_n$,

$$f = \sum_\alpha f_\alpha \mathbf{x}^\alpha \in \mathbb{C}\{\mathbf{x}\}, \quad F := \mathrm{supp}(f) = \{\alpha \in M \mid f_\alpha \neq 0\} \subseteq M.$$

Suppose that $\mathcal{J}_f$ is generated by monomials in terms of the variables $\mathbf{x}$. Then J_F is the set of exponents of monomials in $\mathcal{J}_f$, and $\mathrm{Min}(J_F)$ is the subset of exponents of minimal monomial generators of $\mathcal{J}_f$. Since $\mathcal{J}_f$ is generated by n many partial derivatives, its minimal number of generators is bounded by

$$\mu(J_F) = \mu(\mathcal{J}_f) \leqslant n.$$

By Nakayama's Lemma, any minimal generators $\frac{\partial f}{\partial x_i}$, $i \in I$, of $\mathcal{J}_f$ map to a basis of the $\mathbb{C}$-vector space $\mathcal{J}_f/\langle \mathbf{x}\rangle \mathcal{J}_f$, which has a monomial basis with exponents in $\mathrm{Min}(J_F)$. Gaussian Elimination yields a bijection $\psi : I \to \mathrm{Min}(J_F)$ such that $\mathbf{x}^{\psi(i)}$ is a monomial of $\frac{\partial f}{\partial x_i}$, and hence, $\psi(i) \in \delta_i(F)$ for all $i \in I$. Thus, ψ is a partial transversal. It follows that

$$\mu(\mathcal{J}_f) = |I| \leqslant \mathrm{rk}\, \mathsf{M}_F \leqslant \mu(J_F) = \mu(\mathcal{J}_f)$$

is an equality. This makes J_F a Jacobian semigroup ideal.

The impression that Jacobian semigroup ideals are quite special is further supported by our main combinatorial result. It replaces F by the support of a Thom–Sebastiani polynomial leaving J_F unchanged. We first illustrate its proof in the simple case of the Whitney umbrella from Example 1.8.

Example 2.3. For $f = x^2 + y^2 z$ with $J_f = \langle x, yz, y^2\rangle$, we obtain

$$\begin{aligned} F &= \{(2,0,0),(0,2,1)\},\\ \delta_1(F) = \{(1,0,0)\},\quad \delta_2(F) &= \{(0,1,1)\},\quad \delta_3(F) = \{(0,2,0)\},\\ \delta((2,0,0)) = \{(1,0,0)\},\quad \delta((0,2,1)) &= \{(0,1,1),(0,2,0)\},\\ \mathrm{Min}(J_F) &= \{(1,0,0),\ (0,1,1),\ (0,2,0)\}. \end{aligned}$$

We can reconstruct F from J_F as follows: Choose

$$\psi(1) := (1,0,0) \in \mathrm{Min}(J_F),\quad \psi(1) = \delta_1(\alpha),\quad \alpha := (2,0,0) \in F,$$

to obtain a partial transversal $\psi : [1] \hookrightarrow \mathrm{Min}(J_F)$. Set $F' := \{\alpha\}$. Since

$$\mathrm{rk}(\psi) = |[1]| = 1 < 3 = \mu(J_F) = \mathrm{rk}\, \mathsf{M}_F,$$

ψ extends to [2] by

$$\psi(2) \in \delta_2(F) = \{\delta_2((0,2,1))\},\quad (0,2,1) := \alpha',$$

and $F' \cup \{\alpha'\} = F$. Then

$$\delta(F') = \psi([2]) \sqcup \{\delta_3(\alpha')\},\quad \delta_3(\alpha') =: \psi(3),$$

extends ψ to [3] and the process terminates.

We now develop the approach of Example 2.3 into a general argument.

Proposition 2.4. *Let $F \subseteq M$ be a subset such that J_F is a Jacobian semigroup ideal. Then $J_F = J_{F'}$ for some subset $F' \subseteq F \setminus \{\mathbf{0}\}$ whose elements $\alpha \in F'$ have disjoint supports $[\alpha]$ and contribute minimal generators $\delta(\alpha) \subseteq \mathrm{Min}(J_F)$ of J_F.*

Proof. For increasing ℓ, we construct partial transversals

$$[\ell] \overset{\psi}{\hookrightarrow} \operatorname{Min}(J_F),$$

enumerating $\operatorname{Min}(J_F)$ by increasing degree, together with subsets $F' \subseteq F$ such that

$$[F'] = \bigsqcup_{\alpha \in F'} [\alpha] = [\ell], \quad \delta(F') = \psi([\ell]).$$

The claim is proven when equality has been reached in

$$\operatorname{rk}(\psi) = \ell \leqslant \mu(J_F) = \operatorname{rk} \mathsf{M}_F.$$

Otherwise, an $\alpha \in F \setminus \{\mathbf{0}\}$ extends ψ to a $k \in [\alpha] \setminus [\ell]$ by a new minimal generator

$$\beta := \psi(k) = \delta_k(\alpha) \in \operatorname{Min}(J_F) \setminus \psi([\ell]).$$

Since ψ is part of a basis of M_F, the degree $|\beta|$ can be chosen minimal. Suppose that, for some $i \in [\alpha]$, $\delta_i(\alpha) \notin \operatorname{Min}(J_F) \setminus \psi([\ell])$ is not a new minimal generator. Since ψ enumerates $\operatorname{Min}(J_F)$ by increasing degree, this means that $\delta_i(\alpha) \in \langle \psi([\ell]) \rangle$. Then there is a $j \in [\ell]$ such that

$$\alpha - \mathbf{e}_i = \delta_i(\alpha) \geqslant \psi(j) =: \gamma.$$

Using that

$$[\alpha] \ni k \notin [\ell] = [F'] \supseteq [\delta(F')] = [\psi([\ell])] \supseteq [\gamma] \Rightarrow \alpha_k > \gamma_k,$$

this leads to the contradiction

$$\operatorname{Min}(J_F) \ni \beta > \beta - \mathbf{e}_i = \alpha - \mathbf{e}_i - \mathbf{e}_k \geqslant \gamma \in J_F.$$

It follows that all $|[\alpha]|$ elements of $\delta(\alpha) \subseteq \operatorname{Min}(J_F) \setminus \psi([\ell])$ are pairwise different new minimal generators of J_F of minimal degree $|\alpha| - 1 = |\beta|$. In particular, ψ extends to $[\ell] \cup [\alpha]$. Suppose that $i \in [\ell] \cap [\alpha]$, and hence,

$$\operatorname{rk}(\psi) = |[\ell] \cup [\alpha]| < \ell + |[\alpha]| \leqslant \mu(J_F) = \operatorname{rk} \mathsf{M}_F.$$

An extension of ψ yields a $j \notin [\ell] \cup [\alpha]$ and an $\alpha' \in F$ such that $\delta_i(\alpha) = \delta_j(\alpha')$. Replace α by α' repeatedly to decrease $|\alpha|_{[\ell]}|$ until $[\ell] \cap [\alpha] = \emptyset$. Then reorder $\ell + 1, \dots, n$ such that $[\ell] \cup [\alpha] = [\ell + |[\alpha]|]$. Now including α in F' increases ℓ by $|[\alpha]|$. Iterating this procedure until $\ell = \mu(J_F)$ yields the claim. □

3 | MATHER–YAU UNDER STRONG EULER HOMOGENEITY

In this section, we discuss a Mather–Yau-type theorem for strongly Euler homogeneous power series. We first recall the definition (see [7, p. 769]).

Definition 3.1. A power series $f \in \mathcal{O}_Y = \mathbb{C}\{\mathbf{x}\}$ is called *(strongly) Euler homogeneous* if $f \in \mathcal{J}_f$ ($f \in \mathfrak{m}_Y \mathcal{J}_f$ where $\mathfrak{m}_Y = \langle \mathbf{x} \rangle$). In this case, a hypersurface singularity $X = V(f) \subseteq Y$ is called *(strongly) Euler homogeneous*.

Remark 3.2.

(a) Euler homogeneity of f is equivalent to $\mathcal{J}_f = \tilde{\mathcal{J}}_f$.
(b) If $X = V(f)$ is strongly Euler homogeneous, then so is f. That is, strong Euler homogeneity is invariant under contact equivalence.
(c) If $X \cong X' \times (\mathbb{C}, 0)$, then X is strongly Euler homogeneous if and only if X' is so (see [7, Lemma 3.2]).
(d) For X with trivial Iso X Euler homogeneity must be strong (see [9, Theorem 1.(4')]). Note also that any X with non-trivial Iso X is already Euler homogeneous. Indeed, for f independent of x_n, $X = V(\exp(x_n)f)$ and $\exp(x_n)f = \frac{\partial \exp(x_n)f}{\partial x_n} \in \mathcal{J}_{\exp(x_n)f}$.

Euler homogeneity and Theorem 1.5 are linked by the following

Remark 3.3. Suppose that the hypersurface singularity $X = V(f) \subseteq Y$ is not smooth, that is, $\mathbf{0} \in \operatorname{Sing} X$. Then there is a modified singular locus

$$\operatorname{Sing}^* X := V(TK(f)), \quad TK(f) := \mathfrak{m}_Y \mathcal{J}_f + \langle f \rangle \trianglelefteq \mathcal{O}_Y,$$

defined by the tangent space $TK(f)$ at f to the orbit of f under the contact group. The underlying reduced space germs of $\operatorname{Sing} X$ and $\operatorname{Sing}^* X$ agree. Note that $TK(f) = \mathfrak{m}_Y \tilde{\mathcal{J}}_f$ if X is strongly Euler homogeneous.

Thus, for strongly Euler homogeneous hypersurface singularities X and X', $\operatorname{Sing} X \cong \operatorname{Sing} X'$ implies $\operatorname{Sing}^* X \cong \operatorname{Sing}^* X'$. By Gaffney and Hauser (see [5, Part I]), this further implies that $X \cong X$ if $\dim X = \dim X'$.

However, Euler homogeneity is not a consequence of harmonicity. In fact, isolated hypersurface singularities are trivially harmonic. Correspondingly, Euler homogeneity does not suffice for the conclusion of Theorem 1.5 due to Remark 3.2.(d) and an example of Gaffney and Hauser (see [5, §4]).

Strongly Euler homogeneous power series satisfy a Mather–Yau Theorem for right equivalence (see [12, Theorem 9.1.10]).

Theorem 3.4. *Let $f, f' \in \mathbb{C}\{\mathbf{x}\}$ be strongly Euler homogeneous power series. Then the following statements are equivalent.*

(a) *The power series f and f' are right equivalent.*
(b) *The power series f and f' are contact equivalent.*
(c) *The $\mathbb{C}$-algebras $\mathbb{C}\{\mathbf{x}\}/\mathcal{J}_f$ and $\mathbb{C}\{\mathbf{x}\}/\mathcal{J}_{f'}$ are isomorphic.*

Proof. (a) implies (b) by Definition 1.3, (b) implies (c) due to Remarks 1.1 and 3.2.(a). It remains to show that (c) implies (a).

An isomorphism $\mathbb{C}\{\mathbf{x}\}/\langle\mathbf{x}\rangle\mathcal{J}_f \cong \mathbb{C}\{\mathbf{x}\}/\langle\mathbf{x}\rangle\mathcal{J}_{f'}$ induced by (c) is trivially one of algebras over $\mathbb{C}\{f\} \cong \mathbb{C}\{f'\}$ because the respective classes of f and f' are zero. This implies (a) (see [12, Theorem 9.1.10]). For the sake of self-containedness, we prove this latter implication:

By Remark 1.1, we may assume that

$$\mathcal{J}_f = \mathcal{J}_{f'} =: \mathcal{J}$$

and consider this as an equality of ideal sheaves on some common domain of convergence $\mathbf{0} \in U \subseteq Y$. Consider the homotopy from f toward f'

$$H := f + t \cdot (f' - f) \in \mathcal{O}_{U\times\mathbb{C}}$$

depending on a parameter $t \in \mathbb{C}$ with its relative Jacobian ideal sheaf

$$\mathcal{J}_H := \left\langle \frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n} \right\rangle \trianglelefteq \mathcal{O}_{U\times\mathbb{C}}.$$

By abuse of notation, we consider $\mathcal{J} \trianglelefteq \mathcal{O}_{U\times\mathbb{C}}$. Then $\mathcal{J}_H \subseteq \mathcal{J}$ and

$$\frac{\partial f}{\partial x_i} \equiv t \cdot \left(\frac{\partial f'}{\partial x_i} - \frac{\partial f}{\partial x_i} \right) \in t\mathcal{J} \mod \mathcal{J}_H \text{ for all } i = 1, \dots, n.$$

Thus, $\mathcal{J}/\mathcal{J}_H = t(\mathcal{J}/\mathcal{J}_H)$, and Nakayama's Lemma yields equality

$$\mathcal{J}_H = \mathcal{J} \trianglelefteq \mathcal{O}_{U\times\mathbb{C},(\mathbf{0},0)} = \mathbb{C}\{\mathbf{x}, t\}. \tag{3.1}$$

In other words, the support Z of the quotient sheaf $\mathcal{J}/\mathcal{J}_H$ does not contain $(\mathbf{0}, 0)$, and by symmetry not $(\mathbf{0}, 1)$ either. Due to coherence of the sheaf, $Z \subseteq U \times \mathbb{C}$ is an analytic subset (see [12, Corollary 6.2.9]), and $Z \cap (\{0\} \times \mathbb{C})$ identifies with a discrete subset $D \subseteq \mathbb{C} \setminus \{0, 1\}$ (see [12, Theorem 3.1.10]). For any $\tau \in \mathbb{C} \setminus D$, strong Euler homogeneity of f and f' yields that

$$\frac{\partial H}{\partial(t-\tau)} = f' - f \in \langle\mathbf{x}\rangle\mathcal{J} = \langle\mathbf{x}\rangle\mathcal{J}_H \trianglelefteq \mathcal{O}_{U\times\mathbb{C},(\mathbf{0},\tau)} = \mathbb{C}\{\mathbf{x}, t-\tau\}.$$

By local triviality (see [12, Corollary 9.1.6]), it follows that

$$f_\tau := H(\mathbf{x}, \tau) \in \mathcal{O}_Y = \mathbb{C}\{\mathbf{x}\}$$

has locally constant right equivalence class for $\tau \in \mathbb{C} \setminus D$. Pick a continuous path $\gamma : [0, 1] \to \mathbb{C} \setminus D$ from $\gamma(0) = 0$ to $\gamma(1) = 1$. By compactness, the right equivalence class of f_τ is then constant for $\tau \in \gamma([0, 1])$. In particular, $f = f_{\gamma(0)}$ and $f' = f_{\gamma(1)}$ are right equivalent, and hence, $X \cong X'$ as claimed. □

4 | QUASIHOMOGENEOUS JACOBIAN ALGEBRAS

In this section, we deduce strong Euler homogeneity for a hypersurface singularity from a positive *analytic* grading on the singular locus, generalizing a result of Xu and Yau (see [20, Theorem 1.2]).

The geometric meaning of strong Euler homogeneity is rather subtle in general. It becomes more transparent in the following special case.

Definition 4.1. An *Euler derivation* on a space germ X is a $\mathbb{C}$-linear derivation $\chi : \mathcal{O}_X \to \mathcal{O}_X$ of the form

$$\chi = \sum_{i=1}^{n} w_i x_i \frac{\partial}{\partial x_i}, \quad \mathbf{w} = w_1, \dots, w_n \in \mathbb{Q}_{>0}, \tag{4.1}$$

where $\mathbf{x} = x_1, \dots, x_n$ minimally generates $\mathfrak{m}_X$. If it exists, then X and all eigenvectors of χ are called *quasihomogeneous*.

Remark 4.2. If $X = V(\mathcal{I}_{X/Y}) \subseteq Y$ with Y smooth and $\dim Y = \operatorname{edim} X$, then $\mathbf{x}$ are coordinates on Y and χ lifts to a $\mathbb{C}$-linear derivation $\chi : \mathcal{O}_Y \to \mathcal{O}_Y$ with $\chi(\mathcal{I}_{X/Y}) \subseteq \mathcal{I}_{X/Y}$ (see [19, (2.1)]). Conversely, any such *logarithmic Euler derivation* long $\mathcal{I}_{X/Y}$ induces an Euler derivation on X.

Remark 4.3. Any Thom–Sebastiani polynomial $f = \sum_{i=1}^{k} \mathbf{x}^{\alpha_i} \in \mathcal{O}_Y$ is quasihomogeneous. In fact, setting

$$w_i := \begin{cases} \frac{1}{|\alpha_j|}, & i \in [\alpha_j], \\ 1, & i \notin \bigsqcup_{j=1}^{k} [\alpha_j], \end{cases}$$

in (4.1) yields an Euler derivation χ on Y such that $\chi(f) = f$.

Remark 4.4. Any quasihomogeneous $f \in \mathfrak{m}_Y$ is (strongly) Euler homogeneous since $\mathbb{Q}_{>0} \cdot f \ni \chi(f) \in \mathfrak{m}_Y \mathcal{J}_f$ if $f \neq 0$. The converse holds true for isolated hypersurface singularities due to a result of K. Saito (see [16]).

Remark 4.5. In more intrinsic terms, quasihomogeneity of X means that $\mathfrak{m}_X$ is generated by eigenvectors $\mathbf{x} = x_1, \dots, x_n$ of χ with eigenvalues $\mathbf{w} \in \mathbb{Q}^n_{>0}$. By clearing denominators such that $\mathbf{w} \in \mathbb{Z}^n_{>0}$, this becomes equivalent to a *positive analytic grading* over $(\mathbb{Z}, +)$ in the sense of Scheja and Wiebe on the analytic algebra $(\mathcal{O}, \mathfrak{m}) := (\mathcal{O}_X, \mathfrak{m}_X)$, whose kth homogeneous part is the k-eigenspace of χ,

$$\mathcal{O}_k = \langle \mathbf{x}^\alpha \mid \langle \mathbf{w}, \alpha \rangle = k \rangle_{\mathbb{C}}, \quad \langle \mathbf{w}, \alpha \rangle := \sum_{i=1}^{n} w_i \alpha_i,$$

spanned by monomials of $\mathbf{w}$-weighted degree k (see [19, §1–3]). This means that the vector spaces $\mathcal{O}_k \subseteq \mathcal{O}$, $k \in \mathbb{Z}$, induce a grading of $\mathcal{O}/\mathfrak{m}^\ell$, for all $\ell \in \mathbb{Z}_{\geqslant 0}$, compatible with the canonical surjections $\mathcal{O}/\mathfrak{m}^\ell \to \mathcal{O}/\mathfrak{m}^{\ell'}$, for all $\ell \geqslant \ell'$. Note, however, that $\mathcal{O} \neq \bigoplus_{l \in \mathbb{Z}} \mathcal{O}_k$ in general.

Using Remark 4.5, we can generalize a result of Xu and Yau (see [20, Theorem 1.2]) as follows.

Theorem 4.6. *If a hypersurface singularity $X = V(f)$ has quasihomogeneous singular locus* Sing X, *then f is strongly Euler homogeneous.*

Proof. To reduce to the case where $\operatorname{Iso} X$ is trivial in (1.1), apply Lemma 4.7 to $\operatorname{Sing} X$ and use Remark 3.2.(b). Then, by Remark 3.2.(d), it suffices to to show that f is Euler homogeneous. This follows from the argument of Xu and Yau (see proof of [20, Theorem 1.2]) using Lemma 4.8 and a positive *analytic* grading on $\mathcal{O}_{\operatorname{Sing} X} = \mathcal{O}_Y/\tilde{J}_f$ (see Remark 4.5). □

Lemma 4.7. *If a Cartesian product $X = X' \times Z$ of space germs is quasihomogeneous and Z is smooth, then X' is quasihomogeneous.*

Proof. Quasihomogeneity of X yields an Euler derivation χ as in (4.1). By the Implicit Mapping Theorem (see [12, Theorem 3.3.6]), reordering $\mathbf{x}$ yields

$$X' \cong X'' := V(x_1, \dots, x_k) \subseteq X, \quad k = \dim Z.$$

The derivation induced by χ makes X'' and hence X' quasihomogeneous. □

Lemma 4.8. *Let $\mathcal{O}$ be an analytic algebra with maximal ideal $\mathfrak{m}$, and χ an Euler derivation as in (4.1). Then χ induces a $\mathbb{C}$-linear automorphism on any χ-invariant ideal $\mathcal{I} \subseteq \mathfrak{m}$.*

Proof. By Remark 4.2, clearing denominators of $\mathbf{w}$ and reordering variables, we may assume that $\mathcal{O} = \mathbb{C}\{\mathbf{x}\}$ in the situation of Remark 4.5 with

$$w_1 \leqslant \cdots \leqslant w_n \Rightarrow w_1|\alpha| \leqslant \langle \mathbf{w}, \alpha \rangle \leqslant w_n|\alpha|. \tag{4.2}$$

Expanding an element

$$f = \sum_{k \in \mathbb{Z}_{>0}} f_k \in \mathfrak{m} = \langle \mathbf{x} \rangle, \quad f_k \in \mathcal{O}_k,$$

in terms of $\mathbf{w}$-weighted homogeneous parts, one finds a unique preimage

$$\int_\chi f := \sum_{k \in \mathbb{Z}_{>0}} \frac{f_k}{k}, \quad \chi\Big(\int_\chi f\Big) = f.$$

Writing $f = \sum_{|\alpha|>0} f_\alpha \mathbf{x}^\alpha$ in terms of monomials, then using (4.2) we obtain

$$\left\| \int_\chi f \right\|_{\mathbf{t}} = \sum_{|\alpha|>0} \frac{|f_\alpha|}{\langle \mathbf{w}, \alpha \rangle} \mathbf{t}^\alpha \leqslant \frac{1}{w_1} \sum_{|\alpha|>0} \frac{|f_\alpha|}{|\alpha|} \mathbf{t}^\alpha \leqslant \frac{1}{w_1} \sum_{|\alpha|>0} |f_\alpha| \mathbf{t}^\alpha = \frac{\|f\|_{\mathbf{t}}}{w_1} < \infty$$

for all $\mathbf{t} \in \mathbb{R}^n_+$, and hence, $\int_\chi f \in \mathfrak{m}$ (see [6, §1.2, Satz 3', §3.3]).

If $\mathcal{I} \subseteq \mathfrak{m}$ is a χ-invariant and hence $\mathbf{w}$-weighted homogeneous ideal, then $\int_\chi$ leaves all homogeneous parts $\mathcal{I}_k$, $k \in \mathbb{Z}_{>0}$, and hence $\mathcal{I}$ itself invariant. □

5 | MONOMIAL JACOBIAN IDEALS

In this section, we combine Proposition 2.4 and Theorems 3.4 and 4.6 to prove our main Theorem 1.10.

We consider the following extremal variant of quasihomogeneity given by a maximal number of linearly independent weight vectors.

Definition 5.1. We call an ideal of an analytic algebra $\mathcal{O}$ *monomial* if it is generated by monomials in terms of some minimal generators of the maximal ideal $\mathfrak{m} \lhd \mathcal{O}$. A space germ $X \subseteq Y$ is *normal crossing* if its defining ideal $\mathcal{I}_{X/Y} \trianglelefteq \mathcal{O}_Y$ is monomial. Note that such space germs are quasihomogeneous.

Remark 5.2. In more intrinsic terms, maximal quasihomogeneity of X means that $\mathrm{Aut}_{\mathbb{C}}\, \mathcal{O}_X$ contains an algebraic torus of dimension $\operatorname{edim} X$ (see Remark 1.2) as a subgroup in the sense of Hauser and Müller (see [10, §1)]). Indeed, such a torus acts linearly in terms of suitable coordinates and lifts to any smooth space germ $Y \supseteq X$ with $\dim Y = \operatorname{edim} X$ (see [10, Satz 6.i)]). The dimension condition is redundant because a general such embedding is isomorphic to $X \times Z \subseteq Y \times Z$ with Z smooth. The torus invariant defining ideal $\mathcal{I}_{X/Y} \trianglelefteq \mathcal{O}_Y$ is then generated by monomials. The converse implication holds trivially.

We are ready to prove our main result.

Proof of Theorem 1.10. Sufficiency is due to Remark 1.1. Suppose that $\tilde{\mathcal{J}}_f$ is monomial for some $0 \neq f \in \langle \mathbf{x} \rangle$. Then $\operatorname{Sing} X = V(\tilde{\mathcal{J}}_f)$ is quasihomogeneous, and hence, f is strongly Euler homogeneous by Theorem 4.6. By Remark 2.2, the support $F := \operatorname{supp}(f)$ of f defines a Jacobian semigroup ideal J_F. Then F' obtained from Proposition 2.4 is the set of exponents of a Thom–Sebastiani polynomial

$$f' := \sum_{\alpha \in F'} \mathbf{x}^{\alpha} \in \mathbb{C}\{\mathbf{x}\}, \quad \mathcal{J}_f = \mathcal{J}_{f'},$$

which is strongly Euler homogeneous by Remarks 4.3 and 4.4. Then f and f' are right equivalent due to Theorem 3.4, proving the claim. □

6 | LOGARITHMIC DERIVATIONS AND HOLONOMICITY

In this section, we describe the logarithmic derivations along hypersurface singularities defined by Thom–Sebastiani polynomials and show that they define a finite logarithmic stratification in the sense of K. Saito (see [17]).

Definition 6.1. The $\mathcal{O}_X$-module of *logarithmic derivations* along the hypersurface singularity $X = V(f) \subseteq Y$ (see Remark 4.2 and [17, (1.4)]),

$$\operatorname{Der}(-\log X) \subseteq \operatorname{Der}_{\mathbb{C}}\, \mathcal{O}_Y =: \Theta_Y,$$

consists of all $\mathbb{C}$-linear derivations $\delta : \mathcal{O}_Y \to \mathcal{O}_Y$ with $\delta(f) \subseteq \langle f \rangle$.

The *logarithmic stratification* of X on Y by smooth connected immersed submanifolds is characterized by the fact that, for all $y \in Y$, the tangent space at y of the stratum containing y is spanned by the evaluations of all elements of $\operatorname{Der}(-\log X)$ at p (see [17, (3.3)]). If this stratification is finite, then X is called *holonomic* (see [17, (3.8)]).

Remark 6.2.

(a) Replacing f by its squarefree part, that is, X by the associated reduced space germ X^{red}, does not affect logarithmic derivations and stratification.
(b) The complement $Y \setminus X$ and the connected/irreducible components of $X \setminus \mathrm{Sing}(X^{\mathrm{red}})$ are (finitely many) logarithmic strata (see [17, (3.4) iii)]).
(c) The derivations annihilating f form an $\mathcal{O}_X$-submodule

$$\mathrm{Der}(-\log X) \supseteq \mathrm{Der}(-\log f) := \mathrm{ann}_{\Theta_Y}(f) \cong \mathrm{syz}(\mathcal{J}_f),$$

isomorphic to the syzygy module of $\mathcal{J}_f$. Euler homogeneity of $X = V(f)$ yields a logarithmic vector field χ such that $\chi(f) = f$. If suitably chosen, this yields a direct sum decomposition

$$\mathrm{Der}(-\log X) = \mathcal{O}_Y \cdot \chi \oplus \mathrm{Der}(-\log f).$$

Remark 6.3. Consider a monomial (Thom–Sebastiani polynomial) $f = \mathbf{x}^\alpha$ defining a normal crossing divisor $X = V(f)$. With the Euler derivation χ from Remark 4.3, one verifies that

$$\mathrm{ann}_{\Theta_Y}(f) = \langle x_i \frac{\partial}{\partial x_i} - \alpha_i \cdot \chi \mid i \in [\alpha] \rangle + \langle \frac{\partial}{\partial x_i} \mid i \in [n] \setminus [\alpha] \rangle.$$

Since $X^{\mathrm{red}} = \bigcup_{i\in[\alpha]} V(x_i)$ it follows with Remark 6.2.(c) that

$$\mathrm{Der}(-\log X)|_{X^{\mathrm{red}}} = \mathrm{ann}_{\Theta_Y}(f)|_{X^{\mathrm{red}}} = \langle x_i \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \mid i \in [\alpha], j \in [n] \setminus [\alpha] \rangle|_{X^{\mathrm{red}}}.$$

This shows that $\mathrm{ann}_{\Theta_Y}(f)$ defines the logarithmic stratification of X on X, and that the strata are relative complements of coordinate subspaces. In particular, X is holonomic by Remark 6.2.(b).

Proposition 6.4. *Any Thom–Sebastiani polynomial defines a holonomic hypersurface singularity.*

Proof. Let $f = \sum_{i=1}^k \mathbf{x}^{\alpha_i}$ be a Thom–Sebastiani polynomial with support $F := \{\alpha_i \mid i \in [k]\}$, defining a hypersurface singularity $X = V(f) \subseteq Y$. In the case where $k = 1$, the claim is due to Remark 6.3. The monomials of f define (normal crossing) hypersurface singularities (see (2.1))

$$X_i = V(\mathbf{x}^{\alpha_i}) \subseteq (\mathbb{C}^{[\alpha_i]}, \mathbf{0}) =: Y_i, \quad i = 1, \dots, k,$$

such that (see (2.2))

$$X' := V(f) \subseteq \prod_{i=1}^k Y_i =: Y', \quad X = X' \times Z \subseteq Y' \times Z = Y, \quad Z := (\mathbb{C}^{[n]\setminus[F]}, \mathbf{0}),$$

and all logarithmic strata of X are products of strata of X' with Z. We may thus assume that $Z = \{\mathbf{0}\}$, that is, $[F] = [n]$. Then

$$\mathrm{Sing}\, X = \prod_{i=1}^k \mathrm{Sing}\, X_i.$$

The Euler derivation from Remark 4.3 restricts to that in Remark 6.3 on each Y_i. The syzygies $\mathrm{syz}(\mathcal{J}_f)$ are generated by all $\mathrm{syz}(\mathcal{J}_{\mathbf{x}^{\alpha_i}})$ and the Koszul relations. These latter vanish on $\mathrm{Sing}\, X$. By

Remarks 6.2.(c) and 6.3, it follows that the logarithmic strata of X in $\operatorname{Sing} X$ are products of finitely many strata of the X_i in $\operatorname{Sing} X_i$. Thus, X is holonomic by Remark 6.2.(b). □

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JOURNAL INFORMATION

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