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AN EXISTENCE THEOREM FOR THE UNMODIFIED VLASOV EQUATION

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Abstract

The initial value problem for the modified Vlasov equation with a mollification parameter $\delta > 0$, as introduced by Batt, has a unique global solution in the weak sense whenever $f_0 \in L^1$ and $f_0 \geqq 0$ λ-a.e. Assuming boundedness of f_0 and boundedness of the kinetic energy, it is shown that, as $\delta \to 0$, there are subsequences $\delta_n \to 0$ such that the corresponding solutions converge weakly in the measure-theoretical sense. The limits are shown to be global weak solutions of the initial value problem for Vlasov's equation, and these solutions are seen to be weakly continuous with respect to t. For the plasma physical case, boundedness of the kinetic energy is a consequence of energy conservation.

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An Existence Theorem for the unmodified Vlasov Equation

Reinhard Illner and Helmut Neunzert

The mathematical description of the state of a stellar system or collisionless electrostatic plasma leads to an initial value problem for the so-called Vlasov equation. In the stellar dynamical case, this equation was first derived by Jeans in 1915. The initial value problem amounts to finding a function $f = f(t,x,v)$ such that

$$(1) \qquad \frac{\partial f}{\partial t} + \langle v, \nabla_x f\rangle + \langle K, \nabla_v f\rangle = 0, \quad f(0,x,v) = f_o(x,v)$$

for $t \geq 0$, $(x,v) \in \mathcal{X} \times \mathcal{V} = \mathbb{R}^6$, where

$$(2) \qquad \begin{aligned} K(t,x) &:= - \int_{\mathcal{X}} \frac{\gamma(x-y)}{|x-y|^3} [\rho(t,y) - n(y)]dy \\ \rho(t,y) &:= \int_{\mathcal{V}} f(t,y,v)dv \end{aligned}$$

and γ and n are given. $f(t,\cdot)$ can be interpreted as the density of a stellar or electron gas in phase space $\mathcal{X} \times \mathcal{V}$ at time t, f_o being a given initial value. The case $\gamma>0$, $n=0$ corresponds to the stellar dynamical problem, where the force K results from the Newtonian potential associated with the spatial density ρ. For $\gamma<0$, (1) and (2) describe the evolution in time of an electron gas with a fixed ion background n, and K is the Coulomb force due to the spatial charge density $\rho-n$. The initial value problem (1)-(2) has led to a series of existence investigations, started by Kurth [7], who proved an existence theorem locally-in-time. Later, Batt [2] considered a modified problem, for which existence and uniqueness

results could be obtained under various conditions on f_o (see, e.g., Batt [2], Neunzert [9]).
More recently, Batt [3] and Horst [in his dissertation, Munich 79] have formulated global existence theorems for the stellar dynamical problem, assuming certain symmetries for the initial values and hence for the solutions. Moreover, Arsen'ev [1] has shown the globally-in-time-existence of a weak solution for the plasma physical problem.

In this paper we start with the "mollified" or modified equation introduced by Batt [2], which results from (2) by mollifying the potential $\frac{1}{|x|}$ in a δ-neighborhood of the origin. Then, using the fact that for any $\delta>0$ the mollified problem has (weak) solutions f_δ under rather weak conditions imposed on f_o (see [9]), the globally-in-time existence of a (weak) solution of (1)-(2) is shown by letting $\delta\to 0$. As a crucial assumption, we will suppose that the kinetic energy

$$\int_{X\times V} v^2 f_\delta(t,x,v)\,dxdv$$

of the system remains bounded for all t uniformly in δ. As we shall see, this assumption is always true in the plasma physical case ($\gamma<0$). Our existence result is hence similar to Arsen'ev's. The conditions imposed on f_o are comparable, but, in our case, the obtained solution is smoother with respect to t. The methods used are largely different from Arsen'ev's and we believe that our proof is easier to comprehend and more closely related to older results concerning the modified equation.

In chapter 1 we present the needed notation and results for the mollified problem. In chapter 2, assuming boundedness of the kinetic energy, we show the existence of a sequence $\delta_n \to 0$ such that the measures $\mu_t^{(n)}$ associated with the densities $f_{\delta_n}(t,\cdot)$ converge weakly to a limit measure μ_t for all t. μ_t is seen to have a density $f(t,\cdot)$, and the mapping $t \to \mu_t$ is proven to be weakly continuous. Chapter 3 contains the proof that this density is a weak solution of the problem (1)-(2).

In the last chapter we deal with energy conservation and its consequences for the plasma physical case.

§ 1 Results for the mollified case.

We start with some notation. A point in phase space $\mathbb{R}^6$ is denoted by $P=(x,v)$, where $x \in \mathcal{X} = \mathbb{R}^3$, $v \in \mathcal{V} = \mathbb{R}^3$ stand for the position and velocity coordinate respectively. $|P|, |x|, |v|$ denotes the euclidean norm in $\mathbb{R}^6$ and $\mathbb{R}^3$ respectively.

Let $\mathcal{M}_1$ be the set of all finite Borel measures normed to 1 (probability measures) on $\mathbb{R}^6$; By C^b we denote the space of all continuous bounded functions on $\mathbb{R}^6$, by C_o^n the space of all n-times continuously differentiable functions with compact support, and by L^1 the space of all absolutely integrable functions.

A sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures $\mu_n \in \mathcal{M}_1$ is said to converge weakly to $\mu \in \mathcal{M}_1$, $\mu_n \Rightarrow \mu$, if

$$(1.1) \qquad \lim_{n\to\infty} \int \varphi d\mu_n = \int \varphi d\mu \quad \text{for all } \varphi \in C^b.$$

As is well-known, $\mu_n \Rightarrow \mu$ holds if and only if (1.1) holds for every $\varphi \in C_o^o$. As every $\varphi \in C_o^o$ can be approximated by elements of C_o^1 in the maximum norm, it even follows that $\mu_n \Rightarrow \mu$ holds if (1.1) is true for every $\varphi \in C_o^1$.

$S \subset \mathcal{M}_1$ is called tight if, for any $\varepsilon > 0$, there is a compact set $K \subset \mathbb{R}^6$ such that $\mu(K) > 1-\varepsilon$ for all $\mu \in S$. One then has Prohorov's Theorem [compare, e.g. [Billingsley], Thm. 6.1]: If $S \subset \mathcal{M}_1$ is tight, then S is relatively compact.

Let us now specify the problem.

Let $T > 0$ be arbitrary but fixed. We call $f:[0,T] \times \mathbb{R}^6 \longrightarrow \mathbb{R}$ weak solution of the IVP (1)-(2), if

(i) $f(t,\cdot) \in L^1$, $\|f(t,\cdot)\|_{L^1} = 1$, $f(t,P) \geq 0$ λ-almost everywhere for all $t \in [0,T]$. and $f(t,\cdot)$ is weakly continuous with respect to $t \in [0,T]$.

(ii) for $K(t,x)$, defined by

$$(1.2) \qquad K(t,x) := \int_{\mathbb{R}^3} G(x,y)[\rho(t,y) - n(y)]dy$$

with

$$(1.3) \qquad G(x,y) = -\frac{\gamma(x-y)}{|x-y|^3}, \quad \rho(t,y) = \int_{\mathbb{R}^3} f(t,y,v)dv,$$

and for all $\varphi \in C_o^1([0,T] \times \mathbb{R}^6)$, the equation

$$(1.4) \qquad \int_0^T \left(\int_{\mathbb{R}^6} f\left[\frac{\partial \varphi}{\partial t} + \langle v, \nabla_x \varphi \rangle + \langle K, \nabla_v \varphi \rangle\right] dP \right) dt + \int_{\mathbb{R}^6} \varphi(0,P) f_o(P) dP = 0$$

holds, where $f_o \in L^1$ is the given initial density, $f_o \geq 0$ λ-a.e., and n is a given nonnegative bounded function in $L^1(\mathfrak{X})$.

Remarks to (i)-(ii)

(i) Weak continuity of $t \longrightarrow f(t,\cdot)$ is defined by the corresponding weak continuity of the measures μ_t with the densities $f(t,\cdot)$; i.e., $t \longrightarrow f(t,\cdot)$ is weakly continuous iff the mapping

$$t \longrightarrow \int_{\mathbb{R}^6} f(t,\cdot)\varphi \, dP \quad , \quad t \in [0,T],$$

is continuous for any $\varphi \in C^b$.

(ii) As $f(t,\cdot) \in L^1$, $\rho(t,x)$ exists λ-a.e. and $\rho(t,\cdot) \in L^1(\mathbb{R}^3)$; therefore, by Fubini's Theorem, $K(t,\cdot) \in L^1_{loc}(\mathfrak{X})$. n represents in the plasma physical case the ion background; a frequent requirement in this context is

$$\int_{\mathbb{R}^3} n \, dx = \int_{\mathbb{R}^6} f_o \, dP \;;$$

then the whole system is initially (and thus for all $t \in [0,T]$) electrically neutral.

(1.4) is the weak formulation of the initial value problem (1)-(2); this form is common for conservation equations (see, e.g., Lax [8]).

Next we define the modified problem.
Let $\delta > 0$ be arbitrary. The "mollifier" $\omega_\delta : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is defined by

$$\omega_\delta(x) = \begin{cases} 0 & \text{for } |x| \geq \delta \\ c_\delta \exp\left(-1 + \frac{|x|^2}{\delta^2}\right)^{-1} & \text{for } |x| < \delta \end{cases},$$

where c_δ is such that $\int_{\mathbb{R}^3} \omega_\delta \, dx = 1$.

Let

(1.3') $\quad G_\delta(x,y) := \int_{\mathbb{R}^3} G(x,z)\omega_\delta(z-y)dz.$

We call a function f_δ: $[0,T]\times\mathbb{R}^6 \longrightarrow \mathbb{R}$ weak solution of the modified problem, if f_δ satisfies (i) and (ii), where in (1.4) K has to be replaced by

$$(1.2') \quad K_\delta(t,x) := \int_{\mathbb{R}^3} G_\delta(x,y)[\rho_\delta(t,y)-n(y)]dy.$$

Note that f_o and n are independent of δ.

The following results are available for the modified problem (see Neunzert [9], section 3. Only the case where n=0 is dealt with there; however, all the proofs can be transferred without any changes to the general case):

1. For $f_o \in L^1$, $f_o(P) \geq 0$ λ-a.e. and $\int_{\mathbb{R}^6} f_o dP = 1$, the modified problem has a uniquely determined weak solution f_δ.

2. This solution can be described in the following convenient way:
To every weakly continuous function $\mu_. : t \rightarrow \mu_t \in \mathcal{M}_1$, $t \in [0,T]$, we assign a vector field $K_\delta[\mu_.]$ by

$$(1.2'') \quad K_\delta[\mu_.](t,x) = \int_{\mathbb{R}^6} G_\delta(x,y)d\mu_t(y,v) - \int_{\mathbb{R}^3} G_\delta(x,y)n(y)dy.$$

The initial value problem

$$(1.5) \quad \dot{x} = v, \ \dot{v} = K_\delta[\mu_.](t,x) \quad \text{with}$$

$$P(s) = (x(s),v(s)) := Q, \quad s \in [0,T], \ Q = (x_o,v_o) \in \mathbb{R}^6$$

then has for all $t \in [0,T]$ a uniquely determined solution, which we denote by

$$(1.6) \quad P(t) = T^\delta_{t,s}[\mu_.]Q .$$

$T^\delta_{t,s}[\mu_.]$ is a two-parametric set of diffeomorphic mappings of $\mathbb{R}^6$ onto itself (i.e. every $T^\delta_{t,s}$ has an inverse and both are of class C^1), with $T^\delta_{t,s} \circ T^\delta_{s,r} = T^\delta_{t,r}$, $T^\delta_{s,s}$ = id for all $s,t,r \in [0,T]$; in particular, ${T^{\delta}_{t,s}}^{-1} = T^{\delta}_{s,t}$. Moreover it follows from Liouville's Theorem that $T^{\delta}_{t,s}$ is measure-preserving with respect to the Lebesgue measure λ, because the right side of (1.5) is divergence-free.

Using this concept, the solution of the modified problem can be written in the form

(1.7) $\quad f_{\delta}(t,P) = f_{0}(T^{\delta}_{0,t}[\mu^{\delta}_{\cdot}]P)$.

Please observe that $f_{\delta}(t,\cdot)$ is the density of μ^{δ}_{t}; it is shown in [9] that, if $f_{\delta}(t,\cdot)$ is a weak solution of the modified problem, then $\mu^{\delta}_{\cdot}$ obeys the equation

(1.8) $\quad \mu^{\delta}_{t} = \mu_{0} \circ T^{\delta}_{0,t}[\mu^{\delta}_{\cdot}]$, $\quad t \in [0,T]$.

Since $T^{\delta}_{0,t}$ is measure-preserving, (1.7) is an immediate consequence of (1.8).

3. (1.7) has the following obvious consequences:
If $0 \leq f_{0}(P) \leq M$ λ-a.e. in $\mathbb{R}^6$, then also

(1.9) $\quad 0 \leq f_{\delta}(t,P) \leq M$ $\quad \lambda$-a.e. in $\mathbb{R}^6$ for all $t \in [0,T]$.

Furthermore,

(1.10) $\quad \int_{\mathbb{R}^6} f_{\delta}(t,P)dP = \int_{\mathbb{R}^6} f_{0}(P)dP = 1.$

§ 2 Convergence of solutions of the modified problem.

The initial density $f_{0} \in L^1$ will henceforth be assumed to have the properties

(2.1) $\quad$ There is an M such that $0 \leq f_{0}(P) \leq M$ λ-a.e.

(2.2) $\quad \int_{\mathbb{R}^6} |P|^2 f_{0}(P)dP < \infty.$

The main purpose of this chapter is to prove

Theorem 1 Let f_o satisfy (2.1) and (2.2). Besides we suppose that the kinetic energies

(2.3) $$E_\delta(t) := \int_{\mathbb{R}^6} |v|^2 f_\delta(t,x,v)\,dxdv$$

belonging to the solutions f_δ of the modified problem are uniformly bounded:

(2.4) $E_\delta(t) \leq E < \infty$ for all $t \in [0,T]$ and all $\delta > 0$.

Then there is a sequence $(\delta_n)_{n\in\mathbb{N}}$, $\delta_n \searrow 0$, such that the measures $\mu_t^{\delta_n}$ associated with $f_{\delta_n}(t,\cdot)$ converge weakly to a measure $\mu_t \in \mathcal{M}_1$. The function $\mu_.$ defined in this way is weakly continuous and μ_t is absolutely continuous with respect to λ (the density of μ_t will be denoted by $f(t,\cdot)$).

Remark In chapter 4 it will be shown that (2.4) is always true in the plasma physical case.

Proof of Theorem 1 The proof requires several steps which we shall formulate as lemmata.

Lemma 1 The set of solutions $\{\mu_t^\delta;\ t \in [0,T],\ \delta > 0\}$ of the modified problem is tight in $\mathcal{M}_1$.

Proof (2.2) implies

(2.5) $$\int_{\mathbb{R}^6} |x|^2 f_o(x,v)\,dxdv < \infty .$$

We denote the position and velocity component of $T_{t,s}^\delta$ by $X_{t,s}^\delta$ and $V_{t,s}^\delta$ respectively. Then, in view of (1.7),

$$h_\delta(t) := \int_{\mathbb{R}^6} |x|^2 f_\delta(t,x,v)\,dxdv = \int_{\mathbb{R}^6} |x|^2 f_o(T^\delta_{o,t}[\mu^\delta_\cdot]P)\,dxdv$$

$$(2.6) \qquad = \int_{\mathbb{R}^6} |X^\delta_{t,o}[\mu^\delta_\cdot]P|^2 f_o(P)\,dP \ ,$$

and we have used that $T^\delta_{t,o}$ is measure-preserving.
We show that $h_\delta(t)$ exists for all $t\in[0,T]$. Indeed, for every $\delta>0$ there is a constant γ_δ such that

$|G_\delta(x,y)| \leq \gamma_\delta$ for all x,y. Therefore,

$|K_\delta(t,x)| \leq C_\delta$ for all $t\in[0,T]$ and all $x\in\mathbb{R}^3$, and consequently $|V^\delta_{t,o}P-v| \leq C_\delta t$, $|x^\delta_{t,o}P-x| \leq \frac{C_\delta}{2} t^2+|v|t$.

Thus, $h_\delta(t)$ can be majorized by a linear combination of integrals $\int_{\mathbb{R}^6} |x|^\alpha |v|^\beta f_o(x,v)\,dxdv$, $\alpha,\beta\in\{0,1,2\}, \alpha+\beta\leq 2$, with coefficients depending on t and δ. These integrals converge in view of (2.1) and (2.2).
Moreover, $h_\delta(t)$ is uniformly (with respect to δ) bounded in $[0,T]$. In fact, one shows similarly as above that h_δ is differentiable and that differentiation and integration can be interchanged. Hence

$$|h'_\delta(t)| = \left|\int_{\mathbb{R}^6} \frac{\partial}{\partial t} |X^\delta_{t,o}P|^2 \cdot f_o(P)\,dP\right| =$$

$$=2\left|\int_{\mathbb{R}^6} <X^\delta_{t,o}P, V^\delta_{t,o}P>\ f_o(P)\,dP\right| \leq$$

$$\leq 2 \int_{\mathbb{R}^6} |X^\delta_{t,o}P| \cdot \sqrt{f_o(P)} \cdot |V^\delta_{t,o}P| \sqrt{f_o(P)}\,dP \leq$$

$$\leq 2\left(\int_{\mathbb{R}^6} |X^\delta_{t,o}P|^2 f_o(P)\,dP\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^6} |V^\delta_{t,o}P|^2 f_o(P)\,dP\right)^{\frac{1}{2}} =$$

$$=2h_o^{1/2}(t) \cdot \left(\int_{\mathbb{R}^6} |v|^2 f_\delta(t,x,v)\,dxdv\right)^{\frac{1}{2}} \leq 2E^{\frac{1}{2}} h_\delta^{\frac{1}{2}}(t) := Ch_\delta^{\frac{1}{2}}(t).$$

As $h_\delta(t)>0$ for all $t\in[0,T]$, it follows that

$$\int_0^t \frac{h_\delta'(\tau)}{h_\delta^{1/2}(\tau)}\,d\tau = 2h_\delta^{\frac{1}{2}}(t)-2h^{\frac{1}{2}}(0)\le Ct \qquad \text{or}$$

$$0 \le h_\delta(t) \le [\tfrac{1}{2}Ct+h^{\frac{1}{2}}(0)]^2 =: C_t' \ .$$

The bound C_t' does not depend on δ, because C and f_o (and hence $h(0)$) did not depend on δ.

Now choose $\varepsilon>0$ arbitrarily and let $R>0$ be such that $\frac{C_T'+E}{R^2}<\varepsilon$. Let $K_R = \{P\in \mathbb{R}^6 \ ; \ |P|\le R\}$. Then

$$R^2\cdot\mu_t^\delta(\mathbb{R}^6\setminus K_R)=R^2\int_{\mathbb{R}^6\setminus K_R} f_\delta(t,P)dP \le \int_{\mathbb{R}^6\setminus K_R} |P|^2 f_\delta(t,P)dP \le$$

$$\le \int_{\mathbb{R}^6} |P|^2 f_\delta(t,P)dP = h_\delta(t) + E_\delta(t) \le C_t' + E < \varepsilon R^2,$$

i.e. $\mu_t^\delta(\mathbb{R}^6\setminus K_R) < \varepsilon$ for sufficiently large R and all $\delta>0$. Thus the set $\{\mu_t^\delta;\ t\in[0,T],\delta>0\}\subset\mathcal{M}_1$ is tight, and Lemma 1 is proved.

Now let $T'=\{t_\nu;\nu\in\mathbb{N}\}$ be a countable dense subset of $[0,T]$. By Prohorov's Theorem, Lemma 1 and a common diagonal scheme there is a monotone sequence $(\delta_n)_{n\in\mathbb{N}}$, $\delta_n\searrow 0$, such that $\mu_{t_\nu}^{\delta_n} \Rightarrow \mu_{t_\nu}\in\mathcal{M}_1$ for all $\nu\in\mathbb{N}$. We will show that $\mu_t^n:=\mu_t^{\delta_n}$ converge weakly even if $t\notin T'$, and we will see that the limit μ_t defines a weakly continuous function $t\longrightarrow\mu_t, t\in[0,T]$. First we prove a more technical Lemma which is also needed for later applications.

<u>Lemma 2</u> Let $\Gamma\subset\mathbb{R}^6$ be a compact set and let $G_n:=G_{\delta_n}$, $n\in\mathbb{N}$. Then the functions

(2.7) $$S_n(t,y):=\int_\Gamma |G_n(x,y)|\,d\mu_t^n(x,v)$$

are uniformly bounded with respect to $y\in\mathfrak{X}$, $t\in[0,T]$ and $n\in\mathbb{N}$:

(2.8) $$0\le S_n(t,y) \le A \quad \text{for all } y\in\mathfrak{X},\ n\in\mathbb{N},\ t\in[0,T].$$

<u>Proof</u> There is an R>O, such that

$$\Gamma \subset \{(x,v): |x| \leq R, |v| \leq R\}.$$

For $f_n := f_{\delta_n}$ (2.1) and (1.9) imply

(2.9) $0 \leq f_n(t,P) \leq M$ for all $n \in \mathbb{N}, t \in [0,T], P \in \mathbb{R}^6$.

By Fubini's Theorem

$$0 \leq S_n(t,y) = \int_\Gamma |G_n(x,y)| f_n(t,x,v)dxdv \leq$$

$$\leq \int_{|x|\leq R} \left(\int_{|v|\leq R} |G_n(x,y)| f_n(t,x,v)dv \right) dx =$$

$$= \int_{|x|\leq R} |G_n(x,y)| \hat{\rho}_n(t,x)dx, \quad \text{where}$$

$$\hat{\rho}_n(t,\cdot) := \int_{|v|\leq R} f_n(t,\cdot,v)dv \quad \in L^1(\mathfrak{X}).$$

In view of (2.9), $\hat{\rho}_n$ satisfies the inequality

(2.10) $0 \leq \hat{\rho}_n(t,x) \leq \frac{4\pi}{3} R^3 \cdot M$ for all $t \in [0,T]$, $x \in \mathfrak{X}$.

By (1.3') $G_n(x,y) = \int_{\mathbb{R}^3} G(x,z)\omega_{\delta_n}(z-y)dz$; for the components $G_n^{(j)}$, j=1,2,3 of G_n this yields

$$|G_n^{(j)}(x,y)| \leq \int_{\mathbb{R}^3} |G^{(j)}(x,y)|\omega_n(z-y)dz,$$

where $\omega_n := \omega_{\delta_n}$. Using (1.3), we get for the Norm $|G_n|$

$$|G_n(x,y)| \leq \sqrt{3} \int_{\mathbb{R}^3} |G(x,z)|\omega_n(z-y)dz = \sqrt{3}|\gamma| \int_{\mathbb{R}^3} \frac{1}{|x-z|^2} \cdot$$

$$\cdot \omega_n(z-y)dz.$$

Hence

$$S_n(t,y) \leq \sqrt{3}|\gamma| \int_{|x|\leq R} \left(\int_{\mathbb{R}^3} \frac{1}{|x-z|^2}\omega_n(z-y)dz \right) \hat{\rho}_n(t,x)dx \leq$$

$$\sqrt{3}|\gamma| \int_{\mathbb{R}^3} \omega_n(z-y) \left(\int_{\mathbb{R}^3} \frac{1}{|x-z|^2} \hat{\rho}_n(t,x)dx \right) dz =$$

$$\sqrt{3}|\gamma| \int_{\mathbb{R}^3} \omega_n(z-y) \left(\int_{\mathbb{R}^3} \frac{1}{|x|^2} \hat{\rho}_n(t,x+z)dx \right) dz.$$

As a consequence of (2.10) we find

$$\int_{\mathbb{R}^3} \frac{1}{|x|^2} \hat{\rho}_n(t,x+z)dx \leq \int_{|x|\leq 1} \frac{1}{|x|^2}\cdot\frac{4\pi}{3}R^3 M\,dx + \int_{|x|>1} \hat{\rho}_n(t,x+z)dx$$

(2.11)

$$\leq \frac{16\pi^2}{3} R^3 M + \int_{\mathbb{R}^3} \hat{\rho}_n(t,x+z)dx \leq \frac{16\pi^2 R^3}{3} M + 1 \ .$$

Collecting terms yields

$$S_n(t,y) \leq \sqrt{3}|\gamma|\left(\frac{16\pi^2 R^3}{3} M+1\right) \int_{\mathbb{R}^3} \omega_n(z-y)dz := A \ ,$$

which concludes the proof of Lemma 2.

Lemma 3 Assume $\varphi \in C_0^1(\mathbb{R}^6)$. Then the set of functions

$$t \longrightarrow \phi_n(t) := \int_{\mathbb{R}^6} \varphi d\mu_t^n \ , \quad t \in [0,T]$$

is equicontinuous.

Proof Abbreviating $\overset{\delta}{T}{}^n_{t,s}[\mu^n_\cdot] =: T^n_{t,s}$, using that $T^n_{t,s}$ preserves the Lebesgue measure and recalling (1.7), we get

$$\phi_n(t) = \int_{\mathbb{R}^6} \varphi \circ T^n_{t,0} d\mu_0$$

and hence

$$|\phi_n(t)-\phi_n(s)| = \left|\int_{\mathbb{R}^6} \varphi(T^n_{t,0}P)\cdot f_0(P)dP - \int_{\mathbb{R}^6} \varphi(T^n_{s,0}P) f_0(P)dP\right| =$$

$$= \left|\int_s^t \left(\frac{d}{d\tau} \int_{\mathbb{R}^6} \varphi(T^n_{\tau,0}P) f_0(P)dP\right)d\tau\right| =$$

$$= \left|\int_s^t \left(\int_{\mathbb{R}^6} \left\langle \nabla_P\varphi(T^n_{\tau,0}P), \frac{\partial}{\partial\tau} T^n_{\tau,0}P \right\rangle f_0(P)dP\right)d\tau\right| .$$

Let Γ be the compact support of φ. Then obviously $\nabla\varphi(Q) = 0$ if $Q \notin \Gamma$. Thus

$$|\phi_n(t)-\phi_n(s)|=|\int_s^t\Big(\int_{T^n_{o,\tau}(\Gamma)}[<\nabla_x\varphi(T^n_{\tau,o}P),\ V^n_{\tau,o}P>+$$

$$<\nabla_v\varphi(T^n_{\tau,o}P),\ \int_{\mathbb{R}^3}G_n(X^n_{\tau,o}P,z)(\rho_n(t,z)-n(z))dz>]f_o(P)dP\Big)d\tau|$$

Substituting $Q=\binom{y}{w}=T^n_{\tau,o}P=\begin{pmatrix}X^n_{\tau,o}P\\ V^n_{\tau,o}P\end{pmatrix}$ yields

$$|\phi_n(t)-\phi_n(s)|=|\int_s^t\Big(\int_\Gamma[<\nabla_x\varphi(Q),w>+<\nabla_v\varphi(Q),\int_{\mathbb{R}^3}G_n(y,z)\cdot$$

$$\cdot(\rho_n(t,z)-n(z))dz>]f_n(t,Q)dQ\Big)d\tau|\ \leq$$

$$\leq\sup_{P\in\Gamma}|\nabla_P\varphi(P)|\cdot\int_s^t\Big(\int_\Gamma[|w|+\sqrt{3}\int_{\mathbb{R}^3}|G_n(y,z)|(\rho_n(t,z)+n(z))dz]\cdot$$

$$\cdot\ f_n(t,Q)dQ\Big)d\tau.$$

Again we choose $R>0$ such that $\Gamma\subset\{(y,w);|y|\leq R,|w|\leq R\}$.

Then

$$\int_s^t\Big(\int_\Gamma|w|f_n(t,Q)dQ\Big)d\tau\ \leq\ R\cdot|t-s|.$$

From Lemma 2 we get

$$\int_\Gamma\Big(\int_{\mathbb{R}^3}|G_n(y,z)|\rho_n(t,z)dz\Big)f_n(t,Q)dQ\ =$$

$$=\int_{\mathbb{R}^6}\Big(\int_\Gamma|G_n(y,z)|d\mu^n_t(y,w)\Big)d\mu^n_t(z,v)\ =$$

$$=\int_{\mathbb{R}^6}S_n(t,z)d\mu^n_t(z,v)\ \leq\ A$$

for all $n\in\mathbb{N}$, $t\in[0,T]$. Finally, as by assumption $n\in L^1(\mathfrak{X})\cap L^\infty(\mathfrak{X})$, it follows that

$$\int_{\mathbb{R}^3}|G_n(y,z)|n(z)dz\ \leq\ B$$

for all $y\in\mathfrak{X}$. Summarizing, we have shown that

$$|\phi_n(t)-\phi_n(s)| \leq \sup_{P\in\Gamma} |\nabla_P \varphi(P)| (R+\sqrt{3}(A+B)) |t-s| = : D|t-s|$$

where D depends only on φ. This concludes the proof of Lemma 3.

Now we use the following well-known theorem (see, e.g., Smirnow IV, p. 47): If the sequence of functions $(\phi_n)_{n\in\mathbb{N}}$ is equicontinuous on $[0,T]$ and pointwise convergent on a dense subset of $[0,T]$, then it converges uniformly on $[0,T]$. Therefore, as $\phi_n(t_i) \rightarrow \int_{\mathbb{R}^6} \varphi d\mu_{t_i}$ for $t_i \in T'$, it follows from Lemma 3 that $\phi_n(t) = \int_{\mathbb{R}^6} \varphi d\mu_t^n$ converges uniformly in $[0,T]$ if $\varphi \in C_o^1(\mathbb{R}^6)$. On the other hand, by Prohorov's Theorem there is for every $t \notin T'$ a subsequence $(n_j)_{j\in\mathbb{N}}$ and a measure $\mu_t \in \mathcal{M}_1$ such that $\mu_t^{n_j} \Rightarrow \mu_t$. From the convergence of $\phi_n(t)$ we conclude

$$\lim_{n\to\infty} \phi_n(t) = \lim_{j\to\infty} \phi_{n_j}(t) = \int_{\mathbb{R}^6} \varphi d\mu_t, \quad \varphi \in C_o^1 .$$

Consequently, to every $t \in [0,T]$ there is a $\mu_t \in \mathcal{M}_1$ with

$$\int_{\mathbb{R}^6} \varphi d\mu_t^n \rightarrow \int_{\mathbb{R}^6} \varphi d\mu_t \quad , \text{ i.e. } \mu_t^n \Rightarrow \mu_t .$$

Furthermore, the function $\mu_.$ defined in this way is weakly continuous. To see this, let $(t_k)_{k\in\mathbb{N}}$ be an arbitrary sequence in $[0,T]$ with $\lim_{k\to\infty} t_k = t$. As $\phi = \lim \phi_n$ is continuous (as uniform limit of continuous functions), it follows that

$$\phi(t_k) := \int_{\mathbb{R}^6} \varphi d\mu_{t_K} \rightarrow \phi(t) \qquad \text{as } k\to\infty ,$$

which means that $\mu_{t_k} \Rightarrow \mu_t$. We get a

<u>Corollary to Lemma 3</u> To every $t \in [0,T]$, there is a $\mu_t \in \mathcal{M}_1$ so that $\mu_t^n \Rightarrow \mu_t$. The function

$$\mu_. : t \rightarrow \mu_t, \quad t \in [0,T] \quad \text{is weakly continuous.}$$

The last point still missing in the proof of Theorem 1 is now the absolute continuity of μ_t with respect to λ.

<u>Lemma 4</u> μ_t, $t \in [0,T]$, is absolutely continuous. The density $f(t,\cdot)$ of μ_t is essentially bounded by M.

<u>Proof</u> For all $\varphi \in C_0^0$, we have

$$\int_{\mathbb{R}^6} \varphi d\mu_t^n = \int_{\mathbb{R}^6} \varphi(P) f_n(t,P) dP \longrightarrow \int_{\mathbb{R}^6} \varphi d\mu_t \quad ,$$

and, by (1.9),

$$\left| \int_{\mathbb{R}^6} \varphi d\mu_t^n \right| \leq M \int_{\mathbb{R}^6} |\varphi| dP = M \|\varphi\|_{L^1} \ .$$

Hence

$$\left| \int_{\mathbb{R}^6} \varphi d\mu_t \right| \leq M \|\varphi\|_{L^1} \qquad \text{for all } \varphi \in C_0^0,\ t \in [0,T],$$

i.e., for any given fixed t, the functional $\varphi \longrightarrow \int_{\mathbb{R}^6} \varphi d\mu_t$ is linear and bounded on the dense subset C_0^0 of L^1. Therefore, it has a unique continuation as a bounded linear functional on L^1 with norm $\leq M$. Since $L^{1\prime} = L^\infty$, there is an $f(t,\cdot) \in L^\infty$ with $\|f(t,\cdot)\|_\infty \leq M$ such that

$$\int_{\mathbb{R}^6} \varphi d\mu_t = \int_{\mathbb{R}^6} \varphi \cdot f(t,\cdot) dP$$

for all $\varphi \in C_0^0$. It is then easily shown that $f(t,\cdot)$ actually is the density of μ_t.

The proof of Theorem 1 is complete.

§ 3 The existence theorem

Theorem 2 Suppose that the assumptions of Theorem 1 hold. Then the densities $f(t,\cdot)$, $t \in [0,T]$, of the measure μ_t constructed in Theorem 1 are a weak solution of the initial value problem (1)-(2).

Proof We have to verify (i) and (ii) from § 1.

(i) follows immediately from Theorem 1.

(ii) We know that f_n is the weak solution of the modified problem with mollification parameter δ_n. Thus, for $\varphi \in C_0^1([0,T) X \mathbb{R}^6)$

$$\int_0^T \left(\int_{\mathbb{R}^6} f_n \left[\frac{\partial \varphi}{\partial t} + \langle v, \nabla_x \varphi \rangle + \langle K_n, \nabla_v \varphi \rangle \right] dP \right) dt + \int_{\mathbb{R}^6} \varphi(0,P) f_0 dP = 0.$$

We want to show that all terms depending on n converge, as $n \to \infty$, to the corresponding terms containing f and K. This will imply (1.4).

Obviously $\chi(t,P) := \frac{\partial \varphi}{\partial t}(t,P) + \langle v, \nabla_x \varphi \rangle \in C_0^0([0,T] x \mathbb{R}^6)$.

Therefore, there is an $R>0$ such that $\chi(t,P)=0$ if $|P|>R$ and $t \in [0,T]$. Moreover, $\chi(t,\cdot) \in C_0^0(\mathbb{R}^6)$.

Hence $\mu_t^n \Rightarrow \mu_t$ implies

$$\int_{\mathbb{R}^6} f_n \cdot \chi(t,\cdot) dP \longrightarrow \int_{\mathbb{R}^6} f \cdot \chi(t,\cdot) dP.$$

Furthermore, as $\left| \int_{\mathbb{R}^6} f_n \chi(t,\cdot) dP \right| \leq M \left(\sup_{[0,T] x K_R} |\chi| \right) \cdot \lambda(K_R)$,

this (time-independent) constant is an integrable majorant; by the dominant convergence theorem

$$\int_0^T \left(\int_{\mathbb{R}^6} f_n \cdot \chi dP \right) dt \rightarrow \int_0^T \left(\int_{\mathbb{R}^6} f \cdot \chi dP \right) dt.$$

We still have to consider the expression

$$\int_0^T \Big(\int_{\mathbb{R}^6} \langle K_n, \psi \rangle f_n dP \Big) dt, \text{ where } \psi = \nabla_v \varphi .$$

The components of ψ all belong to $C_o^o([0,T)\times\mathbb{R}^6)$.

We recall

$$\int_{\mathbb{R}^6} \langle K_n, \psi \rangle f_n dP = \int_{\mathbb{R}^6} \Big(\int_{\mathbb{R}^3} \langle G_n(x,y), \psi(t,x,v) \rangle [\rho_n(t,y) - n(y)] dy \Big) f_n(t,x,v) dx dv$$

and show in detail that

$$\int_0^T \Big(\int_{\mathbb{R}^6} \Big(\int_{\mathbb{R}^3} \langle G_n(x,y), \psi(t,x,v) \rangle \rho_n(t,y) dy \Big) f_n(t,x,v) dx dv \longrightarrow$$

$$\longrightarrow \int_0^T \Big(\int_{\mathbb{R}^6} \Big(\int_{\mathbb{R}^3} \langle G(x,y), \psi(t,x,v) \rangle \rho(t,y) dy \Big) f(t,x,v) dx dv$$

as $n\to\infty$.

Convergence of the term where $\rho_n(t,y)$ is replaced by $n(y)$ can be proven in exactly the same way, with some simplifications.

Now

$$W_n(t) := \int_{\mathbb{R}^6} \Big(\int_{\mathbb{R}^6} \langle G_n(x,y), \psi(t,x,v) \rangle f_n(t,y,w) dy dw \Big) f_n(t,x,v) dx dv =$$

$$= \int_{\mathbb{R}^6} \Big(\int_{\mathbb{R}^6} \langle G_n(x,y), \psi(t,x,v) \rangle f_n(t,x,v) dx dv \Big) f_n(t,y,w) dy dw =$$

$$=: \int_{\mathbb{R}^6} \hat{\psi}_n(t,y) f_n(t,y,w) dy dw \quad , \quad \text{where}$$

(3.1) $$\hat{\psi}_n(t,y) := \int_{\mathbb{R}^6} \langle G_n(x,y), \psi(t,x,v) \rangle f_n(t,x,v) dx dv.$$

For the sake of simplicity, we write $G_o := G$, $f_o := f$ and define $\hat{\psi} := \hat{\psi}_o$ also by (3.1).

We note that the Φ_n, $n \in \mathbb{N}_o$ are uniformly bounded on $[0,T]\times\mathfrak{X}$. In fact, there is an $R>0$ such that $\psi(t,P)=0$ for $|P|>R$ and all $t\in[0,T]$, and as $|\psi(t,P)|\le C$ for all $P\in\mathbb{R}^6$ and all $t\in[0,T]$, (2.7) yields

$$|\Phi_n(t,y)| \le C\cdot\int_{|P|\le R}|G_n(x,y)|f_n(t,P)dP =$$

$$= C\,S_n(t,y) \le A\cdot C \qquad \text{for all } n\in\mathbb{N}_o.$$

We proceed by showing

<u>Lemma 5</u> Φ_n converges uniformly to Φ with respect to $(t,y)\in[0,T]\times\mathfrak{X}$.

<u>Proof</u> We need two steps. Let $\varepsilon>0$ be arbitrary but fixed.

(a) There is a $d>0$ such that

$$(3.2) \qquad \Big|\int_{\mathbb{R}^3}\Big(\int_{|x-y|<d} < G_n(x,y),\psi(t,x,v) > f_n(t,x,v)dx\Big)dv\Big| < \frac{\varepsilon}{3}$$

holds for all $t\in[0,T]$, $y\in\mathfrak{X}$ and $n\in\mathbb{N}_o$. We show this for the "limit" case $n=0$. The left side of (3.2) is then bounded by

$$C\cdot|\gamma|\int_{|y-x|<d}\frac{1}{|x-y|^2}\hat\rho(t,x)dx, \text{ where } \hat\rho(t,x)=\int_{|v|\le R}f(t,x,v)dv.$$

$\hat\rho$ is uniformly bounded, as $f\le M$ by Theorem 1, and thus

$$C\cdot|\gamma|\int_{|y-x|<d}\frac{1}{|x-y|^2}\hat\rho(t,x)dx \le A'\int\frac{1}{|x-y|^2}dx \le A''d ,$$

from which (3.2) follows. The case $n\neq 0$ is similar.

(b) The mappings $x \longrightarrow G_n(x,y)$, $n \in \mathbb{N}_o$, y fixed, are continuous in $\mathbb{R}^3 \setminus K_d(y) := \{x; |x-y| \geq d\}$ and $|G_n(x,y) - G(x,y)| \to 0$ uniformly for all (x,y) with $|x-y| \geq d$. Hence, the functions

$$(x,v) \longrightarrow \langle G_n(x,y), \psi(t,x,v)\rangle, \; n \in \mathbb{N}_o,$$

defined on $(\mathbb{R}^3 \setminus K_d(y)) \times \mathbb{R}^3$, are of class C_o^o and converge uniformly on their domain to $\langle G(\cdot,y), \psi(t,\cdot)\rangle$. The convergence is even uniform with respect to t and y, because, for a given $\eta > 0$, the corresponding n depends only on d.
By Lemma 3, we already know that $\mu_t^n \Longrightarrow \mu_t$ uniformly with respect to t, i.e.

$$\int_{\mathbb{R}^6} \varphi d\mu_t^n \longrightarrow \int_{\mathbb{R}^6} \varphi \, d\mu_t$$

uniformly in t for every $\varphi \in C_o^1(\mathbb{R}^6)$. The same is then true for $\varphi \in C_o^o(\mathbb{R}^6)$ and even if $\mathbb{R}^6$ is replaced by $(\mathbb{R}^3 \setminus K_d(y)) \times \mathbb{R}^3$, as is easily seen. A simple application of the triangle inequality now shows that there is an $n_o = n_o(\varepsilon)$ (independent of t and y) such that

$$(3.3) \quad \Big| \int_{(\mathbb{R}^3 \setminus K_d(y)) \times \mathbb{R}^3} [\langle G_n(x,y), \psi(t,x,v)\rangle \, d\mu_t^n(x,v) - \langle G(x,y), \psi(t,x,v)\rangle \, d\mu_t(x,v)] \Big| \leq \frac{\varepsilon}{3}$$

for all $n \geq n_o(\varepsilon)$. Together with (3.2), this yields the statement of Lemma 5.

<u>Corollary</u> As Φ_n is continuous for all $n \in \mathbb{N}$, Lemma 5 implies continuity of ψ with respect to t and y.

Now, since $\Phi_n \in C^b([0,T] \times \mathfrak{X})$ for all $n \in \mathbb{N}_o$ and $\|\Phi_n(t,\cdot) - \Phi(t,\cdot)\|_\infty \longrightarrow 0$ as $n \to \infty$ uniformly in t, we get at once that

$$|W_n(t)-W_o(t)| = |\int_{\mathbb{R}^6} \Phi_n(t,\cdot)d\mu_t^n - \int_{\mathbb{R}^6} \Phi(t,\cdot)d\mu_t| \longrightarrow 0$$

as $n\to\infty$, uniformly in t. But then also

$$\int_0^T W_n(t)dt \longrightarrow \int_0^T W_o(t)dt \qquad \text{as } n\to\infty,$$

which proves Theorem 2.

§ 4 The Plasma Physical Case. Convergence of Simulation Methods.

We return for the moment to the modified case with an arbitrary but fixed mollification parameter $\delta>0$, and, only for simplicity, vanishing ion background, i.e. $n=0$. Let us recall that the kinetic energy $E_\delta(t)$ was defined by

$E_\delta(t) := \int_{\mathbb{R}^2} v^2 f_\delta(t,x,v)dxdv.$

As the force K in (2) resulted from the Newtonian potential $u(x-y) := -\frac{\gamma}{|x-y|}$, we define the potential energy V(t), as usual, by

$$(4.1) \qquad V(t) := \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} u(x-y)\rho(t,y)dy)\rho(t,x)dx,$$

if this integral exists. Similarly, in the modified case, K_δ results from the "mollified potential" $u_\delta(x-y) := \int_{\mathbb{R}^3} u(x-z)\omega_\delta(z-y)dz$, because then $G_\delta(x,y) = -\nabla_x u_\delta(x-y)$. Here the potential energy $V_\delta(t)$ is defined by

$$(4.1') \qquad V_\delta(t) := \int_{\mathbb{R}^3} (\int_{\mathbb{R}^3} u_\delta(x-y)\rho_\delta(t,y)dy)\rho_\delta(t,x)dx \ ,$$

and the total energy is $E_\delta(t) + V_\delta(t)$.

It is certainly reasonable to start with initial densities such that both E(O) and V(O) are finite. For simplicity, let us assume that, in addition to the conditions (2.1) and (2.2), f_o is such that there is a constant M' with

(4.2) $\quad 0 \leq \rho_o(y) \leq M' \quad$ for all $y \in \mathcal{X}$ λ-a.e.,

where $\rho_o(y) := \int_{\mathbb{R}^3} f_o(y,w)dw$. Then, as is easily seen, V(O) and $V_\delta(O)$ are bounded by a constant which can be chosen independent of δ.

Remark The last statement above is what we really need. (4.2) is a very simple condition such that this statement is true. There are weaker conditions, but they are more complicated.

We can now formulate energy conservation as

Theorem 3 For any given $\delta > 0$,

$$E_\delta(t) + V_\delta(t) = E(O) + V_\delta(O)$$

holds for the solution $f_\delta(t,\cdot)$ of the modified problem, $t \in [O,T]$.

Proof This is, e.g., proven in [5] (Horst). We give a brief outline, using (1.5), (1.6), (1.7).

Actually

$$\begin{aligned} V_\delta(t) &= \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^6} u_\delta(x-y) f_\delta(t,y,w)dydw\right) f_\delta(t,x,v)dxdv = \\ &= \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^6} u_\delta(X^\delta_{t,o}P - X^\delta_{t,o}Q) f_o(Q)dQ\right) f_o(P)dP = \\ &= \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^6} \left[u_\delta(x-y) + \int_0^t \nabla u_\delta(X^\delta_{s,o}P - X^\delta_{s,o}Q)\cdot(V^\delta_{s,o}P - V^\delta_{s,o}Q)ds\right]\cdot \right. \\ &\qquad\qquad \left. \cdot f_o(Q)dQ\right) f_o(P)dP = \end{aligned}$$

$$= V_\delta(0)+\int_0^t \int_{\mathbb{R}^6}\int_{\mathbb{R}^6} \nabla u_\delta(X^\delta_{s,o}P-X^\delta_{s,o}Q)\cdot(V^\delta_{s,o}P-V^\delta_{s,o}Q)\cdot$$

$$\cdot f_o(Q)dQf_o(P)dP.$$

Similarly,

$$E_\delta(t) = \int_{\mathbb{R}^6} v^2 f_\delta(t,x,v)dxdv =$$

$$= \int_{\mathbb{R}^6} (V^\delta_{t,o}P)^2 f_o(P)dP =$$

$$= \int_{\mathbb{R}^6} \left[v^2-2\int_0^t V^\delta_{s,o}P\int_{\mathbb{R}^6} \nabla u_\delta(X^\delta_{s,o}P-X^\delta_{s,o}Q)\cdot f_o(Q)dQds\right] f_o(P)dP=$$

$$= E(0)-2\int_0^t \int_{\mathbb{R}^6}\int_{\mathbb{R}^6} \nabla u_\delta(X^\delta_{s,o}P-X^\delta_{s,o}Q)\cdot(V^\delta_{s,o}P)f_o(Q)f_o(P)\cdot$$

$$\cdot dQdPds =$$

$$= E(0)-\int_0^t \int_{\mathbb{R}^6}\int_{\mathbb{R}^6} \nabla u_\delta(X^\delta_{s,o}P-X^\delta_{s,o}Q)(V^\delta_{s,o}P-V^\delta_{s,o}Q)f_o(Q)f_o(P)\cdot$$

$$\cdot dQdPds ,$$

and by the first part of our calculations this equals $E(0) - V_\delta(t) + V_\delta(0)$, proving the theorem.

<u>Remark</u> If, in the general case with $n \neq 0$, we define

$$V_\delta(t) := \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} u_\delta(x-y)(\rho_\delta(t,y)-n(y))dy\right)(\rho_\delta(t,x)-n(x))dx,$$

the energy conservation is proven in exactly the same way. Only the occuring expressions are longer.

<u>Corollary to Theorem 3</u> For the plasma physical case, where $u(x-y) \geqq 0$ whenever $x \neq y$, the initial value problem (1)-(2) has a weak solution whenever f_o satisfies (2.1), (2.2) and (4.2).

<u>Proof</u> Under the quoted conditions $V_\delta(t)$ is uniformly bounded from below and the total energy remains constant. Hence the boundedness condition concerning $E_\delta(t)$ in Theorem 1 is satisfied.

The weaknesses of our existence theorem are that it is not constructive and yields no information about uniqueness of solutions. A general uniqueness theorem is, to our knowledge, not available in this context, and we will not make any conjecture. There is, however, some hope to prove that the (unique) solutions μ_t^δ of the modified problem will converge weakly to the solution μ_t delivered by Theorem 1. This would not only show uniqueness of μ_t in the class of solutions which can be approximated by solutions μ_t^δ of the modified problem, but also convergence of simulation procedures to our solution [cf. Neunzert [9]].