

Moduli spaces for torsion free modules on curve singularities I

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Introduction

Let R denote the complete local ring of a reduced curve singularity over an algebraically closed field of characteristic 0. The aim of this article is to construct coarse moduli spaces for torsion free R -modules or, which amounts to the same, for Cohen-Macaulay representations of R .

It is well known and to a certain extent surprising that representation properties and, e.g. deformation properties, of R are closely related, at least for plane curve singularities. In particular we know that the moduli space for CM -representations of R has dimension 0 resp. 1 if and only if R dominates a simple (= elliptic) resp. parabolic plane curve singularity (cf. [GK] and the survey article [Gr]). On the other hand, there are many open problems and it seems to be desirable to have a deeper understanding of the classification of torsion free R -modules. We approach this problem via methods of geometric invariant theory (cf. [Mu], [Ne]) aiming at the construction and description of coarse moduli spaces for such modules.

One knows from moduli problems in projective geometry, e.g. moduli for coherent sheaves on a projective variety (the Maruyama scheme may be considered as the global analogue of the moduli scheme constructed in this paper) that one has to fix certain invariants in order to obtain a moduli space in the category of algebraic varieties. It is one of the main points of this paper to find invariants of R -modules which behave well in families of modules and which have the property that there exists a moduli space for modules with these invariants fixed. In the present paper we treat the case of irreducible R and modules of rank 1. The general case which is technically more involved, as well as questions about compactifications, will be treated in a subsequent paper.

To start with one needs a bounded family, i.e. a scheme of finite type, which parametrizes all torsion free R -modules up to isomorphism, but not necessarily in a 1-1 way. This is what we call the “module variety” and it is constructed and studied in chapter 1. In order to be able to do this, we have to rigidify the problem by “sandwiching” the modules and show that sandwiched modules have actually a fine moduli space (chapter 1.5). Moreover, we show that the universal family for sandwiched modules is locally versal for modules with fixed δ -invariant (chapter 3.1). As a byproduct of our construction we get that any torsion free R -module of rank 1 can be obtained as a deformation of the normalization \bar{R} . The module variety is a closed subvariety of a certain Grassmannian. There is a natural action, with respect to a given flag, of a unipotent algebraic group, the Jordan group J , on this Grassmannian which respects the module variety such that the orbits are exactly the isomorphism classes of modules. The Schubert cells relative to the given flag are invariant and we study this action, or rather the

action of the corresponding Lie algebra, in a very explicit way in chapter 2. This allows us to apply the results [GP 1] on geometric quotients of unipotent group actions to the given situation. We construct a stratification $\{W_{\Gamma,E}\}$ of a given Schubert cell W_{Γ} such that the strata are invariant under the action of J and such that on each stratum the geometric quotient exists. By intersecting these strata with the module variety we obtain a stratification $\{M_{\Gamma,E}\}$ of the module variety into finitely many strata such that the geometric quotients $M_{\Gamma,E}/J$ exist.

In chapter 3 we introduce numerical invariants for torsion free R -modules such that each stratum $M_{\Gamma,E}$ is characterized by fixing these invariants. Some of these invariants are similar to invariants of curve singularities but others are completely different and new. Any torsion free rank 1 R -module M can be embedded into the normalization \bar{R} such that $R \subset M \subset \bar{R}$. Then $\delta(M) = \dim_K(\bar{R}/M)$ is like the δ -invariant for curve singularities and $\Gamma(M)$, the value set of M with respect to the valuation of \bar{R} is similar to the semigroup of R hence might be considered as the “topological type” of M . Modules with fixed Γ are parametrized by the intersection of the Schubert cell W_{Γ} with the module variety. The E -invariant E_M is defined as the Hilbert function of $\bar{R}/\text{End}_R(M)$ with respect to a certain filtration $\bar{R} = \text{End}^0(M) \supset \text{End}^1(M) \supset \dots \supset \text{End}(M)$ which is induced by the K^* -action on \bar{R} . Now $M_{\Gamma,E}$ parametrizes all modules M with $\Gamma(M) = \Gamma$ and $E(M) = E$. $M_{\Gamma,E}/J$ will be the desired coarse moduli space for such modules (3.4).

In order to define a natural scheme structure on $M_{\Gamma,E}$ we have to define Γ -constant and E -constant families of modules over a possibly non-reduced base space. This can be done by requiring that certain sheaves are flat. Since the results of [GP 1] do not require the spaces to be reduced and since the geometric quotients are locally trivial with respect to the Zariski topology, the moduli spaces carry a natural scheme structure which may not be reduced. Nevertheless, our approach is very explicit. The proofs of the theorems provide a method for a given Γ to compute the module variety M_{Γ} and for a given E the stratum $M_{\Gamma,E}$ as well as the quotient $M_{\Gamma,E}/J$. We include an example to illustrate this fact.

We treat the case of algebroid and complex analytic curve singularities simultaneously. The module variety and the moduli spaces, however, will be separated algebraic schemes of finite type over the ground field. Hence, when considering families of modules, we encounter objects which are algebraic along the base but analytic or formal along the fibre. Since a base change has to preserve this property we have to define and use a kind of “fibre-analytic” tensor product. A problem is that it does not commute with localization. But it turns out that there is a faithfully flat relation which is sufficient for the applications. The necessary foundational properties are developed in an appendix.

The construction of moduli spaces for objects in local algebraic geometry has attracted broad attention only recently, although there appear to have been many attempts before in the literature, most notably those by Zariski in [Za] (cf. also [LP] for moduli of plane curve singularities with one Puiseux pair and a short historical account). The fundamental difference to the analog problem in projective algebraic geometry is that the algebraic groups describing isomorphism classes are not reductive but rather solvable or, as in our case, unipotent. And the basic obstacle for showing the existence of a moduli space except for a few specific examples seems to have been the lack of sufficiently general and workable criteria for the existence of geometric quotients for such groups (like the Hilbert-Mumford criterion for reductive groups).

Such criteria are now available and have been developed in [GP 1]. These allow not only the defining of an open dense subset of “stable” points on which the geometric quotient exists but also a complete stratification of the parameter space such that on each stratum the quotient exist. The definition of the stratification is functorial as well as explicit in terms of the given action of the group. The explicitness allows the characterizing of the strata by fixing certain numerical invariants while the functoriality allows us to show that the quotient is indeed a coarse moduli space in the sense of Mumford. For our situation this is done in chapter 3.

We should like to mention that the results of [GP 1], which are basic for the present paper, were actually developed in connection with this paper and the examples provided by moduli spaces for modules were essential to obtain the results of [GP1]. On the other hand, those results have a much broader scope of applications in singularity theory, e.g. for the construction of moduli spaces for singularities itself (cf. [GP 2]).

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1 A bounded family for torsionfree modules of rank 1

1.1 Invariants of curve singularities

Throughout this section R denotes the analytic local ring of a reduced and irreducible analytic curve singularity over a field K which we assume to be of characteristic 0 and algebraically closed from section 1.3 on. \bar{R} denotes the **normalization** of R , i.e. the integral closure of R in its quotient field. More precisely, we assume $\bar{R} = K\langle t \rangle$ to be the ring of formal power series (resp. convergent power-series if K has a nontrivial valuation) in one variable. The **quotient field** Q of \bar{R} is equal to $K\langle t \rangle[t^{-1}]$ and $R \subset \bar{R}$ is a sub- K -algebra of finite codimension over K . Note that Q and hence \bar{R} and R have a canonical **valuation** $v : Q \rightarrow \mathbb{Z} \cup \{\infty\}$ given by the subdegree in t . We have the following ideals and numerical invariants associated with R :

\mathfrak{m} the maximal ideal of R , $\mathfrak{c} = An(\bar{R}/R)$ the **conductor** ideal, $c = c(R) = \dim_K(\bar{R}/\mathfrak{c})$ the **multiplicity of the conductor** and $\delta = \delta(R) = \dim_K(\bar{R}/R)$ the **δ -invariant**.

As usual, Q^* , \bar{R}^* and R^* denote the corresponding groups of units. For any non-empty subset $M \subset Q$ let $v(M) := \{v(m) \mid m \in M\}$ the **value set** and $\text{mult}(M) := \inf v(M)$ the **multiplicity** of M . $\Gamma(R) := v(R)$ is called the **semi-group** of R .

1.2 Invariants of modules

Let M be a finitely generated torsion-free R -module of rank 1. This means that $M \otimes_R Q \cong Q$ and the canonical map $M \rightarrow M \otimes_R Q$ is an injection. This injection factors through the **normalization** of M ,

$$\bar{M} := M \otimes_R \bar{R} / \text{torsion}.$$

We define the **δ -invariant of M** to be

$$\delta(M) := \dim_K(\bar{M}/M).$$

The letter δ without any specification is exclusively used for $\delta(R)$.

In order to keep notations short, we write $\mathbf{Mod}(R)$ for the category of finitely generated, torsion-free R -modules of rank 1.

Note that any fractional ideal M' of R (i.e. an R -module contained in Q) is a torsion-free R -module of rank 1. Conversely we have

Lemma 1.1 (i) Any $M \in \text{Mod}(R)$ is isomorphic to some fractional ideal M' such that $R \subset M' \subset \bar{R}$ and $\delta(M) = \dim_K \bar{R}/M'$. In general for any fractional ideal $0 \neq M \subset \bar{R}$, we have $\delta(M) = \dim_K \bar{R}/M - \dim_K \bar{R}/\bar{R}M$, in particular, $\delta(M) = \dim_K \bar{R}/M \iff M \cap \bar{R}^* \neq \emptyset$.

(ii) Let $M, M' \subset \bar{R}$ be two fractional ideals such that $\dim_K \bar{R}/M = \dim_K \bar{R}/M'$, then $M \cong M' \iff \exists u \in \bar{R}^*$ such that $uM = M'$.

(iii) For any $M \in \text{Mod}(R)$ we have $0 \leq \delta(M) \leq \delta(R)$ and each value is attained. Moreover, $\delta(M) = 0$ iff $M \cong \bar{R}$ and $\delta(M) = \delta(R)$ iff $M \cong R$.

(iv) For any $M \in \text{Mod}(R)$, $M \subset \bar{R}$ and $\dim_K \bar{R}/M = d$ we have $t^{d+\delta} \bar{R} \subset M$, and $t^{c+d-\delta(M)} \bar{R} \subset M$.

Proof: Any isomorphism $M \otimes_R Q \cong Q$ maps M onto a fractional ideal $M'' \subset Q$ and any homomorphism between fractional ideals is induced by multiplication with an element of Q . If $m = \text{mult}(M'')$, then $t^{-m}M'' \subset \bar{R}$ and it contains a unit $u \in \bar{R}^*$ hence $M' = u^{-1}t^{-m}M''$ has the required properties of (i). The rest of (i) and (ii) follow easily from the fact that $\dim_K(M/t^p M) = p$. To see (iii) take $M \cong M'$ as in (i) and note that $M' = \bar{R}$. Hence, $\delta(M) = \delta(R) - \dim_K M'/R$, which implies what we wanted to show. Filling gaps of $\bar{R} \setminus M'$ from behind shows that each possible value for $\delta(M)$ is attained.

To see (iv) let $v(M)$ be the value set of M and Γ the semigroup of R . Then $v(M)$ is a Γ -set, i.e. $v(M) + \Gamma \subseteq v(M)$. Let $s = \sup(\mathbb{Z} \setminus v(M))$ then for all $x \in v(M)$, $x \leq s$, $s - x \notin \Gamma$.

This implies

$$s - d + 1 = \#\{x \mid x \in v(M), x \leq s\} \leq \#(\mathbb{Z}_+ \setminus \Gamma) = \delta, \text{ i.e. } s + 1 \leq \delta + d$$

which proves $t^{\delta+d} \bar{R} \subseteq M$. Moreover, by (i) exists an $x \in \bar{R}$, $v(x) = d - \delta(M)$, $M = xM'$ and $R \subset M' \subset \bar{R}$. Hence $t^e \bar{R} \subset M'$, i.e. $t^{c+d-\delta(M)} \bar{R} \subset M$.

Remark 1.2 The easy statements of lemma 1.1 are the basis for our construction of moduli spaces for torsion-free R -modules of rank 1. (i) implies that such modules correspond to points of certain Grassmannians, whilst the fact that $\delta(M)$ is bounded by $\delta(R)$ implies that all $M \in \mathbf{Mod}(R)$ are parametrized by a subvariety of a finite dimensional Grassmannian (“ $\mathbf{Mod}(R)$ is bounded”). Finally, (ii) implies that isomorphism classes of R -modules correspond to orbits of the action of a unipotent algebraic group on this Grassmannian.

1.3 The reduced module variety

From now on assume that the field K is algebraically closed. We introduce the following notations: Let $I \neq 0$ be an \bar{R} -ideal which is contained in the conductor

\mathfrak{c} of R . An R -module M such that $I \subset M \subset \bar{R}$ is called **I -sandwiched** and we refer to I as a **sandwiching ideal**. $\pi : \bar{R} \rightarrow \bar{R}/I$ is the canonical projection. If **colength** $M := \dim_K \bar{R}/M = d$, then

$$\lambda_M := \pi(M)$$

is a closed point in $Gr(\bar{R}/I, d)$, the Grassmannian of d -dimensional quotients of \bar{R}/I . The set of closed points $\lambda \in Gr(\bar{R}/I, d)$ such that

$$M_\lambda := \pi^{-1}(\lambda)$$

is an R -module defines a closed subspace of $Gr(\bar{R}/I, d)$. To see this, notice that the algebraic group $(\bar{R}/I)^*$ acts on $Gr(\bar{R}/I, d)$ via multiplication on \bar{R}/I , and that M_λ is an R -module if and only if λ is a fixed point under the action of the subgroup $(R/I)^*$.

We define

$$M(R; I, d) := Gr(\bar{R}/I, d)^{(R/I)^*}$$

to be the fixed point scheme of the action of $(R/I)^*$ on $Gr(\bar{R}/I, d)$ and call it the **module variety of I -sandwiched R -modules of colength d** . It is shown in [DG] that there is a unique scheme structure on $M(R; I, d)$ as a closed subscheme of $Gr(\bar{R}/I, d)$ which has the obvious universal property. Since K^* acts trivially on $Gr(\bar{R}/I, d)$ we have an action of $J = (\bar{R}/I)^*/K^*$ which is canonically isomorphic to the **Jordan Group** (cf. chapter 2). Since the actions of $(R/I)^*$ and J commute J acts also on $M(R; I, d)$.

Proposition 1.3 (i) $M(R; I, d)$ is a connected, closed subscheme of $Gr(\bar{R}/I, d)$. $\lambda \mapsto M_\lambda$ defines a bijection from the set of closed points of $M(R; I, d)$ onto the set of all I -sandwiched R -modules of colength d .

(ii) J acts regularly on $M(R; I, d)$ and two closed points $\lambda, \mu \in M(R; I, d)$ are in the same orbit if and only if M_λ and M_μ are isomorphic as R -modules. The point $\bar{\lambda} \in M(R; I, d)$ such that $M_{\bar{\lambda}} = \bar{R}t^d$ is in the closure of each J -orbit.

(iii) For two sandwiching ideals $I' \subset I$ we have $M(R; I, d) \subset M(R; I', d)$ and equality holds on the reduced structures if $\dim_K \bar{R}/I \geq d + \delta$. $M(R; I, d)_{red} \cong M(R; t^{2\delta}\bar{R}, \delta)_{red}$ if $d \geq \delta$, $\dim_K \bar{R}/I \geq d + \delta$ and $\lambda \mapsto M_\lambda$ induces a surjection from $M(R; t^{2\delta}\bar{R}, \delta)_{red}$ onto the set of isomorphism classes of elements of $\mathbf{Mod}(R)$.

Proof: Most of (i) and (ii) is obvious or follows from lemma 1.1. Since J is affine and $M(R; I, d)$ is projective, the closure of each J -orbit must contain at least one zerodimensional orbit. But a zerodimensional orbit corresponds to an \bar{R} -ideal in

\bar{R} of colength d and, hence, is equal to $\bar{R}t^d$. This proves (ii) and since the orbits are connected, we also obtain the connectedness of $M(R; I, d)$. (iii): Taking pre-images under the canonical map $\bar{R}/I' \rightarrow \bar{R}/I$ defines a closed embedding of $Gr(\bar{R}/I, d) \subset Gr(\bar{R}/I', d)$ which induces the morphism $M(R; I, d) \rightarrow M(R; I', d)$. Now let $M \subseteq \bar{R}$ be of colength d and $\dim_K \bar{R}/I \geq d + \delta$, then $I \subseteq t^{d+\delta} \bar{R}$. On the other hand, by lemma 1.1(iv), $t^{d+\delta} \bar{R} \subseteq M$, which implies $\lambda_M \in M(R; t^{d+\delta} \bar{R}, d)$. We obtain finally $M(R; t^{d+\delta} \bar{R}, d)_{red} = M(R; I, d)_{red} = M(R; I', d)_{red}$. Now, let M be any object of $\mathbf{Mod}(R)$. By lemma 1.1, M is isomorphic to M' , $R \subset M' \subset \bar{R}$, $\delta(M) = \dim_K(\bar{R}/M')$ and $\delta(M) \leq \delta = \delta(R)$. Then

$$I = t^{2\delta} \bar{R} \subset t^{\delta+\delta(M)} \bar{R} \subset M' \text{ and } t^{\delta-\delta(M)} M' =: M'' \subset \bar{R},$$

$\dim_K \bar{R}/I = 2\delta$, $\dim_K \bar{R}/M'' = \delta$. Hence $M \cong M''$ and $\lambda_{M''} \in M(\bar{R}; I, \delta)$. If $d \geq \delta$ and $\dim_K(\bar{R}/I) \geq d + \delta$ then multiplication with $t^{\delta-d}$ maps $M(R; I, d)_{red}$ isomorphically onto $M(R; t^{\delta-d} I, \delta)_{red}$ which is isomorphic to $M(R; t^{2\delta} \bar{R}, \delta)_{red}$ by the preceding argument.

We call

$$M(R) := M(R; t^{2\delta} \bar{R}, \delta)_{red}$$

the **reduced module variety**.

Remark 1.4 (1) $M(R)$ parametrizes up to isomorphism all torsion-free R -modules of rank 1. Look at the Plücker embedding $Gr(\bar{R}/I, \delta) \hookrightarrow \mathbb{P}(\Lambda^\delta \bar{R}/I)$. The linear action of $(R/I)^*/K^*$ on $Gr(\bar{R}/I, \delta)$ extends to a linear action on $\Lambda^\delta \bar{R}/I$ and since $(R/I)^*/K^*$ is unipotent all characters are trivial. Thus, the fixed point scheme of $(R/I)^*/K^*$ on $Gr(\bar{R}/I, \delta)$ is the intersection of $Gr(\bar{R}/I, \delta)$ in its Plücker embedding with a linear subspace of $\mathbb{P}(\Lambda^\delta \bar{R}/I)$, the fixed point scheme of $(R/I)^*/K^*$ on $\mathbb{P}(\Lambda^\delta \bar{R}/I)$. $M(R)$ is the fixed point set.

(2) Although $M(R)$ is connected, it need not be irreducible. To see this let ω_R be the dualizing module of R . We may assume that ω_R is equal to some fractional ideal ω with $\lambda_\omega \in M(R)$. For $\lambda_M \in M(R)$ let

$$M^\vee := \{x \in Q \mid xM \subseteq \omega\} \cong Hom_R(M, \omega).$$

We claim $\lambda_{M^\vee} \in M(R)$. We have to prove that $M^\vee \subseteq \bar{R}$ and $\dim_K \bar{R}/M^\vee = \delta$. Suppose that $xM \subseteq \omega$ for some $x \in Q$ with $v(x) < 0$ then $\delta = \dim_K \bar{R}/M > \dim_K \bar{R}/xM \geq \dim_K \bar{R}/\omega = \delta$ leads to a contradiction, i.e. $M^\vee \subseteq \bar{R}$. On the other hand $\bar{R}^\vee = t^{2\delta} \bar{R} \subseteq M \subseteq \bar{R}$ and $\dim \bar{R}/M = \delta$ implies $\dim_K M/\bar{R}^\vee = \delta$ and consequently $\dim_K \bar{R}/M^\vee = \delta$.

The functor $M \rightarrow M^\vee$ induces an involution on $M(R)$, which maps λ_R to λ_ω and has $\lambda_{t_\delta \bar{R}}$ as fixed point. Notice that for the value sets $v(M)$ and $v(M^\vee)$ hold

$$v(M^\vee) = \mathbb{Z} \setminus \{2\delta - 1 - v(M)\}.$$

Furthermore $2\delta - 1 \notin v(\omega)$, i.e. $t^{2\delta}\bar{R} = I$ is the maximal possible sandwiching ideal for modules of colength δ in \bar{R} .

(3) We call an irreducible component of $M(R)$ a **smoothing component** if a general point of this component corresponds to a free module. A module that is isomorphic to some M_λ , λ a closed point of some smoothing component, is called **smoothable**. Likewise, a component of $M(R)$ is called **dualizing** if a general point corresponds to a dualizing module. The involution defined above maps a smoothing component onto a dualizing component and vice versa. Since a closed point $\lambda \in M(R)$ corresponds to a free module iff λ belongs to the orbit $J\lambda_R$, which has dimension δ , and since J is irreducible we obtain

Lemma 1.5 *$M(R)$ has a unique smoothing component and a unique dualizing component which are isomorphic and have dimension $\delta(R)$. They coincide if and only if R is Gorenstein.*

Hence $M(R)$ is reducible if R is not Gorenstein. Actually, it follows from results of Rego [Re] that $M(R)$ is irreducible if and only if R is the local ring of a plane curve singularity (see also [AIK] and [KK]).

Note that the tangent space T_λ of $M(R; t^{2\delta}\bar{R}, \delta)$ at λ is isomorphic to $\text{Hom}_R(M_\lambda/I, \bar{R}/M_\lambda)$, where $I = t^{2\delta}\bar{R}$ (cf. corollary 1.13). It follows that $T_\lambda \cong \bar{R}/R$ if $M_\lambda = R$ and $T_\lambda \cong \bar{R}/R$ if $M_\lambda = \omega$. This shows that $M(R; t^{2\delta}\bar{R}, \delta)$ is reduced and smooth at a general point of the smoothing component and of the dualizing component. To see that $T_\lambda = \bar{R}/R$ if $M_\lambda = R$ or ω let $M \subset \bar{R}$ be a module such that $\text{Ext}_R^1(M, M) = 0$. Then

$$0 \rightarrow \text{End}_R(M) \rightarrow \text{Hom}_R(M, \bar{R}) \rightarrow \text{Hom}_R(M, \bar{R}/M) \rightarrow 0$$

is exact. Since $\text{Hom}_R(M, \bar{R}) = t^{-\text{mult}(M)}\bar{R}$ we obtain $\text{Hom}_R(M, \bar{R}/M) = t^{-\text{mult}(M)}\bar{R}/\text{End}_R(M)$.

Hence $\text{Hom}_R(R, \bar{R}/R) = \bar{R}/R$ and $\text{Hom}_R(\omega, \bar{R}/\omega) = t^{c-2\delta}\bar{R}/R$. This implies $\text{Hom}_R(R/I, \bar{R}/R) = \bar{R}/R$ and $\text{Hom}_R(\omega/I, \bar{R}/\omega) = \bar{R}/R$ if $I = t^{2\delta}\bar{R}$.

Example 1.6 $R = R_{\delta+1} = K[[t^{\delta+1}, t^{\delta+2}, \dots]]$. We shall describe the structure of $M(R)$.

For $i = 0, \dots, \delta - 1$ let $\Gamma_i = \{i, i+1, \dots, 2i, \delta+i+1, \delta+i+2, \dots\}$ and W_{Γ_i} the Schubert cell in $\text{Gr}(\bar{R}/t^{2\delta}\bar{R}, \delta)$ defined by Γ_i (cf. chapter 2). The following holds:

1. $M(R) = \bigcup_{i=0}^{\delta-1} \bar{W}_{\Gamma_i}$
2. $\bar{W}_{\Gamma_i} \cong Gr(\bar{R}/t^{\delta+1}\bar{R}, \delta - i)$
3. \bar{W}_{Γ_0} is the smoothing component
4. $\bar{W}_{\Gamma_i}^\vee = \bar{W}_{\Gamma_{\delta-i-1}}$
5. Let $\Delta = \{\delta - 1, \delta + 1, \delta + 2, \dots\}$ then

$$\bar{W}_\Delta = \bigcap_{i=0}^{\delta-1} \bar{W}_{\Gamma_i} \text{ and } \lambda_{t^\delta \bar{R}} \in \bar{W}_\Delta.$$

Proof: (1) Let $\lambda_M \in M(R)$ be a closed point then $\lambda_M \in W_{v(M)}$. Consider the canonical projection $\pi : \bar{R} \rightarrow \bar{R}/t^{2\delta}\bar{R}$ and let $U \subseteq \bar{R}/t^{2\delta}\bar{R}$ be any δ -dimensional subspace. If $U \in \bar{W}_{\Gamma_i}$ for some i then obviously $\pi^{-1}(U)$ is an R -module because $v(\pi^{-1}(U)) = \{\gamma_0, \dots, \gamma_{\delta-1}, 2\delta, 2\delta + 1, \dots\}$ and $\gamma_j \geq i + j$ for $j = 0, \dots, i$, and $\gamma_{i+1} \geq \delta + i + 1$.

On the other hand if for some closed point $\lambda_M \in M(R)$, $mult(M) = i$ then the R -module structure of M requires $\delta + i + 1, \delta + i + 2, \dots \in v(M)$. Furthermore $\dim_K \bar{R}/M = \delta$ implies that $\gamma_1, \dots, \gamma_i \in v(M)$, $i < \gamma_1 < \dots < \gamma_i < \delta + i + 1$. This implies that $\pi(M) \in \bar{W}_{\Gamma_i}$.

(2) is obvious, (3) holds since $\lambda_R \in \bar{W}_{\Gamma_0}$, (4) follows from $v(M^\vee) = \mathbb{Z} \setminus \{2\delta - 1 - v(M)\}$ and (5) is obvious.

1.4 Families of modules

In order to construct moduli spaces we have to consider families of R -modules parametrized by some base space S . We are interested in the cases where S belongs to one of the following two categories:

- K -sch** : category of locally noetherian K -schemes,
- ana** : category of complex analytic spaces.

We say that S is a **base space** if S belongs to one of these categories. The topology of the underlying topological space will be the Zariski topology, resp. the Euclidean topology. If $S = Spec A$, A a local noetherian K -algebra with residue field K we call S a **local base space** and denote it by (S, s) , $s = \mathfrak{m}_A$. If A is complex analytic, $(S, s) = Spec A$ denotes the complex germ defined by A and S a sufficiently small representative of (S, s) .

If $S \in \mathbf{ana}$ and if X is a representative of the complex germ (X, x) defined by R then \mathcal{X}_S denote the germ of $X \times S$ along S , i.e. \mathcal{X}_S is the ringed space $(S, \mathcal{O}_{\mathcal{X}_S})$ where

$$\mathcal{O}_{\mathcal{X}_S} := R_S := \sigma^{-1}\mathcal{O}_{X \times S},$$

the topological restriction of $\mathcal{O}_{X \times S}$ to the canonical section $\sigma : S \rightarrow \{x\} \times S \hookrightarrow X \times S$.

If $S \in K\text{-sch}$, we define $\mathcal{X}_S = (S, \mathcal{O}_{\mathcal{X}_S})$ to be the formal completion of $X_S := \text{Spec}(R) \times S$ along S . $R_S := \mathcal{O}_{\mathcal{X}_S}$ is the sheaf on S associated to the presheaf

$$\mathcal{O}_{\mathcal{X}_S}(\mathcal{U}) = R_S(\mathcal{U}) = R\widehat{\otimes}_K \mathcal{O}_S(\mathcal{U}),$$

$\mathcal{U} \subset S$ open (cf. the appendix for the definition of $\widehat{\otimes}_K$ and properties concerning formal schemes). If $S = \text{Spec} A$ we have $R_A = \Gamma(S, R_S)$ and $R_S = R_A^\sim$, the completion of the sheafification of R_A on $\text{Spec}(R_A)$ along S .

If \mathcal{F} is an arbitrary R_S -sheaf we write \mathcal{F}_s for the stalk of \mathcal{F} at s and $\mathcal{F}(s)$ for the **fibre** over s , i.e.

$$\mathcal{F}(s) = \mathcal{F}_s \widehat{\otimes}_{\mathcal{O}_{S,s}} \kappa(s) = \mathcal{F}_s \otimes_{\mathcal{O}_{S,s}} \kappa(s)$$

where $\kappa(s)$ denotes the residue field of the local ring $\mathcal{O}_{S,s}$ (cf. appendix, lemma 3).

$\mathcal{F}(s)$ is an $R(s)$ -module where

$$R(s) := R_S(s) = R\widehat{\otimes}_K \kappa(s).$$

More generally, if $\varphi : T \rightarrow S$ is any morphism of base spaces and \mathcal{F} an R_S -sheaf, we have the **base change**

$$\varphi^* \mathcal{F} := \varphi^{-1} \mathcal{F} \otimes_{\varphi^{-1} R_S} R_T.$$

Definition: Let S be a base space. A **family of R -modules (of rank 1)** over S is a coherent R_S -module \mathcal{M} on S which is flat as an \mathcal{O}_S -module such that for each $s \in S$ the fibre $\mathcal{M}(s)$ is a finitely generated, torsion-free $R(s)$ -module of rank 1. The category of such R_S -modules is denoted by $\mathbf{Mod}(R/S)$, the morphisms being morphisms of R_S -modules. A family over $\text{Spec} A$ is also called a family over A .

If $\mathcal{M} \in \mathbf{Mod}(R/S)$ and $\varphi : T \rightarrow S$ is a morphism of base spaces then $\varphi^* \mathcal{M} \in \mathbf{Mod}(R/T)$.

For any R -module M we define

$$M_S := M \otimes_R R_S,$$

where M and R are considered as constant sheaves on S . We call M_S the **constant family** associated to M . Special constant families are R_S , \bar{R}_S and Q_S . We have $\bar{R}_S(s) = \kappa(s)\langle t \rangle$ and $Q_S(s) = \kappa(s)\langle t \rangle[t^{-1}]$ (cf. appendix, remark 12).

Lemma 1.7 *Let \mathcal{M} be an R_S -module. Then $\mathcal{M}(s)$ is a torsion-free $R(s)$ -module for each $s \in S$ iff $\mathcal{M}(s)$ is a torsion free R -module for each closed point $s \in S$. In particular, $M_S \in \mathbf{Mod}(R/S)$ iff $M \in \mathbf{Mod}(R)$.*

Proof: Let κ be a field extension of K and M a torsion-free R -module. We have to show that $M_\kappa = M \otimes_R (R \widehat{\otimes}_K \kappa)$ is torsion-free over $R_\kappa = R \widehat{\otimes}_K \kappa$, i.e. that $M_\kappa \rightarrow M_\kappa \otimes_{R_\kappa} Q_\kappa$ is injective where $Q_\kappa = Q \widehat{\otimes}_K \kappa = \text{Quot}(R_\kappa)$. But $M_\kappa \otimes_{R_\kappa} Q_\kappa = M \otimes_R R_\kappa \otimes_{R_\kappa} Q_\kappa = M \otimes_R Q \widehat{\otimes}_K \kappa = M \otimes_R Q \otimes_R R_\kappa$; the required injectivity follows since $M \rightarrow M \otimes_R Q$ is injective and since R_κ is flat over R by proposition 7 of the appendix. The second statement follows from this and the fact that M_S is flat over S by appendix, lemma 8.

Definition:

1. A family $\mathcal{M} \in \mathbf{Mod}(R/S)$ together with an injection $\mathcal{M} \hookrightarrow \bar{R}_S$ is called an **embedded family**. If \bar{R}_S/\mathcal{M} is locally free of finite rank d we call \mathcal{M} an **embedded family of corank d** .
2. Let $I \subset \bar{R}$ be a sandwiching ideal and $I_S \subset \bar{R}_S$ the associated constant family. An embedded family $\mathcal{M} \hookrightarrow \bar{R}_S$ together with an injection $I_S \hookrightarrow \mathcal{M}$ is called an **I -sandwiched family** if the composition $I_S \hookrightarrow \mathcal{M} \hookrightarrow \bar{R}_S$ coincides with the inclusion $I_S \subset \bar{R}_S$.
3. A morphism of embedded resp. I -sandwiched families \mathcal{M} and \mathcal{M}' is a morphism $\mathcal{M} \hookrightarrow \mathcal{M}'$ of R_S -modules such that the diagram

$$\begin{array}{ccccc}
 \mathcal{M} & \hookrightarrow & \bar{R}_S & & I_S & \hookrightarrow & \mathcal{M} & \hookrightarrow & \bar{R}_S \\
 \downarrow & & \parallel & \text{resp.} & \parallel & & \downarrow & & \parallel \\
 \mathcal{M}' & \hookrightarrow & \bar{R}_S & & I_S & \hookrightarrow & \mathcal{M}' & \hookrightarrow & \bar{R}_S
 \end{array}$$

commutes.

We shall frequently use the following standard lemma from local algebra (cf. [A-K], lemma 4.1, p. 142):

Lemma 1.8 *Let $A \rightarrow B$ be a morphism of local noetherian rings, K the residue field of A and $\varphi : M \rightarrow N$ a B -homomorphism of finite B -modules. If N is A -flat, then the following conditions are equivalent:*

- (i) φ is injective with A -flat cocernel,
- (ii) $\varphi \otimes 1 : M \otimes_A K \rightarrow N \otimes_A K$ is injective.

Lemma 1.9 *Let $\mathcal{M} \subset \bar{R}_S$ be a coherent R_S -module. The following are equivalent:*

- (i) $\mathcal{M} \in \mathbf{Mod}(R/S)$,
- (ii) \bar{R}_S/\mathcal{M} is \mathcal{O}_S -flat,
- (iii) $\mathcal{M}_s \mathcal{O}_{S,s} \langle t \rangle \cap \mathfrak{m}_{S,s} \mathcal{O}_{S,s} \langle t \rangle \subset \mathcal{M}_s \mathfrak{m}_{S,s} R_{\mathcal{O}_{S,s}} \forall s$.

Proof: $\mathcal{M} \in \mathbf{Mod}(R/S)$ iff for each $s \in S$ \mathcal{M}_s is $\mathcal{O}_{S,s}$ -flat and the fibre $\mathcal{M}(s)$ is torsion free of rank 1 over $R(s)$. Since \bar{R}_S is \mathcal{O}_S -flat by lemma 1.7 the flatness of \mathcal{M} follows from the flatness of \bar{R}_S/\mathcal{M} . Hence, the equivalence of (i) and (ii) follows from lemma 1.8 applied to $\mathcal{O}_{S,s} \hookrightarrow R_{S,s}$. The equivalence of (ii) and (iii) follows from lemma 1.8, the snake lemma applied to

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathfrak{m}_{S,s} \mathcal{M}_s & \rightarrow & \mathcal{M}_s & \rightarrow & \mathcal{M}(s) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathfrak{m}_{S,s} \bar{R}_{S,s} & \rightarrow & \bar{R}_{S,s} & \rightarrow & \bar{R}_S(s) & \rightarrow & 0 \end{array}$$

and from appendix, remark 11.

1.5 The universal family of sandwiched modules

Consider a sandwiching ideal $I \subset \bar{R}$ and the Grassmannian $Gr(\bar{R}/I, d)$ of d -codimensional subspaces of \bar{R}/I . Let \mathcal{U}' denote the locally free sheaf on $Gr(\bar{R}/I, d)$ associated to the universal subbundle. \mathcal{U}' is a subsheaf of

$$\bar{R}/I \otimes_K \mathcal{O}_{Gr(\bar{R}/I, d)} = \bar{R}/I \hat{\otimes}_K \mathcal{O}_{Gr(\bar{R}/I, d)}$$

and we denote by \mathcal{U} the pre-image of \mathcal{U}' under the canonical map

$$\bar{R}_{Gr(\bar{R}/I, d)} = \bar{R} \hat{\otimes}_K \mathcal{O}_{Gr(\bar{R}/I, d)} \rightarrow \bar{R}/I \hat{\otimes}_K \mathcal{O}_{Gr(\bar{R}/I, d)}.$$

Moreover, if $i : M(R; I, d) \hookrightarrow Gr(\bar{R}/I, d)$ is the inclusion, we set

$$\mathcal{U}(R; I, d) := i^* \mathcal{U}.$$

$\mathcal{U}(R; I, d)$ is called the **universal family of I -sandwiched R -modules of colength d** .

Proposition 1.10 *The family $\mathcal{U}(R; I, d)$ over $M(R; I, d)$ represents the functor which associates to any base space S , the set of isomorphism classes of I -sandwiched families of R -modules of colength d over S . Hence, for any I -sandwiched family \mathcal{M} on S of colength d exists a unique morphism $\varphi : S \rightarrow M(R; I, d)$ such that $\mathcal{M} \cong \varphi^* \mathcal{U}(R; I, d)$.*

Proof: Let $M = M(R; I, d)$ and $\mathcal{U} = \mathcal{U}(R; I, d)$. By construction $I_M \subset \mathcal{U} \subset \bar{R}_M$ and \bar{R}_M/\mathcal{U} is locally free of rank d , hence \mathcal{U} is I -sandwiched of colength d . If $I_S \hookrightarrow \mathcal{M} \hookrightarrow \bar{R}_S$ is any such family on S , then the image of \mathcal{M}/I_S in \bar{R}_S/I_S is independent of the isomorphism class and defines a subbundle of $\bar{R}_S/I_S = (\bar{R}/I)_S$. Moreover, since \mathcal{M} is an R_S -module, the group scheme $(R/I)_S^*$ acts trivially on \mathcal{M}/I_S . Conversely, having a subbundle $\mathcal{V} \subset \bar{R}_S/I_S$ of codimension d on which $(R/I)_S^*$ acts trivially, we obtain a family of I -sandwiched R -modules of colength d by taking the preimage of \mathcal{V} under $\bar{R}_S \rightarrow \bar{R}_S/I_S$. Therefore, the result is a consequence of the universal property of the Grassmannian and the fixed point scheme.

Remark 1.11 We shall see in chapter 1.7 that each family \mathcal{M} over a local base space (S, s) is isomorphic (as abstract family) to an embedded family over (S, s) but there may exist no I such that \mathcal{M} is I -sandwiched.

Indeed, we have the following criterion: Let $S = \text{Spec}(A)$, A a noetherian K -algebra, Q_S the constant family over S associated to $Q = \text{Quot}(R)$ and $\mathcal{M} \in \mathbf{Mod}(R/S)$. Then \mathcal{M} is isomorphic to an I -sandwiched family for some I iff $\mathcal{M} \otimes_{R_S} Q_S \cong Q_S$.

Proof: “ \implies ”. Let $I_S = t^a \bar{R}_S$, $a \geq c$ and \mathcal{M} I -sandwiched. Then

$$t^a \bar{R}_S \subset \mathcal{M} \subset \bar{R}_S \subset t^{-a} \mathcal{M}.$$

Tensoring with Q_S yields morphisms

$$Q_S \rightarrow \mathcal{M} \otimes_{R_S} Q_S \rightarrow Q_S \rightarrow t^{-a} \mathcal{M} \otimes_{R_S} Q_S,$$

where the composition of the first resp. the second two is an isomorphism. This shows that $\mathcal{M} \otimes_{R_S} Q_S \cong Q_S$.

“ \impliedby ”. Let $\varphi : \mathcal{M} \otimes_{R_S} Q_S \rightarrow Q_S$ be an isomorphism and consider $i : \mathcal{M} \rightarrow \mathcal{M} \otimes_{R_S} Q_S$, $m \mapsto m \otimes 1$. Since $\mathcal{M}(s)$ is torsionfree, i is an injection by lemma 1.8. Let $\varphi^{-1}(1) = \sum m_j \otimes q_j$ and $a \geq c$ such that $t^a q_j \in R_S$, then $t^a \varphi^{-1}(1) = (\sum m_j t^a q_j) \otimes 1 =: m \otimes 1 = i(m)$. We get $\mathcal{M} \cong \varphi i(\mathcal{M}) \ni t^a$, $\varphi i(\mathcal{M}) Q_S = Q_S$ and there exists a b such that $\mathcal{M}' = t^b \varphi i(\mathcal{M}) \subset \bar{R}_S$ (since \mathcal{M} is finitely generated over R_S). Finally $t^{a+b+c} \bar{R}_S \subset \mathcal{M}' \subset \bar{R}_S$.

Note that a and b depend in general on A : Consider $A = K[x]/(x^n)$, $R = K[[t^2, t^3, \dots]]$ and $\mathcal{M} = (t^2 + x) \bar{R}[x]/(x^n)$. The smallest power of t contained in \mathcal{M} is t^{2n-4} .

1.6 Infinitesimal structure on the module variety

Definition:

1. Let $M \in \mathbf{Mod}(R)$ and A a local, noetherian K -algebra with residue field K . A **deformation of M over A** is a finitely generated R_A -module \mathcal{M} , flat over A , together with an isomorphism $M \xrightarrow{\cong} \mathcal{M} \hat{\otimes}_A K$. Two deformations \mathcal{M} and \mathcal{M}' of M over A are isomorphic if an isomorphism $\varphi : \mathcal{M} \xrightarrow{\cong} \mathcal{M}'$ of R_A -modules exists such that the diagram

$$\begin{array}{ccc}
 & & \mathcal{M} \hat{\otimes}_A K \\
 & \cong \nearrow & \\
 M & & \cong \downarrow \quad \varphi \hat{\otimes}_A K \\
 & \cong \searrow & \\
 & & \mathcal{M}' \hat{\otimes}_A K
 \end{array}$$

commutes. $Def_M(A)$ denotes the set of isomorphism classes of deformations of M over A (cf. appendix for the definition of R_A and $\hat{\otimes}$).

2. Let $M \subset \bar{R}$ be an embedded module. An **embedded deformation of M over A** is a deformation \mathcal{M} of M over A together with an injection $j : \mathcal{M} \hookrightarrow \bar{R}_A$ such that the diagram

$$\begin{array}{ccc}
 M & \subset & \bar{R} \\
 \cong \searrow & & \nearrow \\
 & \mathcal{M} \hat{\otimes}_A K & j \hat{\otimes}_A K
 \end{array}$$

commutes. Two embedded deformations of M over A are isomorphic if they are isomorphic as deformations by an isomorphism which commutes with the given injections into \bar{R}_A . $Def_{M \subset \bar{R}}(A)$ denotes the set of isomorphism classes of embedded deformations of M over A .

3. Let I be a sandwiching ideal and $I \subset M \subset \bar{R}$ an I -sandwiched module. An **I -sandwiched deformation** of M over A is an embedded deformation $\mathcal{M} \hookrightarrow \bar{R}_A$ of M together with an injection $I_A \hookrightarrow \mathcal{M}$ such that the composition $I_A \hookrightarrow \bar{R}_A$ equals the inclusion induced by $I \subset \bar{R}$. An isomorphism of I -sandwiched deformations is an isomorphism of embedded deformations which commutes with the given injections of I_A . The set of such isomorphism classes is denoted by $Def_{I \subset M \subset \bar{R}}(A)$.

Let $K[\epsilon] = K[x]/(x^2)$.

Lemma 1.12 *Let $I \subset M \subset \bar{R}$ be an I -sandwiched R -module.*

(i) The following diagram with canonical horizontal morphisms commutes:

$$\begin{array}{ccccc} Def_{I \subset M \subset \bar{R}}(K[\epsilon]) & \hookrightarrow & Def_{M \subset \bar{R}}(K[\epsilon]) & \rightarrow & Def_M(K[\epsilon]) \\ \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\ Hom_R(M/I, \bar{R}/M) & \hookrightarrow & Hom_R(M, \bar{R}/M) & \rightarrow & Ext_R^1(M, M). \end{array}$$

(ii) $Def_{I \subset M \subset \bar{R}}(K[\epsilon]) \xrightarrow{\cong} Def_{M \subset \bar{R}}(K[\epsilon])$ if $\dim_K(\bar{R}/I) \geq e := \min\{c + 2(d - \delta(M)), a\}$ where $d = \dim_K(\bar{R}/M)$ and $a = \max\{c + d - \delta(M), 2d + \delta - \delta(M)\}$. $Def_{M \subset \bar{R}}(K[\epsilon]) \rightarrow Def_M(K[\epsilon])$ is surjective if $\dim_K \bar{R}/M \geq c$.

Proof: (i) Let $I \xrightarrow{k} M \xrightarrow{j} \bar{R}$ be injections and $p : \bar{R} \rightarrow \bar{R}/M$ the canonical projection. If $\varphi \in Hom_R(M, \bar{R}/M)$ such that $\varphi \circ k = 0$ then the corresponding element $[E_\varphi] \in Ext_R^1(M, M)$ is represented by a commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{i} & E_\varphi & \xrightarrow{\pi} & M \\ \parallel & & \downarrow \psi & & \downarrow \varphi \\ M & \xrightarrow{j} & R & \xrightarrow{p} & \bar{R}/M, \end{array}$$

where E_φ is the fibre product of p and φ .

E_φ is an $R[\epsilon]$ -module by $\epsilon e := i \circ \pi(e)$ and we obtain a $K[\epsilon]$ -embedding $\phi : E_\varphi \rightarrow \bar{R}[\epsilon]$ defined by $\phi := j \circ \pi + \epsilon \cdot \psi$. Define $\kappa : I + \epsilon I \rightarrow E_\varphi$ by $\kappa(a + \epsilon b) = i \circ k(b) + e$, $e \in E_\varphi$ uniquely determined by $\pi(e) = k(a)$ and $\psi(e) = 0$. Then we obtain

$$I + \epsilon I \xrightarrow{\kappa} E_\varphi \xrightarrow{\phi} \bar{R}[\epsilon], \quad \phi \circ \kappa(a + \epsilon b) = j \circ k(a) + \epsilon \cdot j \circ k(b),$$

which is an I -sandwiched deformation if $I \hookrightarrow M \hookrightarrow \bar{R}$ is a sandwich.

This way we obtain a map

$$Hom_R(M/I, \bar{R}/M) \rightarrow Def_{I \subset M \subset \bar{R}}(K[\epsilon])$$

and at the same time the other vertical maps of the lemma such that the diagram commutes.

In order to construct the inverse maps let

$$\begin{array}{ccccc} I + \epsilon I & \xrightarrow{\kappa} & E & \xrightarrow{\phi} & \bar{R} + \epsilon \bar{R} \\ \downarrow \pi'_1 & & \downarrow \pi & & \downarrow \pi''_2 \\ I & \xrightarrow{k} & M & \xrightarrow{j} & \bar{R} \end{array}$$

be a commutative diagram defining an element from $Def_{I \subset M \subset \bar{R}}(K[\epsilon])$, i.e. E is an $R[\epsilon]$ -module, κ and ϕ are $R[\epsilon]$ -homomorphisms such that π is surjective and $Ker \pi = \epsilon E$ and $\phi \circ \kappa(a + \epsilon b) = j \circ k(a) + \epsilon j \circ k(b)$. Let $\pi''_2 : \bar{R}[\epsilon] \rightarrow \bar{R}$ be defined by $\pi''_2(r + \epsilon r') = r'$ and $\psi := \pi''_2 \circ \phi$ and $\lambda : \epsilon E \rightarrow M$ by $\lambda(\epsilon e) = \pi(e)$ then ψ and

λ define a homomorphism $\varphi : M \rightarrow \bar{R}/M$ such that $\varphi \circ k = 0$ and the following diagram is commutative

$$\begin{array}{ccccc} \epsilon E & \hookrightarrow & E & \xrightarrow{\pi} & M \\ \lambda \downarrow & & \downarrow \psi & & \downarrow \varphi \\ M & \xrightarrow{j} & \bar{R} & \xrightarrow{p} & \bar{R}/M \end{array}$$

(here we use that $\phi_2'' \circ \phi(\epsilon e) = j \circ \pi(e)$). It follows easily that E is the fibre product of p and φ and hence we obtain the inverse map

$$Def_{I \subseteq M \subseteq \bar{R}}(K[\epsilon]) \rightarrow Hom_R(M/I, \bar{R}/M)$$

and also for the other functors. This proves (i).

(ii) By lemma 1.1 (i) there exists an $x \in \bar{R}$, $v(x) = d - \delta(M)$, $M = xM'$ and $R \subset M' \subset \bar{R}$. Let $\varphi \in Hom_R(M, \bar{R}/M)$ then $\varphi(t^{c+2(d-\delta(M))}r) = t^{c+d-\delta(M)}ur\varphi(x) = 0$, where $r \in \bar{R}$ and $u \in \bar{R}^*$, since $x = x \cdot 1 \in M$ and $t^{c+d-\delta(M)}\bar{R} \subset M$ by Lemma 1.1 (iv). Moreover, $t^a r = t^b x r'$ where $b = \max\{c, d + \delta\}$ for some $r' \in \bar{R}$. Therefore $\varphi(t^a r) = t^b r' \varphi(x) = 0$ since $t^{d+\delta}\bar{R} \subset M$.

The concepts of “deformation” and “family” are related as follows. Let (S, s) be the local base space, defined by A (cf. 1.4). If \mathcal{M} is a deformation of M over A , then the sheaf \mathcal{M}^\sim is a family of R -modules over (S, s) and $\mathcal{M}_s^\sim = \mathcal{M}$. Conversely, for any family $\mathcal{M} \in \mathbf{Mod}(R/(S, s))$, the stalk \mathcal{M}_s is a deformation of $\mathcal{M}_s \hat{\otimes}_A K$ over A (cf. appendix and lemma 1.7).

Hence, a deformation over A is the same as a family over (S, s) .

Corollary 1.13 *Let λ be a closed point of the module variety $M(R; I, d)$. Then, for the Zariski tangent space at λ , we have:*

- (i) $T_\lambda M(R; I, d) = Hom_R(M_\lambda/I, \bar{R}/M_\lambda)$,
- (ii) $T_\lambda M(R; I, d) = Hom_R(M_\lambda, \bar{R}/M_\lambda)$ if $\dim_K(\bar{R}/I) \geq e$, e.g. if $\dim_K(\bar{R}/I) \geq c + 2d$.

Proof: This follows from proposition 1.10 and lemma 1.12.

1.7 Comparison to general deformations of modules

Let $M \in \mathbf{Mod}(R)$ and consider the dualizing module ω of R as embedded in \bar{R} such that $\dim_K \bar{R}/\omega = c - \delta$. Let $I \subset \bar{R}$ be a sandwiching ideal.

In this chapter we want to study the canonical maps of the different deformation functors on the category of local noetherian K -algebras with residue field K :

$$\begin{array}{ccc}
Def_{ICMC\bar{R}} & & \\
\downarrow & & \\
Def_{MC\bar{R}} & \longrightarrow & Def_M \\
\uparrow & & \\
Def_{MC\omega} & &
\end{array}$$

Proposition 1.14 (1) Let $M \subset \omega$ then $Def_{MC\omega} \rightarrow Def_M$ is smooth.

(2) Let $M \subset \bar{R}$ and $\dim \bar{R}/M \geq c$ then $Def_{MC\bar{R}} \rightarrow Def_M$ is surjective.

(3) Let $I = t^{2c+d}\bar{R} \subset M \subset \bar{R}$ and $\dim \bar{R}/M = d \geq c$ then for any factorial local K -algebra A , $Def_{ICMC\bar{R}}(A) \rightarrow Def_M(A)$ is surjective.

We use the following lemma, which is essentially proved in [J-S]:

Lemma 1.15 Let $A \rightarrow B$ a morphism of local noetherian rings with residue field K , $J \subseteq A$ an ideal and $\bar{A} = A/J$. Let N, M be finitely generated B -modules which are A -flat and assume $Ext_{B \otimes_A K}^1(N \otimes_A K, M \otimes_A K) = 0$. Then

(i) $Hom_B(N, M)$ is A -flat,

(ii) $Hom_B(N, M) \otimes_A \bar{A} = Hom_{B \otimes_A \bar{A}}(N \otimes_A \bar{A}, M \otimes_A \bar{A})$.

Proof: The case $J = \mathfrak{m}_A$ is proved in [J-S]. This implies (i).

Furthermore, we get (ii) for the case $J = \mathfrak{m}_A$ and also that $Hom_{B \otimes_A \bar{A}}(N \otimes_A \bar{A}, M \otimes_A \bar{A})$ is \bar{A} -flat. Consider now the canonical map

$$\varphi : Hom_B(N, M) \otimes_A \bar{A} \longrightarrow Hom_{B \otimes_A \bar{A}}(N \otimes_A \bar{A}, M \otimes_A \bar{A})$$

Because of (ii) for the case $J = \mathfrak{m}_A$ we know that $\varphi \otimes_{\bar{A}} K$ is an isomorphism. This implies by Lemma 1.8 that φ is injective and $coker \varphi$ is \bar{A} -flat. But $coker \varphi \otimes_{\bar{A}} K = 0$ and $coker \varphi$ is a $B \otimes_A \bar{A}$ -module. Using Nakayama's lemma we obtain $coker \varphi = 0$ which proves the lemma.

Proof of Proposition 1.14: We apply the lemma to the following situation: Let $A \rightarrow R_A$ be the canonical morphism and let $[\mathcal{M}] \in Def_M(A)$ such that $\mathcal{M} \otimes_A \bar{A}$ defines an element in $Def_{MC\omega}(\bar{A})$. Because ω is the dualizing module we know that $Ext_R^1(M, \omega) = 0$. The lemma yields

$$Hom_{R_A}(\mathcal{M}, \omega_A) \otimes_A \bar{A} = Hom_{R_{\bar{A}}}(\mathcal{M} \otimes_A \bar{A}, \omega_{\bar{A}})$$

and we obtain an embedding $\mathcal{M} \subset \omega_A$ which defines an element in $Def_{MC\omega}(A)$. This proves (1).

To prove (2) let $M \subset \bar{R}$ and $\dim_K \bar{R}/M \geq c$ then there exist $x \in \bar{R}$ such that $M \subset x\omega$. Namely because $\dim \bar{R}/M = \dim M^\vee/t^c \bar{R} \geq c$ there is an $x \in \bar{R}$ such that $R \subset xM^\vee \subset \bar{R}$, i.e. $M \subseteq x\omega$. Let A be a noetherian local K -algebra and $[\mathcal{M}] \in \text{Def}_M(A)$. Because $M \subseteq x\omega$ we may choose $\mathcal{M} \in [\mathcal{M}]$ such that $\mathcal{M} \otimes_A K \subseteq x\omega$ and apply lemma 1.15 to lift this to an embedding $\mathcal{M} \subset x\omega_A \subset \bar{R}_A$.

To prove (3) let $[\mathcal{M}] \in \text{Def}_M(A)$. Because of (2) we may assume that $\mathcal{M} \subset \bar{R}_A$. Let $\bar{\mathcal{M}} = \mathcal{M} \cdot \bar{R}_A$. Because A is factorial $\bar{\mathcal{M}} = h \cdot \mathfrak{a}$, $h \in \bar{R}_A$, $\mathfrak{a} \subseteq \bar{R}_A = A\langle t \rangle$ an ideal whose associated primes have height ≥ 2 . Because \bar{R}_A/\mathcal{M} is A -flat $h, \mathfrak{a} \not\subseteq \mathfrak{m}_A \bar{R}_A$. Because $\mathfrak{a} \not\subseteq \mathfrak{m}_A A\langle t \rangle$ and $ht\mathfrak{a} \geq 2$ we obtain $\mathfrak{a} \cap A \neq \{0\}$. Therefore we can choose $a \in A$, $a \neq 0$ such that $a \in \mathfrak{a}$. Then $a \cdot h \in \bar{\mathcal{M}}$ and consequently $ah t^c \in \mathcal{M}$. On the other hand \bar{R}_A/\mathcal{M} is a locally free A -module and $a \cdot [ht^c] = 0$ implies $ht^c \in \mathcal{M}$. Because $\mathcal{M} \subseteq \bar{\mathcal{M}} = h \cdot \mathfrak{a} \subseteq h\bar{R}_A$ we obtain $t^c \in \mathcal{M}' := h^{-1}\mathcal{M} \subseteq \bar{R}_A$.

This implies $t^{2c}\bar{R}_A \subseteq \mathcal{M}' \subseteq \bar{R}_A$ and $\text{rank}_A \bar{R}/\mathcal{M}' \leq d$. There is $y \in \bar{R}$, $v(y) \leq d$, such that $y\mathcal{M}'$ defines the required element in $\text{Def}_{ICM\subset\bar{R}}(A)$.

Remark 1.16 1. $\text{Def}_{M\subset\bar{R}} \longrightarrow \text{Def}_M$ is not smooth in general.

2. $\text{Def}_{ICM\subset\omega} \longrightarrow \text{Def}_{M\subset\omega}$ resp. $\text{Def}_{ICM\subset\bar{R}} \longrightarrow \text{Def}_M$ are not surjective in general.

3. Using the general Artin approximation theorem (Popescu, Rotthaus) we can show a strengthening of 1.14(2):

If A is excellent and henselian, $\mathcal{M} \in \mathbf{Mod}(R/A)$ and $s \in \text{Spec } A$ then there exists a neighbourhood U of s such that $\mathcal{M} \mid U$ is isomorphic to an embedded family. Since we do not need this fact we omit a proof.

Examples: Let $R = K[[t^2, t^3]]$ and $M = t^2\bar{R}$. To prove (1) we consider the following example:

$$\bar{\mathcal{M}} = t^2 R[\bar{\epsilon}] + (t^3 + \bar{\epsilon}t)R[\bar{\epsilon}], \bar{\epsilon}^2 = 0,$$

$$\mathcal{M} = t^2 R[\epsilon] + (t^3 + \epsilon t + \epsilon^2 t^{-1})R[\epsilon], \epsilon^3 = 0.$$

It is not difficult to see that $[\bar{\mathcal{M}}] \in \text{Def}_{M\subset\bar{R}}(K[\bar{\epsilon}])$ and $[\mathcal{M}] \in \text{Def}_M(K[\epsilon])$ is a lifting of $\bar{\mathcal{M}}$ considered as an element of $\text{Def}_M(K[\bar{\epsilon}])$. An easy calculation shows that it is not possible to embed \mathcal{M} into $\bar{R}[\bar{\epsilon}]$ such that this embedding is compatible with the given embedding $\bar{\mathcal{M}} \subset \bar{R}[\bar{\epsilon}]$, i.e. $\text{Def}_{M\subset\bar{R}} \longrightarrow \text{Def}_M$ is not smooth.

Notice that $(1 - \bar{\epsilon}t^{-2})\bar{\mathcal{M}} \subseteq \bar{R}[\bar{\epsilon}]$ can be lifted to $(1 - \epsilon t^{-2})\mathcal{M} \subseteq \bar{R}[\epsilon]$.

To prove (2) we consider the following example:

$$A = K[[x^2, x^3]] \text{ and } \mathcal{M} = (t^2 - x^2)R_A + (t^3 + x^3)R_A$$

then $[\mathcal{M}] \in \text{Def}_{M\subset\omega}(A)$ by lemma 1.9 because $\mathcal{M} \cap \mathfrak{m}_A \bar{R}_A = (x^2, x^3)\mathcal{M}$. We shall see that for every embedding $i : \mathcal{M} \rightarrow \bar{R}$ always $i(\mathcal{M}) \cap R = \{0\}$, i.e. $t^a \bar{R}_A \not\subseteq i(\mathcal{M})$ for all a . This implies that $\text{Def}_{I\subset M\subset\omega}(A) \rightarrow \text{Def}_{M\subset\omega}(A)$ and $\text{Def}_{I\subset M\subset \bar{R}}(A) \rightarrow \text{Def}_M(A)$ is not surjective whatever we choose for I .

To see this we consider the relations $(t^3 + x^3, -(t^2 - x^2))$ and $(t^4 + t^2x^2 + x^4, -(t^3 - x^3))$ of $t^2 - x^2, t^3 + x^3$.

Let $i : \mathcal{M} \rightarrow \bar{R}_A$ be an embedding and $i(t^2 - x^2) = a, i(t^3 + x^3) = b$ then $b = (t + \frac{x^2}{t-x})a$. This implies $a = \alpha(t^2 - x^2) + (t-x)x^2u$ and $b = \alpha(t^3 + x^3) + (t^2 - xt + x^2)x^2u$ for suitable $\alpha \in \bar{R}, u \in \bar{R}_A$. Now it is clear that $(a, b)\bar{R}_A \cap \bar{R} = \{0\}$.

2 Jordan group acting on the Grassmannian

2.1 Definition of the action

Let V be a c -dimensional K -vectorspace with a given basis $\{e_0, \dots, e_{c-1}\}$. Let J be the **Jordan group** defined by the matrices $\underline{a} = [a_1, \dots, a_{c-1}]$ of the form

$$\underline{a} = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{c-1} \\ 0 & \ddots & \ddots & & \vdots \\ & \ddots & & & a_2 \\ & & & & a_1 \\ 0 & \cdots & & 0 & 1 \end{bmatrix}.$$

J is an abelian and unipotent subgroup of $GL(c, K)$ and it acts on V by matrix-multiplication from the right:

$$e_i \circ [a_1, \dots, a_{c-1}] = e_i + a_1 e_{i+1} + \dots + a_{c-1-i} e_{c-1}.$$

The choice of the basis $\{e_0, \dots, e_{c-1}\}$ leads to a flag

$$\begin{aligned} (0) &= V_c \subset V_{c-1} \subset \dots \subset V_1 \subset V_0 = V, \\ V_i &= Ke_i + \dots + Ke_{c-1}, \end{aligned}$$

and a K^* -action on V defined by $e_i \mapsto c^i e_i$ for $c \in K^*$.

Let $Gr(k+1, V) = \{U \subset V \mid \dim U = k+1\}$ be the Grassmannian of $(k+1)$ -

dimensional subspaces of V . The actions of J and K^* on V define actions of J and K^* on $Gr(k+1, V)$. These actions leave the Schubert cells (with respect to the flag defined above) invariant. We will now study the actions of J and K^* on a Schubert cell (if $\lambda \in Gr(k+1, V)$ let $U_\lambda \subset V \otimes \kappa(\lambda)$ denote the corresponding subspace):

For $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_k, c, c+1, \dots\} \subset \mathbb{N}$ such that $0 \leq \gamma_0 < \gamma_1 < \dots < \gamma_k < c$ let

$$W_\Gamma := \{\lambda \in Gr(k+1, V) \mid \dim V_{\gamma_{k+1-i}} \cap U_\lambda = i \text{ and} \\ \dim V_j \cap U_\lambda < i \text{ if } j > \gamma_{k+1-i}, i = 1, \dots, k+1\}$$

be the open Schubert cell defined by Γ and let $U_\Gamma \subset V \times W_\Gamma$ be the restriction to W_Γ of the universal subbundle on $Gr(k+1, V)$. U_Γ is a trivial bundle.

We want to describe a stratification of W_Γ such that the strata $W_{\Gamma, \underline{\Gamma}}$ are J -invariant and the geometric quotient $W_{\Gamma, \underline{\Gamma}}/J$ exists. In order to be able to apply the results of [GP 1] we have to describe the corresponding Lie algebra action on W_Γ explicitly in terms of coordinates of W_Γ .

2.2 Description of the action in terms of coordinates

W_Γ can be parametrized as follows:

Any subspace $U \subset V$, $U \in W_\Gamma$, has a unique basis $\{w_0, \dots, w_k\}$, $w_i = \sum_{j=0}^{c-1} \lambda_{i,j-\gamma_i} e_j$

with the following properties:

- (1) $\lambda_{l,m} = 0$ if $m < 0$,
- (2) $\lambda_{l,\gamma_i-\gamma_l} = \begin{cases} 1 & i = l \\ 0 & i \neq l. \end{cases}$

Namely, consider a matrix whose rows are an arbitrary basis of U and apply the Gauss algorithm via elementary row operations in order to obtain a matrix of the form: $M(\underline{\lambda}) =$

$$\begin{bmatrix} 0 \dots 1 & \lambda_{0,1} \lambda_{0,2} \dots \lambda_{0,\gamma_1-\gamma_0-1} & 0 & \lambda_{0,\gamma_1-\gamma_0+1} \dots & 0 & \lambda_{0,\gamma_i-\gamma_0+1} \dots & 0 & \lambda_{0,\gamma_k-\gamma_0+1} \dots \lambda_{0,c-\gamma_0-1} \\ 0 \dots 0 & & 0 & 1 & \lambda_{1,1} & & 0 & \\ \vdots & & & 0 & \vdots & & \vdots & \\ & & & & 1 & \lambda_{i,1} & & \\ & & & \vdots & 0 & & 0 & \\ 0 \dots 0 & & 0 & 0 & 0 & 0 & 1 & \lambda_{k,1} \dots \lambda_{k,c-1-\gamma_k} \end{bmatrix}$$

The coefficients of w_i constitute the rows of this matrix. The $\lambda_{i,j}$, which are neither 0 nor 1, are coordinates of W_Γ and we identify W_Γ with the corresponding

affine space, i.e. we put

$$\begin{aligned} W_\Gamma &= \text{Spec } K[\underline{\lambda}], \\ \underline{\lambda} &= \{\lambda_{i,j}\}_{(i,j) \in I}, \\ I &= \{(i,j) | 0 \leq i \leq k, j > 0, j + \gamma_i \notin \Gamma\}. \end{aligned}$$

Note that $\dim W_\Gamma = \#I = \frac{(c-1+\delta)(c-\delta)}{2} - \sum_{i=0}^k \gamma_i = (k+1)(c-1) - \sum_{j=0}^k (\gamma_j + j)$,

where $\delta = \#(N - \Gamma)$ is the number of gaps of Γ .

Let $\mathcal{U}'_\Gamma \subset V \otimes_K K[\underline{\lambda}] =: \mathcal{V}$ be the free $K[\underline{\lambda}]$ -module associated to the universal subbundle $U_\Gamma \subset V \times W_\Gamma$. In terms of the basis $\{e_0, \dots, e_{c-1}\}$ we have

$$\mathcal{U}'_\Gamma = \sum_{i=0}^k w_i K[\underline{\lambda}] \subset \sum_{i=0}^{c-1} e_i K[\underline{\lambda}] = \mathcal{V}$$

with $w_i = e_{\gamma_i} + \sum_{j+\gamma_i \notin \Gamma} \lambda_{ij} e_{j+\gamma_i}$.

$\mathcal{V}/\mathcal{U}'_\Gamma$ is a free $K[\underline{\lambda}]$ -module generated by the classes of e_j , $j \notin \Gamma$.

The Jordan group J acts on W_Γ and in terms of the above coordinates this action is given by multiplying $M(\underline{\lambda})$ with $[a_1, \dots, a_{c-1}]$ from the right and then applying the Gauss algorithm (i.e. multiplication by a suitable element of $GL(k+1, K)$ from the left) to obtain the normal form. Hence, in terms of the $\lambda_{i,j}$, the action is of the form

$$W_\Gamma \times J \rightarrow W_\Gamma, (\lambda_{i,j}) \circ [a_1, \dots, a_{c-1}] =: (\varphi_{i,j}(\underline{a}, \underline{\lambda})),$$

where

$$\begin{aligned} \text{(I)} \quad \varphi_{i,j}(\underline{a}, \underline{\lambda}) &= \lambda_{i,j} + \sum_{\nu=0}^{j-1} a_{j-\nu} \lambda_{i,\nu} - \sum_{\nu=i+1}^k \sum_{\sigma=0}^{\gamma_\nu - \gamma_i - 1} a_{\gamma_\nu - \gamma_i - \sigma} \lambda_{i\sigma} \lambda_{\nu, j+\gamma_i - \gamma_\nu} \\ &+ \text{higher order terms with respect to } \underline{a}. \end{aligned}$$

Similarly, for the action of K^* on W_Γ we obtain $(\lambda_{i,j}) \circ c = (c^j \lambda_{i,j})$.

2.3 The corresponding Lie algebra action

Consider the following diagram

$$\begin{array}{ccc} J & \xrightarrow{\varphi} & \text{Aut}^o(K[\underline{\lambda}]) \\ \text{exp} \uparrow & & \uparrow \text{exp} \\ \text{Lie}(J) & \xrightarrow{\varphi_*} & \text{Der}_K^{\text{nil}}(K[\underline{\lambda}]), \end{array}$$

where Aut^o is the connected component of the identity and Der^{nil} the set of derivations acting nilpotently on $K[\underline{\lambda}]$. φ is the representation of J defined by

the action of J on W_Γ and φ_* the induced representation of $\text{Lie}(J)$. We have

$$\begin{aligned}\varphi([a_1, \dots, a_{c-1}])(\lambda_{i,j}) &= \varphi_{i,j} \quad \text{and} \\ \varphi_*([a_1, \dots, a_{c-1}])(\lambda_{i,j}) &= \sum_{\nu=1}^{c-1} a_\nu \frac{\partial \varphi_{i,j}}{\partial a_\nu} \Big|_{\underline{a}=0},\end{aligned}$$

where, by abuse of notation, the argument $[a_1, \dots, a_{c-1}]$ of φ_* stands for the matrix with 0's instead of 1's in the diagonal.

Let L be the image of the abelian Lie algebra $\text{Lie}(J)$ in $\text{Der}^{nil}(K[\underline{\lambda}])$. L is an abelian Lie algebra and generated (as K -vectorspace) by the vector fields

$$\delta_l := \varphi_*([0, \dots, 1, \dots, 0]) = \sum_{(i,j) \in I} \frac{\partial \varphi_{i,j}}{\partial a_l} \Big|_{\underline{a}=0} \frac{\partial}{\partial \lambda_{i,j}}.$$

If $h_{l,i,j} := \frac{\partial \varphi_{i,j}}{\partial a_l} \Big|_{\underline{a}=0} = \delta_l \lambda_{i,j}$, (I) of 2.2 provides the basic formula

$$(II) \quad \boxed{h_{l,i,j} = \lambda_{i,j-l} - \sum_{\nu=i+1}^k \lambda_{i,\gamma_\nu - \gamma_i - l} \lambda_{\nu,j + \gamma_i - \gamma_\nu}}$$

Remark 2.1 (1) $h_{l,i,j} = 0$ if $l > j$

(2) $h_{j,i,j} = 1$ if $(i,j) \in I$

Proof: (1) and (2) follow directly from 2.2.

The explicit formula for $\delta_l \lambda_{i,j}$ will be the basis for all considerations in the rest of this chapter. An invariant interpretation of the matrix $(h_{l,i,j})$ via Kodaira-Spencer will be given in chapter 3.3.

2.4 Blockstructure and rank function

Consider the matrix

$$h(\underline{\lambda}) = (h_{l,i,j}(\underline{\lambda})) = (\delta_l \lambda_{i,j}),$$

$l = 1, \dots, c-1$ and $(i,j) \in I$. The rows generate L over $K[\underline{\lambda}]$ as a submodule of the free module $\text{Der}_K(K[\underline{\lambda}])$ with respect to the basis $\partial/\partial \lambda_{i,j}$, $(i,j) \in I$. If we order the coordinates λ_{ij} such that the partial ordering defined by the K^* -action is refined by putting

$$(i,j) < (m,n) \text{ if } j < n \text{ or if } j = n \text{ and } i < m,$$

then $h(\underline{\lambda})$ has the following shape:

$$\begin{array}{c}
 (*, 1) \dots (*, 1) \quad \dots \quad (*, j) \dots (i, j) \dots (*, j) \quad \dots \quad (0, d) \\
 1 \left(\begin{array}{c} \boxed{1 \quad \dots \quad 1} \\ \vdots \\ \boxed{1 \quad \dots \quad 1} \\ \vdots \\ \underbrace{\boxed{1 \quad \dots \quad 1 \quad \dots \quad 1}}_{w_j} \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right) \\
 j \\
 d \\
 c-1
 \end{array}$$

where:

- d is maximal such that $d \notin \Gamma$,
- $(*, j)$ occurs if $j \notin \Gamma_0$, $\Gamma_0 := \{n \in \mathbb{N} \mid n + \gamma \in \Gamma \quad \forall \gamma \in \Gamma\}$,
- $w_j := \#\{\gamma \in \Gamma \mid j + \gamma \notin \Gamma\}$,
- $h_j - 1 := \#\{\alpha \in \Gamma_0 \mid k < \alpha < j\}$, k the predecessor of j in $\mathbb{N} \setminus \Gamma_0$.

Let B_j be the j -th block of $h(\lambda)$, i.e. the submatrix of $h(\underline{\lambda})$ consisting of columns with second index equal to j . Straightforward checking shows that w_j is the number of columns of B_j and h_j is the height of step from B_{j-1} to B_j . Since only columns with $(i, j) \in I$ do occur, $B_j = \emptyset$ iff $j \in \Gamma_0$.

For any point $\lambda \in W_\Gamma$ define the **rank function** $R(\lambda) : \mathbb{N} \rightarrow \mathbb{N}$ by

$$R(\lambda)(n) := \text{rank}(h_{l, i, j}(\lambda))_{j < n},$$

which is the rank of the union at the first blocks including B_{n-1} , evaluated at λ .

Lemma 2.2 $R(\lambda)(n) = R(\lambda)(n + 1)$ if $n \in \Gamma_0$. If $a = \min(\Gamma_0 \setminus \{0\})$, then $R(\lambda)(a) = a - 1$.

Proof: $n \in \Gamma_0$ implies $B_n = \emptyset$, hence $R(\lambda)(n) = R(\lambda)(n+1)$. $R(\lambda)(a) = a-1$ holds because $1, \dots, a-1 \notin \Gamma_0$ and $h_j = 1, 1 \leq j \leq a-1$.

Since the rows of $h(\underline{\lambda})$, evaluated at $\lambda \in W_\Gamma$, generate the tangent space at the orbit of λ , we have

$$R(\lambda)(c) = \dim \text{orbit}(\lambda) = \dim_{\kappa(\lambda)} J \otimes \kappa(\lambda) / \text{Aut}(\lambda),$$

where $\text{Aut}(\lambda)$ denotes the stabilizer of λ .

We shall interpret the other values of $R(\lambda)$ in a similar way: We define for $\lambda \in \text{Gr}(k+1, V)$ and $n \geq 0$,

$$\text{Aut}^n(\lambda) := \{g \in J \otimes \kappa(\lambda) \mid g(U_\lambda \cap V_i + V_{n+i}) \otimes \kappa(\lambda) = (U_\lambda \cap V_i + V_{n+i}) \otimes \kappa(\lambda), \forall i\}.$$

Then $\text{Aut}^0(\lambda) = J \otimes \kappa(\lambda)$ and $\text{Aut}^n(\lambda) = \text{Aut}(\lambda)$ if $n \geq c$.

Moreover, if $\lambda \in W_\Gamma$, consider the filtration of the tangent space $T_\lambda W_\Gamma$ induced by the K^* -action on W_Γ , given by

$$F_n(T_\lambda W_\Gamma) = \sum_{\substack{(i,j) \in I \\ j \geq n}} \kappa(\lambda) \partial / \partial \lambda_{ij}, \quad n \geq 0.$$

Lemma 2.3

$$\begin{aligned} R(\lambda)(n) &= \dim J \otimes \kappa(\lambda) / \text{Aut}^n(\lambda) \\ &= \dim_{\kappa(\lambda)} T_\lambda(J\lambda) - \dim_{\kappa(\lambda)} T_\lambda(J\lambda) \cap F_n(T_\lambda W_\Gamma). \end{aligned}$$

where $T_\lambda(J\lambda)$ is the tangent space at λ of the orbit $J\lambda$.

Remark 2.4 Thus, fixing $R(\lambda)$ is equivalent to fixing the dimension of all the subgroups

$$J \otimes \kappa(\lambda) \supset \text{Aut}^1(\lambda) \supset \dots \supset \text{Aut}^c(\lambda) = \text{Aut}(\lambda).$$

It is also equivalent to fixing the orbit dimension $d(\lambda)$ of $J\lambda$ and the dimension of the intersection of $T_\lambda(J\lambda)$ with the (non-complete) flag $F_n(T_\lambda W_\Gamma)$ in $T_\lambda W_\Gamma$. This means that $T_\lambda(J\lambda)$ belongs to a fixed Schubert cell of the Grassmannian $\text{Gr}(d(\lambda), T_\lambda W_\Gamma)$ with respect to the flag $F_n(T_\lambda W_\Gamma)$.

Proof of Lemma 2.3: By definition we have $R(\lambda)(n) = \text{rank}(h_{l,i,j}(\lambda))_{j < n}$.

Now the isomorphism

$$T_\lambda(J\lambda) / T_\lambda(J\lambda) \cap F_n(T_\lambda W_\Gamma) \simeq (T_\lambda(J\lambda) + F_n(T_\lambda W_\Gamma)) / F_n(T_\lambda W_\Gamma) \text{ gives}$$

$$R(\lambda)(n) = \dim_{\kappa(\lambda)} T_\lambda(J\lambda) - \dim_{\kappa(\lambda)} T_\lambda(J\lambda) \cap F_n(T_\lambda W_\Gamma).$$

On the other hand,

$$\dim T_\lambda(J\lambda) = \dim \text{Lie}(J) \otimes \kappa(\lambda) / \text{Lie}(\text{Aut}(\lambda)).$$

It remains to prove that

$$\dim \text{Lie}(\text{Aut}^n(\lambda)) / \text{Lie}(\text{Aut}(\lambda)) = \dim T_\lambda(J\lambda) \cap F_n(T_\lambda W_\Gamma).$$

Now (with the notations of 2.2) $\underline{a} = [a_1, \dots, a_{c-1}] \in \text{Aut}^n(\lambda)$ iff $\varphi_{ij}(\underline{a}, \underline{\lambda}) = \lambda_{ij}$ for all i and $j < n$. Consider the embedding

$$\text{Lie}(\text{Aut}^n(\lambda)) / \text{Lie}(\text{Aut}(\lambda)) \hookrightarrow T_\lambda W_\Gamma.$$

The image of this embedding is given by the set of all vector fields $\delta \in L$ such that $\delta(\lambda_{ij})(\lambda) = 0$ for all i and $j < n$, i.e. $\delta(\underline{\lambda})(\lambda) \in T_\lambda(J\lambda) \cap F_n(T_\lambda W_\Gamma)$. This proves the lemma.

2.5 Induced grading and algorithmic stratification

Fix a Schubert cell W_Γ and let, as in 2.4,

$$\Gamma_0 = \{n \in \mathbb{N} \mid n + \gamma \in \Gamma \quad \forall \gamma \in \Gamma\}.$$

Γ_0 is the maximal semigroup acting on Γ . Let $a = \min(\Gamma_0 \setminus \{0\})$ and consider the sub-Lie algebras of $L = \sum_{i=1}^{c-1} K\delta_i$,

$$L^{(o)} := \sum_{i \geq a} K\delta_i, \quad L^{(1)} := \sum_{i < a} K\delta_i.$$

The K^* -action on W_Γ induces a grading on $K[\underline{\lambda}]$ such that $\deg \lambda_{ij} = j$. Define $\deg \partial / \partial \lambda_{ij} = -j$ such that the action of $\text{Der}_K K[\underline{\lambda}]$ on $K[\underline{\lambda}]$ is homogeneous. The next lemma prepares for the application of the main result of [GP 1].

Lemma 2.5 (1) δ_i is homogeneous of degree $-i$,

(2) $H^1(L^{(1)}, K[\underline{\lambda}]) = 0$ and $H^1(L^{(1)}, K[\underline{\lambda}]^{L^{(o)}}) = 0$, where H^1 denotes the first Lie algebra cohomology group.

Proof: (1) Formula (II) of 2.3 shows that $\deg h_{l,i,j} = j - l$, and this implies $\deg \delta_l = -l$.

(2) By definition of Γ_0 , there exists some $i(n)$ for each $n \notin \Gamma_0$ such that $\gamma_{i(n)} \in \Gamma$ and $n + \gamma_{i(n)} \notin \Gamma$. Remark 2.1 implies $\delta_n(\lambda_{i(n),n}) = 1$ and the blockstructure of $h(\underline{\lambda})$ shows that $\delta_m(\lambda_{i(n),n}) = 0$ if $m > n$. This implies in particular that $\lambda_{i(1),1}, \dots, \lambda_{i(a-1),a-1} \in K[\underline{\lambda}]^{L^{(o)}}$. Moreover, for $k \geq n$,

$$\delta_k \delta_m(\lambda_{i(n),n}) = \delta_m \delta_k(\lambda_{i(n),n}) = 0.$$

Therefore, condition (4) of Theorem 3.10 of [GP 1] is fulfilled and (2) follows.

The gradings on $K[\underline{\lambda}]$ and $L^{(o)}$ induce a filtration $F^\bullet K[\underline{\lambda}]$: let $K[\underline{\lambda}] = \bigoplus_{i \geq 0} A_i$, A_i the K -vector space of homogeneous polynomials (with respect to the K^* -action) of degree i . $L^{(o)}$ is graded and let $-a := \max\{\deg \delta \mid \delta \in L^{(o)}\}$. Define

$$F^i K[\underline{\lambda}] := \bigoplus_{\nu \leq (i+1)a-1} A_\nu.$$

Since $L^{(o)}$ is abelian, we use the trivial filtration $Z_\bullet(L^{(o)})$, $Z_0(L^{(o)}) = L^{(o)} \supset Z_1(L^{(o)}) = 0$ of $L^{(o)}$. Let $W_\Gamma = \bigcup W_{\Gamma, \underline{r}}$ be the algorithmic stratification of W_Γ with respect to the filtrations $F^\bullet(K[\underline{\lambda}])$, $Z_\bullet(L^{(o)})$ and the action of $L^{(o)}$ defined after theorem 4.7 and lemma 4.10 in [GP 1].

Theorem 2.6 (1) $W_{\Gamma, \underline{r}}$ is quasi-affine and invariant under the action of L , and $W_{\Gamma, \underline{r}} \rightarrow W_{\Gamma, \underline{r}}/L$ is a geometric quotient which is quasi-affine.

(2) Let $d : K[\underline{\lambda}] \rightarrow \text{Hom}_K(L, K[\underline{\lambda}])$ be the differential defined by $df(\delta) = \delta(f)$ then $W_\Gamma = \cup W_{\Gamma, \underline{r}}$ is the flattening stratification of the modules $\text{Hom}_K(L, K[\underline{\lambda}])/K[\underline{\lambda}]dF^{\nu a}K[\underline{\lambda}]$ and if $\underline{r} = (r_1, \dots, r_l)$, then $\lambda \in W_{\Gamma, \underline{r}}$ iff $R(\lambda)(\nu a) = r_{\nu-1} + a - 1$, $\nu = 2, \dots, [\frac{d}{a}] + 1$, d maximal such that $d \notin \Gamma$.

(3) Let $\underline{r} = (r_1, \dots, r_l)$, $l = [\frac{d}{a}]$, d as in (2), and $r_i = \#\{n \mid n \leq (i+1)a, n \notin \Gamma_0\}$, then $W_{\Gamma, \underline{r}}$ and $W_{\Gamma, \underline{r}}/L$ are affine and $W_{\Gamma, \underline{r}}$ consist of all points of W_Γ with minimal orbit dimension $\delta(\Gamma_0) = \#\mathbb{N} \setminus \Gamma_0$.

Proof: (1) is a consequence of theorem 4.7 and corollary 4.12 of [GP 1]. By lemma 2.5 the assumptions are satisfied.

(2) is a consequence of remark 4.6 of [GP 1].

(3) holds because $R(\lambda)((i+1)a) \geq r_i$, $i = 1, \dots, [\frac{d}{a}]$, for all $\lambda \in W_\Gamma$, and the condition $R(\lambda)((i+1)a) = r_i$ is given by the vanishing of suitable determinants of the matrix $h(\underline{\lambda})$.

2.6 Examples

(1) In the following example we construct the stratification of a fixed Schubert cell in detail, and for each stratum we calculate the quotient after the Jordan group. On two of the strata the orbit dimension is the same, but the quotient does not exist on the union of these strata:

$$\Gamma = \Gamma_0 = \{0, 2, 4, 6, 8, 9, \dots\}, c = 8, k = 3.$$

The coordinates of the Schubert cell are:

$$M(\underline{\lambda}) = \begin{bmatrix} 1 & \lambda_{01} & 0 & \lambda_{03} & 0 & \lambda_{05} & 0 & \lambda_{07} \\ 0 & 0 & 1 & \lambda_{11} & 0 & \lambda_{13} & 0 & \lambda_{15} \\ 0 & 0 & 0 & 0 & 1 & \lambda_{21} & 0 & \lambda_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_{31} \end{bmatrix}.$$

The ordering of the variables $\lambda_{i,j}$ ($(i,j) \in I$) of the Schubert cell W_Γ is given by the ordered set of indices

$$I = \{(0,1), (1,1), (2,1), (3,1), (0,3), (1,3), (2,3), (0,5), (1,5), (0,7)\}.$$

The matrix $H(\underline{\lambda}) = (h_{l,i,j}(\underline{\lambda}))$ of the coefficients of the vector fields $\delta_1, \dots, \delta_7$ is:

$$\begin{pmatrix} 1111 & -\lambda_{11}\lambda_{01} & -\lambda_{21}\lambda_{11} & -\lambda_{31}\lambda_{21} & -\lambda_{21}\lambda_{03} - \lambda_{13}\lambda_{01} & -\lambda_{31}\lambda_{13} - \lambda_{23}\lambda_{11} & * \\ & \lambda_{01} - \lambda_{11} & \lambda_{11} - \lambda_{21} & \lambda_{21} - \lambda_{31} & \lambda_{03} - \lambda_{13} & \lambda_{13} - \lambda_{23} & * \\ & 1 & 1 & 1 & -\lambda_{21}\lambda_{01} & -\lambda_{31}\lambda_{11} & * \\ & & & & \lambda_{01} - \lambda_{21} & \lambda_{11} - \lambda_{31} & * \\ & & & & 1 & 1 & * \\ & & & & & & \lambda_{01} - \lambda_{31} \\ & & & & & & 1 \end{pmatrix}$$

We have $m(\Gamma_0) = 2, d = 7$.

$W_{\Gamma, \mathbf{r}} \neq \emptyset$ iff $\mathbf{r} \in \{(3,5,6), (2,4,5), (3,4,5), (2,3,4)\}$.

$$\begin{aligned} W_{\Gamma, (3,5,6)} &= \{\lambda \mid \lambda_{01} - \lambda_{11} - \lambda_{21} + \lambda_{31} \neq 0\} \\ W_{\Gamma, (3,4,5)} &= \{\lambda \mid \lambda_{01} - \lambda_{11} - \lambda_{21} + \lambda_{31} = 0, 2\lambda_{11} - \lambda_{01} - \lambda_{21} \neq 0\} \\ W_{\Gamma, (2,4,5)} &= \{\lambda \mid \lambda_{01} - \lambda_{11} - \lambda_{21} + \lambda_{31} = 2\lambda_{11} - \lambda_{01} - \lambda_{21} = 0, \\ &\quad 2\lambda_{31} - \lambda_{03} - \lambda_{23} + (\lambda_{01} - \lambda_{11})(\lambda_{11}\lambda_{31} - \lambda_{01}\lambda_{21}) \neq 0\} \\ W_{\Gamma, (2,3,4)} &= \{\lambda \mid \lambda_{01} - \lambda_{11} - \lambda_{21} + \lambda_{31} = 2\lambda_{11} - \lambda_{01} - \lambda_{21} = 0, \\ &\quad 2\lambda_{13} - \lambda_{03} - \lambda_{23} + (\lambda_{01} - \lambda_{11})(\lambda_{11}\lambda_{31} - \lambda_{01}\lambda_{21}) = 0\} \end{aligned}$$

Let $L_1 := K\delta_1 + K\delta_3 + K\delta_5 + K\delta_7$, then $K[\underline{\lambda}] = K[\underline{\lambda}]^{L_1} [\lambda_{01}, \lambda_{03}, \lambda_{05}, \lambda_{07}]$ and $K[\underline{\lambda}]^{L_1} = K[\bar{\lambda}_{11}, \bar{\lambda}_{21}, \bar{\lambda}_{31}, \bar{\lambda}_{13}, \bar{\lambda}_{23}, \bar{\lambda}_{15}]$,

$$\begin{aligned} \bar{\lambda}_{11} &= \lambda_{11} - \lambda_{01} \\ \bar{\lambda}_{21} &= \lambda_{21} - \lambda_{01} \\ \bar{\lambda}_{31} &= \lambda_{31} - \lambda_{01} \\ \bar{\lambda}_{13} &= \lambda_{13} - \lambda_{03} - \lambda_{01}(\lambda_{11}\lambda_{01} - \lambda_{11}\lambda_{21}) + \frac{1}{2}\lambda_{01}^2(\lambda_{01} - \lambda_{21}) \\ \bar{\lambda}_{23} &= \lambda_{23} - \lambda_{03} - \lambda_{01}(\lambda_{11}\lambda_{01} - \lambda_{21}\lambda_{31}) + \frac{1}{2}\lambda_{01}^2(\lambda_{11} + \lambda_{01} - \lambda_{21} - \lambda_{31}) \end{aligned}$$

$$\begin{aligned}\bar{\lambda}_{15} &= \lambda_{15} - \lambda_{05} - (\lambda_{01}\lambda_{21} - \lambda_{11}\lambda_{31})\lambda_{03} + \lambda_{01}(\lambda_{31} - \lambda_{01})(\lambda_{13} - \lambda_{03}) \\ &\quad + \lambda_{01}\lambda_{11}(\lambda_{23} - \lambda_{03}) - \frac{1}{2}\lambda_{01}^2(\lambda_{23} - \lambda_{03}) + \text{polynomial in } \lambda_{01}, \lambda_{11}, \lambda_{21}, \lambda_{31}.\end{aligned}$$

We have

$$\delta_2|K[\underline{\lambda}]^{L_1} = (2\bar{\lambda}_{11} - \bar{\lambda}_{21})\frac{\partial}{\partial\bar{\lambda}_{13}} + (\bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21})\frac{\partial}{\partial\bar{\lambda}_{23}} + (2\bar{\lambda}_{13} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2\bar{\lambda}_{31})\frac{\partial}{\partial\bar{\lambda}_{15}}$$

$$\delta_4|K[\underline{\lambda}]^{L_1} = (\bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21})\frac{\partial}{\partial\bar{\lambda}_{15}}$$

$$\delta_6|K[\underline{\lambda}]^{L_1} = 0.$$

Let $\bar{W}_{\Gamma, \underline{r}} = \varphi(W_{\Gamma, \underline{r}})$, $\varphi : W_{\Gamma} \rightarrow W_{\Gamma}/L_1 = \text{Spec } K[\bar{\lambda}]^{L_1}$ be the quotient map and $L_o = K\delta_2 + K\delta_4$. Then

$$\begin{aligned}\bar{W}_{\Gamma, (3,5,6)} &= \{\bar{\lambda} \mid \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} \neq 0\} \\ \bar{W}_{\Gamma, (3,4,5)} &= \{\bar{\lambda} \mid \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} = 0, 2\bar{\lambda}_{11} - \bar{\lambda}_{21} \neq 0\} \\ \bar{W}_{\Gamma, (2,4,5)} &= \{\bar{\lambda} \mid \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} = 2\bar{\lambda}_{11} - \bar{\lambda}_{21} = 0, 2\bar{\lambda}_{13} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2\bar{\lambda}_{31} \neq 0\}, \\ \bar{W}_{\Gamma, (2,3,4)} &= \{\bar{\lambda} \mid \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} = 2\bar{\lambda}_{11} - \bar{\lambda}_{21} = 2\bar{\lambda}_{13} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2\bar{\lambda}_{31} = 0\} \\ W_{\Gamma, (3,5,6)}/L &= \bar{W}_{\Gamma, (3,5,6)}/L_o \\ &= \text{Spec } K[\bar{\lambda}_{11}, \bar{\lambda}_{21}, \bar{\lambda}_{31}, \bar{\lambda}_{13}(\bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21}) - \bar{\lambda}_{23}(2\bar{\lambda}_{11} - \bar{\lambda}_{21})]_g \\ g &= \bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21}. \\ W_{\Gamma, (3,4,5)}/L &= \bar{W}_{\Gamma, (3,4,5)}/L_o \\ &= \text{Spec } K[\bar{\lambda}_{11}, \bar{\lambda}_{21}, \bar{\lambda}_{23}, \bar{\lambda}_{15}(2\bar{\lambda}_{11} - \bar{\lambda}_{21}) - \bar{\lambda}_{13}(2\bar{\lambda}_{13} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2 \\ &\quad (\bar{\lambda}_{11} + \bar{\lambda}_{21})) + \bar{\lambda}_{13}^2]_h \\ h &= 2\bar{\lambda}_{11} - \bar{\lambda}_{21} \\ W_{\Gamma, (2,4,5)}/L &= \bar{W}_{\Gamma, (2,4,5)}/L_o \\ &= \text{Spec } K[\bar{\lambda}_{21}, \bar{\lambda}_{13}, \bar{\lambda}_{23}]_{2\bar{\lambda}_{13} - \bar{\lambda}_{23} - 3\bar{\lambda}_{21}^3} \\ W_{\Gamma, (2,3,4)}/L &= \bar{W}_{\Gamma, (2,3,4)} \\ &= \text{Spec } K[\bar{\lambda}_{21}, \bar{\lambda}_{23}, \bar{\lambda}_{15}].\end{aligned}$$

Remark 2.7 Let $V := W_{\Gamma, (2,4,5)} \cup W_{\Gamma, (3,4,5)}$.

Then $V = \{\underline{\lambda} \in W_{\Gamma} \mid \text{orbit dimension at } \lambda \text{ is } 5\}$ and even $\Gamma_{\underline{\lambda}} = \{0, 2, 4, 8, 9, \dots\}$ is constant. The geometric quotient V/L does not exist, neither in the algebraic nor in the analytic category.

The proof of the remark can be reduced to example 2.2 of [GP 1] because in our case we have to study the action of the vector field $(2\bar{\lambda}_{11} - \bar{\lambda}_{21})\frac{\partial}{\partial\bar{\lambda}_{13}} + (2\bar{\lambda}_{13} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2(\bar{\lambda}_{11} + \bar{\lambda}_{21}))\frac{\partial}{\partial\bar{\lambda}_{15}}$ on the open set in $\text{Spec } K[\bar{\lambda}_{11}, \bar{\lambda}_{21}, \bar{\lambda}_{13}, \bar{\lambda}_{23}, \bar{\lambda}_{15}]$ defined by the ideal $(2\bar{\lambda}_{11} - \bar{\lambda}_{21}, 2\bar{\lambda}_{13} - \bar{\lambda}_{23} - \bar{\lambda}_{11}^2(\bar{\lambda}_{11} + \bar{\lambda}_{21}))$.

(2) Let $\Gamma = \Gamma_0 = \langle 0, 3, 6, 9, 10, \dots \rangle$, $c = 9, k = 2$.

The coordinates of the Schubert cell are

$$\begin{bmatrix} 1 & \lambda_{01} & \lambda_{02} & 0 & \lambda_{04} & \lambda_{05} & 0 & \lambda_{07} & \lambda_{08} \\ 0 & 0 & 0 & 1 & \lambda_{11} & \lambda_{12} & 0 & \lambda_{14} & \lambda_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_{21} & \lambda_{22} \end{bmatrix}$$

$$I = \{(0, 1), (1, 1), (2, 1), (0, 2), (1, 2), (2, 2), (0, 4), (1, 4), (0, 5), (1, 5), (0, 7), (0, 8)\}.$$

The coefficient matrix of the vector field is

$$\begin{pmatrix} 1 & 1 & 1 & * & * & * & * & * & * & * & * & * \\ & & & 1 & 1 & 1 & * & * & * & * & * & * \\ & & & & & & \lambda_{01} - \lambda_{11} & \lambda_{11} - \lambda_{21} & \lambda_{02} - \lambda_{12} & \lambda_{12} - \lambda_{22} & * & * \\ & & & & & & 1 & 1 & \lambda_{01} & \lambda_{11} & * & * \\ & & & & & & & & 1 & 1 & * & * \\ & & & & & & & & & & * & * \\ & & & & & & & & & & * & * \\ & & & & & & & & & & 1 & * \\ & & & & & & & & & & & 1 \end{pmatrix}.$$

We have $m(\Gamma_0) = 4, d = 8$.

$W_{\Gamma, \mathbf{r}} \neq \emptyset$ iff $\mathbf{r} \in \{(5, 7), (4, 6)\}$.

$$\begin{aligned} W_{\Gamma, (5,7)} &= \{\lambda \in W_{\Gamma} \mid \text{rank} \begin{pmatrix} \lambda_{01} - \lambda_{11} & \lambda_{11} - \lambda_{21} & \lambda_{02} - \lambda_{12} & \lambda_{12} - \lambda_{22} \\ 1 & 1 & \lambda_{01} & \lambda_{11} \\ 0 & 0 & 1 & 1 \end{pmatrix} = 3\} \\ &= \{\lambda \in W_{\Gamma} \text{ with maximal orbit dimension}\}. \end{aligned}$$

$$W_{\Gamma, (4,6)} = W_{\Gamma} \setminus W_{\Gamma, (5,7)} = \{\lambda \in W_{\Gamma} \text{ with minimal orbit dimension}\}.$$

In this example the stratification is the stratification by orbit dimension.

Notice that not all 3-minors of the matrix defining $W_{\Gamma, (5,7)}$ are invariant under the group action but $W_{\Gamma, (5,7)}$ itself is.

3 A moduli space for modules with fixed Γ - and E -invariant.

3.1 δ - and Γ -constant families

In this section K denotes an arbitrary field.

For the construction of moduli spaces we have to fix certain invariants of modules.

In this section we consider for $M \in \mathbf{Mod}(R)$ the coarse invariants

$$\delta(M) = \dim_K(\bar{M}/M), \quad \bar{M} = M \otimes_R \bar{R}/\text{torsion} \text{ and}$$

$$\Gamma(M) = \{\gamma \in \mathbb{N} \mid \exists m \in M \text{ s.t. } m \in \bar{M}_{\gamma} \setminus \bar{M}_{\gamma+1}\}$$

where $\bar{M}_n := M \otimes_R \bar{R}_n/\text{torsion}$ and $\bar{R}_n = \{r \in \bar{R} \mid v(r) \geq n\}$, v the valuation on $\bar{R} = K\langle t \rangle$ given by the subdegree of power series in t . We call $\delta(M)$ the δ -invariant and $\Gamma(M)$ the Γ -invariant of M .

Let $M_n := M + \bar{M}_n$ and $Gr_n M = M_n/M_{n+1}$. We see at once that $Gr_n M$ is either 0 or 1-dimensional and that

$$\Gamma(M) = \{n \in \mathbb{N} \mid Gr_n M = 0\}.$$

If $M \subset \bar{R}$ we have defined the **value set** and the **multiplicity** of M in \bar{R} as $v(M) = \{v(m) \mid m \in M\}$ and $mult(M) = \min v(M)$.

It is easy to see that $\Gamma(M) = \{v(m) - mult(M) \mid m \in M\} = \{\gamma_0, \gamma_1, \dots, \gamma_k, c, c+1, \dots\}$, where $0 = \gamma_0 < \gamma_1 < \dots < \gamma_k < c$, c the multiplicity of the conductor. In particular, $\delta(M) = \#(\mathbb{N} - \Gamma(M)) = c - 1 - k$.

Note that if R is the local ring of a complex **plane** curve singularity, then the embedded topological type of R determines and is determined by the semigroup $\Gamma(R)$. One might consider $\Gamma(M)$ as something like the “topological type of M ”.

Let S be a base space as in 1.4 and $\mathcal{M} \in \mathbf{Mod}(R/S)$ a family of R -modules over S . If S is reduced, we call \mathcal{M} a δ -constant resp. Γ -constant family if $\delta(\mathcal{M}(s))$ resp. $\Gamma(\mathcal{M}(s))$ is independent of $s \in S$. Note that $\mathcal{M}(s)$ is a module over $R(s) \subset \bar{R}(s) = \kappa(s)\langle t \rangle$ and the definitions of δ and Γ for $\mathcal{M}(s)$ refer to the residue field $\kappa(s)$. Of course, Γ -constant implies δ -constant.

For a not necessarily reduced base space S consider the injection $\mathcal{M} \hookrightarrow \mathcal{M} \otimes_{R_S} Q_S$ and define the **normalization of the family \mathcal{M}** as

$$\bar{\mathcal{M}} := \mathcal{M} \otimes_{R_S} \bar{R}_S / t - \text{torsion}$$

where $t - \text{torsion}$ denotes the submodule of elements which are annihilated by some power of t . Note that $t^c \bar{\mathcal{M}} \subset \mathcal{M}$.

Lemma 3.1 *Let A be a reduced noetherian local K -algebra with residue field K and \mathcal{M} a deformation of $M \in \mathbf{Mod}(R)$ over A . The following are equivalent*

- (i) $\delta(\mathcal{M}(s))$ is independent of $s \in S = \text{Spec } A$,
- (ii) $\bar{\mathcal{M}} \otimes_A K$ is torsion free, i.e. $\bar{\mathcal{M}} \otimes_A K = \bar{M}$,
- (ii)' $\bar{\mathcal{M}}$ is a deformation of \bar{M} ,
- (iii) $\bar{\mathcal{M}} \cong \bar{R}_A$.

The equivalence of (ii), (ii)' and (iii) holds, even if A is not reduced.

Proof: By proposition 1.14(2) there exists an injection $\mathcal{M} \hookrightarrow \bar{R}_A$ such that \bar{R}_A/\mathcal{M} is locally free of finite rank d . Tensoring with \bar{R}_A and dividing out the t -torsion we obtain an exact sequence

$$0 \rightarrow \bar{\mathcal{M}} \rightarrow \bar{R}_A \rightarrow \bar{R}_A/\bar{\mathcal{M}} \rightarrow 0$$

where $\bar{R}_A/\bar{\mathcal{M}}$ is finitely generated over A . Since A is reduced, $\bar{R}_A/\bar{\mathcal{M}}$ is generically free over S . Since $\dim_{\kappa(s)}(\bar{R}_A/\bar{\mathcal{M}})(s) = d - \delta(\mathcal{M}(s))$ by lemma 1.1, (i) \Leftrightarrow (ii) follows. (ii)' is just a reformulation of (ii) and (iii) \Rightarrow (ii) is obvious. (ii) implies that $\bar{\mathcal{M}}$ is a deformation of \bar{M} as \bar{R} -module. Since such deformations are trivial we get (iii).

Remark 3.2 1. Let A be reduced and noetherian but not necessarily local. Let $\mathfrak{m} \subset A$ be a maximal ideal, $A/\mathfrak{m} = K$ and assume that for $\mathcal{M} \in \mathbf{Mod}(R/S)$, $S = \text{Spec } A$, $\delta(\mathcal{M}(p))$ is constant for all points p in a neighbourhood of \mathfrak{m} . We claim that there exists a neighbourhood U of 0 such that $\bar{\mathcal{M}}|_U \cong \bar{R}_S|_U$. This is not just an obvious consequence of lemma 3.1 since the stalk $\bar{R}_{S,\mathfrak{m}}$ is not equal to $\bar{R}_{A,\mathfrak{m}}$. To prove the claim, we notice that for a non-closed point, $p \in \mathfrak{m}$ the fibres $\mathcal{M}_p(p)$ and $(\mathcal{M}_p \otimes_{R_{S,p}} R_{A,\mathfrak{m}})(p)$ are the same. Applying lemma 3.1 to the latter module, we get that $\overline{\mathcal{M}_m \otimes_{R_{S,m}} R_{A,\mathfrak{m}}} = \bar{\mathcal{M}}_m \otimes_{R_{S,m}} R_{A,\mathfrak{m}}$ is isomorphic to $\bar{R}_{A,\mathfrak{m}}$. Consider a minimal free presentation of \mathcal{M}_m as $\bar{R}_{S,m}$ -module. Applying $- \otimes_{R_{S,m}} R_{A,\mathfrak{m}}$ we get a minimal free presentation of $\bar{\mathcal{M}}_m \otimes_{R_{S,m}} R_{A,\mathfrak{m}}$ as $\bar{R}_{A,\mathfrak{m}}$ -module, since $\bar{R}_{A,\mathfrak{m}}$ is flat over $\bar{R}_{S,m}$ (appendix). Necessarily $\bar{\mathcal{M}}_m \cong \bar{R}_{S,m} = \varinjlim_{f \notin \mathfrak{m}} (A_f[[t]])$ and this proves the claim.

2. It follows from the proof of the lemma and the fact that $\dim_{\kappa(s)}(\bar{R}_S/\bar{\mathcal{M}})(s)$ is upper semicontinuous, that $\delta(\mathcal{M}(s))$ is lower semicontinuous, i.e. the set where δ is smaller than a given value is closed in S for any reduced base space.

For any base space S and $\mathcal{M} \in \mathbf{Mod}(R/S)$ define

$$\begin{aligned} \bar{\mathcal{M}}_n &:= \mathcal{M} \otimes_{R_S} (\bar{R}_n)_S / t - \text{torsion}, \\ \mathcal{M}_n &:= \mathcal{M} + \bar{\mathcal{M}}_n, \\ \text{Gr}_n \mathcal{M} &:= \mathcal{M}_n / \mathcal{M}_{n+1}, \end{aligned}$$

where we consider \mathcal{M} as a submodule of $\bar{\mathcal{M}}$ via $\mathcal{M} \hookrightarrow \bar{\mathcal{M}}$.

Lemma 3.3 *Let $\mathcal{M} \in \mathbf{Mod}(R/S)$, S reduced and connected and $\delta(\mathcal{M}(s))$ independent of $s \in S$. The following are equivalent:*

- (i) $\Gamma(\mathcal{M}(s))$ is independent of $s \in S$,
- (ii) $\text{Gr}_n \mathcal{M}$ is flat over $S \forall n$.

Proof: By the remark 3.2 we may assume that $S = \text{Spec}A$ is local and by lemma 3.1 that $\bar{\mathcal{M}} = \bar{R}_A$ and hence $\bar{\mathcal{M}}_n = (\bar{R}_n)_A = A\langle t^n, t^{n+1}, \dots \rangle$. Now $Gr_n\mathcal{M}$ is flat over A if and only if it is free over A . Since $(Gr_n\mathcal{M})(s) = Gr_n(\mathcal{M}(s))$ we have

$$\Gamma(\mathcal{M}(s)) = \{n \mid Gr_n\mathcal{M}(s) = 0\},$$

and since $Gr_n\mathcal{M}$ is either of rank 0 or of rank 1 over A , the result follows.

We extend the notion of δ -constant and Γ -constant families to possibly non-reduced base spaces by the following

Definition: Let S be a base space and $\mathcal{M} \in \mathbf{Mod}(R/S)$. \mathcal{M} is called **δ -constant** resp. **Γ -constant** if each point $s \in S$ has a neighbourhood $U \subset S$ such that $\mathcal{M}|_U \cong \bar{R}_U$ resp. if \mathcal{M} is δ -constant and $Gr_n\mathcal{M}$ is flat over S for all n . Let us denote by Def^δ resp. Def^Γ the subfunctor of Def defined by δ -constant resp. Γ -constant deformations.

Now fix d , $0 \leq d \leq \delta$, let $I = \mathfrak{c} = t^c\bar{R}$ be the conductor ideal and consider the universal family $\mathcal{U}(R; I, d)$ of I -sandwiched R -modules of colength d over $M(R; I, d)$, defined in section 1.5. Let

$$M^\delta(R; I, d) := \{\lambda \in M(R; I, d) \mid 0 \in v(M_\lambda)\}$$

which is open in $M(R; I, d)$ and let $\mathcal{U}^\delta(R; I, d)$ be the restriction of $\mathcal{U}(R; I, d)$ to $M^\delta(R; I, d)$. Recall that $M_\lambda = \pi^{-1}(\lambda)$ where $\pi : \bar{R} \rightarrow \bar{R}/I$ is the residue map.

Proposition 3.4 (i) $\mathcal{U}^\delta(R; I, d)$ over $M^\delta(R; I, d)$ represents the functor of I -sandwiched, δ -constant families of colength d with δ -invariant equal to d .

(ii) Let $M \in \mathbf{Mod}(R)$, $\delta(M) = d$ and \mathcal{M} a δ -constant family over S , $s \in S$ such that $\mathcal{M}(s) = M$. There exists a neighbourhood U of s in S and a morphism $\varphi : U \rightarrow M^\delta(R; I, d)$ such that $\mathcal{M}|_U \cong \varphi^*\mathcal{U}^\delta(R; I, d)$.

(iii) Assume moreover that $I \subset M \subset \bar{R}$. The natural transformation of functors

$$Def_{I \subset M \subset \bar{R}}^\delta \rightarrow Def_M^\delta$$

is smooth.

In order to have short notation for the statement (ii) we say that $\mathcal{U}^\delta(R; I, d)$ is **locally versal** for δ -constant families (and similar for other invariants instead of δ).

The proposition says that $M^\delta(R; I, d)$ is an algebraic representative of a versal δ -constant deformation for each $M \in \mathbf{Mod}(R)$, $\delta(M) = d$. Since any versal deformation is smooth over the semiuniversal deformation, we get (i) of the following

Corollary 3.5 (i) For any $M \in \mathbf{Mod}(R)$, there exists a semiuniversal algebraic δ -constant deformation of M .

(ii) If $R = R_c = K\langle t^c, t^{c+1}, \dots \rangle$ then, for any $M \in \mathbf{Mod}(R)$, the base space of the semiuniversal δ -constant deformation of M is smooth.

Proof: (ii): Since $I = t^c \bar{R} = \mathfrak{m}_{R_c}$, any I -sandwiched family of vector spaces is already a family of R_c -modules. Hence, $M^\delta(R; I, d)$ is an open subset of the Grassmannian $Gr(\bar{R}/I, d)$ which is smooth.

Proof of the proposition: (i) follows from proposition 1.10 and lemma 1.1 (i).

(ii) We may assume that $S = \text{Spec} A$, A a noetherian K -algebra, $\kappa(s) = K$ and we may also assume (after shrinking S) that $\mathcal{M} \subset \bar{R}_A = \bar{\mathcal{M}}$ by remark 3.2, hence $t^c \bar{R}_A \subset \mathcal{M}$. This proves that \mathcal{M} is I -sandwiched of colength d and the claim follows from (i).

(iii) Let A be a noetherian local K -algebra with residue field K and \bar{A} a factor ring of A . Let

$$I_{\bar{A}} \subset \mathcal{M}_{\bar{A}} \subset \bar{R}_{\bar{A}}$$

be an I -sandwiched δ -constant deformation of M over \bar{A} and \mathcal{M}_A a δ -constant deformation of M over A such that $\mathcal{M}_A \otimes_A \bar{A} = \mathcal{M}_{\bar{A}}$. By lemma 3.1 there exist isomorphisms $i_{\bar{A}} : \bar{\mathcal{M}}_{\bar{A}} \xrightarrow{\cong} \bar{R}_{\bar{A}}$ and $i_A : \bar{\mathcal{M}}_A \xrightarrow{\cong} \bar{R}_A$. $(i_A \otimes_A \bar{A}) \circ (i_{\bar{A}})^{-1}$ is an $R_{\bar{A}}$ -isomorphism of $\bar{R}_{\bar{A}}$ hence equal to the multiplication with a unit of $\bar{R}_{\bar{A}}$. But this can be lifted to a unit u of \bar{R}_A and we obtain a commutative diagram

$$\begin{array}{ccc} \bar{\mathcal{M}}_A & \xrightarrow[u^{-1}i_A]{\cong} & \bar{R}_A \\ \downarrow & & \downarrow \\ \bar{\mathcal{M}}_{\bar{A}} & \xrightarrow[i_{\bar{A}}]{\cong} & \bar{R}_{\bar{A}}. \end{array}$$

Since $t^c \bar{\mathcal{M}}_A \subset \mathcal{M}_A$, \mathcal{M}_A is I -sandwiched and the digramme

$$\begin{array}{ccccc} I_A & \subset & \mathcal{M}_A & \xrightarrow[u^{-1}i_A]{\hookrightarrow} & \bar{R}_A \\ \downarrow & & \downarrow & & \downarrow \\ I_{\bar{A}} & \subset & \mathcal{M}_{\bar{A}} & \xrightarrow[i_{\bar{A}}]{\hookrightarrow} & \bar{R}_{\bar{A}} \end{array}$$

commutes. This had to be shown.

Again let $I = t^c \bar{R}$. Fix a subset of \mathbb{Z} ,

$$\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_k, c, c+1, \dots\}$$

such that $0 \leq \gamma_0 < \gamma_1 < \dots < \gamma_k < c$ and consider the corresponding Schubert cell in $Gr(\bar{R}/I, d)$,

$$W_\Gamma = \{\lambda \in Gr(\bar{R}/I, d) \mid v(M_\lambda) = \Gamma\}$$

where $d = c - (k + 1)$ (with respect to the basis $1, t, t^2, \dots, t^{c-1}$ of \bar{R}/I). Let

$$M_\Gamma(R; I, d) := W_\Gamma \cap M(R; I, d) \text{ and}$$

$$\mathcal{U}_\Gamma(R; I, d) := i_\Gamma^* \mathcal{U}(R; I, d)$$

where $i_\Gamma : M_\Gamma(R; I, d) \hookrightarrow Gr(\bar{R}/I, d)$ is the inclusion map. Note that $\lambda \in M_\Gamma(R; I, d) \Leftrightarrow \lambda \in M(R; I, d)$ and $\Gamma(M_\lambda) = \Gamma - \gamma_0$; hence $\Gamma(M_\lambda) = \Gamma$ implies $\gamma_0 = 0$.

Proposition 3.6 *Proposition 3.4 and corollary 3.5 remain true if we replace “ δ ” by “ Γ ”, “ δ -invariant equal to d ” by “ Γ -invariant equal to Γ ” and “ $\delta(M) = d$ ” by “ $\Gamma(M) = \Gamma$ ”.*

Proof: Since Γ -constant implies δ -constant, we have only to check that the scheme structure on $M_\Gamma(R; I, d)$ is defined by the flattening stratification of $M(R; I, d)$ with respect to $Gr_n \mathcal{U}(R; I, d)$, $n \geq 0$ (set theoretically this is clear). By the universal property of the fixed point scheme, it suffices that $\{W_\Gamma\}_\Gamma$ is the flattening stratification of $Gr(\bar{R}/I, d)$ with respect to the sheaves

$$\mathcal{U}_n / \mathcal{U}_{n+1}, \quad 0 \leq n \leq c-1,$$

where $\mathcal{U}_n := \mathcal{U}' + (t^n \bar{R}/I) \otimes_K \mathcal{O}_{Gr(\bar{R}/I, d)}$ (cf. 1.5).

Consider an affine chart U of $Gr(\bar{R}/I, d)$ of the form

$$\begin{pmatrix} 0 & \gamma_0 & & \gamma_1 & & \gamma_k & & c-1 \\ * \dots * & 1 & * \dots * & 0 & & 0 & * \dots * \\ * \dots * & 0 & * \dots * & 1 & & 0 & * \dots * \\ * \dots * & 0 & * \dots * & 0 & & 0 & * \dots * \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * \dots * & 0 & * \dots * & 0 & & 0 & * \dots * \\ * \dots * & 0 & * \dots * & 0 & & 1 & * \dots * \end{pmatrix}$$

where the *’s are the variables of the chart and the rows w_i of the matrix are a basis of the (trivial) bundle $\mathcal{U}'|_{U \subset \bar{R}/I} \otimes_K \mathcal{O}_U = \sum_{i=0}^{c-1} t^i \mathcal{O}_U$. W_Γ is the subspace of this chart, defined by putting the first $\gamma_i - 1$ *’s in the $(i + 1)$ st row

equal to zero. This is the same as requiring that the rank of the module spanned by the first n columns increases by 1 exactly for $n \in \Gamma$, i.e. $\mathcal{U}_n = \mathcal{U}_{n+1}$ iff $n \in \Gamma$, $0 \leq n \leq c-1$. This proves the claim since W_Γ as well as the flattening stratum of $\{\mathcal{U}_n/\mathcal{U}_{n+1} \mid 0 \leq n \leq c-1\}$ which corresponds to Γ are defined by the same subminors of the above matrix.

The remaining statements follow by the universal property of flattening stratifications, since the formation of $Gr_n \mathcal{M}$ commutes with base change and since W_Γ is smooth.

Remark: Let $\varphi : T \rightarrow S$ be a morphism of base spaces and \mathcal{M} a δ -constant family over S . Then $\varphi^* \bar{\mathcal{M}} = \overline{(\varphi^* \mathcal{M})}$ and $\varphi^* \mathcal{M}$ is also δ -constant. Moreover, if \mathcal{M} is Γ -constant then so is $\varphi^* \mathcal{M}$.

3.2 E -constant families

We introduce now a more subtle invariant for modules $M \in \mathbf{Mod}(R)$. As in the definition of Γ we use the filtration induced by that of \bar{R} , but in a more essential way. For any two modules N_1, N_2 over some ring A with decreasing filtrations $\{F^i N_1\}_{i \in \mathbb{Z}}$ let

$$Hom_A^{[n]}(N_1, N_2) := \{\varphi \in Hom_A(N_1, N_2) \mid \varphi(F^i N_1) \subset F^{i+n} N_2 \forall i\}.$$

We consider M as a submodule of \bar{M} which is filtered by $\bar{M}_n = M \otimes \bar{R}_n / \text{torsion}$ and define on M resp. \bar{M}/M the submodule resp. quotient filtration, i.e.

$$\begin{aligned} F^i M &:= M \cap \bar{M}_i, \\ F^i \bar{M}/M &:= \bar{M}_i + M/M. \end{aligned}$$

$Hom_R^{[n]}(M, \bar{M}/M)$ is defined with respect to these filtrations. We have a canonical map $\bar{R} = Hom_R(M, \bar{M}) \rightarrow Hom_R(M, \bar{M}/M)$ which factors through $Hom_R^{[0]}(M, \bar{M}/M)$ and hence a morphism

$$End^n(M) := Ker(Hom_R(M, \bar{M}) \rightarrow Hom_R^{[0]}(M, \bar{M}/M) / Hom_R^{[n]}(M, \bar{M}/M)).$$

By lemma 1.12 (ii) $Hom_R(M, \bar{M}/M) = Hom_R(M/t^c \bar{M}, \bar{M}/M)$. Moreover, since R -linear maps inject into K -linear maps we see that

$$End^n(M) = Ker(\beta_n), \text{ where}$$

$$\beta_n : Hom_R(M, \bar{M}) \rightarrow Hom_K^{[0]}(M/t^c \bar{M}, \bar{M}/M) / Hom_K^{[n]}(M/t^c \bar{M}, \bar{M}/M).$$

This remark will be essential when we consider families. Another definition of $End^n(M)$ is the following: let

$$M_{n,i} := M \cap \bar{M}_i + \bar{M}_{n+i}, \quad i, n \geq 0,$$

then

$$\begin{aligned} \text{End}^n(M) &:= \{r \in \bar{R} \mid rM_{n,i} \subset M_{n,i} \forall i \geq 0\} \\ &= \bigcap_{i \geq 0} \text{End}_R(M_{n,i}). \end{aligned}$$

We obtain a filtration

$$\bar{R} = \text{End}^0(M) \supset \text{End}^1(M) \supset \dots \supset \text{End}^c(M) = \text{End}_R(M).$$

By lemma 3.7 below and lemma 2.2 we have $\text{End}^n(M) = \text{End}^{n+1}(M)$ if $n \in \Gamma_\circ$.

If we set

$$\text{Aut}^n(M) := \{g \in J \mid gM_{n,i} = M_{n,i}\}$$

where $J = 1 + t\bar{R}/t^c\bar{R}$ is the Jordan group, we have $\text{Lie}(J)/\text{Lie}(\text{Aut}^n(M)) = t\bar{R}/t\text{End}^n(M) \cong \bar{R}/\text{End}^n(M)$. Lemma 2.3 of chapter 2 implies

Lemma 3.7 *Let $M(R; I, d)$ be the module variety of I -sandwiched R -modules of colength d defined in §1.3, $\lambda \in M(R; I, d)$ and M_λ the corresponding module. Then*

$$R(\lambda)(n) = \dim_K \bar{R}/\text{End}^n(M_\lambda).$$

Let

$$\Gamma_0(M) := \{n \in \mathbb{N} \mid n + \Gamma(M) \subset \Gamma(M)\}$$

be the maximal semigroup contained in $\Gamma(M)$ and

$$a(M) := \text{mult}(\Gamma_0(M) \setminus \{0\}).$$

We define for $n \geq 0$

$$E_M(n) := \dim_K \bar{R}/\text{End}^{na(M)}(M),$$

i.e. we evaluate the function $n \mapsto \dim_K \bar{R}/\text{End}^n(M)$ only at the multiples of $a(M)$. We call the function $E_M : \mathbb{N} \rightarrow \mathbb{N}$ the **E -invariant** of M ; it is the Hilbert function of $\bar{R}/\text{End}(M)$ with respect to the filtration $\{\text{End}^{na(M)}/\text{End}(M)\}_n$.

E_M can be computed by

$$\dim_K \bar{R}/\text{End}^n(M) = \#(\mathbb{N} \setminus \Gamma(\text{End}^n M))$$

which is the δ -invariant of the ring $End^n(M)$. By lemma 2.2 and 3.7 we have $E_M(1) = a - 1$, and, of course, $E_M(0) = 0$.

Any family $\mathcal{M} \in \mathbf{Mod}(R/S)$ may be considered, via the canonical map $\mathcal{M} \hookrightarrow \mathcal{M} \otimes_{R_S} Q_S$, as an R_S -submodule of $\mathcal{M} \otimes_{R_S} Q_S$ which is a Q_S -module. We define $\mathcal{M} : \mathcal{M}$ to be the subsheaf of Q_S defined by the presheaf

$$\mathcal{M} : \mathcal{M}(U) = \{q \in Q_S(U) \mid q\mathcal{M}|_U \subset \mathcal{M}|_U\},$$

$U \subset S$ open. $\mathcal{M} : \mathcal{M}$ is in a canonical way a subsheaf of $End_{R_S}(\mathcal{M})$.

Lemma 3.8 (i) *If \mathcal{M} can be sandwiched, then $\mathcal{M} : \mathcal{M} = End_{R_S}(\mathcal{M})$. Moreover, for any embedding $i : \mathcal{M} \hookrightarrow \mathcal{M} \otimes_{R_S} Q_S \cong Q_S$, we have $\mathcal{M} : \mathcal{M} = i\mathcal{M} : i\mathcal{M}$.*

(ii) *If \mathcal{M} is a δ -constant family, then $\mathcal{M} : \mathcal{M} \subset \bar{R}_S$ and $Aut_{R_S}(\mathcal{M}) \subset (\bar{R}_S)^*$. If \mathcal{M} is δ -constant, we usually identify $End_{R_S}(\mathcal{M})$ with $\mathcal{M} : \mathcal{M} \subset \bar{R}_S$.*

Proof: (i) By remark 1.11, $\mathcal{M} \otimes_{R_S} Q_S \cong Q_S$. Since any endomorphism φ of \mathcal{M} extends to an R_S -linear endomorphism $\tilde{\varphi}$ of Q_S , it is given by multiplication with $\tilde{\varphi}(1)$. The second statement of (i) follows since two embeddings of \mathcal{M} in Q_S differ by multiplication with a unit of Q_S .

(ii) Any endomorphism φ of \mathcal{M} extends to an endomorphism of $\bar{\mathcal{M}}$ which is locally isomorphic to \bar{R}_S and by (i) φ determines a unique endomorphism $\bar{\varphi}$ of \bar{R}_S . Any endomorphism of \bar{R}_S is multiplication with $\bar{\varphi}(1)$. If φ is an automorphism, then $\bar{\varphi}$ too and the global section $\bar{\varphi}(1)$ of \bar{R}_S is a unit in $\bar{R}_{S,s}$ for each $s \in \bar{R}$.

Let $\mathcal{M} \in \mathbf{Mod}(R/S)$. We consider \mathcal{M} , $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}}_i$ as submodules of $\mathcal{M} \otimes_{R_S} Q_S$ and define

$$\mathcal{M}_{n,i} := \mathcal{M} \cap \bar{\mathcal{M}}_i + \bar{\mathcal{M}}_{n+i}$$

as a submodule of $\mathcal{M} \otimes_{R_S} Q_S$. moreover, we put

$$(\mathcal{M} : \mathcal{M})^n := \bigcap_{i \geq 0} (\mathcal{M}_{n,i} : \mathcal{M}_{n,i})$$

where the intersection is taken in Q_S . If \mathcal{M} is a δ -constant family, we have canonically

$$(\mathcal{M}_{n,i} : \mathcal{M}_{n,i}) = End(\mathcal{M}_{n,i}) \subset \bar{R}_S$$

and hence

$$End^n(\mathcal{M}) := (\mathcal{M} : \mathcal{M})^n = \bigcap_{i \geq 0} End(\mathcal{M}_{n,i}) \subset \bar{R}_S.$$

As in the absolute case we define on \mathcal{M} , $\mathcal{M}/t^c\bar{\mathcal{M}}$, $\bar{\mathcal{M}}/\mathcal{M}$ the submodule resp. quotient filtrations induced from the filtration $\bar{\mathcal{M}}_i$ of $\bar{\mathcal{M}}$. If \mathcal{M} is δ -constant, then $\bar{\mathcal{M}}$ is locally on S isomorphic to \bar{R}_S and hence $\text{Hom}_{R_S}(\mathcal{M}, \bar{\mathcal{M}})$ is locally isomorphic to \bar{R}_S (cf. 3.13), i.e. $\text{Hom}_{R_S}(\mathcal{M}, \bar{\mathcal{M}}) = \text{Hom}_{R_S}^{[0]}(\mathcal{M}, \bar{\mathcal{M}})$. Now it follows easily that for δ -constant \mathcal{M} ,

$$\text{End}^n(\mathcal{M}) = \text{Ker}(\delta_n) \text{ where}$$

$$\delta_n : \text{Hom}_{R_S}(\mathcal{M}, \bar{\mathcal{M}}) \rightarrow \text{Hom}[0]_S(\mathcal{M}/t^c\bar{\mathcal{M}}, \bar{\mathcal{M}}/\mathcal{M}) / \text{Hom}_S^{[n]}(\mathcal{M}/t^c\bar{\mathcal{M}}, \bar{\mathcal{M}}/\mathcal{M}).$$

Note that Hom_S denotes \mathcal{O}_S -linear maps.

Definition: Let S be a base space and $\mathcal{M} \in \mathbf{Mod}(R/S)$ a Γ -constant family. Let $a = \text{mult}(\Gamma_0 - \{0\})$. We call \mathcal{M} an **E -constant family** if $\bar{R}_S/\text{End}^{na}(\mathcal{M})$ is flat over S for all $n \geq 0$ and if the formation of $\text{End}^{na}(\mathcal{M})$ is compatible with base change, i.e. if $\varphi^*\text{End}^{na}(\mathcal{M}) = \text{End}^{na}(\varphi^*\mathcal{M})$ for each morphism $\varphi : T \rightarrow S$ of base spaces.

Of course, the notion “ E -constant” is compatible with base change.

Remark 3.9 1. Let $\mathcal{M} \in \mathbf{Mod}(R/S)$ be δ -constant. It follows from the definition that \mathcal{M} is Γ -constant iff $(\mathcal{M} + \bar{\mathcal{M}}_i)/\bar{\mathcal{M}}_i$ is flat over S for all $i \geq 0$. From the exact sequence

$$0 \rightarrow \mathcal{M} \cap \bar{\mathcal{M}}_i \rightarrow \mathcal{M} \rightarrow (\mathcal{M} + \bar{\mathcal{M}}_i)/\bar{\mathcal{M}}_i \rightarrow 0$$

follows that \mathcal{M} is Γ -constant iff $\mathcal{M} \cap \bar{\mathcal{M}}_i$ is flat over S and $\varphi^*(\mathcal{M} \cap \bar{\mathcal{M}}_i) = (\varphi^*\mathcal{M}) \cap (\varphi^*\bar{\mathcal{M}}_i)$ for all i and each base change $\varphi : T \rightarrow S$. Hence \mathcal{M} is Γ -constant iff $\mathcal{M}_{n,i}$ is flat and $\varphi^*(\mathcal{M}_{n,i}) = (\varphi^*\mathcal{M})_{n,i}$ for all n, i, φ .

2. **Claim:** If $\mathcal{M} \in \mathbf{Mod}(R/S)$ is a Γ -constant family then $\text{Hom}_S^{[n]}(\mathcal{M}/t^c\bar{\mathcal{M}}, \bar{\mathcal{M}}/\mathcal{M}) \subset \text{Hom}_S^{[0]}(\mathcal{M}/t^c\bar{\mathcal{M}}, \bar{\mathcal{M}}/\mathcal{M})$ are locally free \mathcal{O}_S -modules and compatible with base change such that the quotient is also locally free and compatible with base change.

Proof: Since the statement is local, we may assume that $S = \text{Spec } A$ is local on $\bar{\mathcal{M}} = \bar{R}_A$. Then $\mathcal{M}/t^c\bar{R}_A$ resp. \bar{R}_A/\mathcal{M} are free over the local ring A of rank say $k+1$ resp. $c-k-1 = \#(\mathbb{N} - \Gamma)$. Since \mathcal{M} is Γ -constant, $F^i\mathcal{M}/t^c\bar{R}_A = \mathcal{M}\hat{t}^i\bar{R}_A/t^c\bar{R}_A$ is a free and split A -submodule of $\mathcal{M}/t^c\bar{R}_A$, see (1). Hence we can find an A -basis $\{m_i\}_{0 \leq i \leq k}$ of $\mathcal{M}/T^c\bar{R}_A$,

$$m_i = t^{\gamma_i} + \sum_{j \geq 0, j+\gamma_i \notin \Gamma} a_{i,j} t^{\gamma_i+j},$$

where $\Gamma = \{\gamma_0, \dots, \gamma_k, c, c+1, \dots\}$, $0 = \gamma_0 \leq \gamma_1 \leq \dots \leq \gamma_k \leq c$, and $a_{i,j} \in A$. \bar{R}_A/\mathcal{M} has a basis $\{t^j\}_{j \notin \Gamma}$. Hence, $\text{Hom}_A(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M})$ is A -free with basis $\{\varphi_{i,j}\}_{0 \leq i \leq k, j \notin \Gamma}$, $\varphi_{i,j}(m_r) = \delta_r^i t^j$ where δ_r^j is the Kronecker symbol. The submodule $\text{Hom}_A^{[0]}(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M})$ is then easily seen to be free with basis $\{\sigma_{i,j}\}_{(i,j) \notin I}$ where $\sigma_{i,j} = \varphi_{i,j+\gamma_i}$ and $I = \{(i,j) \mid 0 \leq i \leq k, 0 \leq j, \varphi_i + j \notin \Gamma\}$. Moreover, $\text{Hom}_A^{[n]}(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M})$ is free with basis $\{\sigma_{i,j}\}_{(i,j) \notin I, j \geq n}$. The compatibility with base change is obvious.

Lemma 3.10 *Let $\mathcal{M} \in \mathbf{Mod}(R/S)$ be a δ -constant family. The following are equivalent*

- (i) $\text{End}_{R_S}(\mathcal{M})$ is compatible with base change and $\bar{R}_S/\text{End}_{R_S}(\mathcal{M})$ is flat over S .
- (ii) $\text{Cok}(\text{Hom}_{R_S}(\mathcal{M}, \bar{\mathcal{M}}) \xrightarrow{\beta} \text{Hom}_S(\mathcal{M}, \bar{\mathcal{M}}/\mathcal{M}))$ is flat over S .
- (iii) $\text{Cok}(\text{Hom}_{R_S}(\mathcal{M}/\bar{\mathcal{M}}) \rightarrow \text{Hom}_S^{[0]}(\mathcal{M}/t^c \bar{\mathcal{M}}/\mathcal{M}))$ is flat over S .

Proof: Since the statements are local, we may assume that $(S, s) = S$ is local and that $\mathcal{M} \subset \bar{R}_S$ and $\bar{\mathcal{M}} = \mathcal{M}\bar{R}_S = \bar{R}_S$. It follows easily that $\text{Hom}_{R_S}(\mathcal{M}, \bar{\mathcal{M}}) = \text{Hom}_{R_S}(\bar{R}_S, \bar{R}_S) = \bar{R}_S$. Letting β be the composition $\beta : \bar{R}_S = \text{Hom}_{R_S}(\mathcal{M}, \bar{R}_S) \rightarrow \text{Hom}_{R_S}(\mathcal{M}, \bar{R}_S/\mathcal{M}) = \text{Hom}_{R_S}(\mathcal{M}/t^c \bar{R}_S, \bar{R}_S/\mathcal{M}) \hookrightarrow \text{Hom}_S(\mathcal{M}/t^c \bar{R}_S, \bar{R}_S/\mathcal{M})$ (the first resp. middle equality are proved in lemma 3.13 resp. in the proof of 3.14 (3)) we get an exact sequence

$$0 \rightarrow \text{End}_{R_S}(\mathcal{M}) \xrightarrow{\alpha} \bar{R}_S \xrightarrow{\beta} \text{Hom}_S(\mathcal{M}/t^c \bar{R}_S, \bar{R}_S/\mathcal{M}) \rightarrow \text{Cok}(\beta) \rightarrow 0$$

Since \bar{R}_S/\mathcal{M} is locally free of finite rank over S we see that the flatness of $\text{Cok}(\beta)$ implies the flatness of $\text{Cok}(\alpha)$ and also the flatness of $\text{End}_{R_S}(\mathcal{M})$. Moreover, if $\varphi : T \rightarrow S$ is any morphism of local base spaces we have $\varphi^* \bar{R}_S = \bar{R}_T$ and $\varphi^* \text{Hom}_S(\mathcal{M}/t^c \bar{R}_S, \bar{R}_S/\mathcal{M}) = \text{Hom}_T(\varphi^* \mathcal{M}/t^c \bar{R}_S, \bar{R}_T/\varphi^* \mathcal{M})$. By the five-lemma we obtain that $\varphi^* \text{End}_{R_S}(\mathcal{M}) = \text{End}_{R_T}(\varphi^* \mathcal{M})$. Conversely, if (i) holds, we have, if \mathfrak{m} denotes the maximal ideal of the local ring of S , $(\text{Im} \beta) \otimes_{\mathcal{O}_{S,s}} \kappa(s) = \text{Im}(\beta \otimes_{\mathcal{O}_{S,s}} \kappa(s)) \hookrightarrow \text{Hom}_{\kappa(s)}(\mathcal{M}(s)/t^c \bar{R}(s), \bar{R}/\mathcal{M}(s))$, i.e. $\text{Im}(\beta)$ and $\text{Cok}(\beta)$ are flat over S by lemma 1.8.

The proof of (i) \Leftrightarrow (iii) is the same as for (i) \Leftrightarrow (ii) since $\text{Hom}_S^{[0]}(\mathcal{M}/t^c \bar{\mathcal{M}}, \bar{\mathcal{M}}/\mathcal{M})$ is locally free over S and compatible with base change by 3.9 (2).

Let $\mathcal{M} \in \mathbf{Mod}(R/S)$ be δ -constant and consider the exact sequence

$$0 \rightarrow \text{End}^n(\mathcal{M}) \rightarrow \text{Hom}_{R_S}(\mathcal{M}, \bar{\mathcal{M}}) \xrightarrow{\delta_n} \text{Hom}_S^{[0]}(\mathcal{M}/t^c \bar{\mathcal{M}}, \bar{\mathcal{M}}/\mathcal{M}) / \text{Hom}_S^{[n]}(\mathcal{M}/t^c \bar{\mathcal{M}}, \bar{\mathcal{M}}/\mathcal{M}).$$

Lemma 3.11 1. Let $\mathcal{M} \in \mathbf{Mod}(R/S)$ be Γ -constant. Then \mathcal{M} is E -constant iff $Cok(\beta_{na})$ is S -flat for all $n \in \{1, \dots, \lfloor \frac{c-1}{a} \rfloor + 1\}$ (equivalently, for all $n \geq 0$).

2. Let $\mathcal{M} \in \mathbf{Mod}(R/S)$, S reduced and connected. Then

(i) there exists an open dense subset of S where \mathcal{M} is E -constant.

(ii) \mathcal{M} is E -constant on S iff the following holds:

- $\Gamma(\mathcal{M}(s))$ is independent of $s \in S$,
- $E_{\mathcal{M}(s)}$ is independent of $s \in S$.

Proof: 1. Since δ_n is a morphism between locally free \mathcal{O}_S -modules which are compatible with base change, this follows as in lemma 3.10.

2. (i) follows from 1 since S is reduced and since flatness and “ Γ -constant” are Zariski-open properties.

The necessity of the condition of (ii) is clear, the sufficiency follows also from the exact sequence above (cf. the proof of lemma 3.10) observing that over a reduced base (ii) implies the freeness of $Im(\beta_{na})$ and hence of $Cok(\beta_{na})$.

Let $I = t^c \bar{R}$, $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_k, c, c+1\} \subset \mathbb{N}$, $d = c - (k+1)$ and consider $\mathcal{U}_\Gamma := \mathcal{U}_\Gamma(R; I, d)$ over $M_\Gamma := M_\Gamma(R; I, d)$, the universal family of Γ -constant I -sandwiched R -modules. As above consider (note that $Hom_{R_{M_\Gamma}}(\mathcal{U}_\Gamma, \bar{\mathcal{U}}_\Gamma) = \bar{R}_{M_\Gamma}$)

$$\beta_n : \bar{R}_{M_\Gamma} \rightarrow Hom_{M_\Gamma}^{[0]}(\mathcal{U}'_\Gamma, \bar{\mathcal{U}}_\Gamma/\mathcal{U}_\Gamma) / Hom_{M_\Gamma}^{[n]}(\mathcal{U}'_\Gamma, \bar{\mathcal{U}}_\Gamma/\mathcal{U}_\Gamma)$$

where $\mathcal{U}'_\Gamma = \mathcal{U}_\Gamma/t^c \bar{\mathcal{U}}_\Gamma$ and the flattening stratification of $Cok(\beta_{na})$ for all $n = 2, \dots, \lfloor \frac{c-1}{a} \rfloor + 1$, $a = mult(\Gamma_0 \setminus \{0\})$. By 3.11, the underlying set of each stratum is defined by the condition that $E_{\mathcal{U}_\Gamma(\lambda)}$ is independent of λ , but we endow each stratum with the scheme structure defined by the flattening property.

For any nonincreasing function $E : \mathbb{N} \rightarrow \mathbb{N}$ define

$$M_{\Gamma, E} := M_{\Gamma, E}(R; I, d)$$

as the **stratum of the flattening stratification** of $Cok(\beta_{na})$ on which $E_{\mathcal{U}_\Gamma(\lambda)} = E$. $\mathcal{U}_{\Gamma, E} := \mathcal{U}_{\Gamma, E}(R; I, d)$ denotes the restriction of \mathcal{U}_Γ to $M_{\Gamma, E}$.

Proposition 3.12 Let R be an irreducible reduced curve singularity, $I = t^c \bar{R}$. Then

- (i) $\mathcal{U}_{\Gamma, E}(R; I, d) \in \mathbf{Mod}(R/M_{\Gamma, E}(R; I, d))$ represents the functor of I -sandwiched E -constant families of colength d with δ -invariant equal to d , Γ -invariant equal to Γ and E -invariant equal to E ,

(ii) $\mathcal{U}_{\Gamma,E}(R; I, d)$ is locally versal for E -constant families,

(iii) let $I \subset M \subset \bar{R}$ and denote by Def^E the subfunctor of Def defined by E -constant deformations. Then $\text{Def}_{I \subset M \subset \bar{R}}^E \rightarrow \text{Def}_M^E$ is smooth.

Proof: This follows from proposition 3.6, the preceding discussion and the universal property of flattening stratifications.

3.3 Kodaira-Spencer map

In this section we interpret the action of the Jordan group in terms of the Kodaira-Spencer map.

Any Schubert cell $W_\Gamma \subset \text{Gr}(\bar{R}/t^c\bar{R}, d)$, $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_k, c, c+1, \dots\}$, $d = \#(\mathbb{N} - \Gamma)$, $0 = \gamma_0 < \gamma_1 < \dots < \gamma_k < c$, contains a special point $\lambda = 0$ which is the origin with respect to the coordinates defined in chapter 2.2. The corresponding module

$$M_0 = \sum_{i=0}^k t^{\gamma_i} K + t^c \bar{R}$$

is called the **monomial module** with value set Γ . M_0 is a module over any monomial ring $R' = \sum_{\gamma \in \Gamma'} t^\gamma K + t^c \bar{R}$, where Γ' is a semigroup which acts on Γ . We have two canonical choices for Γ' , the minimal semigroup $\Gamma_c = \{0, c, c+1, \dots\}$ which leads to the ring $R_c = K\langle t^c, t^{c+1}, \dots \rangle$ and the maximal semigroup $\Gamma_0 = \{n \in \mathbb{N} \mid n + \Gamma \subset \Gamma\}$ which was introduced in 2.4. Note that Γ_0 is just the semigroup of the ring $\text{End}(M_0)$ and that $a = \text{mult}(\Gamma_0 - \{0\})$ was used in the definition of the E -invariant.

We shall now consider M_0 as $R = R_c$ -module and consider the modules corresponding to other points of W_Γ as deformations of M_0 . In proposition 3.6 we proved that M_0 has a versal Γ -constant deformation $\mathcal{M} = \mathcal{U}_\Gamma(R_c; t^c \bar{R}, d)$ with base space W_Γ . This family can be described explicitly as follows (cf. 2.2):

$$\begin{aligned} A &= K[\underline{\lambda}], \quad \underline{\lambda} = \{\lambda_{ij}\}_{(i,j) \in I}, \\ I &= \{(i, j) \mid 0 \leq i \leq k, j > 0, j + \gamma_i \notin \Gamma\}, \\ \mathcal{M} &= \sum_{i=0}^k m_i R_A + t^c \bar{R}_A, \quad m_i = t^{\gamma_i} + \sum_{j+\gamma_i \notin \Gamma} \lambda_{ij} t^{j+\gamma_i} \end{aligned}$$

We want to study the **Kodaira-Spencer map**

$$\rho : \text{Der}_K A \rightarrow \text{Ext}_{R_A}^1(\mathcal{M}, \mathcal{M})$$

of the family \mathcal{M} of R_A -modules.

Using the exact sequence $\mathcal{M} \xrightarrow{i} \bar{R}_A \xrightarrow{\pi} \bar{R}_A/\mathcal{M}$ the Kodaira-Spencer map ρ of the family \mathcal{M} factors through the Kodaira-Spencer map

$$\rho_e : Der_K A \rightarrow Hom_{R_A}(\mathcal{M}, \bar{R}_A/\mathcal{M}), \quad \rho_e(\delta)(m) = \text{class of } \delta(m)$$

of the embedded family $\mathcal{M} \subset \bar{R}_A$, where $\delta \in Der_K(A)$ is lifted to $\bar{R}_A = A\langle t \rangle$ by $\delta(t) = 0$. There is an exact sequence

$$Hom_{R_A}(\mathcal{M}, \bar{R}_A) \xrightarrow{\pi_*} Hom_{R_A}(\mathcal{M}, \bar{R}_A/\mathcal{M}) \xrightarrow{\partial} Ext_{R_A}^1(\mathcal{M}, \mathcal{M})$$

and we have $\rho = \partial \circ \rho_e$.

Lemma 3.13 $Hom_{R_A}(\mathcal{M}, \bar{R}_A) = \bar{R}_A$, $t^c \bar{R}_A \subseteq Ker \pi_*$.

Proof: Let $\varphi : \mathcal{M} \rightarrow \bar{R}_A$, $m \in \mathcal{M}$, then

$$t^{2c} \varphi(m) = \varphi(t^{2c} m) = t^c m \cdot \varphi(t^c)$$

because t^{2c} , $t^c m \in R_A$. This implies $\varphi(m) = t^{-c} \varphi(t^c) \cdot m$. On the other hand m_0 is a unit in \bar{R}_A , i.e. $t^{-c} \varphi(t^c) = m_0^{-1} \varphi(m_0) \in \bar{R}_A$, and φ is multiplication with $m_0^{-1} \varphi(m_0)$. Obviously for $r \in \bar{R}_A$ the map $m \mapsto r \cdot m$ is in $Hom_{R_A}(\mathcal{M}, \bar{R}_A)$.

Lemma 3.14 1. ρ_e is injective

2. $Ker \partial \subseteq Im \rho_e$

3. $Hom_A(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M}) = Hom_{R_A}(\mathcal{M}, \bar{R}_A/\mathcal{M})$

Proof: 1. \bar{R}_A/\mathcal{M} is a free A -module generated by $\{t^j\}_{j \notin \Gamma}$. On the other hand if $\delta \in Der_K A$ then $\rho_e(\delta)(m_i) = \sum_{j+\gamma_i \notin \Gamma} \delta(\lambda_{ij}) t^{j+\gamma_i}$, i.e. $\rho_e(\delta) = 0$ iff $\delta(\lambda_{ij}) = 0$ for

$(i, j) \in I$, i.e. $\delta = 0$.

2. $Ker \partial = Im \pi_*$ and we have to prove that $Im \pi_* \subseteq Im \rho_e$. We know by the previous lemma that $Hom_{R_A}(\mathcal{M}, \bar{R}_A) = \bar{R}_A$. Let $r \in \bar{R}_A$ and $\sigma_r : \mathcal{M} \rightarrow \bar{R}_A$ defined by $\sigma_r(m) = rm$. Let class $(\sigma_r(m_i)) = \sum_{j+\gamma_i \notin \Gamma} h_{ij} t^{j+\gamma_i} \in \bar{R}_A/\mathcal{M}$ then

$$\delta_{\sigma_r} := \sum_{(i,j) \in I} h_{ij} \frac{\partial}{\partial \lambda_{ij}} \in Der_K A \text{ and } \rho_e(\delta_{\sigma_r}) = \pi_*(\sigma_r).$$

3. Let $\varphi \in Hom_{R_A}(\mathcal{M}, \bar{R}_A/\mathcal{M})$ then $\varphi(t^c r) = \varphi(t^c r m_0^{-1} m_0) = t^c r m_0^{-1} \varphi(m_0) = 0$ since $t^c \bar{R}_A \subset \mathcal{M}$. Moreover, since $R = R_c$ we have $Hom_{R_A}(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M}) = Hom_A(\mathcal{M}/t^c \bar{R}_A, \bar{R}/\mathcal{M})$. Moreover, since $R = R_c$ we have $Hom_{R_A}(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M}) = Hom_A(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M})$.

Using lemma 3.14 and lemma 3.15 we obtain the following commutative diagram:

$$\begin{array}{ccc}
\bar{R}_A & \xrightarrow{\pi_*} & \text{Hom}_{R_A}(\mathcal{M}, \bar{R}_A/\mathcal{M}) \\
p \downarrow & & \parallel \\
\bar{R}_A/t^c \bar{R}_A & \xrightarrow{\bar{\pi}_*} & \text{Hom}_A(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M}) \\
\uparrow & & \uparrow \rho_0 \\
t\bar{R}_A/t^c \bar{R}_A & \xrightarrow{\varphi_*} & \text{Der}_K A
\end{array}$$

Because $\rho = \partial \circ \rho_e$ and since ρ_e is injective we obtain $\mathcal{L} := \text{Ker} \rho = \text{Im} \varphi_*$, i.e. the kernel of the Kodaira-Spencer map consists of all vector fields which correspond to the multiplication with elements from $t\bar{R}/t^c \bar{R}_A$ in $\text{Hom}_A(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M})$. Especially, \mathcal{L} is generated as A -module by $\{\varphi_*(t^i)\}_{i=1, \dots, c-1}$.

We want to apply now the results from chapter 2. In order to use the same notations let $V = \bar{R}/t^c \bar{R}$ and $e_i = t^i$, $i = 0, \dots, c-1$. Then $W_\Gamma = \text{Spec} A$, $\bar{R}_A/t^c \bar{R}_A \simeq \mathcal{V}$, $\mathcal{M}/t^c \bar{R}_A \simeq \mathcal{U}_\Gamma$ and the action of the Jordan group J is the canonical action of the multiplicative group $1 + t\bar{R}/t^c \bar{R}$ by multiplication on $\bar{R}/t^c \bar{R}$. With this identification we have $\text{Lie}(J) = t\bar{R}/t^c \bar{R}$.

Let $L \subseteq \text{Der}_K A$ be the image of the Lie algebra $\text{Lie}(J)$ of the Jordan group acting on $W_\Gamma = \text{Spec} A$ under the corresponding representation and $\delta_\ell = \sum_{(i,j) \in I} h_{\ell,ij} \frac{\partial}{\partial \lambda_{ij}}$, $h_{\ell,ij} = \lambda_{i,j-\ell} - \sum_{\nu=i+1}^k \lambda_{i,\gamma_\nu - \gamma_i - \ell} \lambda_{\nu,j+\gamma_i - \gamma_\nu}$ its generators as described in chapter 2.

Proposition 3.15 1. $\varphi_*(t^i) = \delta_i$,

$$2. AL = \sum_{\ell=1}^{c-1} A\delta_\ell = \mathcal{L}.$$

Proof: (2) is a consequence of (1) because $\{\varphi_*(t^i)\}$ generate \mathcal{L} by the previous proof. (1) is a consequence of the fact that the action of $\text{Lie}(J)$ on the global sections of the tangent-bundle of the Grassmannian restricted to W_Γ , $\text{Hom}_A(\mathcal{U}_\Gamma, \mathcal{V}_\Gamma/\mathcal{U}_\Gamma)$, factors over the global sections of the tangent bundle of W_Γ , $\text{Der}_K A$:

$$\begin{array}{ccc}
\text{Lie}(J) \otimes_K A & \rightarrow & \text{Hom}_A(\mathcal{U}_\Gamma, \mathcal{V}_\Gamma/\mathcal{U}_\Gamma) \\
\varphi_* \searrow & & \nearrow \\
& & \text{Der}_K A.
\end{array}$$

In our situation

$$\begin{array}{ccc}
t\bar{R}_A/t^c \bar{R}_A & \rightarrow & \text{Hom}_A(\mathcal{M}/t^c \bar{R}_A, \bar{R}_A/\mathcal{M}) \\
\searrow & & \nearrow \\
& & \text{Der}_K A
\end{array}$$

is just the corresponding diagram.

Note that $t\bar{R}_A/t^c\bar{R}_A$ resp. $\mathcal{M}/t^c\bar{R}_A$ resp. \bar{R}_A/\mathcal{M} are A -free with basis $\{t^\ell\}_{1 \leq \ell \leq c-1}$ resp. $\{\mathbf{m}_i\}_{0 \leq i \leq k}$ resp. $\{t^j\}_{j \notin \Gamma}$. Hence $\text{Hom}_A(\mathcal{M}/t^c\bar{R}_A, \bar{R}/\mathcal{M})$ is A -free with basis $\{\varphi_{i,j}\}_{0 \leq i \leq k, j \notin \Gamma}$, $\varphi_{i,j}(\mathbf{m}_r) = \delta_r^i \cdot t^j$. The submodule $\text{Hom}_A^F(\mathcal{M}/t^c\bar{R}_A, \bar{R}/\mathcal{M})$ consisting of those homomorphisms which respect the filtration induced from the valuation of \bar{R} is A -free with basis $\{\sigma_{i,j}\}_{(i,j) \in I}$ where $\sigma_{i,j} = \varphi_{i,j+\gamma_i}$. With respect to the bases $\{t^\ell\}_{1 \leq \ell \leq c-1}$ of $t^c\bar{R}_A$ and the basis $\{\sigma_{i,j}\}_{(i,j) \in I}$ constructed in remark 3.9(2) the map $t\bar{R}_A/t^c\bar{R}_A \rightarrow \text{Hom}_A^{[0]}(\mathcal{M}/t^c\bar{R}_A, \bar{R}_A/\mathcal{M})$ is given by the transpose of the matrix $(h_{\ell,i,j})$ considered in chapter 2.

This gives an invariant description of the fundamental matrix $(h_{\ell,i,j})$ in terms of the Kodaira-Spencer map. Moreover, $\{\sigma_{i,j}\}_{(i,j) \in I_n}$, $I_n = \{(i,j) \in I \mid j < n\}$, is a basis of $\text{Hom}_A^{[0]}(\mathcal{M}/t^c\bar{R}_A, \bar{R}_A/\mathcal{M})/\text{Hom}_A^{[n]}(\mathcal{M}/t^c\bar{R}_A, \bar{R}_A/\mathcal{M})$ and the matrix $(h_{\ell,i,j})_{1 \leq \ell \leq c-1, (i,j) \in I_n}$ describes the morphisms $\bar{\beta}_n$ from $t\bar{R}_A/t^c\bar{R}_A$ to this module, i.e. $\bar{\beta}_n$ is the restriction of beta_n (described after lemma 3.8) restricted to $t\bar{R}_A/t^c\bar{R}_A$. Note that β_n and $\bar{\beta}_n$ have the same cokernel.

Now let R denote the local ring of an arbitrary reduced curve singularity, $M_\Gamma = M_\Gamma(R; I, d) = W_\Gamma \cap M(R; I, d)$ the universal base of Γ -constant I -sandwiched families and $\mathcal{U}_\Gamma = \mathcal{U}_\Gamma(R; I, d)$ the universal family on $M_\Gamma(R; I, d)$ (cf. chapter 3.1). Using the notations of the beginning of this section we have $W_\Gamma = \text{Spec}(A)$ and $\mathcal{U}_\Gamma = i^*\mathcal{M}$ where $i : M_\Gamma \hookrightarrow W_\Gamma$ is the inclusion. Let B denote the affine coordinate ring of M_Γ . Since the actions of J and $(R/I)^*$ on W_Γ commute, and since M_Γ is just the fixed point scheme $W_\Gamma^{(R/I)^*}$, J acts on M_Γ . We have a commutative diagram

$$\begin{array}{ccccccc} t\bar{R}_A/t^c\bar{R}_A & \xrightarrow{\varphi_*} & \text{Der}_K(A) & \xrightarrow{\rho_*} & \text{Hom}_{R_{cA}}(\mathcal{M}, \bar{R}_A/\mathcal{M}) & \xrightarrow{\partial} & \text{Ext}_{R_{cA}}^1(\mathcal{M}, \mathcal{M}) \\ \downarrow & & \cup \downarrow & & \downarrow & & \downarrow \\ t\bar{R}_B/t^c\bar{R}_B & \xrightarrow{\varphi_*^B} & \text{Der}_K(B) & \xrightarrow{\rho_*^B} & \text{Hom}_{R_B}(\mathcal{U}_\Gamma, \bar{R}_B/\mathcal{U}_\Gamma) & \xrightarrow{\partial^B} & \text{Ext}_{R_B}^1(\mathcal{U}_\Gamma, \mathcal{U}_\Gamma) \end{array}$$

where $\rho^B = \partial^B \circ \rho_e^B$ is the Kodaira-Spencer map of $\mathcal{U}_\Gamma \in \mathbf{Mod}(R/M_\Gamma)$. Hence

$$\mathcal{L}^B := \text{kern } \rho^B = \text{Im } \varphi_*^B,$$

i.e. \mathcal{L}^B is generated as B -module by $\{\varphi_*^B(t^i)\}_{i=1, \dots, c-1}$.

We call $L^B \subset \text{Der}_K(B)$ the image of $\text{Lie}(J)$ under this representation. As in proposition 3.15 we obtain that L^B generates \mathcal{L}^B as B -module. Consequently we get

Proposition 3.16 *The action of the kernel \mathcal{L}^B of the Kodaira-Spencer map ρ_B of the universal Γ -constant family $\mathcal{U}_\Gamma(R; I, d)$ over $M_\Gamma(R; I, d)$ and the action of*

the Jordan group J on $M_\Gamma(R; I, d)$ have the same orbits. In particular, two modules of the family \mathcal{U}_Γ are isomorphic iff the corresponding points of M_Γ belong to the same integral manifold of $\text{Ker}(\rho_B)$.

3.4 Construction of the moduli space

In this chapter we construct a coarse moduli space for all modules $M \in \mathbf{Mod}(R)$ with fixed Γ -invariant and fixed E -invariant.

Consider the two functors $F_{\Gamma, E}$ and $F_{\Gamma, E}^I$ from the category of base spaces to the category of sets, where $F_{\Gamma, E}(S)$ (resp. $F_{\Gamma, E}^I(S)$) denotes the set of isomorphism classes of families (resp. of I -sandwiched families, $I = t^c \bar{R}$) over S with Γ -invariant equal to Γ and E -invariant equal to E .

In 3.2 we constructed the family $\mathcal{U}_{\Gamma, E} = \mathcal{U}_{\Gamma, E}(R; I, d)$ over $M_{\Gamma, E} = M_{\Gamma, E}(R; I, d)$, $d = \delta(M_\lambda) \forall \lambda \in M_{\Gamma, E}$. Proposition 3.12 says that $M_{\Gamma, E}$, together with $\mathcal{U}_{\Gamma, E}$, is a fine moduli space for the functor $F_{\Gamma, E}^I$. The functor $F_{\Gamma, E}$ does not have a fine moduli space but it has a coarse moduli space as we shall see. Let J be the Jordan group acting on $M_{\Gamma, E}$ (cf. chapter 2).

Remark 3.17 $M_{\Gamma, E}(R; I, d) = W_{\Gamma, E} \cap M(R; I, d) = W_{\Gamma, E}^{(R/I)^*}$ where $W_{\Gamma, E} = M_{\Gamma, E}(R_c; I, d)$ and $R_c = K\langle t^c, t^{c+1}, \dots \rangle$. Here W_Γ is the Schubert cell of $Gr(\bar{R}/I, d)$ belonging to Γ and we claim that $W_{\Gamma, E} = W_{\Gamma, \bar{E}}$ is a stratum of the algorithmic stratification of W_Γ where $E(n) = r_{n-1} + a - 1$ (cf. theorem 2.6).

Proof: By theorem 2.6 and lemma 3.7 we know already that $W_{\Gamma, E} = W_{\Gamma, \bar{E}}$ set theoretically. We have to prove that they carry the same scheme structure. $W_{\Gamma, E}$ is defined by the flattening stratification of the modules $\text{Cok}(\bar{\beta}_n)$, $\bar{\beta}_n : t\bar{R}_A/t^c\bar{R}_A \rightarrow \text{Hom}_A^{[0]}(\mathcal{U}'_\Gamma, \bar{R}_A/\mathcal{U}_\Gamma)/\text{Hom}_A^{[n]}(\mathcal{U}'_\Gamma, \bar{R}_A/\mathcal{V}_\Gamma)$ and $W_{\Gamma, \bar{E}}$ is defined by the flattening stratification of the modules $\text{Hom}_K(L, A)/\text{Ad}F^n A$ (cf. theorem 2.6), $n \in a\mathbb{Z}$. Now we have the following presentations (cf. after proposition 3.15, proposition 2.6) of these modules ($\mathcal{U}'_\Gamma = \mathcal{U}_\Gamma/t^c\bar{\mathcal{U}}_\Gamma$):

1.

$$\begin{array}{ccccc} t\bar{R}_A/t^c\bar{R}_A & \rightarrow & \text{Hom}_A^{[0]}(\mathcal{U}'_\Gamma, \bar{R}_A/\mathcal{U}_\Gamma)/\text{Hom}_A^{[n]}(\mathcal{U}'_\Gamma, \bar{R}_A/\mathcal{U}_\Gamma) & \rightarrow & \text{Cok}(\beta_n), \\ \uparrow \cong & & & & \uparrow \cong \\ A^{c-1} & & & & A^{\#I_n} \end{array}$$

$I_n = \{(i, j) \in I \mid j < n\}$, with presentation-matrix ${}^t(h_{\ell, ij})_{\ell=1, \dots, c-1, (i, j) \in I_n}$ corresponding to the bases $\{t, \dots, t^{c-1}\}$ resp. $\{\sigma_{ij}\}_{(i, j) \in I_n}$.

2.

$$\begin{array}{ccccc} \Omega_{A|K}^{[n]} & \longrightarrow & \text{Hom}_K(L, A) & \longrightarrow & \text{Hom}_K(L, A)/\text{Ad}F^n A \\ \uparrow \cong & & \uparrow \cong & & \\ A^{\#I_n} & & A^{c-1} & & \end{array}$$

with presentation-matrix $(h_{\ell,ij})_{\ell=1,\dots,c-1,(i,j)\in I_n}$ corresponding to the bases $\{d\lambda_{ij}\}_{(i,j)\in I_n}$ resp. the base induced by $L = \sum_{i=1}^{c-1} K\delta_i$. (Where $\Omega_{A|K}^{[n]} = \sum_{(i,j)\in I_n} \text{Ad}\lambda_{ij} \subseteq \Omega_{A|K}$ and $\Omega_{A|K} \rightarrow \text{Hom}_K(L, A)$ is the canonical map defined by $L \subseteq \text{Der}_K A$.) I.e. the presentation matrices are the transpose of each other which proves $W_{\Gamma,E} = W_{\Gamma,E}$.

Let J be the Jordan group acting on W_Γ and $W_{\Gamma,E}$ (cf. chapter 2). J acts on $M_{\Gamma,E}$, the fixed point scheme of $W_{\Gamma,E}$ under $(R/I)^*$ since the actions of J and $(R/I)^*$ commute.

Theorem 3.18 *Let R be an irreducible, reduced curve singularity,*

$$M_{\Gamma,E} = M_{\Gamma,E}(R; I, d), \quad I = t^e \bar{R}, \quad d = \delta(M_\lambda) \quad \forall \lambda \in M_{\Gamma,E}.$$

The following holds:

1. $M_{\Gamma,E} \rightarrow M_{\Gamma,E}/J$ is a locally trivial geometric quotient.
2. $M_{\Gamma,E}/J$ is a quasiffine algebraic variety of dimension $\dim M_{\Gamma,E} - E([\frac{c-1}{a}] + 1)$
3. $M_{\Gamma,E}/J$ is a coarse moduli space for the functor $F_{\Gamma,E}$.

Proof: It is enough to prove the theorem for $R_c = K\langle t^c, t^{c+1}, \dots \rangle$ since $M_{\Gamma,E}(R; I, d) = M_{\Gamma,E}(R_c; I, d)^{(R/I)^*}$ is closed in $M_{\Gamma,E}(R_c; I, d)$ and invariant under J . Hence (1) and (2) are a consequence of theorem 2.6 and remark 3.17. The dimension statement follows since

$$E([\frac{c-1}{a}] + 1) = \dim_K \bar{R}/\text{End}(M)$$

is equal to the orbit dimension of J .

(3) follows by general principles (cf. [Ne]) from the following three facts:

- $\mathcal{U}_{\Gamma,E}$ is a locally versal family for the functor $F_{\Gamma,E}$ by proposition 3.12,
- isomorphism classes of modules in $\mathcal{U}_{\Gamma,E}$ correspond to orbits of the action of J on $M_{\Gamma,E}$ by lemma 1.1,
- $M_{\Gamma,E}/J$ is a geometric quotient by theorem 2.6.

Remark 3.19 1. The stratum $M_{\Gamma, E_{\max}}$ on which E takes maximal values is open in M_{Γ} . We may call modules belonging to M_{Γ} **stable**. $M_{\Gamma, E_{\max}}$ is contained in the subset of M_{Γ} where the orbit dimension of J is maximal; we have no example where this subset is proper. But for non-maximal E it is definitely not sufficient to fix the orbit dimension in order to get a geometric quotient; see example 2.6.

2. The stratum $M_{\Gamma, E_{\min}}$ where E takes minimal values is closed in M_{Γ} and hence affine. This stratum is described in theorem 2.6(3).

Example: Let $R = K[[t^4, t^9]]$ then $\delta(R) = 12$ and for $M \in \text{Mod}(R)$ we have $0 \leq \delta(M) \leq 12$.

Let us consider modules with $\delta(M) = 4$. We have the following possibilities for $\Gamma(M)$:

$$\begin{aligned} &\langle 0, 1, 4, 5, 6, 8, 9, 10, 12, 13, \dots \rangle \\ &\langle 0, 1, 4, 5, 8, 9, \dots \rangle \\ &\langle 0, 2, 4, 6, 8, 9, \dots \rangle \\ &\langle 0, 3, 4, 5, 7, 8, 9, 11, 12, \dots \rangle \\ &\langle 0, 3, 4, 7, 8, \dots \rangle \\ &\langle 0, 4, 5, 6, 8, 9, \dots \rangle \\ &\langle 0, 4, 5, 7, 8, \dots \rangle \\ &\langle 0, 4, 6, 7, \dots \rangle. \end{aligned}$$

Let $\Gamma = \langle 0, 2, 4, 6, 8, 9, \dots \rangle$ then the coordinates of the corresponding Schubert cell W_{Γ} in $Gr(\bar{R}/t^{24}\bar{R}, 4)$ are

$$M(\lambda) = \begin{pmatrix} 1 & \lambda_{01} & 0 & \lambda_{03} & 0 & \lambda_{05} & 0 & \lambda_{07} & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \lambda_{11} & 0 & \lambda_{13} & 0 & \lambda_{15} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \lambda_{21} & 0 & \lambda_{23} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \lambda_{31} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

This example was essentially treated in 2.6. since we may consider W_{Γ} as a Schubert cell in $Gr(\bar{R}/t^8\bar{R}, 4)$.

Let $M \in \text{Mod}(R)$ and $\Gamma(M) = \Gamma$ and consider the E -invariant $E_M : \mathbb{N} \rightarrow \mathbb{N}$ then $E_M(1) = 1$ and $E_M(n) = E_M(4)$ if $n \geq 4$, i.e. E_M is determined by $(E_M(2), E_M(3), E_M(4))$. In Example 2.6 we obtained $W_{\Gamma, E} \neq \emptyset$ iff $(E(2), E(3), E(4)) \in \{(3, 5, 6), (2, 4, 5), (3, 4, 5), (2, 3, 4)\}$ and gave an explicit description of $W_{\Gamma, E}$.

On the other hand it is not difficult to see that the module variety $M_\Gamma(R; t^{24}\bar{R}, 4) = W_\Gamma^{(R/t^{24}R)^*}$ is the closed subset of W_Γ defined by the following equations: $\lambda_{01} - \lambda_{21} = \lambda_{11} - \lambda_{31} = \lambda_{03} - \lambda_{23} = 0$. This implies (cf. example 2.6):

$$\begin{aligned} M_{\Gamma,(3,5,6)} &= \emptyset \\ M_{\Gamma,(3,4,5)} &= \{\underline{\lambda} \mid \lambda_{01} - \lambda_{21} = \lambda_{11} - \lambda_{31} = \lambda_{03} - \lambda_{23} = 0, \\ &\quad \lambda_{11} - \lambda_{21} \neq 0\} \\ M_{\Gamma,(2,4,5)} &= \{\underline{\lambda} \mid \lambda_{01} - \lambda_{21} = \lambda_{11} - \lambda_{21} = \lambda_{31} - \lambda_{21} = \lambda_{03} - \lambda_{23} = 0, \\ &\quad \lambda_{13} - \lambda_{23} \neq 0\} \\ M_{\Gamma,(2,3,4)} &= \{\underline{\lambda} \mid \lambda_{01} - \lambda_{21} = \lambda_{11} - \lambda_{21} = \lambda_{31} - \lambda_{21} = \lambda_{03} - \lambda_{23} = \lambda_{13} - \lambda_{23} = 0\} \end{aligned}$$

In example 2.6 we obtained the geometric quotients $W_{\Gamma,E} \rightarrow W_{\Gamma,E}/L$ in two steps.

$\bar{W}_\Gamma = W_\Gamma/L_1 = \text{Spec } K[\bar{\lambda}_{11}, \bar{\lambda}_{21}, \bar{\lambda}_{31}, \bar{\lambda}_{13}, \bar{\lambda}_{23}, \bar{\lambda}_{15}]$. $\bar{M}_\Gamma = M_\Gamma/L_1$ is the closed subset defined by $\bar{\lambda}_{11} - \bar{\lambda}_{31} = \bar{\lambda}_{21} = \bar{\lambda}_{23} = 0$.

This implies, because $\bar{\lambda}_{11} - \bar{\lambda}_{31} + \bar{\lambda}_{21} = 0$ in $\bar{W}_{\Gamma,E}$ if $(E(2), E(3), E(4)) \neq (3, 5, 6)$, that $M_{\Gamma,E}/L$ is the closed subset of $W_{\Gamma,E}/L$ defined by $\lambda_{21} = \lambda_{23} = 0$. Especially the moduli space for modules $M \in \mathbf{Mod}(R)$ with $\Gamma(M) = \langle 0, 2, 4, 6, 8, 9, \dots \rangle$ decomposes into the following three smooth affine pieces (corresponding to the three possibilities for E):

$$\begin{aligned} M_{\Gamma,(3,4,5)}/L &\simeq \text{Spec } K[x, y]_x, \\ M_{\Gamma,(2,4,5)}/L &\simeq \text{Spec } K[x]_x, \\ M_{\Gamma,(2,3,4)}/L &\simeq \text{Spec } K[x]. \end{aligned}$$

We can write down explicit families of modules over the quotients:

$$\begin{aligned} \mathcal{M}_{\Gamma,(3,4,5)} &= R_A + t^2(1 + xt + yt^5)R_A + t^8\bar{R}_A, \quad A = K[x, y]_x \\ \mathcal{M}_{\Gamma,(2,4,5)} &= R_A + t^2(1 + xt^3)R_A + t^8\bar{R}_A, \quad A = K[x]_x \\ \mathcal{M}_{\Gamma,(2,3,4)} &= R_A + t^2(1 + xt^5)R_A + t^8\bar{R}_A, \quad A = K[x]. \end{aligned}$$

Giving values for x and y we get all modules with $\Gamma = \langle 0, 2, 4, 6, 8, 9, \dots \rangle$ and for different values the corresponding modules are not isomorphic.

Appendix: Local rings of formal schemes

Let K be a field and R a local, noetherian K -algebra with residue field K . By the structure theorem for complete local rings [Na], its \mathfrak{m}_R -adic completion \widehat{R} is isomorphic to $K[[\underline{x}]]/\mathfrak{i}$ where \mathfrak{i} is a proper ideal in the formal power series ring $K[[\underline{x}]] = K[[x_1, \dots, x_n]]$.

Let A be an arbitrary K -algebra and let $\mathfrak{M} = \mathfrak{m}_R(R \otimes_K A)$. We define

$$R_A := R \widehat{\otimes}_K A := \varprojlim (R \otimes_K A) / \mathfrak{M}^n,$$

the \mathfrak{M} -adic completion of $R \otimes_K A$.

Remark 1: $R_A = \widehat{R}_A \cong A[[\underline{x}]]/\mathfrak{i}A[[\underline{x}]]$. This follows since $\widehat{R} \otimes_K A / \mathfrak{m}^n \widehat{R} \otimes_K A = \widehat{R} / \mathfrak{m}^n \otimes_K A = R / \mathfrak{m}^n \otimes_K A = R \otimes_K A / \mathfrak{M}^n$ and $\widehat{R} / \mathfrak{m}^n \otimes_K A \cong A[[\underline{x}]] / (\mathfrak{i} + (\underline{x})^n)A[[\underline{x}]]$.

There are natural morphisms $R \rightarrow R_A$ and $A \rightarrow R_A$ of K -algebras. R_A is an A -algebra with section, i.e. the composition $A \rightarrow R_A \rightarrow R_A / \mathfrak{m}_R R_A = A$ is the identity.

Any morphism $R \rightarrow S$ of analytic K -algebras lifts to $R_A \rightarrow S_A$. If $S = R/\mathfrak{j}$ for some ideal $\mathfrak{j} \neq R$ we have $S_A = S \otimes_R R_A$. More generally

Lemma 2: *Let $R \rightarrow S$ be a morphism of analytic K -algebras, such that S is a finitely generated R -module. Then, for each K -algebra A , S_A is a finitely generated R_A -module and*

$$S_A = S \otimes_R R_A$$

Proof: By remark 1 we may assume that $R = K[[\underline{x}]]$ and, since S is of finite type over R that S is of the form $R[\underline{y}]/\mathfrak{j}$, $\underline{y} = (y_1, \dots, y_m)$, $y_j^N \in \mathfrak{j}$ for some N . Hence, $K[[\underline{x}]][\underline{y}]/\mathfrak{j} = K[[\underline{x}, \underline{y}]]/\mathfrak{j}K[[\underline{x}, \underline{y}]]$ and

$$\begin{aligned} S_A &= (K[[\underline{x}]][\underline{y}]/\mathfrak{j}) \widehat{\otimes}_K A = (K[[\underline{x}, \underline{y}]]/\mathfrak{j}K[[\underline{x}, \underline{y}]]) \widehat{\otimes}_K A \\ &= A[[\underline{x}, \underline{y}]]/\mathfrak{j}A[[\underline{x}, \underline{y}]] = A[[\underline{x}]][\underline{y}]/\mathfrak{j}A[[\underline{x}]][\underline{y}] \\ &= K[[\underline{x}]][\underline{y}]/\mathfrak{j} \otimes_{K[[\underline{x}]]} A[[\underline{x}]] = S \otimes_R R_A \end{aligned}$$

The latter is certainly finitely generated over R_A .

If M is any R -module, we define

$$M_A := M \widehat{\otimes}_K A := M \otimes_R R_A$$

For any morphism $A \rightarrow B$ of K -algebras and for any R_A -module \mathcal{M} define the **base change**

$$\mathcal{M} \widehat{\otimes}_A B := \mathcal{M} \otimes_{R_A} R_B.$$

The functor $-\widehat{\otimes}_A B$ is right exact and commutes with direct sums. Hence, if \mathcal{M} is finitely generated (resp. presented) over R_A so is $\mathcal{M} \widehat{\otimes}_A B$ over R_B .

Lemma 3: *If $A \rightarrow B$ is a morphism of K -algebras such that B is a finitely generated A -module and if \mathcal{M} is an R_A -module then*

$$R_B = R_A \widehat{\otimes}_A B = R_A \otimes_A B,$$

$$\mathcal{M} \widehat{\otimes}_A B = \mathcal{M} \otimes_A B.$$

Proof: Again we may assume that $R = K[[\underline{x}]]$. Since B is finitely generated over A we may assume the $B = A[\underline{y}]/\mathfrak{b} = A[[\underline{y}]]/\mathfrak{b}A[[\underline{y}]]$. Then $R_A \otimes_A B = A[[\underline{x}]] \otimes_A A[\underline{y}]/\mathfrak{b} = A[[\underline{x}]][\underline{y}]/\mathfrak{b}A[[\underline{x}]][\underline{y}] = A[[\underline{x}, \underline{y}]]/\mathfrak{b}A[[\underline{x}, \underline{y}]] = (A[[\underline{y}]]/\mathfrak{b}A[[\underline{y}]])[[\underline{x}]] = R_B$.

The second statement follows immediately from the first.

Lemma 4: *If A is noetherian, so is R_A .*

Proof: $A[[\underline{x}]]$ is noetherian by [Bo], III.2.10 cor.6 and this is sufficient by remark 1.

From now on assume A to be noetherian.

Lemma 5: *$\mathfrak{m}_R R_A$ is contained in the Jacobson radical of R_A , hence, the pair $(R_A, \mathfrak{m}_R R_A)$ is a Zariski ring.*

Proof: It is clear that for each $a \in (\mathfrak{x})A[[\underline{x}]]$, $1 + a$ is a unit in $A[[\underline{x}]]$. Therefore $1 + [a]$ is a unit in R_A for each $[a] \in \mathfrak{m}_R R_A$. This proves the lemma.

Let $\mathfrak{p} \subset A$ be a prime ideal and $A_{\mathfrak{p}}$ the localization at \mathfrak{p} . The algebra $R_{A_{\mathfrak{p}}} = R \widehat{\otimes}_K A_{\mathfrak{p}} = A_{\mathfrak{p}}[[\underline{x}]]/iA_{\mathfrak{p}}[[\underline{x}]]$ is local with maximal ideal $(\mathfrak{p} + \mathfrak{m}_R)R_{A_{\mathfrak{p}}}$. $(\mathfrak{p} + \mathfrak{m}_R)R_A$ is a prime ideal (resp. maximal ideal if \mathfrak{p} is maximal) of R_A and any maximal ideal \mathfrak{q} of R_A is of this form. Consider the localization of R_A with respect to $(\mathfrak{p} + \mathfrak{m}_R)R_A$, $R_{A,(\mathfrak{p} + \mathfrak{m}_R)}$. We have an inclusion $R_{A,(\mathfrak{p} + \mathfrak{m}_R)} \subset R_{A_{\mathfrak{p}}}$ which is strict in general (e.g. $\sum_{\nu \geq 0} y^{-\nu} x^{\nu} \in K[y]_{(y)}[[x]]$ but $\notin (K[y]_{(y)})_{(x,y)}$).

Let $S = \text{Spec } A$ and $X_S = \text{Spec}(R \otimes_K A)$ which is a scheme over S and as such has a canonical section, defined by the ideal \mathfrak{M} , which we identify with S . Let

\mathcal{X}_S denote **the formal completion of X_S along S** , i.e. it is the ringed space $(S, \mathcal{O}_{\mathcal{X}_S})$ where the topological space is S and

$$\mathcal{O}_{\mathcal{X}_S} = \varprojlim \mathcal{O}_{X_S} / \mathfrak{M}^n \mathcal{O}_{X_S}$$

(cf. [Ha], II.9). For any affine open subset $U = \text{Spec } B \subset S$ we have $R_B = \Gamma(U, \mathcal{O}_{\mathcal{X}_S})$ in particular

$$R_A = \Gamma(S, \mathcal{O}_{\mathcal{X}_S}).$$

Let \mathcal{M} be an R_A -module and $\tilde{\mathcal{M}}$ be the associated sheaf on $\text{Spec } R_A$. We define the sheaf \mathcal{M}^\sim on \mathcal{X}_S to be the completion of $\tilde{\mathcal{M}}$ along S ,

$$\mathcal{M}^\sim := \varprojlim \tilde{\mathcal{M}} / \mathfrak{M}^n \tilde{\mathcal{M}}.$$

Any coherent sheaf on \mathcal{X}_S is by definition locally of this form with \mathcal{M} finitely generated over R_A . By [Ha], II.9.7 we have

- $\Gamma(S, \mathcal{M}^\sim) = \mathcal{M}$ for all \mathcal{M} , finitely generated over R_A ,
- $\Gamma(S, \mathcal{F})^\sim = \mathcal{F}$ for \mathcal{F} coherent on \mathcal{X}_S .

In particular $R_A^\sim = \mathcal{O}_{\mathcal{X}_S}$. If $A \rightarrow B$ is a morphism of K -algebras and $\varphi : T = \text{Spec } B \rightarrow S$ the induced morphism, then

$$\varphi^* \mathcal{M}^\sim := \varphi^{-1} \mathcal{M}^\sim \otimes_{\varphi^{-1} R_A^\sim} R_B^\sim = (\mathcal{M} \otimes_{R_A} R_B)^\sim.$$

Let $\mathfrak{p} \in S$ be a prime ideal of A . We get for the stalk of \mathcal{M}^\sim at \mathfrak{p} ,

$$\mathcal{M}_{\mathfrak{p}}^\sim = \varinjlim_B \mathcal{M} \otimes_{R_A} R_B = \mathcal{M} \otimes_{R_A} R_{A, \mathfrak{p}}^\sim$$

where $\text{Spec } B$ runs through the affine open neighbourhoods of \mathfrak{p} . For the stalk of R_A^\sim at \mathfrak{p} we get

$$R_{A, \mathfrak{p}}^\sim \cong \varinjlim_{f \notin \mathfrak{p}} A_f[[\underline{x}]] / iA_f[[\underline{x}]],$$

which is a local K -algebra.

Let $A_{\mathfrak{p}}$ resp. $R_{A, (\mathfrak{p} + \mathfrak{m}_R)}$ be the localization of A at \mathfrak{p} resp. of R_A at $(\mathfrak{p} + \mathfrak{m}_R)R_A$.

Let $\kappa(\mathfrak{p})$ be the residue field of $A_{\mathfrak{p}}$ and $\hat{A}_{\mathfrak{p}}$ the $\mathfrak{p}A_{\mathfrak{p}}$ -adic completion.

Lemma 6:

- (i) We have natural inclusions of local rings with maximal ideal generated by $\mathfrak{p} + \mathfrak{m}_R$ and residue field $\kappa(\mathfrak{p})$,

$$R \otimes_K A_{\mathfrak{p}} \subset R_{A,(\mathfrak{p}+\mathfrak{m}_R)} \subset R_{A,\mathfrak{p}}^{\sim} \subset R_{A_{\mathfrak{p}}}.$$

- (ii) Each ring is noetherian. Its completion with respect to the maximal ideal is equal to $R_{\hat{A}_{\mathfrak{p}}}$. In particular, each inclusion is faithfully flat.

Proof: (i) is straightforward.

(ii) We have only to show that $R_{A,\mathfrak{p}}^{\sim}$ is noetherian, for the other rings this is obvious or follows from lemma 4. Note that $A_{\mathfrak{p}}[[\underline{x}]]/iA_{\mathfrak{p}}[[\underline{x}]] = \varinjlim A_f[[\underline{x}]]/iA_f[[\underline{x}]]$, $f \notin \mathfrak{p}$, is flat over $A_f[[\underline{x}]]/iA_f[[\underline{x}]]$ by the local criterion of flatness applied to the ideal (\underline{x}) ([Bo], III. 5.2, theorem 1). Hence $R_{A,\mathfrak{p}}^{\sim} \rightarrow R_{A_{\mathfrak{p}}}$ is flat and therefore faithfully flat. This proves that $R_{A,\mathfrak{p}}^{\sim}$ is noetherian.

If we take any of the four rings and divide out the n -th power of the maximal ideal, we see immediately that this is equal to $R_{\hat{A}_{\mathfrak{p}}}/(\mathfrak{p} + \mathfrak{m}_R)^n R_{\hat{A}_{\mathfrak{p}}}$. This shows that all rings have as completion $R_{\hat{A}_{\mathfrak{p}}}$. By the completion criterion ([Bo], III. 5.4, prop. 4) we obtain faithful flatness.

Proposition 7: R_A is faithfully flat over A as well as over R .

Proof: By localization of flatness, ([Bo], II. 3.4, prop. 15), R_A is flat over A iff $R_{A,(\mathfrak{m}+\mathfrak{m}_R)}$ is flat over $A_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset A$. By the completion criterion and by lemma 6 this is equivalent to $R_{\hat{A}_{\mathfrak{m}}}$ being flat over $\hat{A}_{\mathfrak{m}}$. Therefore, we may assume that A is local and complete and hence, by the structure theorem for complete local rings ([Na]), $A \cong K'[[\underline{y}]]/\mathfrak{b}$ where K' is some coefficient field containing K and $\underline{y} = (y_1, \dots, y_m)$. By lemma 2 we may assume that $A = K'[[\underline{y}]]$ and hence $R_A = \bar{K}'[[\underline{y}, \underline{x}]]/iK'[[\underline{y}, \underline{x}]]$. Since $i \subset K[[\underline{x}]]$, it follows that y_1, \dots, y_m is a regular sequence for R_A and hence R_A is A -flat.

For general A and $\mathfrak{m} \subset A$ maximal we have a surjection $R_A/\mathfrak{m}R_A \rightarrow R_A/(\mathfrak{m} + \mathfrak{m}_R)R_A = A/\mathfrak{m} \neq 0$, whereby R_A is faithfully flat over A .

To see the flatness over R we may assume, by lemma 1 and the completion criterion that $R = K[[\underline{x}]]$. But $R_A = A[[\underline{x}]]$ is certainly faithfully flat over $K[[\underline{x}]]$.

Lemma 8: Let M be a finitely generated R -module. Then M_A is flat over A .

Proof: As in the proof of the preceding lemma we may reduce to the case where $A = K'[[\underline{y}]]$ and have to show that y_1, \dots, y_m is a regular sequence for M_A . But tensoring the exact sequence

$$0 \rightarrow R_{K'[[\underline{y}]]} \xrightarrow{y_m} R_{K'[[\underline{y}]]} \rightarrow R_{K'[[y_1, \dots, y_{m-1}]]} \rightarrow 0$$

over R with M gives again an exact sequence since $\text{Tor}_1^R(M, R_{K'[[y_1, \dots, y_{m-1}]]}) = 0$ by the preceding proposition. The result follows by induction.

Let \mathcal{M} be an R_A -module and \mathfrak{m} a maximal resp. prime ideal of A . In the following two lemmas let $\mathcal{M}_{\mathfrak{m}}$ one of the following modules

$$\mathcal{M} \otimes_{R_A} R_{A,(\mathfrak{m}+\mathfrak{m}_R)}, \mathcal{M} \otimes_{R_A} R_{A,\mathfrak{m}}^{\sim}, \mathcal{M} \otimes_{R_A} R_{A\mathfrak{m}}, \mathcal{M} \otimes_{R_A} R_{\widehat{A}_{\mathfrak{m}}}.$$

(If A is local and \mathfrak{m} the maximal ideal, then the first three modules coincide with \mathcal{M} .)

Lemma 9: *Let \mathcal{M}, \mathcal{N} be finitely generated R_A -modules and $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ R_A -linear. Then φ is injective (surjective, bijective, zero) if and only if $\varphi_{\mathfrak{m}} : \mathcal{M}_{\mathfrak{m}} \rightarrow \mathcal{N}_{\mathfrak{m}}$ is injective (surjective, bijective, zero) for each maximal ideal (equivalently each prime ideal) \mathfrak{m} of A . In particular, a sequence*

$$0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$$

of finitely generated R_A -modules is exact iff the sequence

$$0 \rightarrow \mathcal{M}'_{\mathfrak{m}} \rightarrow \mathcal{M}_{\mathfrak{m}} \rightarrow \mathcal{M}''_{\mathfrak{m}} \rightarrow 0$$

is exact for each maximal (resp. prime) ideal \mathfrak{m} of A .

Proof: The statement follows after localization and completion from lemma 6 and the fact that completion is faithfully flat (cf. [Bo] III. 3.5, cor. 5).

Lemma 10: *Let \mathcal{M} be a finitely generated R_A -module. Then \mathcal{M} is A -flat iff $\mathcal{M}_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$ -flat for all maximal (equivalently, prime) ideals \mathfrak{m} of A .*

Proof: Let $0 \rightarrow \mathcal{N}' \rightarrow \mathcal{N} \rightarrow \mathcal{N}'' \rightarrow 0$ be an exact sequence of finitely generated A -modules. By lemma 10, the sequence of R_A -modules

$$0 \rightarrow \mathcal{N}' \otimes_A \mathcal{M} \rightarrow \mathcal{N} \otimes_A \mathcal{M} \rightarrow \mathcal{N}'' \otimes_A \mathcal{M} \rightarrow 0$$

is exact iff

$$0 \rightarrow (\mathcal{N}' \otimes_A \mathcal{M})_{\mathfrak{m}} \rightarrow (\mathcal{N} \otimes_A \mathcal{M})_{\mathfrak{m}} \rightarrow (\mathcal{N}'' \otimes_A \mathcal{M})_{\mathfrak{m}} \rightarrow 0$$

is exact for all \mathfrak{m} . Since $(\mathcal{N} \otimes_A \mathcal{M})_{\mathfrak{m}} = \mathcal{N} \otimes_A \mathcal{M}_{\mathfrak{m}}$ the result follows.

Remark 11: In the text we write R_S for the sheaf R_A^{\sim} and $R_{S,s}$ for the stalk $R_{A,\mathfrak{m}}^{\sim}$ if $s = \mathfrak{m} \in S = \text{Spec } A$. Then $R_{S,s} \subset R_{\mathcal{O}_{S,s}} = R_{A_{\mathfrak{m}}}$ and it follows from proposition 7 and lemma 6 that $R_{S,s}$ is faithfully flat over $\mathcal{O}_{S,s}$ and over R . Moreover, for

any coherent R_S -sheaf \mathcal{M} , the stalk \mathcal{M}_s is flat over $\mathcal{O}_{S,s}$ iff $\mathcal{M}_s \otimes_{R_{S,s}} R_{\mathcal{O}_{S,s}}$ is flat over $R_{\mathcal{O}_{S,s}}$.

Let us conclude with the following remark concerning constant families of curves.

Remark 12: Let $R = K[[x_1(t), \dots, x_n(t)]] \subset \bar{R} = K[[t]]$ and A any noetherian K -algebra. Then $R_A = A[[x_1(t), \dots, x_n(t)]] \subset \bar{R}_A = A[[t]]$. This follows by applying $\otimes_R R_A$ to $R \subset \bar{R}$ and using lemma 3 and proposition 7. We obtain also $Q_A = A \widehat{\otimes}_K A = K[[t]][t^{-1}] \otimes_R R_A = A[[t]][t^{-1}]$.

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