On moduli spaces of semiquasihomogeneous singularities

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Contents

Introduction

Let $A = \mathbf{C}[[x_1, \ldots, x_n]]/(f)$ be the complete local ring of a hypersurface singularity. A is called **semiquasihomogeneous** with weights $w_1, \ldots, w_n$ if $f = f_0 + f_1,\ f_0$ a quasihomogeneous polynomial defining an isolated singularity and deg $f_0 <$ deg f_1. We assume that $w_1, \ldots, w_n$ are positive integers and let deg always denote the weighted degree, i.e. deg $X^\alpha = w_1\alpha_1 + \cdots + w_n\alpha_n$ for a monomial $X^\alpha = X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n}$. For an arbitrary power series f, deg f denotes the smallest weighted degree of a monomial occurring in f. By definition, all monomials of a quasihomogeneous polynomial have the same degree. The singularity with local ring $A_0 = \mathbf{C}[[x_1, \ldots, x_n]]/(f_0)$ is called the **principal** part of A. If the moduli stratum of A_0 has dimension 0, i.e. the τ–constant stratum in the semiuniversal deformation of A_0 is a reduced point, then A_0 is uniquely determined by the weights. Let $H^i = H^i(\mathbf{C}[[x_1, \ldots, x_n]])$ be the ideal generated by all quasihomogeneous polynomials of degree $\geq iw,\ w := \min\{w_1, \ldots w_n\}$. This (weighted) degree-filtration defines a Hilbert-function $\underline{\tau}$ on the Tjurina algebra of A by

$$\tau_i(A) := \dim_{\mathbf{C}} \mathbf{C}[[x_1, \ldots, x_n]]/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, H^i).$$

We call f or A a **semi Brieskorn** singularity if the principal part is of Brieskorn–Pham type, i.e. $f_0 = x_1^{m_1} + \ldots + x_n^{m_n},\ gcd(m_i, m_j) = 1$ for $i \neq j$. Then f_0 is quasihomogeneous with weight $\underline{w} = (w_1, \ldots, w_n)$, where $w_i = m_1 \cdot \ldots \cdot \hat{m}_i \cdot \ldots \cdot m_n$, and degree $d = m_1 \cdot \ldots \cdot m_n$, the moduli stratum is zero-dimensional and hence f_0 is uniquely determined by its weights (cf. [LP]). We are mainly interested in the classification of such singularities with respect to contact equivalence, i.e. in isomorphism classes of the local algebra A. With respect to this equivalence relation we shall prove:

Theorem *There exists a coarse moduli space $\mathcal{M}_{\underline{w},\underline{\tau}}$ for all semiquasihomogeneous singularities with fixed principal part A_0, weight $\underline{w}$ and Hilbert function $\underline{\tau}$. $\mathcal{M}_{\underline{w},\underline{\tau}}$ is an algebraic variety, locally closed in a weighted projective space.*

We follow the general method to construct such moduli spaces (cf. [LP], [GP 1]):

1. We prove that the versal μ–constant deformation $\tilde{X}_\mu \to \underline{H}_\mu$ of A_0 contains already all isomorphism classes of semiquasihomogeneous singularities with principal part A_0. (If we take the quotient of $\underline{H}_\mu$ by a natural action of the group of d–th roots of unity we obtain already a coarse moduli space with respect to right equivalence.)

2. This family contains analytically trivial subfamilies. They are the integral manifolds of a Lie–algebra V_μ, the kernel of the Kodaira–Spencer map of the family. We prove that two singularities are isomorphic iff they are in one integral manifold of V_μ.

3. The integral manifolds of the (infinite dimensional) Lie–algebra V_μ can be identified with the orbits of a solvable algebraic group G. Now the results of [GP 2]

can be applied. We prove that the stratification $\{\underline{H}_{\mu,\underline{\tau}}\}$ of $\underline{H}_\mu$ by fixing the Hilbert function has the properties required in [GP 2], i.e. $\underline{H}_{\mu,\underline{\tau}} \to \underline{H}_{\mu,\underline{\tau}}/G$ is a geometric quotient and a coarse moduli space of all semiquasihomogeneous singularities with weight $\underline{w}$, Hilbert function $\underline{\tau}$ and principal part A_0.

1 Versal μ–constant deformations and kernel of the Kodaira–Spencer map

In this part we recall some known facts about the versal μ–constant deformation and the kernel of the Kodaira–Spencer map.

Let $f_0 = x_1^{m_1} + \ldots + x_n^{m_n}$, $n \geq 2$, $m_i \geq 2$ and $gcd(m_i, m_j) = 1$ if $i \neq j$.

Let $w_i = m_1 \cdot \ldots \cdot \hat{m}_i \cdot \ldots \cdot m_n$, $i = 1, \ldots, n$ and $d = m_1 \cdot \ldots \cdot m_n$ then f_0 is a quasihomogeneous polynomial with weight $\underline{w} = (w_1, \ldots, w_n)$ of degree d. Let $A_0 = \mathbf{C}[[x]]/(f_0)$, $x = (x_1, \ldots, x_n)$ and consider the deformation functor $Def_{A_0 \to \mathbf{C}}$ which consists of isomorphism classes of deformations of the residue morphism $A_0 \to \mathbf{C}$. Geometrically, an element of $Def_{A_0 \to \mathbf{C}}$ is represented by a "deformation with section" of the singularity defined by f_0 (cf. [Bu]). It is not difficult to see (cf. [LP]) that $Def_{A_0 \to \mathbf{C}}(\mathbf{C}[\epsilon]) = (x)/(f_0 + (x)(\frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n}))$ where (x) denotes the ideal generated by $x_1, \ldots, x_n$. This vector space has a unique monomial base $\{x^\alpha | \alpha \in B\}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $x^\alpha = x^{\alpha_1} \cdot \ldots \cdot x^{\alpha_n}$ where $B = \{\alpha \in \mathbf{N}^n \backslash \{0\} | \alpha_i \leq m_i - 2\} \cup \{(0, \ldots, m_i - 1, 0, \ldots) | i = 1, \ldots, n\}$:

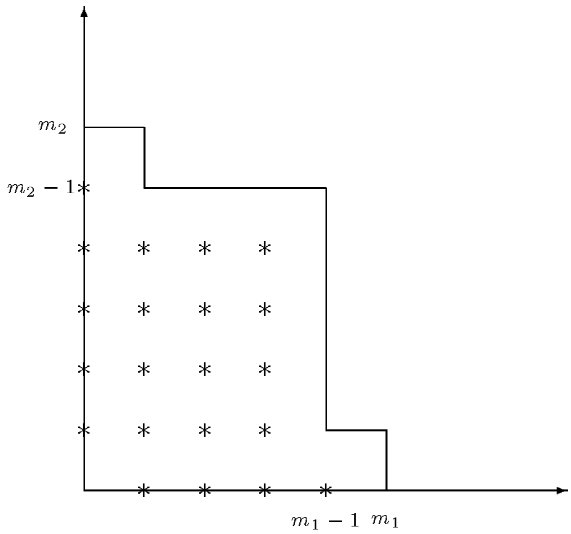

Figure 1: B $(n = 2)$

$Def_{A_0 \to \mathbf{C}}$ has a hull, the semiuniversal deformation, given on the ring level by $H \to H[[x]]/F$ with

$$\begin{aligned} F = F(T) &= f_0 + \sum_{\alpha \in B} T_{d-|\alpha|} x^\alpha \\ H &= \mathbf{C}[T], \end{aligned}$$

$T = (T_{d-|\alpha|})_{\alpha\in B}$ and $|\alpha| = \sum_{i=1}^{n} w_i\alpha_i$ which is by definition the degree of x^α.

Notice that F is quasihomogeneous if we define deg $T_i = i$. We put $\underline{H} := Spec\ H \cong \mathbf{C}^N$, $N = \#B = \prod_{i=1}^{n}(m_i - 1) + n - 1$, the base space of the semiuniversal deformation.

The moduli stratum, i.e. the τ–constant stratum, is the zero point in $\underline{H}$.

Let $Def_{A_0\to\mathbf{C},\mathbf{C}^*}$ denote the functor of $\mathbf{C}^*$-equivariant deformations of $A_0 \to \mathbf{C}$ (cf. [Pi]) and let $Def^{\mu}_{A_0\to\mathbf{C}} = Im(Def_{A_0\to\mathbf{C},\mathbf{C}^*} \to Def_{A_0\to\mathbf{C}})$. $Def^{\mu}_{A_0\to\mathbf{C}}$ gives the μ–constant deformations over a reduced base space. The functor $Def_{A_0\to\mathbf{C},\mathbf{C}^*}$ has a hull, the semiuniversal μ–constant deformation, given by

$$H_\mu \to H_\mu[[x]]/(F_\mu) \text{ with}$$

$$\begin{aligned} F_\mu = F_\mu(T) &= f_0 + \sum_{\alpha\in B_-} T_{d-|\alpha|}x^\alpha \\ H_\mu &= \mathbf{C}[\{T_{d-|\alpha|}\}_{\alpha\in B_-}] \end{aligned}$$

where $B_- = \{\alpha \in B,\ d - |\alpha| < 0\}$:

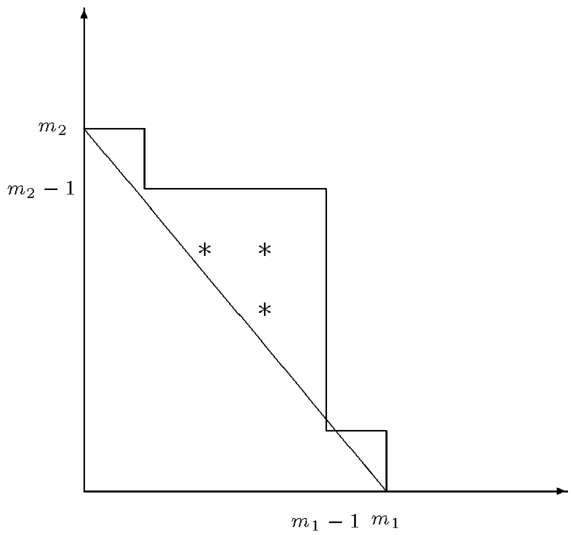

Figure 2: B_- $(n = 2)$

Remark 1.1 (1) The assumption $gcd(m_i, m_j) = 1$ implies that except on the axes there are no extra integral points on the hyperplane $|\alpha| - d = 0$, i.e. f_0 has no moduli. Moreover, it follows that on each hyperplane $|\alpha| = d'$, $\alpha \in B$, there is at most one monomial x^α, hence the elements of B can be numbered by degree which turns out to be very convenient.

(2) For any $t \in \underline{H}_\mu := Spec\, H_\mu$ we have that $F_\mu(t) = f_0 + f_1 \in \mathbf{C}[[x]]$ is semiquasihomogeneous, with principal part f_0. The natural $\mathbf{C}^*$–actions, $c \circ x = (\ldots, c^{w_i} x_i, \ldots)$ and $c \circ t = (\ldots, c^j t_j, \ldots)$, $c = \mathbf{C}^*$, have the property $F_\mu(c \circ t)(c \circ x) = c^d F_\mu(t)(x)$, in particular, $F_\mu(c \circ t)$ and $F_\mu(t)$ are right equivalent if $c^d = 1$.

(3) The action of μ_d on $\underline{H}_\mu - \{0\}$ is faithful since μ_d acts with degree 0 and the T_i have different degrees. This implies: if $\mathcal{X} \to S$ is any μ–constant deformation of $A = \mathbf{C}[[x]]/(f_0 + f_1)$, then there is an open covering $\{\mathcal{U}_i\}$ of S such that $\mathcal{X}|_{\mathcal{U}_i}$ is obtained via some base change $\varphi_i : \mathcal{U}_i \to \underline{H}_\mu$. By the following proposition $\varphi_i \circ \varphi_j^{-1}$ is equal to the $\mathbf{C}^*$–action given by some d–th root of unity c_{ij}. Since μ_d acts faithfully $\{c_{ij}\}$ defines a 1–Čech cocycle of μ_d on S. Hence, if $H^1(S, \mathbf{Z}/d\mathbf{Z}) = 0$, the φ_i can be glued such that $\mathcal{X} \to S$ is globally obtained by some base change $S \to \underline{H}_\mu$.

Proposition 1.2 *1. For any semiquasihomogeneous polynomial $f = f_0 + f_1$ with principal part f_0 there is an automorphism $\varphi \in Aut\, \mathbf{C}[[x]]$ and $t \in \underline{H}_\mu$ such that $\varphi(f) = F_\mu(t)$.*

2. If $F_\mu(t)$ and $F_\mu(t')$ are right equivalent for $t, t' \in \underline{H}_\mu$ then there is a d–th root of unity c, such that $c \circ t = t'$.

Corollary 1.3 *Let μ_d denote the group of d–th roots of unity acting on $\underline{H}_\mu$ as above, then $\underline{H}_\mu/\mu_d$ is a coarse moduli space for semiquasihomogeneous polynomials f with principal part f_0 and with respect to right equivalence.*

For the notion of (coarse) moduli spaces see [MF] and [Ne]. The fact that 1.3 is a corollary of 1.2 follows from general principals (cf. [Ne]; the assumption made there that all spaces are reduced is not necessary). See also remark 3.5.

Proof of 1.2: (1) is proved in [AGV], 12.6, theorem (p. 209).
(2) First notice that roots of unity cannot be avoided: take $f = x^5 + y^{11} + xy^9$, $c_1^5 = c_2^{11} = 1$ and $c = c_1 c_2$. The automorphism $x \mapsto c^{11}x$, $y \mapsto c^5 y$ maps f to $x^5 + y^{11} + c^{56}xy^9$.

The statement of (2) will follow from the following two lemmas:

Lemma 1.4 *Let f, g be semiquasihomogeneous with principal part f_0 as above and $\varphi \in Aut\, \mathbf{C}[[x]]$ such that $\varphi(f) = g$. Then there is a d–th root of unity c such that*

$$\varphi(x_i) = c^{w_i} x_i + h_i,\ \deg\, h_i > w_i.$$

Proof: Let $w_1 < \ldots < w_n$. By proposition 3.2 we have deg $\varphi \geq 0$, hence

$$\varphi(x_i) = \sum_{j \geq i} c_{ij} x_j + \text{ higher order terms.}$$

Since φ is an automorphism, $\prod_i c_{ii} \neq 0$, and $\varphi(x_i) = c_{ii}x_i + h_i, \deg h_i > w_i$. From $x_1^{m_1} + \ldots + x_n^{m_n} = c_{11}^{m_1} x_1^{m_1} + \ldots + c_{nn}^{m_n} x_n^{m_n}$ we deduce $c_{ii}^{m_i} = 1$ and putting $c = \prod_i c_{ii}$ we obtain the result.

Lemma 1.5 *Let $\varphi \in Aut\, \mathbf{C}[[x]]$, $\deg \varphi > 0$, and $t, t' \in \underline{H}_\mu$ such that $\varphi(F_\mu(t)) = F_\mu(t')$. Then $t = t'$.*

Proof: By lemma 1.4, $\varphi(x_i) = x_i + h_i$. Hence $\varphi_s(x_i) := x_i + sh_i$ is a family of automorphisms of positive degree which connects φ with the identity. Then $\varphi_s(F_\mu(t))$ is a $\mathbf{C}^*$–equivariant family of isolated singularities, joining $F_\mu(t)$ and $F_\mu(t')$. This family may not be contained in $\underline{H}_\mu$ but it can be induced from $\underline{H}_\mu$ by a suitable base change (remark 1.1). But since $\underline{H}_\mu$ is everywhere miniversal and does, therefore, not contain trivial subfamilies with respect to right equivalence, $t = t'$ as desired.

The Kodaira–Spencer map (cf. [LP]) of the functor $Def_{A_0 \to \mathbf{C}, \mathbf{C}^*}$ and of the family $H_\mu \to H_\mu[[x]]/F_\mu$,

$$\rho : Der_{\mathbf{C}} H_\mu \longrightarrow (x) H_\mu[[x]] / \left(F_\mu + (x)(\frac{\partial F_\mu}{\partial x_1}, \ldots, \frac{\partial F_\mu}{\partial x_n}) \right),$$

is defined by $\rho(\delta) = class(\delta F_\mu) = class(\sum_{\alpha \in B_-} \delta(T_{d-|\alpha|}) x^\alpha)$.

Let $\mathbf{V}_\mu$ be the kernel of ρ. $\mathbf{V}_\mu$ is a Lie–algebra and along the integral manifolds of $\mathbf{V}_\mu$ the family is analytically trivial (cf. [LP]).

In our situation it is possible to give generators of $\mathbf{V}_\mu$ as H_μ–module:

Let $I_\mu = (x)H_\mu[[x]]/(x)(\frac{\partial F_\mu}{\partial x_1}, \ldots, \frac{\partial F_\mu}{\partial x_n})$, then I_μ is a free H_μ–module and $\{x^\alpha\}_{\alpha \in B}$ is a free basis.

Multiplication by F_μ defines an endomorphism of I_μ and $F_\mu I_\mu \subseteq \oplus_{\alpha \in B_-} x^\alpha H_\mu$.

Especially, for $\alpha \in B$, define $h_{i,j}$ by

$$x^\alpha F_\mu = \sum_{\beta \in B_-} h_{|\alpha|, d-|\beta|} x^\beta \text{ in } I_\mu.$$

Then h_{ij} is homogeneous of degree $i + j$. This implies $h_{ij} = 0$ if $i + j \geq 0$, in particular $h_{ij} = 0$ if $i \geq (n-1)d - 2\sum w_i$. For $\alpha \in B$ and $|\alpha| < (n-1)d - 2\sum w_i$ let $\delta_{|\alpha|} := \sum_{\beta \in B_-} h_{|\alpha|, d-|\beta|} \frac{\partial}{\partial T_{d-|\beta|}}$.

Proposition 1.6 *(cf. [LP], proposition 4.5):*

1. *$\delta_{|\alpha|}$ is homogeneous of degree $|\alpha|$.*
2. *$\mathbf{V}_\mu = \sum_\alpha H_\mu \delta_{|\alpha|}$.*

Now there is a non–degenerate pairing on I_μ (the residue pairing) which is defined in our situation by $\langle h,k\rangle = hess(h\cdot k)$. Here for $h=\sum_{\alpha\in B} h_\alpha x^\alpha \in I_\mu$, $hess(h) = h_{(m_1-2,\dots,m_n-2)}$ which is the coefficient belonging to the Hessian of f.

Let the numbering of the elements of $B_- = \{\alpha_1,\dots,\alpha_k\}$, be such that $|\alpha_1| < \dots < |\alpha_k|$ and denote by $\beta_i = \alpha^\vee_{k-i+1}$, $i=1,\dots,k$, the dual exponents induced by the pairing, i.e. if $\gamma=(\gamma_1,\dots,\gamma_n)$ then $\gamma^\vee = (m_1-2-\gamma_1,\dots,m_n-2-\gamma_n)$.

Using the pairing one can prove the following

Proposition 1.7 *There are homogeneous elements $m_1,\dots,m_k \in H_\mu[[x]]$ with the following properties:*

1. deg $m_i = |\beta_i|$
2. *If* $m_i F_\mu = \sum_{j=1}^k \tilde{h}_{ij} x^{\alpha_j}$ *in* I_μ *then* $\tilde{h}_{ij} = \tilde{h}_{k-j+1,k-i+1}$
3. *If* $\tilde{\delta}_{|\beta_i|} := \sum_{j=1}^k \tilde{h}_{ij}\frac{\partial}{\partial T_{d-|\alpha_j|}}$ *then* $\tilde{\delta}_{|\beta_i|}$ *is homogeneous of degree* $|\beta_i|$ *and* $\mathbf{V}_\mu = \sum_{i=1}^k H_\mu \tilde{\delta}_{|\beta_i|}$.

In [LP] (proposition 5.6) this proposition is proved for $n=2$. The proof can easily be extended to arbitrary n. The important fact is the symmetry, expressed in 2.

Let L be the Lie–algebra generated (as Lie–algebra) by $\{\tilde{\delta}_{|\beta_1|},\dots,\tilde{\delta}_{|\beta_k|}\}$. Then L is finite dimensional and solvable. $L_0 := [L,L]$ is nilpotent and $L/L_0 = \mathbf{C}\tilde{\delta}_{|\beta_1|}$, where $\tilde{\delta}_{|\beta_1|} = \sum_{i=1}^k (|\alpha_i|-d) T_{d-|\alpha_i|}\frac{\partial}{\partial T_{d-|\alpha_i|}}$ is the Euler vector field (cf. [LP]).

Corollary 1.8 *The integral manifolds of* $\mathbf{V}_\mu$ *coincide with the orbits of the algebraic group* $exp(L)$.

Now consider the matrix $M(T) := (\tilde{\delta}_{|\beta_i|}(T_{d-|\alpha_j|}))_{i,j=1,\dots,k} = (\tilde{h}_{ij})_{i,j=1,\dots,k}$. Evaluating this matrix at $t\in \underline{H}_\mu$ we have

$$\begin{aligned}\text{rank}M(t) &= \text{dimension of a maximal integral manifold of}\\ &\quad \mathbf{V}_\mu \text{ (resp. of the orbit of exp(L)) at } t\\ &= \mu - \tau(t),\end{aligned}$$

where $\tau(t)$ denotes the Tjurina number of the singularity defined by t i.e. of $F(x,t)$.

2 Existence of a geometric quotient for fixed Hilbert function of the Tjurina algebra

We want to apply theorem 4.7 from [GP 2] to the action of L_0 on $\underline{H}_\mu$.

Theorem 2.1 *([GP 2]) Let A be a noetherian $\mathbf{C}$–algebra and $L_0 \subseteq Der^{nil}_{\mathbf{C}} A$ a finite dimensional nilpotent Lie algebra. Suppose A has a filtration*

$$F^\bullet : \; 0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \dots$$

by subvector spaces $F^i(A)$ such that

$$(\mathbf{F}) \qquad \delta F^i(A) \subseteq F^{i-1}(A) \; \text{for all } i \in \mathbf{Z},\; \delta \in L_0.$$

Suppose moreover, L_0 has a filtration

$$Z_\bullet : L_0 = Z_0(L_0) \supseteq Z_1(L_0) \supseteq \dots \supseteq Z_e(L_0) \supseteq Z_{e+1}(L_0) = 0$$

by sub Lie algebras $Z_j(L_0)$ such that

$$(\mathbf{Z}) \qquad [L_0, Z_j(L_0)] \subseteq Z_{j+1}(L_0) \, \text{for all} \, j \in \mathbf{Z}.$$

Let $d : A \to Hom_{\mathbf{C}}(L_0, A)$ be the differential defined by $d(a)(\delta) = \delta(a)$ and let $Spec\, A = \cup U_\alpha$ be the flattening stratification of the modules

$$Hom_{\mathbf{C}}(L_0, A)/Ad(F^i(A)) \quad i = 1, 2, \dots$$

and

$$Hom_{\mathbf{C}}(Z_j(L_0), A)/\pi_j(A(dA)) \quad j = 1, \dots, e,$$

where π_j denotes the projection $Hom_{\mathbf{C}}(L_0, A) \to Hom_{\mathbf{C}}(Z_j(L_0), A)$.
Then U_α is invariant under the action of L_0 and $U_\alpha \to U_\alpha / L_0$ is a geometric quotient which is a principal fibre bundle with fibre $\exp(L_0)$.

To apply the theorem we have to construct these filtrations and interpret the corresponding stratification in terms of the Hilbert function of the Tjurina algebra.

There are natural filtrations $H^\bullet(\mathbf{C}[[x]])$ resp. $F^\bullet(H_\mu)$ on $\mathbf{C}[[x]]$ resp. H_μ defined as follows:

Let $F^i(H_\mu) \subseteq H_\mu$ be the $\mathbf{C}$–vectorspace generated by all quasihomogeneous polynomials of degree $> -(i+1)w$ and $H^i(\mathbf{C}[[x]])$ be the ideal generated by all quasihomogeneous polynomials of degree $\geq iw$, where

$$w := \min\{w_1, \dots, w_n\}.$$

For $t \in \underline{H}_\mu$ the **Hilbert function of the Tjurina algebra**

$$\mathbf{C}[[x]]/(F_\mu(t), \frac{\partial F_\mu(t)}{\partial x_1}, \dots, \frac{\partial F_\mu(t)}{\partial x_n})$$

corresponding to the singularity defined by t with respect to $H^\bullet$ is by definition the function,

$$n \mapsto \tau_n(t) := \dim_{\mathbf{C}} \mathbf{C}[[x]]/(F_\mu(t), \frac{\partial F_\mu(t)}{\partial x_1}, \dots, \frac{\partial F_\mu(t)}{\partial x_n}, H^n).$$

Notice that $\tau_n(t) = \tau(t)$ if n is large and $\tau_n(t)$ does not depend on t for small n. On the other hand, $\mu_n := \mu_n(t) := \dim_{\mathbf{C}} \mathbf{C}[[x]]/(\frac{\partial F_\mu(t)}{\partial x_1}, \dots, \frac{\partial F_\mu(t)}{\partial x_n}, H^n)$ does not depend on $t \in H_\mu$ and

$$\mu_n - \tau_n(t) = rank(\tilde{\delta}_{|\beta_i|}(T_{d-|\alpha_j|})(t))_{|\alpha_j|<nw}.$$

This is an immediate consequence of the following fact:
Let

$$T^n := H_\mu[[x]]/(F_\mu, \frac{\partial F_\mu}{\partial x_1}, \dots, \frac{\partial F_\mu}{\partial x_n}, H^n),$$

then the following sequence is exact and splits:

$$\begin{array}{rcccl} 0 \to \bigoplus_{\substack{\alpha\in B\\ |\alpha|\le d}} H_\mu x^\alpha & \to & T^{\frac{d}{w}+i} & \to & Der_{\mathbf{C}} H_\mu / \left(\mathbf{V}_\mu + \sum_{|\beta|\ge d+iw} H_\mu \frac{\partial}{\partial T_{d-|\beta|}}\right) \to 0 \\ x^\alpha & \mapsto & class(x^\alpha) & & \\ & & class(x^\beta) & \mapsto & class\left(\frac{\partial}{\partial T_{d-|\beta|}}\right), \end{array}$$

and with the identification $\sum_{|\beta|<d+iw} H_\mu \frac{\partial}{\partial T_{d-|\beta|}} \simeq H_\mu^{N_i}$ we get $Der_{\mathbf{C}} H_\mu / \left(\mathbf{V}_\mu + \sum_{|\beta|\ge d+iw} H_\mu \frac{\partial}{\partial T_{d-|\beta|}}\right) \simeq H_\mu^{N_i}/M_i$, where M_i is the H_μ–submodule generated by the rows of the matrix $(\tilde{\delta}_{|\beta_\ell|}(T_{d-|\alpha_j|}))_{|\alpha_j|<d+iw}$.

The filtration $F^\bullet(H_\mu)$ has the property (**F**) because every homogeneous vector field of L_0 is of degree $\geq w$ (since L/L_0 is the Euler vector field, cf. §1) and $H_\mu dH_\mu = H_\mu dF^s H_\mu$, $s = \left[\frac{(n-1)d-2\sum w_i}{w}\right]$ (since $nd - 2\sum w_i$ is the degree of the Hessian of f and $T_{d-(nd-2\sum w_i)}$ is the variable of smallest degree).

To define $Z_i(L_0)$ we use the duality defined in chapter 1:

$$\alpha \mapsto \alpha^\vee = (m_1 - 2 - \alpha_1, \dots, m_n - 2 - \alpha_n),$$

and set $Z_i(L_0) :=$ the Lie algebra generated by

$$\left\{\tilde{\delta}_{|\alpha|} \in L_0 \mid T_{d-|\alpha^\vee|} \in F^{s-i}\right\}$$

$Z_\bullet(L_0)$ has the property (**Z**) (for the definition of $\tilde{\delta}_{|\alpha|}$, see proposition 1.3). We have $F \in H^n$, hence $\mu_n = \tau_n$, if $n \le \frac{d}{w}$ and $H^n \subset (\frac{\partial F_\mu}{\partial x_1}, \ldots, \frac{\partial F_\mu}{\partial x_n})$ hence $\mu_n - \tau_n(t)$ is independent of n if $n \ge \frac{d}{w} + s + 1,\ s = [\frac{(n-1)d - 2\Sigma w_i}{w}]$ and equal to $\mu - \tau(t)$.

Therefore, we have $s+1$ relevant values for τ_i, and we denote

$$\begin{aligned} \underline{\tau}(t) &:= (\tau_{\frac{d}{w}+1}(t), \ldots, \tau_{\frac{d}{w}+s+1}(t)), \\ \underline{\mu} &:= (\mu_{\frac{d}{w}+1}, \ldots, \mu_{\frac{d}{w}+s+1}). \end{aligned}$$

Moreover, let $S = \{\underline{r} := (r_1, \ldots, r_{s+1}) \mid \exists t \in \underline{H}_\mu \text{ s.t. } \underline{\mu} - \underline{\tau}(t) = \underline{r}\}$ and $\underline{H}_\mu = \cup_{\underline{r} \in S} U_{\underline{r}}$ be the flattening stratification of the modules $T^{\frac{d}{w}+1}, \ldots, T^{w+1}$ i.e. $\{U_{\underline{r}}\}$ is the stratification of $\underline{H}_\mu$ defined by fixing the Hilbert function $\underline{\tau} = \underline{\mu} - \underline{r}$ with the scheme structure defined by the flattening property. We obtain:

Lemma 2.2 *1. $(0, \ldots, 0, 1)$ and $(0, \ldots, 0) \in S$. $U_{(0,\ldots,0)} = \{0\}$ is a smooth point and $U_{(0,\ldots,1)}$ is defined by $T_{d-|\beta|} = 0$ for $|\beta| < nd - 2\sum w_i$ and $T_{2\sum w_i - (u-1)d} \neq 0$ (and hence is smooth).*

2. Let $\bar{S} = S \backslash \{(0, \ldots, 0)\}$ and for $\underline{r} \in \bar{S}$ put

$$\bar{U}_{\underline{r}} = \begin{cases} U_{\underline{r}} & if\ \ \underline{r} \neq (0, \ldots, 0, 1) \\ U_{(0,\ldots 0,1)} \cup U_{(0,\ldots,0)} & if\ \ \underline{r} = (0, \ldots, 0, 1) \end{cases}$$

Then
$\{\bar{U}_{\underline{r}}\}_{\underline{r} \in \bar{S}}$ is the flattening stratification of the modules $\{Hom_{\mathbf{C}}(L_0, H_\mu)/H_\mu dF^i H_\mu\}$ and $\{Hom_{\mathbf{C}}(Z_i(L_0), H_\mu)/\pi_i(H_\mu dF H_\mu)\}$.

As a corollary we obtain the following theorem (recall that $\mathbf{V}_\mu$ denotes the kernel of the Kodaira Spencer map, cf. §1):

Theorem 2.3 *For $\underline{r} \in S$, $\bar{U}_{\underline{r}}$ is invariant under the action of $\mathbf{V}_\mu$ and $\bar{U}_{\underline{r}} \to \bar{U}_{\underline{r}}/\mathbf{V}_\mu$ is a geometric quotient. $\bar{U}_{\underline{r}}/\mathbf{V}_\mu$ is locally closed in a weighted projective space.*

Proof: Using the lemma and theorem 2.1 we obtain that $\bar{U}_{\underline{r}}$ is invariant under the action of L_0 and $\bar{U}_{\underline{r}} \to \bar{U}_{\underline{r}}/L_0$ is a geometric quotient. $L/L_0 = \mathbf{C}\delta_0$ acts on $\bar{U}_{\underline{r}}/L_0$. By corollary 1.4, $\bar{U}_{\underline{r}}/V_\mu = \bar{U}_{\underline{r}}/L$. If $\underline{r} \neq (0, \ldots, 1)$, then $\bar{U}_{\underline{r}}/L_0 \to \bar{U}_{\underline{r}}/L$ is a geometric quotient embedded in the corresponding weighted projective space. If $\underline{r} = (0, \ldots, 1)$ then $\bar{U}_{\underline{r}}/L_0 = \bar{U}_{\underline{r}}$ and the geometric quotients $U_{(0,\ldots,1)} \to U_{(0,\ldots,1)}/\mathbf{V}_\mu$, $U_{(0,\ldots,0)} \to U_{(0,\ldots,0)}/\mathbf{V}_\mu$ exist as smooth points.

It remains to prove the lemma.

Proof of lemma 2.2: Because of the exact sequence above the flattening stratification of the modules $\{T^{\frac{d}{w}+i}\}$ is also the flattening stratification of $\{Der_{\mathbf{C}}H_\mu/\left(\mathbf{V}_\mu+\sum_{|\beta|\geq d+iw}H_\mu\frac{\partial}{\partial T_{d-|\beta|}}\right)\}$ resp. the flattening stratification of $\{H_\mu^{N_i}/M_i\}$, M_i the submodule generated by the rows of the matrix $(\tilde{\delta}_{|\beta_\ell|}(T_j))_{-iw<j}$. Now we have

$$\tilde{\delta}_{|\beta|}(T_{d-|\alpha|})=\tilde{\delta}_{|\alpha^\vee|}(T_{d-|\beta^\vee|}).\quad (*)$$

By definition of $Z_i(L_0)$ we have

$$H_\mu Z_i(L_0)=\sum_{T_{d-|\alpha^\vee|}\in F^{s-i}}H_\mu\tilde{\delta}_{|\alpha|}$$

and with the identification

$$\sum_{\beta\in B_-}H_\mu\frac{\partial}{\partial T_{d-|\beta|}}=H_\mu^N,$$

and M^i the submodule generated by the rows of the matrix $(\tilde{\delta}_{|\alpha|}(T_j))_{d+(s-i+1)w>|\alpha^\vee|}$ we obtain

$$Der_{\mathbf{C}}H_\mu/H_\mu Z_i(L_0)\cong H_\mu^N/M^i.$$

(*) implies that the flattening stratification of the modules $\{T^{\frac{d}{w}+1},\ldots,T^s\}$, which is $\underline{H}_\mu=\cup_{\underline{r}\in\bar{S}}\bar{U}_{\underline{r}}$, is the flattening stratification of the modules $\{Der_{\mathbf{C}}H_\mu/H_\mu Z_i(L_0)\}_{i=1,\ldots,s}$.

On the other hand, the flattening stratification of the modules $\{H_\mu^N/M^i\}_{i=1,\ldots,s}$ is the flattening stratification of the modules

$$\{Hom_{\mathbf{C}}(Z_i(L_0),H_\mu)/\pi_i(H_\mu dH_\mu)\}$$

because

$$H_\mu Z_i(L_0)=\sum_{T_{d-|\alpha^\vee|}\in F^{s-i}}H\tilde{\delta}_{|\alpha|}.$$

Furthermore the modules $\{Hom_{\mathbf{C}}(L_0,H_\mu)/H_\mu dF^iH_\mu\}$ and $\{Der_{\mathbf{C}}H_\mu/H_\mu L_0+\sum_{|\beta|\geq d+iw}H_\mu\frac{\partial}{\partial T_{d-|\beta|}}\}$ have the same flattening stratification and they are flat on $U_{\underline{r}}$, because

$$0\to H_\mu\to Der_{\mathbf{C}}H_\mu/H_\mu L_0+\sum_{|\beta|\geq d+iw}H_\mu\frac{\partial}{\partial T_{d-|\beta|}}\to Der_{\mathbf{C}}H_\mu/\mathbf{V}_\mu+\sum_{|\beta|\geq d+iw}H_\mu\frac{\partial}{\partial T_{d-|\beta|}}\to 0$$

is exact and splits on $\underline{H}_\mu\setminus\{0\}$.

This proves the lemma.

Remark 2.4 The main point of the lemma is that the flattening stratification of the modules $\{Hom_{\mathbf{C}}(L_0,H_\mu)/H_\mu dF^iH_\mu\}$ is contained in the flattening stratification of the modules $\{Hom_{\mathbf{C}}(Z_j(L_0),H_\mu)/\pi_i(H_\mu dH_\mu)\}$, hence is defined by the Hilbert function of the Tjurina algebra alone without any reference to the action of L. This is a consequence of the symmetry, expressed in proposition 1.3.

3 The automorphism group of semi Brieskorn singularities

In this chapter we prove that the automorphism group of a semi Brieskorn singularity with principal part $f_0 = x_1^{m_1} + \ldots + x_n^{m_n}$, $\gcd(m_i, m_j) = 1$ for $i \neq j$, has no automorphisms of negative degree. A consequence of this result is that two points in $\underline{H}_\mu$ correspond to isomorphic singularities iff they are in one integral manifold of $\mathbf{V}_\mu$. Again, $d = m_i \cdot \ldots \cdot m_n$ denotes the degree of f_0.

Let $\mathbf{C}[[x]]_m$ denote the ideal of $\mathbf{C}[[x]]$ generated by power series of degree $\geq m$. An automorphism φ of $\mathbf{C}[[x]]$ has degree m if

$$(\varphi - id)\mathbf{C}[[x]]_i \subset \mathbf{C}[[x]]_{i+m}$$

for any i. For $c \in \mathbf{C}^*$ let $\varphi_c : \mathbf{C}[[x]] \to \mathbf{C}[[x]]$,

$$\varphi_c(x_i) = c^{w_i} x_i, \; i = 1, \ldots, n$$

denote the $\mathbf{C}^*$–action which is an automorphism of degree 0.

Proposition 3.1 *Let $f = f_0 + \sum_{|\alpha|>d} a_\alpha x^\alpha$, $g = f_0 + \sum_{|\alpha|>d} b_\alpha x^\alpha$, $\varphi \in Aut\,\mathbf{C}[[x]]$ and $u \in \mathbf{C}[[x]]$ a unit such that $uf = \varphi(g)$. Then $\deg \varphi \geq 0$.*

Remark 3.2 Let $\varphi \in Aut\,\mathbf{C}[[x]]$ be of degree ≥ 0, f, g as above and $f = u\varphi(g)$ for some unit u. Then $\varphi(x_i) = c_i x_i + h_i$, deg $h_i > w_i$, $u(0)c_i^{m_i} = 1$. Putting u_i some m_i–th root of $u(0)$ and $c = \Pi_{i=1}^m u_i c_i$ we obtain $c^d = 1$, $c^{w_i} = u_i c_i$, hence $u(0)\varphi(g) = \tilde{\varphi} \circ \varphi_c(g)$ and $f = \tilde{u}\tilde{\varphi} \circ \varphi_c(g)$ where deg $\tilde{\varphi} > 0$ and $\tilde{u}$ is a unit with $\tilde{u}(0) = 1$.

Proof: We prove the proposition by induction on n, the case $n = 1$ being trivial. We may assume that $m_n < \ldots < m_1$. Then we can write $\varphi(x_1) = \alpha_1 x_1 + h_1$, $\alpha_1 \in \mathbf{C}$ and $\deg h_1 > w_1 = \min\{w_1, \ldots, w_n\}$.

First of all we shall see that $\alpha_1 \neq 0$. Assume $\alpha_1 = 0$ then there is an $i > 1$ such that $\varphi(x_i) = \beta x_1 + h_i$ and $\deg h_i > w_1$, $\beta \neq 0$.

Using an automorphism of non–negative degree we may assume $\varphi(x_i) = x_1$.
Now

$$uf\mid_{x_1=0} = g(\varphi(x_1)\mid_{x_1=0}, \ldots, \varphi(x_{i-1})\mid_{x_1=0}, 0, \varphi(x_{i+1})\mid_{x_1=0}, \ldots)$$

and

$$\varphi\mid_{x_1=0}: \mathbf{C}[[x_1, \ldots, \hat{x}_i, \ldots, x_n]] \to \mathbf{C}[[x_2, \ldots, x_n]]$$

$$x_k \mapsto \varphi(x_k)\mid_{x_1=0}$$

is an isomorphism. Hence

$$\varphi\mid_{x_1=0}(g(x_1,\ldots,x_{i-1},0,x_{i-1},\ldots,x_n))=uf\mid_{x_1=0}.$$

But $g(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)$ and $f(0,x_2,\ldots,x_n)$ define isolated singularities with different Milnor numbers (they are semiquasihomogeneous with weights $w_1,\ldots,\hat{w}_i,\ldots,w_n$ resp. $w_2,\ldots,w_n$ and degree d). This is a contradiction and implies $\alpha_1\neq 0$. Using $\varphi_{\alpha^{-1}}$ and an automorphism of positive degree we may assume now $\varphi(x_1)=x_1$.

Let us consider again the automorphism $\varphi|_{x_1=0}$ of $\mathbf{C}[[x_2,\ldots,x_n]]$. Using the induction hypothesis we may assume $deg\,\varphi|_{x_1=0}\geq 0$. Since the inverse is also of non–negative degree we may assume that $\varphi|_{x_1=0}$ is the identity, i.e.

$$\varphi(x_1)=x_1 \text{ and } \varphi(x_i)=x_i+x_1h_i,\ i=2,\ldots,n.$$

Using again an automorphism of non–negative degree we may assume now that h_i has only terms of degree $<w_i-w_1$. We have to prove that $h_i=0$.

If h_i has only terms of degree $<w_i-w_1$ then h_i does not depend on $x_i,\ldots,x_n$. We prove now that $h_n=0$.

We may assume that $g=x_n^{m_n}+x_n^{m_n-2}a_2+\ldots+a_{m_n}$, $a_i\in\mathbf{C}[[x_1,\ldots,x_{n-1}]]$. Indeed by the Weierstrass preparation theorem $g\cdot unit=x_n^{m_n}+a_1x_n^{m_n-1}+\ldots$. This equality implies $\deg a_1x_n^{m_n-1}=(m_n-1)w_n+\deg a_1>d$ and consequently the automorphism defined by $x_n\to x_n-\frac{1}{m_n}a_1$ has positive degree. We may assume $a_1=0$ but this changes $\varphi(x_n)$ to $\varphi(x_n)=x_n+x_1h_n-\frac{1}{m_n}a_1$. Now $f\cdot u=\varphi(x_n^{m_n}+x_n^{m_n-2}a_2+\ldots)=x_n^{m_n}+(m_nx_1h_n-a_1)x_n^{m_n-1}+\ldots$ and $\deg m_nx_1h_nx_n^{m_n-1}<d$. But this is only possible if $h_n=0$ because this term cannot be cancelled (the other h_i do not depend on x_n). This implies $h_n=0$.

Now $f\cdot u\mid_{x_n=0}=f(x_1,x_2+x_1h_2,\ldots,x_{n-1}+x_1h_{n-1},0)$ because the h_i do not depend on x_n. Using again the induction hypothesis we obtain $h_i=0, i=2,\ldots,n-1$. This proves the proposition.

Corollary 3.3 *If $t,t'\in\underline{H}_\mu$ define isomorphic singularities then t and t' are in the same maximal integral manifold of $\mathbf{V}_\mu$.*

Proof: Let $F_\mu(t)=u\varphi(F_\mu(t'))$, $u\in\mathbf{C}[[x]]$ a unit and $\varphi\in Aut\,\mathbf{C}[[x]]$. By the proposition $\deg\varphi\geq 0$. Using remark 3.2 there is a d'th root of unity c such that $F_\mu(x,t)=u\varphi(F_\mu(c^{-1}\circ x,t'))=u\varphi(F_\mu(x,c\circ t'))$ and such that $\deg\varphi>0$ and $u(0)=1$. Then

$$G(z):=u(z^{w_1}x_1,\ldots,z^{w_n}x_n)\cdot F_\mu((\frac{1}{z^{w_1}}\varphi(z^{w_1}x_1),\ldots,\frac{1}{z^{w_n}}\varphi(z^{w_n}x_n),c\circ t')$$

is an unfolding of $G(0) = F_\mu(x, c \circ t')$. This unfolding can be induced by the universal unfolding by remark 1.1, i.e. there exists a family of coordinate transformations $\underline{\psi}(z, -)$ and a path v in $\underline{H}_\mu$ such that

$$G(z) = F_\mu(\psi_1(z,x), \ldots, \psi_n(z,x),\ v(z))$$

and $v(0) = c \circ t', \psi_i(0,x) = x$. By [AGV] we may assume that $\underline{\psi}(z, -)$ has positive degree.

Because $F_\mu(x,t) = F_\mu(\psi(1,x), v(1))$ we obtain $v(1) = t$ by lemma 1.5. This implies that t and $c \circ t'$ are in an analytically trivial family, i.e. in an integral manifold of $\mathbf{V}_\mu$ which contains the $\mathbf{C}^*$–orbits (cf. §1). Hence the result.

This finishes the second step of the approach. Together with the theorem of chapter 2 we obtain the theorem stated in the introduction:

Theorem 3.4 *There exists a coarse moduli space* $\mathcal{M}_{\underline{w},\underline{\tau}} = \bar{U}_{\mu-\underline{\tau}}/\mathbf{V}_\mu$ *of all semi-quasihomogeneous hypersurface singularities* $A = \mathbf{C}[[\underline{x}]]/(f)$ *with fixed principal part* $A_0 = \mathbf{C}[[\underline{x}]]/(f_0)$*, weight* $\underline{w}$ *and Hilbert function* $\underline{\tau}$*.* $\mathcal{M}_{\underline{w},\underline{\tau}}$ *is an algebraic variety, locally closed in a weighted projective space.*

Remark 3.5 To be more precise, first of all $\mathcal{M}_{\underline{w},\underline{\tau}}$ is a coarse moduli space for the functor which associates to any complex space germ S the set of isomorphism classes of flat families over S of quasihomogeneous hypersurface singularities with fixed principal part A_0, weight $\underline{w}$ and Hilbert function τ. The category of base spaces is that of germs since we constructed $\mathcal{M}_{\underline{w},\underline{\tau}}$ from the versal family over $\underline{H}_\mu$ which has the versality property only for germs. But by remark 1.1(3) we can actually enlarge the category of base spaces to all complex spaces S for which $H^1(S, \mathbf{Z}/d\mathbf{Z}) = 0$. The same applies to the coarse moduli space $\underline{H}_\mu/\mu_d$ for functions with respect to right equivalences (cf. corollary 1.3).

4 Problems

We use the notations of chapter 1.

4.1 In the case $n = 2$ (plane curves) the following holds (cf. [LP]): let $\{S_\tau\}$ be the stratification of $\underline{H}_\mu$ by constant Tjurina number, then

(i) $S_\tau \neq \emptyset$ if $\tau_{min} \leq \tau \leq \mu$ (i.e. all possible Tjurinia numbers occur).

(ii) $\dim S_\tau/\mathbf{V}_\mu \geq \dim S_{\tau'}/\mathbf{V}_\mu$ if $\tau \leq \tau'$ (i.e. the number of moduli decreases when τ becomes more special).

(iii) $S_{\tau_{min}}/\mathbf{V}_\mu$ is a quasismooth algebraic variety.

In [LP] is an example showing that (i) and (ii) are wrong in higher dimension.

Problem 1: Does (iii) hold in higher dimension?

Problem 2: Find the dimensions of $\underline{H}_\mu/\mathbf{V}_\mu$.

4.2 In chapter 3 we proved that for semi Brieskorn singularities with principal part $f_0 = x_1^{m_1} + \ldots + x_n^{m_n}$, $\gcd(m_i, m_j) = 1$ for $i \neq j$ the automorphisms have non–negative degree.

Problem 3: Is this true for all quasihomogeneous singularities with zero-dimensional moduli stratum?

A solution of this problem would solve the moduli problem for this class of semi-quasihomogeneous singularities.

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