Equianalytic and equisingular families of curves on surfaces

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July 7, 1998

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Introduction

We consider flat families of reduced curves on a smooth surface S such that for each member C of the family the number of singular points of C and for each singular point $x \in C$ the "singularity type" of (C, x) is fixed. Fixing these data imposes conditions on the space of all curves and we obtain in this way a locally closed subscheme of the Hilbert scheme H_S of S. We are mainly concerned with the study of the equianalytic (H_S^{ea}) resp. the equisingular Hilbert scheme (H_S^{es}), which are defined by fixing the analytic resp. the (embedded) topological type of the singularities. We show that fixing the analytical (resp. topological) type of (C, x) imposes, at most, $\tau(C, x)$ (resp. $\mu(C, x) - \text{mod}(C, x)$) conditions with equality if $H^1(C, \mathcal{N}^{ea}_{C/S})$ (resp. $H^1(C, \mathcal{N}^{es}_{C/S})$) vanish. Here $\mathcal{N}^{ea}_{C/S}$ (resp. $\mathcal{N}^{es}_{C/S}$) denote the equianalytic (resp. equisingular) normal bundle, while τ, μ denote the Tjurina resp. the Milnor number and mod the modality in the sense of [AGV]. The vanishing of H^1 implies the independence of the imposed conditions and the smoothness of H_S^{ea} (resp. H_S^{es}) at C (cf. §3).

In Theorem 3.6 we prove sufficient conditions for the vanishing of $H^1(C, \mathcal{N}^{ea}_{C/S})$ (resp. $H^1(C, \mathcal{N}^{es}_{C/S})$); for the special case $S = \mathbb{P}^2$ we obtain an additional criterion in Corollary 3.9. For the proof we use a vanishing theorem of [GrK] which is an improvement upon the usual vanishing theorem for sheaves which are not locally free. The local isomorphism defect $\text{isod}_x(\mathcal{N}^{ea}_{C/S}, \mathcal{O}_C)$, which is introduced in §3, measures how much $\mathcal{N}^{ea}_{C/S}$ fails to be free at x, and similar for $\mathcal{N}^{es}_{C/S}$. In many cases of interest, in particular for $\mathcal{N}^{es}_{C/S}$ and related sheaves, the isomorphism defect is quite big and gives a considerable improvement of the desired vanishing results. Therefore, we make some effort to compute resp. estimate it for certain classes of singularities in §4. In §5 we give some explicit examples and applications.

The present work is, in some sense, a continuation of some part of [GrK], where only equianalytic families were considered. Our results about the smoothness of $H^{es}_{\mathbb{P}^2}$, which are valid for arbitrary singularities, contain the previously known facts for curves with only ordinary multiple points (cf. [Gia]) as a special case; concerning the smoothness of $H^{ea}_{\mathbb{P}^2}$ they are an improvement of [Sh1]. For concrete applications of the theorems of this paper it is important to have good estimates for the isomorphism defects. Most of the formulas concerning these, together with further refinements and detailed proofs, appeared in [Lo]. Our results for $\mathbb{P}^2$ seem to be quite sharp for small d but are asymptotically weaker than those of Shustin [Sh3] which are quadratic in d. However, the methods presented here work for arbitrary surfaces and may be combined with Shustin's to provide asymptotically optimal results for curves on (some classes of) rational surfaces. This will be the subject of a forthcoming joint paper.

1 Equisingular deformations of plane curve singularities

In this paragraph we recall some definitions and results due to J. Wahl in the framework of formal deformation theory (cf. [Wa2]), transfer them to the complex analytic category and obtain some additional results which are used later.

Let $(C,0) \subset (\mathbb{C}^2,0)$ be a reduced plane curve singularity with defining local equation $f \in \mathcal{O}_{\mathbb{C}^2,0} = \mathbb{C}\{u,v\}$. Moreover, let $m = \mathrm{mult}_0(C)$ denote the multiplicity of $(C,0)$, that is $f \in \mathfrak{m}^m_{\mathbb{C}^2,0} \backslash \mathfrak{m}^{m+1}_{\mathbb{C}^2,0}$ where $\mathfrak{m}_{X,x}$ denotes the maximal ideal of a germ (X,x).
Consider a deformation $\varphi : (\mathcal{C},0) \to (T,0)$ of $(C,0)$ over an arbitrary complex germ $(T,0)$ together with a section $\sigma : (T,0) \to (\mathcal{C},0)$. Without loss of generality we may assume φ to be embedded, that is φ is given by a commutative diagram

$$\begin{array}{ccccc} (C,0) & \hookrightarrow & (\mathcal{C},0) & \hookrightarrow & (\mathbb{C}^2 \times T,0) \\ \downarrow & & \sigma \uparrow \downarrow \varphi & \swarrow pr & \\ 0 & \in & (T,0) & & \end{array}$$

where *pr* is the (natural) projection, σ maps to the trivial section and $(\mathcal{C},0)$ is a hypersurface germ of $(\mathbb{C}^2 \times T,0)$ defined by a power series $F \in \mathcal{O}_{\mathbb{C}^2 \times T,0}$. Let $I_{\sigma(T)}$ denote the ideal of $\sigma(T,0) \subset (\mathbb{C}^2 \times T,0)$, then we call the deformation with section (φ,σ) (resp. the section σ) *equimultiple*, if $F \in I^m_{\sigma(T)}$ (which is, of course, independent of the choice of the embedding and the choice of F). An embedded deformation φ (without section)

$$\begin{array}{ccc} (\mathcal{C},0) & \hookrightarrow & (\mathbb{C}^2 \times T,0) \\ \varphi \downarrow & \swarrow pr & \\ (T,0) & & \end{array}$$

is called *equisingular* if there exists a sequence of blowing up subspaces

$$\mathcal{M}_N \xrightarrow{\pi_N} \mathcal{M}_{N-1} \longrightarrow \cdots \longrightarrow \mathcal{M}_1 \xrightarrow{\pi_1} \mathcal{M}_0 = (\mathbb{C}^2 \times T,0) \qquad (1)$$

inducing a minimal resolution of $(C,0) \subset (\mathbb{C}^2 \times \{0\},0)$ and a (compatible) system of equimultiple sections through all infinitely near points of $(C,0)$ (for details cf. [Wa2, Zar]). J. Wahl has proven that the system of equimultiple sections is uniquely determined. The equimultiple section $\sigma : (T,0) \to (C,0)$ is called a *singular section* of φ.

The following theorem is basically due to J. Wahl ([Wa2], Theorem 7.4):

Theorem 1.1 *Let $\varphi : (\mathcal{C},0) \to (T,0)$ be any equisingular deformation of the reduced plane curve singularity $(C,0) \subset (\mathbb{C}^2,0)$.*

a) Let $\phi : \mathcal{C}_{(C,0)} \to S_{(C,0)}$ be the semiuniversal deformation of $(C,0)$. Then there exists a smooth subgerm $S^{es}_{(C,0)} \subset S_{(C,0)}$ such that if φ is induced from ϕ via the base change $\psi : (T,0) \to S_{(C,0)}$, then ψ factors through $S^{es}_{(C,0)}$. In particular, the restriction of ϕ to $S^{es}_{(C,0)}$ is a semiuniversal equisingular deformation *of $(C,0)$.*

b) Let $T_\varepsilon := Spec(\mathbb{C}[\varepsilon]/\varepsilon^2)$ be the base space of first order infinitesimal deformations. The set

$$I^{es}(C,0) := \left\{ g \in \mathbb{C}\{u,v\} \;\middle|\; \begin{array}{l} F = f + \varepsilon g \text{ defines an equisingular} \\ \text{deformation of } (C,0) \text{ over } T_\varepsilon \end{array} \right\}$$

is an ideal, the equisingularity ideal *of $(C,0)$. Especially it contains the Jacobian ideal*

$$j(C,0) = \left(f, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \right) \cdot \mathbb{C}\{u,v\}$$

and the vector space $I^{es}(C,0)/j(C,0)$ is isomorphic to the tangent space of $S^{es}_{(C,0)} \subset S_{(C,0)}$.

Proof. Wahl considers only deformations over Artinian spaces $(T,0)$ but the above facts follow easily from his results:

a) The existence of a smooth formal semiuniversal equisingular deformation of $(C,0)$ was proved by Wahl. The existence of a convergent representative can be deduced from his result by applying Artin's and Elkik's algebraization theorems. A simple direct proof, using the deformation of the parametrization, is given in [Gr].

b) follows directly from ([Wa2], Proposition 6.1). □

Proposition 1.2 *Openness of versality holds for equisingular deformations, that is if $\varphi : (\mathcal{C},0) \to (T,0)$ is an equisingular deformation of $(C,0)$, then for any equisingular representative $\varphi : \mathcal{C} \to T$ together with the singular section $\sigma : T \to \mathcal{C}$ the set of points $t \in T$ such that $(\mathcal{C},\sigma(t)) \to (T,t)$ is a versal deformation of $(\varphi^{-1}(t),\sigma(t))$ is a Zariski–open subspace of T.*

Proof. This follows quite formally from a criterion for openness of versality due to Flenner ([Fl], Satz 4.3). □

Let $\mu(C,0)$ (resp. $\tau(C,0)$) denote the Milnor (resp. Tjurina) number of $(C,0)$. It is well–known that a deformation of $(C,0)$ over a reduced base $(T,0)$ is equisingular if and only if (for small good representatives) the Milnor number is constant along the (unique) singular section (cf. [Te],§5). Hence $S^{es}_{(C,0)}$, being smooth, coincides with the μ–constant stratum of $S_{(C,0)}$. The codimension of $S^{es}_{(C,0)}$ in $S_{(C,0)}$ is (by Theorem 1.1) equal to

$$\tau^{es}(C,0) = \dim_{\mathbb{C}}(\mathbb{C}\{u,v\}/I^{es}(C,0)).$$

Together with a result of Gabrielov ([Gab]), which states that the *modality* $\mathrm{mod}(f)$ of the function f with respect to right equivalence (cf. [AGV]) is equal to the dimension of the μ–constant stratum of f in the (μ–dimensional) semiuniversal unfolding of f, we obtain the following

Lemma 1.3 *For any reduced plane curve singularity $(C,0)$ with local equation $f \in \mathbb{C}\{u,v\}$, we have $\tau^{es}(C,0) = \mu(C,0) - mod(f)$.*

If $\sim$ denotes any equivalence relation of plane curve singularities, a *$\sim$–singularity type* is a (not ordered) tuple $\mathcal{S} = ((C_1,x_1)/\sim, \ldots, (C_m,x_m)/\sim)$ of equivalence classes with m a non–negative integer. In this paper we are mainly interested in the following two cases:

- analytic equivalence (isomorphism of complex space germs), the corresponding singularity type is called *analytic type* and denoted by $\mathcal{A}$.
- topological equivalence (embedded homeomorphism of complex space germs), the corresponding singularity type is called *equisingularity type* or *topological type* and denoted by $\mathcal{T}$. For a reduced plane curve singularity it can be described by the system of multiplicity sequences resp. the resolution graph (cf. [BrK, Zar]).

If (C,x) is a reduced plane curve singularity, then $\mathcal{S}(C,x) = (C,x)/\sim$ denotes its singularity type and if C is a reduced curve with finitely many singular points $x_1,\ldots,x_m$ which are all planar, then $\mathcal{S}(C) = ((C,x_1)/\sim, \ldots, (C,x_m)/\sim)$ is the $\sim$–singularity type of C. For $\mathcal{S} = \mathcal{A}$ we obtain $\mathcal{A}(C)$, the *equianalytic type* of C, and for $\mathcal{S} = \mathcal{T}$ we obtain $\mathcal{T}(C)$, the *equisingular type* of C.

As equisingular deformations preserve the topological type, the equianalytic deformations preserve the analytic type of each fibre, where a deformation $\varphi : (\mathcal{C},0) \to (T,0)$ of $(C,0)$ is called *equianalytic* if $(\mathcal{C},0)$ is analytically isomorphic to $(C \times T, 0)$ over $(T,0)$, that is, φ is analytically trivial.

2 The equianalytic and equisingular Hilbert scheme

Let S be a smooth surface, T a complex space, then by a *family of embedded (reduced) curves over* T we mean a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \stackrel{j}{\hookrightarrow} & S\times T \\ \varphi\searrow & & \swarrow pr \\ & T & \end{array}$$

where φ is a proper and flat morphism such that for all points $t \in T$ the fibre $\varphi^{-1}(t)$ is a *curve* (that is a reduced pure 1–dimensional complex space), moreover, $j : \mathcal{C} \hookrightarrow S \times T$ is a closed embedding and *pr* denotes the natural projection. Such a family is called *equianalytic* (resp. *equisingular*) if for all $t \in T$ the induced (embedded) deformation of each singular point of $\varphi^{-1}(t)$ over (T,t) is equianalytic (resp. equisingular) — along the unique singular section σ.

The Hilbert functor $\mathcal{H}ilb_S$ on the category of complex spaces defined by

$$\mathcal{H}ilb_S(T) := \{\text{ subspaces of } S\times T, \text{ proper and flat over } T\}$$

is well–known to be representable by a complex space H_S (cf. [Bin]). This means there is a universal family

$$\begin{array}{ccc} \mathcal{U} & \stackrel{j}{\hookrightarrow} & S\times H_S \\ \varphi\searrow & & \swarrow pr \\ & H_S & \end{array}$$

such that each element of $\mathcal{H}ilb_S(T)$, T a complex space, can be induced from φ via base change by a *unique* map $T \to H_S$. We define the *equianalytic* (resp. *equisingular*) *Hilbert functor* $\mathcal{H}ilb_S^{ea}$ (resp. $\mathcal{H}ilb_S^{es}$) to be the subfunctor of $\mathcal{H}ilb_S$ with

$$\begin{aligned} \mathcal{H}ilb_S^{ea}(T) &:= \{\text{equianalytic families of embedded curves over } T\} \\ \mathcal{H}ilb_S^{es}(T) &:= \{\text{equisingular families of embedded curves over } T\} \end{aligned}$$

Moreover, fixing the analytic (resp. topological) singularity type of the fibres to $\mathcal{A}$ (resp. $\mathcal{T}$), we define the functor $\mathcal{H}ilb_S^{\mathcal{A}}$ (resp. $\mathcal{H}ilb_S^{\mathcal{T}}$).

Proposition 2.1 *Let $\mathcal{A}$ be an analytic singularity type corresponding to the topological type $\mathcal{T}$, then the functors $\mathcal{H}ilb_S^{\mathcal{A}}$ and $\mathcal{H}ilb_S^{\mathcal{T}}$ are representable by locally closed subspaces $H_S^{\mathcal{A}} \subset H_S^{\mathcal{T}} \subset H_S$.*

Corollary 2.2 *The functor $\mathcal{H}ilb_S^{ea}$ (resp. $\mathcal{H}ilb_S^{es}$) is representable by a complex space H_S^{ea} (resp. H_S^{es}) which is given as the disjoint union of all $H_S^{\mathcal{A}}$ (resp. $H_S^{\mathcal{T}}$).*

Remark 2.3 If $S = \mathbb{P}^2$ and if we fix the degree of all fibres to be d, then $H^{es,d}_{\mathbb{P}^2} \subset \mathbb{P}^N$ (where $N = (d^2 + 3d)/2$) is given as a *finite* disjoint union of locally closed subspaces, while in general $H^{ea,d}_{\mathbb{P}^2} \subset \mathbb{P}^N$ is an *infinite* union.

For the proof of Proposition 2.1 we need the following Lemma, which, for the equianalytic case, is proven in ([GrK], Lemma 1.4). A proof for the equisingular case is given in [Gr].

Lemma 2.4 *Let (C, x) be the germ of an isolated plane curve singularity and $\varphi : (\mathcal{C}, x) \to (B, b)$ a deformation of (C, x), then there are unique closed subgerms $(B^{ea}, b) \subset (B^{es}, b) \subset (B, b)$ such that for any morphism $f : (T, t) \to (B, b)$:*

$f^*\varphi$ is an equianalytic deformation if and only if $f(T, t) \subset (B^{ea}, b)$
$f^*\varphi$ is an equisingular deformation if and only if $f(T, t) \subset (B^{es}, b)$

Moreover, if $\phi : \mathcal{C}_{(C,x)} \to S_{(C,x)}$ denotes the semiuniversal deformation of (C, x) and if $\psi : (B, b) \to S_{(C,x)}$ is any morphism inducing φ via pull–back, then $(B^{ea}, b) = (\psi^{-1}(0), b)$ and $(B^{es}, b) = (\psi^{-1}(S^{es}_{(C,x)}), b)$.

Proof of Proposition 2.1. First we have to remark that the condition for all fibres to be reduced curves defines an open subspace $H'_S \subset H_S$. Now, let $b \in H'_S$ be such that the fibre $\varphi^{-1}(b)$ in the universal family has topological type $\mathcal{T}$, then by Lemma 2.4 for each $x \in \varphi^{-1}(b)$ there is a unique closed subspace $(H^{es}_x, b) \subset (H'_S, b)$ such that a morphism $f : (T, t) \to (H'_S, b)$ factors through (H^{es}_x, b) if and only if $f^*\varphi$ is an equisingular deformation. Let

$$(H^{es}, b) := \bigcap_{x \in \varphi^{-1}(b)} (H^{es}_x, b) \subset (H'_S, b)$$

and $H^{es}(b) \subset H'_S$ be a small (unique) representative, then $\cup H^{es}(b)$, where the union is taken over all b whose fibre $\varphi^{-1}(b)$ has topological type $\mathcal{T}$, defines a locally closed subspace of H_S which obviously represents $\mathcal{H}ilb^{\mathcal{T}}_S$. The statement for $\mathcal{H}ilb^{\mathcal{A}}_S$ follows in the same manner. □

3 Completeness of the equianalytic and equisingular characteristic linear series

Let S be a smooth complex surface and $C \subset S$ a reduced compact curve. Then a *deformation of* C/S over the pointed complex space T, $0 \in T$, is a triple $(\mathcal{C}, \tilde{i}, j)$ such that we obtain a Cartesian diagram

$$\begin{array}{ccc} C & \stackrel{j}{\hookrightarrow} & \mathcal{C} \\ i \cap & & \cap\, \tilde{i} \\ S & \hookrightarrow & S \times T \\ \downarrow & & \downarrow \pi \\ 0 & \in & T \end{array} \qquad \Phi : \mathcal{C} \to T$$

where j is a closed embedding and the composed morphism $\Phi := \pi \circ \tilde{i}$ is flat ($S \hookrightarrow S \times T$ denotes the canonical embedding with image $S \times \{0\}$ and π is the projection). Two deformations $(\mathcal{C}, \tilde{i}, j)$ and $(\mathcal{C}', \tilde{i}', j')$ of C/S over T are *isomorphic* if there exists an isomorphism $\mathcal{C} \simeq \mathcal{C}'$ such that the obvious diagram (with the identity on $S \times T$) commutes. $\mathcal{D}ef_{C/S}$ denotes the deformation functor from pointed complex spaces to sets defined by

$$\mathcal{D}ef_{C/S}(T) := \{\text{isomorphism classes of deformations of } C/S \text{ over } T\}$$

and we have the natural forgetful morphism $\mathcal{D}ef_{C/S} \to \mathcal{D}ef_C$, where $\mathcal{D}ef_C$ denotes the functor of isomorphism classes of deformations of C (forgetting the embedding). Furthermore, for each point $x \in C$, we consider the morphism $\mathcal{D}ef_C \to \mathcal{D}ef_{C,x}$ where $\mathcal{D}ef_{C,x}$ denotes the functor of isomorphism classes of deformations of the analytic germ (C, x). Let

$$T_\varepsilon := \operatorname{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$$

be the base space of first order infinitesimal deformations. We turn our attention to a subfunctor $\mathcal{D}ef'_{C,x} \subset \mathcal{D}ef_{C,x}$ such that

$$(T^1)' := \mathcal{D}ef'_{C,x}(T_\varepsilon)$$

is an ideal in $\mathcal{D}ef_{C,x}(T_\varepsilon) \cong \mathbb{C}\{u, v\}/j(C, x)$ and the corresponding "global" subfunctor $\mathcal{D}ef'_{C/S} \subset \mathcal{D}ef_{C/S}$ where $\mathcal{D}ef'_{C/S}(T)$ consists exactly of all those elements of $\mathcal{D}ef_{C/S}(T)$ which are mapped to $\mathcal{D}ef'_{C,x}(T)$ for all points $x \in C$.

Examples 3.1 a) $\mathcal{D}ef^{ea}_{C/S}$ the subfunctor of $\mathcal{D}ef_{C/S}$ consisting of all isomorphism classes of equianalytic deformations of C/S, that is of those deformations whose induced deformations of the analytic germs (C, x) happen to be equianalytic for all $x \in C$. Here $\mathcal{D}ef^{ea}_{C,x}(T_\varepsilon) = 0$ in $\mathbb{C}\{u, v\}/j(C, x)$.

b) $\mathcal{D}ef^{es}_{C/S}$ the subfunctor consisting of all isomorphism classes of equisingular deformations of C/S, that is of those deformations whose induced deformations of the (C,x) are equisingular. Here $\mathcal{D}ef^{es}_{C,x}(T_\varepsilon) = I^{es}(C,x)/j(C,x)$.

c) Further examples are the equimultiple, equigeneric and equiclassical deformation functors (cf. [DH]).

Remark 3.2 $\mathcal{D}ef^{ea}_{C/S}$ coincides with $\mathcal{D}ef^{es}_{C/S}$ if (and only if) C has only simple (ADE–)singularities.

Now, let J_C be the ideal sheaf of C in $\mathcal{O}_S$, then we have the natural exact sequence

$$0 \to J_C/J_C^2 \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathcal{O}_C \to \Omega^1_C \to 0$$

and its dual

$$0 \to \theta_C \to \theta_S \otimes_{\mathcal{O}_S} \mathcal{O}_C \stackrel{\Psi}{\to} \mathcal{N}_{C/S} \to \mathcal{T}^1_C \to 0.$$

Here $\mathcal{N}_{C/S} = \mathcal{O}_S(C) \otimes \mathcal{O}_C$ denotes the normal sheaf of C in S and $\mathcal{T}^1_C$ is a skyscraper sheaf concentrated in the singular points of C with $H^0(C, \mathcal{T}^1_C) \cong \mathcal{D}ef_C(T_\varepsilon)$ and with stalk in $x \in C$ equal to

$$\mathcal{T}^1_{C,x} \cong \mathcal{D}ef_{C,x}(T_\varepsilon) = T^1_{(C,x)}$$

(for details cf. [Art]). Furthermore, for each subfunctor $\mathcal{D}ef'_{C/S}$ as above, let $(\mathcal{T}^1_C)'$ denote the subsheaf of $\mathcal{T}^1_C$ with stalk in $x \in C$ isomorphic to $(T^1)' \subset T^1$ and

$$\mathcal{N}'_{C/S} := \operatorname{Ker}(\mathcal{N}_{C/S} \to \mathcal{T}^1_C/(\mathcal{T}^1_C)').$$

In particular, this defines the sheaves

$$\mathcal{N}^{ea}_{C/S} = \operatorname{Ker}(\mathcal{N}_{C/S} \to \mathcal{T}^1_C) \quad \text{and} \quad \mathcal{N}^{es}_{C/S} = \operatorname{Ker}(\mathcal{N}_{C/S} \to \mathcal{T}^1_C/(\mathcal{T}^1_C)^{es}),$$

where $\mathcal{T}^1_{C,x} \cong \mathbb{C}\{u,v\}/j(C,x)$, $(\mathcal{T}^1_C)^{es}_x \cong I^{es}(C,x)/j(C,x)$.

Lemma 3.3 *There is a canonical isomorphism*

$$\Phi: \quad \mathcal{D}ef'_{C/S}(T_\varepsilon) \xrightarrow{\cong} H^0(C, \mathcal{N}'_{C/S}).$$

Proof. Each representative of an element in $\mathcal{D}ef'_{C/S}(T_\varepsilon)$ is given by local equations $(f_i + \varepsilon g_i = 0)_{i\in I}$ (where $f_i, g_i \in \Gamma(U_i, \mathcal{O}_S)$ for an open covering $(U_i)_{i\in I}$ of S), which satisfy

- $(f_i = 0)_{i\in I}$ are local equations for $C \subset S$

- $f_i + \varepsilon g_i = (a_{ij} + \varepsilon b_{ij})(f_j + \varepsilon g_j)$ on $U_i \cap U_j =: U_{ij}$ with a_{ij} a unit in $\Gamma(U_{ij}, \mathcal{O}_S)$ and $b_{ij} \in \Gamma(U_{ij}, \mathcal{O}_S)$
- the germ $g_{i,x}$ of g_i projects to an element of $\mathcal{D}ef'_{C,x}(T_\varepsilon) = (T^1)'$ for all $x \in C \cap U_i$.

For the induced sections $g_i/f_i \in \Gamma(U_i, \mathcal{O}_S(C))$ it follows immediately

$$\frac{g_i}{f_i} - \frac{g_j}{f_j} = \frac{a_{ij}g_j + b_{ij}f_j}{a_{ij}f_j} - \frac{g_j}{f_j} = \frac{b_{ij}}{a_{ij}} \equiv 0 \in \Gamma(U_{ij}, \mathcal{N}_{C/S})$$

and g_i/f_i maps to an element of $(\mathcal{T}^1_C)' \subset \mathcal{T}^1_C$. Hence, $(g_i/f_i)_{i \in I}$ defines a global section in $\mathcal{N}'_{C/S}$. It is easy to check that in this way we get the isomorphism we were looking for (cf. [Mu], [Lo]). □

Let C be a compact reduced curve, $\mathcal{F}$ and $\mathcal{G}$ coherent torsion–free sheaves on C, which have rank 1 on each irreducible component C_i of C and $x \in C$. Then we define the *local isomorphism defect* of $\mathcal{F}$ in $\mathcal{G}$ in x as

$$\mathrm{isod}_x(\mathcal{F}, \mathcal{G}) := \min\left(\dim_{\mathbb{C}} \mathrm{Coker}(\varphi : \mathcal{F}_x \to \mathcal{G}_x)\right)$$

where the minimum is taken over all (injective) local homomorphisms φ. In particular, $\mathrm{isod}_x(\mathcal{F}, \mathcal{G})$ is a non–negative integer and not zero only in finitely many points (in [GrK] $\mathrm{isod}_x(\mathcal{F}, \mathcal{G})$ was denoted by $-\mathrm{ind}_x(\mathcal{F}, \mathcal{G})$). We call

$$\mathrm{isod}(\mathcal{F}, \mathcal{G}) := \sum_{x \in C} \mathrm{isod}_x(\mathcal{F}, \mathcal{G})$$

the *total (local) isomorphism defect* of $\mathcal{F}$ in $\mathcal{G}$. For an irreducible component C_i of C and $x \in C_i$ we set $\mathcal{F}_{C_i} := \mathcal{F} \otimes \mathcal{O}_{C_i}$ modulo torsion and

$$\mathrm{isod}_{C_i,x}(\mathcal{F}, \mathcal{G}) := \min\left(\dim_{\mathbb{C}} \mathrm{Coker}(\varphi_{C_i} : \mathcal{F}_{C_i,x} \to \mathcal{G}_{C_i,x})\right)$$

where the minimum is taken over all φ_{C_i}, which are induced by local homomorphisms $\varphi : \mathcal{F}_x \to \mathcal{G}_x$, and

$$\mathrm{isod}_{C_i}(\mathcal{F}, \mathcal{G}) := \sum_{x \in C_i} \mathrm{isod}_{C_i,x}(\mathcal{F}, \mathcal{G}).$$

Note that this is again a non–negative integer. In Chapter 4 we present some explicit calculations.

Proposition 3.4 *([GrK], Proposition 5.2) Let S be a smooth surface, $C \subset S$ a compact reduced curve and $\mathcal{F}$ a torsion–free coherent $\mathcal{O}_C$–module which has rank 1 on each irreducible component C_i of C $(i = 1, \ldots, s)$. Then $H^1(C, \mathcal{F}) = 0$ if for $i = 1, \ldots, s$*

$$\chi(\mathcal{F}_{C_i}) > \chi(w_{C,C_i}) - isod_{C_i}(\mathcal{F}, w_C).$$

Here $\chi(\mathcal{M}) = \dim H^0(C, \mathcal{M}) - \dim H^1(C, \mathcal{M})$ for a coherent sheaf $\mathcal{M}$ on C and w_C denotes the dualizing sheaf, $w_{C,C_i} := w_C \otimes \mathcal{O}_{C_i}$.

Remark 3.5 Using Riemann–Roch and the adjunction formula, the condition above reads

$$\deg(\mathcal{F}_{C_i}) > (K_S + C)\cdot C_i - \mathrm{isod}_{C_i}(\mathcal{F}, \mathcal{O}_C)$$

where K_S is the canonical divisor on S. Since isod is a local invariant and since C has planar singularities, we can replace w_C by $\mathcal{O}_C$.

Theorem 3.6 *Let S be a smooth complex surface and $C \subset S$ a reduced compact curve, H_S^{ea} resp. H_S^{es} be the representing spaces for the equianalytic resp. equisingular Hilbert functor.*

a) Let $\tau(C)$ denote the total Tjurina–Number and $\tau^{es}(C) := \sum \tau^{es}(C,x)$ with $\tau^{es}(C,x) := \dim_{\mathbb{C}}(\mathcal{O}_{C,x}/I^{es}(C,x))$, then

$$\dim(H_S^{ea}, C) \geq C^2 + 1 - p_a(C) - \tau(C)\ ,$$
$$\dim(H_S^{es}, C) \geq C^2 + 1 - p_a(C) - \tau^{es}(C)$$

b) If $H^1(C, \mathcal{N}^{ea}_{C/S}) = 0$ (respectively $H^1(C, \mathcal{N}^{es}_{C/S}) = 0$) then H_S^{ea} (respectively H_S^{es}) is smooth at C of dimension $C^2 + 1 - p_a(C) - \tau(C)$ (resp. $C^2 + 1 - p_a(C) - \tau^{es}(C)$).

c) Let $C = C_1 \cup \ldots \cup C_s$ be the decomposition into irreducible components, then

- *$H^1(C, \mathcal{N}^{ea}_{C/S}) = 0$ if for $i = 1, \ldots, s$*

$$-K_S \cdot C_i > D \cdot C_i + \tau(C_i) - isod_{C_i}(\mathcal{N}^{ea}_{C/S}, \mathcal{O}_C)$$

- *$H^1(C, \mathcal{N}^{es}_{C/S}) = 0$ if for $i = 1, \ldots, s$*

$$-K_S \cdot C_i > \sum_{x \in \, Sing(C)} \dim_{\mathbb{C}}((\mathcal{O}_{C,x}/I^{es}(C,x)) \otimes \mathcal{O}_{C_i,x}) - isod_{C_i}(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C)$$

where $D = \cup_{j \neq i} C_j$ and K_S denotes the canonical divisor on S. Moreover, the isomorphism defects $isod_{C_i}(\mathcal{N}^{ea}_{C/S}, \mathcal{O}_C)$ (resp. $isod_{C_i}(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C)$) have the lower bound $\#(C_i \cap Sing\ C)$.

Remark 3.7 a) If all singularities of C are quasi–homogeneous or *ordinary k–tuple points* (all branches are smooth with distinct tangents) then we obtain as an equivalent criterium for the vanishing of $H^1(C, \mathcal{N}^{es}_{C/S})$

$$-K_S \cdot C_i > D \cdot C_i + \tau^{es}(C_i) - \mathrm{isod}_{C_i}(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C).$$

b) By the adjunction formula, we obtain: $-K_S \cdot C_i = C_i^2 - 2p_a(C_i) + 2$.

c) If C is *irreducible*, the vanishing conditions in c) read as

$$-K_S C > \tau(C) - \mathrm{isod}(\mathcal{N}^{ea}_{C/S}, \mathcal{O}_C) \text{ resp.} -K_S C > \tau^{es}(C) - \mathrm{isod}(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C)\,.$$

Proof. Most parts of the proof are identical for the equianalytic (ea) and the equisingular (es) case, there we use again the notation H' resp. $\mathcal{N}'_{C/S}$ as above:

b) Let $H^1(C, \mathcal{N}'_{C/S}) = 0$ and $A \to A/(\eta) = \bar{A}$ be a small extension of Artinian $\mathbb{C}$–algebras. For the smoothness of (H', C), we have to show that each equianalytic (resp. equisingular) family $\bar{\mathcal{C}}$ over $\bar{A}$ lifts to an equianalytic (resp. equisingular) family $\mathcal{C}$ over A.

   $\bar{\mathcal{C}}$ is given by local equations $\bar{F}_i \in \Gamma(U_i, \mathcal{O}_S \otimes \bar{A})$, where $(U_i)_{i\in I}$ is an open covering of S, such that

   - on $U_{ij} := U_i \cap U_j$, $\bar{F}_i = \bar{G}_{ij} \cdot \bar{F}_j$ with a unit $\bar{G}_{ij}$
   - the image $F_i^{(0)}$ of $\bar{F}_i$ in $\Gamma(U_i, \mathcal{O}_S \otimes \mathbb{C})$ is a local equation for $C \subset S$
   - the germs $\bar{F}_{i,x}$ describe an equianalytic (resp. equisingular) deformation of (C, x).

   On the other hand, we know that the equianalytic (resp. equisingular) functor E', which associates to each Artinian local $\mathbb{C}$–algebra the set of all equianalytic (resp. equisingular) deformations of (C, x) over Spec A, is smooth and has a very good deformation theory (cf. [Wa1], for the equisingular case). Using the results of M. Schlessinger ([Schl], Remark 2.17), this guarantees in particular

   - the existence of an equianalytic (es) lifting $F_i \in \Gamma(U_i, \mathcal{O}_S \otimes A)$ of $\bar{F}_i$
   - for any lifting $G_{ij} \in \Gamma(U_{ij}, \mathcal{O}_S \otimes A)$ of $\bar{G}_{ij}$ the existence of sections $h_{ij} \in \Gamma(U_{ij}, \mathcal{O}_S \otimes A)$ with $F_i = G_{ij}F_j + \eta h_{ij}$ and $(h_{ij})_x \in j(C, x)$ (resp. $(h_{ij})_x \in I^{es}(C, x)$).

   To obtain the lifted family we are looking for, we have to modify the F_i and G_{ij} in a suitable way, such that the h_{ij} become 0. We know

$$\begin{aligned} \eta h_{ij} + \eta G_{ij} h_{jk} &= F_i - G_{ij}F_j + G_{ij}(F_j - G_{jk}F_k) \\ &= \eta h_{ik} + (G_{ik} - G_{ij}G_{jk})F_k \end{aligned}$$

   where $(G_{ik} - G_{ij}G_{jk}) \in \Gamma(U_{ijk}, \mathcal{O}_S \otimes (\eta))$ and $(\eta) \cdot \mathfrak{m}_A = 0$. As sections in $\mathcal{O}_S \otimes A/\mathfrak{m}_A = \mathcal{O}_S \otimes \mathbb{C}$ we obtain

$$h_{ij} + G^{(0)}_{ij} h_{jk} = h_{ik} + \left[\frac{1 - G_{ij}G_{jk}G^{-1}_{ik}}{\eta}\right] G^{(0)}_{ik} F^{(0)}_k .$$

Furthermore, $F_i^{(0)} = G_{ij}^{(0)} F_j^{(0)}$, which implies in $\Gamma(U_{ijk}, \mathcal{N}_{C/S})$ the cocycle condition

$$\frac{h_{ij}}{F_i^{(0)}} + \frac{h_{jk}}{F_j^{(0)}} = \frac{h_{ik}}{F_i^{(0)}}.$$

From the definition of the h_{ij} it follows that $(h_{ij}/F_i^{(0)} \mid i,j \in I)$ represents an element in $H^1(C, \mathcal{N}'_{C/S}) = 0$. Hence, there exist $f_i \in \Gamma(U_i, \mathcal{O}_S \otimes \mathbb{C})$ such that

$$\frac{h_{ij}}{F_i^{(0)}} = \frac{f_j}{F_j^{(0)}} - \frac{f_i}{F_i^{(0)}}$$

as sections in $\mathcal{N}'_{C/S}$, especially $h_{ij} + f_i - f_j G_{ij}^{(0)} \in \Gamma(U_{ij}, J_C)$ and all germs $(f_i)_x$ lie in the Jacobian (resp. equisingularity) ideal.
Defining $g_{ij} := (h_{ij} + f_i - f_j G_{ij}^{(0)})/F_j^{(0)}$, $\tilde{F}_i := F_i + \eta f_i$ and $\widetilde{G}_{ij} := G_{ij} + \eta g_{ij}$, we obtain the lifted family.

a) The germ (H'_S, C) is the fibre over the origin of a (non–linear) obstruction map $H^0(C, \mathcal{N}'_{C/S}) \to H^1(C, \mathcal{N}'_{C/S})$ (cf. [La], Theorem 4.2.4). Hence

$$\begin{aligned} \dim H^0(C, \mathcal{N}'_{C/S}) &\geq \dim\,(H'_S, C) \\ &\geq \dim H^0(C, \mathcal{N}'_{C/S}) - \dim H^1(C, \mathcal{N}'_{C/S}) \\ &= \chi(\mathcal{N}_{C/S}) - \chi(\mathcal{T}_C^1/(\mathcal{T}_C^1)') \\ &= C^2 + 1 - p_a(C) - \tau'(C) \end{aligned}$$

where $\tau'(C)$ denotes the total Tjurina number of C (resp. $\tau^{es}(C)$). Both inequalities become an equality if $H^1(C, \mathcal{N}'_{C/S}) = 0$.

c) Consider the exact sequence

$$0 \to \overline{\mathcal{N}'_{C/S} \otimes \mathcal{O}_{C_i}} \to \mathcal{N}_{C/S} \otimes \mathcal{O}_{C_i} \to (\mathcal{T}_C^1/(\mathcal{T}_C^1)'), \otimes \mathcal{O}_{C_i} \to 0$$

where $\overline{}$ denotes reduction modulo torsion. By Proposition 3.4, the above statement is an immediate consequence of

$$\dim_{\mathbb{C}} H^0(C, \mathcal{T}_C^1/(\mathcal{T}_C^1)^{es} \otimes \mathcal{O}_{C_i}) = \sum_{x \in C} \dim_{\mathbb{C}}((\mathcal{O}_{C,x}/I^{es}(C,x)) \otimes \mathcal{O}_{C_i,x})$$

resp. using the Leibniz rule (with g as the equation of (D,x)) by

$$\dim_{\mathbb{C}} H^0(C, \mathcal{T}_C^1 \otimes \mathcal{O}_{C_i}) = \sum_{x \in C} \dim_{\mathbb{C}}(\mathcal{O}_{C_i,x}/g \cdot j(C_i,x)) = \tau(C_i) + C_i \cdot D\,.$$

3.1 Curves in $\mathbb{P}^2(\mathbb{C})$

Let $C \subset \mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$ be a reduced curve of degree d with (homogeneous) equation $F(X,Y,Z) = 0$, then we define the *polar* of C relative to the point $(\alpha:\beta:\gamma) \in \mathbb{P}^2$ to be the curve $C_{\alpha\beta\gamma}$ with equation

$$\alpha F_X(X,Y,Z) + \beta F_Y(X,Y,Z) + \gamma F_Z(X,Y,Z) = 0 .$$

Lemma 3.8 *The generic polar $C_{\alpha\beta\gamma}$ (with $(\alpha:\beta:\gamma) \in \mathbb{P}^2$ a generic point) is an irreducible curve of degree $d-1$, if and only if C is not the union of $d \geq 3$ lines through the same point.*

Proof. Applying Bertini's theorem, we have irreducibility if there is no algebraic relation between F_X, F_Y and F_Z. Considering such a (homogeneous) relation of minimal degree and differentiating, we obtain a system of equations

$$A \cdot \begin{pmatrix} F_{XX} \\ F_{XY} \\ F_{XZ} \end{pmatrix} + B \cdot \begin{pmatrix} F_{YX} \\ F_{YY} \\ F_{YZ} \end{pmatrix} + \Gamma \cdot \begin{pmatrix} F_{ZX} \\ F_{ZY} \\ F_{ZZ} \end{pmatrix} = 0$$

where A, B and Γ generically do not vanish. Now the lemma follows from the fact that the Hessian covariant vanishes identically only if C is the union of d lines through one point (cf. [He], Lehrsatz 6). □

In the following we choose suitable coordinates such that Sing C lies in the affine plane $Z \neq 0$ and a generic polar $C' \subset P^2$ relative to $(\alpha:\beta:0)$ with equation $\alpha F_X + \beta F_Y = 0$ is irreducible.

Then we have an obvious morphism $\mathcal{O}_{C'}(d) \to \mathcal{T}^1_C$ given by the natural projections

$$\mathcal{O}_{C',x} = \mathcal{O}_{\mathbb{P}^2,x}/(\alpha f_X + \beta f_Y) \to \mathcal{O}_{\mathbb{P}^2,x}/j(C,x) = \mathcal{T}^1_{C,x}$$

where $f(X,Y) = F(X,Y,1)$ is the affine equation of C. We define

$$\tilde{\mathcal{N}}^{ea}_{C'/\mathbb{P}^2} := \operatorname{Ker}(\mathcal{O}_{C'}(d) \to \mathcal{T}^1_C)$$
$$\tilde{\mathcal{N}}^{es}_{C'/\mathbb{P}^2} := \operatorname{Ker}(\mathcal{O}_{C'}(d) \to \mathcal{T}^1_C/(\mathcal{T}^1_C)^{es}).$$

Corollary 3.9 *Let $C \subset \mathbb{P}^2$ be a reduced projective curve of degree d, C_i $(i = 1, \ldots, s)$ its irreducible components and d_i the degree of C_i.*

a) $H^1(C, \mathcal{N}^{ea}_{C/\mathbb{P}^2}) = 0$ if and only if the forgetful morphism $\mathcal{D}ef_{C/\mathbb{P}^2} \to \prod \mathcal{D}ef_{C,x}$ is surjective.

b) If $H^1(C, \mathcal{N}^{ea}_{C/\mathbb{P}^2}) = 0$ (respectively $H^1(C, \mathcal{N}^{es}_{C/S}) = 0$) then $(H^{ea}_{\mathbb{P}^2}, C)$ (respectively $(H^{es}_{\mathbb{P}^2}, C)$) is smooth of dimension $d(d+3)/2 - \tau(C)$ (resp. $d(d+3)/2 - \tau^{es}(C)$).

c) $H^1(C,\mathcal{N}^{ea}_{C/\mathbb{P}^2})=0$ *(resp.* $H^1(C,\mathcal{N}^{es}_{C/S})=0$*), if for* $i=1,\dots,s$

$$3d_i > d_i\cdot(d-d_i)+\tau(C_i)-isod_{C_i}(\mathcal{N}^{ea}_{C/\mathbb{P}^2},\mathcal{O}_C)$$
$$(resp.\;\; 3d_i > d_i\cdot(d-d_i)+\tau^{es}(C_i)-isod_{C_i}(\mathcal{N}^{es}_{C/\mathbb{P}^2},\mathcal{O}_C)).$$

Moreover, $isod_{C_i}(\mathcal{N}^{ea}_{C/\mathbb{P}^2},\mathcal{O}_C)\geq \#\,(Sing(C)\cap C_i)$.

d) If C *is not the union of* $d\geq 3$ *lines through one point and* C' *denotes the generic polar, then* $H^1(C,\mathcal{N}^{ea}_{C/\mathbb{P}^2})$ *(resp.* $H^1(C,\mathcal{N}^{es}_{C/\mathbb{P}^2})$*) vanishes, if*

$$4d > 4+\tau(C)-isod(\tilde{\mathcal{N}}^{ea}_{C'/\mathbb{P}^2},\mathcal{O}_{C'})$$
$$(resp.\;\; 4d > 4+\tau^{es}(C)-isod(\tilde{\mathcal{N}}^{es}_{C'/\mathbb{P}^2},\mathcal{O}_{C'}))\,,$$

Proof. b) and c) follow immediately from Theorem 3.6. To prove d), we consider the exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{N}'_{C/\mathbb{P}^2} & \to & \mathcal{N}_{C/\mathbb{P}^2} & \to & \mathcal{T}^1_C/(\mathcal{T}^1_C)' & \to & 0 \\ 0 & \to & \tilde{\mathcal{N}}'_{C'/\mathbb{P}^2} & \to & \mathcal{O}_{C'}(d) & \to & \mathcal{T}^1_C/(\mathcal{T}^1_C)' & \to & 0 \end{array}$$

and the corresponding long exact cohomology sequences where $'$ represents again both the equianalytic and the equisingular case.

We know that $\mathcal{N}_{C/\mathbb{P}^2}\cong\mathcal{O}_C(d)$ and (by Proposition 3.4) $H^1(C,\mathcal{O}_C(d))=0$. Furthermore, if C is not the union of $d\geq 3$ lines through one point, C' is irreducible and

$$\deg(\tilde{\mathcal{N}}'_{C'/\mathbb{P}^2})-(K_{\mathbb{P}^2}+C')\cdot C' = 4(d-1)-\tau'(C).$$

Hence, applying Proposition 3.4 the above conditions guarantee the vanishing of $H^1(C',\tilde{\mathcal{N}}'_{C'/\mathbb{P}^2})$. Additionally, the exact sequence

$$0\to\mathcal{O}_{\mathbb{P}_2}\to\mathcal{O}_{\mathbb{P}_2}(d)\to\mathcal{O}_C(d)\to 0$$

resp. an analogous sequence for C' induce surjective mappings

$$\Phi: H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(d))\to H^0(C,\mathcal{O}_C(d)),\;\; \Phi': H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(d))\to H^0(C',\mathcal{O}_{C'}(d))$$

which lead to a commutative diagram with exact horizontal rows

$$\begin{array}{ccl} & \nearrow & H^0(C,\mathcal{O}_C(d))\to H^0(C,\mathcal{T}^1_C/(\mathcal{T}^1_C)')\to H^1(C,\mathcal{N}'_{C/\mathbb{P}^2})\to 0 \\ H^0(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(d)) & & \qquad\qquad\qquad\qquad\qquad \| \\ & \searrow & H^0(C',\mathcal{O}_{C'}(d))\to H^0(C,\mathcal{T}^1_C/(\mathcal{T}^1_C)')\to 0. \end{array}$$

This shows that $H^0(C, \mathcal{O}_C(d)) \to H^0(C, \mathcal{T}_C^1/(\mathcal{T}_C^1)')$ is surjective and, hence, $H^1(C, \mathcal{N}'_{C/\mathbb{P}^2}) = 0$. The same argument shows that $H^0(C, \mathcal{O}_C(d)) \to H^0(C, \mathcal{T}_C^1)$ is surjective if and only if $H^1(C, \mathcal{N}^{ea}_{C/\mathbb{P}^2}) = 0$.

Since $\mathcal{D}ef_{C/\mathbb{P}^2}(T_\varepsilon) = H^0(C, \mathcal{O}_C(d))$ and $H^0(C, \mathcal{T}_C^1) \cong \prod \mathcal{D}ef_{C,x}(T_\varepsilon)$ this is equivalent to $\mathcal{D}ef_{C/\mathbb{P}^2}(T_\varepsilon) \to \prod \mathcal{D}ef_{C,x}(T_\varepsilon)$ being surjective (and $\mathcal{D}ef_{C/\mathbb{P}^2}$ being unobstructed). But $\mathcal{D}ef_{C/\mathbb{P}^2}$ and $\prod \mathcal{D}ef_{C,x}$ being unobstructed, the surjectivity on the tangent level implies the surjectivity of the functors. □

Remark 3.10 a) We call the inequalities in 3.9 c) resp. 3.9 d) the $3d$– resp. $4d$–criteria.

b) The use of a generic polar is due to Shustin [Sh1], who obtained (with a different proof) the weaker inequality $4d > 4 + \mu(C)$ instead of 3.9 d), where $\mu(C)$ is the total Milnor number of C.

A generalization: We are also interested in families of curves in $\mathbb{P}^2$ of degree d where for some singularities the analytic type is fixed, for others only the topological type is fixed and for the remaining singularities any deformation is allowed.

Let $C \subset \mathbb{P}^2$ be of degree d and $\mathrm{Sing}(C) = \{x_1, \dots, x_k;\, y_1, \dots, y_\ell;\, z_1, \dots, z_m\}$. We define the subsheaf $(\mathcal{T}_C^1)'$ of $\mathcal{T}_C^1$ by

$$(\mathcal{T}_C^1)'_x = \begin{cases} 0 & \text{if } x \in \{x_1, \dots, x_k\} \\ I^{es}(C,x)/j(C,x) & \text{if } x \in \{y_1, \dots, y_\ell\} \\ T^1_{(C,x)} & \text{else} \end{cases}$$

and put $\tau'(C) := \dim_{\mathbb{C}} H^0(C, \mathcal{T}_C^1/(\mathcal{T}_C^1)')$, that is,

$$\tau'(C) = \sum_{i=1}^{k} \tau(C, x_i) + \sum_{j=1}^{\ell} \tau^{es}(C, y_j).$$

Assume there exists a reduced curve $C' \subset \mathbb{P}^2$ of degree d' with the following properties:

a) C' is irreducible,

b) $\{x_1, \dots, x_k;\, y_1, \dots, y_\ell\} \subset C'$,

c) if f_j is a local equation of (C', x_j) $(1 \le j \le k)$ then $f_j \in j(C, x_j)$; if f_j is a local equation of (C', y_j) $(1 \le j \le \ell)$ then $f_j \in I^{es}(C, y_j)$.

Define $\mathcal{N}' := \mathrm{Ker}\,(\mathcal{O}_C(d) \to \mathcal{T}_C^1/(\mathcal{T}_C^1)')$ and $\widetilde{\mathcal{N}}' := \mathrm{Ker}\,(\mathcal{O}_{C'}(d) \to \mathcal{T}_C^1/(\mathcal{T}_C^1)')$.

Let $\mathcal{A}$ be the analytic singularity type defined by $(C, x_1), \ldots, (C, x_k)$, $\mathcal{T}$ the topological singularity type defined by $(C, y_1), \ldots, (C, y_\ell)$ and let $\mathcal{H}ilb_{\mathbb{P}^2}^{\mathcal{A},\mathcal{T}}$ denote the functor parametrising proper and flat families of reduced curves in $\mathbb{P}^2$ which have k singular points of fixed analytic type $\mathcal{A}$ and ℓ singular points of fixed topological type $\mathcal{T}$. This functor is represented by a locally closed subspace $H_{\mathbb{P}^2}^{\mathcal{A},\mathcal{T}} \subset H_{\mathbb{P}^2}$ (cf. Proposition 2.1).

Proposition 3.11 *Let $C \subset \mathbb{P}^2$ be a reduced projective curve of degree d, $Sing(C) = \{x_1, \ldots, x_k; y_1, \ldots, y_\ell; z_1, \ldots, z_m\}$ and assume that there exists a curve $C' \subset \mathbb{P}^2$ of degree d' satisfying a)–c) above.*

a) If $\ell = 0$, then $H^1(C, \mathcal{N}') = 0$ if and only if $\mathcal{D}ef_{C/\mathbb{P}^2} \to \prod \mathcal{D}ef_{(C,x_i)}$ is surjective.

b) If $H^1(C, \mathcal{N}')$ vanishes, then $H_{\mathbb{P}^2}^{\mathcal{A},\mathcal{T}}$ is smooth at C of dimension $d(d+3)/2 - \tau'(C)$.

c) $H^1(C, \mathcal{N}') = 0$ if $d'(d - d' + 3) > \tau'(C) - isod(\widetilde{\mathcal{N}}', \mathcal{O}_{C'})$.

Proof. Use the same argumentation as for Corollary 3.9. □

Remark 3.12 a) If $\ell = m = 0$ (resp. $k = m = 0$), then a) is the same as 3.9 a) and b) is the same as 3.9 b). If, moreover, C is irreducible we may take $C' = C$ and then c) is also equivalent to 3.9 c). If C is not the union of d lines through one point, we may take C' to be a generic polar and then c) is the same as 3.9 d) for $\ell = m = 0$ (resp. $k = m = 0$).

b) We obtain the best possible result for a curve C' of degree $d' = (d+3)/2$ satisfying a)–c). In [Sh3], Shustin has proven the existence of such an irreducible curve C' of degree

$$d' \leq (2\kappa^2 + \sqrt{\kappa})\sqrt{\mu(C)} + (1 - \kappa^{-1})d,$$

where $\mu(C)$ denotes the total Milnor number and

$$\kappa = \max_{\substack{1 \leq i \leq k \\ 1 \leq j \leq \ell}} \{\mu(x_i, C) + \mathrm{mult}_{x_i}(C),\ \mu(y_j, C) + \mathrm{mult}_{y_j}(C)\} - 1$$

4 Local isomorphism defects of plane curve singularities

Let u, v be local coordinates of the smooth surface S in a singular point x of the reduced compact curve $C \subset S$, $C = C_1 \cup \ldots \cup C_s$ the decomposition into irreducible components and $f(u, v) = 0$ (resp. $f_i(u, v) = 0$) be local equations of (C, x) (resp. (C_i, x)). In the following, we give estimations for the (local) isomorphism defects occurring in Chapter 3:

Lemma 4.1 *For a reduced plane curve singularity $(C,x) \subset (S,x)$ we have:*

a) $isod_x(\mathcal{N}^{ea}_{C/S}, \mathcal{O}_C) = 1$ *if (C,x) is quasihomogeneous.*
$isod_x(\mathcal{N}^{ea}_{C/S}, \mathcal{O}_C) > 1$ *if (C,x) is not quasihomogeneous.*

b) $isod_{C_i,x}(\mathcal{N}^{ea}_{C/S}, \mathcal{O}_C) \geq 1$ *for $i = 1, \ldots, s$.*

Proof.

a) By definition $\operatorname{isod}_x(\mathcal{N}^{ea}_{C/S}, \mathcal{O}_C) = \min \dim_{\mathbb{C}}(\operatorname{coker}\varphi : j(C,x) \to \mathcal{O}_{C,x})$ where the minimum is taken over all $\mathcal{O}_{C,x}$–linear maps. Now the Jacobian ideal $j(C,x)$ of an isolated singularity cannot be generated by a single element and there is an isomorphism $\varphi : j(C,x) \to \mathfrak{m}_{C,x}$ exactly if (C,x) is quasihomogeneous.

b) We have to look for an $\mathcal{O}_{C,x}$–linear map $\varphi : j(C,x) \to \mathcal{O}_{C,x}$ whose restriction to (C_i,x) has minimal cokernel. □

Let (C,x) be quasihomogeneous with positive weight vector $w = (w_1, w_2)$ and (weighted) degree d, then (as an $\mathcal{O}_{C,x}$–ideal) $I^{es}(C,x)$ is generated by the Jacobian ideal $j(C,x)$ and all monomials $u^\alpha v^\beta$ with $w_1\alpha + w_2\beta \geq d$. Furthermore we have the normalization

$$n : \mathcal{O}_{C,x} \hookrightarrow \bar{\mathcal{O}} := \prod_{i=1}^{r} \mathbb{C}\{t\}$$

where r denotes the number of local irreducible components $(C^{(i)}, x)$ of (C,x) (not to be confused with the global components C_i).

In the following, we use the notations $\operatorname{cond}(\mathcal{O})$, $\operatorname{cond}(j)$ resp. $\operatorname{cond}(I^{es})$ for the *conductor ideals* of $\mathcal{O}_{C,x}$, the Jacobian resp. the equisingularity ideal in $\mathcal{O}_{C,x}$, where, for an $\mathcal{O}_{C,x}$–ideal I,

$$\operatorname{cond}(I) := \{\, g \in I \mid g\bar{\mathcal{O}} \subset I \,\}.$$

Furthermore, for all these $\mathcal{O}_{C,x}$–ideals, we denote by $\Gamma(I) \subset \mathbb{N}^r$ the *set of values* of I and by $\underline{c}(I) \in \mathbb{N}^r$ the *conductor* of I, that is $\Gamma(\operatorname{cond}(I)) = \underline{c}(I) + \mathbb{N}^r$.

Lemma 4.2 *Let (C,x) be quasihomogeneous of degree d, then*

a) $isod_x(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C) = \delta(C,x) - \dim_{\mathbb{C}}(I^{es}(C,x)/cond(I^{es}))$*, especially:*
$isod_x(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C) \geq 1$ *with equality if and only if* $j(C,x) = I^{es}(C,x)$.

b) $isod_{C_i,x}(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C) = \dim_{\mathbb{C}}((\mathcal{O}_{C,x}/cond(\mathcal{O})) \otimes \mathcal{O}_{C_i,x})$
$\qquad - \dim_{\mathbb{C}}((I^{es}(C,x)/cond(I^{es})) \otimes \mathcal{O}_{C_i,x})$.

Proof.

a) To calculate $\mathrm{isod}_x(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C)$ we have to consider an $\mathcal{O}_{C,x}$–linear mapping

$$\Psi : I^{es}(C,x) \to \mathcal{O}_{C,x}$$

with minimal cokernel. We know that such a Ψ maps $\mathrm{cond}(I^{es})$ to $\mathrm{cond}(\mathcal{O})$, hence we obtain the estimate

$$\begin{aligned} \mathrm{isod}_x(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C) &\geq \dim_{\mathbb{C}}(\mathcal{O}_{C,x}/\mathrm{cond}(\mathcal{O})) - \dim_C(I^{es}(C,x)/\mathrm{cond}(I^{es})) \\ &= \delta(C,x) - \dim_{\mathbb{C}}(I^{es}(C,x)/\mathrm{cond}(I^{es})) \end{aligned}$$

with equality if and only if there exists a Ψ that maps $\mathrm{cond}(I^{es})$ onto $\mathrm{cond}(\mathcal{O})$, or equivalently (using the isomorpism $\varphi : j(C,x) \to \mathfrak{m}_{C,x}$) if and only if we can find an $\mathcal{O}_{C,x}$–linear mapping

$$\Phi : I^{es}(C,x) \to j(C,x)$$

of weighted degree $\underline{c}(j) - \underline{c}(I^{es})$. Now $\underline{c}(I^{es}) - \underline{1}$ is a maximal (in the sense of [De]) in the semigroup $\Gamma(I^{es}) \supset \Gamma(j)$. Hence, $(\underline{c}(\mathcal{O}) - \underline{c}(j)) + \underline{c}(I^{es}) - \underline{1}$ is a maximal in $\Gamma(\mathcal{O})$ and using the symmetry of $\Gamma(\mathcal{O})$ we see that

$$(\underline{c}(\mathcal{O}) - \underline{1}) - (\underline{c}(\mathcal{O}) - \underline{c}(j) + \underline{c}(I^{es}) - \underline{1}) = \underline{c}(j) - \underline{c}(I^{es}) \in \Gamma(\mathcal{O}).$$

The additional statement is an immediate consequence from the fact that all monomials of degree at least d are contained in $\mathrm{cond}(I^{es})$; thus,

$$\begin{aligned} \dim_{\mathbb{C}}(I^{es}(C,x)/\mathrm{cond}(I^{es})) &= \dim_{\mathbb{C}}(j(C,x)/\mathrm{cond}(I^{es}) \cap j(C,x)) \\ &\leq \dim_{\mathbb{C}}(j(C,x)/\mathrm{cond}(j)) \\ &= \delta(C,x) - 1. \end{aligned}$$

b) Follows from the considerations above. □

Remark 4.3 If (C,x) is quasihomogeneous, then it is easy to see that

$$\begin{aligned} \dim_{\mathbb{C}}((\mathcal{O}_{C,x}/\mathrm{cond}(\mathcal{O})) \otimes \mathcal{O}_{C_i,x}) &= \dim_{\mathbb{C}}(\mathcal{O}_{C_i,x}/\mathrm{cond}(\mathcal{O}_{C_i})) + (D \cdot C_i, x) \\ \dim_{\mathbb{C}}((I^{es}(C,x)/\mathrm{cond}(I^{es})) \otimes \mathcal{O}_{C_i,x}) &\geq \dim_{\mathbb{C}}(I^{es}(C_i,x)/\mathrm{cond}(I^{es}(C_i,x))) \end{aligned}$$

(where C_j, $j = 1, \ldots, s$) are the irreducible components of C and $(D = \bigcup_{j \neq i} C_j)$. Thus we obtain as an upper bound

$$\mathrm{isod}_{C_i,x}(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C) \leq \mathrm{isod}_x(\mathcal{N}^{es}_{C_i/S}, \mathcal{O}_{C_i}) + (D \cdot C_i, x).$$

Furthermore, for integer weights $w_1 \geq w_2$, $\gcd(w_1, w_2) = 1$, the difference is bounded by $(w_1 - 1)/w_2 + 2$.

Examples 4.4 a) For (C,x) an ADE–singularity $\mathrm{isod}_x(\mathcal{N}^{es}_{C/S},\mathcal{O}_C)=1$.

b) If (C,x) is homogeneous of degree $r\geq 3$, then

$$\mathrm{isod}_x(\mathcal{N}^{es}_{C/S},\mathcal{O}_C)=\frac{r\cdot(r-1)}{2}-2=\tau^{es}(C,x)-\mathrm{mult}_x(C)$$

furthermore, let $C=C_i\cup D$ as above, then we obtain for (C_i,x) a smooth branch

$$\mathrm{isod}_{C_i,x}(\mathcal{N}^{es}_{C/S},\mathcal{O}_C)=r-2,$$

while in the singular case

$$\mathrm{isod}_{C_i,x}(\mathcal{N}^{es}_{C/S},\mathcal{O}_C)=(C_i\cdot D,x)+\mathrm{isod}_x(\mathcal{N}^{es}_{C_i/S},\mathcal{O}_{C_i}).$$

More generally, these statements are valid, if (C,x) is an ordinary r–tuple point (r smooth branches with different tangents, $r\geq 3$); in this case each equimultiple deformation is equisingular.

c) If (C,x) has the local equation $u^p-v^q=0$ where $q\geq p\geq 3$ and (C_i,x) consists of $b\leq r=\gcd(p,q)$ irreducible branches, then

$$\mathrm{isod}_{C_i,x}(\mathcal{N}^{es}_{C/S},\mathcal{O}_C)=\frac{b}{2r}\cdot\left(pq\left(2-\frac{b}{r}\right)+(r-p-q)\right)-\left\lceil\frac{q-2}{p}\right\rceil-M,$$

where $M=2$ unless $b=1$ and $q=kp$ $(k\in\mathbb{N})$, then $M=1$.

Now let $S=\mathbb{P}^2$ and $C\subset\mathbb{P}^2$ be different from the union of $d\geq 3$ lines through one point and $C'\subset\mathbb{P}^2$ denote the (irreducible) generic polar of C with affine equation $\alpha f_X+\beta f_Y=0$.

Lemma 4.5 *a) $\mathit{isod}_x(\widetilde{\mathcal{N}}^{ea}_{C'/\mathbb{P}^2},\mathcal{O}_{C'})\geq 0$ with equality if and only if (C,x) is quasihomogeneous.*

b) $\mathit{isod}_x(\widetilde{\mathcal{N}}^{es}_{C'/\mathbb{P}^2},\mathcal{O}_{C'})\geq\delta(C',x)-\dim_{\mathbb{C}}(I^{es}\otimes\mathcal{O}_{C',x}/\mathit{cond}(I^{es}\otimes\mathcal{O}_{C',x}))$.

Proof. By definition, equality in a) is equivalent to the statement that the $\mathcal{O}_{C',x}$–ideal generated by f,f_X and f_Y is generated by one single element. Obviously, this holds exactly if (C,x) is quasihomogeneous, b) follows from the considerations in the proof of Lemma 4.2. □

Remark 4.6 The generic polar C' depends on the whole curve C and not only on the germ (C,x), hence, in general it is *not* enough to know the local equation of (C,x) to be able to calculate $\mathrm{isod}_x(\widetilde{\mathcal{N}}^{es}_{C'/\mathbb{P}^2},\mathcal{O}_{C'})$. For example, if (C,x) is homogeneous of degree $d\geq 6$, (C',x) need not be quasihomogeneous. But, in special cases, we are able to give explicit formulas:

Examples 4.7 a) For (C,x) an ADE–singularity $\mathrm{isod}_x(\widetilde{\mathcal{N}}^{es}_{C'/\mathbb{P}^2}, \mathcal{O}_{C'}) = 0$.

b) If (C,x) has the (homogeneous) local equation $(u^r - v^r = 0)$ $(r \geq 3)$ then (C',x) has an equation $(\tilde{u}^{r-1} - \tilde{v}^{r-1} = 0)$ and

$$\mathrm{isod}_x(\widetilde{\mathcal{N}}^{es}_{C'/\mathbb{P}^2}, \mathcal{O}_{C'}) = \frac{r \cdot (r-3)}{2}.$$

Moreover, in the case of a (not homogeneous) local equation $(u^p - v^q = 0)$ $(q > p \geq 3)$, (C',x) has an equation $(\tilde{u}^{p-1} - \tilde{v}^{q-1} = 0)$ and we obtain the estimate

$$\mathrm{isod}_x(\widetilde{\mathcal{N}}^{es}_{C'/\mathbb{P}^2}, \mathcal{O}_{C'}) \geq \frac{(p-3)(q-1) + 2\,\mathrm{gcd}(p-1,q-1) - \mathrm{gcd}(p,q) - 1}{2} + \varepsilon$$

where $\varepsilon = -\left[\frac{q}{p}\right]$ unless p divides q, then $\varepsilon = 1 - \left[\frac{q}{p}\right]$.

Problem: Do the different isomorphism defects considered above behave (lower) semicontinuous under equianalytic resp. equisingular deformations of C/S?

5 Applications and Examples

Corollary 5.1 *a) Let $C \subset \mathbb{P}^2$ be a curve of degree d with ordinary (k_i–)multiple points ($i = 1, \ldots, N$) as the only singularities, $C = C_1 \cup \ldots \cup C_s$ its decomposition into irreducible components (deg $C_i = d_i$) then $(H^{es}_{\mathbb{P}^2}, C)$ is smooth of dimension*

$$\frac{d(d+3)}{2} - \sum_{j=1}^{N} \left(\frac{k_j \cdot (k_j+1)}{2} - 2 \right),$$

if for i=1,...,s

$$3d_i > \sum_{\substack{x \in C_i \cap Sing\, C \\ mult_x(C) > 2}} mult_x(C_i)$$

where $mult_x(C)$ (resp. $mult_x(C_i)$) denote the multiplicity of C (resp. C_i) at x.

b) Let $C \subset \mathbb{P}^2$ be a curve of degree d whose singularities are all of local equations $(u^{p_i} - v^{q_i} = 0)$ $(q_i \geq p_i)$ or ADE–singularities, then $(H^{es}_{\mathbb{P}^2}, C)$ is smooth of dimension

$$\frac{d(d+3)}{2} - \sum_{i=1}^{N} \left(\frac{(p_i+1)(q_i+1) - gcd(p_i,q_i) - 5}{2} - \left[\frac{q_i}{p_i}\right] + \varepsilon_i \right)$$

where $\varepsilon_i = 0$ *unless* p_i *divides* q_i*, then* $\varepsilon_i = 1$*, if*

$$4d > 4 + \sum_{\{ADE\}} \mu(C,x) + \sum_{\{not\ ADE\}} (p_i + 2q_i - 3 - gcd(p_i - 1, q_i - 1)).$$

Proof. The statements follow immediately from Corollary 3.9 and Example 4.4 b), resp. Example 4.7 b). □

Remark 5.2 The result in (i) was already obtained by C. Giacinti-Diebolt ([Gia]) using vanishing theorems on the normalization of C. It implies the ancient result of Severi [Sev] that for a curve C with no other singularities but ordinary double points $(H^{es}_{\mathbb{P}^2}, C)$ is smooth.

Another consequence of the calculations in Chapter 4 is the following: the contribution of a quasihomogeneous singularity (C,x) with local equation

$$(u^p - v^q + uv\tilde{f}(u,v) = 0)\ ,\ \ q \geq p \geq 3\ ,$$

to the right–hand side in the $3d$–criterion for an irreducible curve C is

$$\tau^{es}(C,x) - \ \mathrm{isod}_x(\mathcal{N}^{es}_{C/S}, \mathcal{O}_C) = p + q - gcd(p,q) - \varepsilon$$

where $\varepsilon = 0$, unless $q \equiv 1 \bmod p$, then $\varepsilon = 1$, while the contribution of an A_k–singularity is $k-1$. This corresponds to the result of E. Shustin in [Sh2].

Nevertheless, in some cases the new $4d$–criterion gives more information:

Example 5.3 A) $(x^4 - x^2z^2 + y^2z^2 + y^3z)\, y\, (x+2y+z)\, (x-2y-z) = 0$ defines a *reducible* curve $C \subset \mathbb{P}^2$ having exactly 3 ordinary triple points lying on 1 line (hence, Corollary 5.1 does *not* apply) and 7 ordinary double points. But $4d = 28 > 23 = 4 + \tau(C)$; hence, $(H^{ea}_{\mathbb{P}^2}, C)$, resp. $(H^{es}_{\mathbb{P}^2}, C)$ are smooth of dimension 16.

In general, it is a difficult problem to determine for a given d whether there exists a projective plane curve of degree d having a fixed number of singularities of given *analytic* type. On the other hand, the local deformations of a plane curve singularity are well understood. Hence, knowing about the existence of one low degree curve with "big" singularities our $4d$–criterion allows us to give positive answers to some of the above existing problems (answers which we did not obtain with the $3d$–criterion in [GrK]).

Remark 5.4 a) The surjectivity statement in 3.9 a) implies: let $C \subset \mathbb{P}^2$ be of degree d such that $H^1(C, \mathcal{N}^{ea}_{C/\mathbb{P}^2}) = 0$ and let $\{x_1, \ldots, x_n\}$ be any subset of $\mathrm{Sing}(C)$. If, for $i = 1, \ldots, n$, the germ (C, x_i) admits a deformation with nearby fibre having singularities $y_i^1, \dots, y_i^{s_i}$, then C admits an embedded deformation with nearby curve $C_t \subset \mathbb{P}^2$ having $y_1^1, \dots, y_1^{s_1}, \dots, y_n^1, \dots, y_n^{s_n}$ as singularities. Hence, there exists a curve of degree d with the y's as singularities.

b) The $4d$–criterion in 3.9 has the advantage that C need not be irreducible. On the other hand, in the $3d$–criterion in 3.6 and 3.9 we can completely forget about A_1–singularities on C. By Lemma 4.1, for a node we have $\tau(C_i, x)-$ $\mathrm{isod}_{C_i,x}(\mathcal{N}^{ea}_{C/\mathbb{P}^2}, \mathcal{O}_C) \le 0$, which can be neglected in the right–hand side of the $3d$–criterion (we actually obtain -1 if the node results from the intersection of two global components of C).

c) Since, for a node $\mathcal{N}^{ea}_{C/S,x} = \mathcal{N}^{es}_{C/S,x}$, $\tau_{C,x} = \tau^{es}_{C,x}$ and since the isomorphism defect is a local invariant, we can neglect nodes also in the $3d$–formulas for es. For the $4d$–criterion, however, nodes have to be counted with 1.

Examples 5.5 B) The irreducible curve $C \subset \mathbb{P}^2$ of degree 6 with affine equation

$$\begin{aligned} f(x,y) \;=\; & y^2-2x^2y+c_1xy^2+c_2y^3+x^4-2c_1x^3y+c_3x^2y^2+c_4xy^3+c_5y^4 \\ & +c_1x^5-(3c_2+2c_3)x^4y-(2c_4+2)x^3y^2-(2c_1+2c_5)x^2y^3+c_6xy^4 \\ & -(c_4+2)y^5+(2c_2+c_3)x^6+(c_4+2)x^5y+(2c_1+c_5)x^4y^2-c_6x^3y^3 \\ & +(c_4+3)x^2y^4+c_1xy^5-(3c_2+c_3+c_6)y^6 \end{aligned}$$

where
α, β are (complex) solutions of $4\alpha^2-30\alpha+55=0$ and $\beta^3=\alpha^2-7\alpha+12$, the constants c_i $(1 \le i \le 6)$ are defined as $c_1 := 16\alpha\beta^2-66\beta^2$, $c_2 := 3\alpha\beta-\frac{23}{2}\beta$, $c_3 := -(\beta+7c_2)$, $c_4 := -4\alpha+13$, $c_5 := \beta^2-c_1$, $c_6 := 24\alpha\beta-92\beta$, has exactly one singularity, which is of type A_{19}. (The equation of this curve was found by H. Yoshihara (cf. [Yos])). Now $4d = 24 > 23 = \tau(C)+4$ and, hence, each combination of A–singularities given by an adjacent subdiagram of A_{19} occurs on a curve of degree 6 (this is a very simple proof of a well–known result which was previously proved by using moduli theory of $K3$–surfaces).

C) The curve $C \subset \mathbb{P}^2$ with homogeneous equation $x^9+zx^8+z(xz^3+y^4)^2=0$ has exactly one singularity at $(0\!:\!0\!:\!1)$ which is of type A_{31}. Again we have $4d = 36 > 35 = \tau(C)+4$, hence $H^{ea}_{\mathbb{P}^2}$ is smooth at C. C is obtained by a small deformation of Luengo's example of a degree 9 curve ($x^9+z(xz^3+y^4)^2=0$, having an A_{35}–singularity) with non–smooth $(H^{ea}_{\mathbb{P}^2}, C)$. Of course, our criterion supports also this non–smoothness since $4d = 36 < 39 = \tau(C)+4$.

D) $x^7+y^7+(x-y)^2x^2y^2z=0$ defines an irreducible curve $C \subset \mathbb{P}^2$ which has 3 transverse cusps (not quasihomogeneous!) at $(0\!:\!0\!:\!1)$ and no other singularities. Since $4d = 28 > 24+4-1 \ge \tau(C)+4-\mathrm{isod}_{(0:0:1)}(\widetilde{\mathcal{N}}^{ea}_{C'/\mathbb{P}^2}, w_{C'})$

we see that every local deformation of 3 transverse cusps can be realized by curves of degree 7.

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