Preprint Nr. 263 (Univ. Kaiserslautern)

Existence of twistor spaces of algebraic dimension two over the connected sum of four complex projective planes

F. Campana
Université de Nancy
F–54 506 Vandœuvre les Nancy
France

B. Kreußler
Universität Kaiserslautern
D–67 653 Kaiserslautern
Germany

April 10, 1995

Abstract

We prove the existence of twistor spaces of algebraic dimension two over the connected sum of four complex projective planes $4\mathbb{CP}^2$. For this purpose we develop a method to distinguish between twistor spaces of algebraic dimension one and two.

1 Introduction

Twistor spaces were introduced by R. Penrose [Pe]. The first rigorous mathematical foundation of these ideas was given by M. Atiyah, N. Hitchin and I. Singer in their now classical paper [AHS]. The twistor construction gives a close relationship between three–dimensional complex geometry and real four–dimensional conformal self–dual geometry.

1991 Mathematics Subject Classification: 32L25, 32J17, 32J20

From our point of view, a twistor space Z is a complex three–manifold with the following additional structure:

- A proper differentiable submersion $\pi : Z \to M$ onto a real differentiable four–manifold M. The fibres of π are holomorphic curves in Z being isomorphic to $\mathbb{CP}^1$ and having normal bundle in Z isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$.
- An anti–holomorphic fixed–point free involution $\sigma : Z \to Z$ with $\pi\sigma = \pi$.

Note that π is not holomorphic (M carries, in general, no almost complex structure). The fibres of π are called "real twistor lines" and the involution σ is called the "real structure". Anything which is invariant under σ will obtain the attribute "real". For example, a complex subvariety $D \subset Z$ is "real" if $\sigma(D) = D$. We shall only consider twistor spaces which are both *compact* and *connected*.

The first classification results are due to N. Hitchin [H2] and T. Friedrich, H. Kurke [FK]. They showed that there exist exactly two compact twistor spaces which are Kählerian. Such a twistor space Z is automatically Fano (that is: $-K_Z$ is ample, so that Z is in particular projective). The corresponding Riemannian four–manifolds are the 4–sphere S^4 and complex projective plane $\mathbb{CP}^2$ (with Fubini–Study metric).

New examples of twistor spaces of algebraic dimension three (that is Moishezon) were first produced by Y.S. Poon [Po1]. The manifold M was $2\mathbb{CP}^2$ there. The corresponding Z are not projective or even Kähler, although they are bimeromorphic to $\mathbb{P}^3$.

A partial generalization of the results of [H2] and [FK] was obtained in [C2], where it is shown that if Z is a twistor space of the class $\mathcal{C}$ (that is bimeromorphic to some compact Kähler manifold), then Z is Moishezon and simply connected. This implies, using deep results of M. Freedman [F] and S. Donaldson [Don], that M is homeomorphic to $n\mathbb{CP}^2$ for some $n \geq 0$.

In complete anology with the Kähler case, it was also established in [Po3] that if Z is a simply connected twistor space, its algebraic dimension is equal to $\kappa(Z, K_Z^{-1})$, the Iitaka dimension of its anti–canonical bundle. Hence, if Z is Moishezon, it is weakly Fano (that means $\kappa(Z, K_Z^{-1}) = 3$). From results of M. Ville and P. Gauduchon follows that "simply connected" can be relaxed to $b_1(Z) = 0$ in the result of [Po3].

Further examples of twistor spaces Z over $M = n\mathbb{CP}^2$ were given in [DonF] for $n = 4$ with $a(Z) = 1$ and for $n \geq 5$ with $a(Z) = 0$ and in [LeB1] and [Ku] for $n \geq 0$ with $a(Z) = 3$. It does not seem to be known whether (for fixed n) these families belong to the same deformation family.

It seems to be natural to consider the following

Problems:

(i) *What are the possible algebraic dimensions of twistor spaces Z over $M = n\mathbb{CP}^2$?*

(ii) *Compute the algebraic dimension of such twistor spaces Z in terms of given divisors on Z.*

There are several results concerning these problems. To explain them, we need the notion of the "type" of a twistor space. To define this notion, we look at twistor spaces from the Riemannian point of view, that is on the conformal class of self–dual metrics on M. By a result of R. Schoen [Sch], every conformal class of a compact Riemannian four–manifold contains a metric of constant scalar curvature. Its sign will be called the *type* of the twistor space. This is an invariant of the conformal class, hence of the twistor space.

In this paper we focus on the case of *positive type*. This is because in this case the vanishing theorem of Hitchin (2.1) can be applied.

The classification results of N. Hitchin [H2], T. Friedrich, H. Kurke [FK], Y.S. Poon [Po1] and B. Kreußler, H. Kurke [KK] imply:

If Z is a twistor space over $n\mathbb{CP}^2$ with $n \leq 3$, then Z has algebraic dimension three if and only if it has positive type.

Since $c_1(Z)^3$ has the same sign as $4-n$, the case $n=4$ is somehow exceptional. According to results of F. Campana [C1] and C. LeBrun, Y.S.Poon [LeBP], in the case $n=4$ there exist twistor spaces Z with $a(Z)=1$. In his paper [Po2] Poon expressed the algebraic dimension of a twistor space in terms of divisors whose restriction to twistor lines have degree one. In case $n=4$, the algebraic dimension is at least one. Poon's methods allow one to decide whether or not the algebraic dimension is three. See also [PP2].

Investigating the anti–canonical system on certain fundamental divisors (that is elements in $|-\frac{1}{2}K|$), we will be able to distinguish between algebraic dimension one and two. Our main theorem (Theorem 3.4) will be

Theorem: *Let Z be a twistor space of positive type over $4\mathbb{CP}^2$ and S a real fundamental divisor. Assume that $|-K_S|$ contains a smooth curve C. Let $N=N_{C|S}$ be the normal bundle of $C\subset S$. Then N is of degree zero in $\mathrm{Pic}\,(C)$. We have:*

$$a(Z)\leq 2 \qquad \textit{and}$$

$$a(Z)=2 \iff N \quad \textit{is of finite order in the group } \mathrm{Pic}^0(C).$$

Using this result we obtain (Theorem 4.1):

Theorem: *There exist twistor spaces over $4\mathbb{CP}^2$ having algebraic dimension two.*

Acknowledgement *The second author thanks the Université de Nancy for its hospitality during his stay in the autumn of 1994. This stay was supported by INTERREG.*

2 Generalities

This section collects known facts about twistor spaces. For proofs and more information the reader is referred to [ES], [H2], [Kr], [Po1].

Let Z be a simply connected twistor space with $h^2(Z,\mathcal{O}_Z)=0$.

Then $H^i(Z,\mathbb{Z})$ is a free $\mathbb{Z}$–module, which vanishes for i odd. We define the integer n

by the equation $n+1 = \text{rank}\, H^2(Z,\mathbb{Z})$. The Chern–numbers of Z are:

$$\begin{aligned} c_1^3 &= 16(4-n) \\ c_1c_2 &= 24 \\ c_3 &= 2(n+2). \end{aligned}$$

Cohomology of sheaves

For any twistor space we have $h^3(Z,\mathcal{O}_Z) = 0$. By assumption $h^i(Z,\mathcal{O}_Z) = 0$ for $i = 1,2$. In particular, we obtain an isomorphism of abelian groups, given by the first Chern–class:

$$\text{Pic}\,(Z) \xrightarrow{\sim} H^2(Z,\mathbb{Z}).$$

There exists a unique line bundle whose first Chern–class is $\frac{1}{2}c_1$. We will denote it by $K^{-\frac{1}{2}}$. Poon calls it the *fundamental* line bundle. The divisors in the linear system $|-\frac{1}{2}K|$ will be called *fundamental divisors.*

The degree of a line bundle $\mathcal{L} \in \text{Pic}\,(Z)$ will be by definition the degree of its restriction to a real twistor line. For example, $\deg(K^{-\frac{1}{2}}) = 2$. We obtain in this way a *surjective* degree map

$$\deg : \text{Pic}\,(Z) \twoheadrightarrow \mathbb{Z}.$$

From the above equations on Chern–numbers we obtain, by applying the Riemann–Roch theorem,

$$\chi(Z, K^{-\frac{m}{2}}) = m + 1 + \tfrac{1}{3}(4-n)m(m+1)(m+2). \tag{1}$$

Vanishing theorem

The most important tool in classifying twistor spaces is Hitchin's vanishing theorem:

Theorem 2.1 (Hitchin [H1]) *If Z is of positive type then we have for any $\mathcal{L} \in$* Pic(Z)

$$\deg(\mathcal{L}) \leq -2 \quad \Rightarrow \quad H^1(Z, \mathcal{L}) = 0.$$

On the other hand, since the twistor lines cover Z, we obtain:

$$\deg(\mathcal{L}) \leq -1 \quad \Rightarrow \quad H^0(Z, \mathcal{L}) = 0.$$

By Serre–duality this gives the following important vanishing results:

$$\deg(\mathcal{L}) \geq -2 \quad \Rightarrow \quad H^2(Z, \mathcal{L}) = 0 \tag{2}$$

$$\deg(\mathcal{L}) \geq -3 \quad \Rightarrow \quad H^3(Z, \mathcal{L}) = 0. \tag{3}$$

3 Computation of the algebraic dimension

Throughout this section we make the following assumptions: Z is a twistor space over $4\mathbb{CP}^2$, which is *simply connected* and of *positive type*. In particular: $h^1(Z, \mathcal{O}_Z) = h^2(Z, \mathcal{O}_Z) = 0$.

From section 2 we know

$$h^0(K^{-\frac{m}{2}}) = m + 1 + h^1(K^{-\frac{m}{2}}) \qquad \forall m \geq -1 \tag{4}$$

in particular $\dim |-\frac{1}{2}K| \geq 1$. The property $(-\frac{1}{2}K)^3 = 0$ will be important in what follows.

Lemma 3.1 *The linear system $|-\frac{1}{2}K|$ contains a smooth real element.*

Proof: This is proved in [PP1, Lemma 2.1] for the case that $|-\frac{1}{2}K|$ contains an irreducible element. If this is not the case, we would have infinitely many reducible elements in this linear system, hence infinitely many divisors of degree 1 in Z. But by results of Kurke [Ku] and Poon [Po2] this occurs only in the case of conic–bundle twistor spaces . For such twistor spaces the Lemma is also true, which completes the proof. □

Remark 3.2 *The proof shows even more: if Z is an arbitrary compact twistor space with* $\dim|-\frac{1}{2}K|\geq 1$, *then there exists a smooth real divisor in* $|-\frac{1}{2}K|$.

Let now $S\in|-\frac{1}{2}K|$ be a real smooth divisor. Using the exact sequence

$$0\to K^{\frac{1}{2}}\to\mathcal{O}_Z\to\mathcal{O}_S\to 0$$

and Hitchin's vanishing theorem we obtain $h^i(\mathcal{O}_S)=0$ for $i>0$. The adjunction formula gives $K_S^{-1}=K^{-\frac{1}{2}}\otimes\mathcal{O}_S$. From the exact sequence $0\to\mathcal{O}_Z\to K^{-\frac{1}{2}}\to K_S^{-1}\to 0$ we obtain

$$H^i(Z,K^{-\frac{1}{2}})\cong H^i(S,K_S^{-1})\qquad\forall i\geq 1$$

and a surjective restriction map $H^0(Z,K^{-\frac{1}{2}})\twoheadrightarrow H^0(S,K_S^{-1})$ with one–dimensional kernel. In particular, $h^0(S,K_S^2)=h^2(S,K_S^{-1})=h^2(Z,K^{-\frac{1}{2}})=0$ and $h^1(S,\mathcal{O}_S)=0$. Hence, by Castelnuovo's criterion, S is a rational surface.
This leads to the following facts:

(i) $|-\frac{1}{2}K|$ and $|-K_S|$ have the same base locus

(ii) $h^0(K_S^{-1})=h^0(K^{-\frac{1}{2}})-1=1+h^1(K_S^{-1})\geq 1$

(iii) $(-K_S)^2=(-\frac{1}{2}K)^3=0$.

Lemma 3.3 *If* $\dim|-\frac{1}{2}K|\geq 2$ *then:*
$|-\frac{1}{2}K|$ *has no basepoints* $\Longleftrightarrow$ $|-K_S|$ *contains a smooth curve.*

Proof: By assumption we have $\dim|-K_S|\geq 1$.
($\Longrightarrow$) is a consequence of Sard's or Bertini's theorem.

($\Longleftarrow$) Assume $|-\frac{1}{2}K|$ has basepoints. Since $(-K_S)^2 = 0$, the system $|-K_S|$ has no isolated basepoints. But since $\dim|-K_S| \geq 1$ there also exist moving components, hence $|-K_S|$ would contain reducible elements only. But all elements of $|-K_S|$ are connected since $h^1(K_S) = h^1(\mathcal{O}_S) = 0$. This contradicts the existence of a smooth curve in $|-K_S|$. $\square$

Therefore, we have to distinguish three cases:

(i) $|-K_S|$ contains a smooth curve

(ii) $\dim|-\frac{1}{2}K| \geq 2$ and it has basepoints

(iii) $\dim|-\frac{1}{2}K| = 1$ and $|-K_S| = \{C\}$ with a non–smooth curve C.

In the rest of this section we will discuss the first case.

Let Z and S be as above and $C \in |-K_S|$ a smooth curve. By adjunction we obtain $K_C = \mathcal{O}_C$, hence C is an *elliptic curve*. Let $N = N_{C|S} = K_S^{-1} \otimes \mathcal{O}_C$ denote the normal bundle of C in$S$. From $(-K_S)^2 = 0$ we obtain: $\deg N = 0$. Let τ be the order of N in $\mathrm{Pic}^0(C)$.

With this notation we obtain the following

Theorem 3.4

$$\tau = \infty \quad \Longleftrightarrow \quad a(Z) = 1 \tag{5}$$

$$\tau < \infty \quad \Longleftrightarrow \quad a(Z) = 2 \tag{6}$$

If $1 \leq m < \tau$ *, we have* $h^0(K^{-\frac{m}{2}}) = m + 1$ *and* $h^1(K^{-\frac{m}{2}}) = 0$.

If $1 \leq m < \tau$ *and* $i \geq 0$*, we have* $h^i(K_S^{-m}) = h^i(\mathcal{O}_S)$.

If $\tau < \infty$*, then we have furthermore:*

$|-\frac{\tau}{2}K|$ *has no basepoints,* $h^0(K^{-\frac{\tau}{2}}) = \tau + 2$ *and* $h^1(K^{-\frac{\tau}{2}}) = h^1(K_S^{-\tau}) = 1$.

Proof: We will basically rely on the result of Poon [Po3] that $a(Z) = \kappa(Z, K^{-1})$ (the Iitaka dimension of the anti–canonical line bundle). Therefore, from $h^0(K^{-\frac{1}{2}}) \geq 2$ we immediately see: $a(Z) \geq 1$.

Consider the following two exact sequences: $(m \in \mathbb{Z})$

$$0 \to K_S^{-(m-1)} \to K_S^{-m} \to N^{\otimes m} \to 0 \tag{7}$$

$$0 \to K^{-\frac{m-1}{2}} \to K^{-\frac{m}{2}} \to K_S^{-m} \to 0 \tag{8}$$

Observe that $h^0(N^{\otimes m}) = h^1(N^{\otimes m}) = 1$ if $N^{\otimes m} \cong \mathcal{O}_C$. In all other cases these numbers vanish. This is because C is elliptic and $\deg N = 0$.

If $\underline{\tau = \infty}$, for all $m \neq 0$ we have $N^{\otimes m} \ncong \mathcal{O}_C$, hence (7) gives:
$h^i(K_S^{-(m-1)}) = h^i(K_S^{-m})$ $\forall m \geq 1$, hence

$$h^i(K_S^{-m}) = h^i(\mathcal{O}_S) \quad \forall m \geq 1.$$

Using $h^1(\mathcal{O}_Z) = 0$, the sequence (8) gives by induction on $m \geq 1$ that

$$h^0(K^{-\frac{m}{2}}) = 1 + h^0(K^{-\frac{m-1}{2}}) \quad \text{and} \quad h^1(K^{-\frac{m}{2}}) = 0.$$

Hence, for all $m \geq 0$ we obtain

$$h^0(K^{-\frac{m}{2}}) = m + 1 \quad \text{and} \quad h^1(K^{-\frac{m}{2}}) = 0.$$

Therefore, we have $a(Z) = \kappa(Z, K^{-1}) = 1$.

If $\underline{\tau < \infty}$, we obtain by the same arguments as above: $\forall 1 \leq m < \tau$

$$h^i(K_S^{-m}) = h^i(\mathcal{O}_S) \quad \forall i$$

$$\text{and} \qquad h^0(K^{-\frac{m}{2}}) = m + 1 \qquad h^1(K^{-\frac{m}{2}}) = 0.$$

But for $m = \tau$ we, therefore, obtain from (7) and(8) the exact sequences:

$$0 \to H^0(K_S^{-(\tau-1)}) \to H^0(K_S^{-\tau}) \to H^0(\mathcal{O}_C) \to H^1(K_S^{-(\tau-1)}) \to$$
$$\to H^1(K_S^{-\tau}) \to H^1(\mathcal{O}_C) \to H^2(K_S^{-(\tau-1)}) = 0$$

and

$$0 \to H^0(K^{-\frac{\tau-1}{2}}) \to H^0(K^{-\frac{\tau}{2}}) \to H^0(K_S^{-\tau}) \to H^1(K^{-\frac{\tau-1}{2}}) \to$$
$$\to H^1(K^{-\frac{\tau}{2}}) \to H^1(K_S^{-\tau}) \to H^2(K^{-\frac{\tau-1}{2}}) = 0.$$

Therefore, we have $h^0(K_S^{-\tau}) = 2, h^1(K_S^{-\tau}) = 1$ and

$$h^0(K^{-\frac{\tau}{2}}) = h^0(K^{-\frac{\tau-1}{2}}) + h^0(K_S^{-\tau}) = \tau + 2$$

$$h^1(K^{-\frac{\tau}{2}}) = h^1(K_S^{-\tau}) = 1.$$

Next we prove in the case of finite τ that $|-\frac{\tau}{2}K|$ has no basepoints. Since $\tau S \in |-\frac{\tau}{2}K|$, we see from the exact sequence

$$0 \to H^0(K^{-\frac{\tau-1}{2}}) \to H^0(K^{-\frac{\tau}{2}}) \to H^0(K_S^{-\tau}) \to 0$$

that $| - \frac{\tau}{2}K|$ and $| - \tau K_S|$ have the same base locus.

But the exact sequence

$$0 \to H^0(K_S^{-(\tau-1)}) \to H^0(K_S^{-\tau}) \to H^0(\mathcal{O}_C) \to 0$$

tells us that there is a section in $H^0(K_S^{-\tau})$ which does not vanish at any point of C.

Hence, the base locus of $| - \tau K_S|$ is empty.

We continue to assume $\tau < \infty$ and show $a(Z) \geq 2$.

Let $\Phi_\tau : Z \to \mathbb{P}^{\tau+1}$ be the morphism defined by the linear system $| - \frac{\tau}{2}K|$. Let $X_\tau \subset \mathbb{P}^{\tau+1}$ be the image of Φ_τ. Assume $a(Z) = 1$, then by $a(Z) = \kappa(Z, K^{-\frac{1}{2}})$ we have $\dim X_\tau = 1$. By definition of Φ_τ the curve X_τ is not contained in a proper linear subspace of $\mathbb{P}^{\tau+1}$. From a well–known classical result we have, therefore, $\deg X_\tau \geq \tau + 1$. Since X_τ is irreducible and reduced, a generic hyperplane section meets this curve at exactly $\deg X_\tau$ distinct points. Hence, the generic element in $| - \frac{\tau}{2}K|$ is the sum of $\deg X_\tau$ disjoint, but algebraically (even linearly, since $\pi_1(Z) = 0$ forces X_τ to be a rational curve) equivalent divisors in Z. Thus, alternatively, because

$\mathrm{Pic}(Z) \cong H^2(Z,\mathbb{Z})$ is a free abelian group, we obtain $-\frac{\tau}{2}K = \delta K_0$. Here K_0 is a fibre of Φ_τ and $\delta = \deg X_\tau \geq \tau + 1$. Computation of the degree gives $2\tau = \delta \deg K_0$. Therefore, $\delta = 2\tau$ and $\deg K_0 = 1$. Then there were infinitely many divisors of degree 1 in Z, leading again to the case of conic–bundle twistor spaces. But for such twistor spaces we have $h^0(K^{-\frac{1}{2}}) = 4$ in contradiction to our computation. This contradiction shows: $\dim X_\tau \geq 2$, hence $a(Z) \geq 2$.

To finish the proof, we should show $a(Z) \leq 2$. We can easily derive this from the sequences (7) and (8) as follows: From (8) we obtain for all $m \in \mathbb{Z}$

$$P(m) := h^0(K^{-\frac{m}{2}}) - h^0(K^{-\frac{m-1}{2}}) \leq h^0(K_S^{-m})$$

and from (7)

$$h^0(K_S^{-m}) - h^0(K_S^{-(m-1)}) \leq h^0(N^{\otimes m}) \leq 1.$$

Thus $P(m)$ grows at most linearly in m. Hence, $h^0(K^{-\frac{m}{2}})$ grows at most quadratically. This means (cf. [C1]): $a(Z) = \kappa(Z, K^{-\frac{1}{2}}) \leq 1 + \kappa(S, K_S^{-1}) \leq 2 + \kappa(C, N) \leq 2$, as desired. $\square$

4 The existence theorem

In this section we apply the results of Section 3 to prove the following theorem:

Theorem 4.1 *There exist twistor spaces over $4\mathbb{CP}^2$ which have algebraic dimension two.*

One method to establish the existence of twistor spaces is to study the deformation theory of known twistor spaces. It was developed by Donaldson–Friedman [DonF] (see also Campana [C1] and LeBrun [LeB2]). We use the following theorem:

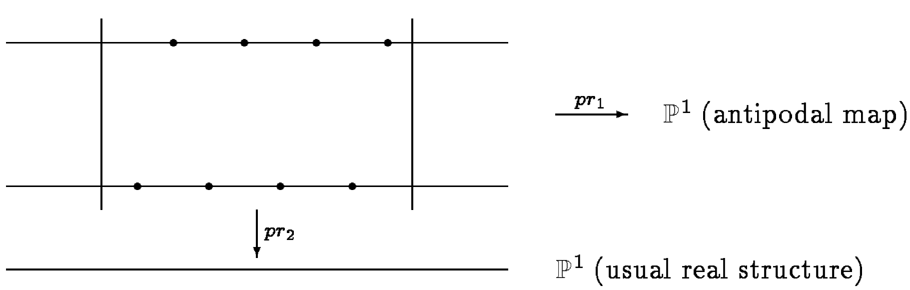
Theorem 4.2 ([C1], [C3], [DonF], [PP2]) *Let Z be a conic–bundle twistor space, $n \geq 4$ and $S \in |-\frac{1}{2}K|$ a smooth real divisor. Then:*

Any real member of a small deformation of Z is again a twistor space. Furthermore, any small deformation of S with real structure is induced by a deformation of Z in the sense that the deformed surfaces are members of the fundamental system of the deformed twistor spaces.

We prove Theorem 4.1 by constructing a small deformation with real structure of a smooth rational surface $S = S_0 \in |-\frac{1}{2}K|$ in a certain twistor space. We will prove that there exist real deformed surfaces containing in their anticanonical system a smooth curve whose normal bundle is a torsion element in the Picard–group. The result then follows by Theorems 3.4 and 4.2.

We will use the conic–bundle twistor spaces, which are discovered by C. LeBrun [LeB1] and investigated by H. Kurke [Ku]. They are described as modifications of conic–bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. Let us recall the structure of general real surfaces $S \in |-\frac{1}{2}K|$ in generic conic–bundle twistor spaces [Ku], [LeB1].

S contains a twistor line $F \subset S$. On S we have $(F^2) = 0$ and $\dim |F| = 1$. Since $(K_S)^2 = 0$, we obtain: the morphism $\pi : S \to \mathbb{P}^1$ defined by $|F|$ exibits S as a successive blow–up of eight points on a ruled surface. More precisely, we know (cf. [PP2]) that π factors over a blow–up $S \to \mathbb{P}^1 \times \mathbb{P}^1$ which is compatible with real structures. The real structure on $\mathbb{P}^1 \times \mathbb{P}^1$ is given by the antipodal map on the first factor and the usual real structure on the second factor. The blown–up set is real and consists of eight distinct points, lying on two conjugate fibres of the first projection. The fibres of the second projection correspond to the elements of $|F|$. This situation is illustrated by the folowing picture:



None of the blown–up points lie on a real fibre of the second projection since all real elements in $|F|$ are irreducible (real twistor lines).

To begin our construction, we fix a blowing–up $S = S_0 \to \mathbb{P}^1 \times \mathbb{P}^1$ of eight points $\{P_1, \ldots, P_8\}$ as described above and choose a curve $C_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ of type $(2,2)$ containing $\{P_1, \ldots, P_8\}$ as smooth points.

Consider the universal family of curves of type $(2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ which is given as $\mathcal{C} = \{(C,x) | x \in C\} \subset |\mathcal{O}(2,2)| \times (\mathbb{P}^1 \times \mathbb{P}^1)$. The family $\mathcal{C} \to B := |\mathcal{O}(2,2)|$ is a deformation of C_0 with basis $B \cong \mathbb{P}^8$. By $0 \in B$ we denote the point corresponding to C_0 and by $C_t \subset \mathbb{P}^1 \times \mathbb{P}^1$ the fibre over $t \in B$.

By $B' \to B$ we denote the 8–fold fibre product $B' = \mathcal{C} \times_B \mathcal{C} \times_B \ldots \times_B \mathcal{C} \to B$. Then, the pulled back family $\mathcal{C}' := \mathcal{C} \times_B B' \to B'$ has eight natural sections $P_1, \ldots, P_8$, given by the projections $pr_i : B' \to \mathcal{C}$.

We denote by $\overline{B}$ the quotient under the natural action of the symmetric group S_8 on the open set of 8–tuples of pairwise distinct points (which is the complement of all twofold diagonals). The family $\mathcal{C}' \to B'$ descends to a family $\overline{\mathcal{C}} \to \overline{B}$. Since the action of S_8 is fibre preserving for $B' \to B$, we obtain a morphism $\overline{B} \to B$. The union of the eight sections $P_1, \ldots, P_8$ is S_8–invariant, hence over $\overline{B}$ we obtain a family of subschemes of $\mathbb{P}^1 \times \mathbb{P}^1$ of length eight $Z = \bigcup_{i=1}^{8} P_i(\overline{B}) \subset \overline{\mathcal{C}} \subset \overline{B} \times (\mathbb{P}^1 \times \mathbb{P}^1) \to \overline{B}$. The given configuration of eight points on C_0 defines a point $0 \in \overline{B}$. If we blow up $\overline{B} \times (\mathbb{P}^1 \times \mathbb{P}^1)$ along Z we obtain a family of surfaces $\mathcal{S} \to \overline{B}$ with fibre $S_0 = S$ over $0 \in \overline{B}$.

The given real structure on $\mathbb{P}^1 \times \mathbb{P}^1$ defines a real structure on B and $\mathcal{C}$, such that $\mathcal{C} \to B$ is equivariant. Therefore, we finally obtain a real structure on $\overline{B}$,$\overline{\mathcal{C}}$, Z and $\mathcal{S} \to \overline{B}$, such that all maps considered before are equivariant.

Hence, by Theorem 4.2 there exists a neighbourhood $\overline{B}^*$ of $0 \in \overline{B}$ in the analytic topology, such that any real S_t sits in a twistor space as a fundamental divisor. We can choose $\overline{B}^*$ in such a way that for all $t \in \overline{B}^*$ the set of eight different points Z_t

consists of smooth points of C_t. Therefore, the strict transform of C_t in S_t will be isomorphic to C_t and we will denote it also by C_t.

We wish now to study the order of the normal bundle $N_t = N_{C_t|S_t}$ of $C_t \subset S_t$.

Let us denote by $\overline{U} \subset \overline{B}^*$ the open subset of those points $t \in \overline{B}^*$ with smooth C_t. Its image $U \subset B$ is also open since projections and flat maps are open. Since $N_t \cong \mathcal{O}(2,2) \otimes \mathcal{O}_{C_t}(-Z_t) \in \operatorname{Pic}^0(C_t)$ we obtain a morphism $\eta : \overline{U} \to \operatorname{Pic}^0(\mathcal{C}|U)$ with $\eta(t) = N_t$. This morphism is compatible with real structures.

To achieve our theorem, we have to show the existence of a *real* point in $\overline{U}(\mathbb{R})$, which is sent by η to a point of finite order. This will follow from

Lemma 4.3 *Let E be an elliptic curve (over $\mathbb{C}$) with a real structure and $\mathcal{L}$ a line bundle on E which has degree 8 and is real. Let $\Delta \subset E \times \ldots \times E/S_8$ be the moduli space of sets of eight distinct (closed) points on E. Let $\eta : \Delta \to \operatorname{Pic}^0(E)$ be the map defined by $\eta(\{x_1, \ldots, x_8\}) = \mathcal{L} \otimes \mathcal{O}_E(-\sum_{i=1}^{8} x_i)$. Then we have:*

(i) The real structure on E induces one on Δ and $\operatorname{Pic}^0(E)$, such that η is equivariant.

(ii) The real points $\operatorname{Pic}^0(E)(\mathbb{R})$ form a real one–dimensional Lie–group. The points of finite order form a dense subset in this group.

(iii) $\eta : \Delta \to \operatorname{Pic}^0(E)$ and $\eta_{\mathbb{R}} : \Delta(\mathbb{R}) \to \operatorname{Pic}^0(E)(\mathbb{R})$ are submersions and are, therefore, open maps.

Proof: Observe first that the precise definition of η as a holomorphic map comes from the universality property of the variety $\operatorname{Pic}^0(E)$.

(i) The real structure on $\operatorname{Pic}(E)$ is defined by $\mathcal{M} \mapsto \sigma^*\overline{\mathcal{M}}$. (We denote by $\overline{\mathcal{M}}$ the complex conjugate line bundle and σ is the antiholomorphic involution on E defining the real structure.) Since $\mathcal{L}$ is by assumption a fixed point of this real structure, the map η is by definition equivariant.

(ii) Since $\mathcal{O}_E \in \mathrm{Pic}^0(E)$ is a real point, the subset $\mathrm{Pic}^0(E)(\mathbb{R})$ of real points forms a real one–dimensional submanifold of $\mathrm{Pic}^0(E)$. Since the formation of tensor products and duals is compatible with the real structure, $\mathrm{Pic}^0(E)(\mathbb{R})$ forms a subgroup of $\mathrm{Pic}^0(E)$. Since $U(1)$ is the unique connected compact one–dimensional real abelian Lie–group, the connected component of $\mathcal{O}_E$ in $\mathrm{Pic}^0(E)(\mathbb{R})$ must be isomorphic to $U(1)$. But the set of points of finite order in this group is dense. By compactness, $\mathrm{Pic}^0(E)(\mathbb{R})$ has only a finite number of connected components. Then it is easy to see that the set of points of finite order is dense in $\mathrm{Pic}^0(E)(\mathbb{R})$.

(iii) Let $\widetilde{\Delta} \subset E \times \ldots \times E$ be the preimage of Δ, that is the set of 8–tuples of pairwise distinct points. Since E is elliptic, by Riemann–Roch there exist eight points $Q_1, \ldots, Q_8 \in E$ with $\mathcal{L} \cong \mathcal{O}_E(\sum_{i=1}^{8} Q_i)$. If we define $\eta_i : E \to \mathrm{Pic}^0(E)$ by $\eta_i(P) := \mathcal{O}_E(Q_i - P)$ and $\widetilde{\eta} = (\eta_1, \ldots, \eta_8)$, we obtain a commutative diagram:

$$\begin{array}{ccc} \widetilde{\Delta} & \xrightarrow{\widetilde{\eta}} & \mathrm{Pic}^0(E) \times \ldots \times \mathrm{Pic}^0(E) \\ \downarrow & & \downarrow \text{mult} \\ \Delta & \xrightarrow{\eta} & \mathrm{Pic}^0(E). \end{array}$$

Since all η_i are isomorphisms, $\widetilde{\eta}$ is an open embedding, hence a submersion. Since multiplication is always a submersion, we have the same property for η.

Finally, since η is equivariant, its tangent map has the same property. But, if a complex linear map is compatible with real structures and surjective, the corresponding map on real subspaces is also surjective, since it can be described by the same matrix. So we obtain that $\eta_{\mathbb{R}}$ is a submersion. Submersions are open by the implicit function theorem. $\square$

Proof: (of Theorem 4.1)

We take any $t \in U(\mathbb{R})$ and apply Lemma 4.3 to $E = C_t$ and $\mathcal{L} = \mathcal{O}(2,2) \otimes \mathcal{O}_{C_t}$. This gives the existence of a point $\overline{t} \in \overline{U}(\mathbb{R})$ over t, defining a torsion point $\eta(\overline{t})$. The result follows now from Theorem 3.4. □

Open Problem: *Which values for τ can be realized by twistor spaces over $4\mathbb{CP}^2$?* Our proof of Theorem 4.1 only shows that large values of τ really occur.

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e-mail: Frederic.Campana@iecn.u-nancy.fr

e-mail: kreusler@mathematik.uni-kl.de