Received: 12 August 2021 | Revised: 17 December 2021 | Accepted: 4 January 2022

DOI: 10.1112/jlms.12606

RESEARCH ARTICLE

Journal of the London Mathematical Society

On the Alperin–McKay conjecture for 2-blocks of maximal defect

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Funding information
SFB, Grant/Award Number: TRR 195; DFG, Grant/Award Number: BR 6142/1-1

Abstract

In this paper, we show that the Alperin–McKay conjecture holds for 2-blocks of maximal defect. A major step in the proof is the verification that principal 2-block of groups of Lie type in odd characteristic is AM-good for the prime 2.

MSC 2020
20C20, 20C33 (primary)

1 | INTRODUCTION

In the representation theory of finite groups some of the most important conjectures predict a very strong relationship between the representations of a finite group G and certain representations of its ℓ-local subgroups, where ℓ is a prime dividing the order of G. One of these conjectures is the Alperin–McKay conjecture. For an ℓ-block b of G we denote by $\mathrm{Irr}_0(G, b)$ its set of height zero characters.

Conjecture 1.1 (Alperin–McKay). *Let b be an ℓ-block of G with defect group Q and B its Brauer correspondent in $\mathrm{N}_G(Q)$. Then*

$$|\,\mathrm{Irr}_0(G, b)| = |\,\mathrm{Irr}_0(\mathrm{N}_G(Q), B)|.$$

In this article we show that the Alperin–McKay conjecture holds for 2-blocks of maximal defect.

Theorem 1.2. *Let b be a 2-block of a finite group G whose defect group is a Sylow 2-subgroup Q and B its Brauer correspondent in $\mathrm{N}_G(Q)$. Then*

$$|\operatorname{Irr}_0(G, b)| = |\operatorname{Irr}_0(\mathrm{N}_G(Q), B)|.$$

Späth [27, Theorem C] showed that the Alperin–McKay conjecture holds for the prime ℓ if all blocks of all finite quasi-simple groups are AM-good for the prime ℓ. It is therefore possible to approach the Alperin–McKay conjecture through the classification of finite simple groups. Thanks to the work of several authors this has been verified for all finite simple groups except in the case where G is a group of Lie type defined over a field of characteristic $p \neq \ell$. Hence, we will focus on the case where G is a group of Lie type defined over a field of odd characteristic. In their seminal paper, Malle–Späth [20] showed that in this case G is McKay-good for the prime 2. For this they constructed a bijection $\operatorname{Irr}_{2'}(G) \to \operatorname{Irr}_{2'}(M)$ between the set of irreducible odd degree characters of G and the corresponding set of characters of a well-chosen subgroup M of G containing $\mathrm{N}_G(Q)$, for Q a Sylow 2-subgroup of G. Based on their bijection we are able to construct an explicit bijection between the height zero characters in the principal blocks of G and $\mathrm{N}_G(Q)$ and show the following.

Theorem 1.3. *Let G be a quasi-simple group of Lie type defined over a field of odd characteristic. Then the principal 2-block of G is AM-good for the prime 2.*

In a previous article, the second author has reduced the verification to quasi-isolated blocks of G [24] and then subsequently for groups of type A to unipotent blocks [25]. Which, in turn, combined with results from [4], lead to a complete verification in type A for primes greater than three. One major hurdle that arises when making use of this reduction, in its current form, is that the possibility to choose a suitable subgroup M as done in Malle–Späth no longer holds. As a consequence of the bijection explicitly constructed to prove Theorem 1.3 and the classification of quasi-isolated elements we obtain the following.

Corollary 1.4. *Let G be a quasi-simple group of classical Lie type. Then every 2-block of G is AM-good for the prime 2.*

Unfortunately, if G is a group of Lie type with exceptional root system, there are many quasi-isolated 2-blocks. However, one can show that the principal 2-block is the unique quasi-isolated 2-block of maximal defect.

Corollary 1.5. *Let G be a quasi-simple group of exceptional Lie type. Then every 2-block of maximal defect of G is AM-good for the prime 2.*

Using this we are able to verify that blocks of maximal defect of finite quasi-simple groups are AM-good for the prime 2, which is then enough to establish Theorem 1.2.

Structure of the paper

In Section 2 we derive some fundamental results on the structure of normalisers of Sylow 2-subgroups of groups of Lie type. This will be used in Section 3 to provide a description of the height zero characters in the principal block of this normaliser. In the same section we moreover give a parametrisation of the height zero characters of the principal block of G in terms of the 1-Harish–Chandra series. In Section 4 and Section 5 we study the action of group automorphisms of G on our parametrisation of characters. This will be used in Section 6 to prove that the principal block is AM-good for the prime 2. In Section 7 we deal with the remaining finite simple groups, and in Section 8 we prove our main results.

2 | SYLOW 2-SUBGROUPS

2.1 | Weyl groups

It is well known that the Sylow 2-subgroups of the symmetric group are self-normalising. That is for $P \in \mathrm{Syl}_2(\mathfrak{S}_n)$, we have that $\mathrm{N}_{\mathfrak{S}_n}(P) = P$. It turns out for all Weyl groups that the Sylow 2-subgroups will be self-normalising. In the following we denote by C_n the cyclic group of order n and to discuss the Weyl group in type C on n nodes we will use $W(B_n)$ to avoid confusion in notation.

Lemma 2.1. *Let W be a Weyl group. Then every Sylow 2-subgroup of W is self-normalising.*

Proof. Since any Weyl group is a direct product of irreducible Weyl groups, we can assume that W is irreducible. The case $W(A_n) \cong \mathfrak{S}_{n+1}$ is well known, which moreover implies the case $W(B_n) \cong \mathrm{C}_2 \wr \mathfrak{S}_n$. The group $W(D_n)$ also follows from the symmetric group as it arises as a normal subgroup of index 2 in $W(B_n)$ isomorphic to $\mathrm{C}_2^{n-1} \rtimes \mathfrak{S}_n$ (which can be constructed as the quotient of $W(B_n)$ by the kernel of the homomorphism $\mathrm{C}_2^n \to \mathrm{C}_2$ which maps $(g_1, \dots, g_n)$ to the product $g_1 \dots g_n$). This only leaves the exceptional cases. The result is immediate for $W(G_2) \cong \mathrm{Dih}_{12}$. For the remaining cases the description of these groups provided in [11, Section 2.12] will be taken.

Observe that $W(F_4)$ arises as the semidirect product of $W(D_4)$ with the automorphism group of the Dynkin diagram of type D_4. The group $W(D_4) \cong (\mathrm{C}_2)^3 \rtimes \mathfrak{S}_4$ is generated by signed permutations $g_1 = (1,2)(-1,-2)$ $g_2 = (2,3)(-2,-3)$, $g_3 = (3,4)(-3,-4)$ and $g_4 = (3,-4)(-3,4)$. Set γ_1 to be the automorphism of order 2 fixing both g_1, g_2 and interchanging g_3 and g_4, while γ_2 denotes the automorphism of order 3 which fixes g_2 and permutes g_1, g_3 and g_4 cyclically. Then $W(F_4) \cong W(D_4) \rtimes \langle \gamma_1, \gamma_2 \rangle$. The group $W(D_4)$ has three Sylow 2-subgroups one of which must be fixed by γ_1. Moreover only one Sylow 2-subgroup of $W(D_4)$ contains g_2, and thus, all three subgroups are fixed by γ_2. In particular, $W(D_4)$ has a Sylow 2-subgroup Q which is fixed by both automorphisms γ_1 and γ_2. Set $P := \langle Q, \gamma_1 \rangle$ which is a Sylow 2-subgroup of $W(F_4)$. As $\mathrm{N}_{W(F_4)}(Q) = \langle Q, \gamma_1, \gamma_2 \rangle$ and $P^{\gamma_2} = \langle Q, \gamma_1^{\gamma_2} \rangle \neq P$, it follows that $\mathrm{N}_{W(F_4)}(P) = P$.

The group $W(E_6)$ contains a subgroup $W^+(E_6) \cong \mathrm{SU}_4(2)$ of index two. In $\mathrm{SU}_4(2)$ the normaliser of a Sylow 2-subgroup Q is a Borel subgroup B, but $B = Q$ as $q = 2$. Hence $W^+(E_6)$ and thus $W(E_6)$ has self-normalising Sylow 2-subgroups. The same argument proves the case of $W(E_7) \cong \mathrm{C}_2 \times \mathrm{Sp}_6(2)$. While for $W(E_8)$ the index two subgroup $W^+(E_8)$ surjects onto $\Omega_8^+(2)$ with kernel $Z(W(E_8))$ of order 2. Thus for G the universal cover of $\Omega_8^+(2)$ with $Z(G) \cong \mathrm{C}_2 \times \mathrm{C}_2$, the same argument as used in E_6 shows that G and consequently also the groups $\Omega_8^+(2)$, $W^+(E_8)$ and $W(E_8)$ have self-normalising Sylow 2-subgroups. □

2.2 | Normalisers of Sylow 2-subgroups

Let H be a finite group and Q be a Sylow 2-subgroup of H. In this section we consider when $\mathrm{N}_H(Q) = \mathrm{C}_H(Q)Q$, for H a group of Lie type. The following remark will be helpful in answering this question.

Remark 2.2. Let H and Q be as above. By Schur–Zassenhaus, we have $\mathrm{N}_H(Q) = Q \rtimes K$ for some subgroup K of $\mathrm{N}_H(Q)$. In particular, $\mathrm{N}_H(Q) = \mathrm{C}_H(Q)Q$ if and only if $K \lhd \mathrm{N}_H(Q)$.

For any central subgroup $Z \leqslant \mathrm{Z}(H)$ let $\overline{U}$ denote the image of any subgroup U of H in the quotient H/Z. Observe that K is the unique Hall $2'$-subgroup of KZ and thus a characteristic subgroup of KZ. In particular, $\mathrm{N}_H(K) = \mathrm{N}_H(KZ)$ and similarly $\mathrm{N}_H(Q) = \mathrm{N}_H(QZ)$. Thus $\overline{K}$ is a complement to $\overline{Q}$ in $\mathrm{N}_{\overline{H}}(\overline{Q}) = \mathrm{N}_H(QZ)/Z$. As $KZ \lhd \mathrm{N}_H(Q)$ if and only if $\overline{K} \lhd \mathrm{N}_{\overline{H}}(\overline{Q})$, it follows that $\mathrm{N}_H(Q) = \mathrm{C}_H(Q)Q$ if and only if $\mathrm{N}_{\overline{H}}(\overline{Q}) = \mathrm{C}_{\overline{H}}(\overline{Q})\overline{Q}$.

We use the following theorem by Malle [17, Theorem 5.19] which is based on work by Aschbacher.

Theorem 2.3. *Let $\mathbf{G}$ be a simple algebraic group and $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism defining an $\mathbb{F}_q$-structure on $\mathbf{G}$. Let d be the order of q modulo 4 and $\mathbf{S}$ a Sylow d-torus of $(\mathbf{G}, F)$. Assume that $\mathbf{G}^F$ is not isomorphic to $\mathrm{Sp}_{2n}(q)$ with $n \geqslant 1$ and $q \equiv 3, 5 \bmod 8$. Then there exists a Sylow 2-subgroup Q of $\mathbf{G}^F$ with $\mathrm{N}_{\mathbf{G}^F}(Q) \leqslant \mathrm{N}_{\mathbf{G}^F}(\mathbf{S})$.*

We can now answer the question posed at the beginning of this section. Note that a similar result for the classical matrix groups was obtained in [15, Theorem 1].

Corollary 2.4. *Keep the assumption of Theorem 2.3 and let Q be a Sylow 2-subgroup of $G := \mathbf{G}^F$. Then $\mathrm{N}_G(Q) = \mathrm{C}_G(Q)Q$. Moreover, for $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ a regular embedding and $\widetilde{Q}$ a Sylow 2-subgroup of $\tilde{G} := \tilde{\mathbf{G}}^F$ with $Q = \widetilde{Q} \cap G$, then $\mathrm{N}_{\tilde{G}}(Q) = \mathrm{N}_{\tilde{G}}(\tilde{Q}) = \mathrm{C}_{\tilde{G}}(\tilde{Q})\tilde{Q}$.*

Proof. As in Remark 2.2, take K a complement to Q in $\mathrm{N}_G(Q)$. According to Theorem 2.3, $K \leqslant \mathrm{N}_G(\mathbf{T})$, where $\mathbf{T} = \mathrm{C}_{\mathbf{G}}(\mathbf{S})$ is a maximal torus of $\mathbf{G}$, see [14, Lemma 3.17]. In particular, K normalises $Q\mathbf{T}^F$. As $\mathbf{S}$ is d-split with $d \in \{1, 2\}$, the group W, where $W := \mathrm{N}_{\mathbf{G}^F}(\mathbf{T})/\mathbf{T}^F$, is again isomorphic to a Weyl group (use [5, page 121] and [21, Corollary B.23]). Hence $Q\mathbf{T}^F/\mathbf{T}^F$, which is a Sylow 2-subgroup of W, is self-normalising in W by Lemma 2.1. Thus $K \leqslant Q\mathbf{T}^F = \mathbf{T}^F_{2'} \rtimes Q$. As K is a $2'$-group, then $K \leqslant \mathbf{T}^F_{2'}$ and so $[K, Q] \leqslant Q \cap \mathbf{T}^F_{2'} = 1$. In other words $K \leqslant \mathrm{C}_H(Q)$. This proves the first statement.

Next observe that $\tilde{\mathbf{G}}^F / \mathrm{Z}(\tilde{\mathbf{G}})^F \cong \mathbf{G}^F_{\mathrm{ad}}$ and the assumption of Theorem 2.3 is always satisfied for $\mathbf{G}^F_{\mathrm{ad}}$. Thus by applying Remark 2.2 it follows that $\mathrm{N}_{\tilde{G}}(\tilde{Q}) = \mathrm{C}_{\tilde{G}}(\tilde{Q})\tilde{Q}$. Therefore it remains to show that $\mathrm{N}_{\tilde{G}}(Q) = \mathrm{N}_{\tilde{G}}(\tilde{Q})$. As any two Sylow 2-subgroups above Q must be conjugate by an element of $\mathrm{N}_{\widetilde{G}}(Q)$, it suffices to consider a fixed $\widetilde{Q} \in \mathrm{Syl}_2(\widetilde{G})$ lying above Q.

For groups of type A this follows from [15, Theorem 1]. In the remaining cases $\tilde{G}/G\,\mathrm{Z}(\tilde{G})$ is either a 2- or a $2'$-group. Note that if $\widetilde{G}/G\,\mathrm{Z}(\widetilde{G})$ is a $2'$-group, then $\widetilde{Q} = Q\,\mathrm{Z}(\widetilde{G})_2$ is the unique Sylow 2-subgroup of $\tilde{G}$ containing Q and so $\mathrm{N}_{\widetilde{G}}(\widetilde{Q}) = \mathrm{N}_{\widetilde{G}}(Q)$. Thus assume that $\widetilde{G}/H\,\mathrm{Z}(\widetilde{G})$ is a 2-group. For $\widetilde{\mathbf{T}} := \mathbf{T}\,\mathrm{Z}(\tilde{\mathbf{G}})$ a maximal torus of $\tilde{\mathbf{G}}$, we have that $\widetilde{Q} := \widetilde{\mathbf{T}}^F_2 Q$ is a Sylow 2-subgroup of $\widetilde{G} = H\,\mathrm{Z}(\widetilde{G})\widetilde{Q}$ and $[K, \widetilde{\mathbf{T}}^F] = 1$. Thus $\mathrm{N}_{\widetilde{G}}(Q) = \mathrm{N}_H(Q)\,\mathrm{Z}(\widetilde{G})\widetilde{Q} = K\,\mathrm{Z}(\widetilde{G})\widetilde{Q} \leqslant \mathrm{C}_{\widetilde{G}}(\widetilde{Q})\widetilde{Q}$. □

2.3 | Groups of Lie type

The following section is used to introduce the setup which will be in place for the remainder of this article. Let $\mathbf{G}$ be a simple algebraic group of simply connected type defined over an algebraic closure of $\mathbb{F}_p$ for some odd prime p. We adopt the notation of [20, Section 2.2]. In particular, $F_0 : \mathbf{G} \to \mathbf{G}$ denotes a field endomorphism inducing an $\mathbb{F}_p$-structure on $\mathbf{G}$ and for every symmetry of the Dynkin diagram associated to $\mathbf{G}$, we have a graph automorphism $\gamma : \mathbf{G} \to \mathbf{G}$. We consider a Frobenius endomorphism $F := F_0^m \gamma$ with γ a (possibly trivial) graph automorphism of $\mathbf{G}$ such that F defines an $\mathbb{F}_q$-structure on $\mathbf{G}$, where $q = p^m$. In addition, we let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be the regular embedding constructed in [20, Section 2.2].

We will also assume until Section 7 that $\mathbf{G}^F$ is not of type $C_n(q)$, $n \geqslant 1$, or ${}^3D_4(q)$ whenever $q \not\equiv 1 \bmod 8$.

Denote by d the order of q modulo 4. We let $\mathbf{T}$ be a maximally split torus of $(\mathbf{G}, F)$ with corresponding Weyl group $\mathbf{W}$. We set $\mathbf{V} := \langle n_\alpha(1) \mid \alpha \in \Phi \rangle \subset \mathrm{N}_{\mathbf{G}}(\mathbf{T})$, and $\mathbf{H} := \mathbf{V} \cap \mathbf{T}$. We define $v := 1$ if $d = 1$ and $v := \tilde{w_0}$ if $d = 2$, where $\tilde{w_0}$ is the canonical representative in $\mathbf{V}$ of the longest element $w_0 \in \mathbf{W}$ as defined in [20, Section 3.1]. We recall [20, Notation 3.3].

Notation 2.5. As before let $F := F_0^m \gamma$ be a fixed Frobenius endomorphism of $\mathbf{G}$. Let E_1 be the subgroup of $\mathrm{Aut}(\mathbf{G})$ generated by the graph automorphisms which commute with γ. Set $e := \mathrm{o}(\gamma)\exp(E_1)\mathrm{o}(v)$. Let $E := \mathrm{C}_{em} \times E_1$ act on $\tilde{\mathbf{G}}^{F_0^{2em}}$ such that the first summand C_{2em} of E acts by $\langle F_0 \rangle$ and the second by the group generated by graph automorphisms. Note that this action is faithful. Let $\widehat{F}_0, \widehat{\gamma}, \widehat{F} \in E$ be the elements that act on $\tilde{\mathbf{G}}^{F_0^{2em}}$ by F_0, γ and F, respectively.

Lemma 2.6. *The torus $\mathbf{T}$ contains a Sylow d-torus $\mathbf{S}$ of $(\mathbf{G}, vF)$. Moreover, $\mathbf{T} = \mathrm{C}_{\mathbf{G}}(\mathbf{S})$ and $N = TV$, where $N := \mathrm{N}_{\mathbf{G}}(\mathbf{S})^{vF}$, $T := \mathbf{T}^{vF}$ and $V := \mathbf{V}^{vF}$.*

Proof. See [20, Lemma 3.2] and [7, Section 5.1]. □

Note that E stabilises N, T, V and hence also $H := \mathbf{H}^{vF}$ and $W := \mathbf{W}^{vF}$. In what follows both the groups $\mathbf{G}^F$ and $\mathbf{G}^{vF}$ will be considered. Therefore, in addition to the notation in Malle–Späth [20] the objects from $G_0 := \mathbf{G}^F$ will be denoted with a subscript 0, for example, $T_0 := \mathbf{T}^F$, $N_0 := \mathrm{N}_{G_0}(\mathbf{S})$ and $W_0 := \mathbf{W}^F$. The following lemma provides a tool to pass between the groups $G_0 = \mathbf{G}^F$ and $G := \mathbf{G}^{vF}$ and compare them.

Lemma 2.7. *Let $g \in \mathbf{G}$ such that $gF(g)^{-1} = v$. Then the map*

$$\iota : \tilde{\mathbf{G}}^{F_q^{2e}} \rtimes E \to \tilde{\mathbf{G}}^{F_q^{2e}} \rtimes E,\ x \mapsto x^{g^{-1}}$$

is an isomorphism which maps $\mathbf{G}^F \rtimes E$ onto $\mathbf{G}^{vF} \rtimes E$.

Proof. See the proof of [7, Proposition 5.3]. □

Since the image of $\hat{F}$ under ι is $v\hat{F}$, we obtain an isomorphism $(\mathbf{G}^F \rtimes E)/\langle \hat{F} \rangle \cong (\mathbf{G}^{vF} \rtimes E)/\langle v\hat{F} \rangle$. From Theorem 2.3 we are now able to explicitly construct a Sylow 2-subgroup of $G := \mathbf{G}^{vF}$. Firstly, we let T_2 and V_2 be a Sylow 2-subgroup of T and V, respectively. We define $P := T_2 V_2$ which forms a Sylow 2-subgroup of G and conclude that $Q := \iota^{-1}(P)$ is a Sylow 2-subgroup of G_0. In the next section, we show that P can be chosen to be E-stable.

2.4 | Automorphisms

Lemma 2.8. *Let W be a Weyl group of irreducible type. If W is of type A_n ($n \geqslant 2$), D_n (n odd) or E_6, then the longest element $w_0 \in W$ acts as the (unique) non-trivial graph automorphism of order 2 on W. In the remaining cases, $w_0 \in \mathrm{Z}(W)$.*

Proof. Follows from remarks following [21, Corollary B.23]. □

For $\mathbf{W}$ and $\mathbf{V}$ as in Section 2.3, it is an obvious question whether the action of the representative $\tilde{w}_0$ of w_0 in $\mathbf{V}$ can be described in a similar way. The next lemma gives a positive answer to this.

Lemma 2.9. *Whenever $w_0 \in \mathrm{Z}(\mathbf{W})$ then we have $\tilde{w}_0 \in Z(\mathbf{V})$. In the remaining cases we have $\mathrm{C}_{\mathbf{V}}(\tilde{w}_0\gamma_0) = V$, where γ_0 is the graph automorphism which acts as w_0 on $\mathbf{W}$.*

Proof. This follows from the citations given in the proof of [20, Lemma 3.2]. □

Lemma 2.10. *There exists an E-stable Sylow 2-subgroup W_2 of $\mathbf{W}^{w_0F}$ with $w_0 \in Z(W_2)$. Moreover, W_2 is a Sylow 2-subgroup of $\mathbf{W}^F$.*

Proof. Let us first assume that $\mathbf{W}$ is not of type D_{2n}. Using the formulas given on the bottom of [5, page 121] together with the well-known order formulas for Weyl groups, we deduce that $|\mathbf{W} : \mathbf{W}^\sigma|$ is odd for any graph automorphism σ. Moreover, w_0 is fixed by σ so we can choose W_2 to be a Sylow 2-subgroup of $\mathbf{W}^\sigma$ with $w_0 \in \mathrm{Z}(W_2)$ by Lemma 2.8.

In type D_{2n} with $2n > 4$, the element w_0 corresponds to a central element of $\mathbf{W}$ and so $\mathbf{W}^{w_0F} = \mathbf{W}^F$. If $2n > 4$, it can be assumed that F is a field automorphism; otherwise, E acts trivially on $\mathbf{W}^F$. It therefore suffices to find a σ-stable Sylow 2-subgroup of $\mathbf{W}$ for σ the graph automorphism. However, σ has order 2, $\mathbf{W}$ has an odd number of Sylow 2-subgroups and so by the orbit-stabiliser theorem one must be fixed by σ.

This leaves the case when $\mathbf{W}$ is of type D_4. As before, it can be assumed that F is a field automorphism; otherwise, the group E acts trivially on $\mathbf{W}^F$. In this case $\mathbf{W}^F = \mathbf{W}$ and it was shown in the proof of Lemma 2.1 that $W(D_4)$ has a Sylow 2-subgroup which is E-stable. □

Let V_2 be the preimage of the Sylow 2-subgroup W_2 from Lemma 2.10 under the natural projection map $V \to W$.

Corollary 2.11. *The Sylow 2-subgroup $P := T_2V_2$ of G is E-stable.*

Proof. The group $\mathbf{H}$ is a normal subgroup of $\mathbf{V}$ with 2-power order and so $H \subseteq V_2$. Since $V/H \cong W$ and the image of V_2 in W is E-stable, it follows that V_2 is E-stable. □

As a consequence of this the Sylow 2-subgroup $Q = \iota^{-1}(P)$ of G_0 is D-stable, where $D := \iota^{-1}(E)/\langle \hat{F} \rangle$.

3 PARAMETRISATIONS OF CHARACTERS

3.1 Duality and character bijections of tori

We recall how duality can be used to provide bijections between certain characters of tori. For $(\mathbf{G}, \mathbf{T}, F)$ from Section 2.3 take $(\mathbf{G}^*, \mathbf{T}^*, F^*)$ to be a triple in duality as in [8, Definition 13.10]. Denote by W_2^* and w_0^* the image of W_2, respectively, w_0 under the isomorphism $\mathbf{W} \to \mathbf{W}^*$ induced by duality. In the following we let v^* be a fixed preimage in $\mathrm{N}_{\mathbf{G}^*}(\mathbf{T}^*)$ of w_0^* whenever $d = 2$, otherwise $v^* := 1$. Moreover, we will denote the images of v and v^* in $\mathbf{W}$, respectively, $\mathbf{W}^*$ by the same symbol.

Proposition 3.1. *Let W_2 be as in Lemma 2.10. Then there exists a bijection*

$$\alpha : \mathrm{Irr}(\mathbf{T}^F)^{W_2} \to \mathrm{Irr}(\mathbf{T}^{vF})^{W_2}.$$

Moreover, if $\sigma : \mathbf{G} \to \mathbf{G}$ is a bijective morphism with $\sigma(\mathbf{T}) = \mathbf{T}$ commuting with F such that $\sigma(v) = v$, then this bijection is equivariant with respect to σ.

Proof. By duality we obtain a $\mathbf{W}^F$-equivariant isomorphism $\mathrm{Irr}(\mathbf{T}^F) \to (\mathbf{T}^*)^{F^*}$. Let σ be a bijective morphism of $\mathbf{G}$ which stabilises $\mathbf{T}$. Then there exists a unique bijective morphism (up to $(\mathbf{T}^*)^{F^*}$-multiplication) $\sigma^* : \mathbf{G}^* \to \mathbf{G}^*$ commuting with F^* and in duality with σ such that this bijection is (σ, σ^*)-equivariant. Then we obtain a bijection $\beta_0 : \mathrm{Irr}(\mathbf{T}^F)^{W_2} \to ((\mathbf{T}^*)^{F^*})^{W_2^*}$.

The triple $(\mathbf{G}, \mathbf{T}, vF)$ is in duality with $(\mathbf{G}^*, \mathbf{T}^*, F^*v^*)$. Thus we similarly obtain a (σ, σ^*)-equivariant bijection $\mathrm{Irr}(\mathbf{T}^{vF}) \to (\mathbf{T}^*)^{F^*v^*}$. Furthermore, since $W_2 \subset \mathrm{C}_W(w_0)$, this induces a bijection $\beta : \mathrm{Irr}(\mathbf{T}^{vF})^{W_2} \to ((\mathbf{T}^*)^{F^*v^*})^{W_2^*}$. However, $v^* \in W_2^*$ and so $((\mathbf{T}^*)^{F^*})^{W_2^*} = ((\mathbf{T}^*)^{F^*v^*})^{W_2^*}$. In particular, we obtain a bijection

$$\alpha := \beta^{-1} \circ \beta_0 : \mathrm{Irr}(\mathbf{T}^F)^{W_2} \to \mathrm{Irr}(\mathbf{T}^{vF})^{W_2},$$

which is σ-equivariant as both β and β_0 are (σ, σ^*)-equivariant. □

Remark 3.2. By [5, Equation (15.2)] duality induces bijections $\mathrm{Z}(\tilde{\mathbf{G}}^*)^{F^*} \to \mathrm{Irr}(\tilde{\mathbf{T}}^F/\mathbf{T}^F)$ and $\mathrm{Z}(\tilde{\mathbf{G}}^*)^{F^*v^*} \to \mathrm{Irr}(\tilde{\mathbf{T}}^{vF}/\mathbf{T}^{vF})$. In particular, if $\theta_0 \in \mathrm{Irr}(\tilde{\mathbf{T}}^F/\mathbf{T}^F)$ is the character corresponding to $z \in \mathrm{Z}(\tilde{\mathbf{G}}^*)^{F^*}$, then $\theta := \theta_0 \circ \iota \in \mathrm{Irr}(\tilde{\mathbf{T}}^{vF}/\mathbf{T}^{vF})$ is the character corresponding to the same central element $z \in \mathrm{Z}(\tilde{\mathbf{G}}^*)^{F^*}$. Thus, we will denote the characters θ and θ_0 by the same symbol $\hat{z}$.

In the following we will employ the notation introduced in Section 2.3 with respect to the dual group $\mathbf{G}^*$. Moreover, for $s \in \mathbf{T}^*$ we denote by $\mathbf{W}^\circ(s)$ the Weyl group of $\mathrm{C}^\circ_{\mathbf{G}^*}(s)$ with respect to the maximal torus $\mathbf{T}^*$ and $\mathbf{W}(s) := \mathrm{C}_{\mathbf{W}^*}(s)$.

Proposition 3.3. *For $s \in (T_2^*)^{W_2^*}$ we have $v^* \in W^\circ(s)$, unless possibly if s is a semisimple element with rational centraliser $(D_d(q)\,{}^2D_{n-d}(q)).2$, for some $2 \leqslant d \leqslant n-1, d \neq n/2$, inside ${}^2D_n(q)$ for even n.*

Proof. It can be assumed that $q \equiv 3 \bmod 4$; otherwise, v^* is trivial. In particular, v^* is our fixed preimage of w_0^* in $\mathrm{N}_{\mathbf{G}^*}(\mathbf{T}^*)$ and s centralises a Sylow 2-subgroup of G^*.

We have $w_0^* \in W(s)$, so $w_0^* \in W^\circ(s)$ whenever $\mathbf{C} := \mathrm{C}_{\mathbf{G}^*}(s)$ is connected. We can therefore assume that $\mathbf{C}$ is disconnected. Let us first suppose that $\mathbf{G}$ is not of type A_n. By the proof of [20, Theorem 8.7], using that s centralises a Sylow 2-subgroup of G^*, the centraliser $\mathbf{C}^\circ$ contains a maximally split torus $\mathbf{S}$ of $(\mathbf{G}^*, F^*v^*)$. As $\mathbf{T}^* \subset \mathbf{C}^\circ$ there exists $x \in \mathbf{C}^\circ$ such that $\mathbf{S} = {}^x\mathbf{T}^*$. Let $h \in \mathbf{G}^*$ such that $F^*(v^*) = hF^*(h^{-1})$. In particular, ${}^{h^{-1}}\mathbf{S}$ is a maximally split torus of $(\mathbf{G}^*, F^*)$.

Assume first that F is untwisted, that is, F^* induces the identity on $\mathbf{W}^*$. Since $\mathbf{T}^*$ is also a maximal 1-split torus of $(\mathbf{G}^*, F^*)$, we have $(h^{-1}x)^{-1}F^*(h^{-1}x) = x^{-1}hF^*(h^{-1})F^*(x) \in \mathbf{T}^*$, see [8, Application 3.23]. Since $x \in \mathbf{C}^\circ, F^*(\mathbf{C}^\circ) = \mathbf{C}^\circ$ and the image of $hF^*(h^{-1})$ in $\mathbf{W}^*$ is w_0^*, we find that $w_0^* \in W^\circ(s)$.

Assume now that F^* is twisted, that is, that $G \cong {}^2D_n(q)$. According to [20, Lemma 8.6] and the proof of [20, Theorem 8.7] the centralizer $\mathbf{C}$ has rational form $(D_d(q)\,{}^2D_{n-d}(q)).2$ for some $2 \leqslant d \leqslant n-1, d \neq n/2$. By comparing the order of the centraliser with the order of G^*, we see that the centraliser of s can only contain a Sylow 2-torus of the ambient group if n is even.

Finally, if $\mathbf{G}^F$ is of type $A_n(\varepsilon q)$, $n > 1$, we use the proof of [19, Theorem 3.4]. As s centralises a Sylow 2-subgroup of G^*, it follows by the arguments given there (together with the information in [10, Table 4.5.1]) that $n+1$ is necessarily a power of 2 and $\mathbf{C}$ is of rational type $(A_{\frac{n-1}{2}}(\varepsilon q) \times A_{\frac{n-1}{2}}(\varepsilon q)).2$ or $A_{\frac{n-1}{2}}(q^2).2$. A calculation shows that $\mathbf{C}$ can only contain a Sylow 2-subgroup of G^* when $\mathbf{C}$ has rational type $(A_{\frac{n-1}{2}}(\varepsilon q) \times A_{\frac{n-1}{2}}(\varepsilon q)).2$. In this case, any element in $W(s) \setminus W^\circ(s)$ permutes the two components of the centraliser. In particular, it cannot centralise a Sylow 2-subgroup of $W(s)$. Therefore, we must necessarily have $w_0^* \in W^\circ(s)$. □

The previous proposition provides a way to compare the characters of $\tilde{\mathbf{T}}^F$ lying over a W_2-stable character of $\mathbf{T}^F$ with the analogous situation arising from $\tilde{\mathbf{T}}^{vF}$. The following result will be used in Section 5.

Proposition 3.4. *Let α be the bijection as in Proposition 3.1. Then there exists a bijection*

$$\tilde{\alpha} : \mathrm{Irr}(\tilde{\mathbf{T}}^F \mid \mathrm{Irr}(\mathbf{T}^F)_2^{W_2}) \to \mathrm{Irr}(\tilde{\mathbf{T}}^{vF} \mid \mathrm{Irr}(\mathbf{T}^{vF})_2^{W_2})$$

such that $\alpha \circ \mathrm{Res}^{\tilde{\mathbf{T}}^F}_{\mathbf{T}^F} = \mathrm{Res}^{\tilde{\mathbf{T}}^{vF}}_{\mathbf{T}^{vF}} \circ \tilde{\alpha}$ and if $\hat{z} \in \mathrm{Irr}(\tilde{\mathbf{T}}^F/\mathbf{T}^F)$, then $\tilde{\alpha}(\hat{z}) = \hat{z}$.

Assume now additionally $\tilde{\lambda} \in \mathrm{Irr}(\tilde{\mathbf{T}}^F \mid \mathrm{Irr}(\mathbf{T}^F)_2^{W_2})$ is v^*-stable. Let $\sigma : \tilde{\mathbf{G}} \to \tilde{\mathbf{G}}$ be a bijective morphism commuting with F such that $\sigma|_{\mathbf{G}}$ is as in Lemma 3.1, then ${}^\sigma\tilde{\lambda} = \tilde{\lambda}\hat{z}$ for some $z \in \mathrm{Z}(\tilde{\mathbf{G}}^*)^{F^*}$ and then we have ${}^\sigma\tilde{\alpha}(\tilde{\lambda}) = \tilde{\alpha}(\tilde{\lambda})\hat{z}$.

Proof. By duality we have a bijection $\mathrm{Irr}(\tilde{\mathbf{T}}^F) \to (\tilde{\mathbf{T}}^*)^{F^*}$. Let $\tilde{s} \in (\tilde{\mathbf{T}}^*)^{F^*}$ be a semisimple element corresponding to a character $\tilde{\lambda} \in \mathrm{Irr}(\tilde{\mathbf{T}}^F \mid \mathrm{Irr}(\mathbf{T}^F)^{W_2})$ under this bijection. The map $i : \mathbf{G} \to \tilde{\mathbf{G}}$ induces by duality a surjective map $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ and the image $s := i^*(\tilde{s})$ of $\tilde{s}$ lies in $((\mathbf{T}^*)^{F^*})^{W_2^*} = ((\mathbf{T}^*)^{F^*v^*})^{W_2^*}$. We deduce that the Lang image under F^*v^* of $\tilde{s}$ is in $\mathrm{Z}(\tilde{\mathbf{G}})$. In particular, we find $z \in \mathrm{Z}(\tilde{\mathbf{G}})$ such that $\beta(\tilde{s}) := \tilde{s}z$ is F^*v^*-stable. We shall assume that $\beta(\tilde{s})$ is chosen such that $\beta(\tilde{s}\tilde{z}) = \beta(\tilde{s})\tilde{z}$ for every $\tilde{z} \in \mathrm{Z}(\tilde{\mathbf{G}}^*)^{F^*} = \mathrm{Z}(\tilde{\mathbf{G}}^*)^{F^*v^*}$.

If $\tilde{\lambda}$ is v^*-stable or equivalently $v^* \in W^\circ(s)$, then $\beta(\tilde{s})$ can be constructed more easily. For this note that the map ι^* yields an isomorphism $W^\circ(\tilde{s}) \cong W^\circ(s)$ and so we deduce that ${}^{v^*}\tilde{s} = \tilde{s}$. In particular, $\tilde{s}$ is F^*v^*-stable and therefore in this case we can set $\beta(\tilde{s}) := \tilde{s}$.

Let $\tilde{\alpha}(\tilde{\lambda}) \in \mathrm{Irr}(\tilde{\mathbf{T}}^{vF})$ denote the character corresponding to $\beta(\tilde{s})$ under the bijection $\mathrm{Irr}(\tilde{\mathbf{T}}^{vF}) \to (\tilde{\mathbf{T}}^*)^{F^*v^*}$. One then checks easily that the so-obtained map

$$\tilde{\alpha} : \mathrm{Irr}(\tilde{\mathbf{T}}^F \mid \mathrm{Irr}(\mathbf{T}^F)^{W_2}) \to \mathrm{Irr}(\tilde{\mathbf{T}}^{vF} \mid \mathrm{Irr}(\mathbf{T}^{vF})^{W_2})$$

is a well-defined bijection which has all the required properties. □

3.2 | Local characters

Recall that P denotes the Sylow 2-subgroup of $G = \mathbf{G}^{vF}$ constructed in Corollary 2.11 and $Q = \iota^{-1}(P)$ its preimage under ι, which is a Sylow 2-subgroup of $G_0 = \mathbf{G}^F$. In this section we make use of the explicit description of P to provide a description of the odd degree characters in the principal 2-block of $\mathrm{N}_{G_0}(Q)$. For a finite group H we denote by $\mathrm{Irr}_{2'}(H)$ its set of irreducible characters of odd degree.

Proposition 3.5. *For $P = T_2V_2$ as in Corollary 2.11, there is a bijection*

$$\mathrm{Irr}_{2'}(P) \to \mathrm{Irr}(T_2)^{W_2} \times \mathrm{Irr}_{2'}(W_2).$$

Proof. Any character of $\mathrm{Irr}_{2'}(P)$ (that is any linear character of P) covers a P-invariant character of the normal subgroup T_2 of P. Since $P/T_2 \cong V_2/H \cong W_2$, the statement follows from [20, Corollary 3.13] and Gallagher's theorem. □

Recall as in Section 3.1 that W_2^* denotes the image of W_2 and $T^* := (\mathbf{T}^*)^{F^*v^*}$ corresponding to $T := (\mathbf{T})^{vF}$ under duality.

Proposition 3.6. *Let B be the principal 2-block of $\mathrm{N}_{G_0}(Q)$. Then for $Z := (T_2^*)^{W_2^*}$, there is a bijection $\mathrm{Irr}_0(B) \to Z \times \mathrm{Irr}_{2'}(W_2)$.*

Proof. By Corollary 2.4 we have $\mathrm{N}_{G_0}(Q) = \mathrm{C}_{G_0}(Q)Q$ and thus by [22, Theorem 9.12] restriction defines a bijection $\mathrm{Irr}_0(B) \to \mathrm{Irr}_{2'}(Q)$. As in the proof of Proposition 3.1, duality provides a bijection $\mathrm{Irr}(T)^{W_2} \to (T^*)^{W_2^*}$, which yields a bijection

$$\mathrm{Irr}(T_2)^{W_2} \to (T_2^*)^{W_2^*}.$$

The result thus follows from Proposition 3.5 using that $P \cong Q$. □

Remark 3.7. Let P^* be a Sylow 2-subgroup of G^*. As for G, it can be obtained as an extension of T_2^* by W_2^*. Therefore, $Z := (T_2^*)^{W_2^*}$ is a central subgroup of P^*. We believe that Z should coincide in most cases with $\mathrm{Z}(P^*)$. For instance, if G^* is of type A, then this is the case by [5, Lemma 13.17(ii)].

3.3 | Global characters

This section focuses on the height zero characters of the principal block for $G_0 = \mathbf{G}^F$ as in Section 2.3. Firstly we count these characters by counting those in $\mathbf{G}^{vF}$ using Malle's parametrisation of $2'$-degree characters.

Lemma 3.8. *The principal 2-block of $G = \mathbf{G}^{vF}$ contains $|Z| \times |\operatorname{Irr}_{2'}(W_2)|$ height zero characters, where W_2 and Z are taken from Lemma 2.10 and Proposition 3.6, respectively.*

Proof. The odd degree characters of G have been parametrised by Malle [17, Proposition 7.3]. However, by the proof of [9, Theorem A], the principal 2-block is the unique unipotent block of maximal defect. Therefore using Malle's explicit parametrisation, it follows that the height zero characters of the principal block of G are in bijection with pairs (s, ϕ), where $s \in Z$ and $\phi \in \operatorname{Irr}_{2'}(W(s))$, where $W(s) := \mathrm{C}_W(s)$. As $s \in Z$, then $W_2 \leqslant \mathrm{C}_W(s)$ and thus by the main result of [20] we have a McKay-bijection $\operatorname{Irr}_{2'}(W(s)) \to \operatorname{Irr}_{2'}(W_2)$. □

Corollary 3.9. *Recall that $\mathbf{G}$ is simple simply connected and F is a Frobenius map with $\mathbf{G}^F \not\cong \{\mathrm{Sp}_{2n}(q), {}^3D_4(q)\}$ whenever $q \not\equiv 1 \bmod 8$. Then the Alperin–McKay conjecture holds for the principal 2-block of $\mathbf{G}^F$.*

Proof. This follows from Proposition 3.6 and Lemma 3.8. □

Define

$$\mathcal{P}_0 := \{(\lambda_0, \eta_0) \mid \lambda_0 \in \operatorname{Irr}(T_0) \text{ and } \eta_0 \in \operatorname{Irr}_{2'}(W_0(\lambda_0))\},$$

where $W_0(\lambda_0) := (N_0)_{\lambda_0}/T_0$. From the proof of [20, Theorem 6.3] there is a surjective map onto the principal Harish–Chandra series

$$\begin{array}{rccc} \Pi_0 : & \mathcal{P}_0 & \to & \bigcup\limits_{\lambda_0 \in \operatorname{Irr}(T_0)} \mathcal{E}(G_0, (T_0, \lambda_0)) \\ & (\lambda_0, \eta_0) & \mapsto & R_{T_0}^{G_0}(\lambda_0)_{\eta_0}, \end{array}$$

which becomes injective on W_0-orbits.

The main aim is to find a suitable subset of $\mathcal{P}_0$ to parametrise the height zero characters of the principal block b of $G_0 = \mathbf{G}^F$. If $R_{T_0}^{G_0}(\lambda_0)_{\eta_0}$ has $2'$-degree, then by [20, Lemma 8.9] it follows that $2 \nmid |W_0 : W_0(\lambda_0)|$. In other words, $W_0(\lambda_0)$ contains a Sylow 2-subgroup of W_0. Furthermore, the principal block of G_0 is a subset of $\mathcal{E}_2(G_0, 1)$, see [5, Theorem 9.12(a)]. However for $s_0 \in T_0^*$ in duality with $\lambda_0 \in \operatorname{Irr}(T_0)$, it follows that s_0 has 2-power order if and only if λ_0 has 2-power order. Therefore if $\chi \in \operatorname{Irr}(b)$ lies in $\mathcal{E}(G_0, (T_0, \lambda_0))$, then λ_0 must have 2-power order. Via the decomposition $T_0 = (T_0)_2 \times (T_0)_{2'}$, the 2-power order characters coincide with the set $\operatorname{Irr}((T_0)_2)$, which can be viewed as the characters of T_0 with $(T_0)_{2'}$ in their kernel. Thus for W_2 the fixed Sylow 2-subgroup of W from Lemma 2.10 defines

$$(\mathcal{P}_0)_2 := \{(\lambda_0, \eta_0) \in \mathcal{P}_0 \mid \lambda_0 \in \operatorname{Irr}((T_0)_2)^{W_2}\}$$

and set Π_{glo} to be the restriction of Π_0 to $(\mathcal{P}_0)_2$.

Theorem 3.10. *Let b be the principal 2-block of G_0. Then the map Π_{glo} yields a bijection*

$$\Pi_{\mathrm{glo}} : (\mathcal{P}_0)_2 \to \operatorname{Irr}_0(b).$$

Proof. Every character of $\mathrm{Irr}_0(b)$ lies in the principal Harish-Chandra series by [19, Theorem 3.3]. That is, $\mathrm{Irr}_0(b) \subset \Pi_0(\mathcal{P}_0)$. If $\chi = R_{T_0}^{G_0}(\lambda_0)_{\eta_0} \in \mathrm{Irr}_0(b)$, then as in the paragraph above, it follows that there is some W_0-conjugate (λ_0', η_0') of (λ_0, η_0) with $\chi = R_{T_0}^{G_0}(\lambda_0')_{\eta_0'}$ and $\lambda_0' \in \mathrm{Irr}((T_0)_2)^{W_2}$; in other words $(\lambda_0', \eta_0') \in (\mathcal{P}_0)_2$. Moreover, as W_2 is self-normalising in W_0 (Lemma 2.1), it follows that λ_0' is the unique character in its W_0-orbit with $W_2 \subseteq W_0(\lambda_0')$. Hence $\mathrm{Irr}_0(b) \subseteq \Pi_{\mathrm{glo}}((\mathcal{P}_0)_2)$ and each $\chi \in \mathrm{Irr}_0(b)$ has a unique preimage in $(\mathcal{P}_0)_2$ under Π_{glo}.

It remains to show that Π_{glo} is indeed a bijection as stated in the theorem. By Lemma 3.8, it suffices to show that $|(\mathcal{P}_0)_2| = |Z| \times |\mathrm{Irr}_{2'}(W_2)|$. From the proof of Proposition 3.1, there is a bijection

$$\mathrm{Irr}((T_0)_2)^{W_2} \to (T_0^*)_2^{W_2^*} = (T_2^*)^{W_2^*} =: Z.$$

Furthermore, for each $\lambda_0 \in (\mathcal{P}_0)_2$, there is a McKay-bijection $\mathrm{Irr}_{2'}(W_0(\lambda_0)) \to \mathrm{Irr}_{2'}(W_2)$ by the main result of [20]. Thus $|(\mathcal{P}_0)_2| = |Z||\,\mathrm{Irr}_{2'}(W_2)|$. □

Example 3.11. Consider $G = \mathrm{SL}_2(q)$ and assume that $q \equiv 3 \bmod 4$. Recall that this case was excluded in Section *2.3*. The principal 2-block b of G has four height zero characters. There are four characters in the principal 1-Harish-Chandra series corresponding to characters in $(\mathcal{P}_0)_2$, but only two of them are of $2'$-degree. On the other hand, all four $2'$-degree characters of b lie in the principal 2-Harish-Chandra series.

Remark 3.12. Assume that $\mathbf{G}$ is not of type A_n, D_{2n+1}, $n > 1$, or E_6 so that the longest element $w_0 \in \mathbf{W}$ acts by inversion on the torus $\mathbf{T}$. It follows from the remarks after [20, Lemma 8.5] that all characters of $2'$-degree lie in the union of Lusztig series $\mathcal{E}(G_0, s)$ with s of 2-power order. By the proof of [9, Theorem A], the principal 2-block is the unique unipotent block of maximal defect. Hence, in these cases the Alperin–McKay conjecture for the principal 2-block is tantamount to the McKay conjecture for the prime 2.

4 | ACTION OF AUTOMORPHISMS

One of the key steps in the proof of Theorem 3.10 was the existence of a McKay-bijection $\mathrm{Irr}_{2'}(W_0(\lambda)) \to \mathrm{Irr}_{2'}(W_2)$. We will now construct such a bijection with suitable equivariance properties. For this we need the following lemma, whose proof follows [18, Lemma 2.1].

Lemma 4.1. *Let H be a finite group and $A \subset \mathrm{Aut}(H)$ a cyclic group of automorphisms stabilizing the normaliser M of a Sylow 2-subgroup of H. Then there exists an A-equivariant McKay bijection* $\mathrm{Irr}_{2'}(H) \to \mathrm{Irr}_{2'}(M)$.

Proof. According to the main result of [20] there exists a McKay bijection. We only need to show that it can be chosen to be A-equivariant. For $i \mid r := |A|$ let a_i (respectively, b_i) be the number of $\theta \in \mathrm{Irr}_{2'}(H)$ (respectively, $\theta \in \mathrm{Irr}_{2'}(M)$) with $|A_\theta| = i$. As A is cyclic, it suffices to show that $a_i = b_i$ for all $i \mid r$. Let q be a prime dividing r and set $s := r/q$. By induction on r we can assume that $a_i = b_i$ for all $i \notin \{r, s\}$ and

$$a_s + a_r = b_s + b_r.$$

Let us first assume that $2 \nmid r$. By Clifford theory we have $|\operatorname{Irr}_{2'}(HA)| = \sum_{i|r} a_i i^2/r$ and similarly $|\operatorname{Irr}_{2'}(MA)| = \sum_{i|r} b_i i^2/r$. Since the McKay-conjecture holds for HA, we have $|\operatorname{Irr}_{2'}(HA)| = |\operatorname{Irr}_{2'}(MA)|$ and so

$$a_s s^2/r + a_r r = b_s s^2/r + b_r r.$$

We therefore have two homogeneous linear equations in the variables $a_s - b_s$ and $a_r - b_r$. As the associated coefficient matrix is invertible, we deduce that $a_s = b_s$ and $a_r = b_r$. Let us now suppose that r is a power of 2. In that case, we obtain $|\operatorname{Irr}_{2'}(HA)| = a_r r = |\operatorname{Irr}_{2'}(MA)| = b_r r$. We again deduce that $a_s = b_s$ and $a_r = b_r$. The general case follows now by using the decomposition $A = A_2 \times A_{2'}$ and coprime arguments. □

Remark 4.2. We note that the existence of an automorphism-equivariant McKay-bijection should also follow from a similar statement as [23, Theorem B]. As we only need the result in the case of a cyclic automorphism group, we have decided not to pursue this.

Take $T_0 = \mathbf{T}^F$ and $N_0 = \mathrm{N}_{G_0}(\mathbf{S})$ as in Section 2.3. For $\tilde{\lambda}_0 \in \operatorname{Irr}(\tilde{T}_0)$ denote $W_0(\tilde{\lambda}_0) := (N_0)_{\tilde{\lambda}_0}/T_0$. Note that if $\tilde{\lambda}_0 \in \operatorname{Irr}(\tilde{T}_0 \mid \lambda_0)$ for some $\lambda_0 \in \operatorname{Irr}(T_0)$, then the factor group $W_0(\lambda_0)/W_0(\tilde{\lambda}_0)$ is an abelian group by the proof of [20, Proposition 3.16].

Lemma 4.3. *Let $\lambda_0 \in \operatorname{Irr}((T_0)_2)^{W_2}$ and let $\tilde{\lambda}_0 \in \operatorname{Irr}((\tilde{T}_0)_2 \mid \lambda_0)$. Then there exists an E_{λ_0}-equivariant bijection*

$$f_{\lambda_0} : \operatorname{Irr}_{2'}(W_0(\lambda_0)) \to \operatorname{Irr}_{2'}(W_2)$$

such that $f_{\lambda_0}(\eta_0\mu_0) = f_{\lambda_0}(\eta_0)\operatorname{Res}^{W_0(\lambda_0)}_{W_2}(\mu_0)$ for every character $\eta_0 \in \operatorname{Irr}_{2'}(W_0(\lambda_0))$ and $\mu_0 \in \operatorname{Irr}(W_0(\lambda_0)/W_0(\tilde{\lambda}_0))$.

Proof. The group

$$W_0(\lambda_0)/W_0(\tilde{\lambda}_0) = \{w \in W_0 \mid {}^w\tilde{\lambda}_0 = \tilde{\lambda}_0 \otimes \nu_0 \text{ for some } \nu_0 \in \operatorname{Irr}(\tilde{T}_0/T_0)\}/W_0(\tilde{\lambda}_0)$$

is always a 2-group since $\tilde{\lambda}_0$ has 2-power order and W_0 acts trivially on $\tilde{T}_0/T_0$. As η_0 is a character of $2'$-degree and the quotient is a 2-group, it follows that η_0 restricts irreducibly to $W_0(\tilde{\lambda}_0)$ [12, Chapter 6]. By Gallagher's theorem, the group $\operatorname{Irr}(W_0(\lambda_0)/W_0(\tilde{\lambda}_0))$ acts fixed point freely on the orbit of $\eta_0 \in \operatorname{Irr}(W_0(\lambda_0))$. On the other hand, every character of $\operatorname{Irr}_{2'}(W_2)$ is linear and thus restricts irreducibly to $W_2(\tilde{\lambda}_0) := W_2 \cap W(\tilde{\lambda}_0)$.

Let us first assume that G_0 is not of type $D_{2n}(q)$. Denote by E_0 the stabiliser of λ_0 in E. Observe that E acts by inner automorphisms on W_0 and centralises W_2 by Lemma 2.8. In particular, every character of $W_0(\lambda_0)$ and W_2 is E_0-stable in this case. Since the Sylow 2-subgroup W_2 is self-normalising in W_0, there exists a McKay bijection $\operatorname{Irr}_{2'}(W_0(\lambda_0)) \to \operatorname{Irr}_{2'}(W_2)$. By the previous discussion it is now easy to construct a bijection $f_{\lambda_0} : \operatorname{Irr}_{2'}(W_0(\lambda_0)) \to \operatorname{Irr}_{2'}(W_2)$ with the required properties.

Let us now assume that G_0 is of type $D_{2n}(q)$. We use the notation of the proof of [20, Theorem 3.17]. Let $\Phi(\tilde{\lambda}_0)$ be the root system associated to the Weyl group $W_0(\tilde{\lambda}_0)$. There exists an E_0-stable base Δ_0 of $\Phi(\tilde{\lambda}_0)$. Denote $A_0 := \operatorname{Stab}_{W_0}(\Delta_0)$ which is E_0-stable as Δ_0 is. By the proof of

[20, Theorem 3.17], $W_0(\lambda_0) = W_0(\tilde{\lambda}_0) \rtimes A_0$. Moreover, A_0 is a 2-group as already observed above. Let $\eta_0 \in \mathrm{Irr}(W_0(\tilde{\lambda}_0))$ which extends to a character of $2'$-degree of $W_0(\lambda_0)$ and set $\delta_0 := \det(\eta_0)$. By [12, Lemma 6.24] there exists a unique extension $\hat{\eta}_0 \in \mathrm{Irr}(W_0(\lambda_0))$, such that $\det(\hat{\eta}_0) = \hat{\delta}_0$, where $\hat{\delta}_0$ is the unique extension of δ_0 with A_0 in its kernel.

Similarly, we have $W_2 = W_2(\tilde{\lambda}_0) \rtimes A_0$. Thus, any character $\eta_0 \in \mathrm{Irr}(W_2(\tilde{\lambda}_0))$ covered by a linear character of W_2 has a unique extension $\hat{\eta}_0 \in \mathrm{Irr}(W_2)$ with A_0 in its kernel.

Let us now first assume that G_0 is not of type $D_4(q)$. Note that $E/\mathrm{C}_E(W_0)$ is cyclic, and thus, by Lemma 4.1 there exists an E_0-equivariant McKay-bijection g_{λ_0} from $\mathrm{Irr}_{2'}(W_0(\lambda_0))$ to $\mathrm{Irr}_{2'}(W_2)$. This induces an E_0-equivariant bijection

$$\begin{array}{rccc} f_0 : & \mathrm{Irr}(W_0(\tilde{\lambda}_0) \mid \mathrm{Irr}_{2'}(W_0(\lambda_0))) & \to & \mathrm{Irr}(W_2(\tilde{\lambda}_0) \mid \mathrm{Irr}_{2'}(W_2)) \\ & \eta_0 & \mapsto & \mathrm{Res}^{W_2}_{W_2(\tilde{\lambda}_0)}(g_{\lambda_0}(\hat{\eta}_0)). \end{array}$$

We then define $f_{\lambda_0} : \mathrm{Irr}_{2'}(W_0(\lambda_0)) \to \mathrm{Irr}_{2'}(W_2)$ by mapping the character $\hat{\eta}_0$ to $\widehat{f_0(\eta_0)}$ and extending this map $\mathrm{Irr}(W_0(\lambda_0)/W_0(\tilde{\lambda}_0))$-equivariantly. As f_0 is E_0-equivariant and A_0 is E_0-stable, so is f_{λ_0}.

Finally, if G_0 is of type $D_4(q)$, then W_2 has index 3 in W_0. Hence, $W_0(\lambda_0) = W_2$ or $W_0(\lambda_0) = W_0$. In the former case, we set f_{λ_0} to be the identity map and in the latter case it is easy to explicitly construct a bijection f_{λ_0} with the required properties. □

We are also interested in the action of automorphisms on local characters. To compute this action we use the following explicit parametrisation of characters. Recall that $T := \mathrm{C}_{\mathbf{G}}(\mathbf{S})^{vF} = \mathbf{T}^{vF}$ and $N := \mathrm{N}_{\mathbf{G}}(\mathbf{S})^{vF}$.

Proposition 4.4. *Let Λ be the extension map from [20, Corollary 3.13] with respect to $T \lhd N$. Then the map*

$$\begin{array}{rccc} \Pi : & \mathcal{P} = \{(\lambda, \eta) \mid \lambda \in \mathrm{Irr}(T), \eta \in \mathrm{Irr}(W(\lambda))\} & \to & \mathrm{Irr}(N) \\ & (\lambda, \eta) & \mapsto & \mathrm{Ind}^N_{N_\lambda}(\Lambda(\lambda)\eta) \end{array}$$

is surjective and satisfies

(1) $\Pi(\lambda, \eta) = \Pi(\lambda', \eta')$ *if and only if there exists some $n \in N$ such that ${}^n\lambda = \lambda'$ and ${}^n\eta = \eta'$.*
(2) ${}^\sigma\Pi(\lambda, \eta) = \Pi({}^\sigma\lambda, {}^\sigma\eta)$ *for all $\sigma \in E$.*
(3) *Let $t \in \tilde{T}$, $\tilde{\lambda} \in \mathrm{Irr}(\langle T, t\rangle \mid \lambda)$ and $\nu_t \in \mathrm{Irr}(N_\lambda/N_{\tilde{\lambda}})$ be the faithful linear character given by ${}^t\Lambda(\lambda) = \Lambda(\lambda)\nu_t$. Then we have ${}^t\Pi(\lambda, \eta) = \Pi(\lambda, \eta\nu_t)$.*

Proof. See [20, Proposition 3.15]. □

Recall that P is a Sylow 2-subgroup of N whose image in N/T is W_2. We denote

$$\mathcal{P}_2 = \{(\lambda, \eta) \mid \lambda \in \mathrm{Irr}(T_2)^{W_2}, \eta \in \mathrm{Irr}_{2'}(W_2)\}.$$

As in the proof of Proposition 3.5 we obtain that the map

$$\begin{array}{rccc} \Pi_{\mathrm{loc}} : & \mathcal{P}_2 & \to & \mathrm{Irr}_{2'}(P) \\ & (\lambda, \eta) & \mapsto & \mathrm{Res}^{N_\lambda}_P(\Lambda(\lambda))\eta, \end{array}$$

is a bijection.

In the following we will compare the parametrisations arising in the two groups $G_0 := \mathbf{G}^F$ and $G := \mathbf{G}^{vF}$. For this, denote by Λ_0 the extension map from [20, Corollary 3.13] with respect to $T_0 \lhd N_0$. To understand the action of automorphisms, recall that $D := \iota^{-1}(E)/\langle \hat{F} \rangle$, see the remarks after Corollary 2.11 and $\tilde{\mathbf{G}} = \tilde{\mathbf{T}}\mathbf{G}$ is the regular embedding as in Section 2.3. Thus for $\tilde{t} \in \tilde{\mathbf{T}}^F$, write $\tilde{t} = tz$ with $t \in \mathbf{T}$ and $z \in \mathrm{Z}(\tilde{\mathbf{G}})$. Then for g from Lemma 2.7 the element ${}^g\tilde{t} \in \tilde{\mathbf{T}}^{vF}$ and we have a decomposition ${}^g\tilde{t} = {}^g t {}^g z = {}^g t z$.

Proposition 4.5. *Let $\lambda_0 \in \mathrm{Irr}(T_0)^{W_2}$ and set $\lambda = \alpha(\lambda_0) \in \mathrm{Irr}(T)^{W_2}$, for α from Lemma 3.1. Suppose that $\nu_0 \in \mathrm{Irr}(W_0(\lambda))$ and $t_0 \in \tilde{T}_0$ satisfy ${}^{t_0}\Lambda_0(\lambda_0) = \Lambda(\lambda_0)\nu_0$. Then we have $\mathrm{Res}^{N_\lambda}_P({}^t\Lambda(\lambda)) = \mathrm{Res}^{N_\lambda}_P(\Lambda(\lambda))\,\mathrm{Res}^{W_0(\lambda)}_{W_2}(\nu_0)$, where $t = \iota(t_0)$.*

Proof. Let $\nu \in \mathrm{Irr}(W(\lambda))$ such that ${}^t\Lambda(\lambda) = \Lambda(\lambda)\nu$. For $n \in V_\lambda$ we have

$$ {}^t\Lambda(\lambda)(n) = \lambda([t,n])\Lambda(\lambda)(n). $$

If $\tilde{\lambda} \in \mathrm{Irr}(\tilde{T})$ is an extension of λ, then we can write $\lambda([t,n]) = \tilde{\lambda}(t)^n\tilde{\lambda}(t^{-1})$. We have ${}^n\tilde{\lambda} = \tilde{\lambda}\hat{z}$ for some linear character $\hat{z} \in \mathrm{Irr}(\tilde{T}/T)$. We conclude that $\nu(w) = \hat{z}(t)$, where w is the image of n in $W(\lambda)$. Since ν is a character of $N_\lambda/T \cong V_\lambda/H$, this uniquely determines ν. Now for $w \in W_2$ the equality ${}^w\tilde{\lambda} = \tilde{\lambda}\nu$ implies by the construction in Proposition 3.4 that ${}^w\tilde{\lambda}_0 = \tilde{\lambda}_0\hat{z}$. The same reasoning as above now equally applies to the extension map Λ_0 with respect to $T_0 \lhd N_0$. Therefore, for $w \in W_2$ we find that $\nu_0(w) = \hat{z}(t_0) = \hat{z}(t) = \nu(w)$. We thus obtain

$$ \mathrm{Res}^{N_\lambda}_P({}^t\Lambda(\lambda)) = \mathrm{Res}^{N_\lambda}_P(\Lambda(\lambda))\,\mathrm{Res}^{W(\lambda)}_{W_2}(\nu) = \mathrm{Res}^{N_\lambda}_P(\Lambda(\lambda))\,\mathrm{Res}^{W(\lambda)}_{W_2}(\nu_0), $$

which finishes the proof. $\square$

We now turn to the action of automorphisms on the global characters.

Theorem 4.6. *Let $x \in \tilde{T}_0 D$ and $\delta_{\lambda_0,x} \in \mathrm{Irr}(W_0({}^x\lambda_0))$ such that $\delta_{\lambda_0,x}\Lambda_0({}^x\lambda_0) = {}^x\Lambda_0(\lambda_0)$. Then*

$$ {}^x(R^{G_0}_{T_0}(\lambda_0)_{\eta_0}) = R^{G_0}_{T_0}({}^x\lambda_0)_{{}^x\eta_0\delta^{-1}_{\lambda_0,x}}. $$

Proof. It follows from the results of [20, Theorem 5.7] as explained in the proof of [20, Proposition 6.3]. $\square$

Remark 4.7. In the following theorem, to compensate for the inversion of $\delta_{\lambda_0,x}$ occurring in Theorem 4.6, a slightly altered version of f_{λ_0} from Lemma 4.3 is required. Fix $\mathcal{T}$ a $\mathrm{Irr}(W_0(\lambda_0)/W_0(\tilde{\lambda}_0)) \rtimes E_{\lambda_0}$-transversal on $\mathrm{Irr}_{2'}(W_0(\lambda_0))$. Then for $\eta_0 \in \mathcal{T}$, $\sigma \in E_{\lambda_0}$ and $\mu_0 \in \mathrm{Irr}(W_0(\lambda_0)/W_0(\tilde{\lambda}_0))$ define

$$ f'_{\lambda_0} : \mathrm{Irr}_{2'}(W_0(\lambda_0)) \to \mathrm{Irr}_{2'}(W_2) $$

by setting $f'_{\lambda_0}({}^\sigma\eta_0\mu_0) := f_{\lambda_0}({}^\sigma\eta_0\mu_0^{-1})$.

It follows from construction for every character $\eta_0 \in \mathrm{Irr}_{2'}(W_0(\lambda_0))$ and $\mu_0 \in \mathrm{Irr}(W_0(\lambda_0)/W_0(\tilde{\lambda}_0))$, then $f'_{\lambda_0}(\eta_0\mu_0) = f'_{\lambda_0}(\eta_0)\,\mathrm{Res}^{W_0(\lambda_0)}_{W_2}(\mu_0^{-1})$. Moreover, the definition implies that f'_{λ_0} is also an E_{λ_0}-equivariant bijection.

Theorem 4.8. *Assume the setting of Section 2.3. For b and B the principal 2-block of G_0, respectively, $\mathrm{N}_{G_0}(Q)$, there exists an $\mathrm{N}_{\tilde{G}_0D}(Q)$-equivariant bijection $\kappa : \mathrm{Irr}_0(b) \to \mathrm{Irr}_0(B)$.*

Proof. Restriction defines an $\mathrm{N}_{\tilde{G}_0D}(Q)$-equivariant bijection $\mathrm{Irr}_0(B) \to \mathrm{Irr}_{2'}(Q)$. Additionally ι induces an equivariant bijection ι' between $\mathrm{Irr}_{2'}(Q)$ and $\mathrm{Irr}_{2'}(P)$, that is, $\iota(\mathrm{N}_{\tilde{G}_0E}(Q)) = \mathrm{N}_{\tilde{G}E}(P)$ and for $x \in \mathrm{N}_{\tilde{G}_0E}(Q)$, then $\iota'({}^x\chi) = {}^{\iota(x)}\iota'(\chi)$. Thus it suffices to produce a bijection

$$\kappa' : \mathrm{Irr}_0(b) \to \mathrm{Irr}_{2'}(P),$$

which in equivariant in the sense that $\kappa'({}^x\chi) = {}^{\iota(x)}\kappa'(\chi)$, for $x \in \mathrm{N}_{\tilde{G}_0D}(Q)$ and $\chi \in \mathrm{Irr}_0(b)$. Note that as P is E-stable, Corollary 2.4 implies $\mathrm{N}_{\tilde{G}E}(P) = \mathrm{C}_{\tilde{G}}(\tilde{P})\tilde{P}E$ for $\tilde{P} = \tilde{T}_2P$. Thus the action on $\mathrm{Irr}_{2'}(P)$ arises from $\tilde{T}_2E$.

By the proof of Theorem 3.10 we have a bijection $\Pi_{\mathrm{glo}} : (\mathcal{P}_0)_2 \to \mathrm{Irr}_0(b)$. On the other hand $\Pi_{\mathrm{loc}} : \mathcal{P}_2 \to \mathrm{Irr}_{2'}(P)$ is a bijection. Finally by combining Proposition 3.1 and Remark 4.7 there is a bijection $(\mathcal{P}_0)_2 \to \mathcal{P}_2$ which sends a pair (λ_0, η_0) to $(\alpha(\lambda_0), f'_{\lambda_0}(\eta_0))$ between parameter sets. More explicitly, combining these yields a bijection

$$\begin{array}{llll} \kappa' : & \mathrm{Irr}_0(b) & \to & \mathrm{Irr}_{2'}(P) \\ & R^{G_0}_{T_0}(\lambda_0)_{\eta_0} & \mapsto & \mathrm{Res}^{N_{\alpha(\lambda_0)}}_{P}(\Lambda(\alpha(\lambda_0)))f'_{\lambda_0}(\eta_0). \end{array}$$

The equivariance of this bijection can be derived by combining the properties of Harish-Chandra induction established in Theorem 4.6, the properties of the parametrisation from Propositions 4.4 and 4.5:

Take $x \in \tilde{T}_0E$ and $\delta_{\lambda_0,x}$ such that ${}^{\tilde{t}_0}\Lambda({}^{\sigma}\lambda_0) = \Lambda({}^{\tilde{t}_0\sigma}\lambda_0)\delta_{\lambda_0,x}$. By Remark 4.7

$$f'_{{}^x\lambda_0}({}^x\eta_0\delta^{-1}_{\lambda_0,x}) = f'_{{}^x\lambda_0}({}^x\eta_0)\,\mathrm{Res}^{W_0({}^x\lambda_0)}_{W_2}(\delta_{\lambda_0,x}) = {}^{\iota(x)}f'_{\lambda_0}(\eta_0)\,\mathrm{Res}^{W_0({}^x\lambda_0)}_{W_2}(\delta_{\lambda_0,x}).$$

While by Proposition 4.4 and Proposition 4.5

$${}^{\iota(x)}\left(\mathrm{Res}^{N_{\alpha(\lambda_0)}}_{P}(\Lambda(\alpha(\lambda_0)))\right) = \mathrm{Res}^{N_{\alpha({}^x\lambda_0)}}_{P}(\Lambda(\alpha({}^x\lambda_0)))\,\mathrm{Res}^{W_0({}^x\lambda_0)}_{W_2}(\delta_{\lambda_0,x}).$$

Thus for $x \in \tilde{T}_0E$, the equivariance follows as

$$\begin{aligned} \kappa'({}^xR^{G_0}_{T_0}(\lambda_0)_{\eta_0}) &= \mathrm{Res}^{N_{\alpha({}^x\lambda_0)}}_{P}(\Lambda(\alpha({}^x\lambda_0)))f_{{}^x\lambda_0}({}^x\eta_0\delta^{-1}_{\lambda_0,x}) \\ &= {}^{\iota(x)}\left(\mathrm{Res}^{N_{\alpha(\lambda_0)}}_{P}(\Lambda(\alpha(\lambda_0)))f_{\lambda_0}(\eta_0)\right) \\ &= {}^{\iota(x)}\kappa'(R^{G_0}_{T_0}(\lambda_0)_{\eta_0}). \end{aligned}$$

□

Corollary 4.9. *Let $\chi \in \mathrm{Irr}_0(G_0) \cap \mathcal{E}(G_0, s)$. Then the $\tilde{G}_0$-orbit of χ has size $|A(s)^{F^*}|$.*

Proof. The size of the $\widetilde{G}_0$-orbit of χ is given by $|\widetilde{T}_0/(\widetilde{T}_0)_\chi|$. Let $\chi = R_{T_0}^{G_0}(\lambda_0)_{\eta_0}$, with $\lambda_0 \in \mathrm{Irr}(T_0)$ and $\eta_0 \in \mathrm{Irr}_{2'}(W_0(\lambda_0))$. Then λ_0 corresponds to the element $s_0 \in T_0^*$, under the (W_0, W_0^*)-equivariant bijection between $\mathrm{Irr}(\widetilde{T}_0)$ and $\widetilde{T}_0^*$, with s_0 being G_0^*-conjugate to s. For any $\tilde{\lambda}_0 \in \mathrm{Irr}(\widetilde{T}_0 \mid \lambda_0)$ we have that $A(s_0)^{F^*} \cong (W_0)_{\lambda_0}/(W_0)_{\tilde{\lambda}_0}$. Moreover there is an isomorphism $\widetilde{T}_0/(\widetilde{T}_0)_{\Lambda_0(\lambda_0)} \cong (N_0)_{\lambda_0}/(N_0)_{\tilde{\lambda}_0}$ which maps an element $\tilde{t}$ to the character $\nu_{\tilde{t}}$ such that ${}^{\tilde{t}}\Lambda_0(\lambda_0) = \Lambda_0(\lambda_0)\nu_{\tilde{t}}$. Thus it suffices to prove that $(\widetilde{T}_0)_\chi = (\widetilde{T}_0)_{\Lambda_0(\lambda_0)}$.

Let $\tilde{t} \in \widetilde{T}_0$. Then by Theorem 4.6 and Proposition 4.4

$$ {}^{\tilde{t}}R_{T_0}^{G_0}(\lambda_0)_{\eta_0} = R_{T_0}^{G_0}(\lambda_0)_{\eta_0\nu_{\tilde{t}}^{-1}}. $$

Since $W_0(\lambda_0)/W_0(\tilde{\lambda}_0)$ is a 2-group and η_0 has $2'$-degree, the character $\mathrm{Res}_{W(\tilde{\lambda}_0)}^{W(\lambda_0)}(\eta_0)$ is irreducible. Hence, $\eta_0\nu_{\tilde{t}}^{-1} = \eta_0$ if and only if $\nu_{\tilde{t}} = 1$. Therefore $(\widetilde{T}_0)_\chi = (\widetilde{T}_0)_{\Lambda_0(\lambda_0)}$. □

5 | CHARACTERS OF $\tilde{G}_0$

In order to check whether a block of $G_0 := \mathbf{G}^F$ is AM-good for the prime 2 we also need information on characters of $\tilde{G}_0 = \tilde{\mathbf{G}}^F$ covering characters of $2'$-degree of G_0. Recall that $T := \mathbf{T}^{vF}$, $N := \mathrm{N}_{\mathbf{G}}(\mathbf{S})^{vF}$ and Λ is an extension map with respect to $T \lhd N$. Additionally $\tilde{T} := \tilde{\mathbf{T}}^{vF}$ and $\tilde{N} := \mathrm{N}_{\tilde{\mathbf{G}}}(\mathbf{S})^{vF}$.

Proposition 5.1. *There exists an NE-equivariant extension map $\tilde{\Lambda}$ with respect to $\tilde{T} \lhd \tilde{N}$ given by sending $\tilde{\lambda} \in \mathrm{Irr}(\tilde{T})$ to the unique common extension of $\tilde{\lambda}$ and $\mathrm{Res}_{N_{\tilde{\lambda}}}^{N_\lambda}(\Lambda(\lambda))$, where $\lambda = \mathrm{Res}_T^{\tilde{T}}(\tilde{\lambda})$.*

Proof. This was shown in the proof of [20, Proposition 3.20]. □

Definition 5.2. We say that $(\lambda_0, \eta_0) \in (\mathcal{P}_0)_2$ (as defined in Section 3.3) is covered by the pair $(\tilde{\lambda}_0, \tilde{\eta}_0)$ if $\tilde{\lambda}_0 \in \mathrm{Irr}(\tilde{T}_0 \mid \lambda_0)$ and $\tilde{\eta}_0 \in \mathrm{Irr}(W_0(\tilde{\lambda}_0) \mid \eta_0)$. Note that by [12, Chapter 6] $\tilde{\eta}_0 = \mathrm{Res}_{W_0(\tilde{\lambda}_0)}^{W_0(\lambda_0)}(\eta_0)$ since $W_0(\lambda_0)/W_0(\tilde{\lambda}_0)$ is a 2-group and η_0 has $2'$-degree.

In the proof of Theorem 4.8, the set $(\mathcal{P}_0)_2$ was used to provide a bijection between the height zero characters of the principal blocks of G_0 and P by mapping $R_{T_0}^{G_0}(\lambda_0)_{\eta_0}$ to $\mathrm{Res}_P^{N_\lambda}(\Lambda(\lambda))f_{\lambda_0}(\eta_0)$, where f_{λ_0} is from Lemma 4.3 and $\lambda := \alpha(\lambda_0)$ for α as defined in Section 3.1. The notion of covering defined for $(\mathcal{P}_0)_2$ can help understand those characters which cover the height zero characters in the principal blocks of G_0 and P under the action of $\tilde{G}_0$, respectively, $\tilde{P} := \tilde{T}_2 P$. Recall that $\tilde{\alpha}$ from Proposition 3.4 is a bijection between $\mathrm{Irr}(\tilde{T}_0 \mid \mathrm{Irr}(T_0)_2^{W_2})$ and $\mathrm{Irr}(\tilde{T} \mid \mathrm{Irr}(T)_2^{W_2})$.

Lemma 5.3. *Let us assume that $\lambda_0 \in \mathcal{E}(T_0, s_0)$ and s_0 is not one of the exceptional cases in Proposition 3.3. Suppose that $(\tilde{\lambda}_0, \tilde{\eta}_0)$ covers $(\lambda_0, \eta_0) \in (\mathcal{P}_0)_2$ as in Definition 5.2 and we assume that $\tilde{\lambda}_0$ has 2-power order. Then the following holds.*

(a) *The character $\tilde{\chi} := R_{\tilde{T}_0}^{\tilde{G}_0}(\tilde{\lambda}_0)_{\tilde{\eta}_0}$ covers $\chi := R_{T_0}^{G_0}(\lambda_0)_{\eta_0}$ and lies in the principal block of $\tilde{G}$.*

(b) *The character $\tilde{\psi} := \mathrm{Ind}_{\tilde{P}_{\tilde{\lambda}}}^{\tilde{P}}(\mathrm{Res}_{\tilde{P}_{\tilde{\lambda}}}^{\tilde{N}_{\tilde{\lambda}}}(\tilde{\Lambda}(\tilde{\lambda}))\,\mathrm{Res}_{W_2(\tilde{\lambda})}^{W_2}(f_{\lambda_0}(\eta_0)))$ covers $\psi := \mathrm{Res}_P^{N_\lambda}(\Lambda(\lambda))f_{\lambda_0}(\eta_0)$, for $\lambda := \alpha(\lambda_0)$ and $\tilde{\lambda} := \tilde{\alpha}(\tilde{\lambda}_0)$.*

In particular, the characters $\tilde{\psi}$ and $\tilde{\chi}$ lie above the same central character of $\mathrm{Z}(\tilde{G})$.

Proof. The first statement of part (a) follows from [2, Theorem 13.9(b)], while part (b) is a consequence of Proposition 5.1 and can be obtained as in [20, Corollary 3.21]. Observe that $\tilde{\psi}$ lies above the character $\mathrm{Res}^{\tilde{T}}_{\tilde{T}_2}(\tilde{\lambda}) \in \mathrm{Irr}(\tilde{T})$ and $\tilde{\chi}$ lies above $\tilde{\lambda}_0 \in \mathrm{Irr}(\tilde{T}_0)$ which is by assumption of 2-power order. By the properties of the bijection in Proposition 3.4 they both lie above the same character of $\mathrm{Z}(\tilde{G}_0) = \mathrm{Z}(\tilde{G})$. In addition, since $\tilde{\lambda}$ is of 2-power order, the character $\tilde{\chi}$ lies in the principal block of $\tilde{G}$ by [9, Theorem A]. □

The following lemma is crucial in verifying whether a block is AM-good.

Lemma 5.4. *Let $\tilde{\chi} \in \mathrm{Irr}(\tilde{G}_0)$ and $\tilde{\psi} \in \mathrm{Irr}(\tilde{P})$ as in Lemma 5.3. Let $\sigma \in E$ and suppose that $\tilde{\chi}^\sigma = \tilde{\chi}\hat{z}$ for some $\hat{z} \in \mathrm{Irr}(\tilde{G}_0/G_0)$. Then we have $\tilde{\psi}^\sigma = \tilde{\psi}\hat{z}$.*

Proof. By [20, Corollary 6.4] there exists a character $\chi \in \mathrm{Irr}(G_0 \mid \tilde{\chi})$ which satisfies $(\tilde{G}_0 E)_\chi = (\tilde{G}_0)_\chi E_\chi$. Therefore, we have $\chi^\sigma = \chi$ and consequently if (λ_0, η_0) is the label in $(\mathcal{P}_0)_2$ of χ, we have $(\lambda_0^\sigma, \eta_0^\sigma) = (\lambda_0, \eta_0)$. We have $\tilde{\lambda}_0^\sigma = \tilde{\lambda}_0\hat{z}$ for some $\hat{z} \in \mathrm{Irr}(\tilde{T}/T)$ and so we obtain that $W_0(\tilde{\lambda}_0)$ is σ-stable. Moreover, $\tilde{\eta}_0^\sigma = \tilde{\eta}_0$. We obtain $\tilde{\Lambda}_0(\tilde{\lambda}_0)^\sigma = \tilde{\Lambda}_0(\tilde{\lambda}_0)\hat{z}$, see Proposition 5.1. Thus,

$$\tilde{\chi}^\sigma = R^{\tilde{G}_0}_{\tilde{T}_0}(\tilde{\lambda}_0^\sigma)_{\tilde{\eta}_0^\sigma} = \hat{z} R^{\tilde{G}_0}_{\tilde{T}_0}(\tilde{\lambda}_0)_{\tilde{\eta}_0} = \hat{z}\tilde{\chi},$$

where the middle equality is derived from [2, Proposition 13.15]. Moreover, $\tilde{\lambda}^\sigma = \tilde{\lambda}\hat{z}$ by Proposition 3.4. On the other hand, $\tilde{\Lambda}(\tilde{\lambda})^\sigma = \tilde{\Lambda}(\tilde{\lambda})\hat{z}$ by [20, Proposition 3.20] and so $\tilde{\psi}^\sigma = \tilde{\psi}\hat{z}$, which finishes the proof. □

6 | THE CONDITION FOR A BLOCK TO BE AM-GOOD

In this section, we show that the principal 2-block of $G_0 := \mathbf{G}^F$ for $(\mathbf{G}, F)$ as in Section 2.3 is AM-good for the prime 2. We make use of the following reformulation in the language of character triples given in [28, Definition 4.12]. For the language of character triples and the definition of the relation $\geqslant_b$, we refer the reader to [24, Section 1.1].

Definition 6.1. Let S be a finite non-abelian simple group with universal covering group G and b an ℓ-block of G with non-central defect group D. Assume for $\Gamma := \mathrm{N}_{\mathrm{Aut}(G)}(D, b)$ there exists

(i) a Γ-stable subgroup M with $\mathrm{N}_G(D) \leqslant M \lneq G$;
(ii) a Γ-equivariant bijection $\Phi : \mathrm{Irr}_0(G, b) \to \mathrm{Irr}_0(M, B)$ where $B \in \mathrm{Bl}(M \mid D)$ is the unique block with $B^G = b$ and
(iii) $\Psi(\mathrm{Irr}_0(b, \nu)) \subseteq \mathrm{Irr}_0(B \mid \nu)$ for every $\nu \in \mathrm{Irr}(Z(G))$ and

$$(G/Z \rtimes \Gamma_\chi, G/Z, \overline{\chi}) \geqslant_b (M/Z \rtimes \Gamma_\chi, M/Z, \overline{\Psi(\chi)}),$$

for every $\chi \in \mathrm{Irr}_0(G, b)$ and $Z = \ker(\chi) \cap Z(G)$, where $\overline{\chi}$ and $\overline{\Phi(\chi)}$ list to χ and $\Phi(\chi)$, respectively. Then we say that b is AM-good for ℓ.

Recall that $Q := \iota^{-1}(P)$ from Section 2.3 is a Sylow 2-subgroup of G_0. In the following b and B denote the principal 2-block of G_0, respectively, $\mathrm{N}_{G_0}(Q)$. We need the following lemma.

Lemma 6.2.

(a) *Let* $\chi \in \mathrm{Irr}_{2'}(G_0)$*. Then* χ *extends to* $G_0 D_\chi$*.*
(b) *Let* $\chi' \in \mathrm{Irr}_0(\mathrm{N}_{G_0}(Q), B)$*. Then* χ' *extends to* $\mathrm{N}_{G_0 D}(Q)_{\chi'}$ *and* $\mathrm{N}_{\tilde{G}_0}(Q)_{\chi'}$*.*

Proof. The first part was proved in [20, Proposition 8.10].

For the first statement of (b) we pass to the Sylow 2-subgroup $P = \iota(Q)$ of $\mathbf{G}^{vF}$. We have $\mathrm{N}_G(P) = \mathrm{C}_G(P)P$ by Theorem 2.3. In particular, any height zero character of the principal block of $\mathrm{N}_G(P)$ is a trivial extension of a linear character of P. In other words, it is enough to show that every linear character $\lambda \in \mathrm{Irr}(P)$ extends to a character $\hat{\lambda} \in \mathrm{Irr}(PE_\lambda)$ with $v\hat{F}$ in its kernel. For this choose a linear character $\nu \in \mathrm{Irr}(E_\lambda)$ with $\nu(\hat{F}) = \lambda(v)^{-1}$ and define $\hat{\lambda}(pe) := \lambda(p)\nu(e)$ for $p \in P$ and $e \in E_\lambda$. The second part follows from Corollary 2.4 and Lemma 5.3(b). □

The following lemma also helps to check whether a block is AM-good.

Lemma 6.3. *Any character in* $\mathrm{Irr}_0(b)$ *or* $\mathrm{Irr}_0(B)$ *has* $Z(G_0)$ *in its kernel.*

Proof. Let $\chi \in \mathrm{Irr}_{2'}(G_0)$. As G_0 is perfect, the linear character $\det\circ\chi$ must be trival. For $g \in \mathrm{Z}(G_0)_2$ there is an $\epsilon \in \mathbb{C}^\times$ with $o(\epsilon)$ a 2-power such that $\chi(g) = \epsilon I_{\chi(1)}$. Therefore as $\epsilon^{\chi(1)} = 1$ it follows that $\epsilon = 1$ and thus $\mathrm{Z}(G_0)_2 \leqslant \ker(\chi)$. Moreover, if χ lies in the principal block, then $\mathrm{Z}(G_0)_{2'} \leqslant \ker(\chi)$ [22, Theorem 6.10]. The local height zero characters were parametrised after Proposition 4.4. Thus, for them the result follows from Lemma 5.3. □

We will use the following theorem to check whether the blocks in question are AM-good. Moreover, for $\chi \in \mathrm{Irr}(H)$, an irreducible character of a finite group H, we denote by $\mathrm{bl}(\chi)$ the 2-block of H to which χ belongs.

Theorem 6.4. *Let* $\chi \in \mathrm{Irr}(G_0, b)$ *and* $\chi' \in \mathrm{Irr}(\mathrm{N}_{G_0}(Q), B)$ *such that the following holds.*

(i) *We have* $(\tilde{G}_0 D)_\chi = (\tilde{G}_0)_\chi D_\chi$ *and* χ *extends to* $(G_0 D)_\chi$*.*
(ii) *We have* $(\mathrm{N}_{\tilde{G}_0}(Q)\mathrm{N}_{G_0 D}(Q))_{\chi'} = \mathrm{N}_{\tilde{G}_0}(Q)_{\chi'}\mathrm{N}_{G_0 D}(Q)_{\chi'}$ *and* χ' *extends to* $\mathrm{N}_{G_0 D}(Q)_{\chi'}$ *and to* $\mathrm{N}_{\tilde{G}_0}(Q)_{\chi'}$*.*
(iii) $(\tilde{G}_0 D)_\chi = G_0(\mathrm{N}_{\tilde{G}_0}(Q)\mathrm{N}_{G_0 D}(Q))_{\chi'}$*.*
(iv) *There exists* $\tilde{\chi} \in \mathrm{Irr}(\tilde{G}_0 \mid \chi)$ *and* $\tilde{\chi}' \in \mathrm{Irr}(\mathrm{N}_{\tilde{G}_0}(Q) \mid \chi')$ *such that the following holds.*
  - *For all* $m \in \mathrm{N}_{G_0 D}(Q)_{\chi'}$*, there exists* $\nu \in \mathrm{Irr}(\tilde{G}_0/G_0)$ *with* $\tilde{\chi}^m = \nu\tilde{\chi}$ *and* $\tilde{\chi}'^m = \mathrm{Res}^{\tilde{G}_0}_{\mathrm{N}_{\tilde{G}_0}(Q)}(\nu)\tilde{\chi}'$*.*
  - *The characters* $\tilde{\chi}$ *and* $\tilde{\chi}'$ *cover the same underlying central character of* $\mathrm{Z}(\tilde{G}_0)$*.*

(v) *The Clifford correspondents* $\tilde{\chi}_0 \in \mathrm{Irr}((\tilde{G}_0)_\chi \mid \chi)$ *and* $\tilde{\chi}'_0 \in \mathrm{Irr}(\mathrm{N}_{\tilde{G}_0}(Q)_{\chi'} \mid \chi')$ *of* $\tilde{\chi}$ *and* $\tilde{\chi}'$*, respectively, satisfy* $\mathrm{bl}(\tilde{\chi}_0) = \mathrm{bl}(\tilde{\chi}'_0)^{(\tilde{G}_0)_\chi}$*.*

Let $Z_0 := \mathrm{Ker}(\chi) \cap \mathrm{Z}(G_0)$*. Then*

$$((\tilde{G}_0 D)_\chi / Z_0, G_0/Z_0, \overline{\chi}) \geqslant_b ((\mathrm{N}_{\tilde{G}_0}(Q)\mathrm{N}_{G_0 D}(Q))_{\chi'}/Z_0, \mathrm{N}_{G_0}(Q)/Z_0, \overline{\chi'}),$$

where $\overline{\chi} \in \mathrm{Irr}(G_0/Z_0)$ *and* $\overline{\chi'} \in \mathrm{Irr}(\mathrm{N}_{G_0}(Q)/Z_0)$ *are the characters which inflate to* χ*, respectively,* χ'*.*

Proof. This is a consequence of [24, Theorem 2.1] and [24, Lemma 2.2]. □

Note that all conditions in Theorem 6.4 except condition (v) only depend on the character theory of G_0 and $\tilde{G}_0$ (together with its associated groups).

Theorem 6.5. *Let* $(\mathbf{G}, F)$ *be as in Section 2.3. Then the principal 2-block b of G_0 is AM-good for the prime 2.*

Proof. We show that the bijection $\varkappa : \mathrm{Irr}_0(b) \to \mathrm{Irr}_0(B)$ from Theorem 4.8 is a strong AM-bijection in the sense of [24, Definition 1.9]. Let $\chi \in \mathrm{Irr}_0(b)$ and $\chi' := \varkappa(\chi)$. By possibly conjugating χ by an element of $\tilde{G}$ we can assume by [24, Theorem 2.11] that the character χ satisfies condition (i) of Theorem 6.4. Using the Butterfly Theorem [28, Theorem 4.6] we see that it is enough to show that χ and χ' satisfy the remaining conditions in Theorem 6.4. Since $\varkappa$ is equivariant, we deduce that conditions (ii) and (iii) hold (the extendibility of the local character follows from Lemma 6.2(b)). Assume now first that $\chi \in \mathcal{E}(G_0, s_0)$ and s_0 is not one of the exceptional cases in Proposition 3.3. Let $\tilde{\chi} \in \mathrm{Irr}(\tilde{G}_0)$ and $\tilde{\psi} \in \mathrm{Irr}(\tilde{P})$ be the characters constructed in Lemma 5.3. Let $\tilde{\chi}'$ be the character obtain by extending $\tilde{\psi} \circ \iota$ trivially to a character of $\mathrm{N}_{\tilde{G}_0}(Q)$. Using Corollary 2.4 we see that $\tilde{\chi}'$ is indeed a well-defined character and lies in the principal block of $\mathrm{N}_{\tilde{G}_0}(Q)$. By Lemma 5.4 we see that the characters $\tilde{\chi}$ and $\tilde{\chi}'$ satisfy condition (iv) of Theorem 6.4. Finally for condition (v) let $\tilde{\chi}_0 \in \mathrm{Irr}((\tilde{G}_0)_\chi \mid \chi)$ and $\tilde{\chi}'_0 \in \mathrm{Irr}(\mathrm{N}_{\tilde{G}_0}(Q)_{\chi'} \mid \chi')$ be the Clifford correspondents of $\tilde{\chi}$ and $\tilde{\chi}'$. By construction $\mathrm{bl}(\tilde{\chi}_0)$ is a block below the principal block of $\tilde{G}_0$. However, the principal block of $(\tilde{G}_0)\chi$ is the unique block below the principal block of $\tilde{G}_0$. Hence, $\mathrm{bl}(\tilde{\chi}_0)$ lies in the principal block of $(\tilde{G}_0)_\chi$, and similarly, $\tilde{\chi}'_0$ lies in the principal block of $\mathrm{N}_{\tilde{G}_0}(Q)_{\chi'}$. By Brauer's second main theorem it follows that $\mathrm{bl}(\tilde{\chi}_0) = \mathrm{bl}(\tilde{\chi}'_0)^{(\tilde{G}_0)_\chi}$ and so condition (v) holds.

Assume finally that s_0 is one of the exceptional cases in Proposition 3.3. By Corollary 4.9 it follows that $(\tilde{G}_0)_\chi = G\,\mathrm{Z}(\tilde{G}_0)$. By Lemma 6.3 we can therefore define $\tilde{\chi}_0$, respectively, $\tilde{\chi}'_0$ as the unique extensions of χ, respectively, χ' with $\mathrm{Z}(\tilde{G})$ in their kernel. From this it is clear that these characters satisfy conditions (iv) and (v). □

7 | THE REMAINING FINITE SIMPLE GROUPS

For the remaining blocks of finite simple groups the following criterion will be helpful.

Lemma 7.1. *Let S be a finite simple non-abelian group and ℓ a prime. Let b be an ℓ-block of the universal covering group $\hat{G}$ of S with defect group Q such that $\mathrm{Out}(\hat{G})_b$ is cyclic. Assume that there exists an $\mathrm{Aut}(\hat{G})_b$-equivariant Alperin–McKay bijection $f : \mathrm{Irr}_0(\hat{G}, b) \to \mathrm{Irr}_0(\mathrm{N}_{\hat{G}}(Q), B)$ preserving central characters of $\mathrm{Z}(\hat{G})$ and that one of the following holds.*

(i) *All characters of $\mathrm{Irr}_0(\hat{G}, b)$ have $\mathrm{Z}(\hat{G})$ in their kernel.*
(ii) *$\mathrm{Out}(\hat{G})_b$ is an ℓ-group.*

Then the block b is AM-good for ℓ.

Proof. We check that the conditions in [28, Definition 4.4] are satisfied. Let $X := \hat{G}/(\mathrm{Ker}(\chi) \cap \mathrm{Z}(\hat{G}))$. There exists an overgroup Y of X such that $Y/\mathrm{C}_Y(X)X \cong \mathrm{Out}(X)_\chi$ and Y/X is cyclic. Let $\chi \in \mathrm{Irr}_0(\hat{G}, b)$ and $\chi' := f(\chi) \in \mathrm{Irr}_0(\mathrm{N}_{\hat{G}}(Q), B)$ considered as characters of X, respectively,

$N_X(Q)$. Assume that we are in case (i) so that $X = \hat{G}/Z(\hat{G}) = S$ is a simple non-abelian group. We can therefore choose an overgroup Y with $C_Y(X) = 1$. There exist extensions $\tilde{\chi} \in \mathrm{Irr}(Y \mid \chi)$ and $\tilde{\chi}' \in \mathrm{Irr}(N_Y(Q) \mid f(\chi))$ such that $\mathrm{bl}(\tilde{\chi}')^Y = \mathrm{bl}(\tilde{\chi})$. As $C_Y(X) = 1$, $\tilde{\chi}$ and $\tilde{\chi}'$ lie over the same central character of $C_Y(X)$. In particular, we have

$$(Y, X, \chi) \geqslant_b (N_Y(Q), N_X(Q), f(\chi))$$

by [28, Proposition 4.4].

In case (ii) we observe that $Y/C_Y(X)X$ is an ℓ-group. In particular, every block of $C_Y(X)X$ is covered by a unique block of Y. By the proof of [28, Lemma 3.16] we find $\tilde{\chi} \in \mathrm{Irr}(Y \mid \chi)$ and $\tilde{\chi}' \in \mathrm{Irr}(N_Y(Q) \mid f(\chi))$ which lie above the same character of $C_Y(X)$. In particular, we have $\mathrm{bl}(\mathrm{Res}^{N_Y(Q)}_{N_X(Q)C_Y(X)}(\tilde{\chi}'))^Y = \mathrm{bl}(\tilde{\chi})$. By [28, Proposition 4.4] this implies that

$$(Y, X, \chi) \geqslant_b (N_Y(Q), N_X(Q), f(\chi)).$$

In both cases, the Butterfly Theorem [28, Theorem 4.6] implies that the block b is AM-good for the prime ℓ. □

We consider now the case excluded in Section 2.3. Together with Theorem 6.5 this completes the proof of Theorem 1.3 from the introduction.

Lemma 7.2. *The principal 2-block of $G \in \{\mathrm{Sp}_{2n}(q), {}^3D_4(q)\}$ is AM-good for the prime 2 whenever q is an odd power of an odd prime.*

Proof. In our case $\mathrm{Out}(G)$ is cyclic and every character of $2'$-degree lies over the trivial character of $Z(G)$. Moreover, $\mathrm{Irr}_{2'}(G) = \mathrm{Irr}_0(B_0(G))$ by Remark 3.12. Let Q be a Sylow 2-subgroup of G. By Lemma 4.1 there exists an $\mathrm{Aut}(G)_Q$-equivariant bijection $\mathrm{Irr}_{2'}(G) \to \mathrm{Irr}_{2'}(N_G(Q))$ preserving the underlying central characters of $Z(G)$. In particular, the principal block of G is AM-good for the prime 2 by Lemma 7.1. □

We say that a simple group S is AM-good for the prime 2 if all 2-blocks of its universal covering group are AM-good for the prime 2.

Lemma 7.3. *Let S be a simple group of Lie type defined over a field of characteristic $p \neq 2$ with exceptional Schur multiplier. Then S is AM-good for the prime 2.*

Proof. As argued in [25, Proposition 14.8] it suffices to consider as S the simple groups ${}^2A_3(3)$ and $B_3(3)$. Let $\hat{G}$ be the universal covering group of S. By [18, Theorem 4.1] there exists a McKay-good bijection $f : \mathrm{Irr}_{2'}(\hat{G}) \to \mathrm{Irr}_{2'}(\hat{M})$, where $\hat{M}$ is the normaliser of a Sylow 2-subgroup of $\hat{G}$. The distribution of 2-blocks of $\hat{G}$ is known by [3]. We observe that for every character $\nu \in \mathrm{Irr}(Z(\hat{G}))$ of $2'$-order there exists a unique 2-block b_ν of $\hat{G}$ of maximal defect associated to it. Moreover, as argued in the proof of [18, Theorem 4.1] we have that $\mathrm{Out}(\hat{G})_\nu$ is a cyclic 2-group for every $1 \neq \nu$ of $2'$-order. The principal block is AM-good for the prime 2 by Theorem 6.5. As a McKay-good bijection preserves central characters, we see that f preserves the block decomposition. We deduce that b_ν, $\nu \neq 1$, is AM-good for the prime 2 by Lemma 7.1. In particular, by [3] the group

${}^2A_3(3)$ is AM-good for the prime 2, as all blocks with non-maximal defect are of central defect. We are left to consider the three blocks b of the universal covering group $\hat{G}$ of $B_3(3)$ of defect 2^2. Let b be one of these blocks. We can use a proof similar to [25, Proposition 14.6]. An inspection of [3] shows that $|\operatorname{Irr}_0(b)| = 2^2$. Moreover, these characters have $\mathrm{Z}(\hat{G})_2$ in their kernel. By [16, Theorem 4.1] we deduce that the Brauer correspondent B of b has also exactly four height zero characters which all have $\mathrm{Z}(\hat{G})_2$ in their kernel. Let $\bar{b}$ and $\bar{B}$ be the images of the blocks b and B in the quotient $\hat{G}/Z(\hat{G})_{3'}$. As $\bar{b}$ has defect 2^3 and precisely 5 ordinary characters (see [3]) its defect group is isomorphic to the dihedral group Dih_8, see [26, Theorem 8.1]. Using [25, Proposition 14.4, Proposition 14.5] we deduce that there exists an $\operatorname{Aut}(\hat{G}')$-equivariant bijection $\operatorname{Irr}(\bar{b}) \to \operatorname{Irr}(\bar{B})$. Thus, b is AM-good for the prime 2 by Lemma 7.1. □

Lemma 7.4. *Let S be a simple group of Lie type defined over a field of characteristic p. Then S is AM-good for the prime p.*

Proof. Let $\hat{G}$ be the universal covering group of S and $G = \hat{G}/Z(\hat{G})_p$. By [22, Theorem 9.10] there exists a bijection between the set of p-blocks of $\hat{G}$ and the set of p-blocks of G. With this observation the statement follows as in the proof of [27, Theorem 8.4]. □

8 | CONSEQUENCES

In this section we derive some consequences of Theorem 6.5. We keep the notation and setup of Section 2.3 but we make no restriction on the type of G.

Corollary 8.1. *Assume that the root system of* **G** *is of classical type. Then every 2-block of G is AM-good for the prime 2.*

Proof. By Lemma 7.3 we can assume that $S := G/\mathrm{Z}(G)$ has non-exceptional Schur multiplier. Observe that every subgraph of a Dynkin diagram of classical type is again of classical type. According to [24, Theorem 3.12] it suffices to prove that all strictly quasi-isolated 2-blocks b of G are AM-good. Suppose first that **G** is not of type A. Using the classification of quasi-isolated elements in [1] together with [5, Theorem 21.14] we deduce that b is the principal block of G. The claim follows therefore from Theorem 6.5. Suppose therefore now that **G** is of type A and b is a quasi-isolated block of G. We go through the proof of [25, Theorem 12.2] and replace 'Cabanes subgroup of the defect group' everywhere by 'defect group'. All arguments apply directly to our setup except for the reference to the proof of [24, Lemma 3.12] which in our situation must be changed to the proof of [24, Lemma 3.10]. Hence, as in [25, Theorem 12.2] we can conclude that it suffices to prove that all isolated 2-blocks (that is the principal 2-block) are AM-good for the prime 2. This again follows from Theorem 6.5. □

Corollary 8.2. *Suppose that the root system of* **G** *is of exceptional type and let b be a quasi-isolated 2-block of G of maximal defect. Then b is the principal block of G.*

Proof. Suppose first that b is a unipotent block of G. Then the claim of the corollary follows from the description of defect groups given in [9]. Suppose now that G is not of type E_6. Any block of maximal defect contains a character of $2'$-degree. According to Remark 3.12 such characters lie in a unipotent block.

Finally for $G = E_6(\pm q)$ the non-unipotent quasi-isolated 2-blocks are given in [13, Table 3]. The order of the defect group is bounded by $|C_{G^*}(s)|_2$, see [13, Lemma 2.6(a)], where $1 \neq s \in G^*$ is the semisimple quasi-isolated element of $2'$-order associated to the block b. Going through the list given in [13] one checks that $|C_{G^*}(s)|_2$ is always smaller than $|G|_2$. □

We can now complete the proof of Theorem 1.2 from the introduction.

Theorem 8.3. *The Alperin–McKay conjecture holds for 2-blocks of maximal defect.*

Proof. By [6, Proposition 2.5] it suffices to establish that every block b of maximal defect of the universal central extension of a finite simple non-abelian group S is AM-good for the prime 2. As explained in the proof of [25, Proposition 14.8], alternating groups, Suzuki and Ree groups and sporadic groups are AM-good for the prime 2. By Lemma 7.3 and Lemma 7.4 we can therefore assume that $S = G/Z(G)$, such that G is a group of Lie type defined over a field of odd characteristic and G is the universal covering group of S. By Corollary 8.1 we can assume that G is an exceptional group of Lie type. By the main result of [24] we can assume that b is a quasi-isolated block. In this case the result follows from Corollary 8.2 and Theorem 6.5. □

ACKNOWLEDGEMENTS

The authors would like to thank both Britta Späth and Gunter Malle for discussions relating to previous drafts of this article. This paper is a contribution via the second author to the SFB TRR 195, and the first author is supported by the DFG (Project: BR 6142/1-1).

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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