

Moduli spaces of semiquasihomogeneous singularities with fixed principal part

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Introduction

One of the important achievements of singularity theory is the explicit classification of certain “generic” classes of isolated hypersurface singularities via normal forms and the analysis of its properties (cf. [AGV]). More complicated singularities deform into a collection of singularities from these classes and deformation theory is a powerful tool in studying specific singularities. For a further classification of more complicated classes of singularities the explicit determination of normal forms seems to be impossible and not appropriate.

The aim of this article is to start towards a classification of isolated hypersurface singularities of any dimension via geometric methods, that is by explicitly constructing a (coarse) moduli space for such singularities with certain invariants being fixed. It is probably too optimistic to hope for a complete solution in general but we believe that the proposed approach can be extended to other interesting classes of singularities. Our method starts from deformation theory and leads to the construction of geometric quotients of quasiaffine spaces by certain algebraic groups whose main part is unipotent. This last part is a major ingredient and uses the general results of [GP 2]. In projective algebraic geometry, the theory of moduli spaces is highly developed but in singularity theory only a few attempts have been made so far, for example by Ebey, Zariski, Laudal, Pfister, Luengo, Greuel (cf. [LP] for a systematic approach and [GP 1] for a short survey). In this paper we consider only semiquasihomogeneous singularities given as a power series $f \in \mathbb{C}\{x_1, \dots, x_n\}$ or as a complex space germ $(X, 0) = (f^{-1}(0), 0) \subset (\mathbb{C}^n, 0)$, together with positive weights w_1, \dots, w_n of the variables such that the principal part f_0 of f (terms of lowest degree) has an isolated singularity.

For the classification we first fix the Milnor number, probably the most basic invariant of an isolated hypersurface singularity. Fixing the Milnor number is known (for $n \neq 3$) to be equivalent to fixing, in a family, the embedded topological type of the singularity. If the Milnor number is fixed, the classification of semiquasihomogeneous singularities falls naturally into two parts. Firstly, the classification of the quasihomogeneous principal parts or, which amounts to the same, the classification of hypersurfaces in a weighted projective space. Secondly, the classification of semiquasihomogeneous hypersurface singularities with fixed principal part. These two parts differ substantially, since the group actions whose orbits describe isomorphism classes of singularities are of a completely different nature. This article is devoted to the second task.

The most important equivalence relations for hypersurface singularities are right equivalence (change of coordinates in the source) and contact equivalence (change of coordinates and multiplication with a unit or, equivalently, preserving the isomorphism class of space germs). It turns out that right equivalence, which is really a classification of functions, is easier to handle. We prove the existence of a finite group E_{f_0} acting on the affine space T_- , the base space of the semiuniversal μ -constant deformation of f_0 of strictly negative weight, such that T_-/E_{f_0} is the desired coarse moduli space. We also show that a fine moduli space almost never

exists. See §1 for definitions and precise statements. Hence, T_-/E_{f_0} classifies, up to right equivalence, semiquasihomogeneous power series with fixed principal part.

An important step in the construction of moduli spaces with respect to right equivalence as well as with respect to contact equivalence is to prove that isomorphisms between two semiquasihomogeneous functions have necessarily non-negative degree. This is proved in §2 and uses the fact that the filtration on the Brieskorn lattice $H_0''(f)$ induced by the weights coincides with the V -filtration defining the mixed Hodge structure, which is independent of the coordinates. The proof relies on an analysis of this filtration given in [He].

In order to obtain a moduli space with respect to contact equivalence we have to fix, in addition to the Milnor number, also the Tjurina number. This is clear because the dimensions of the orbits of the contact group acting on T_- depend on the Tjurina number. But fixing the Tjurina number is not sufficient. The orbit space of the contact group for fixed Tjurina number is, as a topological space, in general not separated, hence, cannot carry the structure of a complex space. It turns out, however, that if we fix the whole Hilbert function of the Tjurina algebra induced by the weights, the orbit space is a complex space and a coarse moduli space which classifies, up to contact equivalence, semiquasihomogeneous hypersurface singularities with fixed principal part and fixed Hilbert function of the Tjurina algebra. For precise statements see §4. These moduli spaces are actually locally closed algebraic varieties in a weighted projective space.

The orbits of the contact group acting on T_- can also be described as orbits of an algebraic group $G = U \rtimes (E_{f_0} \cdot \mathbb{C}^*)$ where E_{f_0} is the finite group mentioned above and U is a unipotent algebraic group. The main ingredient for the proof in the case of contact equivalence is the theorem on the existence of geometric quotients for unipotent groups in [GP 2]. But, in order to give the above simple description of the strata, we have to use, in a non-trivial way, a certain symmetry of the Kodaira-Spencer map (cf. 3.2).

The stratification with respect to the Hilbert function of the Tjurina algebra and the proof for the existence of a geometric quotient are constructive and allow the explicit determination of the moduli spaces and families of normal forms for specific examples.

We conclude the paper with some open problems.

1 Moduli spaces with respect to right equivalence

Let $\mathbb{C}\{x_1, \dots, x_n\} = \mathbb{C}\{x\}$ be the convergent power series ring. Two power series $f, g \in \mathbb{C}\{x\}$ are called **right equivalent** ($\overset{r}{\sim}$) if there exists a $\psi \in \text{Aut}(\mathbb{C}\{x\})$ such that $f = \psi(g)$; f and g are called **contact equivalent** ($\overset{c}{\sim}$) if there exists a $\psi \in \text{Aut}(\mathbb{C}\{x\})$ and $u \in \mathbb{C}\{x\}^*$ such that $f = u\psi(g)$. (Equivalently, the local algebras $\mathbb{C}\{x\}/(f)$ and $\mathbb{C}\{x\}/(g)$ are isomorphic respectively the complex germs $(X, 0) \subset (\mathbb{C}^n, 0)$ and $(Y, 0) \subset (\mathbb{C}^n, 0)$ defined by f and g are isomorphic.)

Let d and w_1, \dots, w_n be any integers. A polynomial $f_0 \in \mathbb{C}[x_1, \dots, x_n] = \mathbb{C}[x]$ is **quasihomogeneous of type** $(d; w_1, \dots, w_n)$ if for any monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ occurring in f_0 ,

$$\deg x^\alpha := |\alpha| := w_1\alpha_1 + \cdots + w_n\alpha_n$$

is equal to d . w_1, \dots, w_n are called **weights** and $\deg x^\alpha$ is called the (weighted) **degree** of x^α .

For an arbitrary power series $f = \sum c_\alpha x^\alpha$, $f \neq 0$, we set

$$\deg f = \inf\{|\alpha| \mid c_\alpha \neq 0\},$$

and call it the degree of f . For a family of power series $F = \sum c_{\alpha,\beta} x^\alpha s^\beta \in \mathbb{C}\{x, s\}$, parametrized by $\mathbb{C}\{s\}$, we put $\deg_x F = \inf\{|\alpha| \mid \exists \beta \text{ such that } c_{\alpha,\beta} \neq 0\}$.

f is called quasihomogeneous if it is a quasihomogeneous polynomial (of some type). f is called **semiquasihomogeneous** of type $(d; w_1, \dots, w_n)$, if

$$f = f_0 + f_1,$$

where f_0 is a quasihomogeneous polynomial of type $(d; w_1, \dots, w_n)$, f_1 is a power series such that $\deg f_1 > \deg f_0$ and, moreover, f_0 has an isolated singularity at the origin. f_0 is called the **principal part** of f . Two right equivalent semiquasihomogeneous power series of the same type have right equivalent principal parts by Theorem 2.1.

Recall ([SaK 1]) that a power series f with isolated singularity is right equivalent to a quasihomogeneous polynomial with respect to positive weights if and only if

$$f \in j(f) := (\partial f / \partial x_1, \dots, \partial f / \partial x_n).$$

Moreover, in this case the **normalized weights** $\bar{w}_i = \frac{w_i}{d} \in \mathbb{Q} \cap (0, \frac{1}{2}]$ are uniquely determined.

We may consider $f \in \mathbb{C}\{x\}, f(0) = 0$ as a map germ $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$. An **unfolding** of f over a complex germ or a pointed complex space $(S, 0)$ is by definition a cartesian diagram

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \hookrightarrow & (\mathbb{C}^n, 0) \times (S, 0) \\ f \downarrow & & \downarrow \phi \\ (\mathbb{C}, 0) & \hookrightarrow & (\mathbb{C}, 0) \times (S, 0) \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & (S, 0). \end{array}$$

Hence, $\phi(x, s) = (F(x, s), s)$ and the unfolding ϕ is determined by $F : (\mathbb{C}^n, 0) \times (S, 0) \rightarrow (\mathbb{C}, 0)$, $F(x, s) = f(x) + g(x, s)$, $g(x, 0) = 0$, and we say that F defines an unfolding of f . Two unfoldings ϕ and ϕ' defined by F and F' over $(S, 0)$ are called right equivalent if there is an isomorphism $\Psi : (\mathbb{C}^n, 0) \times (S, 0) \xrightarrow{\cong} (\mathbb{C}^n, 0) \times (S, 0)$, $\Psi(x, s) = (\psi(x, s), s)$, such that $\phi \circ \Psi = \phi'$.

For the construction of moduli spaces we have to consider, more generally, families of unfoldings over arbitrary complex base spaces. Let S denote a **category of base spaces**, for example the category of complex germs or of pointed complex spaces or of complex spaces. A **family of unfoldings** over $S \in \mathcal{S}$ is a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^n, 0) \times S & \xrightarrow{\phi} & (\mathbb{C}, 0) \times S \\ & \searrow & \swarrow \\ & S & \end{array}$$

Hence, $\phi(x, s) = (G(x, s), s) =: (G_s(x), s)$ and for each $s \in S$, the germ $\phi : (\mathbb{C}^n, 0) \times (S, s) \rightarrow (\mathbb{C}, 0) \times (S, s)$ is an unfolding of $G_s : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$. A morphism of two families of unfoldings ϕ and $\phi' = (G', id_s)$ over S is a morphism $\Psi : (\mathbb{C}^n, 0) \times S \rightarrow (\mathbb{C}^n, 0) \times S$, $\Psi(x, s) = (\psi(x, s), s) = (\psi_s(x), s)$ such that $\phi \circ \Psi = \phi'$ (equivalently : $G_s(\psi(x, s)) = G'_s(x)$). ϕ and ϕ' are called **right equivalent families of unfoldings** if there is a morphism Ψ of ϕ and ϕ' such that for each fixed $s \in S$, $\psi_s \in \text{Aut}(\mathbb{C}^n, 0)$.

From now on let $f_0 \in \mathbb{C}[x_1, \dots, x_n]$ denote a quasihomogeneous polynomial with isolated singularity of type $(d; w_1, \dots, w_n)$ with $w_i > 0$ for $i = 1, \dots, n$.

Consider a power series f which is right equivalent to a semiquasihomogeneous power series of type $(d; w_1, \dots, w_n)$. We say that an unfolding F defines an **unfolding of f of negative weight** over $(S, 0)$ if F is right equivalent to $f'(x) + g(x, s)$ for some semiquasihomogeneous power series f' with $g(x, 0) = 0$ and $\deg_x g > d$. This holds, for instance, if there exists a \mathbb{C}^* -action with (strictly) negative weights on $(S, 0)$ such that $\deg g = d$, with respect to the \mathbb{C}^* -actions on $(\mathbb{C}^n, 0)$ and on $(S, 0)$. By Theorem 2.1 the definition is independent of the choice of f' .

We shall now describe the semiuniversal unfolding of f_0 of negative weight. Let $\{x^\alpha \mid \alpha \in B \subset \mathbb{N}^n\}$, be a monomial basis of the Milnor algebra $\mathbb{C}\{x\}/(\partial f_0/\partial x_1, \dots, \partial f_0/\partial x_n)$ which is of \mathbb{C} -dimension μ (the Milnor number of f_0), and let $\bar{F}(x, t) = f_0(x) + \sum_{\alpha \in B} x^\alpha s_\alpha$, $s = (s_\alpha)_{\alpha \in B} \in \mathbb{C}^\mu$ be the semiuniversal unfolding of f_0 . We are mainly interested in the sub-unfolding over the affine pointed space $T_- = (\mathbb{C}^k, 0)$,

$$F(x, t) = f_0(x) + \sum_{i=1}^k t_i m_i, \quad t = (t_1, \dots, t_k) \in T_-,$$

where the m_i are the “upper” monomials, that is

$$\{m_1, \dots, m_k\} = \{x^\alpha \mid \alpha \in B, |\alpha| > d\}.$$

For fixed $t \in T_-$, $F_t(x) = F(x, t) \in \mathbb{C}[x]$ is a semiquasihomogeneous polynomial with principal part f_0 .

Let $A = \mathbb{C}[(s_\alpha)_{\alpha \in B}]$ and $A_- = \mathbb{C}[t_1, \dots, t_k]$. If we give weights to s_α and t_i by $w(s_\alpha) = d - |\alpha|$ and $w(t_i) = d - \deg(m_i)$, then \bar{F} and F are quasihomogeneous polynomials in $\mathbb{C}[x, s]$ respectively $\mathbb{C}[x, t]$ and F is the restriction of \bar{F} to T_- , the negative weight part of $T = \text{Spec } A$, defined by $\{t_1, \dots, t_k\} = \{s_\alpha \mid w(s_\alpha) < 0\}$.

Example: $f_0 = x^3 + y^3 + z^7$ is quasihomogeneous of type $(d; w_1, w_2, w_3) = (21; 7, 7, 3)$ with Milnor number $\mu = 24$. The upper monomials of a monomial basis of the Milnor algebra $\mathbb{C}\{x, y, z\}/(x^2, y^2, z^6)$ are $m_1 = xz^5$, $m_2 = yz^5$, $m_3 = xyz^3$, $m_4 = xyz^4$, $m_5 = xyz^5$ and, hence, $A_- = \mathbb{C}[t_1, \dots, t_5]$, $T_- = \mathbb{C}^5$,

$$F(x, y, z, t) = f_0 + \sum_{i=1}^5 t_i m_i = f_0 + t_1 xz^5 + t_2 yz^5 + t_3 xyz^3 + t_4 xyz^4 + t_5 xyz^5,$$

$$w(t_1, \dots, t_5) = (-1, -1, -2, -5, -8).$$

Remark 1.1 Fix any $t \in T_-$. F defines an unfolding of F_t of negative weight over the pointed space (T_-, t) . If we restrict this unfolding to the germ (T_-, t) this is actually a semiuniversal unfolding of F_t of negative weight because of the following:

The monomials m_1, \dots, m_k represent certainly a basis of $\mathbb{C}\{x\}/(\frac{\partial F_t}{\partial x_1}, \dots, \frac{\partial F_t}{\partial x_n})$ for t sufficiently close to 0, since $\mu(F_t) = \mu(f_0)$. But, using the \mathbb{C}^* -actions on T_- and on \mathbb{C}^n , we see that any F_t is contact equivalent to some $F_{t'}$, t' close to 0. Hence, $\mathcal{O}_{\mathbb{C}^n \times T_- \times T_-} / (\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})$ is actually free over T_- with basis m_1, \dots, m_k and the result follows.

We call the affine family

$$F : \mathbb{C}^n \times T_- \rightarrow \mathbb{C},$$

$(x, t) \mapsto f_0(x) + \sum_{i=1}^k t_i m_i$ the **semiuniversal family of unfoldings of negative weight of semiquasihomogeneous power series with fixed principal part f_0** .

Lemma 1.2 *The family of unfoldings F has the following property. If f is any semiquasihomogeneous power series with principal part f_0 , then:*

- (i) $T_- = \{0\}$ if and only if f_0 is simple or simple elliptic.
- (ii) There exists a $t \in T_-$ such that $f \stackrel{r}{\sim} F_t$.
- (iii) Let $f \stackrel{r}{\sim} F_t$ and let $G(x, s) = f(x) + g(x, s)$ be any unfolding of f of negative weight over the germ $(S, 0)$. Then there exists a morphism, unique on the tangent level, of germs $\varphi : (S, 0) \rightarrow (T_-, t)$ such that $\varphi^* F$ is right equivalent to G (that is T_- does not contain trivial subfamilies of unfoldings).
- (iv) Assume f_0 is neither simple nor simple elliptic. There exist $t, t' \in T_-, t \neq t'$, arbitrarily close to 0, such that $F_t \stackrel{r}{\sim} F_{t'}$ (that is F is not universal in any neighbourhood of $0 \in T_-$).

Proof:

- (i) is due to Saito [SaK 2].
- (ii) follows from [AGV], 12.6, Theorem (p. 209).
- (iii) If T_- would contain trivial subfamilies of unfoldings there must be a $t \in T_-$ with $\mu(F_t) < \mu(f_0)$, which is not the case.
- (iv) The group μ_d of d -th roots of unit acts on T_- , has 0 as fixed point and a non-trivial orbit for any $t \neq 0$. Since for $\xi \in \mu_d$, $F_{\xi \circ t}(\xi \circ x) = \xi^d F_t(x) = F_t(x)$, two different points of an orbit of μ_α correspond to right equivalent functions, we obtain (iv).

Let us introduce the notion of a fine and coarse moduli space for unfoldings of negative weight with principal part f_0 (the weights w_1, \dots, w_n and f_0 are given as above): let \mathcal{S} be a category of base spaces. For $S \in \mathcal{S}$, a **family of unfoldings of negative weight with principal part f_0** over S is a family of unfoldings

$$\phi : (\mathbb{C}^n, 0) \times S \rightarrow (\mathbb{C}, 0) \times S, (x, s) \mapsto (G(x, s), s) = (G_s(x), s)$$

such that: for any $s \in S$, $G_s : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ is right equivalent to a semi-quasihomogeneous power series with principal part f_0 and the germ of G at s , $G : (\mathbb{C}^n, 0) \times (S, s) \rightarrow (\mathbb{C}, 0)$, is an unfolding of G_s of negative weight. For any morphism of base spaces $\varphi : T \rightarrow S$, the induced map $\varphi^* \phi : (\mathbb{C}^n, 0) \times T \rightarrow (\mathbb{C}, 0) \times T$, $(x, t) \mapsto (G(x, \varphi(t)), t)$, is an unfolding of negative weight with principal part f_0 over T . Hence, we obtain a functor

$$\text{Unf}_{f_0}^- : \mathcal{S} \rightarrow \text{sets}$$

which associates to $S \in \mathcal{S}$ the set of right equivalence classes of families of unfoldings of negative weight with principal part f_0 over S . If $pt \in \mathcal{S}$ denotes the base space consisting of one reduced point, then

$$\text{Unf}_{f_0}^-(pt) = \{ \text{right equivalence classes of power series } f \in \mathbb{C}\{x_1, \dots, x_n\} \text{ which are right equivalent to a semi-quasihomogeneous power series with principal part } f_0 \}.$$

A **fine moduli space** for the functor $\text{Unf}_{f_0}^-$ is, by definition, a pair (T, ψ) where T is a base space and ψ a natural transformation of functors

$$\psi : \text{Unf}_{f_0}^- \rightarrow \text{Hom}(-, T)$$

such that (T, ψ) represents the functor $\text{Unf}_{f_0}^-$.

The pair (T, ψ) is a **coarse moduli space** for $\text{Unf}_{f_0}^-$ if

- (i) if $\psi(pt)$ is bijective, and

(ii) given the solid arrows (natural transformations) in the following diagram

$$\begin{array}{ccc}
 & \text{Unf}_{f_0}^- & \\
 & \swarrow \quad \searrow & \\
 \text{Hom}(-, T) & \text{-----} > & \text{Hom}(-, T'),
 \end{array}$$

there exists a unique dotted arrow (natural transformation) making the diagram commutative.

A fine moduli space is, of course, coarse.

The definitions of fine and coarse moduli spaces still depend on the category of base spaces \mathcal{S} . If \mathcal{S} is the category of complex germs and if $(S, 0) \in \mathcal{S}$, then $\text{Hom}((S, 0), T)$ denotes the set of morphisms of germs $(S, 0) \rightarrow (T, t)$ where t may be any point of T . In this case, if (T, ψ) is a fine moduli space, given any $t \in T$, there exists a unique (up to right equivalence) universal unfolding of negative weight with principal part f_0 over the germ (T, t) which corresponds to $\text{id} \in \text{Hom}((T, t), (T, t))$. But we may not have a universal family over all of T . If \mathcal{S} is the category of all complex spaces, the existence of a fine moduli space implies the existence of a global universal family over T . But we shall see that even for complex germs as base spaces a fine moduli space may not exist. A coarse moduli space, however, does exist even if \mathcal{S} is the category of all complex spaces. The reason is that for a coarse moduli space we do not require any kind of a universal family.

Theorem 1.3 *Let E_{f_0} be the finite group defined in Definition 2.6, acting on T_- . The geometric quotient T_-/E_{f_0} is a coarse moduli space for the functor $\text{Unf}_{f_0}^- : \text{complex spaces} \rightarrow \text{sets}$.*

Proof: Since E_{f_0} is finite, and the action is holomorphic, the geometric quotient T_-/E_{f_0} exists as a complex space. According to Theorem 2.1, Proposition 2.3 and Corollary 2.6, for any semiquasihomogeneous power series f with principal part f_0 there exists a unique point $\underline{t} \in T_-/E_{f_0}$ such that if $f_t \stackrel{r}{\sim} f$, $t \in T_-$ maps to \underline{t} . In this way we obtain a bijection $\psi(pt)$ from the set of right equivalence classes of semiquasihomogeneous power series with principal part f_0 to T_- .

Now let $G : (\mathbb{C}^n, 0) \times S \rightarrow (\mathbb{C}, 0)$ define an element of $\text{Unf}_{f_0}^-(S)$ for some complex space S . We may cover S by open sets U_i such that there exist morphisms $\varphi_i : U_i \rightarrow T_-$ with $\varphi_i^* F \stackrel{r}{\sim} G|_{U_i}$. Even if the φ_i are not unique, by the properties of a quotient the compositions $U_i \xrightarrow{\varphi_i} T_- \rightarrow T_-/E_{f_0}$ glue together to give a morphism $S \rightarrow T_-/E_{f_0}$. This construction is functorial and provides the desired natural transformation $\text{Unf}_{f_0}^- \rightarrow \text{Hom}(-, T_-/E_{f_0})$. This finishes the proof of Theorem 1.3 (for further details for construction of moduli spaces via geometric quotients cf. [Ne]).

Remark 1.4 (i) If f_0 is simple or simple elliptic, then the coarse moduli space constructed above consists of one reduced point. Hence, it is even a fine moduli space.

(ii) If f_0 is neither simple nor simple elliptic, $\text{Unf}_{f_0}^-$ does not admit a fine moduli space, even not if we take complex germs as base spaces. This can be seen as follows: assume there exists such a fine moduli space then, since it is also coarse, it must be isomorphic to T_-/E_{f_0} . Moreover, there exists a universal unfolding over the germ $(T_-/E_{f_0}, 0)$ which can be induced from the semiuniversal unfolding F over the germ $(T_-, 0)$ and vice versa. Since T_- does not contain trivial subfamilies, the semiuniversal family F over $(T_-, 0)$ would be universal, which contradicts Lemma 1.2 (iv).

Example: Let $f_0(x, y) = x^4 + y^5$. We obtain $T_- = \mathbb{C}$ and $F(x, y, t) = x^4 + y^5 + tx^2y^3$, $(d; w_1, w_2; w(t)) = (20; 4, 5; -2)$. In this case $E_{f_0} = \mu_d$ and the ring of invariant functions on T_- is $\mathbb{C}[t^{10}]$, hence $T_-/E_{f_0} \cong \mathbb{C}$. We give a computational argument that a fine moduli space does not exist:

A local universal family over $(T_-/E_{f_0}, 0)$ would be given by $G : (\mathbb{C}^n, 0) \times (T_-/E_{f_0}) \rightarrow (\mathbb{C}, 0)$, $(x, y, s) \mapsto G(x, y, s)$. The proof of Theorem 1.3 shows that then F would be induced from G by the canonical map $T_- \rightarrow T_-/E_{f_0}$, which is not an isomorphism. Moreover, the fibre $F^{-1}(0)$ would be isomorphic to $G^{-1}(0)$ under the map $(x, y, t) \mapsto (x, y, s = t^{10})$. The image of this map can be computed by eliminating t from $F(x, y, t) = 0$, $s - t^{10} = 0$. The result is the hypersurface defined by $G = (x^4 + y^5)^{10} - sx^{20}y^{30}$. The special fibre for $s = 0$ has a non-isolated singularity, hence is not isomorphic to $f_0 = 0$.

Remark 1.5 Since the group E_{f_0} acts even algebraically on T_- by Proposition 2.4, T_-/E_{f_0} is an algebraic variety. We may take the category of base spaces \mathcal{S} to be the category of (separated) algebraic spaces and define (families of) unfoldings in the same manner as above, replacing the analytic local ring $\mathbb{C}\{x\}$ by the henselization of $\mathbb{C}[x]$. With the same proof as above we obtain that T_-/E_{f_0} is a coarse moduli space for the functor

$$\text{Unf}_{f_0}^- : \text{algebraic spaces} \rightarrow \text{sets.}$$

2 Isomorphism of semiquasihomogeneous singularities

We fix weights $w_1, \dots, w_n \in \mathbb{N}$ and a degree $d \in \mathbb{N}$ such that the normalized weights $\bar{w}_i = \frac{w_i}{d}$ fulfill $0 < \bar{w}_i \leq \frac{1}{2}$. The weights induce a filtration on $\mathbb{C}\{x\}$. An automorphism $\varphi \neq id$ of $\mathbb{C}\{x\}$ has degree $m = \deg \varphi$ if m is the maximal number such that

$$\deg(\varphi(x_i) - x_i) \geq w_i + m \quad \forall i = 1, \dots, n.$$

The automorphisms of degree ≥ 0 form the group $\text{Aut}_{\geq 0}(\mathbb{C}\{x\})$ of all automorphisms of $\mathbb{C}\{x\}$ which respect the filtration. The automorphisms of degree > 0 form a normal subgroup $\text{Aut}_{> 0}(\mathbb{C}\{x\})$ in $\text{Aut}_{\geq 0}(\mathbb{C}\{x\})$. Automorphisms will be called quasihomogeneous if they map each quasihomogeneous polynomial to a quasihomogeneous polynomial of the same degree. They form a group $G_w \subset \text{Aut}_{\geq 0}(\mathbb{C}\{x\})$, which is isomorphic to the quotient $\text{Aut}_{\geq 0}(\mathbb{C}\{x\})/\text{Aut}_{> 0}(\mathbb{C}\{x\})$.

The image $\varphi(f)$ of a semiquasihomogeneous power series f of degree d by an automorphism φ of $\mathbb{C}\{x\}$ is semiquasihomogeneous of the same degree if $\deg \varphi \geq 0$. The converse is true, too:

Theorem 2.1 *Let f and g be semiquasihomogeneous of degree d , and let φ be an automorphism of $\mathbb{C}\{x\}$ such that $\varphi(f) = g$. Then $\deg \varphi \geq 0$.*

Proof: The proof uses some facts which come from the Gauss–Manin connection for isolated hypersurface singularities ([SS], [SaM], [AGVII], [He]). The main idea is the following: in the case of a semiquasihomogeneous singularity the weights \bar{w}_i induce a filtration on $\mathbb{C}\{x\}$ and a filtration on the Brieskorn lattice $H_0''(f)$. This last filtration coincides with the V -filtration and is independent of the coordinates.

The Brieskorn lattice $H_0''(f)$ is

$$H_0'' = \Omega^n / df \wedge d\Omega^{n-1}.$$

Here $\Omega^k = \Omega_{\mathbb{C}^n, 0}^k$ denotes the space of germs of holomorphic k -forms. The class of $\omega \in \Omega^n$ in $H_0''(f)$ is denoted by $s[\omega]_0 \in H_0''(f)$. The V -filtration on $H_0''(f)$ is determined by the orders $\alpha_f(\omega) = \alpha_f(s[\omega]_0)$ of n -forms $\omega \in \Omega^n$. The most explicit description of the order $\alpha_f(\omega)$ might be the following ([AGVII], [He]):

$$\begin{aligned}
\alpha_f(\omega) = \min \{ \alpha \mid & \exists \text{ (manyvalued) continuous family of cycles} \\
& \delta(t) \in H_{n-1}(X_t, \mathbb{Z}) \text{ on the Milnor fibers } X_t \\
& \text{of the singularity } f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0), \\
& \text{such that } a_{\alpha, k} \neq 0 \text{ in} \\
& \int_{\delta(t)} \frac{\omega}{df} = \sum_{\beta, k} a_{\beta, k} \cdot t^\beta \cdot (\ln t)^k \\
& \text{for a } k \text{ with } 0 \leq k \leq n-1 \}.
\end{aligned}$$

The description shows that we have

$$\alpha_f(\omega) = \alpha_g(\varphi(\omega)) = \alpha_g(\varphi(h)d\varphi(x))$$

for $\omega = h(x)dx_1 \dots dx_n = hdx \in \Omega^n$.

Since f is semiquasihomogeneous it is possible to give a simple algebraic description of the order $\alpha_f(\omega)$. Indeed, we define mappings

$$\begin{aligned}
\nu_C & : \mathbb{C}\{x_1, \dots, x_n\} \rightarrow \mathbb{Q}_{\geq 0} \cup \{\infty\}, \\
\nu_\Omega & : \Omega^n \rightarrow \mathbb{Q}_{> -1} \cup \{\infty\}, \\
\nu_f & : H_0^n(f) \rightarrow \mathbb{Q}_{> -1} \cup \{\infty\}
\end{aligned}$$

by

$$\nu_C(x^\alpha) = \sum_{i=1}^n \bar{w}_i \alpha_i, \quad \nu_C(0) = \infty, \quad \nu_C\left(\sum b_\alpha x^\alpha\right) = \min\{\nu_C(x^\alpha) \mid b_\alpha \neq 0\}$$

and

$$\nu_\Omega(hdx) = \nu_C(hx_1 \dots x_n) - 1$$

and

$$\nu_f(s[\omega]_0) = \nu_f(\omega) = \max\{\nu_\Omega(\eta) \mid s[\eta]_0 = s[\omega]_0\}.$$

Then, from [He], Chapter 2.4, it follows that

$$\nu_f(\omega) = \alpha_f(\omega) = \alpha_g(\varphi(\omega)) = \nu_g(\varphi(\omega)).$$

For all $\eta \in \Omega^{n-2}$ we have

$$\nu_f(df \wedge d\eta) \geq -1 + \sum_j \bar{w}_j + (1 - \max(\bar{w}_i)) \geq \sum_j \bar{w}_j - \frac{1}{2}.$$

For ω with

$$\min\{\nu_\Omega(\omega), \nu_f(\omega), \nu_g(\varphi(\omega)), \nu_\Omega(\varphi(\omega))\} < \sum_j \bar{w}_j - \frac{1}{2}$$

this implies

$$\nu_\Omega(\omega) = \nu_f(\omega) = \nu_g(\varphi(\omega)) = \nu_\Omega(\varphi(\omega)).$$

We obtain

$$\sum_j \bar{w}_j - 1 = \nu_\Omega(dx) = \nu_f(dx) = \nu_g(d\varphi(x)) = \nu_\Omega(d\varphi(x)).$$

For i with $\bar{w}_i < \frac{1}{2}$ we obtain

$$\begin{aligned} \bar{w}_i + \nu_\Omega(dx) &= \nu_C(x_i) + \nu_\Omega(dx) = \nu_\Omega(x_i dx) = \nu_f(x_i dx) \\ &= \nu_g(\varphi(x_i) d\varphi(x)) = \nu_\Omega(\varphi(x_i) d\varphi(x)) = \nu_C(\varphi(x_i)) + \nu_\Omega(d\varphi(x)) \\ &= \nu_C(\varphi(x_i)) + \nu_\Omega(dx) \end{aligned}$$

and $\nu_C(\varphi(x_i)) = \bar{w}_i$.

For i with $\bar{w}_i = \frac{1}{2}$ the equality $\nu_\Omega(x_i dx) = \sum \bar{w}_i - \frac{1}{2}$ implies

$$\nu_\Omega(\varphi(x_i) d\varphi(x)) \geq \sum \bar{w}_i - \frac{1}{2}$$

and $\nu_C(\varphi(x_i)) \geq \frac{1}{2}$. Therefore, $\nu_C(\varphi(x_i)) \geq \nu_C(x_i) = w_i \quad \forall i = 1, \dots, n$, and thus $\deg \varphi \geq 0$.

Remark 2.2 In the following, Theorem 2.1 will be used to describe a finite group $E_{f_0} \subset \text{Aut}(T_-)$ which operates transitively on each set of parameters in T_- which belong to one right equivalence class. Theorem 2.1 also shows that the Hilbert function of the Tjurina algebra (cf. Chapter 4) is an invariant of the contact equivalence class.

Now let $f_0 \in \mathbb{C}[x_1, \dots, x_n]$ be quasihomogeneous of degree d with an isolated singularity in 0 . Let m_1, \dots, m_k denote the monomials of degree $> d$ in a monomial base of the Milnor algebra of f_0 . Consider the semiuniversal unfolding of f_0 of negative weight,

$$F = f_0 + \sum_{i=1}^k m_i t_i.$$

For a fixed value of t we write $F_t = f_0 + \sum m_i t_i$. With $\deg t_i = w(t_i) = d - \deg m_i < 0$ we obtain a filtration on $\mathbb{C}[t_1, \dots, t_k] = A_-$ such that $F \in \mathbb{C}[x, t]$ is quasihomogeneous of degree d in x and t . We write $T_- = \text{Spec } A_-$ (cf. §1).

Proposition 2.3 *For any semiquasihomogeneous power series f with principal part f_0 there exist an automorphism $\varphi \in \text{Aut}_{>0}(\mathbb{C}\{x\})$ and a parameter $t \in T_-$ such that $\varphi(f) = F_t$. The $t \in T_-$ is uniquely determined.*

Proof: The existence of φ and t is proved in [AGV], 12.6, Theorem (p. 209). The following proves the uniqueness of t .

Let t and $t' \in T_-$ and $\psi \in \text{Aut}_{>0}(\mathbb{C}\{x\})$ be given such that $\psi(F_t) = F_{t'}$. With $\psi_s(x_i) = x_i + s(\psi(x_i) - x_i)$ we obtain a family ψ_s of automorphisms in $\text{Aut}_{>0}(\mathbb{C}\{x\})$. The family $\psi_s(F_t)$ of semiquasihomogeneous functions with principal part f_0 connects $\psi_0(F_t) = F_t$ and $\psi_1(F_t) = F_{t'}$. The family may not be contained in T_- , but can be induced from T_- by a suitable base change: Following the proof of the theorem in [AGV], 12.6 (p. 209), we can find a family χ_s of automorphisms and a holomorphic map $\sigma : \mathbb{C} \rightarrow T_-$ such that $\chi_s \circ \psi_s(F_t) = F_{\sigma(s)}$ and $\chi_s \in \text{Aut}_{>0}(\mathbb{C}\{x\})$ and even $\chi_0 = id = \chi_1$, $\sigma(0) = t$, $\sigma(1) = t'$. But since T_- is part of the semiuniversal deformation, which is miniversal on the μ -constant stratum, and since T_- does not contain trivial subfamilies with respect to right equivalence, $t = t'$ as desired.

Proposition 2.4 1. For any $\varphi \in G_w^{f_0} = \{\psi \in G_w \mid \psi(f_0) = f_0\}$ and any $t \in T_-$ there exist $s = \theta(\varphi)(t) \in T_-$ and an automorphism $\psi \in \text{Aut}_{>0}(\mathbb{C}\{x\})$ such that $\psi \circ \varphi(F_t) = F_s$.

2. The function $\theta(\varphi) : T_- \rightarrow T_-$ is uniquely determined, bijective and fulfills $\theta(\varphi^{-1}) = \theta^{-1}(\varphi)$ and $\theta(\varphi) \circ \theta(\psi) = \theta(\varphi \circ \psi)$ for any $\psi \in G_w^{f_0}$.

3. The components $\theta(\varphi)(t_i)$ are quasihomogeneous polynomials in A_- of degree $\deg(t_i)$.

Proof: The statements 1. and 2. follow from Proposition 2.3 and from the fact that $\text{Aut}_{>0}(\mathbb{C}\{x\})$ is a normal subgroup of $\text{Aut}_{\geq 0}(\mathbb{C}\{x\})$. Statement 3. follows from the proof of the theorem in [AGV], 12.6 (p. 209). Along the lines of this proof one can construct power series $\psi_1, \dots, \psi_n \in \mathbb{C}\{x, t\}$ and a family of automorphisms $\psi(t)$ such that $\psi(t)(x_i) = \psi_i(t)$ with the following properties:

- ψ_i is quasihomogeneous in x and t of degree w_i ,
- $\psi_i - x_i$ has degree $> w_i$ in x ,
- for any fixed t the automorphism $\psi(t) \in \text{Aut}_{>0}(\mathbb{C}\{x\})$ with $\psi(t)(x_i) = \psi_i(t)$ gives $\psi(t) \circ \varphi(F_t) = F_{\theta(\varphi)(t)}$.

The power series $F = f_0 + \sum m_i t_i$, and $\varphi(F) = f_0 + \dots$ and $\psi(t) \circ \varphi(F) = f_0 + \sum m_i \theta(\varphi)(t_i)$ are all quasihomogeneous of degree d with respect to x and t . This proves 3.

The functions $\theta(\varphi)$ are biholomorphic.

Definition 2.5 The image $\theta(G_w^{f_0})$ in $\text{Aut}(T_-)$ will be denoted by E_{f_0} .

Corollary 2.6 The map $\theta : G_w^{f_0} \rightarrow E_{f_0} \subset \text{Aut}(T_-)$ is a group homomorphism. The automorphisms $\theta(\varphi)$ of T_- commute with the \mathbb{C}^* -operation on T_- . Each orbit of E_{f_0} consists of all parameters in T_- which belong to one right equivalence class.

Proof: The first two statements follow from Proposition 2.4, the third statement follows from Theorem 2.1.

Proposition 2.7 1. The group $G_w^{f_0}$ is finite if $\bar{w}_1, \dots, \bar{w}_{n-1} < \frac{1}{2}$ and $\bar{w}_n \leq \frac{1}{2}$.

2. The group E_{f_0} is finite.

Proof:

1. The dimension of the algebraic group G_w is

$$\dim G_w = \sum_{i=1}^n \#(\text{monomials } x^\alpha \text{ of degree } w_i).$$

The group G_w operates on

$$V = \bigoplus_{\deg x^\alpha = d} \mathbb{C} \cdot x^\alpha.$$

Let $j(f_0)$ denote the Jacobi ideal of f_0 and $j_i(f_0)$ the ideal

$$j_i(f_0) = \left(\frac{\partial f_0}{\partial x_1}, \dots, \frac{\partial f_0}{\partial x_{i-1}}, \frac{\partial f_0}{\partial x_{i+1}}, \dots, \frac{\partial f_0}{\partial x_n} \right).$$

The tangent space $T_{f_0} G_w f_0 \subset T_{f_0} V$ of $G_w f_0$ in f_0 is

$$T_{f_0} G_w f_0 \cong j(f_0) \cap V.$$

For any relation

$$0 = \sum_{i=1}^n \sum_{\deg x^\alpha = w_i} a_{\alpha,i} \cdot x^\alpha \cdot \frac{\partial f_0}{\partial x_i} = \sum_{i=1}^n b_i \frac{\partial f_0}{\partial x_i}$$

with $a_{\alpha,i} \in \mathbb{C}$ and $b_i = \sum_{\deg x^\alpha = w_i} a_{\alpha,i} \cdot x^\alpha$ we have $\deg b_i = w_i$ and $\deg \frac{\partial f_0}{\partial x_j} = d - w_j > w_i$ for $j \neq i$. Therefore, $b_i \notin j_i(f_0)$ or $b_i = 0$. But since f_0 has an isolated singularity, the sequence $(\frac{\partial f_0}{\partial x_1}, \dots, \frac{\partial f_0}{\partial x_n})$ is a regular sequence and $\frac{\partial f_0}{\partial x_i}$ is not a zero divisor in $j_i(f_0)$. This implies $b_i = 0$ for any i , and

$$j(f_0) \cap V = \bigoplus_{i=1}^n \bigoplus_{\deg x^\alpha = w_i} \mathbb{C} \cdot x^\alpha \cdot \frac{\partial f_0}{\partial x_i},$$

and

$$\dim G_w^{f_0} = \dim G_w - \dim j(f_0) \cap V = 0.$$

2. One can order the weights w_i such that $\bar{w}_1, \dots, \bar{w}_r < \frac{1}{2}$, $\bar{w}_{r+1}, \dots, \bar{w}_n = \frac{1}{2}$. The generalized Morse lemma and Theorem 2.1 imply the existence of an automorphism $\varphi \in G_w$ and of a quasihomogeneous polynomial $g_0 \in \mathbb{C}[x_1, \dots, x_r]$ of degree d such that $\varphi(f_0) = g_0 + x_{r+1}^2 + \dots + x_n^2$. Now let $\tilde{m}_1, \dots, \tilde{m}_k$ be the monomials of degree $> d$ in a monomial base of the Jacobi algebra of g_0 . Analogously to F we obtain families

$$\begin{aligned}\tilde{G} &= g_0 + \sum_{i=1}^k \tilde{m}_i \tilde{t}_i \\ \text{and } G &= \tilde{G} + x_{r+1}^2 + \dots + x_n^2.\end{aligned}$$

It is well known that G_t and $G_{t'}$ are right equivalent if and only if \tilde{G}_t and $\tilde{G}_{t'}$ are right equivalent. Let \tilde{w} be the tuple of weights $\tilde{w} = (w_1, \dots, w_r)$. The group $G_{\tilde{w}}^{g_0}$ is finite by the first part of this proposition and induces a finite group $\tilde{E}_{\tilde{w}}$ of automorphisms of $\tilde{T}_- = \text{Spec } \mathbb{C}[\tilde{t}]$. In fact this is the largest subgroup of $\text{Aut}(\tilde{T}_-)$ which respects the right equivalence classes. Similarly to Proposition 2.4 one can prove that φ induces a biholomorphic mapping from T_- to \tilde{T}_- which respects the right equivalence classes. This gives an injective (in fact bijective) mapping from E_{f_0} to $\tilde{E}_{\tilde{w}}$. Hence, E_{f_0} is finite.

Example 2.8 $f_0 = x^3 + y^3 + z^7$, $(d; w_1, w_2, w_3) = (21; 7, 7, 3)$, $T_- = \mathbb{C}^5$, $F = f_0 + \sum_{i=1}^5 t_i m_i = f_0 + t_1 x z^5 + t_2 y z^5 + t_3 x y z^3 + t_4 x y z^4 + t_5 x y z^5$, the weights of (t_1, \dots, t_5) are $(-1, -1, -2, -5, -8)$.

$G_w^{f_0}$ contains $6 \cdot 3 \cdot 7$ elements: obviously, $G_w^{f_0} \cong G_{(1,1)}^{g_0} \times \mathbb{Z}_7$ where $g_0 = x^3 + y^3$. The group $G_{(1,1)}^{g_0}$ is isomorphic to a subgroup of $\mathbf{GL}(2, \mathbb{C})$. The image in $\mathbf{PGL}(2, \mathbb{C})$ permutes three points in $\mathbf{P}^1 \mathbb{C}$ and is isomorphic to S_3 , the kernel is isomorphic to $\{id, \xi \cdot id, \xi^2 \cdot id\}$, where $\xi = e^{2\pi i/3}$. Therefore $G_{(1,1)}^{g_0}$ is

$$G_{(1,1)}^{g_0} = (\langle \alpha \rangle \times \langle \beta \rangle) \times \langle \gamma \rangle \times \langle \delta \rangle \cong S_3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$$

with

$$\begin{aligned}\alpha &: (x, y, z) \rightarrow (y, x, z), \\ \beta &: (x, y, z) \rightarrow (\xi x, \xi^2 y, z), \\ \gamma &: (x, y, z) \rightarrow (\xi x, \xi y, z), \\ \delta &: (x, y, z) \rightarrow (x, y, e^{2\pi i/7} z).\end{aligned}$$

The mapping $\theta : G_w^{f_0} \rightarrow E_{f_0}$ is an isomorphism with

$$\begin{aligned}\theta(\alpha) &: (t_1, t_2, t_3, t_4, t_5) \rightarrow (t_2, t_1, t_3, t_4, t_5), \\ \theta(\beta) &: (t_1, t_2, t_3, t_4, t_5) \rightarrow (\xi t_1, \xi^2 t_2, t_3, t_4, t_5), \\ \theta(\gamma) &: (t_1, t_2, t_3, t_4, t_5) \rightarrow (\xi t_1, \xi t_2, \xi^2 t_3, \xi^2 t_4, \xi^2 t_5), \\ \theta(\delta) &: (t_1, t_2, t_3, t_4, t_5) \rightarrow (\zeta^5 t_1, \zeta^5 t_2, \zeta^3 t_3, \zeta^4 t_4, \zeta^5 t_5) \text{ with } \zeta = e^{2\pi i/7}.\end{aligned}$$

Let \mathbb{C}^* denote the group of \mathbb{C}^* -operations on T_- . Then $E_{f_0} \cap \mathbb{C}^* = \langle \theta(\gamma), \theta(\delta) \rangle$ and $E_{f_0} \cdot \mathbb{C}^* \cong \langle \theta(\alpha), \theta(\beta) \rangle \times \mathbb{C}^* \cong S_3 \times \mathbb{C}^*$.

3 Kodaira–Spencer map and integral manifolds

Let f_0 be semiquasihomogeneous of type $(d; w_1, \dots, w_n)$, $w_i > 0$, and $F : \mathbb{C}^n \times T_- \rightarrow \mathbb{C}$, $(x, t) \mapsto f_0(x) + \sum_{i=1}^k t_i m_i$, the semiuniversal family of unfoldings of negative weight as in §1. In order to describe the orbits of the contact group acting on T_- we study the Kodaira–Spencer map of the induced semiuniversal family of deformations (of space germs) defined as follows. Let

$$\mathcal{X} = \{(x, t) \in \mathbb{C}^n \times T_- \mid F(x, t) = 0\}$$

and let $(\mathcal{X}, 0 \times T_-)$ denote the germ of \mathcal{X} along the trivial section $0 \times T_-$ which is a subgerm of $(\mathbb{C}^n \times T_-, 0 \times T_-) = (\mathbb{C}^n, 0) \times T_-$. The composition with the projection gives a morphism

$$\phi : (\mathcal{X}, 0 \times T_-) \hookrightarrow (\mathbb{C}^n, 0) \times T_- \rightarrow T_-$$

such that, for any $t \in T_-$, $(\phi^{-1}(t), (0, t)) \cong (\mathcal{X}_t, 0) \subset (\mathbb{C}^n, 0)$ is a semiquasihomogeneous hypersurface singularity with principal part equal to $(\mathcal{X}_0, 0) = (f_0^{-1}(0), 0) =: (X_0, 0)$. We call this family the **semiuniversal family of deformations of negative weight of semiquasihomogeneous hypersurface singularities with fixed principal part $(\mathbf{X}_0, \mathbf{0})$** (see also §4).

For the study of the Kodaira–Spencer map of $(\mathcal{X}, 0 \times T_-) \rightarrow T_-$ it is more convenient to work on the ring level $A_- \rightarrow A_- \{x\} / F$.

The Kodaira–Spencer map (cf. [LP]) of the family $A_- \rightarrow A_- \{x\} / F$,

$$\rho : \text{Der}_{\mathbb{C}} A_- \rightarrow (x)A_- \{x\} / \left(F + (x) \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right) \right),$$

is defined by $\rho(\delta) = \text{class}(\delta F) = \text{class}(\sum_{i=1}^k \delta(t_i) m_i)$.

Let \mathcal{L} be the kernel of ρ . \mathcal{L} is a Lie–algebra and along the integral manifolds of \mathcal{L} the family is analytically trivial (cf. [LP]).

In our situation it is possible to give generators of \mathcal{L} as A_- –module:

Let $I = A_- \{x\} / (\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})$, then I is a free A_- –module and $\{m_i\}_{i=1, \dots, k}$ can be extended to a free basis.

Multiplication by F defines an endomorphism of I and $FI \subseteq \bigoplus_{i=1}^k m_i A_-$.

Define $h_{\alpha, j}$ by

$$x^\alpha F = \sum h_{\alpha, j} m_j \text{ in } I.$$

Then $h_{\alpha, j}$ is homogeneous of degree $|\alpha| + \deg(t_j) = |\alpha| + d - \deg(m_j)$. This implies $h_{\alpha, j} = 0$ if $|\alpha| + \deg(t_j) \geq 0$, in particular $h_{\alpha, j} = 0$ if $|\alpha| \geq (n-1)d - 2 \sum w_i$. For α and $|\alpha| < (n-1)d - 2 \sum w_i$ let $\delta_\alpha := \sum h_{\alpha, j} \frac{\partial}{\partial t_j}$.

Proposition 3.1 (cf. [LP], Proposition 4.5):

1. δ_α is homogeneous of degree $|\alpha|$.
2. $\mathcal{L} = \sum A_- \delta_\alpha$.

Now there is a non-degenerate pairing on I (the residue pairing) which is defined by $\langle h, k \rangle = \text{hess}(h \cdot k)$. Here $\text{hess}(h)$ is the evaluation of h at the socle (the hessian of f).

Using the pairing one can prove the following:

Proposition 3.2 *There are homogeneous elements $n_1, \dots, n_k \in A_- \{x\}$ with the following properties:*

1. If $n_i F = \sum_{j=1}^k h_{ij} m_j$ in I then $h_{ij} = h_{k-j+1, k-i+1}$.
2. If $\delta_i := \sum_{j=1}^k h_{ij} \frac{\partial}{\partial t_j}$ then δ_i is homogeneous of degree $\deg(n_i)$ and $\mathcal{L} = \sum_{i=1}^k A_- \delta_i$.

In [LP] (Proposition 5.6) this proposition is proved for $n = 2$. The proof can easily be extended to arbitrary n . The important fact is the symmetry, expressed in 1.

Let L_+ be the Lie-algebra of all vector fields of \mathcal{L} of degree $\geq w = \min\{w_i\}$. Then L is finite dimensional and nilpotent. $\delta_2, \dots, \delta_k \in L_+$ and $\delta_1 = \sum_{i=1}^k \deg(t_i) t_i \frac{\partial}{\partial t_i}$ is the Euler vector field (cf. [LP]). Let $L = L_+ \oplus \mathbb{C} \delta_1$ then L is a finite dimensional and solvable Lie-algebra and $\mathcal{L} = \sum A_- L$, $L/L_+ \cong \mathbb{C} \delta_1$.

Corollary 3.3 *The integral manifolds of \mathcal{L} coincide with the orbits of the algebraic group $\exp(L)$.*

Now consider the matrix $M(t) := (\delta_i(t_j))_{i,j=1,\dots,k} = (h_{ij})_{i,j=1,\dots,k}$. Evaluating this matrix at $t \in T_-$ we have

$$\begin{aligned} \text{rank } M(t) &= \text{dimension of a maximal integral manifold of } \mathcal{L} \\ &\quad (\text{resp. of the orbit of } \exp(L)) \text{ at } t \\ &= \mu - \tau(t), \end{aligned}$$

where $\tau(t)$ denotes the Tjurina number of the singularity defined by t i.e. of $F(x, t)$.

Example 3.4 We continue with Example 2.8, $f_0 = x^3 + y^3 + z^7$. Let

$$\begin{aligned}
n_1 &= -21 \\
n_2 &= -21z + \left(\frac{250}{49}t_1^3t_2 + \frac{55}{7}t_1^2t_3 - \frac{250}{49}t_2^4 \right) y - \frac{55}{7}t_2^2t_3x \\
n_3 &= -21z^2 - 30t_2y \\
n_4 &= -21x \\
n_5 &= -21y
\end{aligned}$$

then the matrix defined by Proposition 3.2 is

$$(\delta_i(t_j)) = \begin{pmatrix} t_1 & t_2 & 2t_3 & 5t_4 & 8t_5 \\ 0 & 0 & 0 & 2t_3 - \frac{10}{7}t_1t_2 & 5t_4 \\ 0 & 0 & 0 & 0 & 2t_3 \\ 0 & 0 & 0 & 0 & t_2 \\ 0 & 0 & 0 & 0 & t_1 \end{pmatrix}.$$

We have $\mu = 24$ and

$$\begin{aligned}
\tau = 21 & \quad \text{if and only if } 2t_3 - \frac{10}{7}t_1t_2 \neq 0, \\
\tau = 22 & \quad \text{if and only if } 2t_3 - \frac{10}{7}t_1t_2 = 0 \text{ and } t_1 \neq 0 \text{ or } t_2 \neq 0 \text{ or } t_3 \neq 0 \text{ or } t_4 \neq 0, \\
\tau = 23 & \quad \text{if and only if } t_1 = t_2 = t_3 = t_4 = 0 \text{ and } t_5 \neq 0, \\
\tau = 24 & \quad \text{if and only if } t_1 = t_2 = t_3 = t_4 = t_5 = 0.
\end{aligned}$$

4 Moduli spaces with respect to contact equivalence

In this section we want to construct a coarse moduli space for semiquasihomogeneous hypersurface singularities with fixed principal part with respect to contact equivalence, that is isomorphism of space germs. Such a moduli space does only exist if we fix further numerical invariants. We shall use the Hilbert function of the Tjurina algebra induced by the given weights.

Let us first define the functor for which we are going to construct the moduli space.

A complex germ $(X, 0) \subset (\mathbb{C}^n, 0)$ is called a **quasihomogeneous** (respectively **semiquasihomogeneous**) **hypersurface singularity** of type $(d; w_1, \dots, w_n)$ if there exists a quasihomogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ (respectively a semiquasihomogeneous power series $f \in \mathbb{C}\{x_1, \dots, x_n\}$) of type $(d; w_1, \dots, w_n)$ such that $(X, 0) = (f^{-1}(0), 0)$. If f_0 is the principal part of f then $(X_0, 0) = (f_0^{-1}(0), 0)$ is called the **principal part** of $(X, 0)$. Multiplying f with a unit changes f_0 by a constant, hence the principal part is well-defined. Two power series are contact equivalent if and only if the corresponding space germs are isomorphic.

A **deformation (with section)** of $(X, 0)$ over a complex germ or a pointed complex space $(S, 0)$ is a cartesian diagram

$$\begin{array}{ccc} 0 & \hookrightarrow & (S, 0) \\ \downarrow & & \downarrow \sigma \\ (X, 0) & \hookrightarrow & (\mathcal{X}, 0) \\ \downarrow & & \downarrow \phi \\ 0 & \hookrightarrow & (S, 0) \end{array}$$

such that ϕ is flat and $\phi \circ \sigma = \text{id}$. Two deformations (ϕ, σ) and (ϕ', σ') of $(X, 0)$ over $(S, 0)$ are isomorphic if there is an isomorphism $(\mathcal{X}, 0) \xrightarrow{\cong} (\mathcal{X}', 0)$ such that the obvious diagram commutes. We shall only consider deformations with section.

If $(X, 0) = (f^{-1}(0), 0)$ and if $F : (\mathbb{C}^n, 0) \times (S, 0) \rightarrow (\mathbb{C}, 0)$ is an unfolding of f then the projection $(\mathcal{X}, 0) = (F^{-1}(0), 0) \rightarrow (S, 0)$ is a deformation of $(X, 0) \hookrightarrow (\mathcal{X}, 0)$ with trivial section $\sigma(s) = (0, s)$. Conversely, any deformation of $(X, 0)$ is isomorphic to a deformation induced by an unfolding in this way. A deformation (ϕ, σ) of a hypersurface singularity $(X, 0)$, which is isomorphic to a semiquasihomogeneous hypersurface singularity $(X', 0) = (f^{-1}(0), 0)$ of type $(d; w_1, \dots, w_n)$ over $(S, 0)$, is called **deformation of negative weight** if it is isomorphic to a deformation induced by an unfolding of f of negative weight.

We have to show that the definition is independent of the chosen unfolding: two inducing unfoldings differ by a right equivalence and a multiplication with a unit. We have shown in §1 that the definition depends only on the right equivalence class. Hence, we have to show the following: if $f(x)$ is a semiquasihomogeneous power series, $f(x) + g(x, s)$, $g(x, 0) = 0$, $\deg_x g > d$, an unfolding of negative weight and $u(x, s) \in \mathcal{O}_{\mathbb{C}^n \times S, 0}^*$ a unit, then $u(f + g) \stackrel{r}{\sim} f'(x) + g'(x, s)$

for a suitable semiquasihomogeneous f' with $f^{-1}(0) = f'^{-1}(0)$, $g'(x, 0) = 0$ and $\deg_x g' > d$. Replacing $u(x, s)$ by $(u(x, 0))^{-1}u(x, s)$ we may assume that $u(x, s) = u_0(s) + u_1(x, s)$, $u_0(0) = 1$, $u_1(0, s) = 0$. If $\nu \in \mathcal{O}_{S,0}$ is a d -th root of u_0 and if ψ denotes the automorphism of degree 0, $\psi(x, s) = (\nu(s)^{w_1}x_1, \dots, \nu(s)^{w_n}x_n)$, then $u_0(s)f(x) = f(\psi(x, s)) + s\tilde{f}(x, s)$, $\deg_x \tilde{f} > d$. But this implies $u(f+g) \circ \psi^{-1} = f+g'$ with $g'(x, 0) = 0$ and $\deg g'_x > d$ as desired.

Again, we have to consider not only germs but also arbitrary complex spaces as base spaces. A **family of deformations** of hypersurface singularities over a base space $S \in \mathcal{S}$ is a morphism $\phi : \mathcal{X} \rightarrow S$ of complex spaces together with a section $\sigma : S \rightarrow \mathcal{X}$ such that for each $s \in S$ the morphism of germs $\phi : (\mathcal{X}, \sigma(s)) \rightarrow (S, s)$ is flat and the fibre $(\mathcal{X}_s, \sigma(s)) = (\phi^{-1}(s), \sigma(s))$ is a hypersurface singularity. This is, of course, only a condition on the germ $(\mathcal{X}, \sigma(S))$ of \mathcal{X} along $\sigma(S)$. A morphism of two families (ϕ, σ) and (ϕ', σ') over S is a morphism $\psi : \mathcal{X} \rightarrow \mathcal{X}'$ such that $\phi = \phi' \circ \psi$ and $\sigma' = \psi \circ \sigma$. (ϕ, σ) and (ϕ', σ') are called **contact equivalent** or **isomorphic families of deformations** if there exists a morphism ψ such that for any $s \in S$, ψ induces an isomorphism of the germs of the fibres $(\mathcal{X}_s, \sigma(s)) \cong (\mathcal{X}'_s, \sigma'(s))$.

Let us fix a quasihomogeneous hypersurface singularity $(X_0, 0) \subset (\mathbb{C}^n, 0)$ of type $(d; w_1, \dots, w_n)$. For $S \in \mathcal{S}$, a **family of deformations of negative weight with principal part $(X_0, 0)$** over S is a family of deformations

$$S \xrightarrow{\sigma} (\mathcal{X}, \sigma(S)) \xrightarrow{\phi} S$$

with section such that: for any $s \in S$ the fibre $(\mathcal{X}_s, \sigma(s))$ is isomorphic to a semiquasihomogeneous hypersurface singularity with principal part $(X_0, 0)$ and the germ $(S, s) \xrightarrow{\sigma} (\mathcal{X}, \sigma(s)) \xrightarrow{\phi} (S, s)$ is a deformation of $(\mathcal{X}_s, \sigma(s))$ of negative weight.

For any morphism of base spaces $\varphi : T \rightarrow S$, the induced deformation $T \rightarrow (\varphi^*\mathcal{X}, \varphi^*\sigma(T)) \rightarrow T$ is a family of deformations with negative weight and principal part $(X_0, 0)$. We obtain a functor

$$\text{Def}_{X_0}^- : \mathcal{S} \rightarrow \text{sets}$$

which associates to $S \in \mathcal{S}$ the set of isomorphism classes of families of deformations of negative weight with principal part $(X_0, 0)$ over S . The notations of **fine** and **coarse moduli space** for the functor $\text{Def}_{X_0}^-$ are defined in the same manner as for the functor $\text{Unf}_{f_0}^-$ in §1. The objects we are going to classify are elements of

$$\text{Def}_{X_0}^-(pt) = \{ \text{isomorphism classes of complex space germs } (X, 0) \\ \text{which are isomorphic to a semiquasihomogeneous hy-} \\ \text{persurface singularity with principal part } X_0 \}.$$

Again, as for $\text{Unf}_{f_0}^-$, we cannot expect to obtain fine moduli spaces in general. In order to obtain a coarse moduli space, we have to stratify T_- into G -invariant strata on which the geometric quotient with respect to G exists, where $G = \exp L_+ \rtimes (E_{f_0} \cdot \mathbb{C}^*) \subset \text{Aut}(T_-)$. Once we have this, the proof is the same as for Theorem 1.3.

We want to apply Theorem 4.7 from [GP 2] to the action of L_+ on T_- .

Theorem 4.1 ([GP 2]) *Let A be a noetherian \mathbb{C} -algebra and $L_+ \subseteq \text{Der}_{\mathbb{C}}^{\text{nil}} A$ a finite dimensional nilpotent Lie algebra. Suppose A has a filtration*

$$F^\bullet : 0 = F^{-1}(A) \subset F^0(A) \subset F^1(A) \subset \dots$$

by subvector spaces $F^i(A)$ such that

$$(F) \quad \delta F^i(A) \subseteq F^{i-1}(A) \text{ for all } i \in \mathbb{Z}, \delta \in L_+.$$

Suppose, moreover, L_+ has a filtration

$$Z_\bullet : L_+ = Z_1(L_+) \supseteq Z_2(L_+) \supseteq \dots \supseteq Z_e(L_+) \supseteq Z_{e+1}(L_+) = 0$$

by sub Lie algebras $Z_j(L_+)$ such that

$$(Z) \quad [L_+, Z_j(L_+)] \subseteq Z_{j+1}(L_+) \text{ for all } j \in \mathbb{Z}.$$

Let $d : A \rightarrow \text{Hom}_{\mathbb{C}}(L_+, A)$ be the differential defined by $d(a)(\delta) = \delta(a)$ and let $\text{Spec } A = \cup U_\alpha$ be the flattening stratification of the modules

$$\text{Hom}_{\mathbb{C}}(L_+, A) / \text{Ad}(F^i(A)) \quad i = 1, 2, \dots$$

and

$$\text{Hom}_{\mathbb{C}}(Z_j(L_+), A) / \pi_j(A(dA)) \quad j = 1, \dots, e,$$

where π_j denotes the projection $\text{Hom}_{\mathbb{C}}(L_+, A) \rightarrow \text{Hom}_{\mathbb{C}}(Z_j(L_+), A)$.

Then U_α is invariant under the action of L_+ and $U_\alpha \rightarrow U_\alpha/L_+$ is a geometric quotient which is a principal fibre bundle with fibre $\exp(L_+)$. Furthermore, the closure \bar{U}_α of U_α is affine, $\bar{U}_\alpha = \text{Spec } A_\alpha$, and the canonical map $U_\alpha/L_+ \rightarrow \text{Spec } A_\alpha^{L_+}$ is an open embedding.

To apply the theorem we have to construct these filtrations and interpret the corresponding stratification in terms of the Hilbert function of the Tjurina algebra.

There are natural filtrations $H^\bullet(\mathbb{C}\{x\})$ respectively $F^\bullet(A_-)$ on $\mathbb{C}\{x\}$ respectively A_- defined as follows:

Let $F^i(A_-) \subseteq A_-$ be the \mathbb{C} -vector space generated by all quasihomogeneous polynomials of degree $> -(i+1)w$ and $H^i(\mathbb{C}\{x\})$ be the ideal generated by all quasihomogeneous polynomials of degree $\geq iw$, where

$$w := \min\{w_1, \dots, w_n\}.$$

The filtration $F^\bullet(A_-)$ has the property **(F)** because every homogeneous vector field of L_+ is of degree $\geq w$. We also have $A_-dA_- = A_-dF^sA_-$ with $s = \left\lceil \frac{(n-1)d-2\sum w_i}{w} \right\rceil$, since $nd - 2\sum w_i$ is the degree of the Hessian of f and t_k is the variable of smallest degree.

To define Z_\bullet let $Z_i(L_+) :=$ the Lie algebra generated by the vectorfields $\delta \in L_+$, δ homogeneous and $\deg(\delta) \geq r_i$,

$$r_i := \min\{\deg(\delta_j) \mid t_{k+1-j} \in F^{s-i}(A_-)\}.$$

$Z_\bullet(L_+)$ has the property **(Z)** because $\deg([\delta, \delta']) \geq \deg(\delta) + \deg(\delta')$ for all $\delta, \delta' \in L_+$.

Example 4.2 We continue with Example 3.4, $f_0 = x^3 + y^3 + z^7$.

$w = 3$.

$F^0(A_-)$ is the \mathbb{C} -vector space generated by $t_1, t_2, t_3, t_1^2, t_1t_2, t_2^2$.

$F^1(A_-)$ is the \mathbb{C} -vector space generated by $t_4, \{t_1^\nu t_2^\mu t_3^\lambda\}_{\nu+\mu+2\lambda \leq 5}$.

$F^2(A_-)$ is the \mathbb{C} -vector space generated by $t_5, \{t_1^\nu t_2^\mu t_3^\lambda t_4\}_{\nu+\mu+2\lambda \leq 3}, \{t_1^\nu t_2^\mu t_3^\lambda\}_{\nu+\mu+2\lambda \leq 8}$.

We have $s = 2 = \left\lceil \frac{2 \cdot 21 - 2 \cdot 17}{3} \right\rceil$.

$$A_-dF^0(A_-) = \bigoplus_{i=1}^3 A_-dt_i.$$

$$A_-dF^1(A_-) = \bigoplus_{i=1}^4 A_-dt_i.$$

$$A_-dF^2(A_-) = A_-dA_-.$$

$$r_1 = 3, r_2 = 6.$$

$$L_+ = Z_1(L_+).$$

$Z_2(L_+)$ generated by the homogeneous vector fields $\delta \in L_+$ with $\deg(\delta) \geq 6$.

Especially $A_-Z_2(L_+) = \sum_{i=3}^5 A_-dt_i$.

$$Z_3(L_+) = 0.$$

We can use Theorem 4.1 to obtain a geometric quotient of the action of L_+ on the flattening stratification defined by the filtrations F^\bullet and Z_\bullet . Before doing this we shall prove that this flattening stratification is also the flattening stratification of the modules defining the Hilbert function of the Tjurina algebra.

For $t \in T_-$ the **Hilbert function of the Tjurina algebra**

$$\mathbb{C}\{x\} / \left(F(t), \frac{\partial F(t)}{\partial x_1}, \dots, \frac{\partial F(t)}{\partial x_n} \right)$$

corresponding to the singularity defined by t with respect to H^\bullet is by definition the function

$$m \mapsto \tau_m(t) := \dim_{\mathbb{C}} \mathbb{C}\{x\} / \left(F(t), \frac{\partial F(t)}{\partial x_1}, \dots, \frac{\partial F(t)}{\partial x_n}, H^m \right).$$

Notice that $\tau_m(t) = \tau(t)$ if m is large and $\tau_m(t)$ does not depend on t for small m . On the other hand, $\mu_m := \mu_m(t) := \dim_{\mathbb{C}} \mathbb{C}\{x\} / \left(\frac{\partial F(t)}{\partial x_1}, \dots, \frac{\partial F(t)}{\partial x_n}, H^m \right)$ does not depend on $t \in T_-$ and

$$\mu_m - \tau_m(t) = \text{rank} \left(\delta_i(t_j)(t) \right)_{\deg(t_j) > d - mw}.$$

This is an immediate consequence of the following fact:

Let

$$T^m := A_- \{x\} / \left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}, H^m \right),$$

then the following sequence is exact and splits: let $\{X^\alpha\}_{\alpha \in B}$ be a monomial base of $A_- \{x\} / \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right)$.

$$\begin{aligned} 0 \rightarrow \bigoplus_{\substack{|\alpha| \leq d \\ \alpha \in B}} A_- x^\alpha &\rightarrow T^{\frac{d}{w} + i} \rightarrow \text{Der}_{\mathbb{C}} A_- / \left(\mathcal{L} + \sum_{\deg(t_j) \leq -iw} A_- \frac{\partial}{\partial t_j} \right) \rightarrow 0 \\ x^\alpha &\mapsto \text{class}(x^\alpha) \\ &\mapsto \text{class}\left(\frac{\partial}{\partial t_j}\right), \end{aligned}$$

and with the identification $\sum_{\deg(t_j) > -iw} A_- \frac{\partial}{\partial t_j} \simeq A_-^{N_i}$ we obtain

$\text{Der}_{\mathbb{C}} A_- / \left(\mathcal{L} + \sum_{\deg(t_j) \leq -iw} A_- \frac{\partial}{\partial t_j} \right) \simeq A_-^{N_i} / M_i$, where M_i is the A_- -submodule generated by the rows of the matrix $(\delta_i(t_j))_{\deg(t_j) > -iw}$.

We have $F \in H^m$, hence $\mu_m = \tau_m$, if $m \leq \frac{d}{w}$ and $H^m \subset \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right)$, hence $\mu_m - \tau_m(t)$ is independent of m and equal to $\mu - \tau(t)$, if $m \geq \frac{d}{w} + s + 1$.

Therefore, we have $s + 1$ relevant values for τ_i , and we denote

$$\begin{aligned} \underline{\tau}(t) &:= (\tau_{\frac{d}{w} + 1}(t), \dots, \tau_{\frac{d}{w} + s + 1}(t)), \\ \underline{\mu} &:= (\mu_{\frac{d}{w} + 1}, \dots, \mu_{\frac{d}{w} + s + 1}). \end{aligned}$$

Moreover, let $\Sigma = \{ \underline{r} := (r_1, \dots, r_{s+1}) \mid \exists t \in T_- \text{ so that } \underline{\mu} - \underline{\tau}(t) = \underline{r} \}$ and $T_- = \cup_{\underline{r} \in \Sigma} U_{\underline{r}}$ be the flattening stratification of the modules $T^{\frac{d}{w} + 1}, \dots, T^{\frac{d}{w} + s + 1}$. That is, $\{U_{\underline{r}}\}$ is the stratification of T_- defined by fixing the Hilbert function $\underline{\tau} = \underline{\mu} - \underline{r}$ with the scheme structure defined by the flattening property.

Let us now consider an arbitrary deformation $\phi : (\mathcal{X}, \{0\} \times S) \hookrightarrow (\mathbb{C}^n, 0) \times S \rightarrow S$ of $(X, 0) \subset (\mathbb{C}^n, 0)$ of negative weight over a base space $S \in \mathcal{S}$ where, for each $s \in S$, the ideal of the germ $(\mathcal{X}, (0, s)) \subset (\mathbb{C}^n \times S, (0, s))$ is defined by $F(x, s) = f(x) + g(x, s)$, $g(x, 0) = 0$.

Let us denote by $\mathcal{O}_S\{x\} = \mathcal{O}_{\mathbb{C}^n \times S, 0 \times S}$ the topological restriction of $\mathcal{O}_{\mathbb{C}^n \times S}$ to $0 \times S$, considered as a sheaf on S . Then $J(I_{\mathcal{X}, 0 \times S})$, the Jacobian ideal sheaf of $(\mathcal{X}, \{0\} \times S) \subset (\mathbb{C}^n, 0) \times S$, is locally defined by $(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}) \subset \mathcal{O}_S\{x\}$ and $H_S^m \subset \mathcal{O}_S\{x\}$ is the ideal sheaf generated by $g \in \mathcal{O}_S\{x\}$ such that $\deg_x g \geq mw$, $w = \min\{w_1, \dots, w_n\}$ as above. We say that the **family** ϕ is $\underline{\tau}$ -**constant** if the coherent \mathcal{O}_S -sheaves

$$T_S^m := \mathcal{O}_S\{x\}/J(I_{\mathcal{X}, \{0\} \times S}) + H_S^m$$

are flat for $\frac{d}{w} + 1 \leq m \leq \frac{d}{w} + s + 1$ (equivalently, for all m). Of course, if T_S^m is flat, then

$$\tau_m(s) := \dim_{\mathbb{C}} T_{S,s}^m \otimes \mathcal{O}_{S,s}/\mathfrak{m}_{S,s}$$

is independent of $s \in S$. The converse holds for reduced base spaces:

Lemma 4.3 *If S is reduced, then the sheaf T_S^m is flat if and only if $\tau_m(s)$ is independent of $s \in S$.*

The proof is standard (cf. [GP 3]). Hence, over a reduced base space S , $\underline{\tau}$ -constant means just that the Hilbert function $\underline{\tau}(s) = (\tau_{\frac{d}{w}+1}(s), \dots, \tau_{\frac{d}{w}+s+1}(s))$ of the Tjurina algebra is constant. But for arbitrary base spaces we have to require flatness of the corresponding T_S^m .

Example ($f_0 = x^3 + y^3 + z^7$, continued)

$$\begin{aligned} \underline{\tau}(t) &= (\tau_8(t), \tau_9(t), \tau_{10}(t)) \\ \underline{\mu} &= (\mu_8, \mu_9, \mu_{10}) = (22, 23, 24) \\ \Sigma &= \{(0, 0, 0), (0, 0, 1), (0, 1, 2), (1, 1, 2), (1, 2, 3)\} \\ U_{(1,2,3)} &= D(2t_3 - \frac{10}{7}t_1t_2) \subseteq T_- = \mathbb{C}^5 \\ U_{(1,1,2)} &= V(2t_3 - \frac{10}{7}t_1t_2) \cap D(t_1, t_2) \subseteq T_- \\ U_{(0,1,2)} &= V(t_1, t_2, t_3) \cap D(t_4) \subseteq T_- \\ U_{(0,0,1)} &= V(t_1, t_2, t_3, t_4) \cap D(t_5) \subseteq T_- \\ U_{(0,0,0)} &= \{(0, 0, 0, 0, 0)\}. \end{aligned}$$

Lemma 4.4 1. $(0, \dots, 0, 1)$ and $(0, \dots, 0) \in \Sigma$. $U_{(0, \dots, 0)} = \{0\}$ is a smooth point and $U_{(0, \dots, 1)}$ is defined by $t_1 = \dots = t_{k-1} = 0$ and $t_k \neq 0$.

2. Let $\bar{\Sigma} = \Sigma \setminus \{(0, \dots, 0)\}$ and for $\underline{r} \in \bar{\Sigma}$ put

$$\tilde{U}_{\underline{r}} = \begin{cases} U_{\underline{r}} & \text{if } \underline{r} \neq (0, \dots, 0, 1) \\ U_{(0, \dots, 0, 1)} \cup U_{(0, \dots, 0)} & \text{if } \underline{r} = (0, \dots, 0, 1). \end{cases}$$

Then $\{\tilde{U}_{\underline{r}}\}_{\underline{r} \in \bar{\Sigma}}$ is the flattening stratification of the modules $\{\text{Hom}_{\mathbb{C}}(L_+, A_-)/A_-dF^iA_-\}$ and $\{\text{Hom}_{\mathbb{C}}(Z_i(L_+), A_-)/\pi_i(A_-dA_-)\}$.

Proof of Lemma 4.4: Because of the exact sequence above the flattening stratification of the modules $\{T_{\tilde{w}}^{\frac{d}{w}+i}\}$ is also the flattening stratification of $\{\mathrm{Der}_{\mathbb{C}}A_{-}/(\mathcal{L} + \sum_{\deg(t_j) \leq -iw} A_{-} \frac{\partial}{\partial t_j})\}$ respectively the flattening stratification of $\{A_{-}^{N_i}/M_i\}$, M_i the submodule generated by the rows of the matrix $(\delta_i(t_j))_{\deg(t_j) > -iw}$.

Now we have

$$(*) \quad \delta_i(t_j) = \delta_{k-j+1}(t_{k-i+1}).$$

By definition of $Z_i(L_+)$ we have

$$A_{-}Z_i(L_+) = \sum_{t_{k+1-j} \in F^{s-i}} A_{-}\delta_j$$

and with the identification

$$\sum A_{-} \frac{\partial}{\partial t_j} = A_{-}^k,$$

and M^i the submodule generated by the rows of the matrix $(\delta_{\ell}(t_j))_{\ell \geq r_i}$ we obtain

$$\mathrm{Der}_{\mathbb{C}}A_{-}/A_{-}Z_i(L_+) \cong A_{-}^k/M^i.$$

(*) implies that the flattening stratification of the modules $\{T_{\tilde{w}}^{\frac{d}{w}+1}, \dots, T_{\tilde{w}}^{\frac{d}{w}+s}\}$, which is $T_{-} = \cup_{\underline{r} \in \tilde{\Sigma}} \tilde{U}_{\underline{r}}$, is the flattening stratification of the modules $\{\mathrm{Der}_{\mathbb{C}}A_{-}/A_{-}Z_i(L_+)\}_{i=1, \dots, s}$.

Furthermore the modules $\{\mathrm{Hom}_{\mathbb{C}}(L_+, A_{-})/A_{-}dF^i A_{-}\}$ and $\{\mathrm{Der}_{\mathbb{C}}A_{-}/A_{-}L_+ + \sum_{\deg(t_j) \leq -iw} A_{-} \frac{\partial}{\partial t_j}\}$ have the same flattening stratification and they are flat on $U_{\underline{r}}$, because

$$0 \rightarrow A_{-} \rightarrow \mathrm{Der}_{\mathbb{C}}A_{-}/A_{-}L_+ + \sum_{\deg(t_j) \leq -iw} A_{-} \frac{\partial}{\partial t_j} \rightarrow \mathrm{Der}_{\mathbb{C}}A_{-}/\mathcal{L} + \sum_{\deg(t_j) \leq -iw} A_{-} \frac{\partial}{\partial t_j} \rightarrow 0$$

is exact and splits on $T_{-} \setminus \{0\}$.

This proves the lemma.

Remark 4.5 The main point of the lemma is that the flattening stratification of the modules $\{\mathrm{Hom}_{\mathbb{C}}(L_+, A_{-})/A_{-}dF^i A_{-}\}$ is equal to the flattening stratification of the modules $\{\mathrm{Hom}_{\mathbb{C}}(Z_i(L_+), A_{-})/\pi_i(A_{-}dA_{-})\}$, hence, is defined by the Hilbert function of the Tjurina algebra alone, without any reference to the action of L . This is a consequence of the symmetry expressed in Proposition 3.2.

As a corollary we obtain the following

Theorem 4.6 For $\underline{r} \in \Sigma$, $\tilde{U}_{\underline{r}}$ is invariant under the action of L_+ . Let $\mathrm{Spec} A_{\underline{r}}$ be the closure of $\tilde{U}_{\underline{r}}$ then $\tilde{U}_{\underline{r}} \rightarrow \tilde{U}_{\underline{r}}/L_+$ is a geometric quotient contained in $\mathrm{Spec} A_{\underline{r}}^{L_+}$ as an open subscheme of $\mathrm{Spec} A_{\underline{r}}^{L_+}$.

Example ($f_0 = x^3 + y^3 + z^7$, continued)

$$\begin{array}{ccc}
1) & \tilde{U}_{(1,2,3)} = D(2t_3 - \frac{10}{7}t_1t_2) & \longrightarrow & \tilde{U}_{(1,2,3)}/L_+ = \text{Spec } \mathbb{C}[t_1, t_2, t_3]_{2t_3 - \frac{10}{7}t_1t_2} \\
& \cap | & & \cap | \\
& \text{Spec } \mathbb{C}[t_1, \dots, t_5] & \longrightarrow & \text{Spec } \mathbb{C}[t_1, t_2, t_3] \\
2) & \tilde{U}_{(1,1,2)} & \longrightarrow & \tilde{U}_{(1,1,2)}/L_+ = D(t_1, t_2) \\
& \cap | & & \cap | \\
& \text{Spec } \mathbb{C}[t_1, t_2, t_4, t_5] & \longrightarrow & \text{Spec } \mathbb{C}[t_1, t_2, t_4] \\
& (\text{identifying } \mathbb{C}[t_1, \dots, t_5]/2t_3 - \frac{10}{7}t_1t_2 = \mathbb{C}[t_1, t_2, t_4, t_5].) \\
3) & \tilde{U}_{(0,1,2)} & \longrightarrow & \tilde{U}_{(0,1,2)}/L_+ = D(t_4) \\
& \cap | & & \cap | \\
& \text{Spec } \mathbb{C}[t_4, t_5] & \longrightarrow & \text{Spec } \mathbb{C}[t_4] \\
4) & \tilde{U}_{(0,0,1)} & = & \tilde{U}_{(0,0,1)}/L_+ \\
& \parallel & & \parallel \\
& \text{Spec } \mathbb{C}[t_5] & = & \text{Spec } \mathbb{C}[t_5]
\end{array}$$

Now $L/L_+ \simeq \mathbb{C}\delta_1$ acts on the geometric quotients $\tilde{U}_{\underline{r}}/L_+$ (the \mathbb{C}^* -action defined by the Euler vector field δ_1). Also the group E_{f_0} acts and this action commutes with the \mathbb{C}^* -action (cf. 2.6). If we combine this fact with Theorem 4.6 we obtain the main theorem of this article. In order to formulate it properly let us denote by

$$\text{Def}_{X_0, \underline{\tau}}^- : \mathcal{S} \rightarrow \text{sets}$$

the subfunctor of $\text{Def}_{X_0}^-$ which associates to a base space $S \in \mathcal{S}$ the set of isomorphism classes of $\underline{\tau}$ -constant families of deformations of negative weight with principal part $(X_0, 0)$ over S . For such a family $\underline{\tau}(s)$ is constant and equal to some tuple $\underline{\mu} - \underline{r} \in \mathbb{N}^{s+1}$.

Theorem 4.7 *Let $G = \exp L_+ \rtimes (E_{f_0} \cdot \mathbb{C}^*) \subseteq \text{Aut}(T_-)$.*

1. *The orbits of G are unions of finitely many integral manifolds of \mathcal{L} .*
2. *Let $T_- = \cup_{\underline{r} \in \Sigma} U_{\underline{r}}$ be the stratification fixing the Hilbert function $\underline{\tau}$ of the Tjurina algebra described above. $U_{\underline{r}}$ is invariant under the action of G and the geometric quotient $U_{\underline{r}} \rightarrow U_{\underline{r}}/G$ exists and is locally closed in a weighted projective space.*
3. *$U_{\underline{r}}/G$ is the coarse moduli space for the functor $\text{Def}_{X_0, \underline{\tau}}^- : \text{complex spaces} \rightarrow \text{sets}$ with $\underline{\tau} = \underline{\mu} - \underline{r}$.*

Remark 4.8 As in the case of right equivalence (see Remark 1.5) we may take (separated) algebraic spaces as category of base spaces. That is, $U_{\underline{r}}/G$ is a coarse moduli space for the functor

$$\text{Def}_{\bar{X}_0, \underline{\tau}}^- : \text{algebraic spaces} \rightarrow \text{sets.}$$

Proof (of Theorem 4.7): We first prove that $U_{\underline{r}}$ is invariant under the action of G and that $U_{\underline{r}} \rightarrow U_{\underline{r}}/G$ is a geometric quotient.

To prove that $U_{\underline{r}}$ is invariant under the action of G it is enough by definition of $U_{\underline{r}}$ that it is invariant under the action of E_{f_0} . The Hilbert function $\underline{\tau}$ of the Tjurina algebra is invariant under contact equivalence. This is a consequence of Theorem 2.1 because an automorphism φ of $\mathbb{C}\{x\}$ inducing the isomorphism of two semiquasihomogeneous singularities with principal part f_0 has degree ≥ 0 . More precisely, let f, g be semiquasihomogeneous with principal part f_0 and $uf = \varphi(g)$ for a unit u then $\deg(\varphi) \geq 0$ and consequently $(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, H^m)$ is mapped isomorphically to $(g, \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}, H^m)$ for all m , in particular $\underline{\tau}(f) = \underline{\tau}(g)$.

Moreover, let $\sigma \in E_{f_0}$, then there is a $\varphi : A_- \{x\} \rightarrow A_- \{x\}$, $\deg_x(\varphi) \geq 0$ and $\varphi|_{A_-} = id_{A_-}$ such that

$$\varphi(F(x, t)) \equiv F(x, \sigma(t)) \pmod{A_- H^N} \text{ for sufficiently large } N$$

(cf. proof of Proposition 2.4).

This implies $\sigma(T^m) = T^m$ for all m and proves that E_{f_0} and, therefore, G acts on the strata $U_{\underline{r}}$ of the flattening stratification of the modules $\{T^m\}$.

Now we prove that $U_{\underline{r}} \rightarrow U_{\underline{r}}/G$ is a geometric quotient. First of all it is obvious that the geometric quotients

$$U_{(0, \dots, 0, 1)} \rightarrow U_{(0, \dots, 0, 1)}/G = \{*\}$$

and

$$U_{(0, \dots, 0)} = \{*\} = U_{(0, \dots, 0)}/G = \{*\}$$

exist.

Let $\underline{r} \neq (0, \dots, 0, 1), (0, \dots, 0)$ then $\tilde{U}_{\underline{r}} = U_{\underline{r}}$. Let $U_{\leq \underline{r}} = \text{Spec} A_{\underline{r}}$ be the closure of $U_{\underline{r}}$ then we obtain

$$\begin{array}{ccc} \text{Spec} A_{\underline{r}} & \xrightarrow{\pi} & \text{Spec} A_{\underline{r}}^{L+} \\ \cup | i & & \cup | j \\ U_{\underline{r}} & \xrightarrow{\pi|_{U_{\underline{r}}}} & U_{\underline{r}}/L_+. \end{array}$$

$\pi|_{U_{\underline{r}}}$ defines a geometric quotient and i, j are open embeddings (Theorem 4.6). Notice that π itself is not necessarily a geometric quotient.

Now $\text{Spec} A_{\underline{r}}^{L+}$ is affine and E_{f_0} acts on $\text{Spec} A_{\underline{r}}^{L+}$ and also on $U_{\underline{r}}/L_+$. This implies (cf. [MF]) that

$$\text{Spec} A_{\underline{r}}^{L+} \xrightarrow{\lambda} \text{Spec}(A_{\underline{r}}^{L+})^{E_{f_0}}$$

is a geometric quotient (not necessarily as algebraic schemes since $A_{\underline{r}}^{L+}$ need not be of finite type over \mathbb{C}) and consequently

$$\lambda|_{U_{\underline{r}}/L_+} : U_{\underline{r}}/L_+ \rightarrow (U_{\underline{r}}/L_+)/E_{f_0}$$

is a geometric quotient which is an algebraic scheme. Especially $(U_{\underline{r}}/L_+)/E_{f_0} \subseteq \text{Spec}(A_{\underline{r}}^{L+})^{E_{f_0}}$ is an open subset.

Finally, \mathbb{C}^* acts on $\text{Spec}(A_{\underline{r}}^{L+})^{E_{f_0}}$. It has one fixed point $\{*\}$ corresponding to $U_{(0,\dots,0)} \subseteq \bar{U}_r = \text{Spec}A_{\underline{r}}$. Outside this fixed point the \mathbb{C}^* -action leads to a geometric quotient:

$$\begin{array}{ccc} \text{Spec}(A_{\underline{r}}^{L+})^{E_{f_0}} \setminus \{*\} & \longrightarrow & \text{Proj}(A_{\underline{r}}^{L+})^{E_{f_0}} \\ \cup & & \cup \\ (U_{\underline{r}}/L_+)/E_{f_0} & \longrightarrow & ((U_{\underline{r}}/L_+)/E_{f_0})/\mathbb{C}^* \\ & & \parallel \\ & & U_{\underline{r}}/G. \end{array}$$

This proves part (1) and (2) of the theorem.

It remains to prove that if $t, t' \in T_-$ define isomorphic singularities then t and t' are in the same orbit of G .

Let $F_t = u\varphi(F_{t'})$ for $t, t' \in T_-$, $u \in \mathbb{C}\{x\}^*$ a unit and φ an automorphism of $\mathbb{C}\{x\}$. Using the \mathbb{C}^* -action we find $t'' \in T_-$, $u_1 = \frac{u}{u(0)} \in \mathbb{C}\{x\}^*$ and an automorphism φ_1 of $\mathbb{C}\{x\}$ such that $F_t = u_1\varphi_1(F_{t''})$, $u_1(0) = 1$ and t' and t'' are in one \mathbb{C}^* -orbit. Then

$$G(z) := (1 + z(u_1 - 1))\varphi_1(F_{t''})$$

is an unfolding of $G(0) = F_t$ of negative weight. This unfolding can be induced by the semiuniversal unfolding, that is there exists a family of coordinate transformations $\underline{\psi}(z, -)$ and a path v in T_- such that

$$G(z) = F(\psi_1(z, x), \dots, \psi_n(z, x), v(z))$$

and $v(0) = t$ and $F_{t''} \stackrel{\sim}{\sim} F(\psi(1, x), v(1))$. Now $t = v(0)$ and $v(1)$ are in one orbit of $\exp L$, and $v(1)$ and t'' are in one orbit of E_{f_0} . Hence the result.

Now (3) follows in the same manner as the proof of Theorem 1.3.

Example ($f_0 = x^3 + y^3 + z^7$, continued)

1. $U_{(1,2,3)} \longrightarrow U_{(1,2,3)}/G \simeq \mathbb{C}^2$, $\underline{\tau} = (21, 21, 21)$, $\tau = 21$
normal form: $f_0 + t_1xz^5 + t_2yz^5 + t_3xyz^3$,
 $(t_1 : t_2 : t_3) \in D_+(2t_3 - \frac{10}{7}t_1t_2)/S_3 \subset \mathbb{P}_{(1:1:2)}^2/S_3$
 $(D_+(2t_3 - \frac{10}{7}t_1t_2)/S_3 \simeq \mathbb{C}^2$, the S_3 -action being explained in Example 2.8).
2. $U_{(1,1,2)} \longrightarrow U_{(1,1,2)}/G \simeq \mathbb{P}_{(2,3,5)}^2 \setminus (0 : 0 : 1)$, $\underline{\tau} = (21, 22, 22)$, $\tau = 22$
normal form: $f_0 + t_1xz^5 + t_2yz^5 + \frac{10}{7}t_1t_2xyz^3 + t_4xyz^4$,
 $(t_1 : t_2 : t_4) \in \mathbb{P}_{(1:1:5)}^2/S_3 \simeq \mathbb{P}_{(2,3,5)}^2$

3. $U_{(0,1,2)} \longrightarrow U_{(0,1,2)}/G = \{*\}$, $\underline{\tau} = (22, 22, 22)$, $\tau = 22$
normal form: $f_0 + xyz^4$
4. $U_{(0,0,1)} \longrightarrow U_{(0,0,1)}/G = \{*\}$, $\underline{\tau} = (22, 23, 23)$, $\tau = 23$
normal form: $f_0 + xyz^5$
5. $U_{(0,0,0)} \longrightarrow U_{(0,0,0)}/G = \{*\}$, $\underline{\tau} = (22, 23, 24)$, $\tau = 24$
normal form: f_0

Hence the moduli space of semiquasihomogeneous hypersurface singularities $X = \{(x, y, z) \mid f(x, y, z) = 0\}$ with principal part $X_0 = \{(x, y, z) \mid x^3 + y^3 + z^7 = 0\}$ consists of 5 strata (\mathbb{C}^2 , $\mathbb{P}_{(2,3,5)}^2 \setminus (0 : 0 : 1)$, and 3 isolated points) corresponding to 5 possible Hilbert functions $\underline{\tau}$ of the Tjurina algebra $\mathbb{C}\{x, y, z\}/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. The generic stratum $U_{(1,2,3)}$ (minimal $\underline{\tau}$) is an open subset in \mathbb{C}^5 , the quotient being 2-dimensional, as well as the quotient of the 4-dimensional “subgeneric” stratum $U_{(1,1,2)}$. Note that the families of normal forms are not universal. It just means that each semiquasihomogeneous singularity with principal part f_0 occurs and that different parameters do not give contact equivalent singularities, except modulo the \mathbb{C}^* - and S_3 -action.

We see that $U_{(1,1,2)}/G$ can be compactified by $U_{(0,1,2)}/G$, that is

$$U_{(1,1,2)} \cup U_{(0,1,2)} \rightarrow (U_{(1,1,2)} \cup U_{(0,1,2)})/G = \mathbb{P}_{(2,3,5)}^2$$

is a geometric quotient. So in this example there exist geometric quotients of the strata with constant Tjurina number and, hence, a coarse moduli space for fixed principal part and fixed Tjurina number. In general this is false (cf. [LP], §7).

Remark 4.9 1. The generic stratum $U_{\underline{\tau}_{\min}}$ corresponding to minimal Hilbert function $\underline{\tau}$ (with respect to lexicographical ordering) is an open, quasiasfine subset of T_- and, hence, $U_{\underline{\tau}_{\min}}/L_+$ is smooth by Theorem 4.1. In particular, the generic moduli space $U_{\underline{\tau}_{\min}}/G$ has, at most, quotient singularities (coming from the \mathbb{C}^* -action and the finite group E_{f_0}). It is not known whether the bigger stratum $U_{\tau_{\min}}$ corresponding to minimal Tjurina number τ admits a geometric quotient, except for $n = 2$ (cf. [LP]).

2. We always have two special strata, the most special $U_{(0,\dots,0)} = \{*\}$ (corresponding to f_0) and the “subspecial” or “hessian” stratum $U_{(0,\dots,1)} \cong \mathbb{C} \setminus \{*\}$ (corresponding to the singularity $f_0 + m_k$, m_k generating the socle of $\mathbb{C}\{x\}/j(f_0)$, that is the monomial of maximal degree). The G -quotients of these strata give two reduced, isolated points.

3. As we have seen for $x^3 + y^3 + z^7$, the finite group E_{f_0} need not be abelian. If $f_0 = x_1^{a_1} + \dots + x_n^{a_n}$ is of Brieskorn–Pham type and $\gcd(a_i, a_j) = 1$ for $i \neq j$, then $E_{f_0} \cong \mu_d$, the group of d 'th roots of unity, $d = \deg f_0$.

4. Note that a coarse moduli space is more than just a bijection between its points and the corresponding set of isomorphism classes. For instance, let $U_{\underline{\tau}}/G$

be affine and let $S \xrightarrow{\sigma} (\mathcal{X}, \sigma(S)) \xrightarrow{\phi} S$ be a family of deformations from $\text{Def}_{\bar{X}_0}^-(S)$ with $\underline{\tau}(\mathcal{X}_s, \sigma(s)) = \underline{\mu} - \underline{r}$. If S is compact then ϕ must be locally trivial since any morphism from S to $U_{\underline{r}}/G$ maps S onto finitely many points.

- Problems 4.10**
1. A natural continuation would be to vary the principal part of a semiquasihomogeneous singularity. First, one should consider only quasihomogeneous singularities with respect to fixed weights and degree and try to construct a moduli space. If this is not possible (in the algebraic category) one should find and fix (natural) invariants such that the moduli space exists. If this is done, it should be possible, using the methods in this paper, to construct moduli spaces for all (with respect to fixed weights and degrees) semiquasihomogeneous singularities with certain natural invariants fixed.
 2. Another natural continuation would be to treat the case of semiquasihomogeneous complete intersection singularities (in the spirit of this paper or, more generally, similarly to Problem 1).
 3. Which class of non-semiquasihomogeneous singularities can be studied using the approach of this paper? One idea could be to fix the Newton boundary of a non-degenerate hypersurface singularity, try to describe the μ -constant deformations and study the Kodaira–Spencer map of this family. Here again, it will be necessary to find good invariants.
 4. Possible invariants which generalize the Hilbert function of the Tjurina algebra from semiquasihomogeneous to arbitrary singularities are the following: Let α_{\min} and α_{\max} denote the minimal and maximal spectral number of the V -filtration V^\bullet of the Gauss–Manin connection of an isolated singularity f (cf. [He]). The function

$$\underline{\tau}(\alpha) = \dim H_0''(f) / (\partial_t^{-1} H_0''(f) + t H_0''(f) + V^{>\alpha}),$$

($\alpha_{\min} \leq \alpha \leq \alpha_{\max}$, $e^{2\pi i \alpha}$ eigenvalue of the monodromy) generalizes this Hilbert function.

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