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Moishezon twistor spaces without effective divisors of degree one

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Abstract

We study simply connected compact twistor spaces Z of positive type. Assuming that the fundamental linear system $|-\frac{1}{2}K|$ is at least a pencil, we prove the following theorem: the existence of an irreducible curve $C \subset Z$ which is invariant under the real structure of Z and has the property $C.(-\frac{1}{2}K) < 0$ implies that the twistor space is Moishezon but does not contain effective divisors of degree one. Furthermore, we prove the existence of such twistor spaces with arbitrary Picard number $\rho(Z) \geq 5$. These are the first examples of Moishezon twistor spaces without divisors of degree one.

1 Introduction

After the appearence of the result of Hitchin [H2] and Friedrich, Kurke [FK] stating that precisely two compact twistor spaces are Kählerian, the study of Moishezon twistor spaces was started. A deep result of Campana [C2] shows that such spaces are simply connected. This can be used to deduce that the self–dual Riemannian

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manifold M associated to the twistor space Z is homeomorphic to the connected sum $n\mathbb{CP}^2$ of complex projective planes. In [Po2], Poon has shown in the Moishezon case that M has to be of positive type, that means the scalar curvature of M is positive. On the other hand, Poon's computation of the algebraic dimension [Po2] together with the Riemann–Roch formula and Hitchin's vanishing theorem shows that every twistor space of positive type over $M = n\mathbb{CP}^2$ with $n \leq 3$ is Moishezon. The structure of these manifolds is nowadays fairly well known (see [H2], [FK], [Po1], [KK] and [Po3]). The case $n \geq 4$ (which is equivalent to $c_1(Z)^3 \leq 0$) is much more interesting. In this case Donaldson and Friedman [DonF] have firstly shown the existence of self–dual metrics. They also proved that the generic twistor space has algebraic dimension zero (if $n \geq 5$) respectively one (if $n = 4$). The first explicit examples for any $n \geq 3$ were discovered by LeBrun [LeB1] and studied by Kurke [Ku]. We call these spaces Kurke–LeBrun twistor spaces (see section 3). They are Moishezon spaces.

The goal of this paper is a question stated by Pedersen and Poon in [PP2]:

Question: *If Z is a Moishezon twistor space, does it contain an effective divisor of degree one?*

We shall answer this question with NO!

This is achieved by the following two theorems, which form the main results of this paper:

Theorem 2.1. *Assume $h^0(K^{-\frac{1}{2}}) \geq 2$ and the existence of an irreducible real curve $C_0 \subset Z$ with $C_0.(-\frac{1}{2}K) < 0$. Then the following holds:*

(i) C_0 is a smooth rational curve and $C_0.(-\frac{1}{2}K) = (-\frac{1}{2}K)^3 - 2 = 2(3-n)$, in particular $n \geq 4$.

(ii) $a(Z) = 3$, that is Z is Moishezon.

(iii) The linear system $|-\frac{1}{2}K|$ is two–dimensional and its base locus is precisely

C_0.

(iv) Z does not contain effective divisors of degree one.

Theorem 4.2. *For any $n \geq 4$ there exists a twistor space Z with $c_1(Z)^3 = 16(n-4)$ and containing a smooth rational curve $C_0 \subset Z$ with $C_0.(-\frac{1}{2}K) = 2(3-n)$. Furthermore,* $\dim|-\frac{1}{2}K| = 2$ *and Z fulfills all conditions of Proposition 2.1.*

It is remarkable that we can compute with Theorem 2.1 the algebraic dimension of a twistor space Z from the numerical properties of a single curve in Z. In the work of Poon [Po3] the algebraic dimension was computed from the structure of a divisor on Z. Moreover, his computation of algebraic dimension depends havily on the existence of a divisor of degree one. As we are dealing with twistor spaces not containing a divisor of degree one, we cannot use his results.

It would be interesting to study the question wether in Theorem 2.1 the assumption on the dimension of the linear system $|-\frac{1}{2}K|$ is really necessary. Our results contradict some statements in the paper [PP2]. This will be commented in section 5.

We start now to collect some necessary but well known facts and to introduce terminology. For details, the reader is referred to [AHS], [B], [ES], [H2], [K1], [Ku] and [Po1].

We consider twistor spaces merely from the viewpoint of complex geometry. In this paper, by a *twistor space* we mean a compact complex three–manifold together with the following additional data:

- a proper differentiable submersion $\pi : Z \to M$ onto a real differentiable four–manifold M. The fibres of π are holomorphic curves in Z being isomorphic to $\mathbb{CP}^1$ and having normal bundle in Z isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$;

- an anti–holomorphic fixed point free involution $\sigma : Z \to Z$ with $\pi\sigma = \pi$.

The fibres of π are called "real twistor lines" and the involution σ is called the "real structure". A geometric object will be called "real" if it is σ–invariant. For example, a line bundle $\mathcal{L}$ on Z is real if $\sigma^*\bar{\mathcal{L}} \cong \mathcal{L}$, and a complex subvariety $D \subset Z$ is real if $\sigma(D) = D$. Instead of $\sigma(D)$ we shall often write $\bar{D}$. In particular, by a "real curve" $C \subset Z$ we mean a compact complex subspace which is σ–invariant, that means $C = \bar{C}$.

The twistor–space structure on Z defines a conformal class of self–dual Riemannian metrics on M. By a result of Schoen [Sch] such a conformal class contains a metric with constant scalar curvature. The sign of this constant will be called the *type* of Z. The relations between the type and the algebraic dimension of Z are clarified in [Po2] and [Pon]. Technically important will be the assumption of positive type, because of the following vanishing theorem.

Theorem 1.1 (Hitchin [H1]). *If Z is of positive type then we have for any $\mathcal{L} \in$ Pic (Z)*

$$\deg(\mathcal{L}) \leq -2 \quad \Rightarrow \quad H^1(Z, \mathcal{L}) = 0.$$

The degree $\deg(\mathcal{L})$ of a line bundle $\mathcal{L} \in \mathrm{Pic}\,(Z)$ is by definition the degree of the restriction $\mathcal{L} \otimes \mathcal{O}_F$ to a twistor fibre $F \subset Z$.

This vanishing theorem is what we really need, not the assumption of positive type. Therefore, we introduce the following definition.

Definition: *A twistor space will be called* Hitchin twistor space *if and only if it is compact and the statement of theorem 1.1 holds, that is:* $\deg(\mathcal{L}) \leq -2 \Rightarrow H^1(Z, \mathcal{L}) = 0$.

We can restate Hitchins vanishing theorem by saying: a compact twistor space of positive type is a Hitchin twistor space.

For simply connected twistor spaces of positive type it is well known (see [Po1]) that the associated Riemannian manifold is homeomorphic to the connected sum $n\mathbb{CP}^2$. This remains true if we replace "positive type" by "Hitchin".

Proposition 1.2. *If Z is a simply connected Hitchin twistor space and $Z \to M$ the corresponding twistor fibration, then M is homeomorphic to $n\mathbb{CP}^2$ for some $n \geq 0$.*

Proof: It is well known (see e.g. [ES]) that $H^1(M;\mathbb{C}) \cong H^1(Z;\mathcal{O}_Z)$ and $H^2_-(M;\mathbb{C}) \cong H^2(Z;\mathcal{O}_Z)$ holds. As $\pi_1(Z) \cong \pi_1(M) = \{1\}$ by assumption, we obtain $h^1(\mathcal{O}_Z) = 0$. Hitchins vanishing theorem and $\deg(\mathcal{O}_Z) = 0$ imply $h^2(\mathcal{O}_Z) = 0$, hence $H^2_-(M;\mathbb{C}) = 0$. Therefore, M is a simply connected manifold with positively definite intersection form. From [Don] and [F] the result now follows. $\square$

If Z is a simply connected Hitchin twistor space, then $h^i(\mathcal{O}_Z) = 0$ for $i \neq 0$. Using the exp–sequence this implies that the first Chern class defines an isomorphism of free abelian groups $c_1 : \mathrm{Pic}\,(Z) \xrightarrow{\sim} H^2(Z;\mathbb{Z})$. It is well known, that the degree–morphism $\deg : \mathrm{Pic}\,(Z) \to \mathbb{Z}$ is a surjective homomorphism. The Chern numbers of Z are the following: $c_1^3 = 16(4-n), c_1c_2 = 24, c_3 = 2(n+2)$, where $n+1 = \rho(Z) :=$ rank $\mathrm{Pic}\,(Z)$ is the Picard number of Z. This follows from a detailed description of the cohomology ring of Z which is obtained via Proposition 1.2 and the Laray–Hirsch theorem applied to the twistor fibration $Z \to M$. On Z there exists a unique line bundle, denoted by $K^{-\frac{1}{2}}$, whose square is the anticanonical bundle K_Z^{-1}. Following Poon, we call this *the fundamental line bundle*. The divisors in $|-\frac{1}{2}K|$ are called *fundamental divisors*. The adjunction formula implies $\deg(K^{-\frac{1}{2}}) = 2$. As the real twistor fibres cover Z, an effective divisor must have positive degree. This gives the following vanishing result: $\deg(\mathcal{L}) \leq -1 \Rightarrow H^0(Z,\mathcal{L}) = 0$. By Serre–duality we obtain for any compact twistor space: $\deg(\mathcal{L}) \geq -3 \Rightarrow H^3(Z,\mathcal{L}) = 0$ and for Hitchin twistor spaces: $\deg(\mathcal{L}) \geq -2 \Rightarrow H^2(Z,\mathcal{L}) = 0$.

A typical example of an application of these vanishing results will be the following. Let $S \in |-\frac{1}{2}K|$ be a smooth fundamental divisor. By adjunction we obtain $K_S^{-1} = K^{-\frac{1}{2}} \otimes \mathcal{O}_S$. There exist exact sequences $0 \to K^{\frac{1}{2}} \to \mathcal{O}_Z \to \mathcal{O}_S \to 0$ and $0 \to \mathcal{O}_Z \to K^{-\frac{1}{2}} \to K_S^{-1} \to 0$. As $\deg(K^{\frac{1}{2}}) = -2$ we obtain $h^i(K^{\frac{1}{2}}) = 0$ for all i. The first sequence gives, therefore, $h^i(\mathcal{O}_S) = h^i(\mathcal{O}_Z)$, which is zero for $i \geq 1$. Using $h^1(\mathcal{O}_Z) =$

0 and $h^0(\mathcal{O}_Z) = 1$ we deduce from the second sequence $h^0(K^{-\frac{1}{2}}) = 1 + h^0(K_S^{-1})$. By $a(Z)$ we denote the algebraic dimension of Z. This is by definition the transcendence degree of the field of meromorphic functions of Z over $\mathbb{C}$. If $\dim Z = a(Z)$, then Z is called Moishezon. We need here only the following facts on the algebraic dimension: $\dim Z \geq a(Z)$ and if $f : Z \to \mathbb{P}^N$ is a meromorphic map, then $a(Z) \geq \dim f(Z)$. The reader may find this and many other things on algebraic dimension in [U].

2 Algebraic Dimension

In this section we shall generalize a result of [K2] to the case of arbitray $n \geq 4$. Let Z denote a simply connected Hitchin twistor space with $c_1(Z)^3 \leq 0$. We know $c_1(Z)^3 = 16(4-n)$ and the Riemannian manifold M corresponding to Z is homeomorphic to the connected sum $n\mathbb{CP}^2$.

Theorem 2.1. *Assume $h^0(K^{-\frac{1}{2}}) \geq 2$ and the existence of an irreducible real curve $C_0 \subset Z$ with $C_0.(-\frac{1}{2}K) < 0$. Then the following holds:*

(i) C_0 is a smooth rational curve and $C_0.(-\frac{1}{2}K) = (-\frac{1}{2}K)^3 - 2 = 2(3-n)$, in particular $n \geq 4$.

(ii) $a(Z) = 3$, that is Z is Moishezon.

(iii) The linear system $|-\frac{1}{2}K|$ is two–dimensional and its base locus is precisely C_0.

(iv) Z does not contain effective divisors of degree one.

Proof: First we show the following

Claim: *If $C_0 \subset Z$ is an irreducible real curve with $C_0.(-\frac{1}{2}K) < 0$, then Z does not contain effective divisors of degree one.*

Assume Z contains an effective divisor $D \subset Z$ of degree one. Then $D+\bar{D} \in |-\frac{1}{2}K|$, hence, by assumption, $C_0.D < 0$ or $C_0.\bar{D} < 0$. This implies $C_0 \subset D$ or $C_0 \subset \bar{D}$. But C_0 is assumed to be real, hence $C_0 \subset D \cap \bar{D}$. The intersection of a conjugate pair of divisors of degree one is always a real twistor fibre $F = D \cap \bar{D}$. Hence, $C_0 = F$. But $F.(-\frac{1}{2}K) = 2$ contradicts the assumption $C_0.(-\frac{1}{2}K) < 0$. This proves the claim.

The claim immediately yields statement (iv). In particular, as each effective divisor has positive degree, all fundametal divisors are irreducible. As by assumption $h^0(K^{-\frac{1}{2}}) \geq 1$, we obtain the existence of an irreducible real fundamental divisor $S \in |-\frac{1}{2}K|$. The structure of such divisors is well understood (see e.g. [PP2] or [K2]). In [K2] Lemma 3.3 and Lemma 3.4 the following was shown: S is a smooth rational surface and contains real twistor fibres $F \subset S$, which form the real members of a pencil $|F|$ of curves in S. If we equip $\mathbb{P}^1 \times \mathbb{P}^1$ with the real structure given by the antipodal map on the first factor and the usual real structure on the second factor, then there exists a sequence of n blow–ups $S = S^{(n)} \to S^{(n-1)} \to \ldots \to S^{(0)} = \mathbb{P}^1 \times \mathbb{P}^1$, where at each step a conjugate pair of points is blown up. Each $S^{(j)}$ can be equipped with a unique real structure without real points and being compatible with the morphisms $S^{(j+1)} \to S^{(j)}$. As we do not blow up a real point, the image C_0' of C_0 in $S^{(0)} = \mathbb{P}^1 \times \mathbb{P}^1$ is a curve. By assumption, C_0 is irreducible and $C_0.S < 0$, hence $C_0 \subset S$. By adjunction formula we have $K^{-\frac{1}{2}} \otimes \mathcal{O}_S \cong K_S^{-1}$. Hence, $C_0.(-K_S) = C_0.(-\frac{1}{2}K) < 0$. Therefore, any member of $|-K_S|$ must contain C_0. As $h^0(K_S^{-1}) = h^0(K^{-\frac{1}{2}}) - 1 \geq 1$ by assumption, we have $|-K_S| \neq \emptyset$. As we have seen above, any member of $|-K_S|$ has C_0 as a real component. We can now apply [K2] Prop. 3.6 to obtain the existence of a real member of $|-K_S|$ having the form $C_0 + F$ with a real twistor fibre $F \subset S$. If the blow–down $\sigma : S \to \mathbb{P}^1 \times \mathbb{P}^1$ is chosen appropriately, the image $C_0' = \sigma(C_0)$ is a smooth rational curve being a member of $|\mathcal{O}(2,1)|$.

From [K2] Lemma 3.3 we know that none of the blown–up points lie on a real member of $|\mathcal{O}(0,1)|$. On the other hand, $K_S^{-1} = \sigma^*\mathcal{O}(2,2) \otimes \mathcal{O}_S(-E)$, where E is the exceptional divisor (more precisely, the sum of the pull backs of the exceptional divisors of each step of blow–up $S^{(j+1)} \to S^{(j)}$). Hence, all the blown–up points lie on C_0' and no two of them are infinitesimally near each other. In particular, $C_0^2 = (C_0')^2 - 2n = 4 - 2n$ and $C_0 \subset S$ is the strict transform of the smooth rational curve C_0'. By adjunction formula we obtain $C_0.(-\frac{1}{2}K) = C_0.(-K_S) = C_0^2 + 2 = 6 - 2n$. In particular, $n \geq 4$. This proves the assertion (ii).

Furthermore, we see $|-K_S| = C_0 + |F|$. This implies $\dim|-\frac{1}{2}K| = \dim|-K_S| + 1 = \dim|F| + 1 = 2$. As $h^1(\mathcal{O}_Z) = 0$ the restriction map $H^0(K^{-\frac{1}{2}}) \to H^0(K_S^{-1})$ is surjective. Hence, the linear systems $|-\frac{1}{2}K|$ and $|-K_S|$ have the same base locus. As $|F|$ does not have base–points, the base locus of $|-\frac{1}{2}K|$ is C_0. Thus, we obtained (iii).

To compute the algebraic dimension of Z we study the rational map defined by the two–dimensional linear system $|-\frac{1}{2}K|$. This can be done precisely as in the case $n = 4$ (see [K2] Prop. 5.1). For convenience of the reader, we repeat the argument here. Let $\sigma : \tilde{Z} \to Z$ be the blow–up of the smooth rational curve C_0. By $E \subset \tilde{Z}$ we denote the exceptional divisor. Then we obtain a morphism $\pi : \tilde{Z} \to \mathbb{P}^2$ defined by the linear system $|-\frac{1}{2}K|$ such that $\pi^*\mathcal{O}(1) \cong \sigma^*K^{-\frac{1}{2}} \otimes \mathcal{O}_{\tilde{Z}}(-E)$. As the restriction map $|-\frac{1}{2}K| \to |-K_S|$ is surjective, the restriction $\pi_{|S}$ is given by the linear system $|-K_S| = C_0 + |F|$. This means π exibits S as the blow–up of a ruled surface and $\pi(S)$ is a line in $\mathbb{P}^2$. As $\pi(\tilde{Z})$ is not contained in a linear subspace, π must be surjective. If we equip $\mathbb{P}^2$ with the usual real structure, π becomes compatible with real structures since the linear system $|-\frac{1}{2}K|$ and the blown–up curve C_0 are real. As we have seen above, any real fundamental divisor S is irreducible and smooth. By $\tilde{S} \subset \tilde{Z}$ we denote the strict transform of $S \in |-\frac{1}{2}K|$. As C_0 is a smooth curve in a smooth surface, $\sigma : \tilde{S} \to S$ is an isomorphism. Furthermore, $E \cap \tilde{S}$ will be mapped isomorphically onto $C_0 \subset S$. As $F.C_0 = 2$ and the restriction of π onto $\tilde{S}$ is the map defined by the linear system $|F|$, the restriction of π exibits $E \cap \tilde{S}$ as a double covering over $\pi(S) \cong \mathbb{P}^1$. As real lines cover $\mathbb{P}^2$, the morphism $\pi : E \to \mathbb{P}^2$ does not contract curves and is of degree two.

As generic fibres of π are smooth rational curves, the line bundle $\mathcal{O}_{\tilde{Z}}(E)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(2)$ on such fibres. Hence, after replacing (if necessary) $\mathbb{P}^2$ by the open dense set U of points having smooth fibre, the adjunction morphism $\pi^*\pi_*\mathcal{O}_{\tilde{Z}}(E) \to \mathcal{O}_{\tilde{Z}}(E)$ is surjective. This defines a U–morphism $\Phi : \tilde{Z} \to \mathbb{P}(\pi_*\mathcal{O}_{\tilde{Z}}(E))$. $\pi_*\mathcal{O}_{\tilde{Z}}(E)$ is a locally free sheaf of rank three. The restriction of Φ to smooth fibres coincides with the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ of degree two. Therefore, the image of Φ is a three–dimensional subvariety of the $\mathbb{P}^2$–bundle $\mathbb{P}(\pi_*\mathcal{O}_{\tilde{Z}}(E)) \to U$. Hence, $\tilde{Z}$ is bimeromorphically equivalent to a quasiprojective variety and has, therefore, algebraic dimension three. $\square$

3 Kurke–LeBrun twistor spaces

To prove our existence theorem, we need some knowledge about the Moishezon twistor spaces discovered by LeBrun [LeB1] and studied by Kurke [Ku]. In the sequel we call these twistor spaces, which are bimeromorphic to some conic–bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, Kurke–LeBrun twistor spaces. In this section we merely collect well known properties of such twistor spaces from [LeB1] and [Ku].

The construction starts with a set of n hyperplane sections of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded as a smooth quadric into $\mathbb{P}^3$. This quadric is equipped with the real structure defined by $(x, y) \mapsto (\bar{y}, \bar{x})$, where conjugation denotes the usual real structure on $\mathbb{P}^1$. Let $\varphi_1, \ldots, \varphi_n$ be sections of the line bundle $\mathcal{O}(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, such that their corresponding divisors Δ_i are smooth rational real curves without real points. For our purposes it is enough to assume that these curves Δ_i are in general position, that is the curve $\Delta = \sum_{i=1}^n \Delta_i$ defined by the product $\varphi = \varphi_1 \cdot \ldots \cdot \varphi_n$ has only ordinary nodes as singularities.

On $\mathbb{P}^1 \times \mathbb{P}^1$ we consider the locally free sheaf $\mathcal{E} := \mathcal{O} \oplus \mathcal{O}(1-n,-1) \oplus \mathcal{O}(-1,1-n)$ with constant non–zero sections $z_0 \in H^0(\mathcal{E}), z_1 \in H^0(\mathcal{E}(n-1,1)), z_2 \in H^0(\mathcal{E}(1,n-1))$. Let $p : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \times \mathbb{P}^1$ be the corresponding $\mathbb{P}^2$–bundle (we use Grothendieck's notation, see e.g. [H] II §7). Then there exists a natural isomorphism $H^0(\mathbb{P}^1 \times \mathbb{P}^1, S^2(\mathcal{E}) \otimes \mathcal{O}(n,n)) \cong H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes p^*\mathcal{O}(n,n))$. Hence, the section $F := z_1 z_2 + \varphi \cdot z_0^2 \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, S^2(\mathcal{E}) \otimes \mathcal{O}(n,n))$ defines a divisor $X \subset \mathbb{P}(\mathcal{E})$, whose fibres are conics. The discriminant of the conic bundle $p : X \to \mathbb{P}^1 \times \mathbb{P}^1$, which is the locus of critical values, is the curve $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$. The singularities of X are ordinary nodes and are mapped under p bijectively onto the set of singularities of Δ.

This conic bundle has two sections, namely the divisors $E = \{z_0 = z_1 = 0\}$ and $\bar{E} = \{z_0 = z_2 = 0\}$. These divisors do not contain singular points of X. By choosing certain small resolutions of the singularities of X and appropriate contractions of E and $\bar{E}$ to smooth rational curves B and $\bar{B}$, one obtains a twistor space Z_0.

Let $H_0 \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a real irreducible hyperplane section containing real points, but missing the singular points of Δ. Then, the inverse image of H_0 in X is a blow–up of a smooth ruled surface. The projection to H_0 has $2n$ reducible fibres. These reducible fibres are precisely the fibres over $H_0 \cap \Delta$ and each of them consists of two rational curves. This surface does not meet the set of singularities of X but it intersects E and $\bar{E}$ along sections. The contraction of E and $\bar{E}$ performed to get Z_0 maps this surface isomorphically onto a real fundamental divisor S_0 containing the disjoint curves B and $\bar{B}$. The real fibres of the morphism $S_0 \to H_0$ are precisely the real twistor fibres contained in S_0. The self–intersection number of B and $\bar{B}$ in S_0 is equal to $-n$. If we contract conjugate pairs of irreducible components of reducible fibres of $S_0 \to H_0$, we obtain a morphism $\sigma : S_0 \to \mathbb{P}^1 \times H_0$ which is the blow–up on n pairs of conjugate points on $\mathbb{P}^1 \times H_0$. The two curves $B' = \sigma(B)$ and

$\bar{B}' = \sigma(\bar{B})$ are fibres of the projection $\mathbb{P}^1 \times H_0 \to \mathbb{P}^1$. On B' lie n distinct points which are blown–up under σ. The conjugate set of blown–up points lie, of course, on $\bar{B}'$. As H_0 does not meet the singularities of Δ, the set of $2n$ blown–up points on $\mathbb{P}^1 \times H_0$ is projected onto the $2n$ distinct points $H_0 \cap \Delta$.

4 Existence

In this section we prove the main theorem of our paper, stating for each $n \geq 4$ the existence of a twistor space with the properties of Theorem 2.1. An important tool to achieve this result will be the following theorem on the deformation theory of twistor spaces. This theory was first developed by Donaldson and Friedman [DonF] and later by Campana and LeBrun.

Theorem 4.1 ([C1], [C3], [DonF], [PP2]). *Let Z be a Kurke–LeBrun twistor space, $n \geq 4$ and $S \in |-\frac{1}{2}K|$ a smooth real divisor. Then:*
Any real member of a small deformation of Z is again a twistor space. Furthermore, any small deformation of S with real structure is induced by a deformation of Z in the sense that the deformed surfaces are members of the fundamental system of the deformed twistor spaces.

Theorem 4.2. *For any $n \geq 4$ there exists a simply connected twistor space Z of positive type with $c_1(Z)^3 = 16(n-4)$ and containing a smooth rational curve $C_0 \subset Z$ with $C_0.(-\frac{1}{2}K) = 2(3-n)$. Furthermore,* $\dim|-\frac{1}{2}K| = 2$ *and Z fulfills all conditions of Proposition 2.1.*

Proof: Let Z_0 be a (generic) Kurke–LeBrun twistor space and $S_0 \subset Z_0$ a real fundamental divisor as described in section 3. Then we can choose a real blow–down map $\sigma : S_0 \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the $2n$ blown–up points lie on a conjugate pair of lines B', $\bar{B}' \in |\mathcal{O}(1,0)|$. We can take Z_0, S_0 and σ in such a way that n distinct points on B' and their conjugates on $\bar{B}'$ are blown–up and no member of

$|\mathcal{O}(0,1)|$ contains more than one of these $2n$ points. The real members of $|\mathcal{O}(0,1)|$ do not contain blown–up points. The idea of the proof is to move the $2n$ points on $\mathbb{P}^1 \times \mathbb{P}^1$ such that they lie on a smooth real member of $|\mathcal{O}(2,1)|$ and to apply then Theorem 4.1.

To make this more precise, let $F' \in |\mathcal{O}(0,1)|$ be a real fibre. Define $C_0 := B' + F' + \bar{B}' \in |\mathcal{O}(2,1)|$. Let $T := |\mathcal{O}(2,1)| \cong \mathbb{P}^5$ be the parameter space of the universal family $\mathcal{C} = \{(C,x) \mid x \in C\} \subset |\mathcal{O}(2,1)| \times (\mathbb{P}^1 \times \mathbb{P}^1)$ of curves of type $(2,1)$. Then $\mathcal{C} \to T$ is a flat family being a deformation of C_0. By $0 \in T$ we denote the point, corresponding to C_0. If $t \in T$ is a point, we shall denote the fibre of $\mathcal{C} \to T$ over t by C_t.

Let us equip $\mathbb{P}^1 \times \mathbb{P}^1$ with the real structure given by the antipodal map on the first factor and the usual real structure on the second factor. As this real structure preserves the type of a divisor, we obtain a real structure on T. This induces a real structure on $\mathcal{C}$, such that $\mathcal{C} \hookrightarrow T \times (\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathcal{C} \to T$ are real morphisms. If we set $T_0 := T$, we can recursively define the spaces $T_k := T_{k-1} \times_T \mathcal{C} \to T$ for any $k \geq 1$. We obtain a flat family $T_{k+1} \to T_k$, whose fibres are curves of type $(2,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This family has k natural sections $P_k^{(i)} : T_k \to T_{k+1}$ $(i = 1, \ldots, k)$ given by the k projections $T_k \to \mathcal{C}$ which we obtain recursively.

As we want to blow up $2n$ distinct points on a curve of type $(2,1)$, we introduce the open subset $T_k^\circ \subset T_k$ defined as follows. Let $T_0^\circ := T_0$ and $T_k^\circ \subset T_{k-1}^\circ \times_T \mathcal{C}$ be the complement of the union $\mathcal{P}_{k-1} := \bigcup_{i=1}^{k-1} P_{k-1}^{(i)}(T_{k-1}^\circ)$ of the images of the $k-1$ natural sections of $T_k \to T_{k-1}$ restricted to T_{k-1}°. The closed subvariety $\mathcal{P}_{k+1} \subset T_{k+1}$ is by definition étale of degree k over T_k°. Using the closed embedding $\mathcal{C} \subset T \times \mathbb{P}^1 \times \mathbb{P}^1$ we obtain, recursively, closed embeddings $T_{k+1} \subset T_k \times (\mathbb{P}^1 \times \mathbb{P}^1)$ and $\mathcal{P}_{k+1} \subset T_k^\circ \times (\mathbb{P}^1 \times \mathbb{P}^1)$.

As $\mathcal{P}_{k+1} \to T_k^\circ$ is flat, the blow–up $\mathcal{S}_k \to T_k^\circ \times (\mathbb{P}^1 \times \mathbb{P}^1)$ along $\mathcal{P}_{k+1}$ defines a flat family $\mathcal{S}_k \to T_k^\circ$ whose fibres are surfaces, isomorphic to a blow–up of $\mathbb{P}^1 \times \mathbb{P}^1$ at k

distinct points, lying on a curve of type $(2,1)$. A point $x \in T_k^\circ$ corresponds to an ordered set of k distinct points $(x_1, \dots, x_k)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, which lie on the curve C_t, where t is the image of x under $T_k^\circ \to T_0 = T$.

The given set of $2n$ blown–up points on the given curve $C_0 \in |\mathcal{O}(2,1)|$ defines a point $0 \in T_{2n}^\circ$, such that the fibre of $\mathcal{S}_{2n} \to T_{2n}^\circ$ over 0 is isomorphic to the surface S_0 we started with. Assume we ordered the points $x_1, \dots, x_{2n}$ in such a way that $\overline{x}_i = x_{n+i}$ $(i = 1, \dots, n)$. We then introduce on T_{2n} the real structure given by $(y_1, \dots, y_{2n}) \mapsto (\overline{y}_{n+1}, \dots, \overline{y}_{2n}, \overline{y}_1, \dots, \overline{y}_n)$. With this real structure, $T_{2n}^\circ \subset T_{2n}$ is real and $0 \in T_{2n}^\circ(\mathbb{R})$ is a real point. If we equip $T_{2n} \times (\mathbb{P}^1 \times \mathbb{P}^1)$ with the real structure given by the real structures on both factors, the subvariety $\mathcal{P}_{2n+1}$ is real. Hence, we obtain on $\mathcal{S}_{2n}$ a real structure, such that $\mathcal{S}_{2n} \to T_{2n}^\circ$ is a real morphism. By assumption, the originally blown–up points $x_1, \dots, x_{2n}$ do not lie on F', that means they are smooth points of C_0. Hence, T_{2n}° is smooth at 0. This implies, the subset of real points $T_{2n}^\circ(\mathbb{R})$ is near 0 a real manifold whose real dimension is equal to the (complex) dimension of T_{2n}°.

We can apply Theorem 4.1 to obtain an open analytic neighbourhood of $0 \in T_{2n}^* \subset T_{2n}^\circ$ such that for any real $t \in T_{2n}^*(\mathbb{R})$ there exists a twistor space Z_t containing a fundamental divisor isomorphic to S_t (the fibre of $\mathcal{S}_{2n} \to T_{2n}^\circ$ over t).

By construction, the morphism $T_{2n} \to T$ is flat, hence, open. The image of T_{2n}^* in T is, therefore, open. As the subset of points on T corresponding to non–smooth curves $C_t \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a Zariski–closed subset and $T \cong \mathbb{P}^5$ is irreducible, there exists a Zariski–open dense subset of T_{2n}^* whose image in T corresponds to points parametrizing smooth curves C_t. As $T_{2n}^\circ(\mathbb{R})$ is Zariski–dense in T_{2n}° (at least in a neighbourhood of 0), there exist points $t \in T_{2n}^*(\mathbb{R})$ whose image in $T(\mathbb{R})$, also denoted by t, corresponds to a smooth real divisor $C_t \in |\mathcal{O}(2,1)|$.

This proves the existence of twistor spaces Z_t with a fundamental divisor S_t and a blow–up $S_t \to \mathbb{P}^1 \times \mathbb{P}^1$, such that the $2n$ distinct blown–up points in $\mathbb{P}^1 \times \mathbb{P}^1$ lie on a smooth real curve $C_t \in |\mathcal{O}(2,1)|$. The strict transform $\tilde{C}_t \subset Z_t$ of C_t is a smooth real curve contained in S_t which has there the self–intersection number $4-2n$ and is isomorphic to $\mathbb{P}^1$. Furthermore, $(-K_{S_t})^2 = 8-2n$. By adjunction formula we obtain $\tilde{C}_t.(-\frac{1}{2}K) = \tilde{C}_t.(-K_{S_t}) = \tilde{C}_t^{\,2} + 2 = 2(3-n)$ and $(-\frac{1}{2}K)^3 = (-K_{S_t})^2 = 2(4-n)$. As the Z_t are small deformations of Z_0 they are also simply connected. The twistor space Z_0 is Moishezon, thus of positive type (see [Po2]). A small deformation of a twistor space of positive type is again of positive type. So, we obtain, Z_t is of positive type. As we have by construction $|K_{S_t}| \neq \emptyset$ we obtain $h^0(Z_t, K^{-\frac{1}{2}}) = 1 + h^0(-K_{S_t}) \geq 2$. Hence, we can apply Theorem 2.1 to get the result. $\square$

5 Comment to a paper of Pedersen and Poon

As the twistor spaces we constructed in the previous section are small deformations of Kurke–LeBrun twistor spaces, our result contradicts obviously Theorem 4.7 in [PP2] which claims that any Moishezon twistor space being a small deformation of a Kurke–LeBrun twistor space contains an effective divisor of degree one. So, there are some comments in order.

The solution for this contradiction is, that the proof of Lemma 4.1 in [PP2] has a gap, which appears on page 697. They did not consider in their case (ii) the possibility that a real curve of type $(2,2)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ can split as the sum of two real curves $C_0 + F$ with $C_0 \in |\mathcal{O}(2,1)|$ and $F \in |\mathcal{O}(0,1)|$ a real twistor fibre. (A complete treatment of this situation can be found in [K2].)

As a consequence of this gap, the proofs of Corollary 4.3, Theorems 4.7 and 4.8 in [PP2] are not correct. Our existence theorem shows that these results are even false.

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