Tangent Measure Distributions of Hyperbolic Cantor Sets

by
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Abstract: Tangent measure distributions were introduced by Bandt [2] and Graf [8] as a means to describe the local geometry of self-similar sets generated by iteration of contractive similitudes. In this paper we study the tangent measure distributions of hyperbolic Cantor sets generated by certain contractive mappings, which are not necessarily similitudes. We show that the tangent measure distributions of these sets equipped with either Hausdorff- or Gibbs measure are unique almost everywhere and give an explicit formula describing them as probability distributions on the set of limit models of Bedford and Fisher [5].

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1 Tangent Measure Distributions: Definition and Properties

Tangent measure distributions were introduced by Bandt in [2] and, in the present form, by Graf in [8]. They provide a measure–theoretic tool to describe and understand the local geometry of sets and measures. To each point in the support of a measure, or in a set equipped with some natural measure, we assign a family of probability distributions on the space of measures, or, equivalently, a family of random measures, which reflects the behaviour of the measure as an observer zooms down towards this point. The concept of a tangent measure distribution is an extension of two ideas: On the one hand this is the idea of introducing tangent measures in the process of characterizing the regularity of measures by means of their local behaviour. This idea was used to very high effect by Preiss in his fundamental paper [16]. Recently, tangent measures have been subject of intensive research, for a survey see [11]. On the other hand this is the idea of using an averaging procedure on the set of scales to define, by means of ergodic theory, local characteristics of self-similar sets. This idea is due to Bedford and Fisher (see [4]). Closely related ideas can be found in the work of U. Zähle on self-similar random measures (see [19]), which is continued in a series of joint work with Patzschke and M. Zähle (see e.g. [18], [17]). Bandt joined these ideas and defined the tangent measure distributions of a measure at a point as limiting distributions of sequences of natural probability distributions on (rescaled) enlargements of the measure about this point. Roughly speaking, the weight that a tangent measure distribution assigns to a given set of (tangent) measures depends on the number of scales (in terms of the Haar measure on the multiplicative group of positive reals) for which the corresponding enlargement of the measure is close (in the vague topology) to the given set of measures. More precisely:

Definition and Remarks:
Denote by $\mathcal{M}(\mathbb{R})$ the set of nonnegative Radon measures on $\mathbb{R}$ equipped with the vague topology. The topology of $\mathcal{M}(\mathbb{R})$ is Polish and can be generated by the metric D defined by

$$D(\nu,\mu) = \sum_{i=1}^{\infty} (1/2)^i \cdot \min\{1, D_i(\nu,\mu)\}\,.$$

Here $D_r(\nu,\mu)$ is defined as the supremum of $|\int f\,d\nu - \int f\,d\mu|$ with f running through the set of all $f:\mathbb{R}\to[0,\infty)$ with Lipschitz constant $L_f\le 1$ and $\operatorname{supp} f\subseteq U(0,r)$, where $U(a,r)$ denotes the open interval of length $2r$ centred in a (see [11, Chapter 14]). Let $\mu\in\mathcal{M}(\mathbb{R})$ and suppose that there is some $0\le s\le 1$ such that μ has positive lower and finite upper s-densities, i.e.

$$0<\liminf_{r\downarrow 0}\frac{\mu(U(a,r))}{r^s}\le\limsup_{r\downarrow 0}\frac{\mu(U(a,r))}{r^s}<\infty \text{ for } \mu\text{-almost every } a.$$

Let $a\in\mathbb{R}$. Define the family of measures $(\mu_{a,t})_{t>0}\subseteq\mathcal{M}(\mathbb{R})$, the *enlargements of μ about a*, by $\mu_{a,t}(A)=\mu(a+tA)$ for all Borel sets $A\subseteq\mathbb{R}$. It is not hard to check that the vague topology makes the mapping

$$\begin{array}{rccc}\Lambda: & \mathcal{M}(\mathbb{R})\times\mathbb{R}\times(0,1) & \longrightarrow & \mathcal{M}(\mathbb{R})\\ & (\mu,a,t) & \mapsto & \mu_{a,t}\end{array}$$

continuous. The elements of the set

$$\operatorname{Tan}_s(\mu,a)=\{\nu=\lim_{n\to\infty}\frac{\mu_{a,t_n}}{t_n^s}\text{ in the vague topology for some } t_n\downarrow 0\}\subseteq\mathcal{M}(\mathbb{R})$$

are the *(s-dimensional) tangent measures* of μ at a.
Define probability distributions P_ε^a on $\mathcal{M}(\mathbb{R})$ by

$$P_\varepsilon^a(M)=(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \mathbf{1}_M\Big(\frac{\mu_{a,t}}{t^s}\Big)\,\frac{dt}{t}\quad\text{for Borel sets } M\subseteq\mathcal{M}(\mathbb{R}),$$

where $\mathbf{1}_M$ is the indicator function of M. The set $\mathcal{P}(\mu,a)$ of *(s-dimensional) tangent measure distributions* of μ at a is defined as the set of all limit points of $(P_\varepsilon^a)_{\varepsilon>0}$ as $\varepsilon\downarrow 0$ in the weak topology; if the limit exists, we say that μ has a unique tangent measure distribution at a.

Tangent measure distributions are easily seen to be probability distributions on the set of tangent measures. We now state some general properties of tangent measure distributions for measures μ, ν with positive lower and finite upper s-densities, which will be used in the following investigations. The following properties can be checked easily (for proofs see [12]).

Proposition 1.1 *The closure of $\{\mu_{a,t}/t^s : 0<t<1\}\subseteq\mathcal{M}(\mathbb{R})$ is compact. Consequently, the sets $\operatorname{Tan}_s(\mu,a)$ and $\mathcal{P}(\mu,a)$ are nonempty and P is the unique tangent measure distribution of μ at a if and only if $\mathcal{P}(\mu,a)=\{P\}$.*

Proposition 1.2 *Suppose μ, ν are equivalent measures and $f=\frac{d\mu}{d\nu}$. Then, for ν-almost every a, the tangent measure distributions $P_\nu\in\mathcal{P}(\nu,a)$ are given by $P_\nu(M)=P_\mu(\{f(a)\cdot\varrho : \varrho\in M\})$, $M\subseteq\mathcal{M}(\mathbb{R})$ Borel, where P_μ runs through $\mathcal{P}(\mu,a)$.*

For every $\lambda>0$ we define the rescaling operator $S_\lambda^s:\mathcal{M}(\mathbb{R})\to\mathcal{M}(\mathbb{R})$ by $S_\lambda^s\nu(A)=(1/\lambda^s)\cdot\nu(\lambda A)$ and, for every $u\in\mathbb{R}$, we define the shift operator $T^u:\mathcal{M}(\mathbb{R})\to\mathcal{M}(\mathbb{R})$ by $T^u\nu(A)=\nu(u+A)$. Tangent measure distributions have the following invariance properties:

Proposition 1.3 *Let $0\le s\le 1$ and suppose that $\mu\in\mathcal{M}(\mathbb{R})$ has positive lower and finite upper s-densities μ-almost everywhere.*

(i) *At every $a \in \mathbb{R}$ all tangent measure distributions $P \in \mathcal{P}(\mu, a)$ fulfill*

$$P = P \circ (S^s_\lambda)^{-1} \quad \textit{for all } \lambda > 0.$$

(ii) *At μ-almost every a, all tangent measure distributions $P \in \mathcal{P}(\mu, a)$ fulfill*

$$\iint G(\nu, u)\, d\nu(u)\, dP(\nu) = \iint G(T^u\nu, -u)\, d\nu(u)\, dP(\nu)\,,$$
$$\textit{for all Borel functions } G : \mathcal{M}(\mathbb{R}) \times \mathbb{R} \longrightarrow [0, \infty). \tag{1}$$

Whereas the scaling invariance 1.3(i) is easy to check, the shift-invariance 1.3(ii), the so–called Palm formula, is non-trivial. A proof can be found in [13] or the thesis [12]. We can use these invariance properties to see what happens if we iterate the procedure of taking tangent measure distributions.

Proposition 1.4 *Let $\mu \in \mathcal{M}(\mathbb{R})$ be a measure with positive lower and finite upper s-densities μ-almost everywhere. Then, at μ-almost every point, every tangent measure distribution P has the following property: P-almost every measure ν has a unique tangent measure distribution Q^ν_a at ν-almost all points a and at $a = 0$. Furthermore, let $\mathcal{A}$ be the σ-algebra of all Borel sets of $\mathcal{M}(\mathbb{R})$ that are invariant with respect to the action of the rescaling-group $(S^s_\lambda)_{\lambda>0}$ and let*

$$\begin{array}{rccc} P_{\mathcal{A}} : & \mathcal{M}(\mathbb{R}) & \longrightarrow & \textit{Prob}\ (\mathcal{M}(\mathbb{R})) \\ & \nu & \mapsto & P_{\mathcal{A}}[\nu]\,, \end{array}$$

a conditional distribution of P given $\mathcal{A}$. Then $Q^\nu_a = P_{\mathcal{A}}[T^a\nu]$ at ν-almost every a and at $a = 0$, for P-almost every ν. In particular, if P is ergodic with respect to the action of the rescaling group, then for P-almost every ν, we have $Q^\nu_a = P$ at ν-almost every a and at $a = 0$.

Proof. First of all, observe that the existence of the conditional distribution $P_{\mathcal{A}}$ follows from the fact that $\mathcal{M}(\mathbb{R})$ is a Polish space. Now look at the origin and use Birkhoff's ergodic theorem to calculate, for continuous and bounded $F : \mathcal{M}(\mathbb{R}) \longrightarrow [0, \infty)$,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} |\log \varepsilon|^{-1} \int_\varepsilon^1 F\left(\frac{\nu_{0,t}}{t^s}\right) \frac{dt}{t} &= \lim_{s \uparrow \infty} (1/s) \int_0^s F\left(\frac{\nu_{0,e^{-\tau}}}{e^{-\tau s}}\right) d\tau = \lim_{s \uparrow \infty} (1/s) \int_0^s F \circ S^s_{e^{-\tau}}(\nu)\, d\tau \\ &= \int F\, dP_{\mathcal{A}}[\nu] \end{aligned}$$

for P-almost every ν. As the convergence has only to be checked on a countable, dense set of continuous and bounded functions F, this implies $\lim_{\varepsilon \downarrow 0} P^0_\varepsilon = P_{\mathcal{A}}[\nu]$ for P-almost every ν, which is the statement for the origin. For the other points we define the Borel set

$$M = \{(\nu, a) : \text{ the unique tangent measure distribution of } \nu \text{ at } a \text{ equals } P_{\mathcal{A}}[T^a\nu]\}.$$

We use the Palm formula (1) to get

$$0 = \iint 1_{M^c}(\nu, 0)\, d\nu(a)\, dP(\nu) = \iint 1_{M^c}(T^a\nu, 0)\, d\nu(a)\, dP(\nu).$$

Therefore for ν-almost all a we have $(T^a\nu, 0) \in M$, for P-almost all ν. But $(T^a\nu, 0) \in M$ is equivalent to $(\nu, a) \in M$, which finishes the proof of the main statement. In the ergodic case, finally, the constant kernel P is itself a conditional distribution of P given $\mathcal{A}$. ■

More about tangent measure distributions for general measures can be found in the thesis [12] and in the papers [13] and [14]. Tangent measure distributions of self-similar sets generated by contractive similitudes are investigated in [2] and [8], the random case is studied in [1]. We now turn to the study of tangent measure distributions of a class of Cantor sets generated by contractive mappings, which are not necessarily similitudes, the hyperbolic Cantor sets, or cookie-cutters.

2 Hyperbolic Cantor Sets and their Limit Models

We first give the definition of a hyperbolic Cantor set (occasionally one encounters slightly different definitions, for a comparison see [6]).

Definition: Take $0 < \gamma \le 1$ and an open interval J containing the unit interval $I = [0, 1]$. Let $\varphi_0, \varphi_1 : J \to J$ be $\mathcal{C}^{1+\gamma}$-mappings, i.e. the derivatives φ_0', φ_1' are γ-Hölder-mappings, such that

- $\varphi_0(0) = 0$, $\varphi_1(1) = 1$ and $\varphi_0(J) \cap \varphi_1(J) = \emptyset$,
- there are $0 < \alpha, \beta < 1$ and $c > 0$ such that $\alpha^n < (\varphi_{x_0} \circ \cdots \circ \varphi_{x_{n-1}})'(a) < c\beta^n$ for all $a \in J$, $(x_0, \ldots, x_{n-1}) \in \{0,1\}^n$ and $n \ge 1$.

Then there is a unique nonempty, compact set $C \subseteq J$ such that $C = \varphi_0(C) \cup \varphi_1(C)$, which is the *hyperbolic (or cookie–cutter) Cantor set* associated to φ_0, φ_1.

We denote by $\Sigma^- = \prod_{i=-1}^{-\infty}\{0,1\}$ the space of left–sided sequences in $\{0,1\}$ equipped with the topology coming from the metric d defined by $d(x,y) = \beta^m$, where m is the largest nonnegative integer such that $x_i = y_i$ whenever $|i| < m$. In an analogous way we can equip the spaces $\Sigma^+ = \prod_{i=0}^{\infty}\{0,1\}$ of right–sided sequences and $\Sigma = \prod_{i=-\infty}^{\infty}\{0,1\}$ of two–sided sequences with a metric structure. As $\Sigma = \Sigma^- \times \Sigma^+$ we shall frequently use the notation $\underline{x} = (y, x) \in \Sigma$ for $y \in \Sigma^-$ and $x \in \Sigma^+$. For $x \in \Sigma^+$ we abbreviate $\varphi_{x_0 \ldots x_n} = \varphi_{x_0} \circ \ldots \circ \varphi_{x_n}$ and

$$I_x^n = I_{x_0 \ldots x_{n-1}} = \varphi_{x_0 \ldots x_{n-1}}(I)\,, \quad J_x^n = J_{x_0 \ldots x_{n-1}} = \varphi_{x_0 \ldots x_{n-1}}(J).$$

It is well–known that there is a homeomorphism, the *code–mapping*, $\pi : \Sigma^+ \to C$ such that

$$\{\pi(x)\} = \bigcap_{n=1}^{\infty} I_x^n\,.$$

On the open set $J_0 \cup J_1$ we can define a $\mathcal{C}^{1+\gamma}$–mapping S by $S(a) = \varphi_i^{-1}(a)$ for $a \in J_i$. Observe that $\pi^{-1} \circ S \circ \pi$ is the left–shift on Σ^+. C is also the largest invariant set for the expanding map S.

We shall make use of the following well–known facts about hyperbolic Cantor sets (see [3] or [6] and references therein for proofs). First of all there is the bounded distortion property: There is a constant $K > 0$ such that, for all intervals $E \subseteq J_{x_0 \dots x_{n-1}}$ and $a, b \in E$,

$$\exp(-K\,\delta^\gamma) < \frac{(S^n)'(a)}{(S^n)'(b)} < \exp(K\,\delta^\gamma)\,, \tag{2}$$

where $\delta = |S^n(E)|$ and S^n denotes the n-th iterate of S. We may also find a constant $k > 0$ such that, for all $n \geq 1$ and $x \in \Sigma^+$,

$$k < |I_x^n|(S^n)'(\pi(x)) < k^{-1} \;\text{ and }\; |I_x^{n+1}| \geq k|I_x^n|. \tag{3}$$

Finally, there is a constant $t^* > 0$ such that, for all $n \geq 1$, $a \in C$ and all $t \in \mathbb{R}$ with $0 < \log(S^n)'(a) \leq t$, we have

$$U(a, e^{-(t+t^*)}) \subseteq J^n_{\pi^{-1}(a)}\,. \tag{4}$$

We shall consider two canonical measures ν and $\mu \in \mathcal{M}(\mathbb{R})$ supported by the Cantor set C. Details on the construction of the *Gibbs measure* ν can be found in [7] (see also [3]). There is $\eta > 0$, a unique number $0 < s < 1$ and a unique shift-invariant and ergodic probability measure $\hat\nu$ on Σ such that, for all $x \in \Sigma^+$ and $n \geq 1$,

$$\eta < \hat\nu\{y \in \Sigma \,:\, y_0 = x_0, \dots, y_{n-1} = x_{n-1}\} \cdot \Big[(S^n)'(\pi(x))\Big]^s < \eta^{-1}.$$

$\hat\nu$ is called *Gibbs measure.* Bowen found that the number s is the Hausdorff dimension of C. We can define a measure ν on C by

$$\nu(A) = \hat\nu(\Sigma^- \times \pi^{-1}(A)) \;\text{ for all } A \subseteq C \text{ Borel.}$$

ν is an S-invariant ergodic probability measure on C and is called the *Gibbs measure on* C.

By μ we shall denote the *s–Hausdorff measure* restricted to C. For a diffeomorphism $\Phi : A \to B$ between intervals A, B, a number $r > 0$ and a Radon measure ϱ on A, the (Φ, r)-*conformal transform* of ϱ is the measure $\check\Phi\varrho$ defined by

$$\check\Phi\varrho(E) = \int_{\Phi^{-1}E} |\Phi'|^r \, d\varrho \text{ for } E \subseteq B \text{ Borel.}$$

The conformal transform property $\check\Phi\mathcal{H}^s = \mathcal{H}^s$ of s-Hausdorff measure implies that

$$\mu(S(E)) = \int_E |S'|^s \, d\mu \;\text{ for } E \subseteq J_0,\, J_1 \text{ Borel.}$$

Therefore μ is called the *conformal measure* for (C, S).

It is well–known that the measures μ and ν are equivalent. Moreover there is a constant $\kappa > 0$ such that, for all $a \in C$ and $0 < t \leq 1$,

$$\kappa^{-1} \cdot t^s < \mu(U(a,t)),\, \nu(U(a,t)) < \kappa \cdot t^s. \tag{5}$$

In particular, both μ and ν have positive lower and finite upper s-densities.

Following Bedford and Fisher ([6]) we now introduce the ratio Cantor sets, or limit models, of C. We start with the construction of the scaling function of C, which is due to Sullivan (see [15] and [6]). For this purpose define for a positive integer n a mapping $R^{(n)} : \Sigma^- \to \mathbb{R}^2$ by

$$R^{(n)}(y) = \frac{1}{|I_{y_{-n}\cdots y_{-1}}|} \cdot (|I_{y_{-n}\cdots y_{-1}0}|, |I_{y_{-n}\cdots y_{-1}1}|).$$

For every $y \in \Sigma^-$ there is

$$R(y) = \lim_{n\to\infty} R^{(n)}(y).$$

The resulting γ-Hölder continuous function $R : \Sigma^- \to \mathbb{R}^2$ is called the *scaling function* of C.

For $\mathcal{C}^{1+\gamma}$-hyperbolic Cantor sets C and $\tilde{C}$ with expanding maps S and $\tilde{S}$ the corresponding dynamical systems (C, S) and $(\tilde{C}, \tilde{S})$ are called $\mathcal{C}^{1+\gamma}$*-conjugate* (resp. $\mathcal{C}^1$*-conjugate*) if there is an orientation preserving $\mathcal{C}^{1+\gamma}$ (resp. $\mathcal{C}^1$) diffeomorphism $\Phi : I \to I$ such that $\tilde{S} \circ \Phi = \Phi \circ S$.

If (C, S) and $(\tilde{C}, \tilde{S})$ are $\mathcal{C}^1$-conjugate, it is easy to see that the corresponding shift-invariant measures $\hat{\nu}$ and the dimensions of C and $\tilde{C}$ coincide. Sullivan showed that for two hyperbolic Cantor sets C and $\tilde{C}$ the dynamical systems (C, S), $(\tilde{C}, \tilde{S})$ are $\mathcal{C}^1$-conjugate if and only if their scaling functions coincide, Bedford and Fisher show in [6] that this conjugacy automatically has the same degree of smoothness as the Cantor sets themselves.

For every $y = (y_i) \in \Sigma^-$ we define a sequence $(\Phi_n^y)_{n=1}^\infty$ of mappings $\Phi_n^y : I \to I$ as follows. First define an affine mapping $A_n^y : I_{y_{-n}\cdots y_{-1}} \to I$ by

$$A_n^y(a) = \frac{a - \varphi_{y_{-n}\cdots y_{-1}}(0)}{|I_{y_{-n}\cdots y_{-1}}|},$$

then let $\Phi_n^y = A_n^y \circ \varphi_{y_{-n}\cdots y_{-1}}$. The following theorem is due to Bedford and Fisher and proved in [6].

Theorem 2.1 and Definition: *For every $y \in \Sigma^-$ the limit $\Phi^y = \lim_{n\to\infty} \Phi_n^y$ exists in the $\mathcal{C}^1$-norm $\|f\|_{\mathcal{C}^1} = \|f\|_\infty + \|f'\|_\infty$. More precisely, there is a constant $K' > 0$ such that*

$$\|\Phi_n^y - \Phi^y\|_{\mathcal{C}^1} \leq K' \cdot \beta^{n\gamma} \tag{6}$$

and

$$1/K' < (\Phi_n^y)'(x) < K' \quad \textit{for all } n \geq 1 \textit{ and } (y, x) \in \Sigma, \tag{7}$$

and

$$\|\Phi^y - \Phi^w\|_{\mathcal{C}^1} \leq K' \cdot d(y, w)^\gamma \quad \textit{for all } y, w \in \Sigma^-. \tag{8}$$

The mappings $\Phi^y : I \to I$ are $\mathcal{C}^{1+\gamma}$–diffeomorphisms. For $y \in \Sigma^-$ denote $C^y = \Phi^y(C)$. The family $(C^y)_{y\in\Sigma^-}$ is called the family of ratio Cantor sets *or* limit models *of C. $\mathcal{C}^1$-conjugate Cantor sets have the same limit models. The sets C^y are hyperbolic Cantor sets of the same Hausdorff dimension and (at least) the same degree of smoothness as C and Φ^y defines a $\mathcal{C}^{1+\gamma}$-conjugacy between C and C^y. We denote by $\pi^y = \Phi^y \circ \pi$ the code–mapping, by $\mu^y = \check{\Phi}^y\mu$ the s-Hausdorff measure and by ν^y the Gibbs measure on C^y.*

In the following section we shall give a formula for the tangent measure distributions of both Hausdorff and Gibbs measure on a hyperbolic Cantor set in terms of the scaling function and the limit models defined above.

3 Tangent Measure Distributions of Hyperbolic Cantor Sets

We investigate the s-Hausdorff measure μ on a hyperbolic Cantor set C and start this section with a useful observation.

Lemma 3.1 *The family $(\Lambda_a)_{a\in C}$ defined by $\Lambda_a(t) = \frac{\Lambda(\mu,a,e^{-t})}{e^{-ts}}$ is uniformly equicontinuous with respect to the metric D.*

Proof. Let f be Lipschitz with Lipschitz constant $L_f \le 1$ and supp $f \subseteq U(0,r)$ for some $r>0$ and let $a \in C$. For all $0<\tau_1\le\tau_2$ we thus have

$$\begin{aligned}
&\left|\int f(b)\, d\frac{\mu_{a,\tau_2}}{\tau_2^s}(b) - \int f(b)\, d\frac{\mu_{a,\tau_1}}{\tau_1^s}(b)\right| \\
= \; & \left|\frac{1}{\tau_2^s}\int_{U(a,r\tau_2)} f\left(\frac{b-a}{\tau_2}\right) d\mu(b) - \frac{1}{\tau_1^s}\int_{U(a,r\tau_1)} f\left(\frac{b-a}{\tau_1}\right) d\mu(b)\right| \\
\le \; & \frac{1}{\tau_2^s}\left|\int_{U(a,r\tau_2)} f\left(\frac{b-a}{\tau_2}\right) d\mu(b) - \int_{U(a,r\tau_1)} f\left(\frac{b-a}{\tau_1}\right) d\mu(b)\right| \\
& + \left|\frac{1}{\tau_2^s} - \frac{1}{\tau_1^s}\right| \left|\int_{U(a,r\tau_1)} f\left(\frac{b-a}{\tau_1}\right) d\mu(b)\right| \\
\le \; & \frac{r\tau_2}{\tau_2^s}\left|\frac{\tau_1-\tau_2}{\tau_2\tau_1}\right| \mu(U(a,r\tau_2)) + \left|\frac{\tau_1^s-\tau_2^s}{\tau_2^s\tau_1^s}\right| \|f\|_\infty \mu(U(a,r\tau_1)) \\
\le \; & \kappa r^{s+1}\cdot\left|1-\frac{\tau_2}{\tau_1}\right| + \kappa\|f\|_\infty r^s \cdot \left|1-\frac{\tau_1^s}{\tau_2^s}\right| .
\end{aligned}$$

By definition of the metric D this implies the statement. ■

For $r>0$ we define a mapping $\Pi_r : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$, which restricts a measure on the line to the interval $U(0,r)$, by $\Pi_r(\varrho) = \mathbf{1}_{U(0,r)}\,\varrho$. Clearly, a measure ϱ is uniquely determined if $\Pi_{r_n}(\varrho)$ is known for some sequence $r_n \uparrow \infty$.

The following lemma is a major ingredient in the proof of the following statements.

Lemma 3.2 *For every $r>0$ there is $\lambda_r = ke^{-t^*-\log r} > 0$ with the property that, whenever $x, w \in \Sigma^+$ and $y \in \Sigma^-$ are such that, for a strictly increasing sequence (m_n) of positive integers,*

$$w_{m_n} = y_{-n}, \ldots, w_{m_n+n-1} = y_{-1}, w_{m_n+n} = x_0, \ldots, w_{m_n+2n-1} = x_{n-1}\,,$$

then for all $0<\lambda\le\lambda_r$ and $t_n = |I_w^{m_n+n}|$, the sequence

$$\left(\frac{\mu_{\pi(w),\lambda t_n}}{(\lambda t_n)^s}\right)_{n=1}^{\infty}$$

has a subsequence converging to a limit ϱ fulfilling

$$\Pi_r(\varrho) = \Pi_r\left(\frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s}\right) .$$

Proof. Let $a=\pi(w)\in C$ and for each $n\geq 1$ choose $y^{(n)}\in\Sigma^-$ and $x^{(n)}\in\Sigma^+$ such that

$$\begin{aligned}(y^{(n)}_{-(m_n+n)},\ldots,y^{(n)}_{-1}) &= (w_0,\ldots,w_{m_n+n-1})\\ (x^{(n)}_0,x^{(n)}_1,\ldots) &= (w_{m_n+n},w_{m_n+n+1},\ldots).\end{aligned}$$

Let

$$l_n=\frac{a-\min I_w^{m_n+n}}{|I_w^{m_n+n}|}=A^{y^{(n)}}_{m_n+n}(a).$$

Using Theorem 2.1 we infer from (6) and (8) that

$$\begin{aligned}\|\Phi^{y^{(n)}}_{m_n+n}-\Phi^y\|_{\mathcal{C}^1} &\leq \|\Phi^{y^{(n)}}_{m_n+n}-\Phi^{y^{(n)}}\|_{\mathcal{C}^1}+\|\Phi^{y^{(n)}}-\Phi^y\|_{\mathcal{C}^1}\\ &\leq 2K'\cdot\beta^{n\gamma}\end{aligned}\tag{9}$$

and from (7) and (9) that

$$\begin{aligned}|\Phi^{y^{(n)}}_{m_n+n}(\pi(x^{(n)}))-\Phi^y(\pi(x))| &\leq K'\cdot|\pi(x^{(n)})-\pi(x)|+\|\Phi^{y^{(n)}}_{m_n+n}-\Phi^y\|_{\mathcal{C}^1}\\ &\leq K'c\cdot\beta^n+2K'\cdot\beta^{n\gamma}\,.\end{aligned}$$

Therefore, using that $a=\pi(w)=\varphi_{w_0\ldots w_{m_n+n-1}}(\pi(x^{(n)})$,

$$\lim_{n\to\infty}l_n=\lim_{n\to\infty}A^{y^{(n)}}_{m_n+n}(a)=\lim_{n\to\infty}\Phi^{y^{(n)}}_{m_n+n}(\pi(x^{(n)}))=\Phi^y(\pi(x))=\pi^y(x).\tag{10}$$

Using (5) and the compactness statement of Proposition 1.1 we find a subsequence of (λt_n) (which we denote (λt_n) again) and a measure ϱ, such that

$$\frac{\mu_{a,\lambda t_n}}{(\lambda t_n)^s}\longrightarrow\varrho\,.$$

We now define for every $n\geq 1$ and $y^{(n)}\in\Sigma^-$, the measure

$$\mu[y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}]=\check{\Phi}^{y^{(n)}}_{m_n+n}\mu\,.$$

By (9) we thus have

$$\mu[y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}]=\check{\Phi}^{y^{(n)}}_{m_n+n}\mu\;\longrightarrow\;\check{\Phi}^y\mu=\mu^y\,,$$

and, using the continuity of the mapping Λ and (10), we infer

$$\frac{\mu[y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}]_{l_n,\lambda}}{\lambda^s}\longrightarrow\frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s}\,.\tag{11}$$

We use

$$\Phi^{y^{(n)}}_{m_n+n}=A^{y^{(n)}}_{m_n+n}\circ\varphi_{y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}}$$

and the conformal transform property of s-Hausdorff measure to calculate, for all $E\subseteq I$ Borel,

$$\begin{aligned}\mu[y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}](E) &= \frac{1}{|I_{y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}}|^s}\int_{(\Phi^{y^{(n)}}_{m_n+n})^{-1}(E)}\left[\left(\varphi_{y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}}\right)'\right]^s d\mu\\ &= \frac{1}{(t_n)^s}\,\mu\Big(I_{y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}}\cap(A^{y^{(n)}}_{m_n+n})^{-1}(E)\Big)\,,\end{aligned}$$

recalling that $t_n = |I_{y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}}|$. We observe that

$$\mu_{a,\lambda t_n}\left(\left[\frac{-l_n}{\lambda}, \frac{1-l_n}{\lambda}\right] \cap E\right) = \mu\left(I_{y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}} \cap (A^{y^{(n)}}_{m_n+n})^{-1}(l_n + \lambda E)\right) .$$

Thus, using (11),

$$\frac{(\mu_{a,\lambda t_n})|_{[\frac{-l_n}{\lambda}, \frac{1-l_n}{\lambda}]}}{(\lambda t_n)^s} = \frac{\mu[y^{(n)}_{-(m_n+n)}\cdots y^{(n)}_{-1}]_{l_n,\lambda}}{\lambda^s} \longrightarrow \frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s} .$$

By definition of λ_r and (3), (4), we have $U(a, rt) \subseteq J^{m_n+n}_w$ for all $n \geq 1$ and $t \in \mathbb{R}$ such that $0 < t \leq \lambda_r t_n$. As $0 < \lambda \leq \lambda_r$ we have

$$U(a, r\lambda t_n) \cap C \subseteq J^{m_n+n}_w \cap C = I^{m_n+n}_w \cap C = [a - l_n t_n, a + (1 - l_n)t_n] \cap C ,$$

and we can conclude

$$\Pi_r(\varrho) = \Pi_r\left(\frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s}\right) .$$

■

In the following we describe the tangent measures of hyperbolic Cantor sets. Our result on tangent measures has closest links to Bedford and Fisher's results on the scenery of hyperbolic Cantor sets (cf. [5]).

Lemma 3.3 *For μ-almost all $a \in C$ and all $r > 0$, we have*

$$\Pi_r(\mathrm{Tan}_s(\mu, a)) = \left\{\Pi_r\left(\frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s}\right) : x \in \Sigma^+,\ y \in \Sigma^-,\ 0 < \lambda \leq \lambda_r\right\},$$

where $\lambda_r > 0$ is the constant defined in Lemma 3.2.

Proof. As ν is ergodic and ν and μ are equivalent, we have that, for μ-almost all $a \in C$, every finite sequence in $\{0,1\}$ appears infinitely often in the sequence $w = \pi^{-1}(a)$. We fix $a \in C$ with this property. Suppose $r > 0$ and $x \in \Sigma^+$, $y \in \Sigma^-$ and $0 < \lambda \leq \lambda_r$ are given. For $n \geq 1$ let

$$\begin{aligned} m_1 &= \min\{i > 0 : S^i(a) \in I_{y_{-1}x_0}\} = \min\{i > 0 \ :\ w_i = y_{-1}, w_{i+1} = x_0\} , \\ m_{n+1} &= \min\{i > m_n : S^i(a) \in I_{y_{-(n+1)}\cdots y_{-1}x_0\ldots x_n}\} \\ &= \min\{i > m_n \ :\ w_i = y_{-(n+1)}, \ldots, w_{i+n} = y_{-1}, w_{i+n+1} = x_0, \ldots, w_{i+2n+1} = x_n\} . \end{aligned}$$

Observe that x, w, y and (m_n) fulfill the condition of Lemma 3.2 and therefore we can find a subsequence of $\left(\frac{\mu_{a,\lambda t_n}}{(\lambda t_n)^s}\right)$ converging to a tangent measure ϱ, such that

$$\Pi_r(\varrho) = \Pi_r\left(\frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s}\right) ,$$

as required to prove the first inclusion. In order to prove the other inclusion we assume that

$$\frac{\mu_{a,s_n}}{s_n^s} \longrightarrow \varrho ,$$

for some sequence $s_n \downarrow 0$ and a measure $\varrho \in \mathcal{M}(\mathbb{R})$. We fix an arbitrary $r > 0$. For every s_n there is an integer m such that

$$\lambda_r |I_w^{m+1}| \leq s_n < \lambda_r |I_w^m|.$$

As we can pass to a subsequence, if necessary, we may assume that there are $x \in \Sigma^+$ and $y \in \Sigma^-$ such that, for

$$\begin{aligned} m_1 &= \min\{i > 0 : S^i(a) \in I_{y_{-1}x_0}\}, \\ m_{n+1} &= \min\{i > m_n : S^i(a) \in I_{y_{-(n+1)}\cdots y_{-1}x_0\cdots x_n}\}, \end{aligned}$$

we have

$$\lambda_r |I_w^{m_n+n+1}| \leq s_n < \lambda_r |I_w^{m_n+n}|\,.$$

Denote $t_n = |I_w^{m_n+n}|$. From (3) we infer that the sequence $(s_n/(\lambda_r t_n))$ is bounded away from 0. Hence, passing to a subsequence again, we may assume that there is $0 < \lambda' \leq 1$ such that

$$\frac{s_n}{\lambda_r t_n} \longrightarrow \lambda'\,.$$

Denote $\lambda = \lambda' \lambda_r$. By Lemma 3.2 we can, passing to a subsequence once more, assume that $\frac{\mu_{a,\lambda t_n}}{(\lambda t_n)^s}$ converges to a measure $\tilde{\varrho}$ with

$$\Pi_r(\tilde{\varrho}) = \Pi_r\left(\frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s}\right)\,.$$

But, using $\frac{s_n}{\lambda t_n} \to 1$ and Lemma 3.1, we find that $\varrho = \tilde{\varrho}$ and we have proved the second inclusion. ∎

We may consider the components R_0 and R_1 of the scaling function as functions on the domain Σ by defining $R_i((y,x)) = R_i(y)$. We define a positive, continuous function $f : \Sigma \to \mathbb{R}$ by

$$f(\underline{x}) = f((y,x)) = -\log R_{x_0}(y) \text{ for } \underline{x} = (y,x) \in \Sigma.$$

If $\sigma : \Sigma \to \Sigma$ denotes the left shift, the *flow space* Σ_f *under* f is defined as

$$\Sigma_f = \Sigma \times [0,\infty)/\approx$$

for the equivalence relation $\approx$ generated by $(\underline{x}, f(\underline{x})) \approx (\sigma\underline{x}, 0)$ (see e.g. [3]). It is possible to describe $\mathrm{Tan}_s(\mu, a)$ in terms of Σ_f.

Theorem 3.4 *For μ-almost all $a \in C$ there is a continuous, surjective mapping*

$$\tilde{M} : \Sigma_f \longrightarrow \mathrm{Tan}_s(\mu, a)$$

mapping the class of $((y,x),t)$ onto $\nu \in \mathrm{Tan}_s(\mu,a)$ such that, for all $r > 0$,

$$\Pi_r(\nu) = \Pi_r\left(\frac{(\mu^y)_{\pi^y(x),e^{-t}}}{e^{-ts}}\right),$$

if t is chosen such that $e^{-t} \leq \lambda_r$ and λ_r is defined as in Lemma 3.2.

Proof. Define $M : \Sigma \times [0,\infty) \to \mathcal{M}(\mathbb{R})$ by

$$M(\underline{x},\lambda) = M((y,x),\lambda) = \frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s}.$$

In order to show that the mapping $\tilde{M}$ given in the theorem is well–defined, we only have to prove that, for all $n \geq 1$, $\underline{x} = (y,x) \in \Sigma$ and $0 < \lambda \leq \lambda_r$,

$$\Pi_r\Big(M(\sigma^n \underline{x},\lambda)\Big) = \Pi_r\Big(M(\underline{x},\lambda \textstyle\prod_{i=0}^{n-1} R_{(\sigma^i \underline{x})_0}(\sigma^i \underline{x}))\Big). \tag{12}$$

To prove this we use Lemma 3.2 again. Let $a \in C$ be such that every finite sequence in $\{0,1\}$ appears infinitely often in $w = \pi^{-1}(a)$. For $n \geq 1$ we again pick the numbers m_n such that

$$\begin{aligned} m_1 &= \min\{i > 0 : S^i(a) \in I_{y_{-1}x_0}\}, \\ m_{n+1} &= \min\{i > m_n : S^i(a) \in I_{y_{-(n+1)}\cdots y_{-1}x_0\ldots x_n}\}. \end{aligned}$$

Let

$$t_n = |I_w^{m_n+n}| \quad \text{and} \quad t_n' = |I_w^{m_n+n+1}|.$$

Using Lemma 3.2 we find, for every $0 < \lambda \leq \lambda_r$, a sequence (k_n) of integers and measures $\varrho, \tilde{\varrho}$ such that

$$\frac{\mu_{a,\lambda t_{k_n} R_{x_0}(y)}}{(\lambda t_{k_n} R_{x_0}(y))^s} \longrightarrow \varrho \quad \text{and} \quad \frac{\mu_{a,\lambda t_{k_n}'}}{(\lambda t_{k_n}')^s} \longrightarrow \tilde{\varrho}$$

and

$$\Pi_r(\varrho) = \Pi_r\Big(M(\underline{x},\lambda R_{x_0}(y))\Big) \quad \text{and} \quad \Pi_r(\tilde{\varrho}) = \Pi_r\Big(M(\sigma\underline{x},\lambda)\Big).$$

Using the definition of the scaling function and the bounded distortion property (2) we infer that, for some $\xi, \xi' \in I_w^{m_n+n}$,

$$\frac{t_n'}{t_n R_{x_0}(y)} = \underbrace{\frac{|I_{y_{-n}\cdots y_{-1}x_0}|}{|I_{y_{-n}\cdots y_{-1}}| R_{x_0}(y)}}_{\to 1} \cdot \underbrace{\frac{(S^{m_n})'(\xi)}{(S^{m_n})'(\xi')}}_{\to 1} \longrightarrow 1\,.$$

Therefore, by Lemma 3.1,

$$D\left(\frac{\mu_{a,\lambda t_n R_{x_0}(y)}}{(\lambda t_n R_{x_0}(y))^s}, \frac{\mu_{a,\lambda t_n'}}{(\lambda t_n')^s}\right) \longrightarrow 0$$

and altogether we get

$$\Pi_r\Big(M(\sigma\underline{x},\lambda)\Big) = \Pi_r\Big(M(\underline{x},\lambda R_{x_0}(\underline{x}))\Big),$$

and the full statement of (12) follows inductively. Hence $\tilde{M}$ is well–defined. Moreover, $\tilde{M}$ is continuous by Theorem 2.1 and surjective by Lemma 3.3. ∎

There is a natural semiflow $(\Phi_t)_{t\geq 0}$ on Σ_f mapping the class of $(\underline{x},u)$ onto the class of $(\underline{x},u+t)$. (Φ_t) is called the *flow under the function* f. Every shift–invariant probability measure ν on Σ induces a (Φ_t)–invariant probability measure ν_f on Σ_f by means of

$$\nu_f(A \times B) = \frac{\nu(A)\,\mathcal{L}(B)}{\int f\,d\nu} \text{ for } A \subseteq \Sigma \text{ and } B \subseteq [0, \inf_{x\in A} f(x)],$$

where $\mathcal{L}$ denotes Lebesgue measure.

We can now formulate and prove the main result of this paper, the formula for the tangent measure distributions of the Hausdorff measure μ.

Theorem 3.5 *At μ-almost every point the measure μ has a unique tangent measure distribution P, which does not depend on the point. P is the image under $\tilde{M}$ of the (Φ_t)–invariant measure on Σ_f induced by the Gibbs measure $\hat{\nu}$ on Σ. More explicitly, for $r > 0$ and $\lambda_r > 0$ as in Lemma 3.2 the tangent measure distribution P is given by*

$$P(A) = \frac{-1}{\int \log R_{x_0}(y)\, d\hat{\nu}((y,x))} \int \int_{\lambda_r R_{x_0}(y)}^{\lambda_r} \mathbf{1}_A \left(\frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s} \right) \frac{d\lambda}{\lambda}\, d\hat{\nu}((y,x)) \tag{13}$$

for all $A \subseteq \mathcal{M}(\mathbb{R})$ chosen from the σ-field $\mathcal{B}_r = \{\Pi_r^{-1}(B) \,:\, B \subseteq \mathcal{M}(\mathbb{R}) \text{ Borel}\}$.

Proof. We start with the calculation of the tangent measure distributions of Hausdorff measure on the limit models C^y and afterwards we shall use our knowledge of Lemma 3.3 and Proposition 1.4 to infer the formula for general hyperbolic Cantor sets.

Fix some $r > 0$ and a continuous bounded function $F : \mathcal{M}(\mathbb{R}) \to [0, \infty)$. For $\underline{x} = (y,x) \in \Sigma$ and $0 < \lambda \leq \lambda_r$ we write

$$M(\underline{x}, \lambda) = M((y,x)\,,\lambda) = \frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s}$$

as in Theorem 3.4 and recall equation (12). We define $F_* : \Sigma \to \mathbb{R}$ as

$$F_*(\underline{x}) = \int_{\lambda_r R_{x_0}(y)}^{\lambda_r} F \circ \Pi_r(M(\underline{x}, \lambda))\, \frac{d\lambda}{\lambda}.$$

By Birkhoff's Ergodic Theorem, for $\hat{\nu}$-almost every $\underline{x} = (y,x)$, as $N \to \infty$,

$$\frac{1}{N} \sum_{n=0}^{N-1} F_*(\sigma^n \underline{x}) \longrightarrow \int \int_{\lambda_r R_{u_0}(v)}^{\lambda_r} F \circ \Pi_r(M((v,u),\lambda))\, \frac{d\lambda}{\lambda}\, d\hat{\nu}((v,u)) \tag{14}$$

and

$$\frac{N}{-\log(\lambda_r \prod_{n=0}^{N-1} R_{x_n}(\sigma^n \underline{x}))} \longrightarrow \frac{1}{-\int \log R_{u_0}(v)\, d\hat{\nu}((v,u))}\,. \tag{15}$$

Using the scaling invariance of $\frac{d\lambda}{\lambda}$ and (12) we infer

$$\frac{1}{N} \sum_{n=0}^{N-1} F_*(\sigma^n \underline{x}) = \frac{1}{N} \int_{\lambda_r r_N}^{\lambda_r} F \circ \Pi_r(M(\underline{x}, \lambda))\, \frac{d\lambda}{\lambda},$$

writing $r_N = \prod_{n=0}^{N-1} R_{x_n}(\sigma^n \underline{x})$.
For an arbitrary sequence $\varepsilon_n \downarrow 0$ we find, for every $n \geq 1$, an integer N such that $\lambda_r r_{N+1} < \varepsilon_n \leq \lambda_r r_N$ and thus

$$\int_{\lambda_r r_{N+1}}^{\lambda_r} F \circ \Pi_r(M(\underline{x}, \lambda))\, \frac{d\lambda}{\lambda} \geq \int_{\varepsilon_n}^{\lambda_r} F \circ \Pi_r(M(\underline{x}, \lambda))\, \frac{d\lambda}{\lambda} \geq \int_{\lambda_r r_N}^{\lambda_r} F \circ \Pi_r(M(\underline{x}, \lambda))\, \frac{d\lambda}{\lambda}$$

and $|\log(\lambda_r r_N)| \le |\log \varepsilon_n| \le |\log(\lambda_r r_{N+1})|$. As $|\log(\lambda_r r_N)|/|\log(\lambda_r r_{N+1})| \to 1$, we infer that, whenever $\underline{x}$ fulfills (14) and (15),

$$\frac{1}{|\log \varepsilon_n|}\int_{\varepsilon_n}^{1} F\circ\Pi_r(M(\underline{x},\lambda))\,\frac{d\lambda}{\lambda}\longrightarrow \frac{1}{-\int \log R_{u_0}(v)\,d\hat{\nu}((v,u))}\iint_{\lambda_r R_{u_0}(v)}^{\lambda_r} F\circ\Pi_r(M((v,u),\lambda))\,\frac{d\lambda}{\lambda}\,d\hat{\nu}((v,u))\,. \tag{16}$$

We now show that the set of full measure where (14, 15) and hence (16) hold may be written in the form $\Sigma^- \times A$ with a Borel set $A \subseteq \Sigma^+$ such that $\nu^y(\pi^y(A)) = 1$ for all $y \in \Sigma^-$. For this purpose let $y, z \in \Sigma^-$, $a \in C$, $\lambda > 0$ and $f \ge 0$ with Lipschitz constant $L_f \le 1$. We estimate

$$\begin{aligned}
&\left|\int f\left(\frac{b-\Phi^y(a)}{\lambda}\right)\,d\mu^y(b) - \int f\left(\frac{b-\Phi^z(a)}{\lambda}\right)\,d\mu^z(b)\right|\\
&= \left|\int f\left(\frac{\Phi^y(b)-\Phi^y(a)}{\lambda}\right)|(\Phi^y)'(b)|^s\,d\mu(b)\right.\\
&\qquad\left.-\int f\left(\frac{\Phi^z(b)-\Phi^z(a)}{\lambda}\right)|(\Phi^z)'(b)|^s\,d\mu(b)\right|\\
&\le \left|\int f\left(\frac{\Phi^y(b)-\Phi^y(a)}{\lambda}\right)\left(|(\Phi^y)'(b)|^s - |(\Phi^z)'(b)|^s\right)d\mu(b)\right|\\
&\qquad+\int\left|f\left(\frac{\Phi^y(b)-\Phi^y(a)}{\lambda}\right)-f\left(\frac{\Phi^z(b)-\Phi^z(a)}{\lambda}\right)\right||(\Phi^z)'(b)|^s\,d\mu(b)\\
&\le \|f\|_\infty\cdot\mu(C)\cdot\sup_{b\in C}\left||(\Phi^y)'(b)|^s-|(\Phi^z)'(b)|^s\right|\\
&\qquad+2\,\mu(C)\cdot\sup_{b\in C}|(\Phi^z)'(b)|^s\cdot\sup_{b\in C}\left|\frac{\Phi^z(b)-\Phi^y(b)}{\lambda}\right|.
\end{aligned}$$

Hence, for $x \in \Sigma^+$, $\underline{x} = (y, x)$ and $\underline{x}' = (z, x)$, for $0 < \lambda \le \lambda_r$, using Theorem 2.1,

$$D(M(\sigma^n\underline{x},\lambda), M(\sigma^n\underline{x}',\lambda)) \stackrel{n\to\infty}{\longrightarrow} 0\,. \tag{17}$$

Moreover, for all $n \ge 1$,

$$\begin{aligned}
&|F_*(\sigma^n\underline{x}) - F_*(\sigma^n\underline{x}')|\\
&= \left|\int_{\lambda_r R_{x_n}(\sigma^n\underline{x})}^{\lambda_r} F\circ\Pi_r(M(\sigma^n\underline{x},\lambda))\,\frac{d\lambda}{\lambda} - \int_{\lambda_r R_{x_n}(\sigma^n\underline{x}')}^{\lambda_r} F\circ\Pi_r(M(\sigma^n\underline{x}',\lambda))\,\frac{d\lambda}{\lambda}\right|\\
&\le \sup_{\lambda\in[\lambda_r R_{x_n}(\sigma^n\underline{x}),\lambda_r]} |F\circ\Pi_r(M(\sigma^n\underline{x},\lambda)) - F\circ\Pi_r(M(\sigma^n\underline{x}',\lambda))|\cdot|\log R_{x_n}(\sigma^n\underline{x})|\\
&\qquad+\|F\|_\infty\cdot|\log R_{x_n}(\sigma^n\underline{x}') - \log R_{x_n}(\sigma^n\underline{x})|.
\end{aligned}$$

Note that the components of the scaling function fulfill $0 < k \le R_0, R_1 \le 1$ and the function $F\circ\Pi_r$ is uniformly continuous on the compact set $\{M(\underline{x},\lambda) : \underline{x} \in \Sigma,\ \lambda \in [k\cdot\lambda_r, \lambda_r]\}$. Hence the convergence in (14, 15) is independent of the projection of $\underline{x}$ onto Σ^- and we conclude that the set where the statements (14, 15) and (16) hold can be written as $\Sigma^- \times A$ with $\hat{\nu}(\Sigma^- \times A) = 1$. As all limit models C^y and C are $\mathcal{C}^1$-conjugate, the corresponding measures $\hat{\nu}$ coincide and this implies that

$$\nu^y(\pi^y(A)) = \hat{\nu}(\Sigma^- \times A) = 1 \text{ for all } y \in \Sigma^-.$$

To infer that the measure defined in (13) is the unique tangent measure distribution at μ^y-almost all points of C^y, we observe, using for example [9, Theorem 3.1], that a countable set of functions $F : \mathcal{M}(\mathbb{R}) \to [0,\infty)$ of the form $F(\nu) = h(\nu(f))$, $h : \mathbb{R} \to [0,\infty)$ continuous and bounded, $f : \mathbb{R} \to [0,\infty)$ continuous with compact support, may be picked such that any two probability measures on $\mathcal{M}(\mathbb{R})$ coincide whenever the integrals of these F coincide. Choose $N \subseteq C^y$ as the (countable) union of the ν^y-null-sets $\pi^y(A)$ corresponding to these F. Recalling that μ^y is absolutely continuous with respect to ν^y and that the statement of Proposition 1.1 holds for every $a \in C^y$, we infer that for all $y \in \Sigma^-$ at μ^y-almost every a there is a unique tangent measure distribution P^y of μ^y which is given by (13).

Eventually we return to our principal aim and study the tangent measure distributions of the Hausdorff measure μ. Fix $r > 0$ and let $a \in C$ be such that the statements of Proposition 1.1, Proposition 1.4 and Lemma 3.3 are fulfilled and suppose that P is a tangent measure distribution of μ at a. Then, using the notation of Proposition 1.4, for P-almost all ϱ,

$$Q_u^\varrho = P_{\mathcal{A}}[T^u\varrho] \text{ for } \varrho\text{-almost all } u.$$

By Lemma 3.3 there is $y \in \Sigma^-$, $v \in C^y$ and $\lambda > 0$ such that

$$\Pi_r(\varrho) = \Pi_r\left(\frac{(\mu^y)_{v,\lambda}}{\lambda^s}\right) = \Pi_r \circ S_\lambda^s \circ T^v \mu^y\,.$$

By the scaling invariance of tangent measure distributions we have

$$\mathcal{P}(\mu^y, v + u\lambda) = \mathcal{P}\Big(\frac{(\mu^y)_{v,\lambda}}{\lambda^s}, u\Big)$$

and we can infer from our formula for the tangent measure distributions of the μ^y that

$$Q_u^\varrho = P^y \text{ for } \varrho\text{-almost all } u \in U(0,r).$$

For every set $A \in \mathcal{B}_r$ we thus get, using the Palm formula (1) twice,

$$\begin{aligned}
P^y(A) &= \iint \frac{\mathbf{1}_{U(0,r)}(b)}{\varrho(U(0,r))}\, P^y(A)\, d\varrho(b)\, dP(\varrho) \\
&= \iint \frac{\mathbf{1}_{U(0,r)}(-b)}{T^b\varrho(U(0,r))}\, P_{\mathcal{A}}[T^b\varrho](A)\, d\varrho(b)\, dP(\varrho) \\
&= \iint \frac{\mathbf{1}_{U(0,r)}(b)}{\varrho(U(0,r))}\, P_{\mathcal{A}}[\varrho](A)\, d\varrho(b)\, dP(\varrho) \\
&= \int P_{\mathcal{A}}[\varrho](A)\, dP(\varrho) \\
&= P(A).
\end{aligned}$$

By Proposition 1.1 the set $\mathcal{P}(\mu, a)$ is nonempty and thus it contains only the measure given by (13). Hence this measure is the unique tangent measure distribution of μ at x. ■

Alternatively it is possible to show the uniqueness of the tangent measure distributions of μ by means of techniques analogous to those developed in [4] to prove the existence of order–two densities (see [10]). Then an explicit formula for P may be derived by means of the Palm formula for *unique* tangent measure distributions, which is much easier to prove than the general case (see [12]).

From Theorem 3.5 we also get

Corollary 3.6 *If (C,S) and $(\tilde{C},\tilde{S})$ are $\mathcal{C}^1$-conjugate hyperbolic Cantor sets, then P, the tangent measure distribution at μ-almost all points of C, and $\tilde{P}$, the tangent measure distribution at $\tilde{\mu}$-almost every point of $\tilde{C}$, agree.*

Finally, we infer the formula for the tangent measure distributions of the Gibbs measure ν by means of Proposition 1.2.

Corollary 3.7 *At ν-almost every point $a \in C$ the measure ν has a unique tangent measure distribution P^a. For $r > 0$ and $\lambda_r > 0$ as in Lemma 3.2 the P^a are given by*

$$P^a(A) = \frac{-1}{\int \log R_{x_0}(y)\, d\hat{\nu}((y,x))} \int \int_{\lambda_r R_{x_0}(y)}^{\lambda_r} \mathbf{1}_A \left(\frac{d\nu}{d\mu}(a) \cdot \frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s} \right) \frac{d\lambda}{\lambda}\, d\hat{\nu}((y,x))$$

where $A \subseteq \mathcal{M}(\mathbb{R})$ is chosen from the σ-field $\mathcal{B}_r$.

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