# AVERAGE DENSITIES AND LINEAR RECTIFIABILITY OF MEASURES

P. MÖRTERS

**Abstract:** We show that a measure on $\mathbb{R}^d$ is linearly rectifiable if and only if the lower 1-density is positive and finite and agrees with the lower average 1-density almost everywhere.

## 1 Introduction

Let $\mu$ be a nonnegative, nonzero Radon measure on $\mathbb{R}^d$ and $\alpha \geq 0$. The *lower $\alpha$-density* of $\mu$ at $x$ is the number

$$\underline{d}^\alpha(\mu, x) = \liminf_{t \downarrow 0} \frac{\mu(U(x,t))}{t^\alpha},$$

where $U(x,t)$ denotes the open Euclidean ball centred in $x$ of radius $t$, and the *upper $\alpha$-density* of $\mu$ at $x$ is the number

$$\overline{d}^\alpha(\mu, x) = \limsup_{t \downarrow 0} \frac{\mu(U(x,t))}{t^\alpha}.$$

The geometric regularity of the measure $\mu$ is intimately related to the behaviour of the densities. We say $\mu$ is *$\alpha$-rectifiable* if $\mu$ is absolutely continuous with respect to $\alpha$-Hausdorff measure restricted to a countable family of smooth $\alpha$-manifolds and in the case of $\alpha = 1$ we say that $\mu$ is *linearly rectifiable*. By Marstrand's Theorem (see [Mar64] or [Mat95, Chapter 14]) the equality

$$0 < \underline{d}^\alpha(\mu, x) = \overline{d}^\alpha(\mu, x) < \infty \ \mu\text{-almost everywhere} \tag{1}$$

implies that $\alpha$ must be an integer and by Preiss' Regularity Theorem (see [Pre87] or [Mat95, Chapter 17]) such a measure $\mu$ is even $\alpha$-rectifiable.

A different type of density was introduced by Bedford and Fisher in [BF92], the so called *average density* or *order-two density*. Bedford and Fisher applied a logarithmic average to the density functions and defined the *lower* and *upper average $\alpha$-density* of $\mu$ at $x$ as

$$\underline{D}^\alpha(\mu, x) = \liminf_{\varepsilon \to 0} (|\log \varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu(U(x,t))}{t^\alpha} \frac{dt}{t},$$

and

$$\overline{D}^\alpha(\mu, x) = \limsup_{\varepsilon \to 0} (|\log \varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu(U(x,t))}{t^\alpha} \frac{dt}{t}.$$

We clearly have the following inequalities:

$$\underline{d}^\alpha(\mu, x) \leq \underline{D}^\alpha(\mu, x) \leq \overline{D}^\alpha(\mu, x) \leq \overline{d}^\alpha(\mu, x).$$

It is natural to ask whether one can get statements about the geometric regularity of $\mu$ from weaker inequalities than (1), involving the average densities. This program was started by Falconer and Springer in [FS95] and their results were recently improved by Marstrand (see [Mar96]), who proved the following theorem:

**Theorem 1.1** *Suppose $\mu$ is a nonnegative, nonzero Radon measure on $\mathbb{R}^d$ and $\alpha \geq 0$ such that*

*(i)* $0 < \overline{D}^{\alpha}(\mu, x) = \overline{d}^{\alpha}(\mu, x) < \infty$ *for $\mu$-almost every $x$, or*

*(ii)* $0 < \underline{d}^{\alpha}(\mu, x) = \underline{D}^{\alpha}(\mu, x) < \infty$ *for $\mu$-almost every $x$,*

*then $\alpha$ must be an integer.*

In fact, Marstrand's proof in the case of the second condition needs the additional assumption that $\overline{d}^{\alpha}(\mu, x) < \infty$ for $\mu$-almost every $x$. This assumption is removed in [MP96] and a refinement of the argument given there is contained in Section 2 of this paper.

Do the inequalities above also imply $\alpha$-rectifiability of $\mu$? As Falconer and Springer point out, the answer is clearly no for the first inequality, even in the case $\alpha = 1$, due to an example of O'Neil (see [O'N95]). It is the aim of this paper to give the following partial answer to this question in the case of the second inequality.

**Theorem 1.2** *Suppose $\mu$ is a nonnegative Radon measure on $\mathbb{R}^d$. Then*

$$0 < \underline{d}^1(\mu, x) = \underline{D}^1(\mu, x) < \infty \quad \textit{for } \mu\textit{-almost every } x \tag{2}$$

*if and only if $\mu$ is linearly rectifiable.*

Of course, if $\mu$ is linearly rectifiable it is well known that $0 < \underline{d}^1(\mu, x) = \overline{d}^1(\mu, x) < \infty$ for almost every $x$ and therefore it only remains to prove that (2) implies linear rectifiability of $\mu$. The proof of this statement consists of two parts. In Section 2 we employ the theory of tangent measure distributions to construct, from *(ii)*, at almost every point $x$ an $\alpha$-flat tangent measure $\nu$ such that $\nu(U(0,1)) = \underline{d}^{\alpha}(\mu, x)$ and in Section 3 we finish the proof by showing that in the case $\alpha = 1$ such a tangent measure can only exist if $\mu$ is linearly rectifiable.

## 2 Existence of Flat Tangent Measures

We start by introducing the notion of tangent measures.

**Definition**
Let $\mathcal{M}(\mathbb{R}^d)$ be the set of nonnegative Radon measures on $\mathbb{R}^d$. Equipped with the vague topology, which is generated by the mappings $\mu \mapsto \int \varphi \, d\mu$, $\varphi$ continuous with compact support, $\mathcal{M}(\mathbb{R}^d)$ is a Polish space, see [Mat95, Chapter 14].

Let $0 \leq \alpha \leq d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$. For $r > 0$ define $\mu_{x,r} \in \mathcal{M}(\mathbb{R}^d)$ to be the *enlargement of $\mu$ at $x$ of factor* $1/r$, i.e. the measure defined by $\mu_{x,r}(A) = \mu(x + rA)$. The set $\mathrm{Tan}_{\alpha}(\mu, x)$ of *$\alpha$-dimensional tangent measures of $\mu$ at $x$* is defined as the set of all limit points of $\mu_{x,r}/r^{\alpha}$ in the vague topology as $r \downarrow 0$.

A (tangent) measure $\nu$ is called *$\alpha$-uniform* if, for some $c > 0$, $\nu(U(u, r)) = cr^{\alpha}$ for all $r > 0$ and $u$ in the support of $\nu$. $\nu$ is called *$\alpha$-flat* if $\alpha$ is an integer and there is a linear space $V \subseteq \mathbb{R}^d$ of dimension $\alpha$ and some $c > 0$ such that $\nu = c \cdot \mathcal{H}^{\alpha}|_V$, a multiple of the restriction of $\alpha$-Hausdorff measure to the space $V$.

Preiss introduced the notion of tangent measures in his seminal paper [Pre87], he showed that if $0 < \underline{d}^\alpha(\mu, x) = \overline{d}^\alpha(\mu, x) < \infty$ for $\mu$-almost every $x$, then at $\mu$-almost all $x$ all tangent measures of $\mu$ at $x$ are $\alpha$-flat. This implies that $\mu$ is $\alpha$-rectifiable.

In order to make quantitative statements about the set of tangent measures we introduce a family of probability distributions on $\mathrm{Tan}_\alpha(\mu, x)$, the so-called tangent measure distributions.

**Definition**
Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $0 \le \alpha \le d$. For $x \in \mathbb{R}^d$ and $\varepsilon > 0$ we define probability distributions $P_\varepsilon^x$ on $\mathcal{M}(\mathbb{R}^d)$ by

$$P_\varepsilon^x(M) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_M\Big(\frac{\mu_{x,r}}{r^\alpha}\Big)\, \frac{dr}{r} \quad \text{for Borel sets } M \subseteq \mathcal{M}(\mathbb{R}^d).$$

$\mathcal{P}^\alpha(\mu, x)$ is defined as the set of all limit points of $(P_\varepsilon^x)_{\varepsilon > 0}$ as $\varepsilon \downarrow 0$ in the weak topology, i.e. the topology generated by the mappings $P \mapsto \int F\, dP$, $F$ continuous and bounded. The elements of $\mathcal{P}^\alpha(\mu, x)$ are probability distributions on the set $\mathrm{Tan}_\alpha(\mu, x)$, they are the *$\alpha$-dimensional tangent measure distributions* of $\mu$ at $x$.

Tangent measure distributions were introduced by Bandt ([Ban92]) and Graf ([Gra95]) originally as a tool for the investigation of self-similar sets. They have also turned out to be valuable for the study of more general measures (see [Mör96], [MP96] or the thesis [Mör95]), which is due to the invariance properties described in the following theorem. For every $\lambda > 0$ we define the rescaling operator $S_\lambda^\alpha : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ by $S_\lambda^\alpha \nu(E) = (1/\lambda^\alpha) \cdot \nu(\lambda E)$ and for every $u \in \mathbb{R}^d$ we define the shift operator $T^u : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ by $T^u \nu(E) = \nu(u + E)$.

**Theorem 2.1** *Let $0 \le \alpha \le d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$.*

*(i) At every $x \in \mathbb{R}^d$ every tangent measure distribution $P \in \mathcal{P}^\alpha(\mu, x)$ fulfills*

$$P = P \circ (S_\lambda^\alpha)^{-1} \quad \textit{for all } \lambda > 0. \tag{3}$$

*(ii) At $\mu$-almost every $x$ every tangent measure distribution $P \in \mathcal{P}^\alpha(\mu, x)$ fulfills*

$$\iint G(\nu, u)\, d\nu(u)\, dP(\nu) = \iint G(T^u \nu, -u)\, d\nu(u)\, dP(\nu) \tag{4}$$

*for all Borel functions $G : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow [0, \infty)$.*

Whereas the scaling invariance property (3) is easy to check, the shift invariance property (4) is quite difficult. For a proof, an interpretation and a number of applications of the formula (4) see [MP96]. In this paper we shall make use of the properties of tangent measure distributions by means of the following lemma.

**Lemma 2.2** *Suppose that a probability measure $P$ on $\mathcal{M}(\mathbb{R}^d)$ fulfills the scaling invariance property (3) for some $0 \le \alpha \le d$ and the shift-invariance property (4) and suppose that, with some $c > 0$, $P$-almost every $\nu$ satisfies $\nu(U(0,1)) = c$. Then $\alpha$ is an integer and $P$-almost every measure $\nu$ is $\alpha$-uniform and, in particular, $\alpha$-rectifiable.*

**Proof** From (3) we infer that, for every $r > 0$,

$$\begin{aligned} P(\{\mu : \mu(U(0,r)) = cr^\alpha\}) &= P \circ (S_r^\alpha)^{-1}(\{\mu : \mu(U(0,1)) = c\}) \\ &= P(\{\mu : \mu(U(0,1)) = c\}) = 1 \, . \end{aligned}$$

Thus, using (4) in the last step,

$$\begin{aligned} 1 &= P(\{\mu : \mu(U(0,r)) = cr^\alpha\}) = \frac{1}{cs^\alpha} \int \int_{U(0,s)} \mathbf{1}_{\{\mu : \mu(U(0,r)) = cr^\alpha\}}(\nu)\, d\nu(u)\, dP(\nu) \\ &= \frac{1}{cs^\alpha} \int \int_{U(0,s)} \mathbf{1}_{\{\mu : \mu(U(u,r)) = cr^\alpha\}}(\nu)\, d\nu(u)\, dP(\nu). \end{aligned}$$

As this holds for arbitrary $r$ and $s$ we infer that, for $P$-almost every $\nu$, we have that $\nu(U(u,r)) = cr^\alpha$ for $\nu$-almost every $u$ and every rational $r > 0$. By continuity, $P$-almost every $\nu$ satisfies that $\nu(U(u,r)) = cr^\alpha$ for every $u$ in the support of $\nu$ and every $r > 0$, i.e. $\nu$ is $\alpha$-uniform. We can now use Marstrand's Theorem to conclude that $\alpha$ is an integer and Preiss' Regularity Theorem to conclude that $\nu$ is $\alpha$-rectifiable, but of course much weaker statements (like in [Kir88]) would suffice. ■

**Theorem 2.3** *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a measure such that*

$$0 < \underline{d}^\alpha(\mu, x) = \underline{D}^\alpha(\mu, x) < \infty \quad \textit{for } \mu\textit{-almost every } x.$$

*Then $\alpha$ is an integer and, at $\mu$-almost every $x$, there is an $\alpha$-flat measure $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ such that*

$$\nu(U(0,1)) = \underline{d}^\alpha(\mu, x) \, .$$

**Proof** To make use of the preceding lemma, we prove that at $\mu$-almost every $x$ there is $P \in \mathcal{P}^\alpha(\mu, x)$ such that $P$-almost every $\nu$ satisfies $\nu(U(0,1)) = \underline{d}^\alpha(\mu, x)$.
Fix $x \in \mathbb{R}^d$ such that the density condition is fulfilled and choose $\varepsilon_n \downarrow 0$ such that

$$\lim_{n\to\infty} (|\log \varepsilon_n|)^{-1} \int_{\varepsilon_n}^1 \frac{\mu(U(x,r))}{r^\alpha} \frac{dr}{r} = \underline{D}^\alpha(\mu, x) \, .$$

As, for all $m \geq 1$ and $R > 0$,

$$\begin{aligned} P_{\varepsilon_n}^x(\{\nu : \nu(U(0,m)) > Rm^\alpha\}) &\leq (1/R)\, (|\log \varepsilon_n|)^{-1} \int_{\varepsilon_n}^1 \frac{\mu(U(x,mr))}{(mr)^\alpha} \frac{dr}{r} \\ &\leq (1/R)\, (|\log \varepsilon_n|)^{-1} \int_{\varepsilon_n}^m \frac{\mu(U(x,r))}{r^\alpha} \frac{dr}{r} \end{aligned}$$

we can find, for every $\varepsilon > 0$, a sequence $R_m \uparrow \infty$ such that, for all $n$,

$$P_{\varepsilon_n}^x(\{\nu : \nu(U(0,m)) \leq R_m \text{ for all } m = 1, 2, \dots\}) > 1 - \varepsilon \, .$$

This set is compact in $\mathcal{M}(\mathbb{R}^d)$ and hence, by Prohorov's Theorem, we can choose a convergent subsequence of $(P_{\varepsilon_n}^x)$ and denote the limit $P$. As the mapping $\nu \mapsto \nu(U(0,1))$ is lower semicontinuous we have

$$\int \nu(U(0,1))\, dP(\nu) \leq \lim_{n\to\infty} (|\log \varepsilon_n|)^{-1} \int_{\varepsilon_n}^1 \frac{\mu(U(x,r))}{r^\alpha} \frac{dr}{r} = \underline{D}^\alpha(\mu, x) = \underline{d}^\alpha(\mu, x).$$

But $\nu(U(0,1)) \geq \underline{d}^\alpha(\mu,x)$ for all $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ and therefore we must have $\nu(U(0,1)) = \underline{d}^\alpha(\mu,x)$ for $P$-almost every $\nu$.

By Lemma 2.2 $\alpha$ is an integer and, for $\mu$-almost every $x$, the $P \in \mathcal{P}^\alpha(\mu,x)$ we have constructed has the property that $P$-almost every $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ is $\alpha$-rectifiable and fulfills $\nu(U(y,r)) = \underline{d}^\alpha(\mu,x) \cdot r^\alpha$ for all $r > 0$ and $y$ in the support of $\nu$. Fix such a tangent measure $\nu_0 \in \mathrm{Tan}_\alpha(\mu,x)$ and, using that $\nu_0$ is $\alpha$-rectifiable, fix $y_0$ in its support such that all tangent measures of $\nu_0$ at $y_0$ are $\alpha$-flat. Since, for $\mu$-almost every $x$, $\mathrm{Tan}_\alpha(\nu,y) \subseteq \mathrm{Tan}_\alpha(\mu,x)$ for all $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ and $y$ in the support of $\nu$ (see e.g. [Mat95, Theorem 14.16]), any tangent measure of $\nu_0$ at $y_0$ fulfills the requirements of the theorem. ∎

## 3 The Proof of Theorem 1.2

We now look at the case $\alpha = 1$ and show the following theorem.

**Theorem 3.1** *If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is a measure such that at $\mu$-almost every $x$ there is a 1-flat measure $\nu \in \mathrm{Tan}_1(\mu,x)$ with $\nu(U(0,1)) = \underline{d}^1(\mu,x)$, then $\mu$ is linearly rectifiable.*

Clearly, Theorems 2.3 and 3.1 together imply Theorem 1.2. The following lemma contains the main ingredient of the proof of Theorem 3.1.

**Lemma 3.2** *For any $0 < \xi < 1/2 - \sqrt{1/5}$ and $0 < p \leq 1$ there is an $0 < \varepsilon_0 = \varepsilon_0(\xi,p) < 1$ such that, for all $0 < \varepsilon < \varepsilon_0$, whenever $E$ is a compact subset of the interval $[a,b]$ such that the Lebesgue measure of $[a,b] \setminus E$ is at least $p \cdot (b-a)$ and every connected component of the set $[a,b] \setminus E$ has length at most $\varepsilon \cdot (b-a)$, and whenever $\nu$ is a measure on the line such that $\nu([a,b] \setminus E) \leq \varepsilon \cdot (b-a)$ and*

$$\nu((x-t,x+t)) \geq t \text{ for all } x \in E,\ 0 < t \leq 2\varepsilon \cdot (b-a),$$

*then we have*

$$\nu([a,b]) \geq (b-a) \cdot \left[\frac{1}{2} + \xi p\right] > \frac{b-a}{2}\,.$$

**Proof of Lemma 3.2.** If $\xi$ and $p$ are given, we pick a number $\varrho$ such that $1/2+\xi < \varrho < 1-\sqrt{1/5}$ and numbers $\varepsilon_0, \varepsilon_1 > 0$ such that

$$\left(\varrho - \frac{1}{2}\right) \cdot (1-\varepsilon_1)p - \frac{\varepsilon_1}{2} - 5\varepsilon_0 \geq \xi p. \tag{5}$$

Suppose that $E$ and $\nu$ are given as in the formulation of the lemma and $0 < \varepsilon < \varepsilon_0$. We denote by $\mathcal{I}$ the family of connected components of $[a,b] \setminus E$. We can pick a finite subfamily $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ such that

$$\sum_{I \in \tilde{\mathcal{I}}} |I| \geq (1-\varepsilon_1) \sum_{I \in \mathcal{I}} |I| \geq (1-\varepsilon_1)p \cdot (b-a)\,. \tag{6}$$

For every $I \in \tilde{\mathcal{I}}$ we denote by $\overline{I}$ the interval consisting of all $x \in [a,b]$ such that the distance of $x$ to $I$ is at most $|I|$. We pick any of the longest $I \in \tilde{\mathcal{I}}$ and remove all $J \in \tilde{\mathcal{I}}$ with $J \subseteq \overline{I}$

from the collection. We can go on with this procedure, always starting with one of the longest remaining intervals which has not been considered and, after a finite number of steps, we have a new collection of intervals, which we order from left to right

$$I_1 < I_2 < \cdots < I_{N-1} < I_N .$$

We now show how the statement of the lemma follows from

$$\nu\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) \geq \varrho \cdot \mathcal{L}\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) - 5\varepsilon \cdot (b-a)\,, \tag{7}$$

where $\mathcal{L}$ denotes Lebesgue measure. Using Vitali's Covering Theorem (see e.g. [Mat95, Theorem 2.2]) we can cover $\mathcal{L}$-almost all of the set $E \setminus \bigcup_{i=1}^{N} \overline{I}_i$ by countably many disjoint intervals $[x_i - r_i, x_i + r_i]$ centred in $x_i \in E$ with $r_i < \varepsilon(b-a)$, which are contained in $[a,b] \setminus \bigcup_{i=1}^{N} \overline{I}_i$. Hence

$$\nu\Big([a,b] \setminus \bigcup_{i=1}^{N} \overline{I}_i\Big) \geq \sum_{i=1}^{\infty} \nu([x_i - r_i, x_i + r_i]) \geq \sum_{i=1}^{\infty} r_i \geq \frac{1}{2} \cdot \mathcal{L}\Big(E \setminus \bigcup_{i=1}^{N} \overline{I}_i\Big)\,.$$

We also have

$$\mathcal{L}\Big(([a,b] \setminus E) \setminus \bigcup_{i=1}^{N} \overline{I}_i\Big) \leq \mathcal{L}\Big(\bigcup_{I \in \mathcal{I}} I \setminus \bigcup_{I \in \tilde{\mathcal{I}}} I\Big) \leq \varepsilon_1 \cdot (b-a).$$

Altogether we get

$$\begin{aligned}
\nu([a,b]) &= \nu\Big([a,b] \setminus \bigcup_{i=1}^{N} \overline{I}_i\Big) + \nu\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) \\
&\geq \frac{1}{2} \cdot \mathcal{L}\Big(E \setminus \bigcup_{i=1}^{N} \overline{I}_i\Big) + \varrho \cdot \mathcal{L}\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) - 5\varepsilon(b-a) \\
&\geq \frac{1}{2} \cdot \mathcal{L}\Big([a,b] \setminus \bigcup_{i=1}^{N} \overline{I}_i\Big) + \varrho \cdot \mathcal{L}\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) - \varepsilon_1(b-a)/2 - 5\varepsilon(b-a) \\
&= \frac{b-a}{2} + \Big[\varrho - \frac{1}{2}\Big] \cdot \mathcal{L}\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) - \varepsilon_1(b-a)/2 - 5\varepsilon(b-a) \\
&\geq (b-a)\Big[\frac{1}{2} + \Big(\varrho - \frac{1}{2}\Big) \cdot (1-\varepsilon_1)p - \varepsilon_1/2 - 5\varepsilon_0\Big] \\
&\geq (b-a)\Big[\frac{1}{2} + \xi p\Big]\,,
\end{aligned}$$

using $\mathcal{L}\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) \geq \sum_{I \in \tilde{\mathcal{I}}} |I| \geq (1-\varepsilon_1)p \cdot (b-a)$ in the penultimate step and (5) in the final step.

It remains to show (7). For this purpose fix some $2 \leq k \leq N$. Denote by $Z_k$ the set of points between the intervals $I_{k-1}$ and $I_k$, such that the distance to one of the intervals is less than its length. Let

$$C_k = I_{k-1} \cup Z_k \cup I_k\,.$$

We show that there are $\gamma_2, \ldots, \gamma_N$ with $\sum_{i=2}^N \gamma_i \leq \varepsilon(b-a)$ such that

$$\nu(C_k) \geq \varrho \left[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\right] - \gamma_k . \tag{8}$$

If the distance of $I_{k-1}$ and $I_k$ is larger than the sum of their lengths, then the open interval centred in the right endpoint of $I_{k-1}$ of diameter $2|I_{k-1}|$ and the open interval centred in the left endpoint of $I_k$ of diameter $2|I_k|$ are disjoint and thus, using that the endpoints of these intervals are in $E$,

$$\nu(C_k) \geq |I_{k-1}| + |I_k| = \frac{2}{3}\left[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\right] > \varrho \left[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\right].$$

We may thus suppose that $Z_k$ is an interval with $|Z_k| \leq |I_{k-1}| + |I_k|$ and that

$$\nu(C_k) < \varrho \left[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\right]. \tag{9}$$

Looking at an open interval centred in the endpoint of the larger of the intervals $I_{k-1}$ and $I_k$ we get $\nu(C_k) \geq \frac{1}{2}\Big(|I_{k-1}| + |I_k|\Big)$. Together with (9) we conclude

$$|Z_k| > \Big(\frac{1-\varrho}{2\varrho}\Big)\Big(|I_{k-1}| + |I_k|\Big). \tag{10}$$

If $|Z_k| \geq \varrho/(2-2\varrho)(|I_{k-1}| + |I_k|)$, say $|Z_k| = \lambda(|I_{k-1}| + |I_k|)$ for some $1 \geq \lambda \geq \varrho/(2-2\varrho)$, then

$$\begin{aligned} \nu(C_k) &\geq \lambda|I_{k-1}| + \lambda|I_k| \\ &= \varrho\,|Z_k| + (1-\varrho)\cdot\lambda(|I_{k-1}| + |I_k|) \\ &\geq \varrho\Big[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\Big], \end{aligned}$$

contradicting (9). Hence

$$|Z_k| < \Big(\frac{\varrho}{2-2\varrho}\Big)\Big(|I_{k-1}| + |I_k|\Big). \tag{11}$$

We know from the construction of the $I_k$ that $\zeta_k$, the centre of $C_k$, is in $Z_k$. Let us show that, with $\vartheta = 1/\varrho - 3/2$, the interval $B = (\zeta_k - \vartheta|Z_k|, \zeta_k + \vartheta|Z_k|)$ contains no point of $E$.
For, if $y \in B \cap E$, let

$$t = \frac{1}{2}\cdot\Big(|I_{k-1}| + |I_k|\Big) + \frac{1-2\vartheta}{2}|Z_k| > 0 .$$

Then $(y-t, y+t) \subseteq C_k$ and, using (9),

$$1 \leq \frac{\nu((y-t,y+t))}{t} \leq \frac{\nu(C_k)}{t} < \varrho \cdot \frac{|I_{k-1}| + |I_k| + 2|Z_k|}{|I_{k-1}| + |I_k| + (1-2\vartheta)|Z_k|},$$

and hence, using (11),

$$|Z_k| > \frac{1-\varrho}{2\varrho - 1 + 2\vartheta}\Big(|I_{k-1}| + |I_k|\Big) \geq \frac{(1-\varrho)(2-2\varrho)}{(2\varrho - 4 + 2/\varrho)\varrho}\,|Z_k| = |Z_k|,$$

a contradiction, which implies $B \cap E = \emptyset$.

Observe that $B \subseteq Z_k$, since $\zeta_k \in B \cap Z_k$ but the boundaries of $I_{k-1}$ and $I_k$ are not. Hence there is a connected component $I \in \mathcal{I}$ of $[a,b] \setminus E$ such that

$$B \subseteq I \subseteq Z_k .$$

Define

$$r_k = \min\{|I_{k-1}|, 2\vartheta|Z_k|\} \quad \text{and} \quad s_k = \min\{|I_k|, 2\vartheta|Z_k|\} .$$

As $|I| \geq 2\vartheta|Z_k|$ the intersection of the open interval centred in the right endpoint of $I_{k-1}$ of radius $r_k$ and the open interval centred in the left endpoint of $I_k$ of radius $s_k$ is contained in $I$. Moreover, they are both contained in $C_k$ and therefore

$$\nu(C_k) + \nu(I) \geq r_k + s_k . \tag{12}$$

We choose $\gamma_k = \nu(I)$. We look at three cases that might occur in the definition of $s_k$, $r_k$.

**(1)** $r_k = |I_{k-1}|$ and $s_k = |I_k|$.
From (11) and (12) we derive

$$\nu(C_k) + \gamma_k \geq |I_{k-1}| + |I_k| \geq \frac{\varrho}{2}\Big(|I_{k-1}| + |I_k|\Big) + (1-\varrho/2)\,\frac{2-2\varrho}{\varrho}\;|Z_k| ,$$

and (8) follows since $(1-\varrho/2)\,(2-2\varrho)/\varrho \geq \varrho$.

**(2)** $r_k = |I_{k-1}|$ and $s_k = 2\vartheta|Z_k|$, or $r_k = 2\vartheta|Z_k|$ and $s_k = |I_k|$.
By symmetry we concentrate on the second case. Since $B \subseteq Z_k$ we have

$$|I_{k-1}| \leq \frac{1}{2}\Big(|I_{k-1}| + |I_k| + |Z_k|\Big) - \vartheta|Z_k|$$

and thus

$$|I_k| \geq |I_{k-1}| - (1-2\vartheta)|Z_k| .$$

Using this, (12) and (11) we conclude

$$\begin{aligned} \nu(C_k) + \gamma_k &\geq 2\vartheta|Z_k| + \frac{1}{2}|I_k| + \frac{1}{2}|I_{k-1}| - (1/2-\vartheta)|Z_k| \\ &\geq \frac{\varrho}{2}\Big(|I_{k-1}| + |I_k|\Big) + \Big[(1/2-\varrho/2)\frac{2-2\varrho}{\varrho} + (3\vartheta - 1/2)\Big]\;|Z_k| , \end{aligned}$$

and (8) follows from $(1/2-\varrho/2)(2-2\varrho)/\varrho + (3\vartheta-1/2) \geq \varrho$.

**(3)** $r_k = s_k = 2\vartheta|Z_k|$.
From (12) and (10) we derive

$$\nu(C_k) + \gamma_k \geq 4\vartheta|Z_k| \geq \varrho\,|Z_k| + \Big(\frac{4}{\varrho} - 6 - \varrho\Big)\frac{1-\varrho}{2\varrho}\Big(|I_{k-1}| + |I_k|\Big) ,$$

and (8) follows since $\varrho < 1 - \sqrt{1/5}$ implies $(4/\varrho - 6 - \varrho)(1-\varrho)/(2\varrho) \geq \varrho/2$.

In all three cases we have verified (8) for a $\gamma_k$ such that the $\gamma_2, \ldots, \gamma_N$ fulfill

$$\sum_{i=2}^{N} \gamma_i \le \sum_{I \in \mathcal{I}} \nu(I) = \nu([a,b] \setminus E) \le \varepsilon(b-a) .$$

To finish the proof we estimate, taking special care of the "boundary" intervals $I_1$ and $I_N$,

$$\begin{aligned}
\nu\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) &\ge \sum_{k=2}^{N} \nu(C_k) - \sum_{k=1}^{N} \nu(I_k) \\
&\ge \varrho \cdot \Big[\sum_{k=2}^{N} \frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\Big] - \sum_{k=2}^{N} \gamma_k - \varepsilon \cdot (b-a) \\
&\ge \varrho \cdot \mathcal{L}\Big(\bigcup_{k=1}^{N} \overline{I}_k\Big) - \frac{3}{2}\Big(|I_1| + |I_N|\Big) - 2\varepsilon \cdot (b-a) \\
&\ge \varrho \cdot \mathcal{L}\Big(\bigcup_{k=1}^{N} \overline{I}_k\Big) - 5\varepsilon \cdot (b-a) ,
\end{aligned}$$

which is (7) and thus the proof is finished. ■

We now have the means to carry out the **Proof of Theorem 3.1.**
Denote by $B(x,r)$ the closed Euclidean ball of radius $r$ centred in $x$.
Suppose the statement is false. Then there is some $0 < \eta < 1$ and $\delta > 0$ such that the set

$$\{x \in \mathbb{R}^d \,:\, \delta < \underline{d}^1(\mu, x) < \eta \cdot \overline{d}^1(\mu, x)\}$$

has positive measure. Fix some $0 < \xi < 1/2 - \sqrt{1/5}$ and $1 > 1 - p > \eta$. We can then pick $0 < \varepsilon < \varepsilon_0(\xi, p)$, such that $\varepsilon < \delta$ and $\eta(1 + 5\varepsilon) < 1 - p$ and

$$\frac{1}{2}\Big(1 + \frac{\varepsilon}{\delta - \varepsilon}\Big)\Big(1 + 5\varepsilon\Big) < \frac{1}{2} + \xi p , \tag{13}$$

and we can find a compact set $F \subseteq \mathbb{R}^d$ with $\mu(F) > 0$ and numbers $0 < d < D < \infty$ with $\delta < d < \eta D$ such that there is $R > 0$ such that for all $x \in F$

- $\mu(U(x,r)) > (d - \varepsilon) r$ for all $0 < r < R$,
- there is a sequence $r_n \downarrow 0$ such that the tangent measure

$$\tilde{\mu} = \lim_{n \to \infty} \frac{\mu_{x,r_n}}{r_n}$$

  is 1-flat and $\mu(B(x, r_n)) < d \cdot r_n$, and
- there is a sequence $s_n \downarrow 0$ such that $\mu(U(x, s_n)) > D \cdot s_n$.

Using the Density Theorem (see e.g. [Mat95, Corollary 2.14]) we can fix a density point $y \in F$, i.e. a point $y \in F$ such that $\lim_{r \to 0} \mu(B(y,r) \setminus F)/\mu(B(y,r)) = 0$. Then, in particular,

$$\lim_{n \to \infty} \frac{1}{r_n} \mu(B(y, r_n) \setminus F) = 0 .$$

Denote by $L_y$ the line through $y$ such that $\tilde{\mu} = c \cdot \mathcal{H}^1|_{L_y - y}$ and by $\pi_y$ the orthogonal projection onto $L_y$. Denote by $S(y,r)$ the set of all points $x \in B(y,r)$ such that the distance of $x$ to its projection onto $L_y$ is at most $r\varepsilon/(1+5\varepsilon)$ and, similarly, denote by $S(0,1)$ the set of all $x \in B(0,1)$ such that the distance of $x$ to its projection onto $L_y - y$ is at most $\varepsilon/(1+5\varepsilon)$.Then

$$\limsup_{n\to\infty} \frac{1}{r_n}\, \mu(B(y,r_n) \setminus S(y,r_n)) = \limsup_{n\to\infty} \frac{\mu_{y,r_n}}{r_n}\, (B(0,1) \setminus S(0,1)) = 0 \,.$$

We may partition $(L_y - y) \cap B(0,1)$ into finitely many disjoint intervals $J_1, \ldots, J_k$ with $0 < |J_i| < \varepsilon/(1+5\varepsilon)$ and for each interval we can choose a nonnegative continuous function $f_i$ on $\mathbb{R}^d$ which is positive in some point of $J_i$ with support contained in the set of those points in $S(0,1)$ whose projection onto $L_y - y$ hits $J_i$. We find

$$0 < \int f_i(z)\, d\tilde{\mu}(z) = \lim_{n\to\infty} \frac{1}{r_n} \int_F f_i\Big(\frac{z-y}{r_n}\Big)\, d\mu(z)$$

and conclude that there is $N$ such that for all $n \geq N$ and every $1 \leq i \leq k$ the set $\pi_y^{-1}(y + r_n J_i) \cap S(y, r_n)$ contains points of $F$. Putting these facts together we can find $0 < r < R/4$ such that

- every connected component of the set $\pi_y(B(y,r)) \setminus \pi_y(F \cap S(y,r))$ has length less than $r \cdot 2\varepsilon/(1+5\varepsilon)$,
- there is an open set $B \supseteq B(y,r)$ such that $\mu(B) \leq d \cdot r$,
- $\mu(B(y,r) \setminus S(y,r)) \leq \dfrac{(d-\varepsilon)\cdot\varepsilon}{1+5\varepsilon} \cdot r$,
- $\mu(B(y,r) \setminus F) \leq \dfrac{(d-\varepsilon)\cdot\varepsilon}{1+5\varepsilon} \cdot r$.

Let $s = r/(1+5\varepsilon)$. We define a compact subset of the line $L_y$ by

$$E_y = \pi_y\,(F \cap S(y,r)) \cap B(y,s) \subseteq L_y \,.$$

We now show that the hypotheses of Lemma 3.2 are fulfilled with $E_y$ in the rôle of $E$ and $\pi_y(B(y,s))$ in the rôle of $[a,b]$. By choice of $s$ the connected components of $\pi_y(B(y,s)) \setminus E_y$ have length less than $\varepsilon(2s)$.

By Besicovitch's Covering Theorem (see e.g. [Mat95, Theorem 2.8]) we can cover $\mu$-almost all of $F \cap B(y,r)$ with a countable family $(B(x(i), s(i)))$ of disjoint balls centred in $F$ and contained in $B$ such that $\mu(B(x(i),s(i))) \geq D \cdot s(i)$. We infer that

$$\sum_{i=1}^{\infty} s(i) \leq \frac{\mu(B)}{D} \leq \frac{dr}{D} < \eta r \,.$$

Hence the set $E_y$ covers at most a proportion of $\eta(1+5\varepsilon) < 1-p$ of the length of $\pi_y(B(y,s))$. In other words,

$$\mathcal{L}\Big(\pi_y(B(y,s)) \setminus E_y\Big) \geq 2s(1 - \eta(1+5\varepsilon)) \geq p(2s).$$

Now define a measure on $L_y$ by

$$\nu = \frac{1}{d-\varepsilon} \cdot \mu|_{B(y,r)} \circ \pi_y^{-1} \,.$$

We have

$$\nu(\pi_y(B(y,s))\setminus E_y) \le 1/(d-\varepsilon)\cdot\Big[\mu(B(y,r)\setminus F)+\mu(B(y,r)\setminus S(y,r))\Big] \le \varepsilon(2s)\ .$$

Finally, for all $x \in E_y$ and $t \le 4\varepsilon s < R$, there is $\tilde{x} \in F \cap S(y,r) \cap \pi_y^{-1}(x)$. As $U(\tilde{x},t) \subseteq U(x,5\varepsilon s) \subseteq B(y,r)$ we get

$$\nu(U(x,t)\cap L_y) \ge \frac{1}{d-\varepsilon}\cdot\mu(U(\tilde{x},t)) \ge t\,,$$

and thus the hypotheses of Lemma 3.2 are fulfilled. Therefore

$$\nu(\pi_y(B(y,s))) \quad \ge \quad (2s)\cdot\Big[\frac{1}{2}+\xi p\Big]\,. \tag{14}$$

On the other hand, from the construction of $\nu$, we get

$$\begin{aligned}\nu(\pi_y(B(y,s))) \quad &\le \quad \frac{1}{d-\varepsilon}\cdot\mu(B(y,r)) \\ &\le \quad \frac{d}{d-\varepsilon}(1+5\varepsilon)s\,. \qquad (15)\end{aligned}$$

Now (14) and (15) together imply

$$\frac{1}{2}\Big(1+\frac{\varepsilon}{d-\varepsilon}\Big)\Big(1+5\varepsilon\Big) \ge \Big[\frac{1}{2}+\xi p\Big],$$

which contradicts (13) and finishes the proof. ∎

**Remarks:**

(i) There are alternative ways to prove Theorem 3.1 from Lemma 3.2. For example one could, instead of showing the existence of densities, use the fact that, if a set $F$ does not contain a linearly rectifiable subset of positive measure, the projections onto almost all lines have Lebesgue measure zero by the Besicovitch-Federer Projection Theorem (see [Mat95, Chapter 18]) and then apply Lemma 3.2 with $p = 1$.

(ii) Our result naturally raises the question for which dimensions (other than 1) a result like Theorem 3.1 and hence Theorem 1.2 holds. This seems to be a delicate question.

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Dr. P. Mörters
Universität Kaiserslautern
Fachbereich Mathematik
67663 Kaiserslautern
Germany.

28A75: MEASURE AND INTEGRATION: Classical measure theory; Length, area, volume, other geometric measure theory.