Functions of bounded semivariation and countably additive vector measures

P. Vieten

Fachbereich Mathematik, Universität Kaiserslautern,

Erwin-Schrödinger-Straße, 67663 Kaiserslautern, Germany

eMail: vieten@mathematik.uni-kl.de, Fax: (0631) 2053052

Abstract

In the Banach space c_0 there exists a continuous function of bounded semivariation which does not correspond to a countably additive vector measure. This result is in contrast to the scalar case, and it has consequences for the characterization of scalar-type operators. Besides this negative result we introduce the notion of functions of unconditionally bounded variation which are exactly the generators of countably additive vector measures.

1. Introduction Let Σ denote the σ-field of Borel subsets of the interval $[0,1]$. If $\mu : \Sigma \to \mathbf{C}$ is a complex Borel measure then the function $\phi : [0,1] \to \mathbf{C}$ given by

$$\phi(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ \mu([0,t)) & \text{if} \quad 0 < t < 1, \\ \mu([0,1]) & \text{if} \quad t = 1 \end{cases} \tag{1}$$

has bounded variation, since μ has bounded variation (see e.g. [7, Theorem 6.4]. Moreover, ϕ is normalized in the sense that $\phi(0) = 0$ and $\phi(t^-) = \phi(t)$ for all $0 < t < 1$.

Conversely, if $\phi : [0,1] \to \mathbf{C}$ is a normalized function of bounded variation then there exists a unique complex Borel measure μ such that ϕ and μ

are connected by (1). This can be seen as follows: For $0 \leq a < b \leq 1$ let $[a, b[$ denote the interval

$$[a,b[= \begin{cases} [a,b) & \text{if } b < 1, \\ [a,1] & \text{if } b = 1. \end{cases}$$

Let

$$\Sigma_0 = \left\{ \bigcup_{k=1}^{n} [a_k, b_k[\ : n \in \mathbf{N}, [a_k, b_k[\subseteq [0,1] \text{ pairwise disjoint} \right\}. \tag{2}$$

Then Σ_0 is a field of subsets of $[0,1]$ and we can define an additive set function μ_0 on Σ_0 by

$$\mu_0 \left(\bigcup_{k=1}^{n} [a_k, b_k[\right) = \sum_{k=1}^{n} [\phi(b_k) - \phi(a_k)].$$

Then $\phi(t) = \phi(t) - \phi(0) = \mu_0([0, t[)$ for all $0 \leq t \leq 1$. Moreover, μ_0 has a unique extension to a measure $\mu : \Sigma \to \mathbf{C}$, since Σ is the σ-field generated by Σ_0. (see e.g. [4, Theorem 1, page 358]). Hence, there exists a one-to-one correspondence between complex Borel measures on $[0,1]$ and normalized functions of bounded variation on $[0,1]$.

In this note we want to study the connection (1) for Banach-space-valued measures and functions. In the sequel, X denotes a complex Banach space, X^* its dual, and $\mathcal{L}(X)$ is the space of bounded operators on X. We recall the following definitions: Let $\tilde{\Sigma}$ be a field of subsets of a set Ω. A function $\mu : \tilde{\Sigma} \to X$ is called *vector measure* if it is an additive set function. A vector measure μ is called *countably additive* if

$$\mu \left(\bigcup_{k=1}^{\infty} E_k \right) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k)$$

for all sequences (E_k) of pairwise disjoint members of $\tilde{\Sigma}$ with $\bigcup_{k=1}^{\infty} E_k \in \tilde{\Sigma}$. If $\sum_{k=1}^{\infty} \mu(E_k)$ converges for every sequence (E_k) of pairwise disjoint members of $\tilde{\Sigma}$ then μ is called *strongly additive*. A function $\phi : [0,1] \to X$ has *bounded variation* if

$$\mathrm{Var}(\phi) := \sup \sum_{k=1}^{n} \|\phi(t_k) - \phi(t_{k-1})\|$$

is finite, where the supremum is taken over all finite sequences $0 \leq t_0 < t_1 < \ldots < t_n \leq 1$. The function ϕ has *finite semivariation* if $x^* \circ \phi$ has finite variation for all $x^* \in X^*$. We call a function ϕ of bounded semivariation to be *weakly normalized* if $x^* \circ \phi$ is normalized for all $x^* \in X^*$.

Now, let $\mu : \Sigma \to X$ be a countably additive vector measure, and define $\phi : [0,1] \to X$ by (1). Then it follows by the scalar case that ϕ is a weakly normalized function of bounded semivariation.

But, if conversely $\phi : [0,1] \to X$ is a weakly normalized function of bounded semivariation then there does not necessarily exist a countably additive vector measure μ with (1). Example 1 below shows that such a vector measure need not exist even if ϕ is a continuous function of bounded semivariation.

In the sequel we say that $\phi : [0,1] \to X$ generates the vector measure $\mu : \Sigma \to X$ if ϕ and μ are connected by (1). In section 2 we give necessary and sufficient conditions for ϕ generating a countably additive vector measure μ.

Let H be a Hilbert space and let $A \in \mathcal{L}(H)$ have its spectrum in $[0,1]$. Then the spectral theorem for self-adjoint operators asserts that A is self-adjoint if and only if A has a spectral representation

$$Ax = \int_0^1 t\, E(dt)x, \quad x \in H, \tag{3}$$

where E is a spectral measure for A (for more details we refer the reader to Dowson [3, Chapter 5 and 7]). The spectral theorem may be formulated also in terms of functions of bounded semivariation, that is: A is self-adjoint if and only if there exists a projection-valued function of bounded semivariation $\Phi : [0,1] \to \mathcal{L}(H)$, with

- $\Phi(0) = 0$, $\Phi(1) = Id$,
- $\Phi(\cdot)x$ is weakly normalized for all $x \in H$,
- Φ is increasing in the Boolean algebra of projections,

and

$$Ax = \int_0^1 t\, d\Phi(t), \quad x \in H. \tag{4}$$

We call Φ the spectral family of A, and (4) its spectral family representation. This version of the spectral theorem may be found in [6, Chapter XVI] or [8, Chapter XI].

Extending the notion of self-adjoint operators to a Banach space X, Dunford and Schwarz [5] defined A on X to be a scalar-type operator on $[0,1]$, if there exists a spectral measure E defined on the Borel subsets of $[0,1]$ such that (3) holds. Is it true that A is a scalar-type operator on $[0,1]$ if A has a spectral family representation on $[0,1]$? Since any operator with spectral family representation on $[0,1]$ admits a $C[0,1]$-functional calculus given by

$$f(A) = \int_0^1 f(t)\, d\Phi(t),$$

it follows from [3, Theorem 6.24] together with [1, Theorem VI.2.15] that the answer is positive if X does not contain an isomorphic copy of c_0. In particular, the answer is positive if X is a Hilbert space. But if c_0 is contained in X then the answer is negative, as is shown in Example 2 below.

2. Two examples First, we give an example of a continuous function of bounded semivariation, which does not generate a countably additive vector measure.

EXAMPLE 1 Define a sequence (x_n) in c_0 as follows:

$$\begin{aligned}
x_1 &= (1,0,0,0,\ldots), & x_2 &= (-1,0,0,0,\ldots),\\
x_3 &= (0,1/2,0,\ldots), & x_4 &= (0,-1/2,0,\ldots),\\
x_5 &= (0,1/2,0,\ldots), & x_6 &= (0,-1/2,0,\ldots),\\
x_7 &= (0,0,1/4,\ldots), & x_8 &= (0,0,-1/4,\ldots),\\
x_9 &= (0,0,1/4,\ldots), & x_{10} &= (0,0,-1/4,\ldots),\\
x_{11} &= (0,0,1/4,\ldots), & x_{12} &= (0,0,-1/4,\ldots),\\
x_{13} &= (0,0,1/4,\ldots), & x_{14} &= (0,0,-1/4,\ldots),
\end{aligned}$$

and so on. To be more precise,

$$x_n = (-1)^{n+1} 2^{-k+1} e_k \quad \text{for } n \in \{2^k - 1, \ldots, 2(2^k - 1)\},$$

where e_k denotes the k-th unit vector in c_0. Then the (formal) series $\sum_{n=1}^{\infty} x_n$ has the following properties:

(i) The series converges towards zero because $\left\|\sum_{n=1}^{N} x_n\right\| = 0$ if N is even, and $\left\|\sum_{n=1}^{N} x_n\right\| = 2^{-k+1}$ if N is odd and contained in $\{2^k-1, \ldots, 2(2^k-1)\}$.

(ii) The series is weakly unconditionally convergent because

$$\sum_{n=1}^{\infty} |\langle x_n, e_k\rangle| = 2 \quad \text{for all } k \in \mathbf{N}.$$

(iii) The series does not converge unconditionally: Let $\pi : \mathbf{N} \to \mathbf{N}$ be the permutation which acts on each of the sets $\{2^k - 1, 2(2^k - 1)\}$ in the following way: π "collects" first the 2^{k-1} odd numbers, and then the 2^{k-1} even numbers which are contained in this set. More precisely, put

$$\pi(n) = \begin{cases} 2^k - 1 + 2(n - 2^k + 1), & n \in \{2^k - 1, 3 \cdot 2^{k-1} - 2\}, \\ 2^k + 2(n - 3 \cdot 2^{k-1} + 1), & n \in \{3 \cdot 2^{k-1} - 1, 2(2^k - 1)\}. \end{cases}$$

Then

$$\sum_{n=1}^{3 \cdot 2^{k-1}-2} x_{\pi(n)} = e_k \quad \text{for all } k \in \mathbf{N}.$$

Hence $\sum_{n=1}^{\infty} x_n$ is not unconditionally convergent.

Now, let $\phi : [0,1] \to c_0$ be defined in the following way:

$$\phi\left(\frac{1}{2} - \frac{1}{2N+2}\right) = \sum_{n=1}^{N} x_n \quad \text{for } N \in \mathbf{N},$$
$$\phi(0) = 0, \ \phi(1/2) = 0 \text{ and } \phi(1) = 0,$$

and let ϕ be linear in the intervals $[1/2 - 1/(2N), 1/2 - 1/(2N+2)]$ for $N \in \mathbf{N}$, and in $[1/2, 1]$. The function ϕ has the following properties:

(iv) ϕ is continuous in the intervals $[0, 1/2)$ and $(1/2, 1]$. Moreover,

$$\lim_{t \to 1/2^-} \phi(t) = \lim_{N \to \infty} \sum_{n=1}^{N} x_n = 0 = \phi(1/2),$$

and $\lim_{t \to 1/2^+} \phi(t) = 0 = \phi(1/2)$. Hence ϕ is continuous in the whole interval $[0, 1]$.

(v) ϕ is of bounded semivariation because, for each natural number k, the real valued function $\phi_k = \langle e_k, \phi \rangle$, where e_k is considered as unit vector in the dual space l_1 of c_0, has variation 2.

(vi) ϕ does not define a countably additive vector measure $\mu : \Sigma \to c_0$: To prove this, assume the contrary to be true. Then $\mu([a,b)) = \phi(b) - \phi(a)$ for all $0 \leq a < b < 1$. Consequently, the series

$$\sum_{n=1}^{\infty} \mu\left(\left[\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} - \frac{1}{2n+2}\right]\right)$$

is unconditionally convergent (see [1, page 7]), which contradicts (iii).

If X contains an isomorphic copy of c_0 then there exist an operator A which has a spectral family representation on $[0,1]$, but is not of scalar type on $[0,1]$. This is the content of the next example.

Example 2 Let X contain an isomorphic copy of c_0. Our example of the operator A constructed below is taken from [2].

Since c_0 is not contained in X there exists a sequence (c_k) in X which is equivalent to the unit vector basis of c_0. Let y_n^* be the unique continuous linear functional on the closed linear span of $\{c_n : n \in \mathbf{N}\}$ which satisfies $y_n^*(c_k) = \delta_{n,k}$. Then, by the Hahn-Banach theorem, y_n^* can be extended to a norm preserving linear functional x_n^* on X. Moreover, since (c_n) is equivalent to the unit vector basis of c_0 it follows that there exists a constant $M > 0$ with $\|x_n^*\| \leq M$. Let

$$Ax = \sum_{k=1}^{\infty} \frac{x_k^*(x)}{k} c_k, \quad x \in X.$$

Then $A \in \mathcal{L}(X)$, and it follows from [3, Example 14.6] (see also [2]) that A is not a scalar-type operator.

Now, define $\Phi : [0,1] \to \mathcal{L}(X)$ by

$$\Phi(t)x = \chi_{(0,1]}(t)x + \sum_{k=1}^{\infty} \left(\chi_{(1/k,1]}(t) - \chi_{(0,1]}(t)\right) x_k^*(x) c_k, \quad 0 \leq t \leq 1. \quad (5)$$

Since $\chi_{(1/n,1]}(t) = 1$ for all $n \geq 1/t$ the series in (5) is a finite sum for all $t > 0$. Consequently, $\Phi(t)x \in X$ for all $x \in X$ and $t \in [0,1]$. By easy calculations it follows that $\Phi(t)$ is a projection, Φ is increasing in the Boolean algebra of projections, $\Phi(0) = 0$, $\Phi(1) = Id$, and the mapping $t \mapsto x^*(\phi(t)x)$ is a normalized function of bounded variation for all $x \in X$ and $x^* \in X^*$. Thus Φ is a spectral family. Moreover,

$$\int_0^1 t\, d\Phi(t)x = \sum_{k=1}^{\infty} x_k^*(x)c_k \int_0^1 t\, d\chi_{(1/k,1]}(t) = \sum_{k=1}^{\infty} \frac{1}{k} x_k^*(x)c_k = Ax.$$

Consequently, A has a spectral family representation.

3. Functions of unconditional bounded semivariation If ϕ generates a countably additive vector measure then we see from [1, page 7] that $\sum_{k=1}^{\infty}[\phi(b_k) - \phi(a_k)]$ should converge unconditionally for every choice of countably many pairwise disjoint intervals $[a_k, b_k)$. Therefore, we have the following definition.

DEFINITION 3 A function $\phi : [0,1] \to X$ is of *unconditional bounded semivariation* if for each choice of countably many pairwise disjoint intervals $[a_n, b_n) \subseteq [0,1]$ the series

$$\sum_{n=1}^{\infty}[\phi(b_n) - \phi(a_n)] \text{ is unconditionally convergent.}$$

In this section (see Theorem 5 below) we show that ϕ generates a countably additive vector measure μ if and only if ϕ is a weakly normalized function of unconditional bounded semivariation.

Before we prove Theorem 5 we show that the set of functions of unconditional bounded semivariation contains the functions of bounded variation and is contained in the set of functions of bounded semivariation. This will be immediately clear from the following

PROPOSITION 4 *Let $\phi : [0,1] \to X$ be any function.*

(i) ϕ is of bounded variation if and only if for each choice of countably many pairwise disjoint intervals $[a_n, b_n) \subseteq [0,1]$ the series

$$\sum_{n=1}^{\infty}[\phi(b_n) - \phi(a_n)] \textit{ is absolutely convergent.}$$

(ii) ϕ is of bounded semivariation if and only if for each choice of countably many pairwise disjoint intervals $[a_n, b_n) \subseteq [0,1]$ the series

$$\sum_{n=1}^{\infty} [\phi(b_n) - \phi(a_n)] \text{ is weakly unconditionally convergent.}$$

Proof. We denote by $\mathcal{I}$ the collection of all sequences $([a_n, b_n))$ of pairwise disjoint subintervals of $[0,1]$. If $I \subseteq [0,1]$ is an interval then $\mathrm{Var}_I(\phi)$ denotes the variation of ϕ on I. Note that a function ϕ has bounded variation on $[0,1]$ iff it has bounded variation on $[0,1)$.

(i) Assume that ϕ is of bounded variation. Let $([a_n, b_n)) \in \mathcal{I}$. Then, for all $N \in \mathbf{N}$, there exist $0 \leq t_0 < t_1 < \ldots < t_k < 1$ so that $\{a_n, b_n : n = 1, \ldots, N\} = \{t_j : j = 0, \ldots, k\}$. Since the $[a_k, b_k)$'s are pairwise disjoint it follows that

$$\sum_{n=1}^{N} \|\phi(b_n) - \phi(a_n)\| \leq \sum_{j=1}^{k} \|\phi(t_j) - \phi(t_{j-1})\| \leq \mathrm{Var}(\phi).$$

Assume now that ϕ does not have bounded variation. We construct first a sequence I_n of pairwise disjoint intervals. In that construction we use the following fact: If ϕ has unbounded variation on an interval K then one can find disjoint intervals $I, J \subseteq K$ with $\mathrm{Var}_I(\phi) \geq 1$ and $\mathrm{Var}_J(\phi) = \infty$.

Since ϕ has unbounded variation on $[0,1]$ we can find disjoint intervals $I_1, J_1 \subseteq [0,1)$ such that $\mathrm{Var}_{I_1}(\phi) \geq 1$ and $\mathrm{Var}_{J_1}(\phi) = \infty$. Assume $I_1, \ldots, I_n$ and $J_1, \ldots J_n$ are intervals with the following properties:

(a) $I_k \cap I_l = \emptyset$ for $k \neq l$.

(b) $I_{k+1} \subseteq J_k$ for $k = 1, \ldots, n-1$.

(c) $J_k \cap \bigcup_{l=1}^{k} I_k = \emptyset$ for $k = 1, \ldots, n$.

(d) $\mathrm{Var}(I_k) \geq 1$ for $k = 1, \ldots n$ and

(e) $\mathrm{Var}(J_k) = \infty$ for $k = 1, \ldots, n$.

If we take disjoint intervals

$$I_{n+1}, J_{n+1} \subseteq J_n \text{ with } \mathrm{Var}_{I_n}(\phi) \geq 1 \text{ and } \mathrm{Var}_{J_n}(\phi) = \infty$$

then it is easily verified that I_k, J_k, $k = 1, \ldots, n+1$ also satisfy (a)-(e).

We can now find intervals $[a_1^{(k)}, b_1^{(k)}), \ldots, [a_{n(k)}^{(k)}, b_{n(k)}^{(k)}) \subseteq I_k$ for each $k \in \mathbf{N}$ such that

$$\sum_{l=1}^{n(k)} \left\| \phi(b_l^{(k)}) - \phi(a_l^{(k)}) \right\| \geq \frac{1}{2}.$$

Consequently, the series

$$\sum_{k=1}^{\infty} \sum_{l=0}^{n(k)} \phi(b_l^{(k)}) - \phi(a_l^{(k)}) \text{ is not absolutely convergent.}$$

(ii) It follows from (i) that

$$\sum_{k=1}^{\infty} \phi(b_k) - \phi(a_k) \text{ is weakly unconditionally convergent}$$

for all sequences $[a_k, b_k) \in \mathcal{I}$ if and only if $x^* \circ \phi$ is of bounded variation for all $x^* \in X^*$. $\equiv$

THEOREM 5 *A function $\phi : [0,1] \to X$ generates a countably additive vector measure μ if and only if ϕ is a weakly normalized function of unconditional bounded semivariation.*

Proof. Let $\mu : \Sigma \to X$ be a countably additive vector measure which is generated by ϕ. Then it follows immediately that ϕ is weakly normalized. Now take a sequence of pairwise disjoint intervals $([a_n, b_n)) \in \mathcal{I}$. Then

$$\sum_{n=1}^{\infty} [\phi(b_n) - \phi(a_n)] = \sum_{n=1}^{\infty} \mu([a_n, b_n)) = \mu\left(\bigcup_{n=1}^{\infty} [a_n, b_n) \right).$$

Consequently, ϕ is of unconditional bounded semivariation.

Conversely, assume that ϕ is a weakly normalized function of unconditional bounded semivariation. Define $\mu_0 : \Sigma_0 \to X$ by

$$\mu_0 \left(\bigcup_{k=1}^{n} [a_k, b_k[\right) = \sum_{k=1}^{n} \phi(b_k) - \phi(a_k).$$

Then μ_0 is a vector measure, and $x^* \circ \mu$ is countably additive for every $x^* \in X^*$ (see the introduction).

We claim that μ_0 is strongly additive. By [1, Theorem I.1.18] it is sufficient to show that

$$\lim_{n\to\infty} \mu_0(E_n) \text{ exists}$$

for each monotone nondecreasing sequence (E_n) in Σ_0. If (E_n) is a monotone nondecreasing sequence in Σ_0 then, for each $n \in \mathbf{N}$, there exist pairwise disjoint intervals $[a_1^{(n)}, b_1^{(n)}[, \ldots, [a_{k(n)}^{(n)}, b_{k(n)}^{(n)}[$ such that

$$E_n = E_{n-1} \cup \bigcup_{k=1}^{k(n)} [a_k^{(n)}, b_k^{(n)}[\ , \text{ and } E_{n-1} \cap \bigcup_{k=1}^{k(n)} [a_k^{(n)}, b_k^{(n)}[= \emptyset,$$

where $E_0 = \emptyset$. It follows that $[a_k^{(n)}, b_k^{(n)}[$ and $[a_l^m, b_l^{(m)}[$ are disjoint if $(k,n) \neq (l,m)$. Consequently,

$$\mu_0(E_n) = \sum_{l=1}^{n} \sum_{k=1}^{k(l)} \mu_0\left([a_k^{(l)}, b_k^{(l)}[\right) = \sum_{l=1}^{n} \sum_{k=1}^{k(l)} \phi(b_k^{(l)}) - \phi(a_k^{(l)})$$

converges as $n \to \infty$, since ϕ is of unconditional bounded semivariation.

Now we are in the position to apply the Caratheodory-Hahn-Kluvanek extension theorem to μ_0 (see [1, Theorem I.5.2]). This theorem states that there exists a countably additive extension μ of μ_0 to the σ-field $\overline{\Sigma}_0$ generated by Σ_0. From $\overline{\Sigma}_0 = \Sigma$ we infer that μ is a countably additive vector measure on the Borel subsets of $[0,1]$, which is generated by ϕ. $\equiv$

REMARK 6 (i) By the Bessage-Pelczinsky theorem [1, Corollary I.4.5] every weakly unconditionally convergent series in X converges unconditionally iff X does not contain an isomorphic copy of c_0. Consequently, if X does not contain c_0 every X-valued function of bounded semivariation is of unconditional bounded semivariation. Conversely, if c_0 is contained in X then it is easy to construct a function of bounded semivariation which is not of unconditional bounded semivariation. Consequently, by Theorem 5, every X-valued function of bounded semivariation generates a countably additive vector measure if and only if c_0 is not contained in X.

(ii) We finally ask the following questions: Is it true that ϕ is of

- bounded variation if the series $\sum[\phi(t_k) - \phi(t_{k-1})]$ is absolutely convergent

- unconditional bounded semivariation if the series $\sum[\phi(t_k) - \phi(t_{k-1})]$ is unconditionally convergent
- bounded semivariation if the series $\sum[\phi(t_k) - \phi(t_{k-1})]$ is weakly unconditionally convergent

for every increasing sequence (t_k) in $[0, 1]$?

References

[1] J. Diestel J., J. J. Uhl, *Vector Measures*, American Mathematical Society, Providence, 1977.

[2] I. Doust, Well-bounded and scalar-type spectral operators on spaces not containing c_0, *Proc. Amer. Math. Soc.* **105** (1989), 367-370.

[3] H. R. Dowson, *Spectral Theory of Linear Operators*, Academic Press, London, 1978.

[4] N. Dinculeanu, *Vector Measures*, Pergamon Press, Oxford, 1967.

[5] N. Dunford, J. T. Schwartz, *Linear operators*, Part III, Wiley-Interscience, New York, 1971.

[6] H. Heuser, *Funktionalanalysis*, B.G. Teubner, Stuttgart, 1992.

[7] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Company, New York, 1987.

[8] K. Yosida, *Functional Analysis*, Springer Verlag, Berlin, 1974.

Peter Vieten
Fachbereich Mathematik
Universität Kaiserslautern
Erwin-Schrödinger-Strasse
67663 Kaiserslautern

e-mail: vieten@mathematik.uni-kl.de