MODULI SPACES OF DECOMPOSABLE MORPHISMS OF SHEAVES AND QUOTIENTS BY NON-REDUCTIVE GROUPS

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Abstract. We extend the methods of geometric invariant theory to actions of non–reductive groups in the case of homomorphisms between decomposable sheaves whose automorphism groups are non–reductive. Given a linearization of the natural action of the group Aut(E)×Aut(F) on Hom(E,F), a homomorphism is called stable if its orbit with respect to the unipotent radical is contained in the stable locus with respect to the natural reductive subgroup of the automorphism group. We encounter effective numerical conditions for a linearization such that the corresponding open set of semi-stable homomorphisms admits a good and projective quotient in the sense of geometric invariant theory, and that this quotient is in addition a geometric quotient on the set of stable homomorphisms.

Contents

1. Introduction

Let X be a projective algebraic variety over the field of complex numbers. Given two coherent sheaves $\mathcal{E},\mathcal{F}$ on X the algebraic group $G = Aut(\mathcal{E}) \times Aut(\mathcal{F})$ acts naturally on the affine space $W = \mathrm{Hom}(\mathcal{E},\mathcal{F})$ by $(g,h).w = h \circ w \circ g^{-1}$. If two morphisms are in the same G–orbit then they have isomorphic cokernels and kernels. Therefore it is natural to ask for good quotients of such actions in the sense of geometric invariant theory. However, in general $\mathcal{E}$ and $\mathcal{F}$ will be decomposable such that G is not reductive. More specifically let $\mathcal{E}$ and $\mathcal{F}$ be direct sums

$$\mathcal{E} = \bigoplus_{1\le i\le r} M_i \otimes \mathcal{E}_i \quad \text{and} \quad \mathcal{F} = \bigoplus_{1\le l\le s} N_l \otimes \mathcal{F}_l,$$

where M_i and N_l are finite dimensional vector spaces and $\mathcal{E}_i$, $\mathcal{F}_l$ are simple sheaves, i.e. their only endomorphisms are the homotheties, and such that $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for $i > j$ and $\mathrm{Hom}(\mathcal{F}_l, \mathcal{F}_m) = 0$ for $l > m$. In this case we call homomorphisms $\mathcal{E} \to \mathcal{F}$ of type (r, s). Then the groups $Aut(\mathcal{E})$ and $Aut(\mathcal{F})$ can be viewed as groups of matrices of the following type. The group $Aut(\mathcal{E})$, say, is the group of matrices

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{21} & g_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

where $g_i \in GL(M_i)$ and $u_{ji} \in \mathrm{Hom}(M_i, M_j \otimes \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j))$.

In the literature on moduli of vector bundles and coherent sheaves many quotients of spaces $\mathbb{P}\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ of type $(1, 1)$ by the reductive group $Aut(\mathcal{E}) \times Aut(\mathcal{F})$ have been investigated, see for example [4], [10], [14], [19]. In case of type (r, s) there are good and projective quotients if one restricts the action to the reductive subgroup

$$G_{red} = \prod GL(M_i) \times \prod GL(N_l).$$

This has been shown recently by A. King in [15]. The quotient problem for $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ of type (r, s) with respect to the full group $Aut(\mathcal{E}) \times Aut(\mathcal{F})$ is however the generic one and of great importance in the theory of moduli of vector bundles. The homomorphisms in a Beilinson complex of a bundle on projective n–space, for example, have in general arbitrary type (r, s) depending on the dimensions of the cohomology spaces of the bundle. If one wants to investigate moduli of such bundles, in particular for small Chern classes, it is desirable to use spaces of such homomorphisms. In several papers, see [18], [23] for example, semi–stable sheaves or ideal sheaves of subschemes of projective spaces, are represented as quotients of injective morphisms of type (r, s), and one should expect that the moduli spaces of such sheaves are isomorphic to a good quotient of an open subset of the corresponding space of homomorphisms. In some cases of type $(2, 1)$ this has been verified for semi–stable sheaves on $\mathbb{P}_2$ in [5].

Unfortunately the by now standard geometric invariant theory (GIT) doesn't provide a direct answer for these quotient problems in case $Aut(\mathcal{E}) \times Aut(\mathcal{F})$ is not reductive. There are several papers dealing with the action of an arbitrary algebraic group like [11], [12], [1], [2] and older ones, but their results are insufficient for the above problem. The conditions of [11] are close to what we need, but they don't allow a concrete description of the set of semi–stable points in our case and they don't guarantee good or projective quotients, see remark 4.2.

Our procedure is very close to standard GIT and we finally reduce the problem of the quotient to the one of a reductive group action. We first introduce polarizations $\Lambda = (\lambda_1, \ldots, \lambda_r, -\mu_1, \ldots, -\mu_s)$ for the space W, consisting of weights λ_i, μ_l for the factors $M_i \otimes \mathcal{E}_i$ and $N_l \otimes \mathcal{F}_l$, which satisfy $\sum \lambda_i \dim(M_i) = \sum \mu_l \dim(N_l)$. We use then the numerical criterion of A. King, [15], as definition for semi–stability with respect to the reductive group G_{red}. An element $w \in W$ is (G_{red}, Λ)–stable if for any proper choice of subspaces $M'_i \subset M_i, N'_l \subset N_l$ such that w maps $\oplus(M'_i \otimes \mathcal{E}_i)$ into $\oplus(N'_l \otimes \mathcal{F}_l)$, we have $\sum \lambda_i \dim(M'_i) < \sum \mu_l \dim(N'_l)$, or semi–stable if equality holds. Let $W^s(G_{red}, \Lambda) \subset W^{ss}(G_{red}, \Lambda)$ denote the set of stable and semi–stable points so defined. If $H \subset G$ is the unipotent radical of G, which is generated by the homomorphisms $\mathcal{E}_i \to \mathcal{E}_j$ and $\mathcal{F}_l \to \mathcal{F}_m$ for $i < j$ and $l < m$, we say that w is (G, Λ)–(semi–)stable if $h.w$ is (G_{red}, Λ)–(semi–)stable for any $h \in H$, see 4.1. We thus have open subsets $W^{ss}(G, \Lambda) \subset W^{ss}(G_{red}, \Lambda)$ and $W^s(G, \Lambda) \subset W^{ss}(G_{red}, \Lambda)$.

The main result of our paper is that there are sufficient numerical and effective bounds for the polarizations Λ such that $W^{ss}(G, \Lambda)$ admits a good and even projective quotient $W^{ss}(G, \Lambda)//G$ and that in addition $W^s(G, \Lambda)$ admits a geometric quotient, which is smooth and quasi–projective, see proposition 6.1 and the results 7.3, 7.7, and section 8.

All this is achieved by embedding the action $G \times W \to W$ into an action $\mathbf{G} \times \mathbf{W} \to \mathbf{W}$ of a reductive group and then imposing conditions for the equality $W^{ss}(G, \Lambda) = W \cap \mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda})$, where $\widetilde{\Lambda}$ is the associated polarization. The quotient is then the quotient of the saturated subvariety $\mathbf{G}W^{ss} \subset \mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda})$. The quotient will be projective if $\overline{\mathbf{G}.W} \smallsetminus \mathbf{G}.W$ doesn't meet $\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda})$. Also for this numerical conditions can be found in section 8.

One would expect that the quotients of W could be obtained by first forming the quotient W/H with respect to the unipotent radical H and then in a second step a quotient of W/H by $G/H \cong G_{red}$. However, the actions of unipotent groups behave generally very badly, [13], and we are not able to prove that the algebra $\mathbb{C}[W/H]$ is finitely generated. This would be an essential step in a direct construction of the quotient. Of course, the main difficulty also in this paper arises from the presence of the group H. The counterexample of M.Nagata, [21], also shows that the finite generatedness depends on the dimensions of the problem. So from a philosophical point of view we are determining bounds for the dimensions involved under which we can expect local affine G-invariant coordinate rings which are finitely generated, and thus to obtain good quasi–projective quotients, even so the bounds might not be the best. The simple example 6.2 shows that $W^s(G, \Lambda)/G$ might not be a geometric quotient if the conditions are not fulfilled.

In the last section 10 we are investigating a few examples in order to test the strength of the bounds. Here we restrict ourselves to small type $(2, 1), (2, 2), (3, 1)$ in order to avoid long computations of the constants which give the bounds for the polarizations. What we discover in varying the polarizations are flips between the moduli spaces, as one has to expect from the general results on the variation of linearizations of group actions, cf. [24], [3], [27]. In example 10.1 we have a very simple effect of a flip, but in example 10.4 the chambers of the polarizations look already very complicated.

The idea of embedding the non–reductive action $G \times W \to W$ into the action $\mathbf{G} \times \mathbf{W} \to \mathbf{W}$ is simply to replace the $\mathcal{E}_i$ by $\mathcal{E}_1$ using the evaluation maps $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i) \otimes \mathcal{E}_1 \to \mathcal{E}_i$. It is explained in 5.1 and 5.2 that this is the outcome when we start to replace the sheaves $\mathcal{E}_i$ step by step and similarly for the sheaves $\mathcal{F}_l$. Since we have to deal everywhere with the dimension of the vector spaces $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$ and $\mathrm{Hom}(\mathcal{F}_l, \mathcal{F}_m)$ which form the components of the unipotent group H, we have translated the whole setup into an abstract multilinear setting and related actions by technical reasons. This gives more general results although we have only applications in the theory of sheaves. The reader should always keep in mind the motivation in 5.1.

The paper is organized as follows: In section 2 we describe our problem in terms of multilinear algebra. In section 3 we recall results of A. King, [15]. The reductive group actions considered in this paper, the action of G_{red} on W and that of $\mathbf{G}$ on $\mathbf{W}$, are particular cases of [15]. We also discuss the relation of Λ–(semi–)stability in W with that in the projective space $\mathbb{P}W$.

But we cannot work solely on the projective niveau, because the embedding $W \subset \mathbf{W}$ is not linear. After defining G–(semi–)stability for the non–reductive group in section 4 we describe the embedding in section 5 and introduce the associated polarizations. Section 6 contains the step of constructing the quotient $W^{ss}(G,\Lambda)//G$ using the GIT–quotient $\mathbf{W}^{ss}(\mathbf{G},\tilde{\Lambda})//\mathbf{G}$ of A. King. Sections 7 and 8 are the hard parts of the paper. Here the conditions of the weights which define good polarizations are derived. It seems that the constants appearing in these estimates had not been considered before. In the most important case of the symmetric algebra some of these constants are estimated. In order to avoid further computations we restrict ourselves to the cases used in the examples in section 10.

By using correspondences between spaces of morphisms, called *mutations*, it is possible to deduce from our results other polarizations such that there exists a good projective quotient (see [7], [8]).

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2. The moduli problem for decomposable homomorphisms

Let $\mathcal{E} = \oplus\, \mathcal{E}_i \otimes M_i$ and $\mathcal{F} = \oplus\, \mathcal{F}_l \otimes N_l$ be semi–simple sheaves as in the introduction. In order to describe the action of $G = Aut(\mathcal{E}) \times Aut(\mathcal{F})$ on $W = \mathrm{Hom}(\mathcal{E},\mathcal{F})$ in greater detail we use the abbreviations

$$\begin{array}{rcl} H_{li} & = & \mathrm{Hom}(\mathcal{E}_i,\mathcal{F}_l) \\ A_{ji} & = & \mathrm{Hom}(\mathcal{E}_i,\mathcal{E}_j) \\ B_{ml} & = & \mathrm{Hom}(\mathcal{F}_l,\mathcal{F}_m)\ , \end{array}$$

such that we are given the natural pairings

$$\begin{array}{ccccclc} H_{lj} & \otimes & A_{ji} & \to & H_{li} & \text{for} & i \le j \\ A_{kj} & \otimes & A_{ji} & \to & A_{ki} & \text{for} & i \le j \le k \\ B_{ml} & \otimes & H_{li} & \to & H_{mi} & \text{for} & l \le m \\ B_{nm} & \otimes & B_{ml} & \to & B_{nl} & \text{for} & l \le m \le n. \end{array}$$

The group G consists now of pairs (g,h) of matrices

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{21} & g_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r1} & \cdots & u_{r,r-1} & g_r \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} h_1 & 0 & \cdots & 0 \\ v_{21} & h_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ v_{s1} & \cdots & v_{s,s-1} & h_s \end{pmatrix}$$

with diagonal elements $g_i \in GL(M_i),\ h_l \in GL(N_l)$ and $u_{ji} \in \mathrm{Hom}(M_i, M_j \otimes A_{ji})$, $v_{ml} \in \mathrm{Hom}(N_l, N_m \otimes B_{ml})$.

Similarly a homomorphism $w \in \mathrm{Hom}(\mathcal{E},\mathcal{F})$ is represented by a matrix $w = (\varphi_{li})$ of homomorphisms $\varphi_{li} \in \mathrm{Hom}(M_i, N_l \otimes H_{li}) = \mathrm{Hom}(H_{li}^* \otimes M_i, N_l)$. Using the natural pairings, the left action $(g,h).w = hwg^{-1}$ of G on W is described by the matrix product

$$\begin{pmatrix} h_1 & 0 & \cdots & 0 \\ v_{21} & h_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ v_{s1} & \cdots & v_{s,s-1} & h_s \end{pmatrix} \circ \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1r} \\ \vdots & & \vdots \\ \varphi_{s1} & \cdots & \varphi_{sr} \end{pmatrix} \circ \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{21} & g_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}^{-1},$$

where the compositions $v_{ml} \circ \varphi_{li}$ and $\varphi_{lj} \circ u_{ji}$ are compositions as sheaf homomorphisms but can also be interpreted as compositions induced by the pairings of the vector spaces above. Thus the group G, the space W and the action are already determined by the vector spaces A_{ji}, B_{ml}, H_{li} and the pairings between them. Therefore, in the following we define G, W and the actions $G \times W \to W$ by abstractly given vector spaces and pairings. The resulting statements can then be applied to systems of sheaves by specifying the spaces as spaces of homomorphisms as above.

2.1. *The abstract setting*

Let r, s be positive integers and let for $1 \le i \le j \le r, \quad 1 \le l \le m \le s$ finite dimensional vector spaces A_{ji}, B_{ml}, H_{li} be given, where we assume that $A_{ii} = \mathbb{C}$ and $B_{ll} = \mathbb{C}$. Moreover we suppose that we are given linear maps, called *compositions*,

$$\begin{array}{lcll} H_{lj} \otimes A_{ji} & \to & H_{li} & \text{for} \quad 1 \le i \le j \le r, \quad 1 \le l \le s \\ A_{kj} \otimes A_{ji} & \to & A_{ki} & \text{for} \quad 1 \le i \le j \le k \le r \\ B_{ml} \otimes H_{li} & \to & H_{mi} & \text{for} \quad 1 \le i \le r, \quad 1 \le l \le m \le s \\ B_{nm} \otimes B_{ml} & \to & B_{nl} & \text{for} \quad 1 \le l \le m \le n \le s. \end{array}$$

We assume that all these maps and the induced maps

$$H_{li}^* \otimes A_{ji} \to H_{lj}^* \quad \text{and} \quad H_{mi}^* \otimes B_{ml} \to H_{li}^*$$

are **surjective**. This is the case when all the spaces are spaces of sheaf homomorphisms as above for which the sheaves $\mathcal{E}_i$ and $\mathcal{F}_l$ are line bundles on a projective space or each of them is a bundle $\Omega^p(p)$.

We may and do assume that these pairings are the identities if $i = j$, $l = m$ etc. . Finally, we suppose that these maps verify the natural associative properties of compositions. This means that the induced diagrams

$$\begin{array}{ccccccc} A_{kj} \otimes A_{ji} \otimes A_{ih} & \longrightarrow & A_{ki} \otimes A_{ih} & \quad & B_{on} \otimes B_{nm} \otimes B_{ml} & \longrightarrow & B_{om} \otimes B_{nl} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A_{kj} \otimes A_{jh} & \longrightarrow & A_{kh} & & B_{on} \otimes B_{nl} & \longrightarrow & B_{ol} \end{array}$$

$$\begin{array}{ccccccc} H_{lk} \otimes A_{kj} \otimes A_{ji} & \longrightarrow & H_{lj} \otimes A_{ji} & \quad & B_{nm} \otimes B_{ml} \otimes H_{li} & \longrightarrow & B_{nl} \otimes H_{li} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{lk} \otimes A_{ki} & \longrightarrow & H_{li} & & B_{nm} \otimes H_{mi} & \longrightarrow & H_{ni} \end{array}$$

$$\begin{array}{ccc} B_{ml} \otimes H_{lj} \otimes A_{ji} & \longrightarrow & H_{mj} \otimes A_{ji} \\ \downarrow & & \downarrow \\ B_{ml} \otimes H_{li} & \longrightarrow & H_{mi} \end{array}$$

are commutative for all possible combinations of indices.

In our setup we also let finite dimensional vector spaces M_i for $1 \leq i \leq r$ and N_l for $1 \leq l \leq s$ be given and we consider finally the vector space

$$W = \bigoplus_{i,l} \operatorname{Hom}(M_i, N_l \otimes H_{li}) = \bigoplus_{i,l} \operatorname{Hom}(H_{li}^* \otimes M_i, N_l)$$

where summation is over $1 \leq i \leq r$ and $1 \leq l \leq s$. This is the space of homomorphisms in the abstract setting. The group G and its action on W are now also given in the abstract setting as follows.

2.2. *The Group G*

We define G as a product $G_L \times G_R$ of two groups where the left group G_L replaces $Aut(\mathcal{E})$ and the right group G_R replaces $Aut(\mathcal{F})$ in our motivation. Let G_L be the set of matrices

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{21} & g_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

with $g_i \in GL(M_i)$ and $u_{ji} \in \operatorname{Hom}(M_i, M_j \otimes A_{ji}) = \operatorname{Hom}(A_{ji}^* \otimes M_i, M_j)$. The group law in G_L is now defined as matrix multiplication where we define the compositions $u_{kj} * u_{ji}$ naturally according to the given pairings as the composition

$$M_i \xrightarrow{u_{ji}} M_j \otimes A_{ji} \xrightarrow{u_{kj} \otimes id} M_k \otimes A_{kj} \otimes A_{ji} \xrightarrow{id \otimes comp} M_k \otimes A_{ki}.$$

Explicitly, if g has the entries g_i, u_{ji} and g' has the entries g_i', u_{ji}' then the product

$$g'' = g'g$$

in G_L is defined as the matrix with the entries $g_i'' = g_i' \circ g_i$ in the diagonal and

$$u_{ki}'' = u_{ki}' \circ g_i + \sum_{i<j<k} u_{kj}' * u_{ji} + (g_k' \otimes id) \circ u_{ki}$$

for $1 \leq i < k \leq r$. The verification that this defines a group structure on G_L is now straightforward.

As a set G_L is the product of all the $GL(M_i)$ and all $\operatorname{Hom}(M_i, M_j \otimes A_{ji})$ for $i < j$ and thus has the structure of an affine variety. Since multiplication is composed by a system of bilinear maps it is a morphism of affine varieties. Hence G_L is naturally endowed with the structure of an algebraic group. The group G_R is now defined in the same way by replacing the spaces M_i and A_{ji} by N_l and B_{ml}. Finally $G = G_L \times G_R$ is defined as an algebraic group.

2.3. *The action of G and W*

We will define a left action of G_R and a right action of G_L on W such that the action of G on W can be defined by $(g,h).w = h.w.g^{-1}$. Both actions are defined as matrix products as described above in the case of sheaf homomorphisms using the abstract compositions as in the definition of the group law.

If w has the entries $\varphi_{li} \in \mathrm{Hom}(H_{li}^* \otimes M_i, N_l)$ and $g \in G_L$ has the entries g_i and u_{ij} then $w.g$ is defined as the matrix product

$$\begin{pmatrix} \varphi'_{11} & \cdots & \varphi'_{1r} \\ \vdots & & \vdots \\ \varphi'_{s1} & \cdots & \varphi_{sr} \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1r} \\ \vdots & & \vdots \\ \varphi_{s1} & \cdots & \varphi_{sr} \end{pmatrix} \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{21} & g_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

with

$$\varphi'_{li} = \varphi_{li} \circ g_i + \sum_{i<j} \varphi_{lj} * u_{ji} \quad \text{(if } i = r \text{ the last sum is 0)},$$

where $\varphi_{lj} * u_{ji}$ is the composition

$$M_i \to M_j \otimes A_{ji} \to N_l \otimes H_{lj} \otimes A_{ji} \to N_l \otimes H_{li}$$

or dually the composition

$$H_{li}^* \otimes M_i \to H_{lj}^* \otimes A_{ji}^* \otimes M_i \to H_{lj}^* \otimes M_j \to N_l.$$

The left action of G_R is defined in the same way. In the next two sections we give an analysis of stability and semi-stability for the action of G and its natural reductive subgroup G_{red}. In the reductive case this is due to A. King.

2.4. *Canonical subgroups of G*

We let $H_L \subset G_L$ and $H_R \subset G_R$ be the maximal normal unipotent subgroups of G_L and G_R defined by the condition that all $g_i = id_{M_i}$ and all $h_l = id_{N_l}$. Then $H = H_L \times H_R$ is a maximal normal unipotent subgroup of G. Similarly we consider the reductive subgroups $G_{L,red}$ and $G_{R,red}$ of G_L and G_R defined by the conditions $u_{ji} = 0$ and $v_{ml} = 0$ for all indices. Then $G_{red} = G_{L,red} \times G_{R,red}$ is a reductive subgroup of G and it is easy to see that $G/H \cong G_{red}$. The restricted action of G_{red} is much simpler and reduces to the natural actions of $GL(M_i)$ on M_i and $GL(N_l)$ on N_l

3. Actions of reductive groups

3.1. *Results of A. King*

Let Q be a finite set, $\Gamma \subset Q\times Q$ a subset such that the union of the images of the two projections of Γ is Q. For each $\alpha \in Q$, let m_α be a positive integer, M_α a vector space of dimension m_α and for each $(\alpha,\beta) \in \Gamma$, let $V_{\alpha\beta}$ be a finite dimensional nonzero vector space. Let

$$W_0 = \bigoplus_{(\alpha,\beta)\in\Gamma} \mathrm{Hom}(M_\alpha \otimes V_{\alpha\beta}, M_\beta).$$

On W_0 we have the following action of the reductive group

$$G_0 = \prod_{\alpha\in Q} GL(M_\alpha)$$

arising naturally in this situation. If $(f_{\beta\alpha}) \in W_0$ and $(g_\alpha) \in G_0$, then

$$(g_\alpha).(f_{\beta\alpha}) = (g_\beta \circ f_{\beta\alpha} \circ (g_\alpha \otimes id)^{-1}).$$

Let $(e_\alpha)_{\alpha\in Q}$ be a sequence of integers such that

$$\sum_{\alpha\in Q} e_\alpha m_\alpha = 0.$$

To this sequence is associated the character χ of G_0 defined by

$$\chi(g) = \prod_{\alpha\in Q} \det(g_\alpha)^{-e_\alpha}.$$

This character is trivial on the canonical subgroup of G_0 isomorphic to $\mathbb{C}^*$ (for every $\lambda \in \mathbb{C}^*$, the element (g_α) of G_0 corresponding to λ is such that $g_\alpha = \lambda.id$ for each α). This subgroup acts trivially on W_0. A point $x \in W_0$ is called χ-*semi-stable* if there exists an integer $n \geq 1$ and a polynomial $f \in \mathbb{C}[W_0]$ which is χ^n-invariant and such that $f(x) \neq 0$ (f is called χ^n-invariant if for every $w \in W_0$ and $g \in G_0$ we have $f(gw) = \chi^n(g)f(w)$). The point x is called χ-*stable* if moreover

$\dim(G_0 x) = \dim(G_0/\mathbb{C}^*)$ and if the action of G_0 on $\{w \in W_0, f(w) \neq 0\}$ is closed.

A. King proves in [15] the following results :

(1) A point $x = (f_{\beta\alpha}) \in W_0$ is χ-semi-stable (resp. χ-stable) if and only if for each family (M'_α), $\alpha \in Q$, of subspaces $M'_\alpha \subset M_\alpha$ which is neither the trivial family (0) nor the given family (M_α) and which satisfies

$$f_{\beta\alpha}(M'_\alpha \otimes V_{\alpha\beta}) \subset M'_\beta$$

for each $(\alpha,\beta) \in \Gamma$, we have

$$\sum_{\alpha\in Q} e_\alpha \dim(M'_\alpha) \ \leq\ 0 \quad (\text{resp. } < 0\ ).$$

(2) Let W_0^{ss} (resp. W_0^s) be the open subset of W_0 consisting of semi-stable (resp stable) points. Then there exists a good quotient

$$\pi : W_0^{ss} \longrightarrow M$$

by $G_0/\mathbb{C}^*$ which is a projective variety.

(3) The restriction of this quotient

$$W_0^s \longrightarrow M^s = \pi(W_0^s)$$

is a geometric quotient and M^s is smooth.

3.2. *Polarizations*

The (semi-)stable points of W_0 remain the same if we replace (e_α) by (ce_α), c being a positive integer. So the notion of (semi-)stability is fully described by the reduced parameters $(\frac{e_\alpha}{t})$, where

$$t \;=\; \sum_{\alpha\in Q, e_\alpha>0} e_\alpha m_\alpha .$$

So we can define the *polarization* of the action of G_0 on W_0 by any sequence $(c_\alpha)_{\alpha\in Q}$ of nonzero rational numbers such that

$$\sum_{\alpha\in Q} c_\alpha m_\alpha = 0 \;, \quad \sum_{\alpha\in Q, c_\alpha>0} c_\alpha m_\alpha = 1.$$

By multiplying this sequence by the smallest common denominator of the c_α we obtain a sequence (e_α) of integers and the corresponding character of G_0. Therefore the loci of stable and semi–stable points of W_0 with respect to G_0 and a polarization $\Lambda_0 = (c_\alpha)$ are well defined and denoted by

$$W_0^s(G_0,\Lambda_0) \quad \text{and} \quad W_0^{ss}(G_0,\Lambda_0).$$

3.3. *Conditions imposed by the non-emptiness of the quotient*

If W_0^s is not empty, the e_α must satisfy some conditions. We will derive this only in the three situations which occur in this paper. Polarizations satisfying these necessary conditions will be called *proper*. The first is that of the action of G_{red} in 2.4 and the second is that of $\mathbf{G}$ and $\mathbf{W}$ in section 5, and the third is the case in between occurring in 7.5.2.

3.3.1. *First case*

Let r, s be positive integers. We take

$$Q = \{\alpha_1 \ldots, \alpha_r, \beta_1, \ldots, \beta_s\}, \qquad \Gamma = \{\alpha_1 \ldots, \alpha_r\} \times \{\beta_1, \ldots, \beta_s\}.$$

This is the case of morphisms of type (r,s). For $1 \leq i \leq r$, let $M'_{\alpha_i} = M_{\alpha_i}$ if $e_{\alpha_i} > 0$, and $\{0\}$ otherwise, and for $1 \leq l \leq s$, let $M'_{\beta_l} = M_{\beta_l}$. Then if one e_{α_i} is not positive, we have

$$\sum_{\alpha\in Q} e_\alpha \dim(M'_\alpha) \;\geq\; 0$$

and $(M'_\alpha) \neq (M_\alpha)$, so in this case no point of W_0 is stable. So we obtain , if W_0^s is non-empty, the conditions

$$e_{\alpha_i} > 0 \;, \text{ for any } i, \quad \text{and} \quad e_{\beta_l} < 0 \;, \text{ for any } l.$$

A proper polarization is in this case a sequence $(\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ of rational numbers such that the λ_i and the μ_l are positive and satisfy

$$\sum_{1\leq i\leq r} \lambda_i m_{\alpha_i} = \sum_{1\leq l\leq s} \mu_l m_{\beta_l} = 1.$$

3.3.2. *Second case*

This case appears when we use a bigger reductive group to define the quotient (this is the case of $\mathbf{W}$ later on). Let r, s be positive integers. Here we take

$$Q = \{\alpha_1 \dots, \alpha_r, \beta_1, \dots, \beta_s\}, \qquad \Gamma = \{(\alpha_i, \alpha_{i-1}), 2 \leq i \leq r, (\alpha_1, \beta_s), (\beta_l, \beta_{l-1}), 2 \leq l \leq s\}.$$

Then the necessary conditions for W_0^s to be non-empty are:

$$\sum_{i\leq j\leq r} e_{\alpha_j} m_{\alpha_j} > 0 \quad \text{for any } i, \quad \text{and} \sum_{1\leq l\leq m} e_{\beta_l} m_{\beta_l} < 0 \quad \text{for any } m.$$

To derive the first set of conditions we consider for any i the family (M'_γ) for which $M'_{\alpha_j} = 0$ if $i \leq j \leq r$ and $M'_\gamma = M_\gamma$ for all other $\gamma \in Q$. Then $f_{\alpha\beta}(M'_\alpha \otimes V_{\alpha\beta}) \subset M'_\beta$ for any $f \in W_0$ and any $(\alpha, \beta) \in \Gamma$. If f is stable we obtain

$$-\sum_{i\leq j\leq r} e_{\alpha_j} m_{\alpha_j} = \sum_{\gamma\in Q} e_\gamma \dim M'_\gamma < 0$$

Moreover, if the family (M'_γ) is defined by $M'_{\alpha_j} = 0$ for $1 \leq j \leq r, M'_{\beta_l} = 0$ if $m < l \leq s$ and $M'_\gamma = M_\gamma$ else, we obtain directly

$$\sum_{1\leq l\leq m} e_{\beta_l} m_{\beta_l} = \sum_{\gamma\in Q} e_\gamma \dim M'_\gamma < 0.$$

A proper polarization in this case is then a sequence $(\rho_1, \dots, \rho_r, -\sigma_1, \dots, -\sigma_s)$ of rational numbers satisfying

$$\sum_{1\leq i\leq r} \rho_i m_{\alpha_i} = \sum_{1\leq l\leq s} \sigma_l m_{\beta_l} = 1.$$

and

$$\sum_{i\leq j\leq r} \rho_j m_{\alpha_j} > 0 \quad \text{for any } i \quad \text{and} \quad \sum_{1\leq l\leq m} \sigma_l m_{\beta_l} > 0 \quad \text{for any } m.$$

We could also drop the normalization condition.

3.3.3. *Third case*

This case is a combination of the first and second case. It appears in the proof of the equivalence of semi–stability in 7.4. Here Q is the same as in the previous cases and

$$\Gamma = \{(\alpha_i, \alpha_{i-1})\ ,\ 2 \leq i \leq r\ ,\ (\alpha_1, \beta_l)\ ,\ 1 \leq l \leq s\}.$$

Now the necessary conditions for W_0^s to be non–empty are:

$$\sum_{i\leq j\leq r} e_{\alpha_j} m_{\alpha_j} > 0 \quad \text{for any } i, \qquad \text{and} \qquad e_{\beta_l} < 0 \qquad \text{for any } l.$$

The first condition follows as in the second case when we consider the family (M'_γ) with $M'_{\alpha_j} = 0$ for $i \leq j \leq r$ and $M'_\gamma = M_\gamma$ for all other $\gamma \in Q$. The second condition follows when all M'_γ

are zero except $M'_{\beta_l} = M_{\beta_l}$ for one l. Again a proper polarization in this case is a sequence $(\rho_1, \ldots, \rho_r, -\mu_1, \ldots, -\mu_l)$ with

$$\sum_{1 \leq i \leq r} \rho_i m_{\alpha_i} = \sum_{1 \leq l \leq s} \mu_l m_{\beta_l} = 1$$

and

$$\sum_{i \leq j \leq r} \rho_j m_{\alpha_j} > 0 \quad \text{for any } i \quad \text{and } \mu_l > 0 \text{ for any } l.$$

3.4. *The action of G_0 on $\mathbb{P}(W_0)$*

We suppose that we are in one of the first two preceding cases and that there exist stable points in W_0. Let P be a nonzero homogeneous polynomial, χ^n-invariant for some positive integer n. The χ^n–invariance implies that P has degree $n.t$ where in case 1 (action of G_{red} on W)

$$t = \sum_{1 \leq i \leq r} e_{\alpha_i} m_{\alpha_i},$$

and in case 2 (action of $\mathbf{G}$ on $\mathbf{W}$)

$$t = \sum_{1 \leq i \leq r} i e_{\alpha_i} m_{\alpha_i} - \sum_{1 \leq l \leq s} (s-l) e_{\beta_l} m_{\beta_l}.$$

To see this let $\lambda \in \mathbb{C}^*$ and let g be given by $g_{\alpha_i} = \lambda^{-1} id$ and $g_{\beta_l} = id$ in the first case and by $g_{\alpha_i} = \lambda^{-i} id$ and $g_{\beta_l} = \lambda^{l-s} id$ in the second case. Then $gx = \lambda x$ and $\chi^n(g) = \lambda^{nt}$ in both cases, such that $P(\lambda x) = \lambda^{nt} p(x)$.

Now we will see that there exists a G_0-line bundle $\mathcal{L}$ on $\mathbb{P}(W_0)$ such that the set W_0^{ss} of semi-stable points is exactly the set of points over $\mathbb{P}(W_0)^{ss}(G_0, \mathcal{L})$, which is the set of semi-stable points in the sense of Geometric Invariant Theory corresponding to

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(W_0)}(t).$$

Here the action of G_0 on $\mathcal{L}$ is the natural action multiplied by χ. More precisely, the action of G_0 on W_0 induces an action of this group on $S^t W_0$ and on $S^t W_0^*$ by:

$$(g.F)(w) = F(g^{-1} w)$$

for all $g \in G_0$, $w \in W_0$ and $F \in S^t W_0^*$, viewed as an homogeneous polynomial of degree t on W_0. The line bundle space L of $\mathcal{L}$ is acted on by G_0 in the same way : if $\xi \in L_{<w>}$ then $g.\xi \in L_{<gw>}$ is the form on $< gw >^{*\otimes t} = L_{<gw>}$ given by $(g.\xi)(y) = \xi(g^{-1} y)$. We modify now the action of G_0 on L (resp. $S^t W_0^*$) by multiplying with $\chi(g)$:

$$g * \xi = \chi(g) g.\xi \quad \text{for } \xi \in L_{<w>}, \quad \text{or} \quad g * F = \chi(g) g.F \quad \text{for } F \in H^0(\mathbb{P}(W_0), \mathcal{L}) = S^t W_0^*.$$

Now $P \in H^0(\mathbb{P}(W_0), \mathcal{L}^{\otimes n})$ is an invariant section if and only if P is a homogeneous polynomial of degree tn which satisfies

$$P(gw) = \chi^n(g) P(w).$$

From the definition of semi-stable points in W_0 and $\mathbb{P}(W_0)$ with respect to the modified G_0-structure on $\mathcal{L} = \mathcal{O}_{\mathbb{P}(W_0)}(t)$, we get immediately

3.5. Lemma: *Assume that* $W_0^s(G_0, \Lambda_0) \neq \emptyset$ *and let* t *be defined as above in the two cases of* W_0. *Then the set* $W_0^{ss}(G_0, \Lambda_0)$ *is the cone of the set* $\mathbb{P}(W_0)^{ss}(G_0, \mathcal{O}_{\mathbb{P}(W_0)}(t))$ *as defined in G.I.T.*

However, the cone of the set of stable points in $\mathbb{P}(W_0)$ does not coincide with W_0^s because every point of $\mathbb{P}(W_0)$ has a stabilizer of positive dimension. In fact there is a subgroup of $G_0/\mathbb{C}^*$ of positive dimension which acts trivially on $\mathbb{P}(W_0)$. In the first case for example such a group is given by $g_{\alpha_i} = \lambda id$ and $g_{\beta_l} = \mu id$ with $\lambda, \mu \in \mathbb{C}^*$. Then the cone over $\mathbb{P}(W_0)^s(G_0/\mathbb{C}^* \times \mathbb{C}^*, \mathcal{O}_{\mathbb{P}(W_0)}(t))$ coincides with W_0^s. If we want this coincidence for one and the same group we would have to consider the action of a smaller reductive group in order to eliminate additional stabilizers. We do this only in the first case.

3.6. *The Group* G'

Let G and W be as in section 2 and let $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ be a proper polarization as in 3.3.1 for the action of G_{red} on W. It would suffice to consider the subgroup of G_{red} consisting of elements $((g_i), (h_l))$ satisfying

$$\prod_{1 \leq i \leq r} det(g_i) = \prod_{1 \leq l \leq s} det(h_l) = 1.$$

It will, however, be more convenient later to use the subgroup G'_{red} of G_{red} consisting of elements $((g_i), (h_l))$ satisfying

$$\prod_{1 \leq i \leq r} det(g_i)^{a_{i1}} = \prod_{1 \leq l \leq s} det(h_l)^{b_{sl}} = 1, \text{ where } a_{ji} = \dim A_{ji} \text{ and } b_{ml} = \dim B_{ml}.$$

We consider the action of G'_{red} on $\mathcal{L}$ induced by the modified χ-action of G_{red}. Now the set $W^s(G_{red}, \Lambda)$ of χ-stable points of W is exactly the cone over the locus $\mathbb{P}(W)^s(G'_{red}, \mathcal{L})$ of stable points of $\mathbb{P}(W)$ in the sense of Geometric Invariant Theory.

4. Semi–stability in the non–reductive case

Let G and W be as in section 2. A character χ on G_{red} as in King's setup can be extended to a character of G. Also the modified action of G_{red} on $\mathcal{L}$ can be extended to an action of G. Let G' be the subgroup of G defined by the same equations as for G'_{red}. It contains H and G'_{red}, and we have $G'/H \simeq G'_{red}$.

In the case of the action of G_{red} on W a proper polarization is given by a sequence $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s$ of positive rational numbers such that

$$\sum_{1 \leq i \leq r} \lambda_i m_i = \sum_{1 \leq l \leq s} \mu_l n_l = 1.$$

More precisely, the polarization is exactly the sequence $(\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$. The parameter λ_i (resp. μ_l) will be called the weight of the vector space M_i (resp. N_l). We see that the dimension of the set of possible proper polarizations is $r + s - 2$. Let t denote the smallest common denominator of the numbers λ_i and μ_l and χ the character of G_{red} defined by the sequence of integers $(-t\lambda_1, \ldots, -t\lambda_r, t\mu_1, \ldots, t\mu_s)$. Let

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(W)}(t) \quad \text{with} \quad t = \sum_{1 \leq i \leq r} m_i t \lambda_i.$$

As we have seen, if we consider the modified action of G_{red} on $\mathcal{L}$, then the χ-semi-stable points of W are exactly those over the semi-stable points of $\mathbb{P}(W)$ in the sense of Geometric Invariant Theory with respect to the action of $G_{red}/\mathbb{C}^*$ on $\mathcal{L}$. The χ^{tn}-invariant polynomials are the G_{red}-invariant sections of $\mathcal{L}^n$.

We are now going to define a notion of (semi–)stability for the points of W with respect to the given action of the non–reductive group G. Let $H \subset G$ be the above unipotent group, see also 2.4.

4.1. Definition: *A point $w \in W$ is called G–semi–stable (resp. G–stable) with respect to the (proper) polarization $\Lambda = (\lambda_1, \ldots, \lambda_r, -\mu_1, \ldots, -\mu_s)$ if every point of Hw is G_{red}–semi–stable (resp. G_{red}–stable) with respect to this polarization.*

We denote these sets by $W^{ss}(G, \Lambda)$ resp. $W^s(G, \Lambda)$.

4.2. Remark: In [11], semi-stability is defined as follows : A point $w \in W$ is semi-stable if there exists a positive integer k and a G'-invariant section s of $\mathcal{L}^k$ such that $s(w) \neq 0$ (there is also a condition on the action of H). It is clear that a semi-stable point in the sense of Fauntleroy is also G-semi-stable with respect to $(\lambda_1, \ldots, \lambda_r, -\mu_1, \ldots, -\mu_s)$. It is proved in [11] that there exists a *categorical quotient* of the open subset of semi-stable points in the sense of [11], but it is not clear that all G-semi-stable points are semi-stable. Moreover, in the general situation of [11] there is no way to impose conditions which would imply that the categorical quotient is a good quotient or even projective. Using definition 4.1 we are able to derive a criterion for the existence of a good and projective quotient of W under the action of G.

5. Embedding into a reductive group action

We will construct an algebraic reductive group $\mathbf{G}$, a finite dimensional vector space $\mathbf{W}$ on which $\mathbf{G}$ acts algebraically, and an injective morphism

$$\zeta : W \longrightarrow \mathbf{W}$$

compatible with a morphism of groups

$$\theta : G \longrightarrow \mathbf{G}.$$

The traces of $\mathbf{G}$-orbits on $\zeta(W)$ will be exactly the G-orbits. The space $\mathbf{W}$ is of the same type as those studied in 3.1. We will associate naturally to any polarization of the action of G on W a character χ of $\mathbf{G}/\mathbb{C}^*$, i.e. a polarization of the action of $\mathbf{G}$ on $\mathbf{W}$. We will prove that in certain cases a point w of W is G–(semi-)stable with respect to the given polarization if and only if $\zeta(w)$ is χ-(semi-)stable with respect to the associated polarization. The existence of a good and projective quotient of the open set of G-semi-stable points will follow from this.

5.1. *Motivation in terms of sheaves*

The idea for the embedding of W into a space $\mathbf{W}$ with a reductive group action is to replace the sheaves $\mathcal{E}_i$ in $\mathcal{E} = \oplus(\mathcal{E}_i \otimes M_i)$ by $\mathcal{E}_1 \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i)$ and dually the sheaves $\mathcal{F}_l$ in $\mathcal{F} = \oplus(\mathcal{F}_l \otimes N_l)$ by $\mathcal{F}_s \otimes \mathrm{Hom}(\mathcal{F}_l, \mathcal{F}_s)^*$ and then to consider the induced composed homomorphisms $\gamma(\Phi)$ for $\Phi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}) = W$

$$\mathcal{E}_1 \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}) \to \mathcal{E} \to \mathcal{F} \to \mathcal{F}_s \otimes \mathrm{Hom}(\mathcal{F}, \mathcal{F}_s)^*$$

in the bigger space $\tilde{W}$ of all homomorphisms between $\mathcal{E}_1 \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E})$ and $\mathcal{F}_s \otimes \mathrm{Hom}(\mathcal{F}, \mathcal{F}_s)^*$. This space is naturally acted on by the reductive group $\tilde{G} = GL(\mathrm{Hom}(\mathcal{E}_1, \mathcal{E})) \times GL(\mathrm{Hom}(\mathcal{F}, \mathcal{F}_s)^*)$. However it is not suitable enough for our purpose by two reasons. It does not allow enough polarizations as in section 3 for direct sums in order to have consistency of (semi-)stability and, secondly the group actions $G \times W \to W$ and $\tilde{G} \times \tilde{W} \to \tilde{W}$ don't have consistent orbits. Both insufficiencies are however eliminated when we consider the following enlargement of $\tilde{W}$. We set

$$P_i = \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}) \quad \text{and} \quad Q_l = \mathrm{Hom}(\mathcal{F}, \mathcal{F}_l)^*,$$

and introduce the auxiliary spaces

$$\mathbf{W}_L = \underset{1<i\leq r}{\oplus} \mathrm{Hom}(P_i \otimes \mathrm{Hom}(\mathcal{E}_{i-1}, \mathcal{E}_i), P_{i-1}), \quad \mathbf{W}_R = \underset{1\leq l<s}{\oplus} \mathrm{Hom}(Q_{l+1} \otimes \mathrm{Hom}(\mathcal{F}_l, \mathcal{F}_{l+1}), Q_l),$$

and define

$$\mathbf{W} = \mathbf{W}_L \oplus \mathrm{Hom}(\mathcal{E}_1 \otimes P_1, \mathcal{F}_s \otimes Q_s) \oplus \mathbf{W}_R.$$

There are distinguished elements

$$(\xi_2, \cdots, \xi_r) \in \mathbf{W}_L, \quad (\eta_1, \cdots, \eta_{s-1}) \in \mathbf{W}_R$$

whose components are the natural composition maps. The embedding of W into $\mathbf{W}$ will be defined as the affine map

$$W \xrightarrow{\zeta} \mathbf{W}, \qquad \Phi \mapsto ((\xi_2, \cdots, \xi_r), \gamma(\Phi), (\eta_1, \cdots, \eta_{s-1})),$$

where $\gamma(\Phi)$ is the above composition for a given $\Phi \in W$. The components of $\mathbf{W}_L$ and $\mathbf{W}_R$ will guarantee a compatible action of a reductive group and at the same time the possibility of choosing enough polarizations for this action.

5.2. Remark*:* One might hope to be able to do induction on r and/or s by simply replacing $M_{r-1} \otimes \mathcal{E}_{r-1} \oplus M_r \otimes \mathcal{E}_r$ by $(M_{r-1} \oplus M_r \otimes \mathrm{Hom}(\mathcal{E}_{r-1}, \mathcal{E}_r)) \otimes \mathcal{E}_{r-1}$ and keeping the other $\mathcal{E}_i$ for $i < r-1$. But then we drop the information about the Homomorphisms $\mathcal{E}_i \to \mathcal{E}_r$. Therefore we are lead to replace all $\mathcal{E}_i, i \geq 2$, by $\mathcal{E}_1$ at a time, i.e. by

$$P_1 \otimes \mathcal{E}_1 = (M_1 \oplus M_2 \otimes A_{21} \oplus \cdots \oplus M_r \otimes A_{r1}) \otimes \mathcal{E}_1,$$

where $A_{ji} = \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$. Moreover, in order to keep the information of the homomorphisms $\mathcal{E}_i \to \mathcal{E}_j$ for $2 \leq i \leq j$ we consider also the spaces

$$P_i = M_i \oplus M_{i+1} \otimes A_{i+1,j} \oplus \ldots \oplus M_r \otimes A_{ri}$$

together with the maps $P_i \otimes A_{i,i-1} \to P_{i-1}$ in the following. The reader may convince himself that only because of this the actions of the original group is compatible with the action of the bigger reductive group. It is a beautiful outcome that then we are able to compare the semi–stability with respect to related polarizations in section 7.

5.3. *The abstract definition of* $\mathbf{W}$

The above motivating definition of the space $\mathbf{W}$ can immediately be turned into the following final definition using the spaces H_{li}, A_{ji} and B_{ml} and the pairings between them. For any possible i and l we introduce the spaces

$$P_i = \underset{i\leq j\leq r}{\oplus} M_j \otimes A_{ji} \quad \text{and} \quad Q_l = \underset{1\leq m\leq l}{\oplus} N_m \otimes B^*_{lm},$$

and we denote by p_i and q_l their dimensions. For $1 < i$ and $l < s$ we let

$$P_i \otimes A_{i,i-1} \xrightarrow{\xi_i} P_{i-1} \quad \text{and} \quad Q_{l+1} \otimes B_{l+1,l} \xrightarrow{\eta_l} Q_l$$

be the canonical morphisms, defined as follows. On the component $M_j \otimes A_{ji}$ of P_i, the map ξ_i is the map

$$(M_j \otimes A_{ji}) \otimes A_{i,i-1} \longrightarrow M_j \otimes A_{j,i-1}$$

induced by the composition map of the spaces A. The map η_l is defined in the same way. As in 5.1 we set

$$\mathbf{W}_L = \underset{1<i\leq r}{\oplus} \mathrm{Hom}(P_i \otimes A_{i,i-1}, P_{i-1}), \qquad \mathbf{W}_R = \underset{1\leq l<s}{\oplus} \mathrm{Hom}(Q_{l+1} \otimes B_{l+1,l}, Q_l),$$

and

$$\mathbf{W} = \mathbf{W}_L \oplus \mathrm{Hom}(P_1, Q_s \otimes H_{s1}) \oplus \mathbf{W}_R.$$

In order to define the embedding ζ we define the operator γ as follows. Given $w = (\phi_{li}) \in W$ with $\phi_{li} \in \mathrm{Hom}(M_i, N_l \otimes H_{li})$, we let

$$\gamma(w) \in \mathrm{Hom}(P_1, Q_s \otimes H_{s1}) = \mathrm{Hom}(P_1 \otimes H^*_{s1}, Q_s)$$

be the linear map defined by the matrix $(\gamma_{li}(w))$, for which each $\gamma_{li}(w)$ is the composed linear map

$$M_i \otimes A_{i1} \longrightarrow N_l \otimes H_{li} \otimes A_{i1} \longrightarrow N_l \otimes H_{l1} \longrightarrow N_l \otimes B^*_{sl} \otimes H_{s1},$$

where the first map is induced by ϕ_{li}, the second by the composition $H_{li} \otimes A_{i1} \to H_{l1}$ and the third by the dual composition $H_{l1} \to B^*_{sl} \otimes H_{s1}$.

The map ζ can now be defined by

$$W \xrightarrow{\zeta} \mathbf{W}, \qquad w \mapsto ((\xi_2, \cdots, \xi_r), \gamma(w), (\eta_1, \cdots, \eta_{s-1})).$$

5.4. Lemma: *The linear map γ is injective and hence the morphism ζ is a closed embedding of affine schemes.*

Proof. From the surjectivity assumptions in 2.1 we find that dually the composition

$$H_{li} \longrightarrow H_{l1} \otimes A^*_{i1} \longrightarrow B^*_{sl} \otimes H_{s1} \otimes A^*_{i1}$$

is injective. Now it follows from the definition of $\gamma_{li}(w)$ that ϕ_{li} can be recovered from $\gamma_{li}(w)$, by shifting A_{i1} to its dual. □

5.5. *The new group* $\mathbf{G}$

We consider now the natural action on $\mathbf{W}$ as described in 3.1 in the general situation, where the group is

$$\mathbf{G} = \mathbf{G}_L \times \mathbf{G}_R, \quad \text{with} \;\; \mathbf{G}_L = \prod_{1\le i\le r} GL(P_i), \quad \mathbf{G}_R = \prod_{1\le l\le s} GL(Q_l).$$

To be precise, this action is described in components by

$$g_{i-1} \circ x_{i-1,i} \circ (g_i \otimes id)^{-1}, \quad h_s \circ \psi \circ (g_1 \otimes id)^{-1} \quad \text{and} \quad h_l \circ y_{l,l+1} \circ (h_{l+1} \otimes id)^{-1},$$

with

$$x_{i-1,i} \in \mathrm{Hom}(P_i \otimes A_{i,i-1}, P_{i-1}), \quad \psi \in \mathrm{Hom}(P_1 \otimes H^*_{s1}, Q_s), \quad y_{l,l+1} \in \mathrm{Hom}(Q_{l+1} \otimes B_{l+1,l}, Q_l)$$

and with

$$g_i \in GL(P_i), \quad h_l \in GL(Q_l).$$

The first and third expression describe the natural actions of $\mathbf{G}_L$ on $\mathbf{W}_L$ and of $\mathbf{G}_R$ on $\mathbf{W}_R$.

There are also natural embeddings of $G_L,\ G_R,\ G$ into $\mathbf{G}_L,\ \mathbf{G}_R,\ \mathbf{G}$ respectively. For that it is enough to describe the embedding of G_L in $\mathbf{G}_L$. Given an element $g \in G_L$,

$$g = \begin{pmatrix} g_1 & 0 & \dots & 0 \\ u_{21} & g_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r1} & \dots & u_{r,r-1} & g_r \end{pmatrix}$$

with $g_i \in GL(M_i)$ and $u_{ji} \in \mathrm{Hom}(M_i, M_j \otimes A_{ji})$ we define $\theta_{L,i}(g) \in GL(P_i)$ as the matrix

$$\theta_{L,i}(g) = \begin{pmatrix} \tilde g_i & 0 & \dots & 0 \\ \tilde u_{i+1,i} & \tilde g_{i+1} & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde u_{r,i} & \dots & \tilde u_{r,r-1} & \tilde g_r \end{pmatrix}$$

with respect to the decomposition of P_i with the following components: $\tilde g_j = g_j \otimes id$ on $M_j \otimes A_{ji}$ and for $i \le j \le k$ the map $\tilde u_{kj}$ is the composition

$$M_j \otimes A_{ji} \longrightarrow M_k \otimes A_{kj} \otimes A_{ji} \longrightarrow M_k \otimes A_{ki},$$

where the second arrow is induced by the given pairing. In case $j = i$ we have $\tilde g_i = g_i$ and $\tilde u_{ki} = u_{ki}$. Now we define the map

$$G_L \xrightarrow{\theta_L} \mathbf{G_L} \quad \text{by} \quad g \mapsto (\theta_{L,1} g, \cdots, \theta_{L,r} g).$$

It is then easy to verify that θ_L is an injective group homomorphism and defines a closed embedding of algebraic groups. With this embedding we consider G_L as a closed subgroup of $\mathbf{G}_L$. In the same way we obtain a closed embedding θ_R of $G_R \subset \mathbf{G}_R$. Finally we obtain the closed embedding $\theta = (\theta_L, \theta_R)$ of $G \subset \mathbf{G}$.

5.6. Lemma: *The subgroup $G_L \subset \mathbf{G}_L$ (respectively $G_R \subset \mathbf{G}_R$) is the stabilizer of the distinguished element $(\xi_2, \dots, \xi_r) \in \mathbf{W}_L$ (respectively $(\eta_1, \dots, \eta_{s-1}) \in \mathbf{W}_R$)*

Proof. It is enough to prove the statement only for G_L because of duality. The fact that G_L stabilizes $(\xi_2, \dots, \xi_r)$ is an easy consequence of the properties of the composition maps. The converse can be proved by induction on r. It is trivial for $r = 1$. Suppose that $r \geq 2$ and that the statement is true for $r-1$. Let $(\gamma_1, \dots, \gamma_r)$ be an element of the stabilizer of $(\xi_2, \dots, \xi_r)$. When we replace the space W by W', corresponding to the spaces $M_2, \dots, M_r$ and the same spaces N_l and similarly $\mathbf{W}_L$ by $\mathbf{W}'_L$, then $(\gamma_2, \dots, \gamma_r)$ is an element of the stabilizer of $(\xi_3, \dots, \xi_r)$, so by the induction hypothesis it belongs to G'_L and there exists an element

$$g' = \begin{pmatrix} g_2 & 0 & \cdots & 0 \\ u_{32} & g_3 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r2} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

such that $(\gamma_2, \dots, \gamma_r) = \theta'_L(g')$. Let now $\gamma_1 \in GL(P_1)$ have the components

$$M_i \otimes A_{i1} \xrightarrow{y_{ji}} M_j \otimes A_{j1} \quad \text{for all} \quad 1 \leq i, j \leq r.$$

The identity $\gamma_1 \circ \xi_2 = \xi_2 \circ \gamma_2$ then shows that $y_{ji} = 0$ for $j < i$, $y_{ii} = g_i$ for $2 \leq i$ and $y_{ji} = u_{ji}$ for $2 \leq j < i$. Now let $g_1 = y_{11}$, $u_{j1} = y_{j1}$, for $2 \leq j \leq r$, which are linear mappings $M_1 \longrightarrow M_j \otimes A_{j1}$. Then

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{21} & g_2 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

is an element of G_L and we have $(\gamma_1, \dots, \gamma_r) = \theta_L(g)$. $\square$

Remark: since the action of $\mathbf{G}_L$ on $\mathbf{W}_L$ is linear, it is clear that we have an isomorphism

$$\mathbf{G}_L/G_L \simeq \mathbf{G}_L(\xi_2, \dots, \xi_r), \quad \text{and similarly} \quad \mathbf{G}_R/G_R \simeq \mathbf{G}_R(\eta_1, \dots, \eta_{s-1}).$$

We will use this fact in section 8.

Using the associativity of the composition maps it is again easy to verify that the actions of G on W and $\mathbf{G}$ on $\mathbf{W}$ are compatible, i.e. that the diagram

$$\begin{array}{ccc} G \times W & \longrightarrow & W \\ {\scriptstyle \theta\times\zeta}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\ \mathbf{G} \times \mathbf{W} & \longrightarrow & \mathbf{W} \end{array}$$

is commutative, in which the horizontal maps are the actions. In addition we have the

5.7. Corollary: *Let $w, w' \in W$. Then w and w' are in the same G-orbit in W if and only if $\zeta(w)$ and $\zeta(w')$ are in the same $\mathbf{G}$-orbit in $\mathbf{W}$.*

Proof. It follows from the compatibility of the actions that if $g.w = w'$ in W then also $\theta(g).\zeta(w) = \zeta(w)$ in $\mathbf{W}$ by the last diagram. Conversely, if $\mathbf{g} \in \mathbf{G}$ and $\mathbf{g}.\zeta(w) = \zeta(w')$ then g stabilizes $(\xi_2, \cdots, \xi_r, \eta_1, \cdots, \eta_{s-1})$ by the definition of ζ in 5.3. By Lemma 5.6 $\mathbf{g} \in G$. ☐

5.8. *The associated polarization*

In 3.3.1 and 3.3.2 we had introduced polarizations for the different types of actions of G_{red} on W and of $\mathbf{G}$ on $\mathbf{W}$. In the following we will describe polarizations on W and $\mathbf{W}$ which are compatible with the morphism $\zeta : W \longrightarrow \mathbf{W}$. Their weight vectors are related by the following matrix equations and determine each other. The entries of the matrices are just the dimensions of the spaces A_{ji} and B_{ml}.

In the sequel we will use the following **notation**: the dimension of a vector space will be the small version of its name. So $m_i = \dim M_i$, $n_l = \dim N_l$, $p_i = dimP_i$, $q_m = \dim Q_m$ $a_{ji} = \dim A_{ji}$, $b_{ml} = \dim B_{ml}$ etc.

A proper polarization of the action of G on W is a tuple $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$, where λ_i and μ_l are positive rational numbers such that

$$\sum_{1 \le i \le r} \lambda_i m_i = \sum_{1 \le l \le s} \mu_l n_l = 1.$$

We define the new sequence of rational numbers $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$ by the conditions

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \vdots \\ \lambda_r \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{r1} & \cdots & a_{r,r-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_r \end{pmatrix}, \quad \begin{pmatrix} \mu_1 \\ \vdots \\ \vdots \\ \mu_s \end{pmatrix} = \begin{pmatrix} 1 & b_{2,1} & \cdots & b_{s1} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{s,s-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \vdots \\ \beta_s \end{pmatrix}.$$

Then we have

$$1 = \sum_{1 \le i \le r} \lambda_i m_i = \sum_{1 \le i \le r} \alpha_i p_i \quad \text{and} \quad 1 = \sum_{1 \le l \le s} \mu_l n_l = \sum_{1 \le l \le s} \beta_l q_l.$$

In particular the tuple $\tilde{\Lambda} = (\alpha_1, \cdots, \alpha_r, -\beta_1, \cdots, -\beta_s)$ is a polarization on $\mathbf{W}$ such that α_i is the weight of P_i and $-\beta_l$ the weight of Q_l. It is called the *associated polarization* on $\mathbf{W}$. It is compatible with ζ in the following sense: If $M_i' \subset M_i$, and $N_l' \subset N_l$ are linear subspaces, and if the subspaces of P_i and Q_l are defined by

$$P_i' = \bigoplus_{i \le j} M_j' \otimes A_{ji}, \quad \text{and} \quad Q_l' = \bigoplus_{l \le m} N_l' \otimes B_{ml}^*$$

respectively then we have

$$\sum_{1\le i\le r}\lambda_i m_i' = \sum_{1\le i\le r}\alpha_i p_i', \quad \text{and} \quad \sum_{1\le l\le s}\mu_l n_l' = \sum_{1\le l\le s}\beta_l q_l'.$$

If the set of stable points in $\mathbf{W}$ with respect to the associated polarization is non-empty then by 3.3.2 the weights satisfy the conditions

$$\sum_{i\le j\le r}\alpha_j p_j > 0 \quad \text{for any } i \quad \text{and} \quad \sum_{1\le l\le m}\beta_l q_l > 0 \quad \text{for any } m.$$

Equivalently the conditions may also be written as

$$\sum_{i\le j\le r}\alpha_j p_j > 0 \quad \text{for } 2\le i\le r \quad \text{and} \quad 1-\sum_{m\le l\le s}\beta_l q_l > 0 \quad \text{for } 2\le m\le s.$$

Substituting the weights of the original polarization on W, we can reformulate these conditions. In the cases treated in the examples they reduce to the following

5.8.1. *Weight conditions:*

Let W be of type (r,s) and let $\Lambda = (\lambda_1,\dots,\lambda_r,-\mu_1,\dots,-\mu_s)$ be a proper polarization of W with positive λ_i and μ_l. If the set $\mathbf{W}^s(\mathbf{G},\tilde\Lambda)$ of stable points of $\mathbf{W}$ with respect to the associated polarization $\tilde\Lambda$ is non-empty, then in case of

type $(2,1)$: $\quad \lambda_2 - a_{21}\lambda_1 > 0,$

type $(3,1)$: $\quad \lambda_3 - a_{32}\lambda_2 + (a_{32}a_{21}-a_{31})\lambda_1 > 0, \quad \lambda_1(m_1 + a_{21}m_2 + a_{31}m_3) < 1,$

type $(2,2)$: $\quad \lambda_2 - a_{21}\lambda_1 > 0, \quad \mu_1 - b_{21}\mu_2 > 0.$

5.9. *Comparison of invariant polynomials*

In the following we assume that $\tilde\Lambda = (\alpha_1,\dots,\alpha_r,-\beta_1,\dots,-\beta_s)$ is the polarization on $\mathbf{W}$ associated to the polarization $\Lambda = (\lambda_1,\dots,\lambda_r,-\mu_1,\dots,-\mu_s)$. The semi–stable locus $\mathbf{W}^{ss}(\mathbf{G},\tilde\Lambda)$ with respect to this polarization is more precisely defined by the character $\mathcal{X}$ associated to it as in 3.1. If q is lowest common denominator of $\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_s$, we have

$$\mathcal{X}(\mathbf{g}) = (\prod_{1\le i\le r} det(\mathbf{g}_i)^{-q\alpha_i})(\prod_{1\le l\le s} det(\mathbf{h}_l)^{q\beta_l})$$

for an element $\mathbf{g}\in\mathbf{G}$ with components $\mathbf{g}_i$ and $\mathbf{h}_l$. By the matrix relations between the polarizations q is also a common denominator of $\lambda_1,\dots,\lambda_r,\mu_1,\dots,\mu_s$, such that, if p denotes the lowest, we have $q = pu$ for some u. The character χ with respect to the given polarization can be defined by

$$\chi(g,h) = \prod_{1\le i\le r} det(g_i)^{-p\lambda_i}\prod_{1\le l\le s} det(h_l)^{p\mu_l},$$

where the g_i resp. h_l are the diagonal components of g resp. h, see 2.2. Now the relations between the polarizations imply by a straightforward calculation that

$$\mathcal{X}(\theta(g,h)) = \chi(g,h)^u.$$

If F is a $\mathcal{X}^m$-invariant polynomial on $\mathbf{W}$ it follows that

$$F(\zeta((g,h).w)) = F(\theta(g,h).\zeta(w)) = \chi(g,h)^{um} F(\zeta(w)),$$

i.e. that $F \circ \zeta$ is a χ^{um}-invariant polynomial on W. As a consequence we obtain the

5.10. Lemma: $\zeta^{-1}(\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda})) \subset W^{ss}(G, \Lambda)$,

i.e. if $w \in W$ and $\zeta(w)$ is $\mathbf{G}$-semi-stable in $\mathbf{W}$ with respect to the polarization $\tilde{\Lambda} = (\alpha_1, \dots, \alpha_r, -\beta_1, \dots, -\beta_s)$ then w is G-semi-stable in W with respect to the polarization $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ (in the sense of 4.1).

Proof. There exists a $\mathcal{X}^m$-invariant polynomial F on $\mathbf{W}$ such that $F(\zeta(w)) \neq 0$. Then

$$F(\zeta((g,h).w)) = F(\zeta(w)) \neq 0$$

for any element (g,h) in the unipotent subgroup $H \subset G$. This means that w is G-semi-stable. □

5.11. Remark: When we consider the subgroup $\mathbf{G}' \subset \mathbf{G}$ defined by the condition

$$\det(\mathbf{g}_1) = \det(\mathbf{h}_s) = 1,$$

we have $\theta(G') \subset \mathbf{G}'$ as follows from the definition of G' in 3.6. With respect to these groups the semi-stable points are those over the semi-stable loci in $\mathbb{P}(W)$ resp. $\mathbb{P}(\mathbf{W})$, with respect to the line bundles

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(W)}(t) \qquad \text{and} \quad \mathbf{L} = \mathcal{O}_{\mathbb{P}(\mathbf{W})}(\mathbf{t}),$$

where t and $\mathbf{t}$ is defined as in 3.4 in the different cases endowed with the modified action defined by the characters. However, we cannot compare $\mathbb{P}(W)$ and $\mathbb{P}(\mathbf{W})$ directly because the morphism ζ does not descend.

We need the analogous statement of Lemma 5.10 also in the case of stable points. For that is is more convenient to use the subspace criterion (1) of A. King in the case of G_{red} and $\mathbf{G}$. This gives also another proof in the semi–stable case.

5.12. Lemma: *With the same notation as in the previous Lemma*

$$\zeta^{-1}(\mathbf{W}^s(\mathbf{G}, \tilde{\Lambda})) \subset W^s(G, \Lambda)$$

Proof. Let $w = (\phi_{li})$ be a point of W with maps $M_i \otimes H_{li}^* \xrightarrow{\phi_{li}} N_l$ and suppose that w is not G–stable with respect to the polarization Λ. We can assume that it is not G_{red}–stable, too. Then there are linear subspaces $M_i' \subset M_i$ and $N_l' \subset N_l$ for all i and l such that the family $((M_i')), (N_l'))$ is proper and such that

$$\phi_{li}(M_i' \otimes H_{li}^*) \subset N_l' \quad \text{and} \quad \sum_i \lambda_i m_i' - \sum_l \mu_l n_l' \geq 0.$$

With these subspaces we can introduce the subspaces $P_i' \subset P_i$ and $Q_l' \subset Q_l$ as

$$P_i' = \underset{i \leq j}{\oplus} M_j' \otimes A_{ji} \quad \text{and} \quad Q_l' = \underset{m \leq l}{\oplus} N_m' \otimes B_{lm}^*.$$

They form a proper family of subspaces and satisfy

$$\xi_i(P'_i \otimes A_{i,i-1}) \subset P'_{i-1} \quad , \quad \gamma(w)(P'_1 \otimes H^*_{s1}) \subset Q'_s \quad , \quad \eta_l(Q'_{l+1} \otimes B_{l+1,l}) \subset Q'_l$$

for the possible values of i and l. But by the definition of the spaces and because $\widetilde{\Lambda}$ is the associated polarization, the formulas of 5.8 imply the dimension formula

$$\sum_i \alpha_i p'_i - \sum \beta_l q'_l = \sum_i \lambda_i m'_i - \sum_l \mu_l n'_l \geq 0.$$

This states that also $\zeta(w)$ is not $\mathbf{G}$–stable. □

In section 7 we will derive sufficient conditions for the equality

$$\zeta^{-1}(\mathbf{W}^s(\mathbf{G}, \tilde{\Lambda})) = W^s(G, \Lambda) \quad \text{and} \quad \zeta^{-1}(\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda})) = W^{ss}(G, \Lambda).$$

In the following section we show how this equality implies the existence of a good and projective quotient $W^{ss}(G, \Lambda)//G$ using the result for $\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda})//\mathbf{G}$ from Geometric Invariant Theory.

6. Construction and properties of the quotient

We keep the notation of the previous sections and let $\widetilde{\Lambda}$ be the polarization on $\mathbf{W}$ associated to the polarization Λ on W. We do not require that they are proper here, but we will do that later for the examples. In addition we introduce the saturation

$$Z = \mathbf{G}\zeta(W) \subset \mathbf{W}$$

of the image of W with respect to the action of $\mathbf{G}$.

6.1. Proposition: *Let W and $\mathbf{W}$ together with their G– and $\mathbf{G}$–structure be as in section 2 and 5, let Λ be a polarization for (W, G) and $\widetilde{\Lambda}$ be the associated polarization for $(\mathbf{W}, \mathbf{G})$.*

(1) If $\zeta^{-1}(\mathbf{W}^s(\mathbf{G}, \tilde{\Lambda})) = W^s(G, \Lambda)$, then there exists a geometric quotient $W^s(G, \Lambda) \to M^s$ of W^s by G, which is a quasi–projective nonsingular variety.

(2) If in addition $\zeta^{-1}(\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda})) = W^{ss}(G, \Lambda)$ and $(\bar{Z} \smallsetminus Z) \cap \mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda}) = \emptyset$, then there exists a good quotient $W^{ss}(G, \Lambda) \xrightarrow{\pi} M$, such that M is a normal projective variety, M^s is an open subset of M, and $W^s(G, \Lambda) \to M^s$ is the restriction of π.

We recall here the definition of a good and a geometric quotient of C.S. Seshadri, see [22], [20]. Let an algebraic group G act on an algebraic variety or algebraic scheme X. Then a pair (φ, Y) of a variety and a morphism $X \xrightarrow{\varphi} Y$ is called a good quotient if

(i) φ is G–equivariant (for the trivial action of G on Y),
(ii) φ is affine, open and surjective,
(iii) If U is an open affine subset of Y then φ^* is an isomorphism $\mathcal{O}_Y(U) \approx \mathcal{O}_Y(\varphi^{-1}U)^G$, where the latter denotes the ring of G–invariant functions,
(iv) If F_1, F_2 are disjoint closed and G–invariant subvarieties of X then $\varphi(F_1), \varphi(F_2)$ are closed and disjoint.

If in addition the fibres of φ are the orbits of the action and all have the same dimension, the quotient (φ, Y) is called a geometric quotient.

As usual we write $X//G$ for a good quotient space and X/G for a geometric quotient space.

Proof. We will prove the second statement first, assuming that the conditions of (1) and (2) are satisfied. We use the abbreviations $W^{ss} = W^{ss}(G, \Lambda), \mathbf{W}^{\mathbf{ss}} = \mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda})$ and similarly $W^s, \mathbf{W}^{\mathbf{s}}$ for the subsets of the stable points. By the result of A. King, 3.1, there exists a good projective quotient of $\mathbf{W}^{ss}$ by the reductive group $\mathbf{G}$. So there exists also a good and projective quotient of the closed invariant subvariety $\bar{Z} \cap \mathbf{W}^{ss}$ which we denote by

$$\bar{Z} \cap \mathbf{W}^{ss} \xrightarrow{\pi_0} M.$$

By assumption (2) $\mathbf{G}\zeta(W^{ss}) = Z \cap \mathbf{W}^{ss} = \bar{Z} \cap \mathbf{W}^{ss}$. We let π be the composition

$$W^{ss} \xrightarrow{\zeta} \mathbf{G}\zeta(W^{ss}) \xrightarrow{\pi_0} M.$$

We know already that M is projective. We will then verify that (π, M) is the good quotient of the proposition. We consider first the commutative diagram

$$\begin{array}{ccc} \mathbf{G} \times W^{ss} & \xrightarrow{\mu} & \mathbf{G}\zeta(W^{ss}) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi_0} \\ W^{ss} & \xrightarrow{\pi} & M \end{array} ,$$

in which p is the projection and μ is defined by $(\mathbf{g}, w) \mapsto \mathbf{g}\zeta(w)$. There is an action of G on $\mathbf{G} \times W^{ss}$ by $g.(\mathbf{g}, w) = (\mathbf{g}\theta(g)^{-1}, g.w)$ and it follows that μ is G–equivariant.

Claim: *The morphism μ is a geometric quotient of $\mathbf{G} \times W^{ss}$ by G.*

Proof of the claim: We show first that the fibres of μ are the G–orbits. So let $(\mathbf{g}, w)$, $(\mathbf{g}', w')$ be two elements in $\mathbf{G} \times W^{ss}$ such that $\mu(\mathbf{g}, w) = \mu(\mathbf{g}', w')$. Then $\zeta(w) = \mathbf{g}^{-1}\mathbf{g}'\zeta(w')$. By Lemma 5.6 $g = \mathbf{g}^{-1}\mathbf{g}' \in G$ and $g.(\mathbf{g}, w) = (\mathbf{g}', w')$. The claim will be proved if we show that μ has local sections. For this it suffices to use the remark following Lemma 5.6 and a local section of the quotient map $\mathbf{G} \to \mathbf{G}/G$.

Now we are going to verify the 4 properties of a good quotient for π. Clearly (i) is satisfied by the definition of π.

Proof of (ii). It is clear that π is surjective. If $U \subset W^{ss}$ is open, then $\pi(U) = \pi_0 \circ \mu(p^{-1}(U))$ is open because π_0 and μ are open as good quotients. The morphism π is also affine because $\pi = \pi_0 \circ \zeta$ and π_0 and ζ are affine.

Proof of (iii). Let $U \subset M$ be an affine open subset. Then

$$\mathcal{O}(U) \subset \mathcal{O}(\pi^{-1}(U))^G$$

since π is G–invariant. Conversely let $f \in \mathcal{O}(\pi^{-1}(U))^G$. The $f \circ p \in \mathcal{O}(\mathbf{G} \times \pi^{-1}(U))^G$, and since μ is a geometric quotient, $f \circ p$ descends to an $\bar{f} \in \mathcal{O}(\mu(\mathbf{G} \times \pi^{-1}(U)))$, which is $\mathbf{G}$–invariant. Now again $\bar{f}$ descends because π_0 is a good quotient. This proves equality $\mathcal{O}(U) = \mathcal{O}(\pi^{-1}(U))^G$.

Proof of (iv). Let F_1, F_2 be disjoint, closed, G–invariant subvarieties of W^{ss}. Then $p^{-1}(F_1), p^{-1}(F_2)$ are disjoint, closed and $\mathbf{G}$–invariant subvarieties of $\mathbf{G} \times W^{ss}$. Since μ is a good quotient, $\mu(p^{-1}(F_1)), \mu(p^{-1}(F_2))$ are disjoint, closed and $\mathbf{G}$–invariant in $\mathbf{G}\zeta(W^{ss})$. Finally, since π_0 is a good quotient, $\pi_0 \circ \mu(p^{-1}(F_1)), \pi_0 \circ \mu(p^{-1}(F_2))$ are disjoint and closed subvarieties of M. But $\pi_0 \circ \mu(p^{-1}(F_i)) = \pi(F_i)$, which proves (iv).

The normality of M follows from the fact that $\mathbf{G}\zeta(W^{ss})$ is smooth and π_0 is a good quotient, [20], with respect to the reductive group $\mathbf{G}$. That π becomes a geometric quotient on the open set W^s of stable points follows from the fact that the $\mathbf{G}$–orbits in $\mathbf{G}\zeta(W^s) = Z \cap \mathbf{W}^s$ intersect W^s in G–orbits. In particular the stabilizers of w in G and of $\zeta(w)$ in $\mathbf{G}$ are isomorphic, such that all orbits have the same dimension.

The proof of (1) is a modification of the above. In any case π_0 induces the geometric quotient $\bar{Z} \cap \mathbf{W}^s \xrightarrow{\pi_0} M_0$ with M_0 open in M. Now $\mathbf{G}\zeta(W^s) = Z \cap \mathbf{W}^s$ is a π_0-saturated open subset of $\bar{Z} \cap \mathbf{W}^s$, such that we obtain a geometric quotient $\mathbf{G}\zeta(W^s) \xrightarrow{\pi_0} M^s$ with $M^s \subset M_0$ open. By the same arguments as above applied to the diagram related to $\mathbf{G} \times W^s \to \mathbf{G}\zeta(W^s)$ we conclude that $W^s \xrightarrow{\pi} M^s$ is a geometric quotient. □

Remarks: 1) The idea of this proof comes from [26], and has already been used in [9] and [5].

2) If the second condition of (2) is not satisfied, we cannot even prove that $W^{ss}(G,\Lambda)$ admits a good quasi–projective quotient, because $Z \cap \mathbf{W}^{ss}$ might not be saturated. Of course the projectivity of the quotient depends on this condition.

6.2. Remark: If the assumption on the correspondence of (semi–)stability is violated then $W^s(G,\Lambda)$ need not necessarily admit a geometric quotient. For an example let W be the space of homomorphisms

$$\mathcal{O}(-2) \oplus \mathcal{O}(-1) \to \mathbb{C}^{2n} \otimes \mathcal{O}(1)$$

over $\mathbb{P}_n$ and let the homomorphism $\phi_0 \in W$ be given by the matrix

$$\begin{pmatrix} z_0^2 z_1 & z_1^2 \\ \vdots & \vdots \\ z_0^2 z_n & z_n^2 \\ z_0 z_1^2 & 0 \\ \vdots & \vdots \\ z_0 z_n^2 & 0 \end{pmatrix}$$

where the z_ν are homogeneous coordinates. The stabilizer of ϕ_0 contains $\mathbb{C}^*$ and the pairs

$$\begin{pmatrix} 1 & 0 \\ az_0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} I_n & -aI_n \\ 0 & I_n \end{pmatrix}$$

in $Aut(\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times GL(\mathbb{C}^{2n})$ and thus has dimension ≥ 2. If $\Lambda = (\lambda_1, \lambda_2, -\mu_1)$ is a polarization with $0 < \lambda_1,\ 0 < \lambda_2 < \frac{1}{2}$, then it is easy to see that ϕ_0 is Λ–stable in the sense of 4.1. For example $(m_1', m_2', n') = (0, 1, n)$ is the dimension vector of a ϕ_0–invariant choice of subspaces with $\lambda_1 m_1' + \lambda_2 m_2' - \mu_1 n' = \lambda_2 - 1/2 < 0$. There are however stable homomorphisms $\phi \in W$ with stabilizer $\mathbb{C}^*$. Therefore $W^s(G,\Lambda)/G$ can never admit the structure of a geometric quotient. In this example $\zeta(\phi_0)$ is not stable with respect to the associated polarization. We will see in 7.3 that a sufficient condition for that in the case of this W is $\lambda_2 > (n+1)\lambda_1$ or $\lambda_2 > \frac{n+1}{n+2}$ because $\lambda_1 + \lambda_2 = 1$.

6.3. *S–equivalence*

We suppose that the hypotheses of proposition 6.1 are satisfied, with polarization Λ for (W, G) and associated polarization $\widetilde{\Lambda}$ for $(\mathbf{W}, \mathbf{G})$.

It is easy to define the *Jordan-Hölder filtration* of $\mathbf{G}$-semi-stable elements of $\mathbf{W}$ with respect to $\widetilde{\Lambda}$ (cf.[15] for a more general situation). Using the preceding results we can also define a *Jordan-Hölder filtration* of a G-semi-stable element of W with respect to Λ. Let $w = (\phi_{li}) \in W^{ss}(G, \Lambda)$. Then there exist a positive integer p, an element $h \in H$ and filtrations

$$M_i^0 = \{0\} \subset M_i^1 \subset \cdots \subset M_i^p = M_i, \quad N_l^0 = \{0\} \subset N_l^1 \subset \cdots \subset N_l^p = N_l,$$

with

$$\sum_i \lambda_i \dim(M_i^j) = \sum_l \mu_l \dim(N_l^j)$$

for each j, such that $h.w = (\phi_{li})$ satisfies

$$\phi_{li}(H_{li}^* \otimes M_i^j) \subset N_l^j,$$

and that if

$$\phi_{li}^j : H_{li}^* \otimes (M_i^j / M_i^{j-1}) \longrightarrow N_l^j / N_l^{j-1}$$

is the induced morphism, then $(\phi_{li}^j)_{li}$ is G-stable with respect to Λ for any j. This filtration and h need not be unique, but p is unique and the (ϕ_{li}^j), too, up to the order and isomorphisms. Conversely, an element of W having such a filtration is G-semi-stable with respect to Λ. We say that two elements (ϕ_{li}) and (ϕ'_{li}) of $W^{ss}(G, \Lambda)$ are *S-equivalent* if they have Jordan-Hölder decompositions (ϕ_{li}^j), $({\phi'}_{li}^{j})$ respectively of the same length, and if there exists a permutation σ of $\{1, \ldots, p\}$ such that $({\phi'}_{li}^{j})$ is isomorphic to $(\phi_{li}^{\sigma(j)})$ for any j.

The following result is also easily deduced from 6.1.

6.3.1. Proposition: *Let $w, w' \in W^{ss}(G, \Lambda)$. Then $\pi(w) = \pi(w')$ if and only if w and w' are S-equivalent.*

It follows that the set of closed points of M is exactly the set of S-equivalence classes of elements of W^{ss}.

7. Comparison of semi–stability

We are going to investigate conditions for the weights of the polarizations under which a (semi–)stable point $w \in W$ is mapped to a (semi–)stable point $\zeta(w) \in \mathbf{W}$. For the estimates we need the following constants which depend on the dimensions m_i and the composition maps $H_{li} \otimes A_{i1} \to H_{l1}$.

7.1. Constants: Let $\mathcal{K}$ be the family of proper linear subspaces

$$K \subset \bigoplus_{2 \leq i} M_i \otimes A_{i1}$$

such that K is not contained in $\underset{2\le i}{\oplus} M_i' \otimes A_{i1}$ for any family $(M_i') \neq (M_i)$ of subspaces. For any l we let the map

$$\underset{2\le i}{\oplus} M_i \otimes A_{i1} \otimes H_{l1}^* \xrightarrow{\delta_l} \underset{2\le i}{\oplus} M_i \otimes H_{li}^*$$

be induced by the maps $A_{i1} \otimes H_{l1}^* \to H_{li}^*$ associated to the composition maps, which are supposed to be surjective, see 2.1. We introduce the constant

$$c_l(m_2, \dots, m_r) = \sup_{K\in\mathcal{K}} \rho_l(K) \quad \text{with} \quad \rho_l(K) = \frac{\text{codim } \delta_l(K \otimes H_{l1}^*)}{\text{codim } K}.$$

Similarly we define the constants $d_i(n_1, \dots, n_{s-1})$ in the dual situation. Let

$$\underset{l<s}{\oplus} N_l^* \otimes H_{li}^* \xleftarrow{\delta_i^\vee} \underset{l<s}{\oplus} N_l^* \otimes B_{sl} \otimes H_{si}^*$$

be induced by the maps $B_{sl} \otimes H_{si}^* \to H_{li}^*$ and let $\mathcal{L}$ be the family of proper subspaces

$$L \subset \underset{l<s}{\oplus} N_l^* \otimes B_{sl}$$

which are not contained in $\underset{l<s}{\oplus} N_l' \otimes B_{sl}$ for any family $(N_l') \neq (N_l^*)$ of subspaces. Then we define

$$d_i(n) = d_i(n_1, \dots, n_{s-1}) = \sup_{L\in\mathcal{L}} \frac{\text{codim } \delta_i^\vee(L \otimes H_{si}^*)}{\text{codim } L}.$$

7.1.1. Lemma: *If $m_i \le \bar{m}_i$ for all $i \ge 2$, then $c_l(m_2, \dots, m_r) \le c_l(\bar{m}_2, \dots, \bar{m}_r)$.*

Proof. It will be sufficient to assume that $m_i = \bar{m}_i$ for all i except one, $m_2 < \bar{m}_2$ say. Then let $\bar{M}_i$ be vector spaces of dimensions $\bar{m}_i$ and suppose that

$$\bar{M}_2 = L_2 \oplus M_2 \quad \text{and} \quad \bar{M}_i = M_i \text{ for } i \ge 3.$$

For any $K \in \mathcal{K}$ we consider the subspace

$$\bar{K} = (L_2 \otimes A_{21}) \oplus K \quad \subset \quad (\bar{M}_2 \otimes A_{21}) \oplus (\underset{2<j}{\oplus} M_j \otimes A_{j1}).$$

Then codim $barK$ = codim K and also codim $\delta_l(\bar{K} \otimes H_{l1}^*)$ = codim $\delta_l(K \otimes H_{l1}^*)$ because δ_l is a direct sum of the surjective operator $A_{j1} \otimes H_{l1}^* \to H_{l1}^*$ such that $\delta_l(L_2 \otimes A_{21} \otimes H_{l1}^*)$ equals $L_2 \otimes H_{l2}^*$ and $\delta_l(\bar{K} \otimes H_{l1}^*) = (L_2 \otimes H_{l2}^*) \oplus \delta_l(K \otimes H_{l1}^*)$. Therefore $\rho_l(K) = \rho_l(\bar{K})$. Once we have shown that also $\bar{K}$ belongs to the analogous family $\bar{\mathcal{K}}$, the Lemma is proved. To see this let $\bar{M}_2' \subset \bar{M}_2$ and $\bar{M}_i' = M_i' \subset M_i$ for $i \ge 3$ be subspaces such that

$$\bar{K} \subset \underset{2\le i}{\oplus} \bar{M}_i' \otimes A_{i,1}.$$

Then in particular

$$L_2 \otimes A_{21} \subset \bar{M}_2' \otimes A_{21}$$

and thus $L_2 \subset \bar{M}_2'$. But then $\bar{M}_2' = L_2 \oplus M_2'$ with $M_2' = \bar{M}_2' \cap M_2$ and it follows that

$$K \subset \underset{2\le i}{\oplus} M_i' \otimes A_{i1}.$$

Since $K \in \mathcal{K}$ we obtain $M_i' = M_i$ for all i and then also $\bar{M}_2' = \bar{M}_2$. ☐

7.2. *Study of the converse I*

Let $\Lambda = (\lambda_1, \dots \lambda_r, -\mu_1, \dots, -\mu_s)$ be a polarization on W and let $\tilde{\Lambda} = (\alpha_1, \dots, \alpha_r, - \beta_1, \dots, -\beta_s)$ be the associated polarization on $\mathbf{W}$. We had shown in 5.10 and 5.12 that if $w \in W$ and $\zeta(w)$ is (semi–)stable in $\mathbf{W}$ with respect to $\mathbf{G}$ and $\tilde{\Lambda}$, then so is w with respect to G and Λ. We are going to derive sufficient conditions for the converse, i.e. whether $\zeta(w)$ is (semi–)stable if w is (semi–)stable.

In the sequel we are going to use the following **notation**: Given a family $M' = (M'_i)$ of subspaces $M'_i \subset M_i$ we set

$$P_i(M') = \underset{i \le j}{\oplus} M'_j \otimes A_{ji}$$

and call a subspace $P'_i \subset P_i$ saturated if there is such a family with $P'_i = P_i(M')$. Note that in this case $\sum\limits_i \alpha_i p'_i = \sum\limits_i \lambda_i m'_i$. Similarly we introduce the spaces $Q_l(N')$ for a subfamily $N' = (N'_l)$ of (N_l) and call them saturated.

Let $w = (\phi_{li})$ be given and assume that $\zeta(w)$ is not semi–stable with respect to $\widetilde{\Lambda}$. Then there exist linear subspaces $P'_i \subset P_i$ and $Q'_l \subset Q_l$ such that

$$\xi_i(P'_i \otimes A_{i,i-1}) \subset P'_{i-1}, \quad \gamma(w)(P'_1 \otimes H^*_{s1}) \subset Q'_s, \quad \eta_l(Q'_{l+1} \otimes B_{l+1,l}) \subset Q'_l$$

and such that

$$\sum_i \alpha_i p'_i - \sum_l \beta_l q'_l > 0,$$

where as before the small characters denote the dimension of the spaces. If there were subspaces $M'_i \subset M_i$ and $N'_l \subset N_l$ with $P'_i = P_i(M')$ and $Q'_l = Q_l(N')$ as in 5.12, then $\gamma(w)(P'_1 \otimes H^*_{s1}) \subset Q'_s$ would imply that $\varphi_{li}(M'_i \otimes H^*_{li}) \subset N'_l$ and we would have

$$\sum_i \lambda_i m'_i - \sum_l \mu_l n'_l = \sum_i \alpha_i p'_i - \sum_l \beta_l q'_l > 0,$$

and w would not be semi–stable. In the following we are going to construct families M'', N'' of subspaces $M''_i \subset M_i$ and $N''_l \subset N_l$ such that $P''_i = P_i(M'')$ and $Q''_l = Q_l(N'')$ are as close to P'_i, Q'_l as possible and such that there is a useful estimate for

$$\sum_i \lambda_i m''_i - \sum_l \mu_l n''_l.$$

Step 1: We can assume that P'_i has a decomposition

$$P'_i = M'_i \oplus X_i \quad \text{in} \quad M_i \oplus (\underset{i<j}{\oplus} M_j \otimes A_{ji})$$

and such that $X_r = 0$. To derive this, we remark that for a subspace S of a direct sum $E \oplus F$ of vector spaces there exists a linear map $E \xrightarrow{u} F$ such that the isomorphism $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right)$ of $E \oplus F$ transforms S into $S' \oplus S''$, where S' is the projection of S in E and $S'' = S \cap F$. Using this and descending induction on i we can find an element $h \in H_L \subset G_L$, see 2.4, such that the truncations $\theta_{L,i}(h) \in GL(P_i)$, see 5.5, map P'_i onto a direct sum $M'_i \oplus X_i$ for any i. Since $\xi_i(P'_i \otimes A_{i,i-1}) \subset P'_{i-1}$ we easily derive that

$$\underset{i<j}{\oplus} M'_j \otimes A_{ji} \ \subset \ X_i \ \subset \ \underset{i<j}{\oplus} M_j \otimes A_{ji}$$

for all possible i. We put

$$\rho_i = \operatorname{codim}(\bigoplus_{i<j} M'_j \otimes A_{ji}, X_i) = \operatorname{codim}(P_i(M'), P'_i).$$

Note that $\rho_r = 0$.

Step 2: Let $M''_1, \dots M''_r$ be subspaces of $M_1, \dots, M_r$ respectively such that

$$P_i(M'') \supset P'_i$$

is minimal over P'_i for any i. Then $M'_i \subset M''_i$ since these spaces are the first components of $P'_i \subset P_i(M'')$ respectively and we have $M'_1 = M''_1$. We let

$$\sigma_i = \sum_{i\leq j}(m''_j - m'_j)a_{ji} = \operatorname{codim}(P_i(M'), P_i(M'')).$$

Step 3: We are going to define the subspaces $N'_l \subset N''_l \subset N_l$ as images.

Let $P_1 \otimes H^*_{l1} \xrightarrow{\gamma_l(w)} N_l$ be the map which is the sum of the composed maps

$$M_i \otimes A_{i1} \otimes H^*_{l1} \to M_i \otimes H^*_{li} \xrightarrow{\phi_{li}} N_l.$$

Then we define

$$N'_l = \gamma_l(w)(P'_1 \otimes H^*_{l1}) = \phi_{l1}(M'_1 \otimes H^*_{l1}) + \gamma_l(w)(X_1 \otimes H^*_{l1})$$

and

$$N''_l = \gamma_l(w)(P_1(M'') \otimes H^*_{l1}) = \phi_{l1}(M''_1 \otimes H^*_{l1}) + \sum_{2\leq j} \phi_{lj}(M''_j \otimes H^*_{lj}).$$

It follows $N'_l \subset N''_l$ for any l.

Step 4: If the weights β_l are supposed to be positive, we may assume that

$$\gamma(w)(P'_1 \otimes H^*_{s1}) = Q'_s \quad \text{and} \quad \eta_l(Q'_{l+1} \otimes B_{l+1,l}) = Q'_l$$

for $l < s$. Otherwise we could choose subspaces $\bar{Q}'_l \subset Q'_l$ by descending induction as images. Then $-\sum_l \beta_l \bar{q}'_l \geq -\sum \beta_l q'_l$ would improve the assumption on the choice of the spaces P'_i and Q'_l. Now it follows that for any l

$$Q'_l \subset Q_l(N'')$$

because $P'_1 \otimes H^*_{s1}$ is mapped to $\bigoplus_{l\leq s} N''_l \otimes B^*_{sl}$ and the maps η_l are the identity on the spaces N''_m. Note that we even have $Q'_l \subset Q_l(N')$ since $\gamma_l \mid P'_1 \otimes H^*_{s1}$ factorises through $\bigoplus_{l\leq s} N'_L \otimes B^*_{sl}$ as follows from the definition of N'_l.

7.2.1. Lemma: *Suppose that all $\beta_1, \dots, \beta_s > 0$, and let $\Delta = \sum_i \lambda_i m''_i - \sum_l \mu_l n''_l$. Then*

$$\Delta > \sum_l \beta_l q'_l - \sum_l \mu_l n'_l + \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_l \mu_l c_l(m_2, \dots, m_r)(\sigma_1 - \rho_1).$$

Proof. Let

$$Y_l = \delta_l(X_1 \otimes H_{l1}^*) \ \subset \ Z_l = \underset{2\leq i}{\oplus} \ M_i'' \otimes H_{li}^*.$$

Since X_1 is not contained in a direct sum with spaces smaller than M_i'' we get

$$\operatorname{codim}(Y_l, Z_l) \leq c_l(m_2'', \dots, m_r'')\operatorname{codim}(X_1, \underset{2\leq j}{\oplus} M_j'' \otimes A_{j1}).$$

By Lemma 7.1.1 and above definitions we get

$$\operatorname{codim}(Y_l, Z_l) \leq c_l(m_2'', \dots, m_r'')(\sum_{1\leq i} m_i'' a_{i1} - p_1') = c_l(m_2, \dots, m_r)(\sum_{2\leq i}(m_i'' - m_i')a_{i1} - \rho_1).$$

The map $\sum\limits_j \phi_{lj}$ sends $(M_1'' \otimes H_{l1}^*) \oplus Z_l$ onto N_l'' by definition of N_l'' and also maps $(M_1' \otimes H_{l1}^*) \oplus \delta_l(X_1 \otimes H_{l1}^*)$ onto N_l'. Therefore, since $M_1' = M_1''$, we have a surjection

$$Z_l / Y_l \to N_l'' / N_l'$$

and the dimension estimate

$$n_l'' - n_l' \leq c_l(m_2, \dots, m_r)(\sum_{2\leq i}(m_i'' - m_i')a_{i1} - \rho_1).$$

Now we can derive the estimate of the Lemma. If there is no summation condition it is understood that the sum has to be taken over all indices of the given interval. We have

$$\begin{aligned}
\Delta &= \sum_i \lambda_i m_i'' - \sum_l \mu_l n_l'' \\
&= \sum_i \lambda_i m_i' - \sum_l \mu_l n_l' + \sum_j \lambda_j (m_j'' - m_j') - \sum_l \mu_l (n_l'' - n_l').
\end{aligned}$$

Substituting for λ_j in the third sum and replacing the first by

$$\sum_i \lambda_i m_i' = \sum_i \alpha_i \dim(\underset{i\leq j}{\oplus} M_j' \otimes A_{ji}) = \sum_i \alpha_i (p_i' - \rho_i)$$

and using the definition of σ_i we get

$$\Delta = \sum_i \alpha_i p_i' - \sum_l \mu_l n_l' + \sum_i \alpha_i (\sigma_i - \rho_i) - \sum_l \mu_l (n_l'' - n_l').$$

Now using the assumed estimate for the first sum and the derived estimate for $n_l'' - n_l'$ we get

$$\Delta > \sum_l \beta_l q_l' - \sum_l \mu_l n_l' + \sum_i \alpha_i (\sigma_i - \rho_i) - \sum_l \mu_l c_l(m_2, \dots, m_r)(\sigma_1 - \rho_1).$$

□

7.3. Corollary: *Suppose that* $s = 1$, *let* $\Lambda = (\lambda_1, \dots, \lambda_r, -\frac{1}{n_1})$ *and let* $\tilde{\Lambda}$ *be the associated polarization* $(\alpha_1, \dots, \alpha_r, -\frac{1}{n_1})$. *If all* $\alpha_i > 0$ *and if*

$$\lambda_2 \geq \frac{a_{21}}{n_1} c_1(m_2, \dots, m_r)$$

then

$$\zeta^{-1}\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda}) = W^{ss}(G, \Lambda) \quad \textit{and} \qquad \zeta^{-1}\mathbf{W}^{s}(\mathbf{G}, \tilde{\Lambda}) = W^{s}(G, \Lambda).$$

Remarks: (1) Note that by the normalization of the polarizations we must have $\mu_1 n_1 = 1$ such that $1/n_1$ is the only possible value for $\mu_1 = \beta_1$.

(2) If all $\alpha_i > 0$, then the necessary conditions for $W^s(G,\Lambda) \neq \emptyset$ and $\mathbf{W}^s(\mathbf{G},\tilde{\Lambda}) \neq \emptyset$ are both satisfied, see 5.8. The condition of the corollary is an extra condition.

Proof. Let us first assume that $\zeta(w)$ is not semi–stable and let the spaces P_i' and Q_1' be as at the beginning of 7.2. The only $\beta_1 = 1/n_1$ is positive. Let the other spaces be chosen as in 7.2. The difference $\sum \beta_l q_l' - \sum \mu_l n_l'$ reduces to $q_1'/n_1 - n_1'/n_1$, and since $N_1' = \gamma(w)(P_1' \otimes H_{11}^*) = Q_1'$, this difference is zero. Therefore

$$\Delta > \sum_i \alpha_i(\sigma_i - \rho_i) - \frac{1}{n_1} c_1(m_2, \dots, m_r)(\sigma_1 - \rho_1).$$

Since all the α_i are positive we have

$$\sum_i \alpha_i(\sigma_i - \rho_i) \geq \alpha_1(\sigma_1 - \rho_1) + \alpha_2(\sigma_2 - \rho_2).$$

Moreover, ξ_2 induces a surjection

$$P_2(M'') \otimes A_{21}/P_2' \otimes A_{21} \to P_1(M'')/P_1'$$

because $M_1' = M_1''$. Therefore we obtain the dimensions estimate $(\sigma_2 - \rho_2)a_{21} \geq \sigma_1 - \rho_1$. It follows that

$$\Delta > (-\frac{1}{n_1} c_1(m_2, \dots, m_r) + \alpha_1 + \frac{\alpha_2}{a_{21}})(\sigma_1 - \rho_1).$$

Since $\lambda_2 = a_{21}\alpha_1 + \alpha_2 \geq \frac{a_{21}}{n_1} c_1(m_2, \dots, m_r)$ the last expression is non–negative. This proves the case of semi–stability. For the case of stability we assume that w is stable and that $\zeta(w)$ is already semi–stable. If $\zeta(w)$ were not stable, we would find subspaces P_i' and N_1' as in 7.2 such that $\sum \alpha_i p_i' - \mu_1 n_1' = 0$ and such that at least one P_i' is different from P_i. Now let the spaces M_i'' and N_I'' be constructed as above. Then we have

$$\Delta \geq \sum_i \alpha_i s_i - \frac{c_1}{n_1} s_1 \geq \sum_{2<i} \alpha_i s_i + (\lambda_2 - \frac{c_1}{n_1} a_{21}) \frac{s_1}{a_{21}} \geq 0,$$

where $s_i = \sigma_i - \rho_i = \dim P_i(M'')/P_i'$, and where we use that $s_2 a_{21} \geq s_1$. If the family M'' is different from M, then $0 > \Delta$, and if it is equal, then $\Delta = 0$. In order to obtain a contradiction we have to show that M'' is different from M. Assume that it is not. Then $s_i = \dim P_i/P_i'$ and we must have $s_i = 0$ for $i \geq 3$ and $s_1(\lambda_2 - \frac{c_1}{n_1} a_{21}) = 0$. If also $s_1 = 0$, then by the above estimate also $s_2 = 0$, contradicting the choice of the P_i'. Therefore $s_1 \neq 0$ and $\lambda_2 = \frac{c_1}{n_1} a_{21}$. But then $\Delta = \alpha_2(s_2 - \frac{s_1}{a_{21}})$ and we have $s_2 a_{21} = s_1$. From this it is easy to see that $P_i' = P_i(\tilde{M})$ where $\tilde{M}_i = M_i$ for $i \neq 2$ and $\tilde{M}_2 = M_2' \neq M_2$. Then we have

$$\sum_i \alpha_i \widetilde{m}_i - \mu_1 n_1' = \sum_i \lambda_i p_i' - \mu_1 n_1' = 0$$

which contradicts the stability of w. $\square$

7.4. *Study of the converse II*

We keep the notation of 7.2 and compare the (semi–)stability of points in W and $\mathbf{W}$ in two steps, each reducing to the case $s = 1$. We consider the intermediate space

$$\mathbf{V} = \mathbf{W}_L \oplus \bigoplus_{1\le l\le s} \mathrm{Hom}(P_1 \otimes H_{l1}^*, N_l)$$

and the maps

$$W \xrightarrow{\zeta_1} \mathbf{V} \xrightarrow{\zeta_2} \mathbf{W}.$$

Here ζ_1 is defined by

$$w \mapsto (\xi_2, \dots, \xi_r, \gamma_1(w), \dots, \gamma_s(w)),$$

where $\gamma_l(w)$ is the map defined by $w = (\phi_{li})$ as in 7.2. The map ζ_2 is defined by

$$(x_2, \dots, x_r, \gamma_1, \dots, \gamma_s) \mapsto (x_2, \dots, x_r, \gamma, \eta_1, \dots, \eta_{s-1}),$$

where now $\gamma : P_1 \otimes H_{s1}^* \to Q_s$ is induced by the tuple $(\gamma_1, \dots, \gamma_s)$ as the sum of the compositions

$$P_1 \otimes H_{s1}^* \to N_l \otimes H_{l1} \otimes H_{s1}^* \to N_l \otimes B_{sl}^*$$

which are induced by the γ_l and the pairings $B_{sl} \otimes H_{l1} \to H_{s1}$. It is obvious that

$$\zeta = \zeta_2 \circ \zeta_1.$$

Note that both ζ_1 and ζ_2 are injective by the same reason as for ζ.

On $\mathbf{V}$ the group $\mathbf{G}_L \times G_R$ acts naturally and we have the embedding

$$G = G_L \times G_R \overset{\theta_L\times id}{\hookrightarrow} \mathbf{G}_L \times G_R,$$

see 5.5. It follows as in section 5 that ζ_1 is compatible with the group actions and that $w, w' \in W$ are on the same G–orbit if and only if $\zeta_1(w), \zeta_1(w')$ are on the same $\mathbf{G}_L \times G_R$ orbit. Similarly we have the group embedding $\mathbf{G}_L \times G_R \hookrightarrow \mathbf{G}_L \times \mathbf{G}_R = \mathbf{G}$ and ζ_2 is equivariant and satisfies the analogous statements for the orbits. Given the polarization $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ for (W, G) we consider the polarization $\overline{\Lambda} = (\alpha_1, \dots, \alpha_r, -\mu_1, \dots, -\mu_s)$ for $(\mathbf{V}, \mathbf{G}_L \times G_R)$ where the α_i are defined as in 5.8. As in 5.10, 5.12 it is easy to show that

$$\zeta_1^{-1}\mathbf{V}^{ss}(\mathbf{G}_L \times G_R, \overline{\Lambda}) \subset W^{ss}(G, \Lambda) \quad \text{and} \quad \zeta_1^{-1}\mathbf{V}^{s}(\mathbf{G}_L \times G_R, \overline{\Lambda}) \subset W^{s}(G, \Lambda)$$

and similarly that

$$\zeta_2^{-1}\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda}) \subset \mathbf{V}^{ss}(\mathbf{G}_L \times G_R, \overline{\Lambda}) \quad \text{and} \quad \zeta_2^{-1}\mathbf{W}^{s}(\mathbf{G}, \tilde{\Lambda}) \subset \mathbf{V}^{s}(\mathbf{G}_L \times G_R, \overline{\Lambda}).$$

Note that as for W^{ss}, W^s, we have unipotent sub-orbits in $\mathbf{V}^{ss}$ and $\mathbf{V}^s$, see 4.1. We are going to show that in all 4 cases equality holds under suitable conditions on the weights of the polarizations. Then the same is true for ζ.

7.5. *Estimate for* $\boldsymbol{\zeta}_1$

Let $w = (\phi_{li})$ in W be given and assume that $\zeta_1(w)$ is not semi–stable. Then there are linear subspaces $P_i' \subset P_i$ and $N_l' \subset N_l$ and a unipotent element $h \in H_R$ such that for $(\gamma_1', \dots, \gamma_s') = h.(\gamma_1, \dots, \gamma_s)$ we have

$$\xi_i(P_i' \otimes A_{i,i-1}) \subset P_{i-1}' \quad \text{and} \quad \gamma_l'(P_l' \otimes H_{l1}^*) \subset N_l'$$

for all $i \geq 2$ and all l, and such that

$$\sum_i \alpha_i p'_i - \sum_l \mu_l n'_l > 0.$$

We may assume that $h = id$ because H_R acts on W in the same way and we can replace w by $h.w$. Moreover, we may assume that all N'_l are equal to $\gamma_l(P'_1 \otimes H^*_{l1})$ since all $\mu_l > 0$. Now we proceed as in 7.2 replacing the spaces Q_l by N_l. Therefore we find subspaces $M'_i \subset M''_i \subset M_i$ such that $M'_1 = M''_1$ and such that

$$P'_i = M'_i \oplus X_i \ , \quad P_i(M') \subset P'_i \subset P_i(M'')$$

and the family M'' is minimal with this property. We denote

$$\rho_i = \operatorname{codim}(P_i(M'), P'_i) \quad , \quad \sigma_i = \operatorname{codim}(P_i(M'), P_i(M''))$$

and let

$$N''_l = \gamma_l(P_1(M'') \otimes H^*_{l1}) \supset N'_l.$$

As in 7.2.1 we consider the surjection

$$Z_l/Y_l \to N''_l/N'_l,$$

where $Y_l \subset Z_l$ are the same, and we get the estimate

$$n''_l - n'_l \leq c_l(m_2, \dots, m_r)(\sigma_1 - \rho_1)$$

for any l. The estimation of the discriminant Δ is now simpler than in 7.2.

7.5.1. Lemma: *With the above notation*

$$\Delta := \sum_i \lambda_i m''_i - \sum_l \mu_l n''_l \ > \ \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_l \mu_l c_l(m)(\sigma_1 - \rho_1)$$

where $c_l(m) = c_l(m_2, \dots, m_r)$.

Proof. By replacing dimensions and inserting the estimate for $n''_l - n'_l$ as in 7.2 we get

$$\begin{aligned} \Delta &= \sum_i \alpha_i p'_i - \sum_l \mu_l n'_l + \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_l \mu_l(n''_l - n'_l) \\ &> \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_l \mu_l c_l(m)(\sigma_1 - \rho_1). \end{aligned}$$

□

7.5.2. Corollary: *Let* $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ *be a polarization for* W *and let* $\bar{\Lambda} = (\alpha_1, \dots, \alpha_r, -\mu_1, \dots, -\mu_s)$ *be the associated polarization for* $\mathbf{V}$ *as in 7.4. If all* $\alpha_i > 0$ *and*

$$\lambda_2 \geq a_{21} \sum_l \mu_l c_l(m)$$

then

$$\zeta_1^{-1}\mathbf{V}^{ss}(\mathbf{G}_L \times G_R, \bar{\Lambda}) = W^{ss}(G, \Lambda) \quad \text{and} \quad \zeta_1^{-1}\mathbf{V}^{s}(\mathbf{G}_L \times G_R, \bar{\Lambda}) = W^{s}(G, \Lambda).$$

Proof. The proof is the same as for 7.3, because the spaces P'_i and $P_i(M'')$ are defined in the same way and we thus get the estimate $(\sigma_2 - \rho_2)a_{21} \geq \sigma_1 - \rho_1$. □

7.6. *Estimate for $\boldsymbol{\zeta}_2$*

The analogous estimate for ζ_2 follows by duality while we can assume that $s = 1$ or $r = 1$. The proof could be done by formally transform it into a dual situation which is similar to that of 7.5, but it is better to keep direct track of the weights. Let $(x_2, \dots, x_r, \gamma_1, \dots \gamma_s)$ be given in $\mathbf{W}_L \oplus V$ and assume that its image under ζ_2 is not semi–stable. Then there are subspaces $P_i' \subset P_i$ and $Q_l' \subset Q_l$ such that

$$x_i(P_i' \otimes A_{i,i-1}) \subset P_{i-1}' \ , \quad \gamma(P_i' \otimes H_{s1}^*) \subset Q_s' \ , \quad \eta_l(Q_{l+1}' \otimes B_{l+1,l}) \subset Q_l',$$

where γ is defined as in 7.4, and such that

$$\sum_i \alpha_i p_i' - \sum_l \beta_l q_l' > 0.$$

We assume that all $\alpha_i \geq 0$, and then we may assume that P_i' is maximal, i.e. the inverse image of $P_{i-1}' \otimes A_{i,i-1}^*$ under $P_i \to P_{i-1} \otimes A_{i,i-1}^*$ for $i \geq 2$, and similarly P_1' in P_1 under $P_1 \to Q_s \otimes H_{s1}$. As in 7.5 we can find subspaces $N_l' \subset N_l$ such that

$$Q_l' = N_l' \oplus X_l' \quad \text{ and hence } (Q_l/Q_l')^* = (N_l/N_l')^* \oplus X_l.$$

We choose subspaces $N_l'' \subset N_l'$ which are maximal such that

$$Q_l(N'') \subset Q_l' \subset Q_l(N').$$

We have $N_s'' = N_s'$. We let P_1'' be the inverse image of $Q_s(N'')$ under $P_1 \to Q_s \otimes H_{s1}$. Then $P_1'' \subset P_1'$. Furthermore we let inductively $P_i'' \subset P_i'$ be the inverse images for $i \geq 2$. Then we have injections

$$(P_i'/P_i'') \otimes A_{i,i-1} \to P_{i-1}'/P_{i-1}''$$

and induced by factorization the images

$$P_i'/P_i'' \otimes A_{i,i-1} \otimes \dots \otimes A_{21} \twoheadrightarrow (P_i'/P_i'') \otimes A_{i1} \rightarrowtail P_1'/P_1''.$$

The induced injections

$$P_i'/P_i'' \rightarrowtail (P_1'/P_1'') \otimes A_{i1}^*$$

imply the dimension estimates

$$p_i' - p_i'' \leq a_{i1}(p_1' - p_1'')$$

for $i \geq 2$. Next we consider the homomorphism

$$Z_1 = \underset{l<s}{\oplus} (N_l/N_l'')^* \otimes H_{l1}^* \xleftarrow{\delta_1^\vee} \underset{l<s}{\oplus} (N_l/N_l'')^* \otimes B_{sl} \otimes H_{s1}^*.$$

We have $X_s \subset \bigoplus\limits_{l<s} (N_l/N_l'')^* \otimes B_{sl}$ and consider the subspace

$$Y_1 = \delta_1^\vee(X_s \otimes H_{s1}^*) \subset Z_1.$$

By the definition of the constant $d_1(n) = d_1(n_1, \dots, n_{s-1})$ we get

$$\dim Z_1/Y_1 \leq d_1(n)\mathrm{codim}(X_s) = d_1(n)(\sigma_s - \rho_s)$$

where

$$\sigma_l = \mathrm{codim}(Q_l^*(N/N'), Q_l^*(N/N'')) \quad \text{ and } \quad \rho_l = \mathrm{codim}(Q_l^*(N/N'), (Q_l/Q_l')^*).$$

Further we have a surjective map

$$Z_1/Y_1 \to (P_1/P_1'')^*/(P_1/P_1')^*$$

which is induced by the map $Q_s^* \otimes H_{s1}^* \to P_1^*$ and the induced surjection $Q_s^*(N/N'') \otimes H_{s1}^* \to (P/P_1'')^*$, since $N_s'' = N_s'$. So we get

$$p_1' - p_1'' \leq d_1(n)(\sigma_s - \rho_s).$$

Now we can estimate the discriminant in

7.6.1. Lemma: *Let all the α_i be non-negative and let $\Delta := \sum_i \alpha_i p_i'' - \sum_l \mu_l n_l''$. Then*

$$\Delta > \sum_l \beta_l(\sigma_l - \rho_l) - \sum_i \alpha_i a_{i1} d_1(n)(\sigma_s - \rho_s).$$

Proof. Since $\sum_i \alpha_i p_i = \sum_l \mu_l n_l$ we also have

$$\Delta = \sum_l \mu_l(n_l - n_l'') - \sum_i \alpha_i(p_i - p_i'').$$

with the same steps as in the previous proofs we get

$$\Delta = \sum_i \alpha_i p_i' - \sum_l \beta_l q_l' + \sum_l \beta_l(\sigma_l - \rho_l) - \sum_i \alpha_i(p_i' - p_i'')$$

Inserting the assumption on the first difference and the estimate for $p_i' - p_i''$ we get the result. □

As in the previous cases we obtain the

7.6.2. Corollary: *In the above notation let all $\alpha_i > 0$, and all $\beta_l > 0$, and let*

$$\mu_{s-1} \geq b_{s,s-1} d_1(n) \sum_i \alpha_i a_{i1}.$$

Then

$$\zeta_2^{-1}\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda}) = \mathbf{V}^{ss}(\mathbf{G}_L \times G_R, \bar{\Lambda}) \quad \textit{and} \quad \zeta_2^{-1}\mathbf{W}^{s}(\mathbf{G}, \tilde{\Lambda}) = \mathbf{V}^{s}(\mathbf{G}_L \times G_R, \bar{\Lambda}).$$

Proof. : In the notation of 7.6 there is a surjection $(Q_{s-1}(N')/Q_{s-1}')^* \otimes B_{s,s-1} \to (Q_s(N')/Q_s')^*$ because $N_s'' = N_s'$. Therefore $(\sigma_{s-1} - \rho_{s-1})b_{s,s-1} \geq \sigma_s - \rho_s$. If the condition of the Corollary is satisfied, then $\Delta > 0$ follows, where we use $\mu_{s-1} = \beta_s b_{s,s-1} + \beta_{s-1}$. □

Combining the results of 7.5.2 and 7.6.2 we get the

7.7. Proposition: *Let $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ be a polarization for (W, G) and let $\widetilde{\Lambda} = (\alpha_1, \dots, \alpha_r, -\beta_1, \dots, -\beta_s)$ be the associated polarization for $(\mathbf{W}, \mathbf{G})$. Suppose that all $\alpha_i > 0$, and all $\beta_l > 0$ and that*

$$\lambda_2 \geq a_{21} \sum_l \mu_l c_l(m) \quad \textit{and} \quad \mu_{s-1} \geq b_{s,s-1} d_1(n) \sum_i \alpha_i a_{i1}.$$

Then

$$\zeta^{-1}\mathbf{W}^{ss}(\mathbf{G}, \tilde{\Lambda}) = W^{ss}(G, \Lambda) \quad \textit{and} \quad \zeta^{-1}\mathbf{W}^{s}(\mathbf{G}, \tilde{\Lambda}) = W^{s}(G, \Lambda).$$

8. Projectivity conditions

The projectivity of the quotient in 6.1 depends on the second condition in (2), i.e. whether the boundary $\bar{Z} \setminus Z$ of the saturated set contains no semi–stable points of $\mathbf{W}$. Again this condition depends on the chosen polarization and conditions for the weights. In order to derive these conditions in some cases we describe the boundary in terms independent of the group action.

8.1. *Saturated boundary.*

The elements of $\mathbf{W}$ are tuples $\mathbf{w} = (x_2, \dots, x_r, \gamma, y_1, \dots, y_{s-1})$ of linear maps

$$P_i \otimes A_{i,i-1} \xrightarrow{x_i} P_{i-1}, \quad P_1 \otimes H_{s1}^* \xrightarrow{\gamma} Q_s, \quad Q_{l+1} \xrightarrow{y_l} B_{l+1,l}^* \otimes Q_l$$

If $\mathbf{w} \in Z$, there are an element $w \in W$ and automorphisms $\rho_i \in Aut(P_i)$, $\sigma_l \in Aut(Q_l)$ such that

$$x_i = \rho_{i-1} \circ \xi_i \circ (\rho_i^{-1} \otimes id), \quad \gamma = \sigma_1 \circ \gamma(w) \circ (\rho_1^{-1} \otimes id), \quad y_l = (id \otimes \sigma_l) \circ \eta_l \circ \sigma_{l+1}^{-1}.$$

Here id stands for the different identities of the spaces A, B and H. We let $\widetilde{x_i}$ respectively $\widetilde{\xi_i}$ be the mapping

$$P_i \otimes A_{i,i-1} \otimes \dots \otimes A_{21} \to P_{i-1} \otimes A_{i-1,i-2} \otimes \dots \otimes A_{21}$$

induced by x_i respectively ξ_i for $i \geq 3$. From the relations between the x_i and ξ_i it follows easily that for each $i \geq 3$ the composition $x_2 \circ \widetilde{x_3} \circ \dots \circ \widetilde{x_i}$ has a factorization

$$\begin{tikzcd} P_i \otimes A_{i,i-1} \otimes \cdots \otimes A_{21} \arrow[r] \arrow[d] & P_1 \\ P_i \otimes A_{i1} \arrow[ur, "x_{1i}"'] & \end{tikzcd}$$

where the vertical map is the surjection induced by the pairings. This follows from the commutative diagrams induced by the automorphism ρ_i and because $\xi_2 \circ \widetilde{\xi_3} \circ \cdots \circ \widetilde{\xi_i}$ admits such a factorization for each $i \geq 3$. We put $x_{21} = x_2$. By the dual description for the maps y_l we are given factorizations

$$\begin{tikzcd} & B_{sl}^* \otimes Q_l \arrow[d, tail] \\ Q_s \arrow[ur, "y_{ls}"] \arrow[r] & B_{s,s-1}^* \otimes \dots \otimes B_{sl}^* \otimes Q_l \end{tikzcd}$$

of the maps $\widetilde{y_l} \circ \dots \circ \widetilde{y}_{s-2} \circ y_{s-1}$ for $l \leq s-2$. By similar arguments there are also factorizations

$$\begin{tikzcd} P_i \otimes A_{i1} \otimes H_{s1}^* \arrow[r, "x_{1i} \otimes id"] \arrow[d] & P_1 \otimes H_{s1}^* \arrow[r, "\gamma"] & Q_s \\ P_i \otimes H_{si}^* \arrow[urr, "\gamma_{si}"'] & & \end{tikzcd} \qquad (L_i)$$

for all $i \geq 2$ and dually factorizations

$$\begin{tikzcd}
 & & Q_l \otimes H_{l1} \arrow[d] \\
P_1 \arrow[r, "\gamma"] \arrow[urr, "\gamma_{l1}"] & Q_s \otimes H_{s1} \arrow[r, "y_{ls}\otimes id"'] & B_{sl}^* \otimes Q_l \otimes H_{s1}
\end{tikzcd} \qquad (R_l)$$

for all l. Moreover, there are further factorizations of the induced composed maps

$$\begin{tikzcd}
P_i \otimes H_{si}^* \otimes B_{sl} \arrow[r, "\gamma_{si}\otimes id"] \arrow[d] & Q_s \otimes B_{sl} \arrow[r, "\widetilde{y}_{ls}"] & Q_l \\
P_i \otimes H_{li}^* \arrow[urr, "\Phi_{li}"'] & &
\end{tikzcd} \qquad (L_{li})$$

and dually

$$\begin{tikzcd}
 & & Q_l \otimes H_{li} \arrow[d] \\
P_i \arrow[r, "\widetilde{x}_{1i}"] \arrow[urr, "\Psi_{li}"] & P_1 \otimes A_{i1}^* \arrow[r, "\gamma_{l1}\otimes id"'] & Q_l \otimes H_{l1} \otimes A_{i1}^*
\end{tikzcd} \qquad (R_{li})$$

All these factorizations are based on mappings induced by the pairings. All factorization conditions are independent of the chosen automorphisms. One can rediscover the original components ϕ_{li} of w from Φ_{li} or Ψ_{li} if $x_j = \xi_j$ and $y_l = \eta_l$ for all j and all l. In fact we have

8.1.1. Lemma: *Let $\mathbf{w} = (x_2, \dots, x_r, \gamma, y_1, \dots, y_{s-1}) \in \mathbf{W}$. Then $\mathbf{w} \in Z$ if and only if*

(1) *rank* $x_i = \sum_{i \leq j} m_j a_{j,i-1}$ *for* $i \geq 2$
(1*) *rank* $y_l = \sum_{k \leq l} b_{l+1,k} n_k$ *for* $l \leq s-1$
(2) $x_2 \circ \widetilde{x}_3 \circ \dots \circ \widetilde{x}_i$ *has a factorization* $P_i \otimes A_{i1} \xrightarrow{x_{i1}} P_1$ *for* $i \geq 3$
(2*) $\widetilde{y}_l \circ \dots \circ \widetilde{y}_{s-2} \circ y_{s-1}$ *has a factorization* $Q_s \xrightarrow{y_{ls}} B_{sl}^* \otimes Q_l$ *for* $l \leq s-2$
(3) $\gamma \circ (x_{1i} \otimes id)$ *has factorizations* (L_i) *and* (L_{li})
(3*) $(y_{ls} \otimes id) \circ \gamma$ *has factorizations* (R_l) *and* (R_{li}).

Proof. If $\mathbf{w} \in Z$, the three conditions are satisfied by the above, where *rank* x_i is the dimension of the image of ξ_i and *rank* y_l is the *rank* of η_l as the map $Q_{l+1} \to B_{l+1,l}^* \otimes Q_l$. Let conversely $\mathbf{w}$ satisfy these conditions. We proceed by descending induction to find automorphisms ρ_i by which the x_i can be identified with the ξ_i. Note that the factorization conditions are maintained under automorphisms. Since x_r has maximal rank it is an injection $M_r \otimes A_{r,r-1} \to M_{r-1} \oplus M_r \otimes A_{r,r-1} = P_{r-1}$. Hence we can find an automorphism ρ_{r-1} of P_{r-1} such that $\rho_{r-1} \circ x_r$ becomes ξ_r. Let us assume now that modulo some automorphisms $\rho_{r-1}, \dots, \rho_i$ we have $x_j = \xi_j$ for $j > i$. We are going to find an automorphism ρ_{i-1} such that $\rho_{i-1} \circ x_i = \xi_i$. Because of the rank condition we can assume that $\oplus_{i \leq j} M_j \otimes A_{j,i-1}$ is the image of x_i in P_{i-1}. Now using all the $x_i \circ \widetilde{\xi}_{i+1} \circ \dots \circ \widetilde{\xi}_k$ we find that x_i has a factorization through the standard map

$$P_i \otimes A_{i,i-1} \to \bigoplus_{i \leq j} M_j \otimes A_{j,i-1} \xrightarrow{\bar{x}_i} M_{i-1} \oplus \bigoplus_{i \leq j} M_j \otimes A_{j,i-1}.$$

induced by the pairings. Now the rank condition implies that $\bar{x}_i$ induces an automorphism on $\oplus_{i\leq j} M_j \otimes A_{j,i-1}$. This can be used to make $\bar{x}_i$ the identity via an automorphism ρ'_{i-1}. Now $x_i = \xi_i$. By the analogous dual procedure we can also find automorphism $\sigma_l \in Aut(Q_l)$ such that we can assume that $y_l = \eta_l$. Finally the factorizations (L_{li}) or (R_{li}) resulting from (3) and (3*) yield mappings Φ_{li} or Ψ_{li} from which we get ϕ_{li} as composition

$$M_i \otimes H_{li}^* \rightarrowtail P_i \otimes H_{li} \xrightarrow{\Phi_{li}} Q_l \twoheadrightarrow N_l.$$

It follows from the special type of the ξ_i and η_l that these are original components of an element $w = (\phi_{li})$ inducing $\gamma(w) = \gamma$. □

8.1.2. Corollary: *With the same notation as in 8.1.1, if* $\mathbf{w} \in \bar{Z} \smallsetminus Z$, *then*

(1) rank $x_i \leq$ *rank* ξ_i *and rank* $y_l \leq$ *rank* η_l *with strict inequality for at least one* i *or* l, *and*

(2), (2), (3), (3*) of 8.1.1 are satisfied.*

Proof. All conditions are closed and thus hold for points in $\bar{Z}$. If $\mathbf{w} \in \bar{Z} \smallsetminus Z$ then by 8.1.1 equality in (1) cannot hold for all i and l. □

We are going to derive effective sufficient conditions for the projectivity of the quotient in the cases $(2,1), (2,2), (3,1)$.

8.2. Proposition: *Let the polarizations* Λ *and* $\widetilde{\Lambda}$ *be as in proposition 7.7 and let* $Z = \mathbf{G}\zeta(W)$. *Then* $\bar{Z} \smallsetminus Z$ *contains no semi–stable point in the following cases*

(i) $(r,s) = (2,1)$ *and* $\lambda_2 \geq c_1(m_2)a_{21}\mu_1$
(ii) $(r,s) = (2,2)$ *and*

$$\lambda_2 \geq (\mu_1 c_1(m_2) + \mu_2(c_2(m_2) - b_{21}c_1(m_2))a_{21}, \qquad \mu_1 \geq (\lambda_1(d_1(n_1)) - d_2(n_1)a_{21}) + \lambda_2 d_2(n_1))b_{21}.$$

Proof. We present only the case (ii), case(i) is an easier version of (ii). Let $(x_2, \gamma, y_1) \in \bar{Z} \smallsetminus Z$ and let us assume that *rank* x_2 is not maximal. Let K be the kernel of $M_2 \otimes A_{21} \xrightarrow{x_2} P_1$ and let $M'_2 \subset M_2$ be the smallest subspace such that K is contained in $M'_2 \otimes A_{21}$. We put $P'_2 = M'_2$,

$$P'_1 = x_2(M'_2 \otimes A_{21}) \ , \ Q'_2 = \gamma(P'_1 \otimes H^*_{21}) \ \text{ and } \ Q'_1 = y_1(Q'_2 \otimes B_{21})$$

and consider

$$\Delta = \alpha_1 p'_1 + \alpha_2 p'_2 - \beta_1 q'_1 - \beta_2 q'_2.$$

By definition $p'_1 = \dim M'_2 \otimes A_{21}/K$. Diagram (L_2) reduces in our case, with M_2 replaced by M'_2, to

$$\begin{tikzcd}
M'_2 \otimes A_{21} \otimes H^*_{21} \arrow[r, two heads, "x_2 \otimes id"] \arrow[d, "\delta_2"'] & P'_1 \otimes H^*_{21} \arrow[r, two heads, "\gamma"] & Q'_2 \\
M'_2 \otimes H^*_{22} \arrow[urr, "\gamma_{22}"'] & &
\end{tikzcd}$$

and γ_{22} vanishes on $\delta_2(K \otimes H^*_{21})$ because K is the kernel of x_2. Therefore

$$q'_2 \leq \dim M'_2 \otimes H^*_{22}/\delta_2(K \otimes H^*_{21}) \leq c_2(m'_2)p'_1.$$

In order to estimate q'_1 we consider diagram (L_{21}) enlarged by the commutative square of induced pairings

$$\begin{tikzcd} M'_2 \otimes A_{21} \otimes H^*_{21} \otimes B_{21} \arrow[r, two heads] \arrow[d] & M'_2 \otimes H^*_{22} \otimes B_{21} \arrow[r, two heads, "\gamma_{22}\otimes id"] \arrow[d] & Q'_2 \otimes B_{21} \arrow[r, two heads, "y_1"] & Q'_1 . \\ M'_2 \otimes A_{21} \otimes H^*_{11} \arrow[r, two heads, "\delta_1"] & M'_2 \otimes H^*_{12} \arrow[urr, "\Phi_{12}"'] & & \end{tikzcd}$$

Again the map Φ_{12} vanishes on $\delta_1(K \otimes H^*_{11})$ and we get

$$q'_1 \leq \dim M'_2 \otimes H^*_{12}/\delta_1(K \otimes H^*_{11}) \leq c_1(m'_2)p'_1.$$

Now we have the estimate

$$\Delta \geq \alpha_2 p'_2 + (\alpha_1 - \beta_1 c_1(m_2) - \beta_2 c_2(m_2))p'_1.$$

Therefore the condition $\alpha_1 \geq \beta_1 c_1(m_2) + \beta_2 c_2(m_2)$ would be sufficient, because $\alpha_2 p'_2 > 0$. We modify the last estimate as follows. Since the weights in case $(2,2)$ are related by

$$\begin{array}{rcl} \lambda_1 & = & \alpha_1 \\ \lambda_2 & = & a_{21}\alpha_1 + \alpha_2 \end{array} \quad \text{and} \quad \begin{array}{rcl} \mu_2 & = & \beta_2 \\ \mu_1 & = & \beta_1 + \beta_2 b_{21} \end{array}$$

and since we have

$$\lambda_2 - a_{21}\lambda_1 > 0 \quad \text{and} \quad p'_2 a_{21} - p'_1 > 0,$$

we get the estimate

$$\Delta > (\frac{\lambda_2}{a_{21}} - \mu_1 c_1(m_2) - \mu_2 c_2(m_2) + \mu_2 c_1(m_2) b_{21})p'_1.$$

This shows that $\Delta > 0$ if x_2 is degenerate and the first condition of (ii) is satisfied. In case $rank\ y_1$ is not maximal the second condition follows by the dual procedure. □

8.3. *The case (3,1)*

In order to derive a similar result in case $(3,1)$ we introduce the additional constant $c'_3(m_3)$ analogous to $c_3(m_3) := c_1(0, m_3)$ in 7.1. Let

$$M_3 \otimes A_{32} \otimes H^*_{12} \xrightarrow{\tau} M_3 \otimes H^*_{13}$$

be the linear map induced by the pairing and let $\mathcal{K}$ be the family of all proper subspaces $K \subset M_3 \otimes A_{32}$ which are not contained in $M'_3 \otimes A_{32}$ for any subspace $M'_3 \subset M_3$ different from M_3. We put

$$c'_3(m_3) = \sup_{K \in \mathcal{K}} \frac{\text{codim } \tau(K \otimes H^*_{12})}{\text{codim } K}$$

For brevity we write $c'_3 = c'_3(m_3)$, $c_3 = c_3(m_3) = c_1(0, m_3)$ and $c_1 = c_1(m_2, m_3)$.

8.3.1. Proposition*: Let* $(r,s) = (3,1)$*, let* $\Lambda = (\lambda_1, \lambda_2, \lambda_3, -\mu_1)$ *be a polarization for* (W, G) *and* $\widetilde{\Lambda} = (\alpha_1, \alpha_2, \alpha_3, -\mu_1)$ *be the associated polarization for* $(\mathbf{W}, \mathbf{G})$*, and assume that all* $\alpha_i > 0$*. (In this case* $\mu_1 = \frac{1}{n_1}$*.) If*

(1) $\alpha_2 c_3 + \lambda_1 c'_3 \geq \mu_1 c_3 c'_3$

(2) $\lambda_2 \geq a_{21}\mu_1 c_1$
(3) $\lambda_3 \geq a_{31}\mu_1 c_1$

then $\bar{Z} \setminus Z$ contains no semi–stable point.

Moreover, condition (1) may be replaced by any of the conditions

(i) $\lambda_3 \geq \mu_1 c_3' a_{32} + a_{31}\lambda_1$
(ii) $\lambda_3 \geq \mu_1 c_3 a_{31} + a_{32}\alpha_2$
(iii) $\lambda_3 \geq \mu_1 c_3 a_{32} a_{21}$

Remark: $\bar{Z} \setminus Z$ contains no semi–stable point also in each of the following cases

(a) $\lambda_1 \geq \mu_1 c_3$
(b) $\alpha_2 \geq \mu_1 c_3'$
(c) $\alpha_3 \geq \mu_1 c_3 a_{31}$ or $\alpha_3 \geq \mu_1 c_3' a_{32}$.

This can be seen by a direct estimate of the discriminant Δ after substituting for q_1' in the following proof.

Proof. Let $(x_2, x_3, \gamma) \in \bar{Z} \setminus Z$. We distinguish the following cases of degeneracy of x_2 and x_3.

case 1: x_3 is injective: Then by the proof of 8.1.1 we can assume that $x_3 = \xi_3$ is the canonical embedding and that x_{13} and x_2 have a factorization $\bar{x}_2$ in the following diagram

$$\begin{tikzcd}
M_3 \otimes A_{32} \otimes A_{21} \arrow[r, tail, "\widetilde{\xi}_3"] \arrow[d] & (M_2 \oplus M_3 \otimes A_{32}) \otimes A_{21} \arrow[r, two heads, "x_2"] \arrow[d] & P_1 \ . \\
M_3 \otimes A_{31} \arrow[r, tail, "\xi_3'"'] \arrow[urr, "x_{13}"] & M_2 \otimes A_{21} \oplus M_3 \otimes A_{31} \arrow[ur, two heads, "\bar{x}_2"'] &
\end{tikzcd}$$

Here also ξ_3' is the canonical embedding. Moreover it is easy to verify that in this case also the composed map $\gamma \circ (\bar{x}_2 \otimes id)$ admits a decomposition

$$(M_2 \otimes A_{21} \otimes H_{11}^*) \oplus (M_3 \otimes A_{31} \otimes H_{11}^*) \xrightarrow{\delta_1} (M_2 \otimes H_{12}^*) \oplus (M_3 \otimes H_{13}^*) \xrightarrow{\bar{\gamma}} Q_1.$$

Here $K = Ker(\bar{x}_2) \neq 0$ since $\bar{x}_2$ cannot be injective by the assumption on its rank. We choose subspaces M_2', M_3' such that

$$K \subset M_2' \otimes A_{21} \oplus M_3' \otimes A_{31}$$

and such that these subspaces are minimal with this property. Now we consider the spaces

$$P_3' = M_3', \quad P_2' = M_2' \oplus (M_3' \otimes A_{32}), \quad P_1' = x_2(P_2' \otimes A_{21}), \quad Q_1' = \gamma(P_1' \otimes H_{11}^*)$$

and their discriminant

$$\Delta = \alpha_1 p_1' + \alpha_2 p_2' + \alpha_3 p_3' - \beta_1 q_1'.$$

By the definition of the constant $c_1(m_2', m_3')$ and the diagram

$$\begin{tikzcd}
(M_2' \otimes A_{21} \oplus M_3' \otimes A_{31}) \otimes H_{11}^* \arrow[r, two heads] \arrow[d] & P_1' \otimes H_{11}^* \arrow[r, two heads] & Q_1' \\
M_2' \otimes H_{12}^* \oplus M_3' \otimes H_{13}^* \arrow[urr] & &
\end{tikzcd}$$

we obtain the estimate

$$q_1' \leq c_1(m_2', m_3')p_1' \leq c_1(m_2, m_3)p_1',$$

where by the definition of P_1' we have $p_1' = m_2'a_2 + m_3'a_{31} - k$. Inserting this we obtain

$$\Delta \geq (\mu_1 c_1 - \lambda_1)k + (\lambda_2 - \mu_1 c_1 a_{21})m_2' + (\lambda_3 - \mu_1 c_1 a_{31})m_3'$$

If $\mu_1 c_1 - \lambda_1 > 0$, conditions (2) and (3) imply that $\Delta > 0$. If, however, $\lambda_1 \geq \mu_1 c_1$ we have the direct estimate

$$\Delta \geq (\lambda_1 - \mu_1 c_1)p_1' + \alpha_2 p_2' + \alpha_3 p_3' > 0.$$

This proves the proposition in the first case.

case 2: x_3 is not injective

Here we let K denote the kernel of x_3 and we choose a subspace $M_3' \subset M_3$ such that $K \subset M_3' \otimes A_{32}$ and M_3' is minimal with this property. Then we consider the subspaces

$$P_3' = M_3', \quad P_2' = x_3(M_3' \otimes A_{32}), \quad P_1' = x_2(P_2' \otimes A_{21}), \quad Q_1' = \gamma(P_1' \otimes H_{11}^*).$$

We have the exact sequences

$$\begin{array}{ccccccccc} 0 & \to & K & \to & M_3' \otimes A_{32} & \xrightarrow{x_2} & P_2' & \to 0 & \\ 0 & \to & L & \to & M_3' \otimes A_{31} & \xrightarrow{x_{13}} & P_1' & \to & 0 \end{array}$$

where L denotes the kernel of x_{13}. From the factorization properties restricted to the spaces P_i' and Q_1' we extract the following commutative diagram of surjections

$$\begin{tikzcd}
 & M_3' \otimes A_{31} \otimes H_{11}^* \arrow[r] \arrow[d, "\delta_1"] & P_1' \otimes H_{11}^* \arrow[d, "\gamma"] \\
M_3' \otimes A_{32} \otimes A_{21} \otimes H_{11}^* \arrow[ur] \arrow[dr] & M_3' \otimes H_{13}^* \arrow[r, "\gamma_{13}"] & Q_1' \\
 & M_3' \otimes A_{32} \otimes H_{12}^* \arrow[r] \arrow[u, "\tau"] & P_2' \otimes H_{12}^* \arrow[u, "\gamma_{12}"] .
\end{tikzcd}$$

From this we get again the estimates

$$q_1' \leq c_3(m_3')p_1' \leq c_3(m_3)p_1' \quad \text{and} \quad q_1' \leq c_3'(m_3')p_2' \leq c_3'(m_3)p_2',$$

where $p_1' = m_3'a_{31} - l$ and $p_2' = m_3'a_{32} - k$. Let $0 < t < 1$ be a real number. Then we have

$$q_1' \leq tc_3'p_2' + (1-t)c_3p_1'.$$

Substituting this into the discriminant we get

$$\Delta \geq (\lambda_1 - (1-t)\mu_1 c_3)p_1' + (\alpha_2 - t\mu_1 c_3')p_2' + \alpha_3 m_3'.$$

Now condition (1) enables us to find t with

$$1 - \frac{\lambda_1}{\mu_1 c_3} \leq t \leq \frac{\alpha_2}{\mu_1 c_3'} \ ,$$

such that the first two terms of the estimate are non–negative. Therefore $\Delta > 0$, and again (x_2, x_3, γ) is not semi–stable.

In order to show that (1) can be replaced by one of (i), (ii) or (iii) we substitute α_i and p'_i and get after cancelation

$$\begin{array}{rcl}\Delta & = & -\lambda_1 l - \alpha_2 k + \lambda_3 m'_3 - \mu_1 q'_1 \\ & \geq & -\lambda_1 l - \alpha_2 k + \lambda_3 m'_3 - \mu_1 c'_3(m'_3 a_{32} - k) \\ & = & -\lambda_1 l + (\mu_1 c'_3 - \alpha_2)k + (\lambda_3 - \mu_1 c'_3 a_{32})m'_3 \quad .\end{array}$$

If $\alpha_2 \geq \mu_1 c'_3$, then by a direct estimate we get $\Delta > 0$. Therefore we may assume that $\mu_1 c'_3 - \alpha_2 > 0$. Since in addition $l \leq m'_3 a_{31}$, we get

$$\Delta > (\lambda_3 - \mu_1 c'_3 a_{32} - a_{31}\lambda_1)m'_3.$$

This shows that (1) can be replaced by (i). In the same way one shows that (1) can be replaced by (ii), using the other estimate of q'_1. That finally (1) can be replaced by (iii) can be shown by substituting first $m'_3 \geq \frac{p'_2}{a_{32}}$ and canceling $\alpha_2 p'_2$ and then substituting $p'_2 \geq \frac{p'_1}{a_{21}}$ to get

$$a_{32}a_{21}\Delta \geq \lambda_1 p'_1(a_{32}a_{21} - a_{31}) + (\lambda_3 - \mu_1 c_3 a_{32} a_{21})p'_1.$$

□

9. Estimation of constants

We are going to estimate the constants in the previous statements for some cases by numerical values. This will be done first for bilinear operators of finite dimensional vector spaces. There doesn't seem to be any statement of this kind in the literature. Let E, F and H be finite dimensional vector spaces and let

$$E \otimes H \xrightarrow{\tau} F$$

be a linear map. If M is any further finite dimensional vector space of dimension m, we are given the induced operator $\tau_m = \tau \otimes id_M$

$$E \otimes H \otimes M \to F \otimes M$$

The constant $c(m)$ is defined as follows. Let $\mathcal{K}$ be the set of all proper linear subspaces $K \subset H \otimes M$ such that K is not contained in $H' \otimes M$ for any proper subspace of H. We call such K generic. Then

$$c(m) := \sup_{K \in \mathcal{K}} \frac{\text{codim } \tau_m(E \otimes K)}{\text{codim } K}.$$

We suppose that for any non–zero element $u \in H$ the restricted map $E \otimes u \to F$ is surjective. We are going to estimate $c(m)$ by induction on m. For this we introduce the **minimal length** of a subspace $K \subset H \otimes M$ to be the minimal integer $d \geq 1$ such that a non–zero element $y \in K$ has a representation

$$y = u_1 \otimes x_1 + \ldots + u_d \otimes x_d$$

with $u_\nu \in H$ and $x_\nu \in M$. If such a representation is minimal the vectors $u_1, \ldots, u_d$ resp. $x_1, \ldots, x_d$ are linearly independent in H resp. M.

9.1. *The bounds $s(U)$ and s_d*

Given a subspace $U \subset H$ we let $E_U \subset E$ be the kernel of the composed operator $E \to F \otimes H^* \to F \otimes U^*$ such that $e \in E_U$ iff $\tau(e \otimes U) = 0$. Then the restricted map τ has a factorization

$$E_U \otimes H \to E_U \otimes H/U \to F,$$

and we define

$$s(U) := sup \ \{\dim F/\tau(E_U \otimes h) \mid h \in H \smallsetminus U\},$$

and

$$s_d := sup \ \{s(U) \mid U \subset H \quad \text{and } \dim U = d\}.$$

9.2. *The bounds $t_d(U)$ and t_d*

If $U \subset H$ is a linear subspace of dimension d we let $t_d(U)$ be the supremum of the values

$$\frac{s(U_1) + \cdots + s(U_d)}{d}$$

over all filtrations $0 = U_0 \subset U_1 \subset \ldots \subset U_d = U$ with $\dim U_\nu = \nu$, and we define

$$t'_d := sup\{t_d(U) \mid dimU = d\} \quad \text{and} \quad t_d := \sup_{1 \le k \le d} t'_k.$$

Then

$$t_d \le \sup_{1 \le k \le d} \frac{s_1 + \ldots + s_k}{k}.$$

9.3. Lemma*: Let $E \otimes H \xrightarrow{\tau} F$ be a linear map such that $E \otimes u \to F$ is surjective for any non–zero $u \in H$, and let the constants $c(m), t_d$ be defined as above, and let $n = \dim H$. Then*

$$\begin{array}{rclcl} c(m) & \le & \frac{t_{m-1}}{n} & \textit{if} & m \le n, \\ c(m) & \le & \frac{t_{n-1}}{n} & \textit{if} & n < m. \end{array}$$

Proof. We can do induction on $m = \dim M$. For $m = 1$ we have $c(1) = 0$ by the assumption on τ. Let now $m > 1$, let $K \subset H \otimes M$ be a proper generic subspace of minimal length d, and let $y \in K$ be an element of minimal length,

$$y = u_1 \otimes x_1 + \cdots + u_d \otimes x_d.$$

We let $\bar{M} = M/\langle x_d \rangle$ and denote by $\bar{K} \subset H \otimes \bar{M}$ the image of K. Then $\bar{K}$ is again non zero generic and has minimal length $d-1$ if $d \ge 2$. Writing also a bar for the image in $F \otimes \bar{M}$, we put

$$L = \tau_m(E \otimes K) \quad \text{and} \quad \bar{L} = \overline{\tau_m(E \otimes K)} = \tau_{m-1}(E \otimes \bar{K}),$$

and

$$cd(K) = \dim(H \otimes M/K), \quad cd(L) = \dim(F \otimes M/L)$$

and similarly for $\bar{K}$ and $\bar{L}$. We show now:

(1) If $d = 1$, then $cd(\bar{L}) = cd(L), \quad cd(\bar{K}) \le cd(K),$

(2) If $d > 1$, then $cd(L) \leq cd(\bar{L}) + s(U_{d-1})$, $cd(K) = cd(\bar{K}) + n$.

If $d = 1$, then $\tau_m(E \otimes y) = F \otimes x_1 \subset L$ and therefore the codimensions of L and $\bar{L}$ are equal. The estimate for $cd(\bar{K})$ results from the standard 9–diagram for the quotient spaces. If $d > 1$, we consider the spaces $U_i = Span(u_1, \dots, u_i)$ and we obtain

$$\tau_m(E_{U_{d-1}} \otimes y) \subset (F \otimes x_d) \cap L,$$

since the components $e \otimes u_i \otimes x_i$ are mapped to 0 for $e \in E_{U_{d-1}}$. By the definition of $s(U_{d-1})$ we obtain

$$\dim(F \otimes x_d)/(F \otimes x_d) \cap L \ \leq \ s(U_{d-1}).$$

Now the 9–diagram of the embedding of the exact sequence

$$0 \to (F \otimes x_d) \cap L \to L \to \bar{L} \to 0$$

yields $cd(L) \leq cd(\bar{L}) + s(U_{d-1})$. Because K is generic, $K \to \bar{K}$ is an isomorphism. It follows again from the obvious 9–diagram that $cd(K) = cd(\bar{K}) + \dim H$. This proves (2).

If $d > 1$ we obtain a quotient space M' of M of dimension $m-d+1$ and a subspace $K' \subset H \otimes M'$ non zero and generic of minimal length 1 and $L' = \tau_{m-d+1}(E \otimes K') \subset F \otimes M'$ such that

$$cd(L) \leq cd(L') + s(U_1) + \cdots + s(U_{d-1}),$$

$$cd(K) = cd(K') + n(d-1).$$

It follows that there exist positive integers $d_1, \dots, d_k$ with

$$d_1 + \ldots + d_k \leq m,$$

and

$$cd(L) \leq \sum_{i=1}^{k} s(U_1) + \ldots s(U_{d_i-1}),$$

$$cd(K) \geq n \sum_{i=1}^{k} (d_i - 1).$$

The estimate of the proposition follows now just from the definition of the bounds t_d. $\square$

9.4. *Constants for the symmetric algebra*

Let us now specialize to the case of the canonical mapping defined as contraction

$$S^{p+1}V^* \otimes V \xrightarrow{\tau} S^pV^*,$$

where $p \geq 1$ and $\dim V = q+1$ with $q \geq 2$. If $v_1, \dots, v_{q+1}$ is any basis of V and if V_i is the span of $v_1, \dots, v_i$, we have

$$E_{V_i} = S^{p+1}V_i^{\perp} \quad \text{and} \quad \tau(E_{V_i} \otimes v_{i+1}) = S^pV_i^{\perp}$$

where $V_i^{\perp} \subset V_i^*$ is the orthogonal complement of V_i. We thus obtain

$$s_i = s(V_i) = \dim S^pV^* - \dim S^pV_i^{\perp} = \tbinom{p+q}{p} - \tbinom{p+q-i}{p}$$

and therefore

$$\begin{array}{lll} c(m) & \leq & \frac{1}{q+1}\binom{p+q}{q} - \frac{1}{(q+1)(m-1)}\sum\limits_{i=1}^{m-1}\binom{p+q-i}{p} \quad \text{for} \quad m \leq q+1, \\ c(m) & \leq & \frac{1}{q+1}\binom{p+q}{q} - \frac{1}{q(q+1)}\sum\limits_{i=1}^{q-1}\binom{p+q-i}{p} \qquad\quad \text{for} \quad m > q+1. \end{array}$$

9.4.1. Remark*:* In case $p = 1$ the above estimates become

$$c(m) \leq \tfrac{m}{2(q+1)} \quad \text{for } m \leq q+1 \qquad \text{and} \qquad c(m) \leq \tfrac{1}{2} \quad \text{for } m > q+1.$$

In this case we can easily obtain

$$c(2) = \tfrac{1}{2q+1} \qquad \text{and} \qquad \tfrac{m(m-1)}{2(m(q+1)-1)} \leq c(m) \quad \text{for } m \leq q+1.$$

We conjecture that for $p = 1$ the constants are precisely

$$c(m) = \tfrac{m(m-1)}{2(m(q+1)-1)} \quad \text{for } m \leq q+1, \quad \text{and that} \quad c(m) \leq \tfrac{q+1}{2(q+2)} \quad \text{if } m > q+1.$$

In case $p = 2$ the precise value of $c(2)$ is $q + 1/2q + 1$.

9.5. Corollary*: In the case of homomorphisms of type*

$$M_1 \otimes \mathcal{O}(-2) \oplus M_2 \otimes \mathcal{O}(-1) \to N_1 \otimes \mathcal{O} \qquad \text{or} \qquad \to N_1 \otimes \mathcal{O} \oplus N_2 \otimes \mathcal{O}(1)$$

on the projective space $\mathbb{P}V$ of dimension q the constant $c_1(m_2)$ is the constant $c(m)$ of the mapping $S^2V \otimes V^* \xrightarrow{\tau} V$ and the constant $c_2(m_2)$ is the constant of the mapping $S^3V \otimes V^* \xrightarrow{\tau} S^2V$. Therefore

$$\begin{array}{llll} c_1(m_2) \leq \frac{m_2}{2(q+1)} & \textit{for} \quad m_2 \leq q+1 & & c_2(m_2) \leq \frac{m_2(3q+5-m_2)}{6(q+1)} \quad \textit{for} \quad m_2 \leq q+1 \\ & & \textit{and} & \\ c_1(m_2) \leq \frac{1}{2} & \textit{for} \quad m_2 > q+1 & & c_2(m_2) \leq \frac{q+2}{3} \qquad\qquad \textit{for} \quad m_2 > q+1. \end{array}$$

Proof. By definition in 7.1 the constants $c_l(m_2)$ arise from the maps

$$M_2 \otimes A_{21} \otimes H_{l1}^* \xrightarrow{\delta_l} M_2 \otimes H_{l2}^*.$$

For the given type of homomorphism we have

$$A_{21} = \mathrm{Hom}(\mathcal{O}(-2),\ \mathcal{O}(-1)) = V^*$$

and similarly

$$H_{11} = S^2V^*, \qquad H_{12} = V^*, \qquad H_{21} = S^3V^*, \qquad H_{22} = S^2V^*.$$

Therefore the constant $c_1(m_2)$ is the constant $c(m)$ for $p = 1$ in 9.4 and $c_2(m_2)$ the same constant in case $p = 2$. In both cases the bounds are obtained by specializing the bounds of 9.4. □

9.6. Lemma*: For homomorphisms of type*

$$M_1 \otimes \mathcal{O}(-d) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \to N_1 \otimes \mathcal{O}$$

on the projective space $\mathbb{P}V$ the constant $c_1(1,1)$ is $\dim V / \dim(S^{d-1}V)$.

Proof. We put $s(p) = \dim S^pV$. The homomorphisms δ_1 of 7.1 reduces here to the canonical map

$$(S^{d-2}V^* \oplus S^{d-1}V^*) \otimes S^dV \to S^2V \oplus V.$$

If K is a proper subspace of $S^{d-2}V^* \oplus S^{d-1}V^*$ which is not contained in one of the summands, it contains elements (f, g) with $f \neq 0$ or elements (f, g) with $g \neq 0$. But since $f \otimes S^dV \to S^2V$ is surjective, the map $\delta(K) \to S^2V$ is surjective. Hence codim $\delta(K) \leq s(1)$. If K contains an element $(0, g)$ with $g \neq 0$, then $\delta(K) = S^2V \oplus V$. For then $\delta(K)$ contains V, and since $\delta(K) \to S^2V$ is surjective, if follows that $\delta(K) = S^2V \oplus V$. Therefore, if codim $\delta(K) > 0$, there is a basis $(f_1, g_1), \dots, (f_k, g_k)$ of K with $f_1, \dots f_k$ linearly independent, i.e. $\dim K \leq s(d-2)$ or codim $K \geq s(d-1)$. Therefore $c_1(1, 1) \leq s(1)/s(d-1)$. But now we can find subspaces which realize this bound. For any $z \in V^*$ we let K be the space of all $(f, fz), f \in S^{d-2}V^*$. Then $K \cong S^{d-2}V^*$ and it follows also that in this case $\delta(K) \cong S^2V$. Then codim $\delta(K)/$codim $K = s(1)/s(d-1)$. □

10. Examples

10.1. *First example of type* $(2, 1)$

We use the abbreviation $m\mathcal{F}$ for $\mathbb{C}^m \otimes \mathcal{F}$ for a sheaf and a positive integer and consider here homomorphisms

$$2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{(\phi_1, \phi_2)} 3\mathcal{O}$$

over $\mathbb{P}_2$ of type $(2, 1)$. The polarization $\Lambda = (\lambda_1, \lambda_2, \mu_1)$ is supposed to be proper for W and $\mathbf{W}$, i.e. $\lambda_i > 0$ and $\alpha_i > 0$ for all i. The only constant involved here is $c_1(m_2) = c(1) = 0$. Therefore the conditions of 7.3 and 8.2 are automatically satisfied by $\alpha_2 = \lambda_2 - 3\lambda_1 > 0$. Hence all the quotients of $W^{ss}(G, \Lambda)$ will be good and projective under this condition. Since $2\lambda_1 + \lambda_2 = 1$ and $3\mu_1 = 1$, we can replace the polarization by the rational number $\rho = \lambda_2\lambda_1^{-1} > 3$. The numerical condition for (semi–)stability then becomes

$$\Delta = 3m_1 + 3\rho m_2 - (\rho + 2)n < 0 \ (\leq 0),$$

where (m_1, m_2, n) is the dimension vector of a (ϕ_1, ϕ_2)–invariant sub-family of vector spaces, such that $m_1 \leq 2$, $m_2 \leq 1$, $n \leq 3$. One can easily check that $\rho = 4$ is the only value for which Δ might be zero, and this is the case for the values $(0, 1, 2)$ and $(2, 0, 1)$. And indeed, the homomorphisms ϕ given by matrices

$$\begin{pmatrix} * & * & 0 \\ * & * & z_2 \\ * & * & z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ * & * & z_3 \end{pmatrix}$$

with generically chosen entries and linear forms z_i are semi–stable and not stable for $\rho = 4$.

10.1.1. *The case* $4 < \rho$

It is easy to show that in this case (ϕ_1, ϕ_2) is ρ–stable if and only if

- ϕ_2 is nowhere zero

- for any $(\psi_1, \psi_2) = h.(\phi_1, \phi_2)$ with $h \in H$ and any 1–dimensional subspace $M_1 \subset \mathbf{C}^2$ we have $\psi_1(M_1(\otimes\mathcal{O}(-2)) \neq 0$.

The first condition says that Coker (ϕ_2) is isomorphic to the universal quotient bundle Q on $\mathbb{P}_2$, and the second that ϕ_1 induces a 2–dimensional subspace of $H^0Q(2)$. It follows that the sets $W^s(\rho)$ of stable points are the same for $4 < \rho$, which we denote by W^s_+. Moreover, from the above characterization of stable homomorphism we deduce that the geometric quotient $M_+ = W^s_+/G$ is isomorphic to the Grassmannian

$$M_+ \cong Gr(2, H^0Q(2))$$

which is smooth of dimension 26. There is an interesting subvariety $Z \subset M_+$ which consists of the images of the homomorphisms

$$\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ * & * & z_3 \end{pmatrix} \qquad (1)$$

which belong to W^s_+. These are those (ϕ_1, ϕ_2) for which the induced homomorphism $2\mathcal{O}(-2) \to Q$ is not injective. We will see next that Z is isomorphic to the non–stable locus of M_0 below and is smooth of dimension 10.

10.1.2. *The case* $\rho = 4$

We write W^{ss}_0 for $W^{ss}(4)$. When considering the matrix representations we find that $W^s_+ \subset W^{ss}_0$ and that the remaining part $W^{ss}_0 \smallsetminus W^s_+$ consists of those homomorphisms for which ϕ_2 is zero in exactly one point. Such homomorphisms are equivalent to matrices

$$\begin{pmatrix} * & * & z \\ * & * & w \\ f & g & 0 \end{pmatrix} \qquad (2)$$

where z, w are independent linear and f, g are independent quadratic forms. Note, however, that W^s_+ intersects the non–stable locus of W^{ss}_0 in matrices equivalent to those of type (1). But the orbit closures in W^{ss}_0 of both types (1) and (2) of matrices contain the direct sums

$$\begin{pmatrix} 0 & 0 & z \\ 0 & 0 & w \\ f & g & 0 \end{pmatrix} \qquad (3)$$

of independent linear and quadratic forms. It follows from that that the induced morphism

$$M_+ \to M_0$$

of the quotients is bijective and moreover an isomorphism by Zariski's main theorem, because both spaces are normal. The points of the non–stable locus $M_0 \smallsetminus M^s_0$ are represented by matrices of type (3). It is again routine to deduce from this observation that

$$M_0 \smallsetminus M^s_0 \cong \mathbb{P}_2 \times Gr(2, H^0\mathcal{O}(2)).$$

The subvariety $Z \subset M_+$ corresponds to this set under the isomorphism. We can also identify the set M^s_0 of stable points with $Gr(2, H^0Q(2)) \smallsetminus Z$.

10.1.3. *The case* $3 < \rho < 4$

Similarly to the case W_+^s we find that here $W_-^s = W^s(\rho)$ is independent of ρ and that $W_-^s \subset W_0^{ss}$. The remaining part consists now of all homomorphisms which are equivalent to a matrix of type (1). Note that now homomorphisms of type (2) are contained in W_-^s. The induced morphism

$$M_- \to M_0$$

is again surjective but not injective over $M_0 \setminus M_0^s$. Let Y be the inverse image of $M_0 \setminus M_0^s$. Then Y consists of the points which are represented by matrices of type (2) which are not equivalent to matrices of type (3). It is easy to check that the restricted morphism

$$M_- \setminus Y \underset{\approx}{\longrightarrow} M_0^s$$

is bijective and therefore also an isomorphism by Zariski's main theorem. We are going to verify that Y is a divisor in M_-. There is a morphism

$$Y \xrightarrow{p} \mathbf{P}_2$$

which assigns to the class of (ϕ_1, ϕ_2) the point x at which ϕ_2 is degenerate. In this case

$$Coker(\phi_2) \cong \mathcal{O} \oplus \mathcal{I}_x(1)$$

where $\mathcal{I}_x$ is the ideal sheaf of x. For such (ϕ_1, ϕ_2) we are given an exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & 2\mathcal{O}(-2) & & & & \mathcal{O} & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \to & 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) & \xrightarrow{(\phi_1,\phi_2)} & 3\mathcal{O} & \to & \mathcal{F} & \to & 0 \\
 & & \uparrow & & \| & & \uparrow & & \\
0 & \to & \mathcal{O}(-1) & \xrightarrow{\phi_2} & 3\mathcal{O} & \to & \mathcal{O} \oplus \mathcal{I}_x(1) & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & 2\mathcal{O}(-2) & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

such that (ϕ_1, ϕ_2) corresponds to a 2–dimensional subspace $\Gamma \subset H^0(\mathcal{O}(2) \oplus \mathcal{I}_x(3))$. The condition of defining a element of Y is that Γ is neither contained in $H^0\mathcal{I}_x(3)$ nor in $H^0(\mathcal{O}(2))s$ for any section s of $\mathcal{O} \oplus \mathcal{I}_x(1)$. We let $U_x \subset Gr(2, H^0(\mathcal{O}(2) \oplus \mathcal{I}_x(3))$ denote the open subvariety of such Γ. By assigning to Γ the class of (ϕ_2, ϕ_2) where ϕ_1 is defined by a lifting in the above diagram, we get a morphism $U_x \to M_-$ whose image is the fibre $Y_x = p^{-1}(x)$. The morphism

$$U_x \twoheadrightarrow Y_x$$

is nothing but the quotient of U_x by the algebraic group $Aut(\mathcal{O} \oplus \mathcal{I}_x(1))$. It follows that Y_x is a variety of dimension 23. Using the techniques of this paper for this quotient, we can even prove that Y is smooth. Finally Y has dimension 25 and thus is a divisor in the irreducible and normal variety M_-.

Remarks: (1) One would like to interpret the matrices of type (2) as representing extensions of the sheaves $Coker(f, g)$ and $\mathcal{I}_x(1) = Coker\binom{z}{w}$. Indeed a matrix of type (2) defines such an extension, but this extension is isomorphic to the direct sum.

(2) The above correspondence between (ϕ_1, ϕ_2) and Γ indicates that the quotient spaces considered here are spaces of coherent systems as in [17].

10.1.4. *The flip*

The diagram $M_- \to M_0 \overset{\approx}{\leftarrow} M_+$ can be interpreted as a flip. It is induced by the inclusions $W_-^s \subset W_0^{ss} \supset W_+^s$. The orbits of stable points of type (2) in W_-^s and of type (1) in W_+^s don't intersect in W_0^{ss} but so do their closures in W_0^{ss}. Thus the fibres of $M_- \to M_0$ and $M_0 \leftarrow M_+$ correspond to the two different types of semi–stable orbits in W_0^{ss} defining the same points in $M_0 \smallsetminus M_0^s$.

10.2. *General homomorphisms of type* $(2,1)$

In a more general situation of type $(2,1)$ we consider homomorphisms

$$m_1\mathcal{O}(-2) \oplus m_2\mathcal{O}(-1) \to n_1\mathcal{O}$$

over $\mathbb{P}_n$. By 7.3 and 8.2 the conditions

$$\lambda_2 > (n+1)\lambda_1 \quad \text{and} \quad \lambda_2 \geq \frac{n+1}{n_1}c_1(m_2)$$

are sufficient for a polarization $\Lambda = (\lambda_1, \lambda_2, -\mu_1)$ for $W^{ss}(G,\Lambda)$ to admit a good and projective quotient. A polarization in this case is determined by the rational number $t = m_2\lambda_2$ with $0 < t < 1$ and $1 - t = m_1\lambda_1$, $\mu_1 = 1/n_1$. A Λ–(semi–)stable homomorphism is then called t–(semi–)stable. We write $W^{ss}(t)$ and $W^s(t)$ for $W^{ss}(G,\Lambda)$ and $W^s(G,\Lambda)$. In terms of t the conditions are

$$1 > t > \frac{(n+1)m_2}{(n+1)m_2 + m_1} \quad \text{and} \quad t \geq \frac{(n+1)m_2}{n_1}c_1(m_2).$$

Such polarizations exist if and only if

$$n_1 > (n+1)m_2c_1(m_2).$$

By the 9.4.1 and 9.5 these conditions are satisfied if

$$n_1 > 0, \quad n_1 > 1, \quad n_1 > \frac{m_2^2}{2}, \quad n_1 > \frac{n+2}{2}m_2$$

in case

$$m_2 = 1, \quad m_2 = 2, \quad 2 < m_2 \leq n, \quad n < m_2$$

respectively. In order to measure t–stability we introduce the numbers

$$r_1 = \frac{m_1'}{m_1}, \quad r_2 = \frac{m_1'}{m_2}, \quad s_2 = \frac{n_1'}{n_1}$$

and call (r_1, r_2, s_1) ϕ–admissible if there are subspaces $M_1' \subset M_1, M_2' \subset M_2, N_1' \subset N_1$ of dimensions m_1', m_2', n_1' such that ϕ maps $M_1' \otimes \mathcal{O}(-2) \oplus M_2' \otimes \mathcal{O}(-1)$ into $N_1' \otimes \mathcal{O}$. Then ϕ is t–(semi–)stable if and only if for any ϕ–admissible proper triple (r_1, r_2, s_1), i.e. a triple which is neither $(0,0,0)$ or $(1,1,1)$, we have

$$\Delta_t = (1-t)r_1 + tr_2 - s_1 < 0 \ (\leq 0).$$

A polarization t is called critical if there are proper triples with $\Delta_t = 0$. Thus the critical values of t are the rational numbers

$$\frac{s_1 - r_1}{r_2 - r_1}\ ,$$

where we may assume $s_1 \neq 0, 1$ and thus $r_1 \neq r_2$. We let t_{max} be the maximal critical value if there are such with $0 < t < 1$ and put $t_{max} = 0$ otherwise. If t is not critical we have $W^s(t) = W^{ss}(t)$.

10.2.1. Lemma: *Suppose that m_2 and n_1 are relatively prime and that $t_{max} < t < 1$. Then $\phi = (\phi_1, \phi_2)$ is t–stable if and only if*

(1) *ϕ_2 is stable with respect to the group $GL(M_2) \times GL(N_1)$.*
(2) *For any 1–dimensional subspace $\mathbb{C} \overset{j}{\hookrightarrow} M_1$, and any $h \in Hom(M_1 \otimes \mathcal{O}(-2), M_2 \otimes \mathcal{O}(-1))$ the map $(\varphi_1 + h \circ \varphi_2) \circ j \ :\ \mathcal{O}(-2) \to N_1 \otimes \mathcal{O}$ is not zero.*

Proof. By the characterization of stability in section 3 the homomorphism ϕ_2 is stable if and only if for any proper pair $M_2' \subset M_2, N_1' \subset N_1$ of ϕ_2–admissible subspaces $r_2 < s_1$. Now let (ϕ_1, ϕ_2) be stable. If ϕ_2 were not stable there would be a proper ϕ_2–admissible pair (r_2, s_1) with $s_1 \leq r_2$. But then $s_1 < r_2$ because m_2, n_1 are supposed to be relatively prime. Then $s_1/r_2 < t$ because s_1/r_2 is a critical value and thus $\Delta_t = r_2 t - s_1 > 0$, contradicting the stability of (ϕ_1, ϕ_2). The condition (2) is trivially satisfied if (ϕ_1, ϕ_2) is t–stable, because otherwise $(1, 0, 0)$ would be admissible with $\Delta_t = 1 - t > 0$. We have to show now that conversely (1), (2) imply that (ϕ_1, ϕ_2) is t–stable. For this let (r_1, r_2, s_1) be a proper (ϕ_1, ϕ_2)–admissible triple. If $r_1 \leq r_2$ and $r_2 = 0$, there is nothing to prove. If $r_2 > 0$ then $r_2 < s_1$ by (1) and we have $t(r_2 - r_1) < s_1 - r_1$ and hence $\Delta_t < 0$. If however $r_2 < r_1$ we have $\Delta_t < 0$ in case $r_1 \leq s_1$. Since the case $s_1 \leq r_2$ is only possible if $s_1 = r_2 = 0$ and then $r_1 = 0$ by (2), we can assume that $r_2 < s_1 < r_1$. But then

$$\frac{r_1 - s_1}{r_1 - r_2} < t$$

because the fraction is a critical value, and last inequality is the inequality $\Delta_t < 0$. □

Now we are able to describe the space $M_+ = W^s(t)/G$ for $t_{max} < t$ which is independent of t. According to the lemma $W^s(t)$ can only be non–empty if there are stable morphisms ϕ_2. This is the case if and only if

$$\frac{1}{\sigma(n)} < \frac{n_1}{m_2} < \sigma(n)$$

where $\sigma(n) = \frac{1}{2}(n + 1 + \sqrt{(n+1)^2 - 4})$, see [4]. We restrict ourselves now to the case where in addition to the previous conditions on n_1, m_2 we have $n_1 \geq n m_2$ and $(n_1, m_2) = 1$. Then a stable ϕ_2 is injective and a subbundle (except at finite number of points in case $n_1 = n m_2$, see [4], [6]). The quotient space of this space of stable homomorphisms by $GL(M_2) \times GL(N_1)$ is denoted by $N = N(n+1, m_2, n_1)$. It is a smooth projective variety and there is a universal sheaf $\mathcal{E}$ on $N \times \mathbb{P}_n$. For $x \in N$ let $\mathcal{E}_x$ denote the fibre sheaf representing x. Since it is the cokernel of the representing homomorphism ϕ_2, we get

$$h^0\mathcal{E}_x(2) = (n+1)\left(\frac{n_1(n+2)}{2} - m_2\right).$$

Therefore $p_*\mathcal{E}(2)$ is locally free on N where p denotes the first projection of $N \times \mathbb{P}_n$. Now M_+ can be non–empty only if

$$m_1 \leq (n+1)\left(\frac{n_1(n+2)}{2} - m_2\right).$$

If conversely this is the case for any stable ϕ_2 and any subspace $M_1 \subset H^0\mathcal{E}_x(2)$ where $x = [\phi_2]$, there is a lifting $\phi_1 : M_1 \otimes \mathcal{O}(-2) \to N_1 \otimes \mathcal{O}$ of $M_1 \otimes \mathcal{O}(-2) \to \mathcal{E}_x$, and (ϕ_1, ϕ_2) satisfies (1), (2) of the lemma. It follows now easily by considering corresponding families that

$$M_+ \cong Gr_N(m_1, p_*\mathcal{E}(2))$$

where Gr_N denotes the relative Grassmannian. It is more difficult to characterize the other moduli spaces $M(t) = W^{ss}(t)/G$ for the intervals between the critical values or for the critical values and to interpret the flips between them.

10.3. *Example of type* $(2,2)$

We consider now a simple example of type $(2,2)$ on $\mathbb{P}_3$ of homomorphisms

$$\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\phi} \mathcal{O} \oplus 3\mathcal{O}(1).$$

Again the polarizations $\Lambda = (\lambda_1, \lambda_2, -\mu_1, -\mu_2)$ are supposed to be proper for W and $\mathbf{W}$ such that we have $\lambda_i > 0,\ \mu_l > 0$ and

$$\lambda_2 > 4\lambda_1 \quad \text{and} \quad \mu_1 > 4\mu_2.$$

All constants $c_l(m_2)$ and $d_i(n_1)$ are again zero, because $m_2 = n_1 = 1$. Then by the above conditions also the conditions for proposition 7.7 and proposition 8.2 are satisfied, such that there exists a good and projective quotient $W^{ss}(G,\Lambda)//G$ for any polarization satisfying the conditions. Since we have $\lambda_1 + \lambda_2 = 1$ and $\mu_1 + 3\mu_2 = 1$, the polarization Λ is determined already by λ_2 and μ_1, for which the above conditions become

$$1 > \lambda_2 > \frac{4}{5} \quad \text{and} \quad \frac{3}{7} > 1 - \mu_1 > 0. \tag{1}$$

Next we derive the conditions for the occurrence of true semi–stable points. If (m_1, m_2, n_1, n_2) is the dimension vector of a ϕ–invariant sub-family we have to consider the equation

$$\Delta = (1-\lambda_2)m_1 + \lambda_2 m_2 - \mu_1 n_1 - \frac{1}{3}(1-\mu_1)n_2 = 0.$$

By inserting all possible dimension vectors we get the 6 conditions

$$1 - \mu_1 = \frac{3}{k}\lambda_2, \quad 1 - \mu_1 = -\frac{3}{k}\lambda_2 + \frac{3}{k} \tag{2}$$

for $k = 1, 2, 3$. If one of these is satisfied, there might be non–stable points in $W^{ss}(G,\Lambda)$. In the following figure 1 the lines with the equations (2) are shown together with the rectangle (1) (lower right), for the points of which we get good and projective quotients.

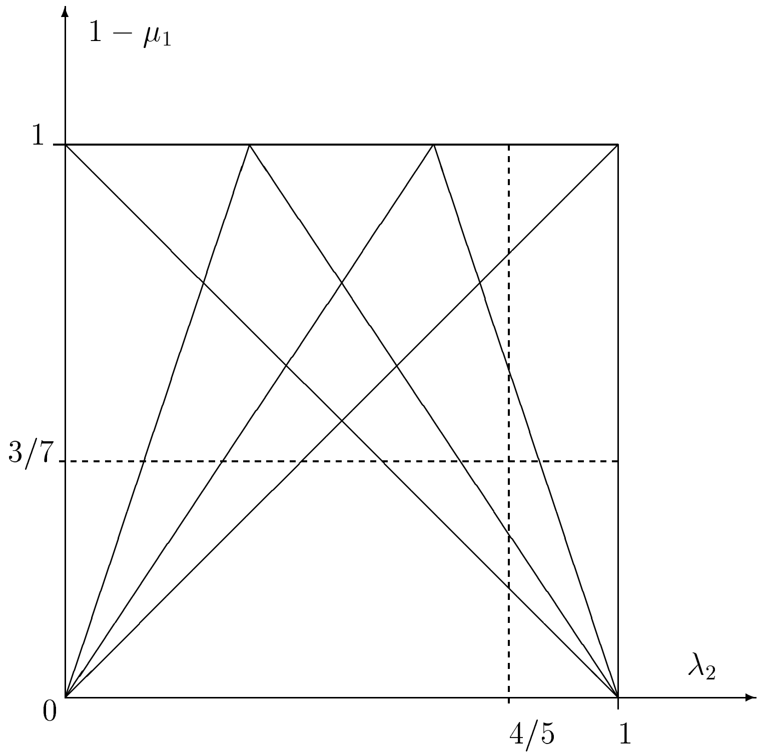

Figure 1

The homomorphism ϕ defined by the matrix

$$\left(\begin{array}{c|c} z_2^2 - z_1 z_3 & z_0 \\ \hline z_0^3 & z_1^2 \\ z_1^3 & z_2^2 \\ z_2^3 & z_3^2 \end{array}\right),$$

where the z_i are homogeneous coordinates of $\mathbb{P}_3$, is easily verified to be G–stable for each polarization Λ in the rectangle (1). Therefore the moduli spaces are not empty. On each of the 3 lines in the rectangle (1) each point defines one and the same open set $W^{ss}(G,\Lambda)$ and hence one and the same moduli space with semi–stable and non–stable points. Similarly, on each of the 4 open triangles we have one and the same moduli space, which is a smooth projective geometric quotient. Each of the 7 spaces has dimension 77. The reader may also verify that the moduli space for an open triangle admits a morphism to the moduli space of each of its edges, thereby defining a chain of flips.

10.4. *More general homomorphisms of type* $(2,2)$

More general homomorphisms for which we know the constants explicitly are homomorphisms of type

$$m_1\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \to 2\mathcal{O} \oplus n_2\mathcal{O}(1)$$

over $\mathbb{P}_3$, say. By remark 9.4.1 the constants are here

$$c_1(2) = d_2(2) = \frac{1}{7} \quad \text{and} \quad c_2(2) = d_1(2) = \frac{4}{7}.$$

Let W be the space of those homomorphisms. A proper polarization $\Lambda = (\lambda_1, \lambda_2, -\mu_1, -\mu_2)$ for W satisfies

$$m_1\lambda_1 + 2\lambda_2 = 1 \ , \quad 2\mu_1 + n_2\mu_2 = 1$$

with $\lambda_1, \lambda_2, \mu_1, \mu_2$ positive. We will also assume that $\alpha_2 > 0$, $\beta_1 > 0$, i.e. $\lambda_2 > 4\lambda_1$ and $\mu_1 > 4\mu_2$. These four conditions can be replaced by

$$\frac{4}{8+m_1} < \lambda_2 < \frac{1}{2} \quad \text{and} \quad \frac{4}{8+n_2} < \mu_1 < \frac{1}{2} \tag{1}$$

10.4.1. Claim: *There are polarizations Λ such that $W^{ss}(G,\Lambda)$ admits a good and projective quotient in the following cases*

(i) $m_1 < 6$ *and* $n_2 < 8$
(i') $m_1 \leq 6$ *and* $n_2 = 8$
(ii) $8 \leq m_1 + 3 \leq n_2$ *and* $8m_1 + 8 < 7n_2$

Proof. The conditions of 7.3 for the equivalence of (semi–)stability become

$$\lambda_2 \geq \frac{4}{7}(\mu_1 + 4\mu_2) \quad \text{and} \quad \mu_1 \geq \frac{16}{7}(4\lambda_2 - 15\lambda_1) \tag{2}$$

and the conditions of 8.2 for the projectivity of the quotient become

$$\lambda_2 \geq \frac{4}{7} \quad \text{and} \quad \mu_1 \geq \frac{4}{7}\lambda_2 \ . \tag{3}$$

The first condition of (3) follows already from the first of (2). After replacing λ_1 and μ_2 conditions (2) and (3) are equivalent to

$$\begin{array}{rcl} \frac{7}{4}n_2\lambda_2 & \geq & (n_2-8)\mu_1 + 4 \\ \frac{7}{16}m_1\mu_1 & \geq & (4m_1+30)\lambda_2 - 15 \\ \mu_1 & \geq & \frac{4}{7}\lambda_2 \end{array} \tag{4}$$

Using (1) for λ_2, we find that (4) has a solution (λ_2, μ_1) if the system

$$\begin{array}{rcl} \frac{7n_2}{8+m_1} & \geq & (n_2-8)\mu_1 + 4 \\ \mu_1 & > & \frac{16}{7(8+m_1)} \end{array} \tag{5}$$

has a solution μ_1. For this we distinguish the cases $n_2 < 8$, $n_2 = 8$, $8 < n_2$. If $n_2 < 8$ the first inequality of (5) has a solution $\mu_1 < \frac{1}{2}$ if $m_1 < 6$. If $n_2 = 8$, then $m_1 \leq 6$, which is case (i'). If $n_2 > 8$, the first inequality of (5) reduces to

$$\frac{7n_2 - 4m_1 - 32}{(n_2-8)(m_1+8)} \geq \mu_1 > \frac{4}{n_2+8}. \tag{6}$$

Then (5) has a solution μ_1 if and only if

$$\begin{array}{rcl} 7n_2 - 4m_1 - 32 & > & 0 \\ (7n_2 - 4m_1 - 32)(n_2+8) & > & 4(n_2-8)(m_1+8) \\ 7(7n_2 - 4m_1 - 32) & > & 16(n_2-8) \end{array}$$

These inequalities reduce to

$$\begin{array}{rcl} 7n_2 & > & 4m_1 + 32 \\ 7n_2 & > & 8m_1 + 8 \\ 33n_2 & > & 28m_1 + 96 \end{array}$$

They are all satisfied if we suppose (ii) of the claim. $\square$

In *figure 2* the lines of the critical values of the polarizations i.e. of the pairs (λ_2, μ_1) are shown together with the small region of those pairs which satisfy the sufficient conditions (4) for the existence of a good and projective quotient, based on the values $m_1 = 3$ and $n_2 = 5$.

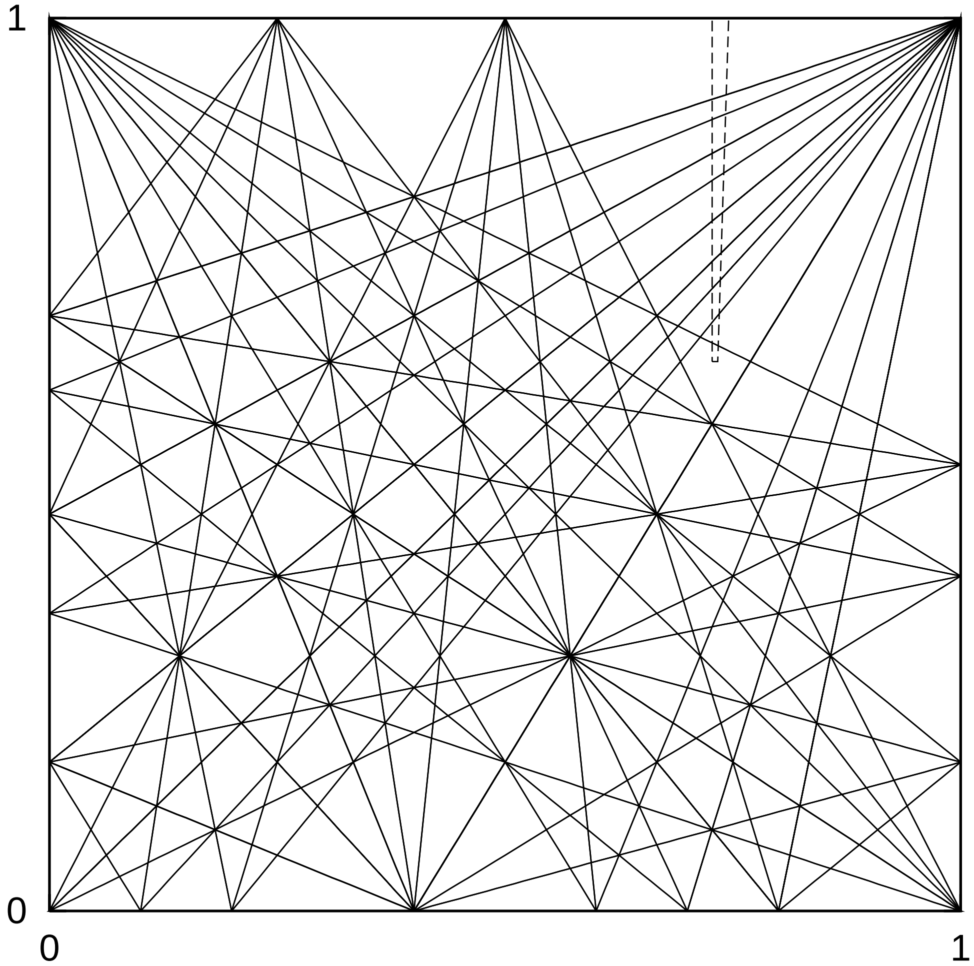

Figure 2
Here the horizontal axis represents $m_2\lambda_2$ and the vertical axis represents $n_1\mu_1$ for $m_1 = 3$ and $n_2 = 5$.

10.5. *Example of type* $(3,1)$

As an example of type $(3,1)$ we consider only the space of homomorphisms

$$\mathcal{O}(-4) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \to 5\mathcal{O}$$

over $\mathbb{P}_3$. We assume again that all λ_i and all α_i are positive. Then the conditions of 7.3 together with the normalization of the polarization are

$$\begin{array}{ll} \lambda_1 + \lambda_2 + \lambda_3 = 1 & \mu_1 = \frac{1}{k} \\ \lambda_2 > 10\lambda_1 & \lambda_2 \geq \frac{4}{5} c_1(1,1) \\ \lambda_3 - 4\lambda_2 + 20\lambda_1 > 0. & \end{array}$$

As additional condition for the projectivity of the quotient we use condition (a) of the remark following proposition 8.3.1. Since in this case both the constants $c_3(1)$ and $c_3'(1)$ are zero, this condition is just $\lambda_1 \geq 0$ and is already satisfied by our assumption.

For homomorphisms of the above type the condition $\lambda_3 < \frac{4}{5}$ is necessary if $W^s(G,\Lambda) \neq \emptyset$. For if $\phi = (\phi_1, \phi_2, \phi_3)$ is an element of W then ϕ_3 has degree 1 and thus contains at most 4 independent components. Then $m_1 = m_2 = 0$ and $m_3 = 1$, $n_1 = 4$ is a choice of dimensions of ϕ–invariant subspaces and the discriminant becomes $\Delta = \lambda_3 - \frac{1}{5}$.

By 9.6 the value of $c_1(1,1)$ is $\frac{1}{5}$. Now it is easy to see that there exist polarizations Λ which satisfy the above inequalities. That $W^s(G,\Lambda)$ is then indeed non–empty follows from the existence of generic matrices as in 10.3. Moreover there are again regions of polarizations for which the sets $W^{ss}(G,\Lambda)$ are the same and which are responsible for flips.

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