Brakhage's implicit iteration method and Information Complexity of equations with operators having closed range

Sergei V. Pereverzev and Eberhard Schock

Abstract

An a posteriori stopping rule connected with monitoring the norm of second residual is introduced for Brakhage's implicit nonstationary iteration method, applied to ill-posed problems involving linear operators with closed range. It is also shown that for some classes of equations with such operators the algorithm consisting in combination of Brakhage's method with some new discretization scheme is order optimal in the sense of Information Complexity.

1 Introduction

The present paper is devoted to methods for approximate solution of linear operator equations of the form

$$Tx = y \tag{1.1}$$

with operators T having closed range. The equations (1.1) will be considered in the Hilbert space X with the usual inner product $(\cdot, \cdot)$ and the usual norm $\|\cdot\|_X$. It is known [9, p.153] that operator $T \in \mathcal{L}(X, X)$ has closed $Range(T)$ if and only if for some $\beta > 0$

$$\inf_{\substack{u \in X \\ u \perp Ker(T), u \neq 0}} \frac{\|Tu\|_X}{\|u\|_X} \geq \beta \tag{1.2}$$

Let $e_1, e_2, \ldots, e_m, \ldots$ be some orthonormal basis of the Hilbert space X and P_m be the orthogonal projector on *span* $\{e_1, e_2, \ldots, e_m\}$, that is

$$P_m u = \sum_{i=1}^{m} (u, e_i) e_i$$

For $r \in (0, \infty)$ we let X^r denote a linear subspace of X which is equipped with the norm

$$\|u\|_{X^r} := \|u\|_X + \|D_r u\|_X,$$

where D_r is some linear (non-bounded) operator acting from X^r to X, and for any $m = 1, 2, \ldots$

$$\|I - P_m\|_{X \to X^r} \leq c_r m^{-r}, \tag{1.3}$$

where I is the identity operator and the constant c_r is independent of m.

We assume that operators T of equations (1.1) have some special structure. Namely,

$$T = B + A \tag{1.4}$$

where B is some fixed operator such that $B, B^* \in \mathcal{L}(X, X) \cap \mathcal{L}(X^r, X^r)$ and

$$A \in \mathcal{H}^r_\gamma := \{A : \|A\|_{X \to X^r} \leq \gamma_1, \ \|A^*\|_{X \to X^r} \leq \gamma_2, \ \|(D_r A)^*\|_{X \to X^r} \leq \gamma_3\},$$

$$\gamma = (\gamma_1, \gamma_2, \gamma_3)$$

where L^* is such that for any $u, v \in X \quad (u, Lv) = (L^* u, v)$. For fixed B such that $B, B^* \in \mathcal{L}(X, X) \cap \mathcal{L}(X^r, X^r)$ and $\beta > 0$ we denote by $\mathcal{K}^r_{B,\beta,\gamma}$ the set of operators $T \in B \oplus \mathcal{H}^r_\gamma$ of the form (1.4) which satisfy the condition (1.2).

Consider some examples of operators $T \in \mathcal{K}^r_{B,\beta,\gamma}$.

Example 1. As Hilbert space X we take the space $L_2(0,1)$ of square-summable functions on (0,1) and for $r = 1$ as X^r we take Sobolev space $W^1_2(0,1)$ of functions f having square-summable derivatives $f' \in L_2(0,1)$. In this case $D_r = \frac{d}{dt}$. In Chapter 6 [9] as an example of equation (1.1) with operator having closed range one takes the Fredholm problem of the second kind

$$Tx(t) := x(t) - \lambda^{-1} \int_0^1 a(t,\tau) x(\tau) d\tau = y(t), \tag{1.5}$$

where λ is eigenvalue of integral operator with kernel $a(t,\tau)$. If $a(t,\tau)$ has mixed square-summable partial derivatives then integral operator from (1.5) belongs to $\mathcal{H}_\gamma^r$ for some γ and $r=1$. It means that operator T of equation (1.5) belongs to $\mathcal{K}_{B,\beta,\gamma}^r$ for $r=1$, $B=I$ and for some $\beta>0$.

EXAMPLE 2. Let $X^r=W_2^r$ be the Sobolev space of 2π-periodic functions having derivatives up to the order r which are square-summable on $[0,2\pi]$. In this case $X=L_2(0,2\pi)$, $D_r=\frac{d^r}{dt^r}$ and orthonormal basis $\{e_i\}$ consists of trigonometric functions. In [6, §3.4.3] one considers singular integral equations with Hilbert kernel

$$Tx(t):=a_1(t)x(t)+\frac{a_2(t)}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}x(\tau)d\tau+\int_0^{2\pi}a_3(t,\tau)x(\tau)d\tau=y(t). \tag{1.6}$$

It is common knowledge that for $a_1(t),a_2(t)\in W_2^r$ singular operators

$$Bx(t)\;:=\;a_1(t)x(t)\;+\;\frac{a_2(t)}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\tau-t}{2}x(\tau)d\tau, \tag{1.7}$$

$$B^*x(t)\;:=\;a_1(t)x(t)\;+\;(2\pi)^{-1}\int_0^{2\pi}\operatorname{ctg}\frac{t-\tau}{2}a_2(\tau)x(\tau)d\tau,$$

act continuously from L_2 to L_2 and from W_2^r to W_2^r. Moreover, if $a_1^2+a_2^2\neq 0$ then operator T from (1.6) has closed range and satisfies the codition (1.2) for some $\beta>0$. On the other hand, if the kernel $a_3(t,\tau)$ has square-summable partial derivatives $\frac{\partial^{i+j}}{\partial t^i\partial t^j}$, $i,j=0,1,\dots,r$, then integral operator

$$Ax(t)\;:=\int_0^{2\pi}a_3(t,\tau)x(\tau)d\tau$$

belongs to $\mathcal{H}_\gamma^r$ for some γ and $X=L_2$, $X^r=W_2^r$. Thus, if coefficients $a_1(t),a_2(t),a_3(t,\tau)$ satisfy the above mentioned conditions and $a_1(t),a_2(t)$ are fixed then the operator T from equation (1.6) belongs to $\mathcal{K}_{B,\beta,\gamma}^r$ for B determined by (1.7). Note that special case of (1.6) when $a_1(t)\equiv 0, a_2(t)\equiv 1$ was considered in [3, §1.5].

Solving the problem (1.1) with operators T having nontrivial $Ker(T)\neq 0$ one usually seeks the unique element that has minimal norm among all solutions of (1.1). If T^+ denotes the Moore-Penrose generalized inverse for operator T, this unique element is given by T^+y, $y\in Range(T)$. On the other hand, as indicated in [9, p.152], if $Range(T) \neq X$ or $Ker(T) \neq 0$ then the problem (1.1) is not well posed in the sence of Hadamard and the crux of the difficulty is that only an approximation $y_\delta \in X$ to $y \in Range(T)$ is available such that $\|y - y_\delta\|_X \leq \delta$, where δ is a known error bound.

In this paper we will study the information complexity of the problem of recovery of T^+y from the equation

$$Tx = y_\delta \tag{1.8}$$

The paper is organized as follows. In Section 2 we propose a new discretization scheme for (1.8). Combining this scheme with Brakhage's implicit iteration method we obtain an approximation to T^+y with the best possible order of accuracy $O(\delta)$. In Section 3 we show that in the sence of Information Complexity our discretization scheme is order optimal for the class of equation (1.1) with operators $T \in \mathcal{K}^r_{B,\beta,\gamma}$ at least in the case when $Range(B)$ is closed and $dim\,Ker(B) < \infty$. Note that these conditions are fulfilled for operators B considered in Examples 1 and 2.

2 Brakhage's implicit iteration method

In 1993 in personal communication H.Brakhage proposed an implicit nonstationary adaptive iteration method for solving linear operator equations of the form (1.1), which has a linear convergence rate. This method consists in the construction of the following sequence of approximate solutions

$$x_k = (I + \alpha_{k-1}T^*T)^{-1}(x_{k-1} + \alpha_{k-1}T^*y), \quad x_0 = 0 \tag{2.1}$$

where

$$\alpha_{k-1} = \frac{\|Tx_{k-1} - y\|^2_X}{\|T^*(Tx_{k-1} - y)\|^2_X} \tag{2.2}$$

Now we present a very nice unpublished result of H.Brakhage concerning method (2.1),(2.2).

Theorem. (H.Brakhage, 1993). *Let T be a linear and injective operator from X to X. Assume that $y \in Range(T)$ is such that the solution $\hat{x}$ of (1.1) belongs to $Range((T^*T)^\nu)$ for some $\nu > 0$. Then there is a constant $c > 0$ such that*

$$\|\hat{x} - x_k\|_X \leq c2^{-\frac{\nu k}{\nu+1}}.$$

Proof. For $g \in Range((T^*T)^p)$ we will use a norm

$$\|g\|_p = \|(T^*T)^p g\|_X.$$

It is common knowledge that

$$\|g\|_X \le \|g\|_{-p}^{\frac{q}{p+q}} \|g\|_q^{\frac{p}{p+q}}, \quad p, q > 0, \tag{2.3}$$

and for $l \ge 0$, $q \ge p$

$$\frac{\|g\|_q}{\|g\|_p} \le \frac{\|g\|_{q+l}}{\|g\|_{p+l}}. \tag{2.4}$$

Let $\theta_k = x_k - \hat{x}$. From (2.1), (2.2) it follows that

$$(I + \alpha_{k-1} T^*T)\theta_k = \theta_{k-1}, \tag{2.5}$$

$$\alpha_{k-1} = \frac{\|\theta_{k-1}\|_{1/2}^2}{\|\theta_{k-1}\|_1^2}. \tag{2.6}$$

If $\hat{x} \in Range((T^*T)^\nu)$ then using (2.5) it is easy to calculate that for $p \ge -\nu$

$$\|\theta_k\|_p^2 + 2\alpha_{k-1}\|\theta_k\|_{p+1/2}^2 + \alpha_{k-1}^2 \|\theta_k\|_{p+1}^2 = \|\theta_{k-1}\|_p^2 \tag{2.7}$$

and

$$\|\theta_k\|_p \le \|\theta_{k-1}\|_p, \quad p \ge -\nu. \tag{2.8}$$

Combining (2.7) and (2.4),(2.6) we have

$$\frac{\|\theta_k\|_1^2}{\|\theta_{k-1}\|_1^2} = \frac{\|\theta_{k-1}\|_{1/2}^2}{\|\theta_{k-1}\|_1^2} \cdot \frac{\|\theta_k\|_1^2}{\|\theta_{k-1}\|_{1/2}^2} =$$

$$= \alpha_{k-1} \frac{\|\theta_k\|_1^2}{\|\theta_k\|_{1/2}^2 + 2\alpha_{k-1}\|\theta_k\|_1^2 + \alpha_{k-1}^2\|\theta_k\|_{3/2}^2} =$$

$$= \frac{\alpha_{k-1}}{\frac{\|\theta_k\|_{1/2}^2}{\|\theta_k\|_1^2} + 2\alpha_{k-1} + \alpha_{k-1}^2 \frac{\|\theta_k\|_{3/2}^2}{\|\theta_k\|_1^2}} \le \frac{\alpha_{k-1}}{\alpha_k + 2\alpha_{k-1} + \alpha_{k-1}^2 \frac{\|\theta_k\|_1^2}{\|\theta_k\|_{1/2}^2}} =$$

$$= \frac{\alpha_{k-1}}{\alpha_k + 2\alpha_{k-1} + \alpha_{k-1}^2/\alpha_k} = \left(\frac{\alpha_k}{\alpha_{k-1}} + 2 + \frac{\alpha_{k-1}}{\alpha_k} \right)^{-1} \le \frac{1}{4}.$$

Then

$$\|\theta_k\|_1 \le 2^{-1}\|\theta_{k-1}\|_1 \le 2^{-2}\|\theta_{k-2}\|_1 \le \cdots \le 2^{-k}\|\theta_0\|_1 = 2^{-k}\|\hat{x}\|_1.$$

Combining this with (2.3), (2.8) for $\hat{x} = (T^*T)^\nu v$, $v \in X$, we arrive at the final inequality

$$\|\hat{x} - x_k\|_X = \|\theta_k\|_X \leq \|\theta_k\|_{-\nu}^{\frac{1}{\nu+1}} \|\theta_k\|_1^{\frac{\nu}{\nu+1}} \leq \|\theta_0\|_{-\nu}^{\frac{1}{\nu+1}} 2^{-\frac{k\nu}{\nu+1}} \|\hat{x}\|_1 =$$

$$= 2^{-\frac{k\nu}{\nu+1}} \|v\|_X^{\frac{1}{\nu+1}} \|v\|_{\nu+1} \leq c 2^{-\frac{k\nu}{\nu+1}},$$

as claimed.

Note that the structure of Brakhage's method (2.1), (2.2) is close to explicit iteration method proposed by V.M. Friedman [1], in which

$$x_k = (I - \alpha_{k-1} T^*T) x_{k-1} + \alpha_{k-1} T^* y, \quad k = 1, 2, \ldots,$$

and α_{k-1} has the form (2.2) too. Originally Friedman's method was proposed for nonperturbed equations (1.1) with operators having closed range. For noisy equations (1.8) with such operators explicit iteration method closed to Friedman's method was studied in [2, §3.3]. In [2] it is also shown that an order-optimal accuracy $O(\delta)$ is attained when a posteriori residual stopping rule is used to determine the iteration number M for which

$$\|y_\delta - T_n x_M\|_X \leq b\,\delta, \tag{2.9}$$

where $b > 1$ and T_n is such that $\|T - T_n\|_{X \to X} \leq \delta$.

Our goal in this section is to establish the order of accuracy $O(\delta)$ for Brakhage's implicit nonstationary method with perturbed data when the iteration number m is selected by discrepancy principle connected with monitoring the norm of second residual. Namely, m will be chosen by

$$\|T_n^*(T_n x_m - y_\delta)\|_X \leq b\,\delta. \tag{2.10}$$

In our opinion it makes sense to use a posteriori stopping criterion (2.10) because usually m selected by (2.10) has value less than M chosen as in (2.9). Note that for the stationary iteration methods stopping criterion (2.10) was studied earlier in [9, p.166].

In the sequel as operator T_n we will take

$$T_n = B + \sum_{k=1}^{n} (P_{2^k} - P_{2^{k-1}}) A P_{2^{n-k}} + P_1 A P_{2^n}.$$

Moreover, we assume that for y from (1.1) $\|y\|_X \leq \rho$.

From Lemma A.2 [5] it follows

Lemma 2.1 *For $T \in \mathcal{K}^r_{B,\beta,\gamma}$ and n such that $n2^n \asymp \delta^{-1/r} \log^{1+1/r} \frac{1}{\delta}$*

$$\|T - T_n\|_{X\to X} \ \le \ \delta.$$

Let us denote by $S_p(L)$ the spectrum of some selfadjoint nonnegative operator $L \in \mathcal{L}(X, X)$.

Lemma 2.2 *For $T \in \mathcal{K}^r_{B,\beta,\gamma}$, $0 < \delta < \frac{\beta}{2}$ and n such that $n2^n \asymp \delta^{-1/r} \log^{1+1/r} \frac{1}{\delta}$*

$$S_p(T_n^*T_n), S_p(T_nT_n^*) \subset [0, \delta^2] \cup [(\beta - \delta)^2, \|T_n\|^2_{X\to X}].$$

The assertion of the Lemma 2.2 follows from Lemma 2.1 and Lemma 1.3 [9, p.154].

Let E_n and F_n be the orthogonal projectors on invariant subspaces of $T_n^*T_n$ and $T_nT_n^*$ respectively corresponding to the part of spectrum which belongs to $[(\beta - \delta)^2, \|T_n\|^2_{X\to X}]$. Moreover, we will consider the orthogonal projectors E and F on closed subspaces $Range(T^*)$ and $Range(T)$ respectively.

From Lemma 2.1 and Lemma 1.5 [9, p.156] we obtain the following statement.

Lemma 2.3 *Asume the condition of Lemma 2.1. Then*

$$\|T_n(I - E_n)\|_{X\to X} \le \delta, \quad \|(I - E_n)E\|_{X\to X} \le \frac{\delta}{\beta - \delta}.$$

Let us apply Brakhage's implicit iteration method to the equation (1.8) and define x_k by

$$x_k \ = \ (I + \alpha_{k-1}T_n^*T_n)^{-1}(x_{k-1} + \alpha_{k-1}T_n^*y_\delta), \tag{2.11}$$

where

$$\alpha_{k-1} \ = \ \frac{\|T_nx_{k-1} - y_\delta\|^2_X}{\|T_n^*(T_nx_{k-1} - y_\delta)\|^2_X}, \quad x_0 = 0, \ T_n^*y_\delta \ne 0. \tag{2.12}$$

From (2.11), (2.12) we see that x_k may be expressed as

$$x_k \ = \ g_k(T_n^*T_n)T_n^*y_\delta,$$

where

$$g_k \ = \ \lambda^{-1}\Big(1 \ - \ \prod_{i=0}^{k-1}(\alpha_i\lambda + 1)^{-1}\Big), \quad \lambda \ge 0.$$

From [7],[10] it follows

Lemma 2.4 *For $\nu \in (0, 1]$*

$$\max_{\lambda\in[0,\infty)} |1 - \lambda g_k(\lambda)|\lambda^\nu \le \nu^\nu \sigma_k^{-\nu},$$

$$\max_{\lambda\in[0,\infty)} g_k(\lambda) = \sigma_k,$$

where

$$\sigma_k := \sum_{i=0}^{k-1} \alpha_i.$$

Let us denote by Δ_k the second residual obtained on k-th iteration of Brakhage's method (2.11), (2.12), that is

$$\Delta_k = T_n^*(T_n x_k - y_\delta).$$

Theorem 2.1 *Let the assumptions of Lemma 2.1 be fulfilled and m be such that for some $b \ge 1$ $\|\Delta_k\|_X > b\delta$, $k = 1, 2, \ldots, m$, but $\|\Delta_{m+1}\|_X \le b\,\delta$. Then*

$$\|T^+y - x_{m+1}\|_X \le c\,\delta,$$

where c depends only on β, γ, r, b and ρ.

Proof. We begin by noting that

$$\Delta_k = T_n^*(T_n g_k(T_n^*T_n)T_n^* y_\delta - y_\delta) =$$

$$= (T_n^*T_n g_k(T_n^*T_n) - I)T_n^* y + (T_n^*T_n g_k(T_n^*T_n) - I)T_n^*(y_\delta - y) =$$

$$= J_1 + J_2. \tag{2.13}$$

Using Lemmas 2.1 and 2.4 we have

$$\|J_1\|_X \le \|(T_n^*T_n g_k(T_n^*T_n) - I)T_n^*T_nT^+y\|_X +$$

$$+ \|(T_n^*T_n g_k(T_n^*T_n) - I)T_n^*(T - T_n)T^+y\|_X \le$$

$$\le \max_{\lambda\ge 0} |1 - \lambda g_k(\lambda)|\lambda\|T^+\|_X + \max_{\lambda\ge 0} |1 - \lambda g_k(\lambda)|\sqrt{\lambda}\,\|T - T_n\|_{X\to X}\|T^+\|_X \le$$

$$\le (\sigma_k^{-1} + \frac{\sqrt{2}}{2}\delta\sigma_k^{-1/2})\|T^+y\|_X, \tag{2.14}$$

$$\|J_2\|_X \;\leq\; \delta \sup_{\lambda\geq 0} |1-\lambda g_k(\lambda)|\sqrt{\lambda} \;\leq\; \frac{\sqrt{2}}{2}\delta\sigma_k^{-1/2} \tag{2.15}$$

By virtue of (1.2) for y such that $\|y\|_X \leq \rho$ we have

$$\|T^+ y\|_X \;\leq\; \rho\,\beta^{-1}. \tag{2.16}$$

Then from (2.13)-(2.16) one sees that for $k = 1, 2, \ldots, m$

$$b\,\delta \;\leq\; \|\Delta_k\|_X \;\leq\; c_1\,\sigma_k^{-1} \;+\; c_2\,\delta\,\sigma_k^{-1/2}, \tag{2.17}$$

where c_1 and c_2 are some constants depending on ρ, β. (In the sequel we will often use the same symbol c for possibly different constants).

Keeping in mind (2.17) it is easy to calculate that

$$\sigma_k^{-1/2} \;\geq\; \frac{-c_2\delta + \sqrt{c_2^2\delta^2 + 4c_1 b\delta}}{2c_1} \;=\; \frac{2b\delta}{\sqrt{\delta}(c_2\sqrt{\delta} + \sqrt{c_2^2\delta + 4c_1 b})} \;\geq$$

$$\geq\; \frac{2b\sqrt{\delta}}{c_2 + \sqrt{c_2^2 + 4c_1 b}} \;\geq\; c\sqrt{\delta}.$$

Using this bound, we obtain for $k = 1, 2, \ldots, m$ the estimate

$$\sigma_k \;\leq\; c\delta^{-1}. \tag{2.18}$$

To complete the proof of the theorem we need the following lemmas.

Lemma 2.5 *Assume the condition of Lemma 2.1. Then for any k*

$$\|(I-E_n)(x_k - T^+ y)\|_X \;\leq\; c\delta(1+\sigma_k\delta).$$

Proof. From the definitions of E, T^+ it follows that $ET^+y = T^+y$ and as in [9, p. 158] we have

$$(I-E_n)(x_k - T^+y) = (I-E_n)(T_n^*T_n g_k(T_n^*T_n) - I)ET^+y+$$

$$+(I-E_n)g_k(T_n^*T_n)T_n^*(y_\delta - T_nT^+y) = J_3 + J_4.$$

From Lemma 2.3 and (2.16) one sees that

$$\|J_3\|_X = \|(T_n^*T_n g_k(T_n^*T_n) - I)(I-E_n)ET^+y\|_X \leq$$

$$\le \sup_{\lambda\ge 0} |1-\lambda g_k(\lambda)| \|(I-E_n)ET^+y\|_X \le$$

$$\le \frac{\delta\rho}{(\beta-\delta)\beta} \sup_{\lambda\ge 0} \prod_{i=0}^{k-1} (\alpha_i\lambda+1)^{-1} \le c\,\delta.$$

On the other hand, from Lemma 2.1 and (2.16) we get the estimate

$$\|y_\delta - T_nT^+y\|_X = \|y_\delta - y + (T-T_n)T^+y\|_X \le \delta(1+\rho\beta^{-1}) \qquad (2.19)$$

Then by virtue of Lemma 2.4 we find

$$\|J_4\|_X \le \|(I-E_n)g_k(T_n^*T_n)T_n^*\|_{X\to X}\|y_\delta - T_nT^+y\|_X \le$$

$$\le c\,\delta \sup_{\lambda\in[0,\delta^2]} g_k(\lambda)\sqrt{\lambda} \le c\,\delta^2\,\sigma_k.$$

Summing up we get the assertion of the lemma.

Lemma 2.6 *Assume the condition of Theorem 2.1. Then there is some constant c depending only on β, ρ, b such that $\alpha_m \le c$.*

Proof. Note that

$$\alpha_m = \frac{\|T_nx_m - y_\delta\|_X^2}{\|T_n^*(T_nx_m - y_\delta)\|_X^2} \le$$

$$\le \left\{ \frac{\|F_n(T_nx_m - y_\delta)\|_X}{\|T_n^*(T_nx_m - y_\delta)\|_X} + \frac{\|(I-F_n)(T_nx_m - y_\delta)\|_X}{b\delta} \right\}^2.$$

By virtue of the definition of F_n for any $k = 1, 2, \ldots$ we have

$$\frac{\|F_n(T_nx_k - y_\delta)\|_X}{\|T_n^*(T_nx_k - y_\delta)\|_X} \le \frac{\|F_n(T_nx_k - y_\delta)\|_X}{\|T_n^*F_n(T_nx_k - y_\delta)\|_X} \le \frac{1}{\beta-\delta} \qquad (2.20)$$

Moreover, from Lemmas 2.3, 2.5 and (2.16), (2.18) it follows that

$$\|(I-F_n)(T_nx_m - y_\delta)\|_X = \|(I-F_n)(y-y_\delta+(T_n-T)T^+y+T_n(x_m-T^+y))\|_X \le$$

$$\le \delta(1+\beta^{-1}\rho) + \|(I-F_n)T_n(x_n - T^+y)\|_X \le$$

$$\le c\,\delta + \|T_n(I-E_n)(I-E_n)(x_m - T^+y)\|_X \le$$

$$\le c\,\delta + \delta\|(I-E_n)(x_m - T^+y)\|_X \le c\,\delta(1+\delta+\sigma_m\delta^2) \le c\,\delta.$$

Using this bound and (2.20) we obtain the assertion of the lemma.

Now we are in a position to complete the proof of our theorem.

It is easy to see that

$$\|T^+y - x_{m+1}\|_X =$$
$$= \|E_n(T^+y - x_{m+1})\|_X + \|(I - E_n)(T^+y - x_{m+1})\|_X. \tag{2.21}$$

By virtue of the definition of E_n (see [9, p. 166])

$$\|E_n(T^+y - x_{m+1})\|_X \leq \frac{1}{\beta - \delta}\|T_n E_n(T^+y - x_{m+1})\|_X =$$
$$= \frac{1}{\beta - \delta}\|F_n T_n(T^+y - x_{m+1})\|_X. \tag{2.22}$$

Keeping in mind that $\|\Delta_{m+1}\|_X \leq b\,\delta$, from (2.20) we find

$$\|F_n(T_n x_{m+1} - y_\delta)\|_X \leq \frac{1}{\beta - \delta}\|\Delta_{m+1}\|_X \leq \frac{b\,\delta}{\beta - \delta}.$$

Using this bound and (2.19) we obtain

$$\|F_n T_n(T^+y - x_{m+1})\|_X \leq \|F_n(T_n T^+y - y_\delta)\|_X +$$
$$+\|F_n(T_n x_{m+1} - y_\delta)\|_X \leq \|T_n T^+ y - y_\delta\|_X + \frac{b\,\delta}{\beta - \delta} \leq c\,\delta. \tag{2.23}$$

On the other hand, from (2.18) and Lemmas 2.5, 2.6 we know that

$$\|(I - E_n)(T^+y - x_{m+1})\|_X \leq c\,\delta(1 + \sigma_{m+1}\delta) =$$
$$= c\,\delta(1 + \sigma_m\delta + \alpha_m\delta) \leq c\,\delta(1 + \delta). \tag{2.24}$$

The assertion of the theorem follows from (2.21)–(2.24).

Remark 2.1 *Since $\alpha_{k-1} \geq \|T_n^*\|_{X\to X}^{-2}$ and $\sigma_k \to \infty$ with $k \to \infty$, it follows from (2.17) that $\|\Delta_k\|_X \to 0$ with $k \to \infty$ and there exists m satisfying stopping criterion (2.10)*

3 The Estimate of Information Complexity

In this section we shall investigate the information complexity of finding approximate solutions of equations (1.1) with operators $T \in \mathcal{K}^r_{B,\beta,\gamma}$ and exact free terms $y \in X^r_\rho := \{g : g \in X^r, \|g\|_{X^r} \leq \rho\}$. The formulation of the problem and terminology are borrowed from the monograph by J. Traub, G. Wasilkowski, H. Wozniakowski [8].

Let $U = \{u_i\}_{i=1}^k$ be some collection of continuous functionals u_i of which $u_1, u_2, \dots, u_j$ are determined on $\mathcal{L}(X, X)$ and $u_{j+1}, u_{j+2}, \dots, u_k$ on X, $card(U) := k$. Assume that we have perfect information about fixed operator B in (1.4). Usually this operator has very simple form (see Examples 1 and 2 in Section 1), Therefore, as discrete information about equations (1.1), (1.8) we will consider numerical vector

$$U(A, y_\delta) = \{u_1(A), u_2(A), \dots, u_j(A), u_{j+1}(y_\delta), \dots, u_k(y_\delta)\}, \tag{3.1}$$

generated by collection U. Any such collection of functionals will be called a method of specifying information.

By the aldorithm φ of an approximate solution of the equations (1.1) we mean the operator assigning to information (3.1) an element $\varphi(U, A, y_\delta) \in X$ as an approximate solution of (1.1). Moreover, for a fixed method of specifying information U we denote by $\Phi(U)$ the set of all algoritms using the information of the form (3.1).

The error of the algorithm $\varphi \in \Phi(U)$ on the class of equations (1.1) with operators $T \in \mathcal{K}^r_{B,\beta,\gamma}$ and exact free terms $y \in Range(T) \cap X^r_\rho$ is defined as

$$e_\delta(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho, \varphi, U) = \sup_{T \in \mathcal{K}^r_{B,\beta,\gamma}} \sup_{y \in Range(T) \cap X^r_\rho} \sup_{\substack{y_\delta \\ \|y - y_\delta\|_X \leq \delta}} \|T^+ y - \varphi(U, A, y_\delta)\|_X.$$

The minimal error which can be achieved using at most n values of information functionals is determined by the quantity

$$\mathcal{R}_{n,\delta}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) = \inf_{\substack{U \\ card(U) \leq n}} \inf_{\varphi \in \Phi(U)} e_\delta(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho, \varphi, U) \tag{3.2}$$

called the n-th minimal radius of information. From the results of Chapter 6 [9] and Chapter 3 [2] it follows that for sufficiently large n

$$\mathcal{R}_{n,\delta}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \asymp \delta$$

Then the quantity

$$N_{\delta,\varkappa}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) = \inf\{n : \mathcal{R}_{n,\delta}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \le \varkappa\delta\},\ \varkappa \ge 1,$$

characterizes the information complexity of recovering solutions T^+y of equations (1.1) with $T \in \mathcal{K}^r_{B,\beta,\gamma}$, $y \in Range(T) \cap X^r_\rho$ from perturbed equations (1.8). This is the minimal amount of discrete information which allows to obtain the best possible order of accuracy $O(\delta)$.

The next lemma ascertains a connection between (3.2) and so-called Babenko's pretabulated n-width

$$\Delta_n(X^r_\mu, X) := \inf_{\pi\in\Pi_n} \sup_{g\in X^r_\mu} \sup_{\substack{g_1,g_2\in X^r_\mu \\ \pi(g_1)=\pi(g_2)=\pi(g)}} \|g_1 - g_2\|_X,$$

where Π_n is the set of all continuous maps π from X^r_μ into n-dimensional Euclidean space.

Lemma 3.1 *If the operator B in (1.4) has a closed range and $dim\,Ker(B) < \infty$ then for sufficiently large $\gamma_1, \gamma_2, \gamma_3, \beta, \rho$*

$$R_{n,0}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \ge \frac{1}{2}\Delta_n(X^r_\mu, X),$$

where μ depends only on β, γ, ρ.

Proof. Denote by σ_r the imbedding constant of X^r in X, i.e. $\|g\|_X \le \sigma_r\|g\|_{X^r}$ for any $g \in X^r$. Let us assume for simplicity that there is $v \in X^r$ such that $\|v\|_X = 1$ and $\|v\|_{X^r} = \sigma_r^{-1}$ (In general case for any arbitrary however small $\varepsilon > 0$ there is v_ε such that $\|v_\varepsilon\|_X = 1,\ \sigma_r^{-1} \le \|v_\varepsilon\|_{X^r} \le \sigma_r^{-1} + \varepsilon$).

Note that if

$$\gamma_1 > \|B^*\|_{X\to X}\sigma_r^{-1},\ \gamma_2 > \|B^*\|_{X^r\to X^r}\sigma_r^{-1},\ \gamma_3 > \|B^*\|_{X^r\to X^r}\sigma_r^{-2},$$

then for any $g \in X^r_\mu$,

$$\mu = \min\{\gamma_1\sigma_r - \|B^*\|_{X\to X}, \gamma_2 - \|B^*\|_{X^r\to X^r}\sigma_r^{-1},$$
$$\gamma_3\sigma_r - \|B^*\|_{X^r\to X^r}\sigma_r^{-1}, (\|B\|_{X^r\to X^r} + \gamma_1)^{-1}\rho\}, \tag{3.3}$$

one can find the operator $T_g \in \mathcal{K}^r_{B,\beta,\gamma}$ such that

$$g = T_g^* v, \tag{3.4}$$

$$y_g = T_g g \in X^r_\rho. \tag{3.5}$$

Indeed, for $g \in X^r_\mu$ consider the operator A_g determined by formula $A_g f = (g - B^* v, f)v$. For any $f \in X$ and μ determined by (3.3) we have

$$\|A_g f\|_{X^r} \leq \|v\|_{X^r}(\|g\| + \|B^*\|_{X\to X}\|v\|_X)\|f\|_X \leq$$

$$\leq \sigma_r^{-1}(\mu + \|B^*\|_{X\to X})\|f\|_X \leq \gamma_1\|f\|_X,$$

$$\|A_g^* f\|_{X^r} = \|g - B^* v\|_{X^r}(v, f) \leq (\mu + \|B^*\|_{X^r\to X^r}\sigma_r^{-1})\|f\|_X \leq \gamma_2\|f\|_X,$$

$$\|(D_r A_g)^* f\|_{X^r} = \|g - B^* v\|_{X^r}(D_r v, f) \leq$$

$$\leq (\mu + \|B^*\|_{X^r\to X^r}\sigma_r^{-1})\|D_r v\|_X\|f\|_X \leq$$

$$\leq (\mu + \|B^*\|_{X^r\to X^r}\sigma_r^{-1})\sigma_r^{-1}\|f\|_X \leq \gamma_3\|f\|_X.$$

This means that for any $g \in X^r_\mu$ $A_g \in \mathcal{H}^r_\gamma$. It is common knowledge that if $Range(B)$ is closed and $dim\,Ker(B) < \infty$, $dim\,Range(A) < \infty$ then $Range(A+B)$ is closed too. Since $dim\,Range(A_g) = 1$ and $A_g \in \mathcal{H}^r_\gamma$ it follows that for some $\beta > 0$ $\quad T_g = B + A_g \in \mathcal{K}^r_{B,\beta,\gamma}$. Moreover,

$$\|y_g\|_{X^r} = \|T_g g\|_{X^r} \leq (\|B\|_{X^r\to X^r} + \|A_g\|_{X\to X^r})\mu \leq$$

$$\leq (\|B\|_{X^r\to X^r} + \gamma_1)\mu \leq \rho,$$

$$T_g^* v = B^* v + A_g^* v = B^* v + (g - B^* v)\|v\|_X^2 = g,$$

and (3.4), (3.5) are proved. From these relatoins it follows that $g \perp Ker(T_g)$ and $g = T_g^+ y_g,\ y_g \in X^r_\rho$. Thus, for any $g \in X^r_\mu$ we can find an equatuon (1.1) with $T \in \mathcal{K}^r_{B,\beta,\gamma}$ and $y \in X^r_\rho$ such that $T^+ y = g$. Then the same steps as in the proof of Lemma 17.1 [4] lead to the assertion of the lemma.

Let Γ_n be the plane set of the form

$$\Gamma_n = \{1\} \times [1, 2^n] \bigcup_{k=1}^{n} (2^{k-1}, 2^k] \times [1, 2^{n-k}].$$

Consider the method of specifying information

$$U_{\Gamma_n}(A, y_\delta) = \{(e_i, Ae_j),\ (i,j) \in \Gamma_n,\ (e_k, y_\delta),\ k = 1, 2, \ldots, 2^n\}$$

and the algorithm $\varphi_m \in \Phi(U_{\Gamma_n})$ within the framework of which we apply Brakhage's method (2.11), (2.12) to equation

$$T_n x = P_{2^n} y_\delta. \tag{3.6}$$

In this algorithm an approximation to the solution $T^+ y$ is given by

$$\varphi_m(U_{\Gamma_n}, A, y_\delta) = g_{m+1}(T_n^* T_n) T_n^* P_{2^n} y_\delta$$

and m is determined in Theorem 2.1, where instead of y_δ we use $P_{2^n} y_\delta$. Keeping in mind that for $n 2^n \asymp \delta^{-1/r} \log^{1+1/r} \frac{1}{\delta}$ and $y \in X_\rho^r$

$$\|y - P_{2^n} y_\delta\|_X \leq \|y - P_{2^n} y\|_X + \|P_{2^n}(y - y_\delta)\|_X \leq c_r \rho 2^{-nr} + \delta \leq c\,\delta,$$

from Theorem 2.1 we obtain the estimate

$$e_\delta(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho, \varphi_m, U_{\Gamma_n}) \leq c\,\delta. \tag{3.7}$$

Theorem 3.1 *If* $\dim Ker(B) < \infty$, *Range*(B) *is closed and for the pretabulated width of the ball* X_μ^r *we have the estimate*

$$\Delta_n(X_\mu^r, X) \geq c\, n^{-r}, \quad n = 1, 2, \ldots, \tag{3.8}$$

then for sufficiently large $\gamma_1, \gamma_2, \gamma_3, \beta, \rho$ *and* $\varkappa$

$$c_1\, \delta^{-1/r} \leq N_{\delta,\varkappa}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \leq c\,\delta^{-1/r} \log^{1+1/r} \frac{1}{\delta}$$

The method of specifying information U_{Γ_n}, $n 2^{-n} \asymp \delta^{-1/r} \log^{1+1/r} \frac{1}{\delta}$ *is order-optimal in the power scale in the sense of the quantity* $N_{\delta,\varkappa}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho)$.

Proof. From Lemma 3.1 and (3.8) for any n such that

$$\mathcal{R}_{n,\delta}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \leq \varkappa\,\delta$$

we have

$$\varkappa\,\delta \geq \mathcal{R}_{n,\delta}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \geq \mathcal{R}_{n,0}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \geq c\, n^{-r},$$
$$n \geq c\,\delta^{-1/r}.$$

Thus $N_{\delta,\varkappa}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \geq c\,\delta^{-1/r}$. On the other hand, from (3.8) it follows that for

$$N = card(U_{\Gamma_n}) \asymp n2^n \asymp c\,\delta^{-1/r}\log^{1+1/r}\frac{1}{\delta},$$

$$\mathcal{R}_{n,\delta}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \leq e_\delta(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho, \varphi_m, U_{\Gamma_n}) \leq c\,\delta$$

and for sufficiently large $\varkappa$

$$N_{\delta,\varkappa}(\mathcal{K}^r_{B,\beta,\gamma}, X^r_\rho) \leq N \leq c\,\delta^{-1/r}\log^{1+1/r}\frac{1}{\delta}.$$

The theorem is proved.

Remark 3.1 *The conditions of Theorem 3.1 are fulfilled for operators B and subspaces X^r indicated in Examples 1 and 2. Thus, for equations considered in these Examples our Theorem gives the exact order of Information Complexity in the power scale.*

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