Clones preserving a quasi-order

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Abstract. It is proved that if a finite non-trivial quasi-order is not a linear order then there exist $2^{\aleph_0}$ clones, which consist of functions preserving the quasi-order and contain all unary functions with this property. It is shown that, for a linear order on a three-element, there are only 7 such clones.

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1. Introduction and results

1° Let A be a finite set, $|A| = k > 1$, and let $O_A^{(n)}$ denote the system of all n-ary functions (or operations) on A, $O_A = \bigcup_{n=1}^{\infty} O_A^{(n)}$. A set $C \subseteq O_A$ is called a clone on A if it is closed under superposition and contains all projections (i.e. operations of the form $e_i^n(x_1, \dots, x_n) = x_i$). All clones on A form w.r.t. inclusion a complete lattice, which is denoted by $\mathcal{L}_A$.

For a two-element set A, a complete description of $\mathcal{L}_A$ was given by E.L.Post [10]; in particular, $\mathcal{L}_A$ is countable. If $k \geq 3$ then we have $|\mathcal{L}_A| = 2^{\aleph_0}$ [17] and a detailed description of the lattice is hardly to find. However, some interesting parts of $\mathcal{L}_A$ turn out to be finite and can be described. For example, G.A.Burle has proved [1] that the clones on A, which contain all unary functions, form a $(k+1)$-element chain. This result lets us pose the following "problem". Given a condition, is the set of all clones, which consist of functions satisfying the condition and contain all unary functions with this property, finite? In the present paper we consider this "problem" for the condition of monotony.

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Let ρ be a quasi-order (or preorder, in other terminology), i.e. a reflexive and transitive binary relation, on A. A function $f \in O_A^{(n)}$ is said to preserve ρ (or to be monotone w.r.t. ρ) if, for all $a_i, b_i \in A$, the condition $a_i \rho b_i$, $i = 1, \ldots, n$, implies $f(a_1, \ldots, a_n) \rho f(b_1, \ldots, b_n)$. All such functions form a clone, which is denoted by $Pol\rho$. The question, this paper is devoted to, is the following one.

Question *What conditions should ρ satisfy to guarantee that the subclones of $Pol\rho$, which contain all unary functions from $Pol\rho$, are finite in number?*

2° Now let us recall some definitions and notation. A variable x_i of a function $f \in O_A^{(n)}$ is said to be ficticious if, for all $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, a, b \in A$, we have $f(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$. Otherwise x_i is called essential.

An m-ary relation on A is just a subset of A^m. Elements of a relation will be written by columns. A function $f \in O_A^{(n)}$ is said to preserve an m-ary relation θ if, for each $m \times n$-matrix X with columns from θ, the column $f(X)$, which is calculated stringwise, belongs to θ as well. The system of all functions preserving a relation θ forms a clone, which is denoted by $Pol\theta$.

For a subset F of O_A, let $F^{(n)} = F \cap O_A^{(n)}$ (we write $Pol_n\theta$ instead of $(Pol\theta)^{(n)}$) and denote by $\langle F \rangle$ the clone generated by F. For a clone C, the system $C^{(1)}$ is a submonoid of the full transformation monoid T_A $(= O_A^{(1)})$. It is well known (see e.g.[15]) that, for an arbitrary submonoid M of T_A, the set $\{C \in \mathcal{L}_A | C^{(1)} = M\}$ is an interval of $\mathcal{L}_A$. Such an interval is denoted by $IntM$ and is called monoidal. Á. Szendrei has posed in [15] the problem of classifying monoidal intervals w.r.t. their cardinalities.

Let C_A denote the system of all constant functions on A. The following fact is well known (see e.g. Proposition II.6.1 [2]). For an equivalence ε on A and $F \subseteq O_A$, we have

$$F \subseteq Pol\varepsilon \text{ if and only if } \langle F \cup C_A \rangle^{(1)} \subseteq Pol_1\varepsilon.$$

The standard proof of this fact (see [2]) does not use symmetry of ε. So the fact stays true if we write "quasi-order" instead of "equivalence". This means, in particular, that if $M = Pol_1\rho$ for a quasi-order ρ then $IntM = [\langle Pol_1\rho \rangle; Pol\rho]$. Hence our question can be considered as a particular case of the problem of Á. Szendrei.

3° By trivial quasi-order on A we mean the equality relation ω or the full relation $A \times A$. A partial order ρ is said to be linear (or total, in other terminology) if, for every $x, y \in A$, we have $x\rho y$ or $y\rho x$.

Theorem 1 *Let ρ be a non-trivial quasi-order, not being a linear order, on A and let $M = Pol_1\rho$. Then $|IntM| = 2^{\aleph_0}$.*

Now let $A = \{0, 1, \ldots, k-1\}$, $\leq$ be the natural order on A and $M = Pol_1 \leq$. It follows from [10] that, for $k = 2$, the interval $IntM$ is the following lattice:

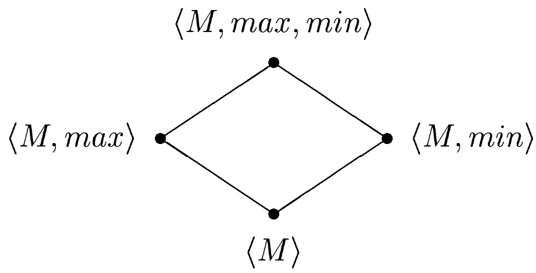

Fig.1

Recall that the clone consisting of all non-surjective functions and all functions from $\langle T_A \rangle$ is called Slupecki clone and is denoted by Sl. The following theorem is based on some ideas from [6].

Theorem 2 *Let $A = \{0, 1, 2\}$ and $M = Pol_1 \leq$. Then the interval $IntM$ is shown in Fig.2.*

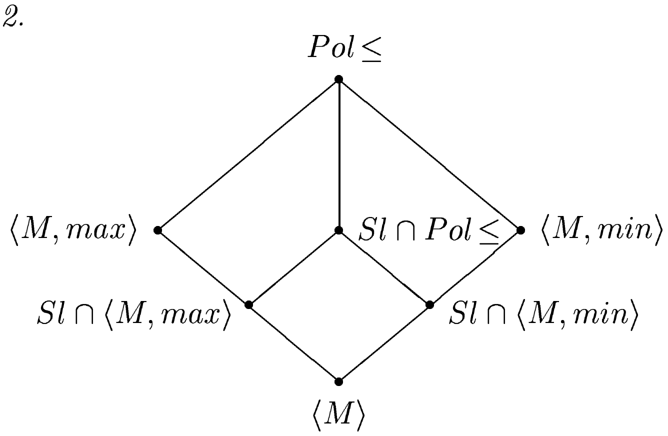

Fig.2

So a nesessary condition, which is asked about in the question, is to be a linear order. For $k \leq 3$, this condition is also sufficient. It is still unknown, whether this is true in general. The authors' hypothesis: it is true.

Problem *Let $|A| \geq 4$, ρ be a linear order on A and $M = Pol_1\rho$. Prove that $IntM$ is finite.*

We will need the following result. Let us fix two different elements $a, b \in A$ so that $a\varepsilon b$ for a non-trivial equivalence ε on A, and let M_0 denote the set $\{f \in Pol_1\varepsilon | Im(f) \subseteq \{a, b\}$ and $(x\varepsilon y \to f(x) = f(y))\}$.

Theorem 3 *[4] If $M_0 \subseteq M \subseteq Pol_1\varepsilon$ for a monoid M then $IntM$ has a subinterval isomorphic to the lattice of all subsets of a countable set. In particular, $|IntM| = 2^{\aleph_0}$.*

2. Technical lemma

Further on, for a string $\overline{x}$, we denote by $\overline{x}^\top$ the column transposed to $\overline{x}$. Let us determine $2m$-ary ($m \geq 5$) relation $\delta_{m,i}$ on A as the set

$$\{(x_1, \ldots, x_{2m})^\top | x_1 = \cdots x_{i-1} = x_{i+1} = \cdots = x_m = x_{m+i}$$

$$\text{and } x_i = x_{m+1} = \cdots x_{m+i-1} = x_{m+i+1} = \cdots = x_{2m}\}$$

and let $\delta_m = \bigcup_{i=1}^m \delta_{m,i}$.

Fix two distinct elements $a, b \in A$ and let

$$W_{n,a} = \{(x_1, \ldots, x_n) | \text{ one of } x_i \text{ is equal to } b \text{ and all other to } a\},$$

$$W_{n,b} = \{(x_1, \ldots, x_n) | \text{ one of } x_i \text{ is equal to } a \text{ and all other to } b\}.$$

Let $\mathcal{X}_{m,n}$ denote the set of all $2m \times n$-matrices X such that each string of X belongs to $W_{n,a} \cup W_{n,b}$ and each column of X belongs to δ_m.

Lemma Let $f \in O_A^{(n)}$ have the property

$$f(x_1, \ldots, x_n) = \begin{cases} a, & \text{if } (x_1, \ldots, x_n) \in W_{n,a}, \\ b, & \text{if } (x_1, \ldots, x_n) \in W_{n,b}. \end{cases}$$

Then $f(X) \in \delta_m$ holds for every $X \in \mathcal{X}_{m,n}$ if and only if $m \neq n$.

Proof.
If $m = n$ then

$$f\left(\begin{pmatrix} b & a & \cdots & a \\ a & b & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & b \\ a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{pmatrix}\right) = \begin{pmatrix} a \\ a \\ \vdots \\ a \\ b \\ b \\ \vdots \\ b \end{pmatrix} \notin \delta_m.$$

Now let $m \neq n$. We take arbitrary $X \in \mathcal{X}_{m,n}$ and prove that $f(X) \in \delta_m$. We need the following two remarks, which can straightforwardly be proved.

Remark 1. The condition $f(X) \in \delta_m$ is invariant under the following transformations of X:

(1) replacing all entries of a by b and all entries of b by a;

(2) arbitrary reordering of columns;

(3) permuting of i-th and j-th and at the same time of $(m+i)$-th and $(m+j)$-th strings where $1 \leq i < j \leq m$.

Remark 2. (1) If two elements of a string of X are the same then the value of f on this string coincides with them. Thus if two columns of X are the same then $f(X)$ coincides with them;

(2) if some three elements in the upper half of a column of X are not the same then they uniquely determine the column;

(3) if all elements in the upper half of a column of X are the same then all elements of the column are the same.

It is clear that that if all strings of X coincide then $f(X) \in \delta_m$. From now on we suppose that not all strings of X are the same.

Assume first that in the upper half of X there are a string from $W_{n,a}$ and a string from $W_{n,b}$. Then using remark 1 one may consider that

$$X = \begin{pmatrix} b & a & a & a & a & \cdots & a \\ a & b & b & b & b & \cdots & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a & b & b & b & b & \cdots & b \\ b & a & a & a & a & \cdots & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \end{pmatrix} \text{ or } X = \begin{pmatrix} a & b & a & a & a & \cdots & a \\ a & b & b & b & b & \cdots & b \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ & & b & b & b & \cdots & b \\ & & a & a & a & \cdots & a \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \end{pmatrix}.$$

In both cases at least two of three elements having (as elements of a matrix) indices (3,3), (3,4) and (3,5) coincide. Therefore, by remark 2.2, the two corresponding columns are the same and we have $f(X) \in \delta_m$ by remark 2.1.

Now suppose that all strings from the upper half of X belong to $W_{n,a}$. Recall that each of these strings contains exactly one entry of b. So, since $m \neq n$, it cannot happen that the upper half of every column of X contains exactly one entry of b. We consider two cases separately.

Case 1. The upper half of each column of X contains at most one entry of b.

One may consider that

$$X = \begin{pmatrix} b & a & a & \cdots & a & \cdots & a \\ a & b & a & \cdots & a & \cdots & a \\ a & a & b & \cdots & a & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ a & a & a & \cdots & b & \cdots & a \\ a & b & b & \cdots & b & \cdots & a \\ b & a & b & \cdots & b & \cdots & a \\ b & b & a & \cdots & b & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots \\ b & b & b & \cdots & a & \cdots & a \end{pmatrix}.$$

In this case the last string of X contains at least two entries as well of a so of b, what contradicts the condition $X \in \mathcal{X}_{m,n}$. So this case is not possible.

Case 2. The upper half of some column of X contains at least two entries of b.

Since not all strings of X are the same, one may consider that

$$X = \begin{pmatrix} b & a & a & \cdots & a \\ a & b & a & \cdots & a \\ b & a & a & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & a & a & \cdots & a \\ a & b & a & \cdots & a \\ b & a & a & \cdots & a \\ a & b & a & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & b & a & \cdots & a \end{pmatrix}.$$

It is clear that in this case $f(X) = (a, \ldots, a)^\top \in \delta_m$.
Lemma is proved. $\square$

3. Proof of theorem 1

It is well known that, for a quasi-order ρ, the relation $\varepsilon = \rho \cap \rho^{-1}$, where $\rho^{-1} = \{(x, y) | (y, x) \in \rho\}$, is an equivalence.

Suppose first that $\varepsilon \neq \omega$. Then we can apply theorem 3. In fact, there are two distinct $a, b \in A$ such that $\{a, b\}^2 \subseteq \rho$. Hence all functions $f \in T_A$ with the property $Im(f) \subseteq \{a, b\}$ belong to M. It is easy to prove that $M \subseteq Pol_1\varepsilon$. So the monoid M satisfies the condition of theorem 3 and we have $|IntM| = 2^{\aleph_0}$.

Further on it is assumed that $\rho \cap \rho^{-1} = \omega$. This means that ρ is a partial order.

A sequence $z_0, \ldots, z_n$ of elements of A is called a path between x and y in ρ if $z_0 = x$, $z_n = y$ and, for $i = 1, \ldots, n$, we have $z_{i-1}\rho z_i$ or $z_i \rho z_{i-1}$. A quasi-order is said to be connected if there is a path between two arbitrary elements of A in ρ.

We consider three cases separately.

Case 1. ρ is not connected.

Consider a relation $\theta = \{(x, y) |$ there is a path between x and y in $\rho\}$. It is easy to see that θ is a non-trivial equivalence and that $M \subseteq Pol_1\theta$. Choose two distinct elements $a, b \in A$ so that $a\rho b$. If a function $f \in Pol_1\theta$ such that $Im(f) \subseteq \{a, b\}$ has the property $x\theta y \to f(x) = f(y)$ for all $x, y \in A$ then f

belongs to M. So in this case the monoid M also satisfies the conditions of theorem 3 and we have $|IntM| = 2^{\aleph_0}$.

<u>Case 2.</u> ρ is bounded.

Denote by 0 and 1 the least and the greatest element of ρ respectively. Let σ_m be the $2m$-ary relation on A consisting of all columns $(x_1, \ldots, x_{2m})^\top$ with the following property: some element of a column is comparable (w.r.t. ρ) with each of its other elements. Further, let $\rho_m = \sigma_m \cup \delta_m$ where δ_m is the relation determined in section 2.

Choose $a, b \in A$ to be incomparable (here we use non-linearity ρ) and define functions $f_n \in O_A^{(n)}$, $n \geq 5$, in the following way:

$$f_n(x_1, \ldots, x_n) = \begin{cases} a, & \text{if } (x_1, \ldots, x_n) \in W_{n,a}, \\ b, & \text{if } (x_1, \ldots, x_n) \in W_{n,b}, \\ 0, & \text{if } (x_1, \ldots, x_n) \text{ is less (as an element of } (A, \rho)^n) \\ & \text{than some element from } W_{n,a} \cup W_{n,b}, \\ 1 & \text{otherwise.} \end{cases}$$

Now we prove that $f_n \in Pol\rho_m$ if and only if $n \neq m$.

If $n = m$ then the construction from lemma shows that $f_n \notin Pol\rho_m$.

Let $n \neq m$ and let X be a $2m \times n$-matrix with columns from ρ_m. If X has a string, which does not belong to $W_{n,a} \cup W_{n,b}$, then $f_n(X)$ contains 0 or 1 and by that $f_n(X) \in \sigma_m \subseteq \rho_m$. If each string of X belongs to $W_{n,a} \cup W_{n,b}$ then every column of X belongs to δ_m and, as follows from lemma, $f_n(X) \in \delta_m \subseteq \rho_m$. Thus $f_n \in Pol\rho_m$.

It it easy to prove that, for each $n \geq 5$, we have $f_n \in Pol\rho$ and, for each $m \geq 5$, $M \subseteq Pol\rho_m$ holds. Denote by C_I the clone $\langle M \cup \{f_i | i \in I\}\rangle$ where $I \subseteq \{n | n \geq 5\}$. Then we have $C_I \subseteq Pol\rho_m$ if and only if $m \notin I$. So the clones of the form C_I are pairwise distinct and all of them belong to $IntM$. Hence $|IntM| = 2^{\aleph_0}$.

Such method of constructing large sets of clones was first used in [3].

<u>Case 3.</u> ρ is connected and not bounded.

Suppose that ρ has no greatest element. Then we can choose two distinct maximal (w.r.t. ρ) elements $a, b \in A$, which have a common lower bound c.

Let relations ρ_m be defined as in case 2 and functions $f_n \in O_A^{(n)}$, $n \geq 5$, as follows:

$$f(x_1,\dots,x_n)=\begin{cases} a, & \text{if } (x_1,\dots,x_n)\in W_{n,a},\\ b, & \text{if } (x_1,\dots,x_n)\in W_{n,b},\\ c & \text{otherwise.}\end{cases}$$

The same consideration as in the preceeding case shows that $f_n \in Pol\rho_m$ if and only if $n \neq m$, that, for each $n \geq 5$, we have $f_n \in Pol\rho$ and that, for each $m \geq 5$, $M \subseteq Pol\rho_m$ holds. Therefore we have $|IntM| = 2^{\aleph_0}$.

If ρ has no least element then the consideration is dual.

So in every case we have $|IntM| = 2^{\aleph_0}$ and theorem 1 is proved. □

4. Proof of theorem 2

In this section A is always equal to $\{0,1,2\}$, $\leq$ denotes the natural order on A, and $M = Pol_1 \leq$. Denote by $\vee$ and $\wedge$ the binary operations of maximum and minimum on A respectively.

In [9], H.Machida has found all maximal subclones of the clone $Pol \leq$, there are 13 of them. Using this list, it is easy to show that the only maximal subclones of $Pol \leq$, which contain $Pol_1 \leq$, are the clones $\langle M, \vee\rangle$, $\langle M, \wedge\rangle$ and $Sl \cap Pol \leq$. D.Lau has described [8] all subclones of $\langle M, \vee\rangle$. It follows from this description that there exist only three subclones of $\langle M, \vee\rangle$, which contain M. These clones are $\langle M, \vee\rangle$, $Sl \cap \langle M, \vee\rangle$ and $\langle M\rangle$. It is clear that then the interval $[\langle M\rangle; \langle M, \wedge\rangle]$ is also a three-element chain (see Fig.2).

So, to prove theorem 2 it suffices to show that if a function f belongs to $(Sl \cap Pol \leq) \setminus (\langle M, \vee\rangle \cup \langle M, \wedge\rangle)$ then $\langle M, f\rangle = Sl \cap Pol \leq$.

It is easy to see that $(A, \vee, \wedge)$ is a lattice and that M is the monoid of all endomorphisms of this lattice. It immediately follows from these facts that

$$Sl \cap \langle M, \vee\rangle = \langle M\rangle \cup \{\bigvee_{i=1}^{n} m_i(x_i) | m_i \in M, Im(m_i) = \{a, b\} \subseteq A, 1 \leq i \leq n\},$$

$$Sl \cap \langle M, \wedge\rangle = \langle M\rangle \cup \{\bigwedge_{i=1}^{n} m_i(x_i) | m_i \in M, Im(m_i) = \{a, b\} \subseteq A, 1 \leq i \leq n\}.$$

Now we prove that if $f(x_1, \dots, x_n) \in (Sl \cap Pol \leq) \setminus \langle M\rangle$ then f can be represented as $\bigvee_{i=1}^{r} f'_i$ and as $\bigwedge_{i=1}^{s} f''_i$ where $r, s \geq 1$, $f'_1, \dots, f'_r \in Sl \cap \langle M, \wedge\rangle$, $f''_1, \dots, f''_s \in Sl \cap \langle M, \vee\rangle$, and the images of all f'_i and f''_i coincide with $Im(f)$.

Let $Im(f) = \{a, b\}$, $a < b$. Consider the set $f^{-1}(b) \subseteq A^n$. This set is an order filter in the poset $(A, \leq)^n$ because the function f is monotone. Let $(b_{11}, \ldots, b_{1n}), \ldots, (b_{r1}, \ldots, b_{rn})$ be the full list of minimal elements of $f^{-1}(b)$. We construct functions $f'_i \in Sl \cap \langle M, \wedge \rangle$, $1 \leq i \leq r$ by the rule $f'_i = \bigwedge_{j=1}^{n} m_{ij}(x_j)$ where

$$m_{ij}(x) = \begin{cases} b, & \text{if } x \geq b_{ij}, \\ a & \text{otherwise.} \end{cases}$$

It can straightforwardly be proved that $f(x_1, \ldots, x_n) = \bigvee_{i=1}^{r} f'_i(x_1, \ldots, x_n)$. Note that this representation of f is irreducible in the sence that none of f'_i may be omitted. One can analogously construct functions f''_i so that $f(x_1, \ldots, x_n) = \bigwedge_{i=1}^{s} f''_i$.

It follows from the above consideration and from the structure of intervals $[\langle M \rangle; \langle M, \vee \rangle]$ and $[\langle M \rangle; \langle M, \wedge \rangle]$ that the clone $Sl \cap Pol \leq$ can be generated by the set $M \cup \{g_1, g_2\}$ where g_1 and g_2 are arbitrary functions from $(Sl \cap \langle M, \vee \rangle) \setminus \langle M \rangle$ and from $(Sl \cap \langle M, \wedge \rangle) \setminus \langle M \rangle$ respectively.

Now we take arbitrary $f \in (Sl \cap Pol \leq) \setminus (\langle M, \vee \rangle \cup \langle M, \wedge \rangle)$ and find such functions g_1, g_2 in the clone $\langle M, f \rangle$. There exist functions $m_1, m_2 \in M$ such that $f = m_1 m_2 f$ and $Im(m_2) = \{0, 1\}$. By that one may consider without loss of generality that $Im(f) = \{0, 1\}$. We have proved that f has an irreducible representation $f = \bigvee_{i=1}^{r} f'_i$ where $f'_i \in \langle M, \wedge \rangle$. Then the condition $f \notin (\langle M, \vee \rangle \cup \langle M, \wedge \rangle)$ implies that $r > 1$ and that $f'_i \notin \langle M \rangle$ for some i. Now choose in the set $\{f'_i | f'_i \notin \langle M \rangle\}$ a function with maximal number of ficticious variables. Let $x_1, \ldots, x_l$, $l \geq 2$, be the list of all essential variables of the chosen function. Consider the function

$$h(x_1, \ldots, x_l) = f(x_1, \ldots, x_l, 0, \ldots, 0)$$

Then h can be represented as $h = h'_1 \vee \cdots \vee h'_t$, where none of h'_i is a constant operation, $h'_i \in \langle M, \wedge \rangle$, $i = 1, \ldots, t$, and each h'_i either essentially depends on $x_1, \ldots, x_l$ or belongs to $\langle M \rangle$. This representation of h is also irreducible because the representation of f is so.

Let us fix denotations for two functions from M. Let $u_1(0) = 0, u_1(1) = u_1(2) = 1$ and $u_2(0) = u_2(1) = 0, u_2(2) = 1$. Note that $u_1 u_1 = u_1, u_1 u_2 = u_2$ and $u_2 u_1 = 0$. Since $u_1 h = h$, one may consider that the function h is constructed only with the help of u_1 and u_2. Moreover, if $h'_i \in \langle M \rangle$ then $h'_i = u_2(x_j)$ for some $1 \leq j \leq l$, since the representation of h is irreducible and, for all $x, y \in A$, we have $u_1(x) \vee (u_2(x) \wedge y) = u_1(x)$. If, in addition, this variable x_j is an essential variable of some other function h_q, $1 \leq q \leq t$, then it appears in h_q as the argument of u_1. Now we substitute $u_1(x_j)$ instead of x_j for each variable x_j with the described above properties. Then, since we have $u_1 u_1 = u_1$ and $u_2 u_1 = 0$, all functions h'_i with $h'_i \in \langle M \rangle$ disappear and the other h'_i do not change. So we may assume that each function h'_i essentially depends on $x_1, \dots, x_l$.

Identifying the variables $x_3, \dots, x_l$ with x_2 we get, because of the equality $u_1(x) \wedge u_2(x) = u_2(x)$, either a function $u_i(x_1) \wedge u_j(x_2) \in (Sl \cap \langle M, \wedge \rangle) \setminus \langle M \rangle$ or the function $h''(x_1, x_2) = (u_1(x_1) \wedge u_2(x_2)) \vee (u_2(x_1) \wedge u_1(x_2))$. Further, $h''(u_1(x_1), x_2) = u_1(x_1) \wedge u_2(x_2) \in (Sl \cap \langle M, \wedge \rangle) \setminus \langle M \rangle$.

So we have found a function from $(Sl \cap \langle M, \wedge \rangle) \setminus \langle M \rangle$ in the clone $\langle M, f \rangle$. Analogous consideration (starting from the representation $f = \bigwedge_{i=1}^{s} f''_i$) shows that there is a function from $(Sl \cap \langle M, \vee \rangle) \setminus \langle M \rangle$ in the clone $\langle M, f \rangle$. Now it follows that $\langle M, f \rangle = Sl \cap Pol \leq$. Theorem 3 is proved. □

5. Concluding remarks

The problem of Á.Szendrei is very hard to solve in general. By that one tries to decide it for monoidal intervals which contain some important clones.

It is well known that a clone lattice is coatomic. A.V.Kuznetsov has proved [7] that the lattice $\mathcal{L}_A$ has only a finite number of maximal clones, i.e. coatoms. It follows from [12] that Sl is a maximal clone and that if $M = C^{(1)}$ for a maximal clone $C \neq Sl$ then $IntM = [\langle M \rangle; C]$. So one can try to solve Á.Szendrei's problem for such monoidal intervals. Note that Sl belongs to the monoidal interval $IntT_A$ described by G.A.Burle [1].

A full description of maximal clones was given by S.V.Yablonskii [16] for $k = 3$ and by I.G.Rosenberg [11] for an arbitrary finite set A. According to the theorem of Rosenberg, the maximal clones on A are the clones $Pol\theta$ where θ is a relation of one of the following types:

($\mathcal{O}$) bounded partial order;

($\mathcal{E}$) non-trivial equivalence;

($\mathcal{C}$) central relation;

($\mathcal{A}$) affine relation (if k is a prime power);

($\mathcal{P}$) graph of fixpointfree permutation of prime order;

($\mathcal{R}$) regular (or h-universal, in other terminology) relation.

For definitions of these relations see e.g. [15].

It was proved in [4] and [5] that if θ is a relation of type $(\mathcal{E})$ and $(\mathcal{C})$ resp. then the monoidal interval which contains $Pol\theta$ has $2^{\aleph_0}$ elements. Monoidal intervals containing a clone $Pol\theta$ where θ is of type $(\mathcal{P})$, if k prime, or of type $(\mathcal{A})$ were described in [14] and [13] resp.; these intervals are finite chains.

So our results can be considered as a contribution to the solution of this particular case of the problem of Á.Szendrei. For $k = 3$, our theorem 2 completes the solution, since, in this case, the only maximal clone determined by a regular relation is Slupecki clone.

The cardinality of a monoidal interval, which contains a maximal clone $Pol\theta$, is yet unknown only if $k \geq 4$ and θ is a linear order, or k is composite and θ is of type $(\mathcal{P})$, or θ is of type $(\mathcal{R})$ and $Pol\theta \neq Sl$. It seems likely that these monoidal intervals are all finite.

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