

# Holomorphic rank-2 vector bundles on primary Kodaira surfaces

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## Introduction

Let  $X$  be a smooth compact complex manifold of dimension  $n$  and consider  $\mathcal{V} \xrightarrow{\pi} X$  be a topological complex vector bundle on  $X$ . A classical problem demands to determine the conditions to be satisfied by  $\mathcal{V}$  ensuring that  $\mathcal{V}$  is the underlying topological bundle of a *holomorphic* vector bundle on  $X$ .

A priori there is no reasonable way to control the existence of holomorphic structures. Then one needs firstly to classify all the topological vector bundles by using some suitable invariants, and then to produce a set of holomorphic vector bundles whose corresponding invariants cover as much as possible.

In the case of surfaces, Wu has proved (cf. [?]) that topological vector bundles are completely characterized by the rank and the Chern classes (for threefolds this stays no longer true as seen in [?], [?]). There is also an obvious necessary condition for the existence problem, namely the first Chern class *must* belong to the Néron-Severi group of  $X$ :  $NS(X) := \text{Ker}(H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathcal{O}_X))$ . Since the rank 1 case is trivial anyway, the existence problem on surfaces translates into the equivalent problem:

**Problem.** Find all the triples  $(r, c_1, c_2) \in \mathbf{N} \times NS(X) \times \mathbf{Z}$ ,  $r \geq 2$  for which there exists a rank  $r$  holomorphic vector bundle  $\mathcal{E}$  on  $X$  with Chern classes  $c_1(\mathcal{E}) = c_1$  and  $c_2(\mathcal{E}) = c_2$ .

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For projective surfaces, Schwarzenberger (cf. [?]) proved that any triple  $(r, c_1, c_2) \in \mathbf{N} \times NS(X) \times \mathbf{Z}$ ,  $r \geq 2$  comes from a rank  $r$  holomorphic vector bundle. In contrast to this situation, for non-projective surfaces, there is a natural *necessary* condition for the existence problem (cf. [?] Theorem 3.1 for the general case; for the rank-2 case see [?] Proposition 1.1, [?]):

$$\Delta(r, c_1, c_2) := \frac{1}{r} \left( c_2 - \frac{r-1}{2r} c_1^2 \right) \geq 0.$$

In the same context there is also a *sufficient* condition (cf [?]; see also [?], [?]):

$$\Delta(r, c_1, c_2) \geq m(r, c_1),$$

where

$$m(r, c_1) := -\frac{1}{2r} \max \left\{ \sum_{i=1}^r \left( \frac{c_1}{r} - \mu_i \right)^2, \mu_1, \dots, \mu_r \in NS(X), \sum_{i=1}^r \mu_i = c_1 \right\},$$

with the sole excepted case:  $X$  a  $K3$  surface with algebraic dimension zero,  $c_1$  divisible by  $r$  in  $NS(X)$  and  $\Delta(r, c_1, c_2) = \frac{1}{r}$ . This result also has a converse, that is (see [?], [?], [?]) any *filtrable* rank  $r$  holomorphic vector bundle  $\mathcal{E}$  on a non-algebraic surface  $X$  with Chern classes  $c_1(\mathcal{E}) = c_1$  and  $c_2(\mathcal{E}) = c_2$  satisfies the inequality

$$\Delta(\mathcal{E}) := \Delta(r, c_1, c_2) \geq m(r, c_1).$$

Therefore the only unknown situations are achieved for the case  $\Delta(r, c_1, c_2) \in [0, m(r, c_1))$ . If  $m(r, c_1) \neq 0$ , then this interval is non-empty, and then, in order to solve the existence problem, one has to construct holomorphic vector bundles having the corresponding discriminant  $\Delta$  lower than  $m(r, c_1)$ . Of course, all these vector bundles will be *non-filtrable* (for more details see, for example [?]).

Restricting to the rank 2 case, Toma has recently proved (cf. [?]) by using the Kobayashi-Hitchin correspondence (see for example [?]) that if  $X$  is a 2-torus, then any couple  $(c_1, c_2) \in NS(X) \times \mathbf{Z}$  with  $\Delta(2, c_1, c_2) \geq 0$  is associated to a holomorphic rank 2 vector bundle on  $X$ . In [?], [?], he had already shown that on a primary Kodaira surface or on a 2-torus any triple  $(r, c_1, c_2)$  with  $\Delta(r, c_1, c_2) = 0$  comes from a rank  $r$  holomorphic vector bundle. The rank-2 case is, in a certain sense, easier to handle with, as well as to understand it. For example, the natural bound  $m(2, c_1)$  computes directly by (cf. [?], [?])

$$m(2, c_1) = -\frac{1}{8} \sup_{\mu \in NS(X)} (c_1 - 2\mu)^2.$$

Moreover, if  $m(2, c_1) > 0$ , then all holomorphic rank 2 vector bundles with  $\Delta \in [0, m(2, c_1))$  are *irreducible* (i.e. without coherent sub-sheaves of lower rank), hence they are *stable* with respect to any Gauduchon metric on the surface  $X$  (see, for example [?]).

From now on, unless otherwise stated, by a *vector bundle* we shall mean a *holomorphic vector bundle*, and a *curve* will always be a smooth, complex projective curve.

The aim of our work is to cover in a unitary way, on a primary Kodaira surface  $X$ , the set of all couples  $(c_1, c_2) \in NS(X) \times \mathbf{Z}$ , such that  $\Delta(2, c_1, c_2) \in [0, m(2, c_1))$ , with Chern classes of rank-2 vector bundles. The main result is the following:

**Theorem 1.** *Let  $X \rightarrow B$  be a primary Kodaira surface over the elliptic curve  $B$ , with fiber the elliptic curve  $E$ . Then for any pair  $(c_1, c_2) \in NS(X) \times \mathbf{Z}$  with  $m(2, c_1) > 0$  and  $\Delta(2, c_1, c_2) \in [0, m(2, c_1))$ , there exists a double covering  $C \xrightarrow{f} B$ , and a line bundle  $L$  on  $Y := X \times_B C$  such that, denoting  $Y \xrightarrow{\varphi} X$  the canonical covering,  $\mathcal{E} := \varphi_* L$  is a rank-2 vector bundle on  $X$  having Chern classes  $c_1(\mathcal{E}) = c_1$  and  $c_2(\mathcal{E}) = c_2$ .*

The case  $\Delta(2, c_1, c_2) = 0$  has been proved by Toma (cf. [?]; see also [?]) by using *unramified* double coverings of  $B$  with another suitable elliptic curve. To solve the other cases, we use *curves of genus two with elliptic differentials*, i.e. curves  $C$  of genus 2 which admit a non-constant morphism

to an elliptic curve, which does not factor over an isogeny of that elliptic curve. These curves were studied extensively by Bolza, Humbert, Picard, Poincaré and many others (see Chapter XI of Krazer's book [?]), and a remarkable thing about them is that the covers occur in pairs. More precisely (for details, see for example [?]), if we consider  $f : C \rightarrow B$  such a double covering, and choose  $F = \text{Ker}(J_C \rightarrow B)$ , then there is a *complementary* natural double cover  $g : C \rightarrow F$ , giving rise an isogeny of degree four  $f^* \times g^* : B \times F \rightarrow J_C$ . Its kernel  $H$  is the graph of an isomorphism  $\psi : B[2] \rightarrow F[2]$  (which is *symplectic* with respect to the Weil pairings).

Conversely, one can get such curves out of a particular case of the "Basic Construction" and "Reducibility Criterion" (cf. [?], see also [?], [?], [?]). Suppose one starts with  $B$  and  $F$  two elliptic curves, and  $\Theta = 0 \times F + B \times 0$  the canonical principal polarization on  $B \times F$  given by the product structure. Assume  $\psi : B[2] \rightarrow F[2]$  is an isomorphism between the 2-torsion subgroups. Then  $\psi$  is automatically symplectic in the sense used in [?], [?], [?]; therefore, by denoting  $H_\psi = \text{Graph}(\psi)$  (which is a maximal isotropic subgroup of  $(B \times F)[2]$ ), and  $p : B \times F \rightarrow (B \times F)/H_\psi$  the canonical projection, it turns out that the surface  $(B \times F)/H_\psi$  carries a principal polarization  $\Theta_\psi$  such that  $p^*\Theta_\psi$  is linear equivalent to  $2\Theta$  (cf. [?], [?]; see also [?]).

**Theorem 2.** (Kani; cf. [?] Theorem 3) *With the previous notations, if  $\psi$  is not the restriction of an isomorphism between  $B$  and  $F$ , then the linear system  $|\Theta_\psi|$  contains a smooth curve  $C$  of genus 2.*

In particular, the principally polarized abelian surface  $((B \times F)/H_\psi, \Theta_\psi)$  is isomorphic to the Jacobian of  $C$ , and  $C$  covers two-to-one the curves  $B$  and  $F$ , via the restrictions to  $C$  of the canonical maps  $(B \times F)/H_\psi \rightarrow B$  and  $(B \times F)/H_\psi \rightarrow F$ , given by  $(x, y) \rightarrow 2x$  and  $(x, y) \rightarrow 2y$  respectively.

Once we proved Theorem 1, we use it together with Theorem 2.3 from [?] to get a complete answer concerning the existence of holomorphic structures in topological rank-2 vector bundles over primary Kodaira surfaces. We mention here that Toma (cf. [?]) has given a partial answer (for the case  $4\Delta(2, c_1, c_2)$  even), by starting with vanishing discriminant, and then using a trick that allows increasing the second Chern class by one unit.

**Corollary 3.** *Let  $X \rightarrow B$  be a primary Kodaira surface over the elliptic curve  $B$ , with fiber  $E$ . Then for any classes  $c_1 \in NS(X)$  and  $c_2 \in \mathbf{Z}$  there exists a rank 2 vector bundle  $\mathcal{E}$  on  $X$  such that  $c_1(\mathcal{E}) = c_1$  and  $c_2(\mathcal{E}) = c_2$  if and only if  $\Delta(2, c_1, c_2) \geq 0$ .*

In order to get to the proof of Theorem 1, we organized several preliminary sections. The first section, put here for the convenience of the reader, gathers together some elementary facts on curves coverings. In section 2, we recall (cf. [?], [?]) the description of the Néron-Severi group of a principal elliptic bundle, we give an explicit formula for computing the intersection in the Néron-Severi group, as well as a formula for the push-forward map between Néron-Severi groups while considering a covering of the base curve. Section 3 computes the Chern classes of a rank two vector bundle, over a primary Kodaira surface  $X$ , which is obtained as a push-forward of a line bundle over a double covering of  $X$ , and finally, the last section is devoted to the proof of Theorem 1.

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## 1. Some remarks on curves coverings

This section is devoted to some useful Lemmas about curves coverings, which will be used in the next sections. The lack of a good reference for these results obliged us to insert the proofs as well, although they are more or less trivial.

**Lemma 1.1.** *Let  $\gamma : B \rightarrow E$  a covering of an elliptic curve with a curve of genus  $g$ . Choose  $\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g\} \in H_1(B, \mathbf{Z})$  a symplectic basis. Then*

$$\deg(\gamma) = \sum_{i=1}^g \gamma_*(\alpha_i) \cdot \gamma_*(\beta_i)$$

where  $\gamma_* : H_1(B, \mathbf{Z}) \rightarrow H_1(E, \mathbf{Z})$  is the natural push-forward map.

*Proof.* Consider the Abel-Jacobi embedding of  $B$  in its Jacobian  $J_B$ , corresponding to a chosen point in  $B$ . Following classical notations (cf. [?], p. 325), the class of  $B$  as a cycle in  $J_B$  is represented by  $\{\widetilde{W}_1\} \in H_2(J_B, \mathbf{Z})$ ,  $\{\widetilde{W}_1\} = -\sum_{i=1}^g \alpha_i \star \beta_i$ , where " $\star$ " denotes the Pontrjagin product.

Consider also real coordinates (cf. [?] p. 104)  $\{x_1, \dots, x_g, y_1, \dots, y_g\} \in V = H^0(B, K_B)^*$ , such that  $\{dx_1, \dots, dx_g, dy_1, \dots, dy_g\} \in H_{DR}^1(J_B)$  is the dual basis of the chosen symplectic basis.

For any divisor  $D$  on  $J_B$  whose Chern class is represented, by means of Appell-Humbert theorem, by an alternating form  $A : V \times V \rightarrow \mathbf{R}$ , we have (cf. [?], p.42-43, p. 104, p.106):

$$\begin{aligned} (\{\widetilde{W}_1\} \cdot \{D\}) &= -\sum_{i=1}^g ((\alpha_i \star \beta_i) \cdot \{D\}) \\ &= -\sum_{i=1}^g \sum_{k,l} A(\beta_i, \alpha_k) \int_{\alpha_i \star \beta_i} dx_k \wedge dy_l \\ &\quad - \sum_{i=1}^g \sum_{k<l} A(\alpha_l, \alpha_k) \int_{\alpha_i \star \beta_i} dx_k \wedge dx_l \\ &\quad - \sum_{i=1}^g \sum_{k<l} A(\beta_l, \beta_k) \int_{\alpha_i \star \beta_i} dy_k \wedge dy_l \\ &= -\sum_{i=1}^g A(\beta_i, \alpha_i), \end{aligned}$$

and therefore

$$(1.1) \quad (\{\widetilde{W}_1\} \cdot \{D\}) = \sum_{i=1}^g A(\alpha_i, \beta_i).$$

If one considers now  $D$  a fiber of the norm map  $Nm(\gamma) : J_B \rightarrow E$ , then the associated alternating form is  $A : V \times V \rightarrow \mathbf{R}$ , defined by  $A(u, v) = \gamma_*(u) \cdot \gamma_*(v)$  for all  $u, v \in H_1(B, \mathbf{Z})$ . Now apply the relation (1.1), and observe that  $\deg(\gamma) = (\{\widetilde{W}_1\} \cdot \{D\})$ , and the proof of the Lemma is over.

*Remark 1.2.* In particular, if we also choose  $\{\varepsilon_1, \varepsilon_2\} \in H_1(E, \mathbf{Z})$  a symplectic basis, and the matrix of  $\gamma_*$  in the two given bases is

$$\gamma_* = \begin{pmatrix} a_{11} & \dots & a_{1g} & b_{11} & \dots & b_{1g} \\ a_{21} & \dots & a_{2g} & b_{21} & \dots & b_{2g} \end{pmatrix},$$

then it easily follows that

$$(1.2) \quad \deg(\gamma) = \sum_i \begin{vmatrix} a_{1i} & b_{1i} \\ a_{2i} & b_{2i} \end{vmatrix}.$$

**Lemma 1.3.** *Let  $E = \mathbf{C}/\Lambda_E$ ,  $F = \mathbf{C}/\Lambda_F$  be two elliptic curves, where  $\Lambda_E = (\mu_E, 1)\mathbf{Z}^2$ ,  $\Lambda_F = (\mu_F, 1)\mathbf{Z}^2$ , and  $\mu_F = n\mu_E$  for some positive integer  $n > 0$ , and denote by  $\delta : F \rightarrow E$  the canonical projection. Consider  $\gamma : F \rightarrow E$  an isogeny, and denote  $\gamma[2] : F[2] \rightarrow E[2]$  the morphism induced between the 2-torsion subgroups. We have:*

- (a) if  $\gamma[2] = \delta[2]$ , then there exists  $\mu : F \rightarrow E$  such that  $\gamma - 2\mu = \delta$   
(b) if  $\gamma[2] \equiv 0$ , then there exists  $\mu : F \rightarrow E$  such that  $\gamma = 2\mu$ .

*Proof.* One can write  $\gamma\mu_F = a\mu_E + c$  and  $\gamma = b\mu_E + d$ , where  $a, b, c, d \in \mathbf{Z}$ .

Obviously, the relation between them is

$$(1.3) \quad b\mu_F\mu_E + d\mu_F = a\mu_E + c,$$

and this is also a sufficient condition for a matrix

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries to be the rational representation of a morphism from  $F$  to  $E$ .

(a) Suppose  $\gamma[2] = \delta[2]$ , and  $n$  is odd. In this case, it follows  $\gamma\mu_F/2 \in \mu_E/2 + \Lambda_E$ , and  $\gamma/2 \in 1/2 + \Lambda_E$ , which implies that  $b$  and  $c$  are even, while  $a$  and  $d$  are odd. Therefore, we can write  $a = 2a' + 1$ ,  $b = 2b'$ ,  $c = 2c'$  and  $d = 2d' + 1$ , with  $a', b', c', d' \in \mathbf{Z}$ .

Replacing  $a, b, c, d$  in the relation (1.3) we get

$$b'\mu_F\mu_E + d'\mu_F = \left(a' - \frac{n-1}{2}\right)\mu_E + c',$$

which means that the matrix

$$R' = \begin{pmatrix} a' - \frac{n-1}{2} & b' \\ c' & d' \end{pmatrix}$$

is the rational representation of a morphism from  $F$  to  $E$ ; this is the morphism  $\mu$  we were searching for.

If  $n$  is even, then we get  $\gamma\mu_F/2 \in \Lambda_E$ , and  $\gamma/2 \in 1/2 + \Lambda_E$ , which implies that  $a, b$  and  $c$  are even, and  $d$  is odd. Then, we write  $a = 2a'$ ,  $b = 2b'$ ,  $c = 2c'$  and  $d = 2d' + 1$ , with  $a', b', c', d' \in \mathbf{Z}$ , and the morphism  $\mu$  has now the rational representation

$$R' = \begin{pmatrix} a' - \frac{n}{2} & b' \\ c' & d' \end{pmatrix}.$$

(b) If  $\gamma[2] \equiv 0$ , then  $\gamma\mu_F/2, \gamma/2 \in \Lambda_E$ , which implies that  $a, b, c$  and  $d$  are all even. Then, we write  $a = 2a'$ ,  $b = 2b'$ ,  $c = 2c'$  and  $d = 2d'$ , with  $a', b', c', d' \in \mathbf{Z}$ ; the matrix with entries  $a', b', c', d'$  is the rational representation of  $\mu$ .

**Lemma 1.4.** *Let  $\gamma : F \rightarrow E$  be an isogeny of elliptic curves, and denote  $\gamma[2] : F[2] \rightarrow E[2]$  the morphism induced between the 2-torsion subgroups. Then  $\gamma[2]$  is an isomorphism if and only if  $\deg(\gamma)$  is odd.*

*Proof.* Let  $n = \deg(\gamma)$ , and  $e = e(\gamma)$  its exponent. Since the exponent is a divisor of the degree, if  $n$  is odd, it follows that  $e$  is odd as well. A well-known result (cf. ex. [?], p. 12) states that there exists a morphism  $\gamma' : E \rightarrow F$  such that  $\gamma \circ \gamma' = e_E$  and  $\gamma' \circ \gamma = e_F$ , where  $e_E$  and  $e_F$  are the morphisms given by multiplication by  $e$ . In particular,  $\gamma[2] \circ \gamma'[2] = e_E[2]$  and  $\gamma'[2] \circ \gamma[2] = e_F[2]$ . Finally, because  $e$  is odd we have  $e_E[2] = id_{E[2]}$  and  $e_F[2] = id_{F[2]}$  and then  $\gamma[2]$  is an isomorphism.

Conversely, we suppose that  $n$  is even, and we want to prove that  $\text{Ker}(\gamma[2]) \neq 0$ . For this, write  $E = \mathbf{C}/\Lambda_E$ ,  $F = \mathbf{C}/\Lambda_F$ , where  $\Lambda_E = (\mu_E, 1)\mathbf{Z}^2$ ,  $\Lambda_F = (\mu_F, 1)\mathbf{Z}^2$ ;  $\gamma\mu_F = a\mu_E + c$  and  $\gamma = b\mu_E + d$ , with  $a, b, c, d \in \mathbf{Z}$ . Since  $n$  is even, Lemma 1.1 and formula (1.2) ensure that  $ad - bc$  is even. We distinguish several cases, somehow similar to each other: if  $a$  and  $c$  are even, then  $\gamma\mu_F/2 \in \Lambda_E$ , if  $b$  and  $d$  are even, then  $\gamma/2 \in \Lambda_E$ , if  $a$  and  $d$  are odd (which implies  $b$  and  $c$  are odd), or  $a$  is even,  $c$  is odd,  $b$  is even,  $d$  is odd, or  $a$  is odd,  $c$  is even,  $b$  is odd,  $d$  is even, then  $\gamma(\mu_F + 1)/2 \in \Lambda_E$ . In any case,  $\text{Ker}(\gamma[2]) \neq 0$ .

## 2. Cohomology of principal elliptic bundles

Start with two curves  $B$  and  $E$  and suppose that  $B$  is of genus  $g$  and  $E$  is elliptic. Consider  $X \xrightarrow{\pi} B$  a principal elliptic bundle over  $B$  with fiber  $E$  (cf. [?], [?], [?] for precise definitions). The topological type of the fibration is given by the *transgression map* (cf. [?])  $\delta : H^1(E, \mathbf{Z}) \rightarrow H^2(B, \mathbf{Z})$ .

If we suppose moreover, that  $X \xrightarrow{\pi} B$  is not topologically trivial i.e.  $\delta \neq 0$  then, as stated in [?], we have the following:

**Theorem 2.1.** (cf. [?])

(1)  $H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z})) \cong H^1(B, \mathbf{Z}) \otimes_{\mathbf{Z}} H^1(E, \mathbf{Z})$ .

(2)  $NS(X)/\text{Tors}(NS(X)) \cong \text{Hom}(J_B, E^\vee)$ , where  $J_B$  denotes the Jacobian variety of  $B$  and  $E^\vee$  is the dual curve of  $E$ .

(3) The torsion of  $H^2(X, \mathbf{Z})$  (as well as of  $NS(X)$ ) is generated by the class of a fiber.

We prove next a formula for computing the intersection form on  $H^2(X, \mathbf{Z})$ . It suffices, of course, to make it explicit for classes in  $H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z}))$ . Throughout the rest of the paper, for an element  $c \in H^2(X, \mathbf{Z})$ , we will denote by  $\hat{c}$  its class modulo  $\text{Tors}(H^2(X, \mathbf{Z}))$ .

Consider then  $\{a_1, \dots, a_g, b_1, \dots, b_g\} \in H_1(B, \mathbf{Z})$  a symplectic basis with Kronecker duals  $a_1^*, \dots, a_g^*, b_1^*, \dots, b_g^* \in H^1(B, \mathbf{Z})$ . There are also the Poincaré duals  $\mathcal{P}_B(a_i) = \tilde{a}_i$ . The two bases are characterized by  $a_i^*(a_j) = \delta_{ij}$ ,  $a_i^*(b_j) = 0$ ,  $b_i^*(b_j) = \delta_{ij}$ ,  $b_i^*(a_j) = 0$ ,  $\tilde{a}_i(b_j) = (a_i \cdot b_j) = \delta_{ij}$ ,  $\tilde{a}_i(a_j) = (a_i \cdot a_j) = 0$ ,  $\tilde{b}_i(b_j) = (b_i \cdot b_j) = 0$ ,  $\tilde{b}_i(a_j) = (b_i \cdot a_j) = -\delta_{ij}$ , for any  $i, j = 1, \dots, g$  and the relationship between them is  $a_i^* = -\tilde{b}_i$  and  $b_i^* = \tilde{a}_i$ . Choose also  $\{e_1, e_2\} \in H^1(E, \mathbf{Z})$  a symplectic basis (so that  $e_1 \wedge e_2 \in H^2(E, \mathbf{Z})$  is the positive generator).

Then by means of Theorem 2.1 any class  $\hat{c} \in H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z}))$  can be expressed as

$$\hat{c} = \sum_{k=1}^2 \sum_{i=1}^g (a_{ki}(a_i^* \otimes e_k) + b_{ki}(b_i^* \otimes e_k)),$$

and the self-intersection computes as

$$\hat{c}^2 = -2 \sum_i (a_{1i}b_{2i} - b_{1i}a_{2i}).$$

Writing the class  $\hat{c}$  as a matrix

$$\hat{c} = \begin{pmatrix} a_{11} & \dots & a_{1g} & b_{11} & \dots & b_{1g} \\ a_{21} & \dots & a_{2g} & b_{21} & \dots & b_{2g} \end{pmatrix},$$

the self-intersection expresses as:

$$(2.1) \quad \hat{c}^2 = -2 \sum_i \begin{vmatrix} a_{1i} & b_{1i} \\ a_{2i} & b_{2i} \end{vmatrix}.$$

Let now  $\hat{c} \in NS(X)/\text{Tors}(NS(X))$ . Choosing a base-point in  $B$ , by means of Theorem 2.1, the cohomology class  $c$  gives rise to a covering map  $\gamma(\hat{c}) : B \rightarrow E$  such that  $\gamma(\hat{c})_* = \hat{c}$ . Piecing together the relations (1.2) and (2.1), we get the following:

**Lemma 2.2.** *With the previous notations,  $\hat{c}^2 = -2\text{deg}(\gamma(\hat{c}))$ .*

Suppose now that  $C \xrightarrow{f} B$  is a covering; in this case,  $Y = X \times_B C \rightarrow C$  is a principal elliptic bundle of topological type  $f^* \circ \delta : H^1(E, \mathbf{Z}) \rightarrow H^2(C, \mathbf{Z})$ . Since  $f^* : H^2(B, \mathbf{Z}) \rightarrow H^2(C, \mathbf{Z})$  is the multiplication by the degree of  $f$ , it follows that  $Y \rightarrow C$  is topologically non-trivial as soon as  $X \rightarrow B$  is topologically non-trivial. In particular, Theorem 2.1 also applies to  $Y \rightarrow C$ .

We give next a description of the push-forward map induced between the cohomology groups by the natural covering map  $Y \xrightarrow{\varphi} X$ . First of all, if we denote by  $F_Y$  and  $F_X$  the fibers of the

fibration maps for  $Y$  and  $X$  respectively, in homology we have  $\varphi_*\{F_Y\} = \{F_X\}$ , and thus, via Poincaré dualities, a similar statement holds in cohomology. Therefore, we need only to describe the push-forward map  $\varphi_*$  induced between  $H^2(Y, \mathbf{Z})/\text{Tors}(H^2(Y, \mathbf{Z}))$  and  $H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z}))$ . For this, let us recall that the dual of the norm map,  $Nm(f)^\vee : J_B^\vee \rightarrow J_C^\vee$ , gives rise, via the natural identifications of the Jacobians with their duals (the rational representations of these identifications are the Poincaré dualities) the pull-back morphism  $f^* : J_B \rightarrow J_C$  (see, for example, [?]). Seeing, as usual, the elements in  $H^2(Y, \mathbf{Z})/\text{Tors}(H^2(Y, \mathbf{Z}))$  as maps from  $H_1(C, \mathbf{Z})$  to  $H_1(E^\vee, \mathbf{Z}) \cong H^1(E, \mathbf{Z})$ , and identifying  $f^*$  with its rational representation, we have the following

**Lemma 2.3.** *With the previous notations,  $\varphi_*(\widehat{c}) = \widehat{c} \circ f^*$ , for any  $\widehat{c} \in H^2(Y, \mathbf{Z})/\text{Tors}(H^2(Y, \mathbf{Z}))$ .*

*Proof.* Let us assume that the genus of  $C$  is  $m \geq g$ , and choose a symplectic basis

$$\{\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m\} \subset H_1(C, \mathbf{Z}),$$

and write down the push-forward map  $f_* : H_1(C, \mathbf{Z}) \rightarrow H_1(B, \mathbf{Z})$ :

$$\begin{aligned} f_*\alpha_j &= \sum_{i=1}^g f_{ij}^{(1)} a_i + \sum_{i=1}^g f_{ij}^{(2)} b_i \\ f_*\beta_j &= \sum_{i=1}^g f_{ij}^{(3)} a_i + \sum_{i=1}^g f_{ij}^{(4)} b_i, \end{aligned}$$

for some  $f_{ij}^{(k)} \in \mathbf{Z}$ ,  $i = 1, \dots, g$ ,  $j = 1, \dots, m$ ,  $k = 1, \dots, 4$ . Otherwords, the rational representation of the norm map  $Nm(f) : J_C \rightarrow J_B$  in the given basis is the matrix:

$$\begin{pmatrix} f^{(1)} & f^{(3)} \\ f^{(2)} & f^{(4)} \end{pmatrix},$$

where  $f^{(k)} = (f_{ij}^{(k)})_{i,j}$ .

It induces, by means of Poincaré duality, the push-forward map

$$H^2(Y, \mathbf{Z})/\text{Tors}(H^2(Y, \mathbf{Z})) \xrightarrow{\varphi_*} H^2(X, \mathbf{Z})/\text{Tors}(H^2(X, \mathbf{Z})),$$

by

$$\begin{aligned} \varphi_*(\alpha_j^* \otimes e_k) &= \varphi_*(-\widetilde{\beta}_j \otimes e_k) = \varphi_*(-\widetilde{\beta}_j) \otimes e_k = -\mathcal{P}_B(\varphi_*(\beta_j)) \otimes e_k \\ &= -\mathcal{P}_B \left( \sum_i f_{ij}^{(3)} a_i + \sum_i f_{ij}^{(4)} b_i \right) \otimes e_k \\ &= - \left( \sum_i f_{ij}^{(3)} b_i^* - \sum_i f_{ij}^{(4)} a_i^* \right) \otimes e_k \\ &= \sum_i f_{ij}^{(4)} (a_i^* \otimes e_k) - \sum_i f_{ij}^{(3)} (b_i^* \otimes e_k). \end{aligned}$$

In a similar way we get

$$\varphi_*(\beta_j^* \otimes e_k) = - \sum_i f_{ij}^{(2)} (a_i^* \otimes e_k) + \sum_i f_{ij}^{(1)} (b_i^* \otimes e_k),$$

and thus, for any class  $\widehat{c} \in H^2(Y, \mathbf{Z})/\text{Tors}(H^2(Y, \mathbf{Z}))$  represented by a matrix as before, the push-forward class  $\varphi_*(\widehat{c})$  will be represented by the matrix

$$(2.2) \quad \varphi_*(\widehat{c}) = \widehat{c} \cdot \begin{pmatrix} {}^t f^{(4)} & -{}^t f^{(3)} \\ -{}^t f^{(2)} & {}^t f^{(1)} \end{pmatrix}.$$

One can easily notice that the rational representation of the morphism  $f^* : J_B \rightarrow J_C$  in the given bases is exactly the matrix

$$\begin{pmatrix} {}_t f^{(4)} & -{}_t f^{(3)} \\ -{}_t f^{(2)} & {}_t f^{(1)} \end{pmatrix},$$

and the proof is over.

### 3. Rank-2 vector bundles via double coverings

The double coverings are well-known and intensively used in the literature for several purposes (see for ex. [?], [?]). As for vector bundles, Schwarzenberger (cf. [?], see also [?]) proved that in the *projective* case, any rank two vector bundle is obtained as a push forward of a line bundle over a suitable double covering.

Suppose for the moment that  $D$  is a smooth divisor on a complex compact surface  $X$  such that  $\mathcal{O}_X(D)$  is divisible by two in  $Pic(X)$  that is there exists  $\mathcal{L} \in Pic(X)$  such that  $\mathcal{L}^{\otimes 2} = \mathcal{O}_X(D)$ . Then one can canonically associate a smooth connected surface  $Y$  together with a double covering  $Y \xrightarrow{\varphi} X$  (see for example [?]).

The Chern classes of the new surface are computed by the formulae:

$$c_1(Y) = \varphi^*(c_1(X)) - \varphi^*c_1(\mathcal{L}),$$

$$c_2(Y) = 2c_2(X) - 2c_1(\mathcal{L}) \cdot c_1(X) + 4c_1^2(\mathcal{L}).$$

If one considers now a line bundle  $L$  on  $Y$ , the push-forward sheaf  $\mathcal{E} = \varphi_*L$  is actually a rank 2 vector bundle on  $X$ . Denoting also by  $\varphi_* : H^2(Y, \mathbf{Z}) \rightarrow H^2(X, \mathbf{Z})$  the map induced by the push-forward on homology and Poincaré dualities on  $Y$  and  $X$ , by applying the Riemann-Roch Theorem for  $\varphi$  (see [?], [?]) one gets the following formulae for the Chern classes of  $\mathcal{E}$ :

$$(3.1) \quad \begin{aligned} c_1(\mathcal{E}) &= \varphi_*c_1(L) - c_1(\mathcal{L}) \\ c_2(\mathcal{E}) &= \frac{1}{2} [(\varphi_*c_1(L))^2 - c_1^2(L) - (\varphi_*c_1(L)) \cdot c_1(\mathcal{L})]. \end{aligned}$$

In particular,

$$(3.2) \quad \Delta(\mathcal{E}) = \frac{1}{8} [(\varphi_*c_1(L))^2 - c_1^2(L) - 2c_1^2(L)].$$

Moreover, applying the relation (3.1), one can easily see that for any two line bundles  $L$  and  $L'$  on  $Y$ , the following formula holds

$$(3.3) \quad c_1(\varphi_*(L \otimes L')) = c_1(\varphi_*L) + \varphi_*c_1(L').$$

We restrict next to the case we are directly interested in. Let then  $X \xrightarrow{\pi} B$  be a primary Kodaira surface, over the elliptic curve  $B$ , with fiber  $E$ ,  $C$  a curve of genus 2, and  $f : C \rightarrow B$  a double covering. Let  $Y = X \times_B C \rightarrow C$  which is a principal elliptic bundle over  $C$ , and covers doubly  $X$  by the natural map  $Y \xrightarrow{\varphi} X$ .

If we consider, as before, a line bundle  $L$  on  $Y$  and we denote  $\mathcal{E} = \varphi_*L$ , then by Theorem 2.1 and the formulae above we have

$$(3.4) \quad \begin{aligned} c_1(\mathcal{E}) &\equiv \varphi_*c_1(L) \text{ modulo } \text{Tors}(NS(X)) \\ \Delta(\mathcal{E}) &= \frac{1}{8} [(\varphi_*c_1(L))^2 - 2c_1^2(L)]. \end{aligned}$$

Now, we want to give *algebraic* interpretations for  $c_1(\mathcal{E})$  and  $4\Delta(\mathcal{E})$ . For this, denote firstly by  $F = \text{Ker}(J_C \rightarrow B)$  (which is an elliptic curve), and consider  $g : C \rightarrow F$  the complementary double cover (see, for example [?], [?], [?]). Then the pull-back maps  $i_B := f^* : B \rightarrow J_C$  and  $i_F := g^* : F \rightarrow J_C$  turn out to be injective, and they give rise an isogeny  $i_B \times i_F : B \times F \rightarrow J_C$ . Therefore, we can write  $J_C \cong (B \times F)/H$ , where  $H = \text{Ker}(i_B \times i_F)$ ; the subgroup  $H \subset (B \times F)[2]$  is in fact the graph of a (symplectic) isomorphism from  $B[2]$  to  $F[2]$ .



**Lemma 3.1.** *In the hypotheses above, for any line bundle  $L$  on  $Y$ ,  $\mathcal{E} = \varphi_* L$  is a rank-2 vector bundle on  $X$  with  $\widehat{c}_1(\mathcal{E}) = \widehat{c}_1(\widehat{L}) \circ i_B$  and  $4\Delta(\mathcal{E}) = \text{deg}(\widehat{c}_1(\widehat{L}) \circ i_F)$ .*

*Proof.* Let  $\widehat{c} \in NS(Y)/\text{Tors}(NS(Y))$  be the class associated to  $c_1(L) \in NS(Y)$  and  $\widehat{c}_1 \in NS(X)/\text{Tors}(NS(X))$  the class associated to  $c_1(\mathcal{E}) \in NS(X)$ . Then the formulae (3.4) read:

$$(3.5) \quad \widehat{c}_1 = \varphi_*(\widehat{c}) \text{ and } \Delta(\mathcal{E}) = \frac{1}{8} (\widehat{c}_1^2 - 2\widehat{c}^2).$$

As shown in [?], we can find suitable symplectic bases such that the period matrix of  $J_C$  is of type

$$\Pi_C = \begin{pmatrix} \mu_B/2 & 1/2 & 1 & 0 \\ 1/2 & \mu_F/2 & 0 & 1 \end{pmatrix},$$

the period matrices of  $B$  and  $F$  are  $\Pi_B = (\mu_B, 1)$  and  $\Pi_F = (\mu_F, 1)$ , and the norm map  $Nm(f) : J_C \rightarrow B$  has the analytical and rational representations  $A = (2 \ 0)$  and respectively

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}.$$

In the notations used in the proof of Lemma 2.3,  $f^{(1)} = (1 \ 0)$ ,  $f^{(2)} = (0 \ 1)$ ,  $f^{(3)} = (0 \ 0)$ ,  $f^{(4)} = (2 \ 0)$ .

In the same time,  $H$  is generated by  $(\mu_B/2, 1/2)$  and  $(1/2, \mu_F/2)$ , and the two canonical inclusions  $i_B : B \hookrightarrow J_C$  and  $i_F : F \hookrightarrow J_C$  have the rational representations

$$R_B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } R_F = \begin{pmatrix} 0 & 0 \\ 2 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$$

respectively.

By means of Theorem 2.1,  $\widehat{c}$  and  $\widehat{c}_1$  are represented by matrices with integer entries

$$\widehat{c} = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{pmatrix}, \widehat{c}_1 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

and the relationship between them is  $c_{11} = 2a_{11} - b_{12}$ ,  $c_{12} = b_{11}$ ,  $c_{21} = 2a_{21} - b_{22}$ ,  $c_{22} = b_{21}$  (i.e.  $\widehat{c}_1 = \widehat{c} \circ i_B$  as shown in Lemma 2.3.).

If we set  $d = \widehat{c} \circ i_F$ , then  $d$  has the rational representation

$$d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

where  $d_{11} = 2a_{12} - b_{11}$ ,  $d_{12} = b_{12}$ ,  $d_{21} = 2a_{22} - b_{21}$ , and  $d_{22} = b_{22}$ . Now, directly from the formulae (2.1) and (3.5) we see that  $-c_1^2(\mathcal{E})/2 = \det(\widehat{c}_1)$  and  $4\Delta(\mathcal{E}) = \det(d)$ .

#### 4. Proof of the Theorem 1.

In the sequel, for the sake of simplicity, we shall not make a clear distinction between  $E$  and its dual curve  $E^\vee$ .

Since the case  $\Delta(2, c_1, c_2) = 0$  has been solved by Toma (cf. [?]), we shall assume, from now on, that  $\Delta(2, c_1, c_2) \in (0, m(2, c_1))$ . We divide the proof of the Theorem into several steps.

To begin with, we set  $n = 4\Delta(2, c_1, c_2) = 2c_2 - c_1^2/2 > 0$ ,  $E = \mathbf{C}/(\mu_E, 1)\mathbf{Z}^2$ , and define the elliptic curve  $F = \mathbf{C}/(\mu_F, 1)\mathbf{Z}^2$ , where  $\mu_F = n\mu_E$ . We denote, as in Lemma 1.3,  $\delta : F \rightarrow E$  the canonical projection. Observe that  $n$  is odd if and only if  $-c_1^2/2 = \text{deg}(\widehat{c}_1)$  is odd.

*Step 1.* The assumptions on the classes  $c_1$  and  $c_2$  imply that  $\widehat{c}_1[2] : B[2] \rightarrow E[2]$  is not identically zero. Indeed, if  $\widehat{c}_1[2]$  vanished identically,  $\widehat{c}_1$  would be divisible by 2 in  $NS(X)/\text{Tors}(NS(X))$  (cf. Lemma 1.3 (b)), which would imply that  $m(2, c_1) = 0$ .

*Step 2.* There exists an isomorphism  $\psi : B[2] \rightarrow F[2]$ , such that  $\widehat{c}_1[2] = \delta[2] \circ \psi$ . Indeed, if  $n$  is odd, then the morphisms induced by  $\widehat{c}_1$  and  $\delta$  between the 2-torsion subgroups  $\widehat{c}_1[2] : B[2] \rightarrow E[2]$  and  $\delta[2] : F[2] \rightarrow E[2]$  are isomorphisms (cf. Lemma 1.4), and we can choose directly  $\psi = \delta[2]^{-1} \circ \widehat{c}_1[2]$ . If  $n$  is even, then  $\text{Ker}(\widehat{c}_1[2]) \neq 0$  (Lemma 1.4), but  $\widehat{c}_1[2] \neq 0$  (cf. Step 1). We can define  $\psi^{-1}$  to map  $\mu_F/2$  (which is the generator of  $\text{Ker}(\delta[2])$ ), to the generator of  $\text{Ker}(\widehat{c}_1[2])$ , and to map  $1/2$  to another non-zero element in  $B[2]$ , which does not lie in  $\text{Ker}(\widehat{c}_1[2])$  (notice that we have two different possible choices).

*Step 3.* There is no isomorphism  $\eta$  between  $B$  and  $F$  such that  $\eta[2] = \psi$ . Indeed, if there is such an isomorphism, we have  $\widehat{c}_1[2] = (\delta \circ \eta)[2]$ , otherwords  $(\widehat{c}_1 \circ \eta^{-1})[2] = \delta[2]$ . By means of Lemma 1.3 (a), there exists then a morphism  $\mu' : F \rightarrow E$  such that  $(\widehat{c}_1 \circ \eta^{-1}) - 2\mu' = \delta$ , and thus, setting  $\mu = \mu' \circ \eta$ , we have  $\widehat{c}_1 - 2\mu = \delta \circ \eta$ . Apply now Lemma 2.2 and see that, in this case,  $m(2, c_1) \leq \text{deg}(\delta \circ \eta)/4 = \Delta(2, c_1, c_2)$ ; this contradicts the assumptions we have made.

*Step 4.* Denote now  $H_\psi = \text{Graph}(\psi) \subset (B \times F)[2]$ . Then, from Kani's "Reducibility Criterion" (Theorem 2) and the previous step, it follows that  $(B \times E)/H_\psi$  is a Jacobian of a curve  $C$  of genus 2 which covers two-to-one  $B$ . Moreover, the morphism  $\widehat{c} : B \times F \rightarrow E$  defined by  $\widehat{c}(x, y) = \widehat{c}_1(x) + \delta(y)$  for any pair  $(x, y) \in B \times F$ , factors through a morphism  $\widehat{c} : (B \times F)/H_\psi \rightarrow E$ . It is clear that  $\widehat{c} \circ i_B = \widehat{c}_1$  and  $\widehat{c} \circ i_F = \delta$ .

If we choose a line bundle  $L$  on  $Y$  whose Chern class modulo torsion of  $NS(Y)$  equals  $\widehat{c}$ , then Lemma 3.1 precisely says that  $\mathcal{E} = \varphi_* L$  is a rank two vector bundle on  $X$  with  $c_1(\mathcal{E}) \equiv c_1$  modulo  $\text{Tors}(NS(X))$  and discriminant  $\Delta(\mathcal{E}) = \text{deg}(\delta)/4 = \Delta(2, c_1, c_2)$ .

Now, Theorem 2.1 tells that in order to get rid of the torsion, we need to add multiples of a class of a fiber. To do this, we consider  $F_Y$  to be a fiber of the projection map  $Y \rightarrow C$ ,  $F_X$  to be a fiber of the projection map  $X \rightarrow B$ , and for any  $m \geq 0$ , we set  $L_m = \mathcal{O}_Y(mF_Y)$ . In homology,  $\varphi_*\{F_Y\} = \{F_X\}$ , and thus  $\varphi_*c_1(L_m) = mc_1(\mathcal{O}_X(F_X))$ . Formula (3.3) reads here  $c_1(\varphi_*(L \otimes L_m)) = c_1(\mathcal{E}) + mc_1(\mathcal{O}_X(F_X))$ , which ends the proof.

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