Locally Maximal Clones II

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November 30, 1999

Abstract

For a general local completeness criterion for algebras on an infinite universe A we need to know all locally maximal clones on A as well the increasing chains of proper local clones whose union is locally complete. The clones in question are all of the form $\mathrm{Pol}\,\rho$ where ρ is a finitary relation on A and $\mathrm{Pol}\,\rho$ is the set of all operations on A preserving ρ. The predecessor paper Local completeness I left the following 5 sets of relations on A to be sorted out: the locally bounded graphs, diagraphs and reflexive digraphs of diameter 2, a set of ternary relations and the pivotal set of all totally reflexive and symmetric relations. We present partial results for each of the 5 types. For graphs they relate to largest infinite cliques and digraphs are restricted to the acyclic ones which either have arctransitive endomorphisms or a certain vanishing-interval property. We restrict the last type to two kinds of relations termed strongly homogeneous and protective and we find many instances of the increasing chains of proper local clones mentioned earlier.

Key words: Local completeness, locally maximal clone, preservation of relations

MR classification: 08 A 40, 08 A 02

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1 Introduction and preliminaries

1.1 The paper is devoted to the search for a general local completeness criterion for universal algebras on a given infinite universe A. Such a criterion is provided by a cofinal subset of the poset of local clones distinct from the greatest clone $\mathcal{O}_A$ of all finitary operations on A. A "small" cofinal set consists of all locally maximal (also called precomplete or preprimal) clones as well as certain towers, (increasing chains of local clones whose union is locally complete).
The paper starts off where [R-Sc 84] ended - namely with 5 types of relations on A listed below - and proceeds by an elimination based on relational constructions. A pivotal part is the elimination in the set T of totally reflexive and totally symmetric relations on A which is likely to produce most of local clones and towers. In §7 a variant of the ideas from [Ro 65,70] leads to a reduction of T to the set of relations called strongly homogeneous and protective (Definition 7.10). Among the remaining 4 types three are provided by the locally bounded graphs, reflexive digraphs of diameter 2 and graphs (§§3-5). The last type consists of nontrivial ternary relations on A of the form $\sigma \cup \{aab : a, b \in A\}$ where $\sigma \subseteq \sigma_3$ (see 6.1). The last 4 types have no analogue in the finite case.
We have been able to produce only very partial results. For graphs they relate to largest infinite cliques (i.e. complete subgraphs) and yield $2^{|A|}$ graphs ρ on A with locally maximal clones $\mathrm{Pol}\,\rho$ (of operations compatible with ρ). For digraphs we narrow the search to strict order-like digraphs of two kinds. The digraphs of the first kind have arc-transitive endomorphisms. We show

that for unbounded chains $<$ with arc-transitive endomorphisms the clone $\mathrm{Pol}\,(<)$ is locally maximal; e.g. this happens for $\mathbb{Q}$ or $\mathbb{R}$ with the natural order. The digraphs of the second kind have a certain vanishing property for intervals. For the reflexive digraphs and the ternary relations we only have very preliminary results.

Perhaps the relevance of locally maximal clones and towers should be addressed. The knowledge of large clones would be of interest because it would allow a certain classification of universal algebras and of some of their basic properties. However, presently there is little hope to gain such knowledge and so local clones could be seen as a reasonable substitute. The local clones on A are exactly the clones determined by sets of finitary relations on A (1.3). Now most compatible relations used explicitely in today's universal algebra, like subalgebras, congruences, tolerances and endomorphisms, are finitary - even often at most binary - leading to the relevancy of local clones. Another reason for the study of large local clones is the fact that some properties of finite maximal clones, like the McKenzie-Gumm theorem, extend to locally maximal clones. Locally maximal clones and towers would also fit well to the large literature on local clones and algebras.

The existential properties of relations on infinite sets arising naturally in this context are less effective than they are on finite universes where eventually one runs out of space. Also relations behave differently on infinite sets than they do on finite sets; e.g. recall the well-known Erdös' result that "almost all" graphs on a countably infinite vertex set are pairwise isomorphic; a fact, however, of little use to us because we may need exactly the "few" exceptional graphs.

While most of the locally maximal clones from [R-Sc 82, R-Sz 84] were related to present universal algebra, the 5 types studied in this paper seem to be located to be outside the realm of today's interests.

It could as well turn out that our task is hopeless but still it would be helpfull to restrict the 5 types as much as possible. From our point of view the relational methodology should be exploited as far as possible. One of the purposes of this paper is to draw attention to the 5 types and we present our very preliminary results with the hope that they might provide the initial push for further research. Of course, it could turn out that several of the results may lead in blind alleys.

We obtained most of the results some time ago and for the above reasons we decided to publish them in spite of their admitted shortcomings and unfinished nature. We have strived to make the presentation reasonably self-contained.

The partial financial support provided to the first author by the University of Kaiserslautern and an NSERC Canada grant as well the hospitality shown

to him by the Mathematics Department during his 1998 one month stay in Kaiserslautern are gratefully acknowledged.

1.2 We briefly outline the background, terminology and notations. Let A be an infinite universe. For a positive integer n denote by $\mathcal{O}_A^{(n)}$ the set of n-ary operations (or functions) on A; i.e. of the maps $f : A^n \to A$. For example $\mathcal{O}_A^n$ contains the i-th n-ary operation e_i^n defined by setting $e_i^n(a_1, ..., a_n) := a_i$ for all $a_1, ..., a_n \in A$. Set $\mathcal{O}_A := \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}$ and for $X \subseteq \mathcal{O}_A$ and $n > 0$ set $X^{(n)} := X \cap \mathcal{O}_A^{(n)}$. A subset C of $\mathcal{O}_A$ is a *clone* on A if it is composition closed and contains all projections (for a more elegant, precise but less intuitive definition see [Ma 66]); equivalently, clones on A are the sets of term operations of universal algebras on A. Denote by L_A the set of all clones on A. The poset $\mathcal{L}_A := (L_A; \subseteq)$ is an algebraic lattice whose greatest element is the clone $\mathcal{O}_A$ and $\cap F$ is the meet of a family F of clones. For $X \subseteq \mathcal{O}_A$ the least clone (in the complete lattice $\mathcal{L}_A$) containing X is the clone *generated by* X.

1.3 The *local closure* Loc on $\mathcal{O}_A$ is defined as follows. Let $X \subseteq \mathcal{O}_A$. Then $f \in \mathcal{O}_A^{(n)}$ belongs to $\operatorname{Loc} X$ if for each finite subset F of A there exists $g \in X^{(n)}$ (depending upon f and F) such that $f \restriction F = g \restriction F$ (here $f \restriction F$ denotes the restriction of f to F; i.e., the map from F^n to A defined by $(f \restriction F)(a_1, ..., a_n) := f(a_1, ..., a_n)$ for all $a_1, ..., a_n \in F$). Such an operation f can be thought off as assembled from the restrictions of operations from $X^{(n)}$ to finite subsets of A; equivalently, somebody with a tunnel vision, allowing him to view only finite parts of A^n, could not discern whether $f \in X^{(n)}$ or not. A clone C is *local* if $\operatorname{Loc} C = C$.

To justify the definition of a local clone we need the following concept. For a positive integer h a subset ρ of A^h is an *h-ary relation* on A. Here the h-tuples from A^h are written as $a_1...a_h$ instead of the usual $(a_1, ..., a_h)$; moreover, we do not view ρ as a predicate and hence do not perceive ρ as a map from A^h into the truth values $\{T, F\}$ or $\{+, -\}$ as it is customary in logics. Denote by $\mathcal{R}_A^{(h)}$ the set of h-ary relations on A (i.e. $\mathcal{R}_A^{(h)} = \mathcal{P}(A^h)$) and set $\mathcal{R}_A := \bigcup_{h=1}^{\infty} \mathcal{R}_A^{(h)}$. We say that $f \in \mathcal{O}_A^{(n)}$ *preserves* $\rho \in \mathcal{R}_A^{(h)}$ if ρ is a subuniverse of $\langle A; f\rangle^h$ (= the h-th power of the universal algebra with a single operation f); explicitly f *preserves* ρ if for every $h \times n$ matrix M whose columns are all in ρ the values of f on the rows of M form an h-tuple from ρ. For example, if $h = 2$ and ρ is an equivalence relation on A, then f preserves ρ iff ρ is a congruence of $\langle A; f\rangle$. Preservation is also known under several other names, like compatibility etc. Set

$$\operatorname{Pol} \rho := \{f \in \mathcal{O}_A^{(n)} : f \text{ preserves } \rho, n \in \mathbb{N}\}$$

and for all $R \subseteq \mathcal{R}_A$ also set $\operatorname{Pol} R := \bigcap_{\rho \in R} \operatorname{Pol} \rho$. (This standard notation

should not be confused with the one used for the set of polynomials of an algebra.)
The following is well known (see e.g. [Rom 77, R-Sc 82] and easy to prove: *A clone* C *on* A is *local if and only if* $C = \operatorname{Pol} R$ *for some* $R \subseteq \mathcal{R}_A$. (This in fact characterizes the Galois-closed subsets of $\mathcal{O}_A$ in the Galois connection induced by the relation "f preserves ρ" between $\mathcal{O}_A$ and $\mathcal{R}_A$.)

1.4 We turn to the Galois closed subsets of $\mathcal{R}_A$ generated by a single relation ρ in the above Galois connection. We say that $\sigma \in \mathcal{R}_A$ *dominates* $\rho \in \mathcal{R}_A$ if $\operatorname{Pol} \rho \subseteq \operatorname{Pol} \sigma$.
Denote by $[\rho]$ the set of all $\sigma \in \mathcal{R}_A$ that dominate ρ (i.e. $[\rho] = \operatorname{Inv} \operatorname{Pol} \rho$ in the standard notation). It is known that $[\rho]$ is closed under arbitrary intersections. For relations ρ and σ on A write $\rho \approx \sigma$ if ρ dominates σ and σ dominates ρ.
How to decide whether σ dominates ρ? To verify directly that $\operatorname{Pol} \rho \subseteq \operatorname{Pol} \sigma$ one needs some knowledge about the operations preserving ρ and this is often time consuming if not outright hard. We describe another way. For $p \geq 1$ a p-ary *resolvent* of an h-ary relation ρ on A is

$$\sigma := \{f(1) \dots f(p) : f \in \operatorname{Hom}(\gamma, \rho)\} \tag{1.1}$$

where γ is an h-ary relation on a set I containing $\{1, \dots, p\}$ and $\operatorname{Hom}(\gamma, \rho)$ denotes the set of *relational homomorphisms* from γ to ρ; i.e. of $f : I \to A$ such that

$$i_1 \dots i_h \in \gamma \Rightarrow f(i_1) \dots f(i_h) \in \rho. \tag{1.2}$$

We denote σ by $\gamma \curvearrowright_p \rho$ and call γ the *auxiliary relation* of the resolvent.

Example. Let ρ be a binary relation on A and

$$\rho \circ \rho := \{xy : xu, uy \in \rho \text{ for some } u\}. \tag{1.3}$$

Set $I := \{1, 2, 3\}$ and $\gamma := \{13, 23\}$. Then $\gamma \curvearrowright_2 \rho = \rho \circ \rho$. This is a prototype for resolvents and reveals the existential character of (1.1) which is due to the fact that for all $i \in I \setminus \{1, \dots, p\}$ the value $f(i)$ exists but does not appear explicitly in σ. We shall often define resolvents in a form similar to (1.3). It is not difficult to prove that every resolvent of ρ dominates ρ; in fact, domination is determined by resolvents. To make this more precise we need two concepts. First an h-ary relation ρ on A is *repetition-free* if for all $1 \leq i < j \leq h$ there exists $a_1 \dots a_h \in \rho$ with $a_i \neq a_j$. It is well known and easy to check that a systematically repeated coordinate in all h-tuples from ρ can be removed without affecting $\operatorname{Pol} \rho$ and so without loss of generality we may assume ρ to be repetition-free. Next recall that $R \subseteq \mathcal{R}_A^{(h)}$ is *directed* if for all $\rho, \rho' \in R$ there exists $\rho'' \in R$ containing $\rho \cup \rho'$. The union of a directed set R is a *directed union*. A basic result from [R-Sz 84] can be formulated as follows. *A p-ary repetition-free relation σ on A dominates a relation ρ on A if and only if σ is the directed union of a set of p-ary resolvents of ρ.* This result is sometimes expressed in terms of predicates, $\exists$ and the equality $=$ but for effective use the rational form (1.1) seems to be more appropriate than a logic one. To show $\operatorname{Pol}\rho \subseteq \operatorname{Pol}\sigma$ the whole trick is to find suitable relations γ in (1.1) and therein actually lies the whole difficulty in domination.

1.5 Denote by Lc_A the set of local clones on A. The poset $(\mathrm{Lc}_A, \subseteq)$ is a complete lattice that is neither algebraic nor a sublattice of $\mathcal{L}_A$. For $X \subseteq \mathcal{O}_A$ the (nonindexed) universal algebra $\langle A; X\rangle$ is *locally complete* if $\operatorname{Loc} X = \mathcal{O}_A$. Set $P := \mathrm{Lc}_A \setminus \{\mathcal{O}_A\}$. A subset $\mathcal{B}$ of P is *generic* if $\mathcal{B}$ is cofinal in the poset $\mathcal{P} := (P, \subseteq)$, i.e. if every local clone C distinct from $\mathcal{O}_A$ is contained in some clone B from $\mathcal{B}$. Evidently every generic set $\mathcal{B}$ provides a local completeness criterion:

Let $X \subseteq \mathcal{O}_A$. Then $\langle A; X\rangle$ is locally complete if and only if $X \subseteq B$ for no $B \in \mathcal{B}$.

Naturally we wish the generic set $\mathcal{B}$ to be as small as possible. Call $C \in P$ *locally maximal* if C is a maximal element of $\mathcal{P}$, i.e., if $C \subset D$ for no $D \in P$. Clearly each generic set contains all locally maximal clones. Unfortunately, it is not true that each $C \in P$ is contained in a locally maximal clone.

A *tower* is a transfinite sequence $\langle \rho_\xi : \xi < \varsigma\rangle$ in $\mathcal{R}_A$ (where ζ is an ordinal) such that:

(i) $\operatorname{Pol}\rho_\theta \subset \operatorname{Pol}\rho_\xi$ whenever $\theta < \xi < \varsigma$, and

(ii) $X := \bigcup_{\xi<\varsigma} \operatorname{Pol}\rho_\xi$ is locally complete; i.e. $\operatorname{Loc} X = \mathcal{O}_A$. The following first example of a tower came up in [R-Sz 84], §3 Remark. Let s be a permutation of A whose cycles are all infinite and let s° denote the graph $\{a\ s(a) : a \in A\}$. Then the countable sequence $\left\langle \left(s^{2^i}\right)^\circ : i < w\right\rangle = \left\langle s^\circ, (s^2)^\circ, (s^4)^\circ, ...\right\rangle$ is a tower. In 7.10 and 7.12 we exhibit a large set of towers.

To resume: Our (perhaps unattainable) goal is to find all locally maximal clones and all towers.

1.6 We conclude with some terminology and notation. Let h be a positive integer. Denote by ι_h the h-ary relation on A consisting of all h-tuples with some repeated coordinate; i.e., of all $a_1...a_h \in A^h$ such that $a_i = a_j$ for some $1 \leq i < j \leq h$ (depending on $a_1...a_h$); e.g. $\iota_1 = \emptyset$, $\iota_2 = \{aa : a \in A\}$, $\iota_3 = \{aab : a, b \in A\} \cup \{aba : a, b \in A\} \cup \{abb : a, b \in A\}$. Set $\sigma_h := A^h \backslash \iota_h$. For an equivalence relation ε on $\{1, ..., h\}$ set

$$\Delta_\varepsilon := \left\{a_1...a_h \in A^h : ij \in \varepsilon \Rightarrow a_i = a_j\right\}.$$

The relations Δ_ε are called *diagonal*. Let ρ be an h-ary relation on A. Call ρ *trivial* if $\rho = \emptyset$ or ρ is diagonal. It is well known that $\operatorname{Pol}\rho = \mathcal{O}_A \Leftrightarrow \rho$ is trivial. Next ρ is *totally reflexive* if $\rho \supseteq \iota_h$. For a permutation p of $\{1, ..., h\}$ set

$$\rho^{(p)} := \left\{ a_{p(1)}...a_{p(h)} : a_1...a_h \in \rho \right\}.$$

Note that $\rho^{(p)} \approx \rho$. Call ρ *totally symmetric* if $\rho^{(p)} = \rho$ for all permutations p of $\{1, ..., h\}$; i.e. if ρ is invariant under all coordinate switches. For all $n \geq 1$ denote by T_n the set of all totally reflexive and totally symmetric nontrivial n-ary relations on A. Call ρ *repellent* if $[\rho]$ is disjoint from all $T_n, \ n = 1, 2, ...$ For $1 \leq i < j \leq h$ set

$$\operatorname{pr}_{ij} \rho := \{a_i a_j : a_1...a_h \in \rho\}$$

and notice that $\operatorname{pr}_{ij} \rho$ dominates ρ.

2 Chains of totally reflexive and symmetric relations

We start with the following simple lemma.

2.1 Lemma.
Let $0 < n_1 < n_2 < ...$ and let ρ_i be totally reflexive n_i-ary relations on A $(i = 1, 2, ...)$ such that for each positive integer h some $\operatorname{Pol} \rho_i$ contains an essentially at least binary operation assuming at least h values.
Then $V := \bigcup_{i=1}^{\infty} \operatorname{Pol} \rho_i$ is locally complete.

Proof. We prove that $\mathcal{O}^{(1)} \subseteq \operatorname{Loc} V$. Let $C \subset A$ be finite, $f \in \mathcal{O}^{(1)}$ and $h := |f(C)|$. Let $n_i > h$, let $c \in f(C)$ and let $g \in \mathcal{O}^{(1)}$ be defined by $g(x) := f(x)$ for all $x \in C$ and $g(x) := c$ otherwise. On account of the total reflexivity of ρ_i and $n_i > h$ we have $g \in \operatorname{Pol} \rho_i \subseteq V$. This proves that $\mathcal{O}^{(1)} \subseteq \operatorname{Loc} V$. Applying [R-Sc 82] 4.2 we obtain the required $\operatorname{Loc} V = \mathcal{O}$. ■

Let B and C be sets and $\alpha : B \to C$ be a surjective map. For an h-ary relation τ on C define:

$$\alpha^{-1}(\tau) := \left\{ (a_1, ..., a_h) \in B^h : (\alpha(a_1), ..., \alpha(a_h)) \in \tau \right\}.$$

We have

2.2 Fact:
Let $\alpha : B \to C$ be surjective and let $\varphi : C \to B$ be such that $\alpha \circ \varphi = id_C$. To $f \in \mathcal{O}_C^{(n)}$ associate $\widehat{f} \in \mathcal{O}_B^{(n)}$ defined by setting

$$\widehat{f}(b_1, ..., b_n) := \varphi(f(\alpha(b_1), ..., \alpha(b_n)))$$

for all $b_1, ..., b_n \in B$. *If* $f \in \operatorname{Pol} \tau$ *then* $\widehat{f} \in \operatorname{Pol} \alpha^{-1}(\tau)$.

Proof. Let X be an $h \times n$ matrix whose columns are all in $\alpha^{-1}(\tau)$. Set $Y = (Y_{ij}) := (\alpha(X_{ij}))$. Clearly all columns of Y are in τ and therefore

$$(\varphi(f(Y_{1*})), ..., \varphi(f(Y_{h*}))) \in \alpha^{-1}(\tau)$$

where Y_{i*} denotes the i-th row of Y. Here $\varphi(f(Y_{i*})) = \widehat{f}(X_{i*})$ $(i = 1, ..., h)$ on account of the definition of $\widehat{f}$, i.e. $\widehat{f}$ preserves $\alpha^{-1}(\tau)$. ■

Recall that for intergers $h > 1$ and $m > 0$
$\iota_h := \left\{(a_1, ..., a_h) \in A^h : a_i = a_j \text{ for some } 1 \leq i < j \leq h\right\}$,
$\iota_h^m := \{((a_{11}, ..., a_{1m}), ..., (a_{h1}, .., a_{hm})) : (a_{1j}, ..., a_{hj}) \in \iota_h, \ j = 1, ..., m\}$.

2.3 Corollary:
Let m *be a positive integer, let* $\alpha : B \to A^m$ *be surjective and let* $\varphi_h := \alpha^{-1}(\iota_h^m)$, $h = 1, 2, ...$. *Then* $V := \bigcup_{h=1}^{\infty} \operatorname{Pol} \varphi_h$ *is a locally complete clone on* B.

Proof. Clearly ι_h^m is totally reflexive and so is ρ_h. It is easy to see that $\operatorname{Pol} \iota_{h+1}^m$ contains an essentially binary operation f with exactly h values. Choose $\varphi : A^m \to B$ so that $\alpha \circ \varphi = \mathrm{id}_{A^m}$. The corresponding operation $\widehat{f}$ (from Fact 2.2) preserves ι_{h+1}. Clearly f assumes h values and thus applying Lemma 2.1 we get the desired result. ■

3 Locally bounded graphs

In this section $(A; \rho)$ is a *simple graph* (i.e. ρ is a binary areflexive and symmetric relation) which is *locally bounded*. This means that each finite subset B of A has a joint or common neighbor u (i.e. $\{u\} \times B \subseteq \rho$). We assume that ρ is repellent. (see 1.6) As usual, a *clique* C is a complete subgraph of ρ (i.e. $c_1c_2 \in \rho$ for all $c_1, c_2 \in C$, $c_1 \neq c_2$). We show first that ρ is covered by countable cliques.

3.1 Lemma:
Each finite clique of ρ, *and in particular each edge of* ρ, *is contained in a countable clique.*

Proof. Let C be an m-element clique of ρ. Let u be a bound of C. Then clearly $C := \{u\} \cup C$ is an $(m + 1)$-element clique. ■

For a subset B of A put $N_B := \{y : xy \in \rho \text{ for all } x \in B\}$. Note that B and N_B are disjoint. The graph ρ_B is obtained from ρ by deleting all edges between B and $A \backslash N_B$. Note that for $|B| = 1$ we have $\rho_B = \rho$ and that for

$|B| > 1$ the set B is an independet set of ρ_B (i.e. $b_1b_2 \notin \rho_B$ for all $b_1, b_2 \in B$). We say that ρ is *finitely transitive* if for every positive integer n, each n-element subset B and arbitrary $a_1, ..., a_n \in A$ there exists $\varphi \in \mathrm{Hom}(\rho_B, \rho)$ (see 1.4) such that $\varphi(b_i) = a_i \quad (i = 1, ..., n)$. We have:**(see 1.4)**

3.2 Lemma:
The relation ρ is finitely transitive.
Proof. For a positive integer n and $B = \{b_1, ..., b_n\} \subset A$ put

$$\tau = \tau_{b_1 \dots b_n} := \{(\varphi(b_1), ..., \varphi(b_n)) : \varphi \in \mathrm{Hom}(\rho_B, \rho)\}.$$

We need the following two claims for $\tau := \tau_{b_1 \dots b_n}$. For notational simplicity denote b_i by i for all $i = 1, ..., n$.

3.3 Claim 1:
The relation τ is totally symmetric.

Proof of the claim. Let π be a permutation of $\{1, ..., n\}$ and $(x_1, ..., x_n) \in \tau$. Then $x_i = \varphi(i) \quad (i = 1, ..., n)$ for some $\varphi \in \mathrm{Hom}(\rho_B, \rho)$. Define a selfmap ψ of A by setting $\psi(i) := \varphi(\pi(i))$ for $i = 1, ..., n$ and $\psi(x) := \varphi(x)$ otherwise. We claim that $\psi \in \mathrm{Hom}(\rho_B, \rho)$. Indeed, it suffices to verify it for an edge ix from ρ_B. By the construction of ρ_B we can assume that $i \in B$ and $x \in N_B$. Then $jx \in \rho_B$ for $j = 1, ..., n$. Since $\varphi \in \mathrm{Hom}(\rho_B, \rho)$, it follows that $\psi(i)\psi(x) = \varphi(\pi(i))\varphi(x) \in \rho$. Using $\psi(i) = \varphi(\pi(i)) = x_{\pi(i)}$ we get the required $\left(x_{\pi(1)}, ..., x_{\pi(n)}\right) \in \tau$. This proves the claim.

In view of Claim 1 for $B = \{b_1, ..., b_n\}$ we write τ_B instead of $\tau_{b_1, \dots, b_n}$.

3.4 Claim 2:
Let $B \subset C \subset A$ be such that $|B| = n$ and let $|C| = n+1$. If $(x_1, ..., x_n) \in \tau_B$ then $(x_1, ..., x_n, x_n) \in \tau_C$.
Proof. For notational simplicity let $B := \{1, ..., n\}$ and $C := \{1, ..., n+1\}$. By the definition of τ_B we have $x_i = \varphi(i)$ for some $\varphi \in \mathrm{Hom}(\rho_B, \rho)$. Define $\psi : A \to A$ by setting $\psi(n+1) := \varphi(n)$ and $\psi(x) := \varphi(x)$ otherwise. We claim that $\psi \in \mathrm{Hom}(\rho_C, \rho)$. Again it is enough to check it for an edge $(n+1)x \in \rho_B$. By the definition of ρ_C clearly $nx \in \rho_B \cap \rho_C$ and, as $\varphi \in \mathrm{Hom}(\rho_B, \rho)$ also $\psi(n+1)\psi(x) = \varphi(n)\varphi(x) \in \rho$ and therefore by the definition of τ_C we get $(x_1, ..., x_n, x_n) \in \rho_C$. This proves the claim.

Now it suffices to prove by induction on n that $\tau_{b_1 \dots b_n} = A^n$ for all $\{b_1, ..., b_n\} \subseteq A$.

(1) Let $n = 1$. Fix $b \in A$. Since $\rho_{\{b\}} = \rho$, we have that $b = id_A(b) \in \tau_b$. Suppose $\tau_{\{b\}} \subset A$. Then the relation $\tau_{\{b\}}$ is a proper unary relation on A and thus $\tau_{\{b\}} \in T_1 \cap [\rho]$ in contradiction to the assumption that ρ is repelling. Thus $\tau_b = A$.

(2) Let $B := \{b_1, b_2\} \subset A$ and let $\tau_{\{b_1\}} = A$. By Claim 2 we have $(x, x) \in \tau_B$ for all $x \in A$. By Claim 1 the relation τ_B is symmetric. Next $b_1 b_2 = \mathrm{id}_A(b_1)\,\mathrm{id}_A(b_2) \in \tau_B$ due to $\mathrm{id}_A \in \mathrm{Hom}(\rho_B, \rho)$. Thus $\tau_B \supset \iota_2 := \{(x, x) : x \in A\}$. Since $\tau_B \in [\rho]$ and ρ is repellent, we have the required $\tau_B = A^2$.

(3) Let $n > 1$, $B \subset C \subset A$ with $|B| = n$, $|C| = n+1$ and $\tau_B = A^n$. In view of Claim 1 the relation τ_C is totally symmetric. By virtue of $\tau_B = A^n$ and Claim 2 we have $(x_1, ..., x_n, x_n) \in \tau_C$ for all $x_1, ..., x_n \in A$. Now τ_C being totally symmetric it is also totally reflexive. Since $\tau_C \in [\rho]$ and ρ is repellent, we get the required $\tau = A^{n+1}$. This completes the proof by induction and the proof of the Lemma. ■

3.5

Let $\mathcal{C}$ denote the set of cliques of ρ. The set $\mathcal{C}$ ordered by $\subseteq$ is obviously closed under unions of chains and so by Zorn's lemma each $C \in \mathcal{C}$ extends to a maximal clique M. Here M is obviously maximal iff for each $x \in A \backslash M$ we have $xm \notin \rho$ for some $m \in M$. Let Γ_ρ denote the set of infinite cardinalities of the cliques of ρ. Note that by Lemma 3.1 clearly $\aleph_0 \in \Gamma_\rho$.

3.6 Corollary

Let $\gamma \in \Gamma_\rho$. Then to each finite subset F of A there exists a clique C of of size γ such that $F \times C \subseteq \rho$ and $|C| = \gamma - |F|$.

Proof. Let E be a clique of ρ with $|E| = \gamma$. Choose $B \subset E$ so that $|B| = n := |F|$ and put $E' := E \backslash B$. Note that $E' \subseteq N_B$ and that the graph ρ_B is obtained from ρ by deleting edges between B and $A \backslash N_B$. Since E' is disjoint from B, no edges between the elements of E' have been deleted and so E' is a clique of ρ_B as well. Moreover $B \times E' \subseteq \rho_B$. Now by Lemma 3.2 there is $\varphi \in \mathrm{Hom}(\rho_B, \rho)$ such that $\varphi(B) = F$. Clearly φ carries the clique E' onto the clique $C := \varphi(E')$ of the same size and $F \times C = \varphi(B) \times \varphi(E') \subseteq \rho$. ■

3.7

We give a sufficient condition for a resolvent τ of ρ (see 1.4) to be either trivial or satisfy $\tau \approx \rho$ (i.e. $\mathrm{Pol}\,\tau = \mathrm{Pol}\,\rho$). It can be verified that without loss of generality τ has the following form.

Let n be a positive integer, $N := \{1, ..., n\}$ and $G = (V, \sigma)$ a graph such that $V \supseteq N$. We say that

$$\tau := \{(\varphi(1), ..., \varphi(n)) : \varphi \in \mathrm{Hom}(\sigma, \rho)\} \qquad (3.1)$$

belongs to G. We show that for G with a "small" chromatic number either $\tau \approx \rho$ or $\tau = A^n$. As usual, an equivalence relation on V is *chromatic* if all its blocks are independent in G. This approach is motivated by the fact that due to areflexivity for each $f \in \mathrm{Hom}(\sigma, \rho)$ the kernel $\ker f := \{vv' \in V^2 : f(v) = f(v')\}$ is a chromatic equivalence relation on V. We start with the following technical lemma.

3.8 Lemma
Let $G = (V, \sigma)$ be a graph with a chromatic equivalence relation ε with χ blocks such that $N := \{1, ..., n\} \subseteq V$ and $ij \in \varepsilon \Leftrightarrow i = j$ for all $i, j \in N$. If $\chi \in \Gamma_\rho$ then τ defined by (3.1) belongs to the subgraph of G induced by N.

Proof. Put $\sigma' := \sigma \cap N^2$ and

$$\mu := \{(x_1, ..., x_n) \in A^n : x_i x_j \in \rho \text{ for all } ij \in \sigma'\}.$$

Since $\sigma' \subseteq \sigma$, it is easy to see that $\tau \in \mu$. For the converse, let $(x_1, ..., x_n) \in \rho$. Let X denote the set whose elements are $x_1, ..., x_n$. By Corollary 3.6 there exists a clique C of cardinality χ such that $X \times C \subseteq \rho$. Let $\{B_i : i \in I\}$ be the set of the blocks of ε where $N \subseteq I$ and $i \in B_i$ for $i = 1, ..., n$. Put $J := I \setminus N$. Since $|J| = \chi$, there is a bijection $\varphi : J \to C$. Define $\psi : V \to A$ as follows: Let $i \in I$ and $v \in B_i$. For $i \in J$ put $\psi(v) := \varphi(i)$ and put $\psi(v) := x_i$ otherwise. We verify that $\psi \in \mathrm{Hom}(\sigma, \rho)$. Suppose to the contrary that there is $uv \in \sigma$ such that $\psi(u)\psi(v) \notin \rho$. As C is a clique of ρ, the inclusion $X \times C \subseteq \rho$ holds and $u \in B_i$, $v \in B_{i'}$ for distinct $i, i' \in I$, we have $i, i' \in N$. Then $uv \in \sigma'$ and $\psi(u)\psi(v) = x_i x_{i'} \in \rho$ contrary to our assumption. Thus $\psi \in \mathrm{Hom}(\sigma, \rho)$, hence $(x_1, ..., x_n) \in \tau$ and $\mu \subseteq \tau$. Thus $\mu = \tau$ and τ belongs to the subgraph of G induced by N. ■

Now we have:

3.9 Proposition.
Let $G = (V, \sigma)$ be a graph with a chromatic equivalence relation ε with χ blocks where $N := \{1, ..., n\}$ meets n blocks of ε and let τ belong to G (i.e. is defined by (3.1). *If $\chi \in \Gamma_\rho$, then*

(i) $ij \in \sigma \cap N^2 \Rightarrow \mathrm{pr}_{ij}\, \tau = \rho$,

(ii) $ij \notin \sigma \cap N^2,\ i \neq j \Rightarrow \mathrm{pr}_{ij}\, \tau = A^2$,

(iii) *$\tau \approx \rho$, whenever $\sigma \cap N^2 \neq \emptyset$ and $\tau = A^n$ otherwise.*

Proof. By Lemma 3.8 the relation τ belongs to the graph (N, σ') where $\sigma' = \sigma \cap N^2$. To prove (i) let $ij \in \sigma'$ and let $\nu := \mathrm{pr}_{ij}\, \tau$. We have $\nu = \{\varphi(i)\varphi(j) : \varphi \in \mathrm{Hom}(\sigma', \rho)\} \subseteq \rho$ due to $ij \in \sigma'$. For the converse let $c_i c_j \in \rho$. By Lemma 3.1 the edge $c_i c_j$ is contained in a countably infinite clique

C and so we can choose elements $c_\ell \in C$ for $\ell \in \{1, ..., n\} \setminus \{i, j\}$ so that $c_k c_m \in \rho$ for all $1 \leqq k,\ m \leqq n,\ k \neq m$. Clearly $(c_1, ..., c_n) \in \tau$ and so $c_i c_j \in \nu$ proving $\rho \subseteq \nu$. Together $\nu = \rho$ and $\rho = \mathrm{pr}_{ij}\,\tau$.
To prove (ii) let $ij \notin \sigma'$, $i \neq j$ and a, a' be arbitrary elements of A. According to Corollary 3.6 the set $\{a, a'\}$ is included in an n-element clique C of (A, ρ). Define $\varphi : N \to C$ so that $\varphi(i) := a$, $\varphi(j) := a'$ while $\varphi(k) \neq \varphi(\ell)$ for all $1 \leq k < \ell \leq n$, $\{k, \ell\} \neq \{i, j\}$. It is easy to see that $f \in \mathrm{Hom}(\sigma', \rho)$ and so $(a, a') = (\varphi(i), \varphi(j)) \in \mathrm{pr}_{ij}\,\tau$. This proves (ii).
Finally (iii) follows from (i) and (ii). ■

As usual the *chromatic number* χ of a graph G is the least cardinality of the set of blocks of a chromatic equivalence relation of G. We have:

3.10 Lemma
Let an n-ary non-trivial relation τ belong to $G = (V, \sigma)$. Then

(i) *the chromatic number χ of G satisfies $\chi \leqq \min\{|\mathrm{im}\, f| : f \in \mathrm{Hom}(\sigma, \rho)\}$; in particular, $\chi \leqq |A|$,*

(ii) *If $|\mathrm{im}\, f| \in \Gamma_\rho$ for some $f \in \mathrm{Hom}(\sigma, \rho)$, then either $\tau = A^n$ or $\tau \approx \rho$.*

Proof. (i) Let $f \in \mathrm{Hom}(\sigma, \rho)$. Clearly $\ker f$ determines a chromatic decomposition ε of V with $|\mathrm{im}\, f|$ blocks. To prove (ii) let $|\mathrm{im}\, f| \in \Gamma_\rho$ for some $f \in \mathrm{Hom}(\sigma, \rho)$. From (i) clearly follows $\chi \in \Gamma_\rho$.
Fix a chromatic equivalence relation η on V with χ blocks. We can subdivide the blocks of η so that N meets n blocks of the resulting equivalence relation ε. Here ε has at most $\chi + N$ blocks where clearly $\chi + N \in \Gamma_\rho$. Now (ii) follows from Proposition 3.9. ■

Now we can settle the case $\alpha := |A| \in \Gamma_\rho$. We say that ρ is *clique coherent* if to every finite subset F of A there exists a clique C of size $\alpha := |A|$ such that $F \times C \subseteq \rho$.

3.11 Proposition
If ρ is clique coherent, then $\mathrm{Pol}\,\rho$ *is a locally maximal clone.*

Proof. Clearly $\alpha \in \Gamma_\rho$. Let an n-ary nondiagonal relation τ on A be the directed union of a family $\{\tau_\ell : \ell \in L\}$ of resolvents of ρ. Set $M := \{ij : 1 \leq i < j \leq n\}$ and for each $\ell \in L$ set

$$K_\ell := \{ij \in M : \mathrm{pr}_{\,ij}\tau_\ell = \rho\}.$$

Clearly no $\tau_\ell = A^n$ and so by Proposition 3.9 each K_ℓ is nonvoid. If the set $K := \cap_{\ell \in L} K_\ell$ is nonempty then for $ij \in K$ clearly $\mathrm{pr}_{ij}\,\tau = \rho$; hence $\tau \approx \rho$

and we are done. Thus assume that for every $ij \in M$ there exists $\ell_{ij} \in L$ such that $\operatorname{pr}_{ij} \tau_{\ell_{ij}} = A^2$. In the directed family $\{\tau_\ell : \ell \in L\}$ there exists $k \in L$ such that $\tau_k \supseteq \cup_{ij \in M} \tau_{\ell_{ij}}$. Now $\operatorname{pr}_{ij} \tau_k = A^2$ for all $ij \in M$. According to Proposition 3.9 the resolvent τ_k equals A^n; thus $\tau \supseteq \tau_k$ also equals A^n and τ is diagonal, contrary to our assumption. ■

It is easy to construct examples of such graphs.

3.12 Example
Let a clique C of ρ satisfy $|C| = \alpha$ and let $(A \backslash C) \times C \subseteq \rho$ (i.e. ρ is contained in the complement of a graph with α isolated vertices). Then ρ is clique coherent and so $\operatorname{Pol} \rho$ is locally maximal. Clearly there are 2^α such graphs. A particular instance is the complete graph $K_A = (A, \neq)$. The fact that $\operatorname{Pol}(\neq)$ is locally maximal (and a partial description of $\operatorname{Pol}(\neq)$) is in [R-Sz 84].
Notice how much the infinite case differs from the finite one. For A finite it is known (folklore) and [Po 76] that $\operatorname{Pol}(\neq)$ is the essentially unary clone generated by the set of all permutations of A; all the clones containing $\operatorname{Pol}(\neq)$ are known [H-R 94] and $\operatorname{Pol}(\neq)$ is far from being a maximal clone.

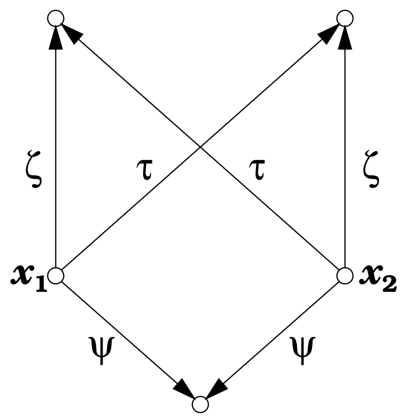
3.13 Remarks

1. In Example 3.12 we had a single clique C of size α. More generally, we may have a family $\{C_\ell : \ell \in L\}$ of cliques of ρ of cardinality α such that
$$A = \cup_{\ell \in L} C_\ell \text{ and } |\cap_{f \in F} C_f| = \alpha$$
for all finite $F \subseteq A$.

2. Let $|A| = \aleph_0$ and (A, ρ) a graph. Then the clone $\operatorname{Pol} \rho$ is locally maximal if and only if ρ is clique coherent.

For the graphs that are not clique coherent we look at some n-ary relations belonging to $G = (A, \rho)$ and derive some symmetries of ρ.

3.14
We say that a finite subset $X = \{x_1, ..., x_n\}$ of A is a *transitivity base* of ρ if
$$A^n = \{(\varphi(x_1), ..., \varphi(x_n)) : \varphi \in \operatorname{End} \rho\}.$$
(where $\operatorname{End} \rho = \operatorname{Hom}(\rho, \rho)$). We say that ρ is *end-transitive* if $\{a\}$ is a transitivity base of ρ for each $a \in A$. A pair $X := \{x_1, x_2\} \subset A$ is *strong for* ρ if there are $\psi, \zeta, \tau \in \operatorname{End} \rho$ such that
$$\psi(x_1) = \psi(x_2), \zeta(x_1) = \tau(x_2) \neq \tau(x_1) = \zeta(x_2)$$



- Fig. 1 -

Note that for a strong pair we have $(x_1, x_2) \notin \rho$ (as $\psi(x_1) = \psi(x_2)$) and if $\varphi(y_i) = x_i \quad (i = 1, 2)$ for some $\varphi \in \operatorname{End}\rho$, then $\{y_1, y_2\}$ is a strong pair as well.

3.15 Proposition
The graph ρ is end-transitive and each strong pair of ρ is a transitivity base.

Proof. Let $a \in A$ and put $\nu := \{\varphi(a) : \varphi \in \operatorname{End}\rho\}$. Clearly ν is a unary relation from $[\rho]$ and $a \in \tau$ in view of $\mathrm{id}_A \in \operatorname{End}\rho$. Since ρ is repellent, we have the required $\nu = A$.
Let $\{x_1, x_2\}$ be strong. Put $\lambda := \{(\varphi(x_1), \varphi(x_2)) : \varphi \in \operatorname{End}\rho\}$. Clearly $\lambda \in [\rho]$. Put $a := \psi(x_1)$. Since ρ is end-transitive, for every $b \in A$ we have $\varphi(a) = b$ for some $\varphi \in \operatorname{End}\rho$. Thus $(b, b) = (\varphi(\psi(x_1)), \varphi(\psi(x_2))) \in \lambda$ proving that λ is reflexive. Put $c := \zeta(x_1)$ and $d := \zeta(x_2)$. From the definition of λ it follows that $(c, d) \in \lambda$ and also $(d, c) = (\tau(x_1), \tau(x_2)) \in \lambda$. Form $\mu := \lambda \cap \lambda^{-1}$. Clearly $\mu \in [\rho]$ is a reflexive and symmetric relation distinct from $\iota_2 := \{(a, a) : a \in A\}$. Taking into account that ρ is repellent we get $\mu = A^2$ i.e. $\{x_1, x_2\}$ is a transitivity base. ■

The concept of a strong pair may be extended. Inductively we define a finite subset $X = \{x_1, ..., x_n\}$ with $n > 2$ to be *strong* if for all $1 \leqq j \leqq n$ the set $X \setminus \{x_j\}$ is strong and for all $1 \leqq i, j \leqq n$ with $i \neq j$ there is $\psi \in \operatorname{End}\rho$ such that $\psi(x_\ell) = x_\ell$ for all $1 \leqq \ell \leqq n$, $\ell \neq j$ and $\psi(x_j) = x_i$.
We have:

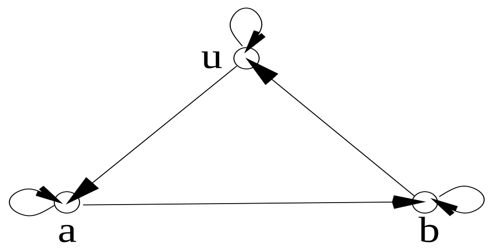
3.16 Proposition
Every strong set X is a transitivity base.

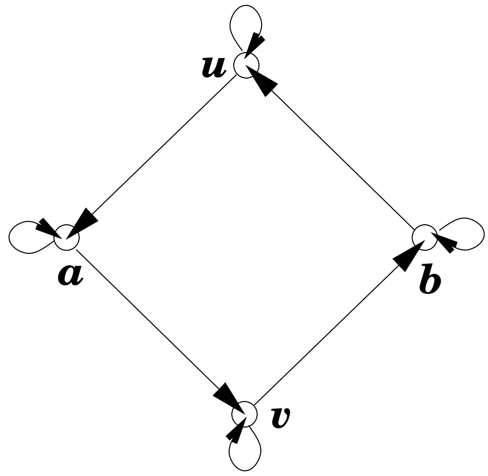
Proof. By induction on $n \geq 2$. By Proposition 3.15 it holds for $n = 2$. Let the statement hold for $n - 1 \geqq 2$ and let $X = \{x_1, ..., x_n\}$ be strong. Put

$\lambda := \{(\varphi(x_1), ..., \varphi(x_n)) : \varphi \in \operatorname{End}\rho\}$. We show that λ is totally reflexive. Let $1 \leqq i < j \leqq n$ and ψ the endomorphism from the definition of strongness. By assumption $X \setminus \{x_j\}$ is strong. In view of the induction hypothesis $X \setminus \{x_j\}$ is a transitivity base; i.e., for arbitrary $y_1, ..., y_{j-1}, y_{j+1}, ..., y_n \in A$ we have $\varphi(x_\ell) = y_\ell$ ($\ell = 1, ..., n,\ \ell \neq j$) for some $\varphi \in \operatorname{End}\rho$. It is immediate from $\varphi \circ \psi \in \operatorname{End}\rho$, that $y_1...y_{j-1}y_iy_{j+1}, ..., y_n \in \lambda$. This proves that λ is totally reflexive. As $\lambda \in [\rho]$ and ρ is repellent we get the required $\lambda = A^n$. ■

4 Reflexive locally bounded diagraphs of diameter 2

In this section ρ always means a binary, reflexive, antisymmetric and locally bounded relation such that $\rho^2 = \rho \circ \rho = A^2$ and ρ is repellent. For simplicity we use both $a \to b$ and $b \leftarrow a$ to denote $(a, b) \in \rho$, similarly both $a \nrightarrow b$ and $b \nleftarrow a$ for $(a, b) \notin \rho$ and finally $a \parallel b$ for $a \nrightarrow b \nrightarrow a$. The condition $\rho \circ \rho = A^2$ means that each arc $a \to b$ is on an oriented 3-cycle (Fig. 2a) while two vertices a, b with $a \parallel b$ are the opposite vertices of an oriented 4-cycle (Fig. 2b).





- Fig 2a,b -

The fact that $a \to b$ means $b \to u \to a$ for some $u \notin \{a, b\}$ implies that ρ is the union of oriented 3-cycles. Let ¶ denote the set of positive integers. We need the following technical fact:

4.1 Fact

Let λ be a reflexive and antisymmetric relation on a set I with $I \cap ¶ = \{1, 2\}$ and let

$$\sigma_2 := \{f(1) f(2) : f \in \mathrm{Hom}(\lambda, \rho)\}. \tag{4.1}$$

Then

(i) *If σ_2 is symmetric and $\sigma_2 \supseteq \rho$, then $\sigma_2 = A^2$.*

(ii) *Suppose that λ also satisfies*

$$1x \in \lambda \Leftrightarrow 2x \in \lambda \qquad x1 \in \lambda \Leftrightarrow x2 \in \lambda$$

for all $x \in I$. For $n > 2$ put $I_n := I \cup \{3, ..., n\}$ and extend λ to a binary relation λ_n on I_n by setting for all $x \in I$ and $j = 3, ..., n$

$$xj \in \lambda_n \text{ if } x1 \in \lambda, \qquad jx \in \lambda_n \text{ if } 1x \in \lambda.$$

If $\sigma_2 = A^2$ then

$$\sigma_n := \{f(1) ... f(n) : f \in \mathrm{Hom}(\lambda_n, \rho)\} = A^n$$

for all $n = 3, 4, ...$

Proof.

(i) Since $\sigma_2 \in [\rho]$, $\sigma_2 \supseteq \rho \supset \iota_2$ and ρ is repellent, we have $\sigma_2 = A^2$.

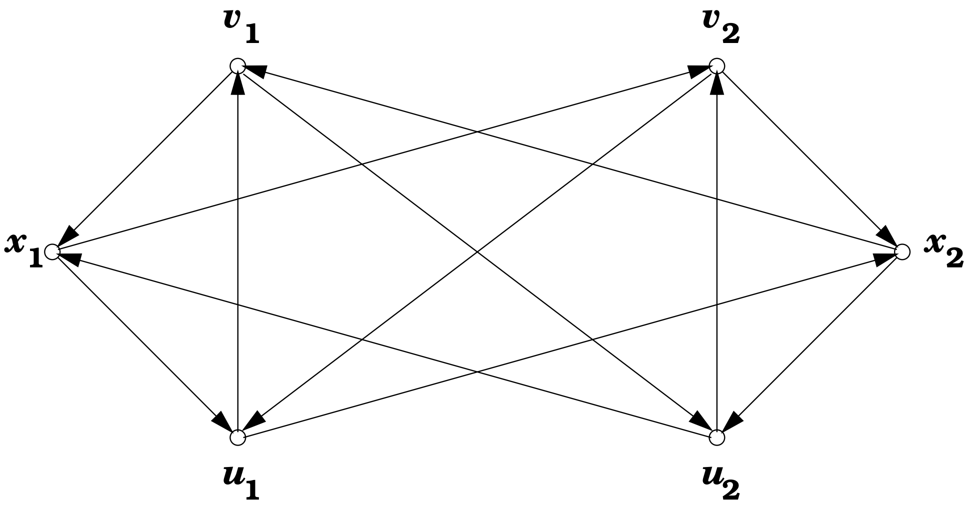
(ii) By induction on $n > 1$. By assumption $\sigma_2 = A^2$. Suppose $n > 2$ and $\sigma_{n-1} = A^{n-1}$. Let $a_1, ..., a_{n-1} \in A$. We show that $a_1 ... a_{n-1} a_{n-1} \in \sigma_n$. Since $\sigma_{n-1} = A^{n-1}$, there is $f \in \mathrm{Hom}(\lambda_{n-1}, \rho)$ such that $f(i) = a_i$ for $i = 1, ..., n-1$. Extend f to $g : I_n \to A$ by setting $g(n) := a_{n-1}$. It suffices to show that $g \in \mathrm{Hom}(\lambda_n, \rho)$. Indeed, let $nj \in \lambda_n$ for some $j \in I_n$. There is nothing to prove if $j = n$. Thus let $j \in I$. Then $(n-1) j \in \lambda_{n-1}$ and so $g(n) g(j) = a_{n-1} f(j) = f(n-1) f(j) \in \lambda_n$. The same argument applies to j, n and so $g \in \mathrm{Hom}(\lambda_n, \rho)$. From the construction of λ_n it follows that σ_n is totally symmetric and so σ_n is totally reflexive. As $\sigma_n \in [\rho]$ and ρ is repellent, we have $\sigma_n = A^n$. ■

4.2 Lemma

For all $x_1, x_2 \in A$ there exist $u_i \to v_i$ such that for $i = 1, 2,$

$$x_i \to u_i \to x_{3-i} \to v_i \to x_i, \ v_i \to u_{3-i} \tag{4.2}$$

(see Fig. 3)



- Fig. 3 -

Proof. Denote by σ the set of all $x_1x_2 \in A^2$ satisfying (4.2) for some $u_1, u_2, v_1, v_2 \in A$. Clearly σ_2 is reflexive and symmetric and so by Fact 4.1 (i) it suffices to show $\rho \subseteq \sigma_2$. Let $x_1, x_2 \in A$, $x_1 \neq x_2$ and $x_1 \to x_2$. Then $x_2 \to w \to x_1$ for some $w \in A$. Setting $v_2 := x_1,\ u_1 := x_2$ and $v_1 = u_2 := w$ we have

$$\begin{aligned} x_1 \to x_2 &= u_1 = x_2 \to w = v_1 \to x_1,\ x_2 \to w = u_2 \to x_1 \to x_1 = v_2 \to x_2, \\ v_1 &= w \to w = u_2,\ v_2 = x_1 \to x_2 = u_1,\ u_1 = x_2 \to w = v_1, \\ & u_2 = w \to x_1 = v_2 \end{aligned}$$

and so $\rho \subseteq \sigma_2$. By Fact 4.1 (i) we have $\sigma_2 = A^2$. ■

4.3 Proposition
Every finite subset B of A has a lower bound ℓ and an upper bound u such that $\ell \to u$.

Proof. Put

$$\sigma_2 := \{x_1x_2 : \ell \to x_i \to u\ (i = 1, 2) \text{ for some } \ell \to u\}$$

Clearly σ_2 is symmetric. To show $\rho \subseteq \sigma_2$ consider $x_1 \to x_2$. Setting $l = x_1,\ u = x_2$ we get $x_1x_2 \in \sigma$. Now we apply Fact 4.2. ■

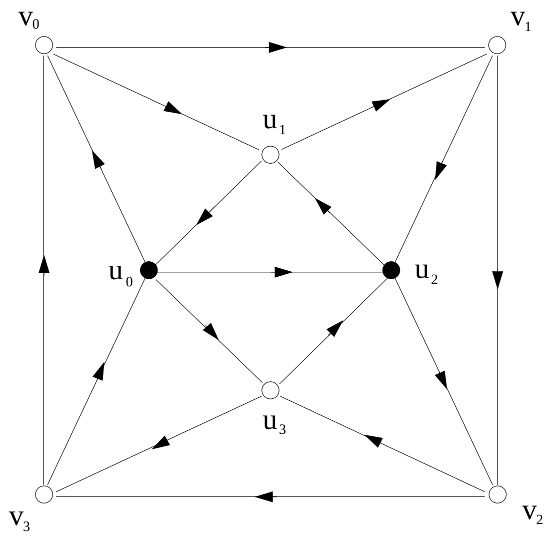
For $x, y \in \mathbf{4} = \{0, ..., 3\}$ denote by $x \dot{+} y$ the element of $\mathbf{4}$ congruent $\mod 4$ to $x + y$.

4.4 Lemma
The relation

$$\xi := \{u_0u_2 : u_i \to v_i \to u_{i\dot{+}1} \to u_i,\ v_i \to v_{i\dot{+}1}\ (i = 0, ..., 3) \quad \textit{for some } u_0, ..., u_3, v_0, ..., v_3 \in A\} \tag{4.3}$$

equals either ι_2 or A^2.

(see Fig. 4)



- Fig 4 -

Proof. The relation ξ is reflexive because ρ is. Assume that $\xi \supset \iota_2$. Since $\xi \in [\rho]$ and ρ is repellent, to show $\xi = A^2$ it suffices to prove the symmetry of ξ. Let $u_0u_2 \in \xi$ and let $u_1, u_3, v_0, ..., v_3$ be the corresponding elements from (4.3). Due to the cyclic (mod 4) nature of the conditions in (4.3), the sequence $u_2, u_3, u_0, u_1, v_2, v_3, v_0, v_1$ also satisfies the conditions of (4.3) and so $u_2u_0 \in \xi$. ■

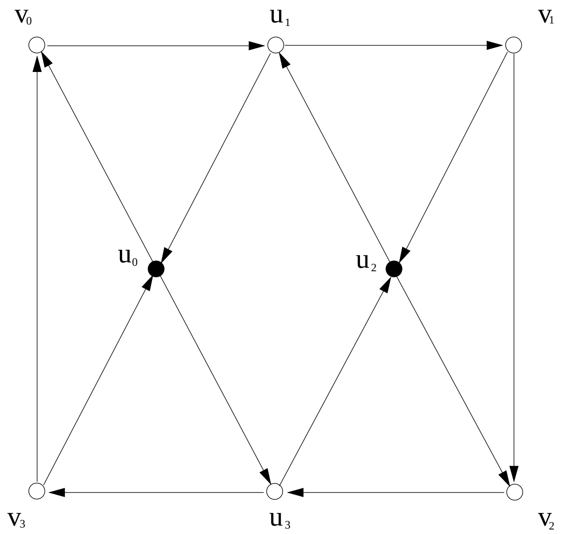
We have a variant of the preceding lemma.

4.5 Lemma

The relation

$$\begin{aligned} \varsigma := \{u_0u_2 : u_i \to v_i \to u_{i+1} \to u_i\ (i = 0, ..., 3),\ v_1 \to v_2, v_3 \to v_0) \\ \textit{for some } u_0, ..., u_3, v_0, ..., v_3\} \end{aligned} \tag{4.4}$$

equals either ι_2 *or* A^2 (The situation is depicted on Fig. 5).



- Fig 5 -

Proof. As in the proof of the preceding lemma, it suffices to show the symmetry of ς. Let $u_0u_2 \in \varsigma$ and let $u_1, u_3, v_0, ..., v_3$ be the corresponding elements from (4.4). Then $u_2, u_3, u_0, u_1, v_2, v_3, v_0, v_1$ also satisfy (4.4) proving the required $u_2u_0 \in \varsigma$. ■

We know that each arc $x_1 \to x_2$ with $x_1 \neq x_2$ is on an oriented 3-cycle $x_1 \to x_2 \to u \to x_1$. We show that either such u is unique or we have the analogue of Proposition 4.4 with $u \to l$.

4.6 Proposition
Either

(i) *each* arc $x \to y$ *with* $x \neq y$ *is on a unique oriented 3-cycle, or*

(ii) *to each finite* $B \subset A$ *there exists* $u \to l$ *so that* $\{l\} \times B \subseteq \rho$ *and* $B \times \{u\} \subseteq \rho$.

Proof. Put

$$\sigma_2 := \{x_1 x_2 : l \to x_i \to u \to l \ (i = 1, 2) \text{ for some } u, l\}.$$

Clearly σ_2 is a reflexive and symmetric binary relation. If $\sigma = \iota_2$ we have the case (i); otherwise $\sigma_2 = A^2$ and we may apply Fact 4.1(ii). ■

In 4.7 - 4.9 we investigate the case (i). Let (i) hold. Then to every $x \to y$ there exists a unique $\varphi_{xy} \in A$ such that $y \to \varphi_{xy} \to x$ (notice that $\varphi_{xx} = x$ on account of the antisymmetry and reflexivity of ρ).

4.7 Lemma
If (i) *holds and* $u_0, ..., u_3 \in A$ *satisfy* $u_0 \neq u_2$ and

$$u_0 \to u_3 \to u_2 \to u_1 \to u_0, \tag{4.5}$$

then for some $0 \leq i < 3$

$$\varphi_{u_i u_{i \dot{-} 1}} \not\to \varphi_{u_{i \dot{+} 1} u_i}.$$

Proof. For $i = 0, ..., 3$ set $v_i := \varphi_{u_i u_{i \dot{-} 1}}$. Suppose to the contrary that $v_0 \to v_1 \to v_2 \to v_3 \to v_0$. Then $u_0 u_2$ belongs to the relation ξ defined in (4.3). Here $u_0 u_2 \notin \iota_2$ and from Lemma 4.5 we obtain $\xi = A^2$. Choose $u_0' \to u_2'$ with $u_0' \neq u_2'$. Since $u_0' u_2' \in A^2 = \xi$, there exist $u_1', u_3', v_0', ..., v_3'$ such that $u_0', ..., u_3', v_0', ..., v_3'$ satisfy the condition of (4.3). In particular, $u_1' = \varphi_{u_0' u_2'}$. It is easy to check that

$$v_0' = \varphi_{u_1' u_0'} = u_2', \qquad v_1' = \varphi_{u_2' u_1'} = u_0'.$$

Now $v_0' \to v_1'$ means $u_2' \to u_0'$. However, $u_0' \to u_2'$ and this contradicts the antisymmetry of ρ. ■

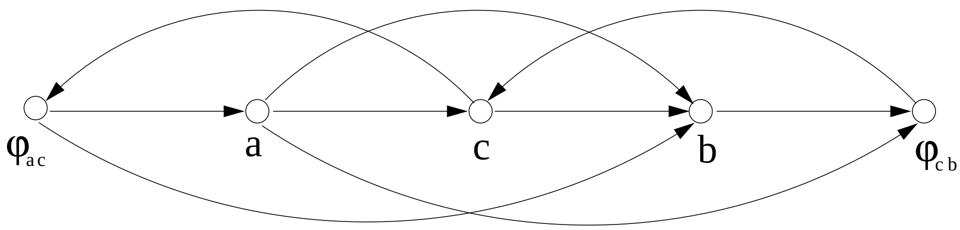
4.8 Lemma
Let (i) *hold and let some* $u_0, ..., u_3 \in A$ satisfy

$$u_0 \neq u_2, \ u_0 \to u_3 \to u_2 \to u_1 \to u_0, \tag{4.6}$$

$$\varphi_{u_2u_1} \to \varphi_{u_3u_2}, \qquad \varphi_{u_0u_3} \to \varphi_{u_1u_0}. \tag{4.7}$$

Then to every $a \to b,\ a \neq b$ *there exists* $c \in A,\ a \neq c \neq b$ such that

$$a \to c \to b, \qquad \varphi_{ac} \to b, \qquad a \to \varphi_{cb} \tag{4.8}$$



- Fig. 6 -

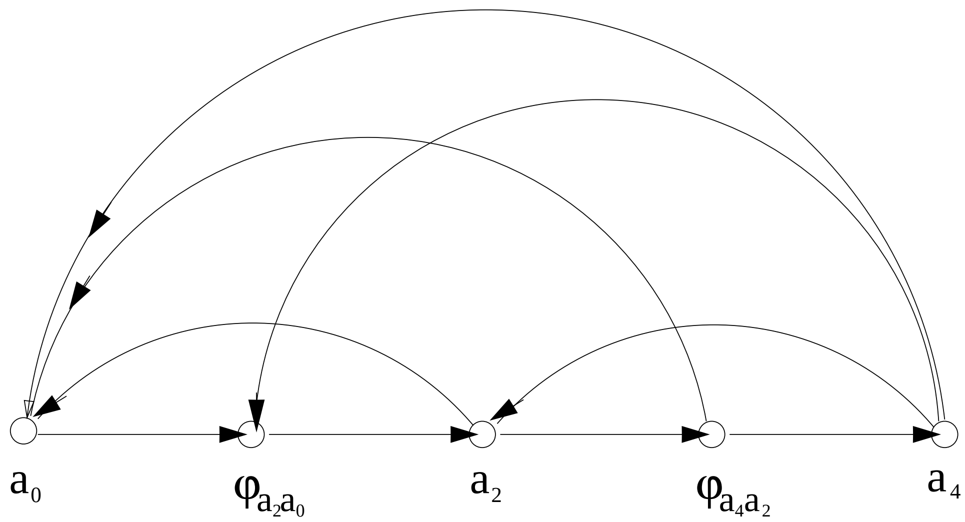
Proof. Clearly u_0u_2 belongs to the relation ς defined by (4.4). By Lemma 4.6 then $\varsigma = A^2$. Let $a \to b,\ a \neq b$ be arbitrary. Clearly $ab \in A^2 = \varsigma$ and so there exist $u_0' := a,\ u_1', u_2' := b,\ u_3',\ v_0', ..., v_3' \in A$ satisfying the conditions from (4.4). Set $d := \varphi_{ab}$. Then $u_1' = d$ and $v_0' = \varphi_{u_1'u_0'} = \varphi_{da} = b$. By the same token $v_1' = \varphi_{u_2'u_1'} = \varphi_{bd} = a$. Set $c := u_3'$. Clearly $a \to c \to b$, $\varphi_{ac} = v_3' \to v_0' = b$ and $a = v_1' \to v_2 = \varphi_{cb}$. Thus (4.8) holds. It remains to show $a \neq c \neq b$. First suppose to the contrary that $a = c$. Then $v_2' = \varphi_{u_3'u_2'} = \varphi_{ab} = d$. Now $u_1' \to v_1'$ means $d \to a$ while $v_1' \to v_2'$ means $a \to d$. By antisymmetry $a = d$ and hence $a = b$. This contradiction shows $a \neq c$. Next suppose to the contrary that $c = b$. Then $v_3' = \varphi_{u_0'u_3'} = \varphi_{ab} = d$ and $v_3' \to v_0'$ means $d \to b$. From $d = \varphi_{ab}$ also $b \to d$; hence again by antisymmetry $d = b$ and $a = b$. This contradiction shows $c \neq b$. ■

4.9 Corollary

The conclusion of Lemma 4.8 *holds provided (i) holds and there exist* $a_0, a_2, a_4 \in A$ *such that*

$$\begin{aligned} a_4 &\to a_2 \to a_0,\ a_4 \to a_0,\ a_0 \neq a_4, \\ a_4 &\to \varphi_{a_2a_0},\quad \varphi_{a_4a_2} \to a_0 \end{aligned}$$

(see Fig. 7)



- Fig. 7 -

Proof. Set

$$u_0 := a_4,\ u_1 := \varphi_{a_4a_0},\ u_2 := a_0,\ u_3 := a_2.$$

Then (4.6) hold. For (4.7) it suffices to notice that

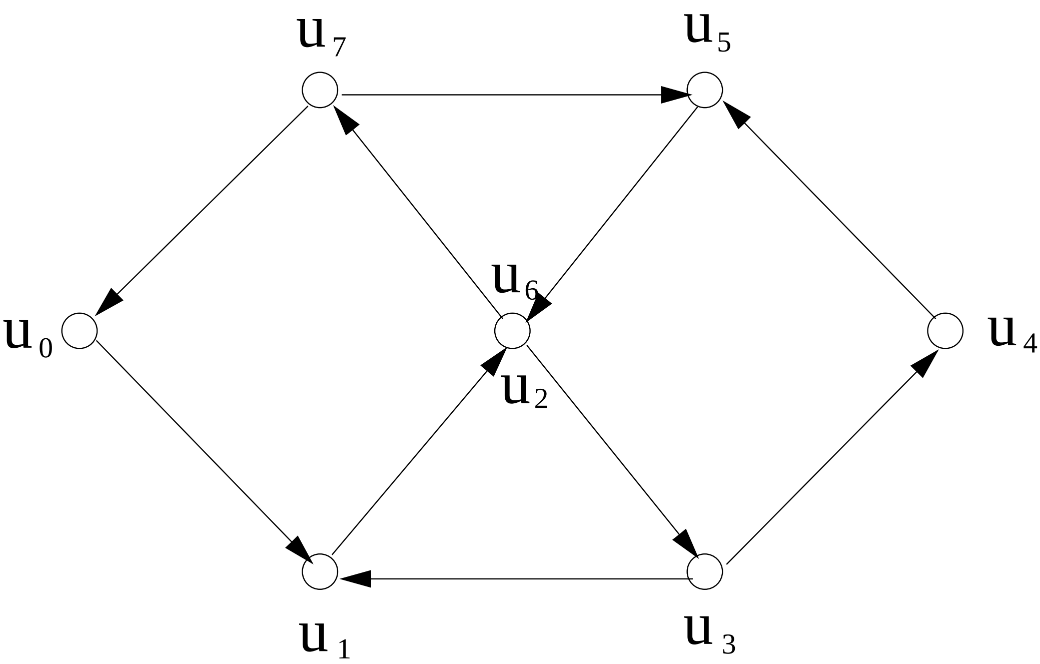
$$\varphi_{u_2u_1} = \varphi_{a_0u_1} = a_4,\ \varphi_{u_3u_2} = \varphi_{a_2a_0},\ \varphi_{u_0u_3} = \varphi_{a_4a_2},\ \varphi_{u_1u_0} = \varphi_{u_1a_4} = a_{0.}\ \blacksquare$$

4.10 Lemma

Let the binary relation μ on A consist of all u_0u_4 for *which there exist $u_1,...,u_3,u_5,...,u_7$ such that*

$$u_0 \to u_1 \to ... \to u_7 \to u_0,\ u_2 = u_6,\ u_3 \to u_1,\ u_7 \to u_5 \qquad (4.9)$$

Then either $\mu = \iota_2$ or $\mu = A^2$(see Fig. 8).



- Fig. 8 -

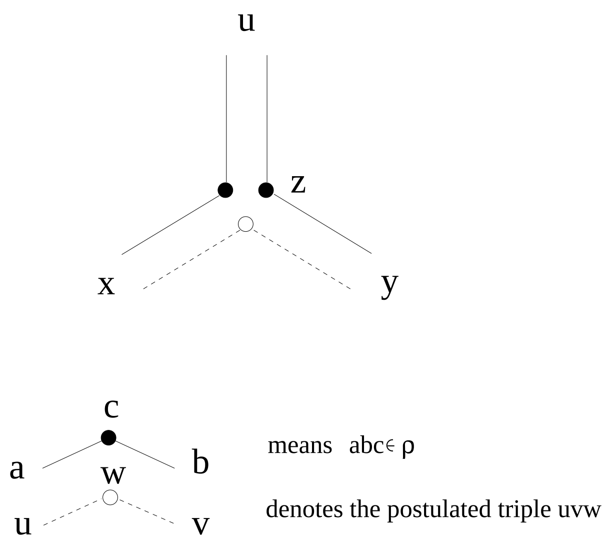
Proof. It suffices to show the symmetry of μ. Let $u_0, ..., u_7$ satisfy (4.9). A direct check shows that $u_4, ..., u_7,\ u_0, ..., u_3$ satisfies (4.9) as well. ■

4.11 Lemma

Suppose ρ satisfies (ii) in Proposition 4.6. Then to each finite $B \subset A$ there exist $l_2, l_1, u_1, u_2 \in A$ such that

$$l_2 \to l_1 \to u_1 \to u_2,\ l_1 \to u_2 \to l_2 \to u_1$$

and $\{l_i\} \times B \subseteq \rho$, $B \times \{u_i\} \subseteq \rho$ $(i = 1.2)$ (see Fig. 9).



- Fig. 9 -

Proof. Consider

$$\sigma_2 := \{x_1x_2 : l_i \to x_j \to u_k \ (i,j,k \in \{1,2\})\,,\ l_2 \to l_1 \to u_1 \to u_2, \\ l_1 \to u_2 \to l_2 \to u_1 \text{ for some } l_1, l_2, u_1, u_2\}.$$

Clearly σ_2 is symmetric and so we only need to prove $\rho \subseteq \sigma_2$. Let $x_1 \to x_2$. By Proposition 4.6(ii) there are $u_2 \to l_2$ so that $l_2 \to x_i \to u_2$ $(i = 1, 2)$. Now it suffices to set $l_1 := x_1$ and $u_1 := x_2$ to obtain $x_1x_2 \in \sigma_2$. Applying Fact 4 we get the required result. ■

5 Locally bounded digraphs

In this section we study locally bounded digraphs ρ. Here ρ is a *digraph* if it is areflexive and asymmetric; i.e., $\rho \cap \rho^{-1} = \emptyset = \rho \cap \iota_2$. As usual, we assume that

(i) ρ is repellent.
It will be convenient to assume that we cannot go over to one of the cases studied in §3-4. More precisely, we assume that $[\rho]$ contains

(ii) no nonempty graph (an areflexive and symmetric relation) and

(iii) no nontrivial reflexive and antisymmetric binary relation.

Nonempty binary relations satisfying the above conditions (i)-(iii) will be termed *paraorders* and we reserve the symbol $\ll$ for paraorders (writing both $x \ll y$ or $y \gg x$ instead of $xy \in \ll$).
We start with the following:

5.1 Fact
If $\ll$ is a paraorder, then each nontrivial binary $\sigma \in [\ll]$ is also a paraorder.

Proof. Suppose $\tau := \{x : xx \in \sigma\} \neq \emptyset$. Since $\tau \in [\ll]$ and $\ll$ is repellent, we have $\tau = A$, i.e. σ is reflexive. Put $\lambda := \sigma \cap \sigma^{-1}$. Then $\lambda \in [\sigma]$ is reflexive, symmetric and $\lambda \subseteq \sigma \subset A^2$ and taking into account that $\ll$ is repellent we get $\lambda = \iota_2$ i.e. σ is antisymmetric. Now σ being nontrivial we have a contradiction to (iii) and so $\tau = \emptyset$. If $\lambda := \sigma \cap \sigma^{-1} \neq \emptyset$ we get a graph in $[\ll]$. Thus by (iii) we get $\sigma \cap \sigma^{-1} = \emptyset$. Clearly $\sigma \in [\ll]$ inherits the other paraorder properties from $\ll$. ■

We recall a fact from [R-Sz 84] Lemma 3.5.

5.2 Fact
A nontrivial binary relation ρ satisfying (i)-(iii) from 5.1 is locally bounded.

We can improve the local bounds of a paraorder.

5.3 Lemma
In a paraorder $\ll$ every finite subset B of A has a lower bound l and an upper bound u such that $l \ll u$.

Proof. Put

$$\sigma_n := \{x_1 ... x_n : l \ll x_1, ..., x_n \ll u \text{ for some } l \ll u\} .$$

Let $x_1, x_2 \in A$. By Fact 5.2 the set $\{x_1, x_2\}$ has an upper bound u, i.e. $x_1 \ll u \gg x_2$. Similarly the set $\{x_1, x_2, u\}$ has a lower bound l, i.e. $x_1 x_2 \in \sigma_2$ proving $\sigma_2 = A^2$. Using the fact that $\ll$ is repellent, it is easy to prove by induction that $\sigma_n = A^n$. ■

We use Lemma 5.3 to prove the existence of infinite chains. A *chain* C is a subset of A such that $\ll \cap C^2$ is a total (or linear) strict order (i.e. $\ll \cap C^2$ is transitive and $c_1 \ll c_2$ or $c_2 \ll c_1$ for all $c_1, c_2 \in C$, $c_1 \neq c_2$). A chain $\{c_i \mid i \in \mathbb{Z}\}$ with $c_i \ll c_j$ iff $i < j$ is of *type* $\mathbb{Z}$ (or $\omega^* + \omega$).

5.4 Lemma
Every $a \in A$ is on a chain of type $\mathbb{Z}$.

Proof. By induction on $n \geqq 0$ we construct a chain $\{a_{-n}, ..., a_n\}$. Put $a_0 := a$. Suppose $n > 0$ and we have constructed a chain $C = \{a_{1-n}, ..., a_{n-1}\}$ with $a_{1-n} \ll ... \ll a_{n-1}$. By Lemma 5.3 the set C has a lower bound a_{-n} and an upper bound a_n such that $a_{-n} \ll a_n$. Clearly $\{a_{-n}, ..., a_n\}$ is the required chain. ■

Recall that a $\ll$-*endomorphism* is a selfmap f of A such that $f(a) \ll f(b)$ whenever $a \ll b$ (equivalently, $f \in \mathrm{Pol} \ll$). We have:

5.5 Lemma
The set End $\ll$ *is transitive (in the monoid* $(\mathcal{O}^{(1)}; \circ, \mathrm{id}_A)$*).*

Proof. Fix $a \in A$ and put $\tau := \{f(a) : f \in \mathrm{End} \ll\}$. In view of $\mathrm{id}_A \in \mathrm{End} \ll$, we have $a \in \tau$ and so $\tau = A$; i.e., for each $b \in A$ we have $f(a) = b$ for some $f \in \mathrm{End} \ll$. ■

We say that $\ll$ is *rigid at* $a \in A$ if for all $f, g \in \mathrm{End} \ll$ we have $f = g$ whenever $f(x) = g(x)$ for all $x \in A \setminus \{a\}$. We show that via $\ll$-endomorphisms each vertex may be replaced by any finite set. (This may be considered as vertex splitting.) For $a, b \in A$ and $\varphi : A \setminus \{a\} \to A$ define $\varphi_{ab} : A \to A$ by $\varphi_{ab}(a) := b$ and $\varphi_{ab}(x) := \varphi(x)$ otherwise. We have:

5.6 Lemma
Let $\ll$ *be not rigid at* $a \in A$. *Then to each finite subset* B *of* A *there exists a map* $\varphi : A \setminus \{a\} \to A$ *such that for each* $b \in B$ *the map* φ_{ab} *is a* $\ll$*-endomorphism.*

Proof. Suppose $I\!P \cap A = \emptyset$. For $n > 0$ put $A_n := (A \setminus \{a\}) \cup N$ where $N := \{1, ..., n\}$. Define a binary relation $\prec$ on A_n as follows. For $x, y \in A \setminus \{a\}$ and $i \in N$ put $x \prec y$ if $x \ll y$, $x \prec i$ if $x \ll a$ and $i \prec x$ if $a \ll x$ (i.e. a is blown up to N). Put

$$\sigma_n := \{f(1) ... f(n) : f \in \mathrm{Hom}(\prec, \ll)\}.$$

Clearly $\sigma_n \in [\ll]$ is totally symmetric. Given the structure of $\prec$ it is enough to show that $\sigma_n = A^n$ for all $n > 0$. For $n = 1$ we have $\sigma_1 = \{f(a) : f \in \mathrm{End} \ll\}$ (as 1 just replaces a) and from Lemma 5.5 the required $\sigma_1 = A$. Consider $n = 2$. The assumption that $\ll$ is not rigid at a guarantees that $\sigma_2 \supseteq \iota_2$. Clearly $\sigma_2 \supset \iota_2$ due to $\sigma_1 = A$. As $\ll$ is repellent, we have $\sigma_2 = A^2$. Suppose $n > 2$ and $\sigma_{n-1} = A^{n-1}$. Then σ_n is totally reflexive. For $n > 2$ the total reflexivity of σ_n implies $\sigma_n = A^n$. ■

5.7 Lemma
Let C *be a chain in* $\ll$ *and* a *a lower set (upper) bound of* C *such that* $\ll$ *is not rigid at* a. *Then for each finite subset* B *of* A *the set of lower (upper) bounds of* B *contains a copy of* C.

Proof. According to Lemma 5.6 for each $b \in B$ the $\ll$-endomorphism φ_{ab} carries C onto a $\ll$-chain $\varphi(C)$ which is a copy of C. Suppose a is an upper bound for C. Then $a \notin C$ and $a \gg c$ for all $c \in C$ shows $b = \varphi_{ab}(a) \gg \varphi_{ab}(c) = \varphi(c)$ for all $c \in C$. This is true for all $b \in B$ and so $\varphi(C)$ is in the set of lower bounds of B. The proof for a a lower bound of B is quite similar. ■

5.8 Lemma
Let $(V,<)$ be a strict (i.e. areflexive) nonempty order, let $v, v' \in V$, $v \neq v'$, and let $g, g' \in H := \mathrm{Hom}(<, \ll)$

(i) *if $vv' \in \ker g$ then for each $x \in A$ we have that $h(v) = h(v') = x$ for some $h \in H$,*

(ii) if *$vv' \in \ker g \backslash \ker g'$ then for all $x, x' \in A$ we have that $h(v) = x$, $h(v') = x'$ for some $h \in H$.*

Proof. (i) Let $vv' \in \ker g$. Set

$$\alpha := \{h(v) : h(v) = h(v') \text{ for some } h \in H\}.$$

Clearly $g(v) \in \alpha \in [\ll]$, whence $\alpha = A$ and (i) holds.
(ii) Let $vv' \in \ker g \backslash \ker g'$. Form

$$\beta := \{h(v)\, h(v') : h \in H\}.$$

In view of (i) clearly $\beta \supseteq \iota_2$. As $g'(v) \neq g'(v)$, clearly $\beta \supset \iota_2$. From the assumptions (i)-(iii) from 5.1 we obtain $\beta = A^2$ and (ii). ■

The existence of chains of type $\mathbb{Z}$ in $\ll$ (Lemma 5.4) raises the question whether there are cycles in $\ll$. As usual, a *cycle of length l* of $\ll$ is a sequence $a_1 \ll ... \ll a_l \ll a_1$ and $\ll$ is termed *acyclic* if it has no cycles of finite length. We prove that $\ll$ is acyclic. First we show:

5.9 Lemma
A paraorder has no cycle of length 2^k $(k = 2, 3, ...)$.

Proof. Suppose $\ll$ has a cycle of length 2^k and let k be the least integer for which this holds. Let $a_1 \ll ... \ll a_{2^k} \ll a_1$. In view that $\ll$ is asymmetric, we have $k > 1$. Put

$$\sigma := \{x_1 x_2 : x_1 \ll u_2 \ll ... \ll u_{2^{k-1}} \ll x_2 \ll u_{2^{k-1}+2} \ll ... \ll u_{2^k} \ll x_1 \\ \text{for some } u_1, ..., u_{2^{k-1}}, u_{2^{k-1}+2}, ... u_{2^k} \in A\}.$$

It is almost immediate that σ is symmetric and that $a_1 a_{2^{k-1}} \in \sigma$. The relation σ is also areflexive because $x_1 \ll u_2 \ll ... \ll u_{2^{k-1}} \ll x_1$ would yield a cycle of length 2^{k-1} in contradiction to the minimality of k. Thus $\sigma \in [\ll]$ is a nonempty graph in contradiction to the definition of a paraorder. ■

Now we can prove:

5.10 Theorem
Every paraorder is acyclic.

Proof. Assume that $\ll$ has a cycle $a_1 \ll ... \ll a_l \ll a_1$. Set $x_1 \prec x_2$ if there are $u_3, ..., u_l \in A$ such that

$$x_1 \ll x_2 \ll u_3 \ll ... \ll u_l \ll x_1. \tag{5.1}$$

In view of $a_1 \prec a_2$ the relation $\prec$ is a nonempty subrelation of $\ll$. Now $\prec \in [\ll]$ and Fact 5.1 show that $\prec$ is a paraorder. We show that each pair $x_1 \prec x_2$ is on a cycle of length l of $\prec$. Indeed, if $x_1 \prec x_2$ then (5.1) holds for some $u_1, ..., u_l \in A$. It follows that $x_1 \prec x_2 \prec u_3 \prec ... \prec u_l \prec x_1$; i.e., $x_1, x_2, u_3, ..., u_l$ is a cycle of length l of $\prec$. Now let $2^k \geqq l$ and put $m := 2^k - l$. By Lemma 5.4 the paraorder $\prec$ contains a chain of type $\mathbb{Z}$ and therefore there are $b_1 \prec ... \prec b_{m+2}$ such that $b_1 \prec b_{m+2}$. We know that $b_1 \prec b_{m+2}$ is on a cycle $b_1 \prec b_{m+2} \prec v_3 \prec ... \prec v_l \prec b_1$ of length l of $\prec$. Together $b_1 \prec ... \prec b_{m+2} \prec v_3 \prec ... \prec v_l$ form a cycle of $\prec$ of length $m + 1 + l - 1 = 2^k$ in contradiction to Lemma 5.9. ■

Recall that the product of binary relations ρ and σ on A is

$$\rho \circ \sigma := \{x_1x_2 : x_1u \in \rho,\ ux_2 \in \sigma \text{ for some } u \in A\}.$$

Next ρ^i is defined inductively by $\rho^1 := \rho$, $\rho^{i+1} := \rho^i \circ \rho \quad (i = 1, 2, ...)$ and the *transitive hull* $\operatorname{tr}\rho$ of ρ is the relation $\bigcup\limits_{i=1}^{\infty} \rho^i$. In view of Theorem 5.10 the relation $\operatorname{tr} \ll$ is always a strict order (i.e. areflexive and transitive). The problem is that $\operatorname{tr} \ll$ need not belong to $[\ll]$. (If the system $\{\ll^i : i = 1, 2, ...\}$ is directed, then $\operatorname{tr} \ll$ belongs to $[\ll]$.) We have:

5.11 Definition
For a fixed $a \ll b$ define a binary relation $\ll_{ab}$ on A by setting $x \ll_{ab} y$ if $x = f(a)$ and $y = f(b)$ for some endomorphism f of $\ll$.

The proof of the following lemma is routine.

5.12 Lemma
If $a \ll b$ then $a \ll_{ab} b$ and $\ll_{ab}$ is a paraorder from $[\ll]$. ■

Denote by $\mathcal{S}$ the set of all sequences $\langle \sqsubset_\xi : \xi < \lambda \rangle$, where λ is a nonzero ordinal such that (i) $\sqsubset_0 := \ll$, (ii) if $\xi + 1 < \lambda$ and $\sqsubset_\xi$ is nonvoid then

$$\sqsubset_{\xi+1} = (\sqsubset_\xi)_{ab} \tag{5.2}$$

for some $a \sqsubset_\xi b$ provided $\sqsubset_\xi \supset \sqsubset_{\xi+1}$ and (iii) $\sqsubset_\xi = \bigcap_{\varsigma<\xi} \sqsubset_\varsigma$ for every limit ordinal $\xi < \lambda$. Let $\langle \sqsubset_\xi : \xi < \lambda \rangle \in \mathcal{S}$. An easy transfinite induction (based on Lemma 5.12 and the fact that $[\ll]$ is closed under arbitrary intersections) shows that a) all nonvoid $\sqsubset_\xi$ are paraorders and b) $\sqsubset_\varsigma \supset \sqsubset_\xi$ whenever $\varsigma < \xi < \lambda$. Call a paraorder $\sqsubset$ *uniform* if $\sqsubset_{ab} = \sqsubset$ for all $a \sqsubset b$. A sequence $\langle \sqsubset_\xi : \xi < \lambda \rangle \in \mathcal{S}$ is called $\ll$-*admissible* if λ is an isolated ordinal and $\sqsubset_{\lambda-1}$ is either uniform or void. Call $\ll$ *weakly vanishing* if every $\ll$-admissible $\langle \sqsubset_\xi : \xi < \lambda \rangle$ satisfies $\sqsubset_{\lambda-1} = \emptyset$.

5.13 Lemma
Without loss of generality we may assume that $\ll$ is either uniform or weakly vanishing.

Proof. From b) above it follows that the lengths λ of sequences from $\mathcal{S}$ are bounded from above and hence each sequence from $\mathcal{S}$ can be prolonged to an $\ll$-admissible one. First suppose that there exists an $\ll$-admissible sequence $\langle \sqsubset_\xi : \xi < \lambda \rangle$ with uniform $\sqsubset_{\lambda-1}$. As $\sqsubset_{\lambda-1}$ is a paraorder from $[\ll]$, we can replace $\ll$ by $\sqsubset_{\lambda-1}$. If no such sequence exists then clearly $\ll$ is vanishing. ■

We introduce a variant of the construction from Definition 5.11.

5.14 Definition
For all $a \ll b$ set

$$\uparrow a := \{x \in A : x \gg a\}\,,\ \downarrow a := \{x \in A : x \ll a\}\,, \tag{5.3}$$

$$A_{ab} := (\uparrow a) \cup (\downarrow b) \tag{5.4}$$

and denote by $\ll'$ the restriction of $\ll$ to A_{ab}. For $x, y \in A$ set $x \ll_a^b y$ if $x = h(a)$ and $y = h(b)$ for some $h \in H_{ab} := \mathrm{Hom}(\ll', \ll)$.

5.15 Lemma
If $a \ll b$ then $\ll_a^b$ is a subrelation of $\ll$ such that $a \ll_a^b b$. If, moreover, $\ll$ is transitive then $\ll_a^b$ is transitive.

Proof. From (5.3) we see that $ab \in A_{ab}^2$. Now from (5.4) we obtain that $\ll^* := \ll_a^b$ is a subrelation of $\ll^*$. Next $a^* \ll b$ follows from $\mathrm{id}_A \in H$. Now assume that $\ll$ is transitive. To show that $\ll'$ is also transitive let $c \ll^* d \ll^* e$. Then for some $h, g \in H$

$$c = h(a)\,,\ d = h(b) = g(a)\,,\ e = g(b)\,.$$

Define $f : A_{ab} \to A$ by setting $f(x) := g(x)$ for all $x \gg a$ and $f(x) := h(x)$ otherwise. To prove that $f \in H$ let $x, y \in A_{ab}$, $x \ll y$. There are three cases. 1) Let $x \gg a$. Then $y \gg a$ and $f(x) = g(x) \ll g(y) = f(y)$. 2) Let

$x, y \notin \uparrow a$. Then $f(x) = h(x) \ll h(y) = f(y)$. 3) Thus let $x \notin \uparrow a$ and $a \ll y$. Now $x \ll b$ by (5.3) and so

$$f(x) = h(x) \ll h(b) = g(a) \ll g(y) = f(y.)$$

Thus $f \in H$ and so $c \ll^* e$. ■

The following definitions parallel those of Lemma 5.12; the difference being that they are based on $\ll_a^b$ rather than on $\ll_{ab}$.

5.16 Definition
Call a paraorder $\ll$ *weakly uniform* if $\ll_a^b$ equals $\ll$ for all $a \ll b$. The set $\mathcal{S}'$ is defined in the same way as $\mathcal{S}$ except that (5.2) is replaced by $\sqsubset_{\xi+1} = (\sqsubset_\xi)_a^b$. Call $\langle \sqsubset_\xi : \xi < \lambda \rangle \in \mathcal{S}'$ *weakly $\ll$-admissible* if λ is an isolated ordinal and $\sqsubset_{\lambda-1}$ is either weakly uniform or empty. Call $\ll$ *vanishing* if every weakly $\ll$-admissible $\langle \sqsubset_\xi : \xi < \lambda \rangle$ satisfies $\sqsubset_{\lambda-1} = \emptyset$.

The proof of the next lemma is a straight-forward adaption of the proof of Lemma 5.15.

5.17 Lemma
Without loss of generality we may assume that $\ll$ is either weakly uniform or vanishing. ■

Remark. Let $a \ll b$. From the definitions it follows easily that each $\ll$-endomorphism belongs to the set H_{ab} and so $\ll_a^b \supseteq \ll_{ab}$. From this we deduce

$$\begin{aligned} \text{uniform} &\Rightarrow \text{weakly uniform}, \\ \text{vanishing} &\Rightarrow \text{weakly vanishing}. \end{aligned}$$

From Lemmas 5.13 and 5.17 and the remark we obtain:

5.18 Lemma
Without loss of generality we may assume that $\ll$ is (i) *uniform* (ii) *vanishing or* (iii) *both weakly uniform and weakly vanishing.*

As usual, $\ll$ is *dense* if $\ll$ is a subrelation of $\ll^2$; i.e., if to each pair $x_1 \ll x_2$ there exists u so that $x_1 \ll u \ll x_2$. We have:

5.19 Proposition
Let $\ll$ be a paraorder. Then either $\ll$ is not dense or the transitive hull of $\ll$ is from $[\ll]$.

Proof. Suppose $\ll \subseteq \ll^2$. Then for every $n > 1$

$$\ll^n = \ll^{n-1} \circ \ll \subseteq \ll^{n-1} \circ \ll^2 = \ll^{n+1}$$

whence $\ll \subseteq \ll^2 \subseteq \ldots$. It follows that the transitive hull $<$ of $\ll$ belongs to $[\ll]$. We show that $<$ is dense. If $x_1 < x_2$ then $x_1 \ll^n x_2$ for some $n \geqq 1$. If $n = 1$ then for some u_1 we get $x_1 \ll u_1 \ll x_2$ by $\ll \subseteq \ll^2$ and if $n > 1$ then for some $u_1, \ldots, u_{n-1}$ we have that $x_1 \ll u_1 \ll \ldots \ll u_{n-1} \ll x_2$. In both cases we have the required $x_1 < u_1 < x_2$. ■

Observe that the strict dense order $<$ is a digraph from $[\ll]$ and hence also a paraorder.

The following lemma gives a sufficient condition for the existence of an order in $[\ll]$.

5.20 Lemma
Suppose $k_1 < k_2 < \ldots$ is an infinite sequence in $\mathbb{N}$ and φ a selfmap of $\mathbb{N}$ such that for each $i \in \mathbb{N}$ and all $l \geq \varphi(i)$ we have that $k_l = k_a + k_b$ for some $a, b \geq i$. If there exist $a, b \in A$ satisfying $a \ll^{k_n} b$ for all $n = 1, 2, \ldots$, then $[\ll]$ contains a transitive paraorder.

Proof. For $n = 1, 2, \ldots$ set

$$\lambda_n := \left\{xy : x \ll^{k_i} y \text{ for all } i \geq n\right\}. \tag{5.5}$$

Clearly $\lambda_n \in [\ll]$ and $\lambda_1 \subseteq \lambda_2 \subseteq \ldots$. Thus $\lambda := \bigcup_{n \in \mathbb{N}} \lambda_n$ is the directed union of resolvents of $\ll$ and so $\lambda \in [\ll]$. By assumption $ab \in \lambda_1 \subseteq \lambda$ and therefore λ is nonvoid. Next λ is a subrelation of the transitive hull of $\ll$ and as such also a paraorder. To show that λ is transitive let $c\ \lambda\ d\ \lambda\ e$ (meaning $c\ \lambda\ d$ and $d\ \lambda\ e$). Then $c\ \lambda_u\ d\ \lambda_v\ e$ for some $u, v \in \mathbb{N}$. Setting $i = \max(u, v)$ we have $c\ \lambda_i\ d\ \lambda_i\ e$. By (5.5) clearly $c \ll^{k_a + k_b} e$ for all $a, b \geq i$. Now the sequence $k_1 < k_2 < \ldots$ is such that $c \ll^{k_l} e$ for all $l \geq \varphi(i)$. This proves that $c\ \lambda_{\varphi(i)}\ e$ and $c\ \lambda\ e$. Thus λ is a transitive paraorder. ■

Remark. For $k \in \mathbb{N}$ the sequence $k < 2k < \ldots$ satisfies the assumptions of Lemma 5.20; in particular, the assumptions hold for the sequence $1 < 2 < \ldots$.

A chain order-isomorphic to $1 < 2 < \ldots < \omega$ (to $-\omega < \ldots < -2 < -1$) is of *type* $\omega + 1$ (of *type* $1 + \omega^*$).

5.21 Corollary
If $\ll$ contains a chain of type $\omega + 1$ or $1 + \omega^*$ then $[\ll]$ contains a transitive paraorder.

5.22
An order $<$ on A is a chain (also a linear or total order) if for all distinct $a, b \in A$ either $a < b$ or $b < a$. We consider uniform unbounded chains. It is well known (Cantor, see e.g. [Ro 82] Theorem 2.8) that every unbounded

dense countable chain is order isomorphic to the chain $(\mathbb{Q},<)$ of the rationals (with the natural order). Another example of such chain is the set $\mathbb{R}$ of the reals. To prove that $\mathrm{Pol}\,(<)$ is locally maximal we need the following general fact.

5.23 Proposition
Let B be a clone on A such that every finite subset F of A satisfies (i) *every map $\varphi : F \to A$ is the restriction of some $h \in B^{(1)}$ to F and* (ii) *there exists $g \in B$ whose restriction to F is essential and takes at least $|F|$ values. Then* $\mathrm{Loc}\,B = \mathcal{O}_A$.

Proof. Let $F \subset A$ be finite and let C consist of the restrictions to F of those $b \in B$ having $\mathrm{im}\, b \subseteq F$. By (i) clearly $\mathcal{O}_F^{(1)} \subseteq C$. Set $G := \mathrm{im}\, g$. It is easy to see that there exists $\varphi : G \to F$ such that $\varphi \circ g$ is essential and surjective. Applying the well-known Slupecki criterion [Sl 38] we obtain that $C = \mathcal{O}_F$. Now let $f \in \mathcal{O}_A^{(n)}$ be arbitrary and G an arbitrary finite subset of A. Set $F := G \cup \mathrm{im}\,(f \restriction G)$ (where $f \restriction G$ denotes the restriction of f to G). Clearly there exists $f^* \in \mathcal{O}_F^{(n)}$ agreeing with f on G. By what has been shown above there exists $b \in B^{(n)}$ agreeing with f^* on F. Thus $\mathrm{Loc}\,B = \mathcal{O}_A$. ■

5.24 Proposition
Let $<$ be an unbounded chain on A such that for all $a < b$ and $c < d$ there exists an order endomorphism φ satisfying $\varphi(a) = c$ and $\varphi(b) = d$. Then $\mathrm{Pol}\,(<)$ *is locally maximal.*

Proof. Let $f \in \mathcal{O}_A^{(n)} \setminus \mathrm{Pol}\,(<)$ be arbitrary. Denote by B the clone generated by $\{f\} \cup \mathrm{Pol}\,(<)$. We need the following 4 claims. For notational simplicity let $0, 1 \in A$ be arbitrary elements of A satisfying $0 < 1$.

Claim 1. *There exists $g \in B^{(1)}$ with $1 = g(0) \geq g(1)$ where $g(1) \in \{0,1\}$.*

Proof of the claim. As $f \notin \mathrm{Pol}\,(<)$, there exist $a_i < b_i$ $(i = 1, ..., n)$ such that

$$f(a_1, ..., a_n) \geq f(b_1, ..., b_n).$$

Since $<$ is uniform, there exist unary $g_i \in \mathrm{Pol}\,(<)$ such that $g_i(0) = a_i$ and $g_i(1) = b_i$ $(i = 1, ..., n)$. Define $h \in \mathcal{O}_A^{(1)}$ by setting for every $x \in A$

$$h(x) := f(g_1(x), ..., g_n(x)).$$

Clearly $h \in B$ and $h(0) \geq h(1)$. By uniformity there exists $i \in \mathrm{Pol}\,(<)$ such that 1) $i(h(0)) = 1$ and 2) $i(h(1)) = 0$ whenever $h(0) > h(1)$. Now $g = i \circ h$ is the required unary operation.

Claim 2. $d(0) = d(1) = 0$ *for some $d \in B^{(1)}$.*

Proof of the claim. Let g be the unary operation from Claim 1. It is easy to verify that $\max \in \mathrm{Pol}(<)$ (where, as usual, $\max(x,y)$ is the greatest element in $\{x,y\}$). For all $x \in A$ set $h(x) := \max(g(x), x)$. Clearly

$$h(0) = \max(g(0), 0) = 1 = \max(g(1), 1) = h(1).$$

Choose $i \in \mathrm{Pol}(<)$ such that $i(1) = 0$. Then $d := i \circ g$ is the required unary operation.

Claim 3. *For every finite subset* $\{a_1, ..., a_n\}$ *of* A *there is* $c \in B^{(1)}$ *such that* $c(a_1) = ... = c(a_n) = 0$.

Proof of the claim. We proceed by induction on $n \geq 0$. For $n = 0$ set $c := \mathrm{id}_A$. Suppose the statement holds for $n-1 \geq 0$ and let $\{a_1, ..., a_n\} \subset A$. By the inductive assumption there exists $c' \in B^{(1)}$ such that $c'(a_1) = ... = c'(a_{n-1}) = 0$. There is nothing to prove if $c'(a_n) = 0$. Thus let $b := c'(a_n) \neq 0$. By uniformity there exists $h \in \mathrm{Pol}(<)$ mapping $\{0, b\}$ onto $\{0, 1\}$. Set $c := d \circ h \circ c'$ (where d is the unary operation from Claim 2). It is easy to see that $c(a_1) = ... = c(a_n) = 0$. This concludes the induction step and proves the claim.

Claim 4. *For all* $a, b \in A$ *with* $b \geq 0$ *there exists* $h \in B^{(2)}$ *such that* $h(a, 0) = b$ *and* $h(x, 0) = 0$ *otherwise.*

Proof. If $b = 0$ choose h to be the projection e_2^2. Thus let $b > 0$. Since $<$ is uniform, there exists an $<$-endomorphism φ such that $\varphi(0) = b$. Define $\psi : A \to A$ by

$$\psi(x) := \begin{cases} \varphi(x) & \text{for all } x > 0, \\ x & \text{otherwise.} \end{cases}$$

As $b \geq 0$, clearly ψ is an $<$-endomorphism. Next set

$$h(x,y) := \begin{cases} b & \text{if } x = a,\ y = 0, \\ 0 & \text{if } x \neq a,\ y = 0, \\ \psi(y) & \text{otherwise.} \end{cases}$$

To show that $h \in \mathrm{Pol}(<)$ let $c < e$ and $d < f$. 1) If $d \neq 0 \neq f$ then $h(c,d) = \psi(d) < \psi(f) = h(e,f)$. 2) Let $d = 0$. Then $f > 0$ and $h(c,0) \leq b < \psi(f) = h(e,f)$. 3) Finally let $f = 0$. Then $d < 0$ and $h(c,d) = \psi(d) < 0 \leq h(e,0)$. This proves the claim.

Claim 5. *If* $a_1, ..., a_n \in A$ *are pairwise distinct and* $b_1, ..., b_n \in A$ *then* $h(a_i) = b_i$ $(i = 1, ..., n)$ *for some* $h \in B^{(1)}$.

Proof of the claim. We can choose the element 0 so that $b_i \geq 0$ for all $i = 1, ..., n$. By Claim 3 there exists $c \in B^{(1)}$ such that $c(a_1) = ... = c(a_n) = 0$.

By Claim 4 for $i = 1, ..., n$ there exists $h_i \in B^{(2)}$ such that $h_i(a_i, 0) = b_i$ and $h_i(x, 0) = 0$ otherwise. The n-ary operation $\max_n$ assigns to all $x_1, ..., x_n \in A$ the greatest element of $\{x_1, ..., x_n\}$. It is easy to see that $\max_n \in \mathrm{Pol}(<)$. For all $x \in A$ set

$$h(x) := \max{}_n(h_1(x, c(x)), ..., h_n(x, c(x))).$$

Clearly $h \in B^{(1)}$. Moreover, for $i = 1, ..., n$ due to $b_i \geq 0$

$$\begin{aligned} h(a_i) &= \max{}_n(h_1(a_i, 0), ..., h_i(a_i, 0), ..., h_n(a_i, 0)) \\ &= \max{}_n(0, ..., 0, b_i, 0, ..., 0) = b_i. \end{aligned}$$

This proves the claim.

The proposition now follows from Claim 5 and Proposition 5.23 (choose $g = \max$). ■

6 The ternary relation $\sigma \cup \Delta_{12}$

6.1

In this section we study special ternary relations on A. Recall that $\Delta_{12} = \{xxy : x, y \in A\}$ and that $\sigma_3 := A^3 \backslash \iota_3$ consists of all $xyz \in A^3$ with $x \neq y \neq z \neq x$. We say that a ternary relation λ on A is 12-, 13- and 23-*symmetric* if $xyz \in \lambda$ implies $yxz \in \lambda$, $zyx \in \lambda$ and $xzy \in \lambda$, respectively. We say that it is 12-, 13- and 23-*antisymmetric* if for every $xyz \in \lambda$

$$yxz \in \lambda \Rightarrow x = y,\ zyx \in \lambda \Rightarrow x = z,\ xzy \in \lambda \Rightarrow z = y.$$

In this section we study the ternary relations $\rho = \sigma \cup \Delta_{12}$ on A such that

(i) $\emptyset \neq \sigma \subseteq \sigma_3$,

(ii) σ is 12-symmetric,

(iii) $uxz, uyz \in \rho \Rightarrow xyz \in \rho$,

(iv) for each finite $B \subset A$ we have $B^2 \times \{u\} \subseteq \rho$ for some $u \in A$,

(v) ρ is repellent.

(For (iii) see Fig. 10)

Such a relation will be called a *trilium*. It was shown in [R-Sz 84] Corollary 2.5 that for the richest trilium $\sigma_3 \cup \Delta_{12}$ the clone Pol$(\sigma_3 \cup \Delta_{12})$ is locally maximal.

6.2 Lemma
If ρ is a trilium then either (i)

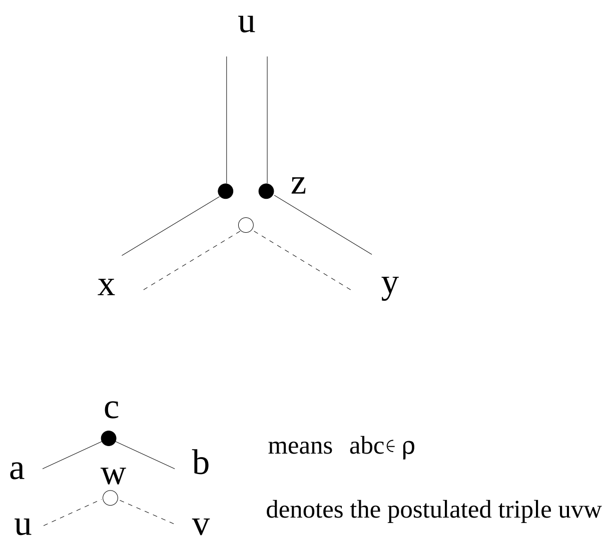
$$uvx, uxv, uvy, uyv \in \rho \Rightarrow x = y \tag{6.1}$$

or (ii) *for every finite subset B of A there are $u, v \in A$ so that $uvb, ubv \in \rho$ for all $b \in B$.*

Proof. For $h = 2, 3, ...$ put

$$\lambda_h := \{x_1...x_h : ux_iv, uvx_i \in \rho \text{ for some } u, v \in A \text{ and all } i = 1, ..., h\}.$$

Clearly all λ_h are totally symmetric. Moreover λ_2 is reflexive and so $\lambda_2 = \iota_2$ or $\lambda_2 = A^2$. In the first case we get (6.1).
In the second case we have $\lambda_2 = A^2$. This means λ_3 is totally reflexive, whence $\lambda_3 = A^3$. Continuing in this way we get $\lambda_h = A^h$ for $h = 2, 3, ...$. In particular for every finite subset B of A, $B = \{b_1, ..., b_h\}$ we get $b_1...b_h \in B^h \subseteq \lambda_h$ which yields the second statement. ■



- Fig 10 -

We say that a trilium ρ is *strong* if to arbitrary $x, y \in A$ we have $xuy \in \rho$ and $uyx \in \rho$ for some $u \in A$. It is *strict* if for all $x, y, z, t \in A$

$$xyz, yzx \in \rho \Rightarrow x = y = z.$$

6.3 Lemma
A trilium is either strong or strict.

Proof. Set

$$\sigma := \{xy : uxy, uyx \in \rho \text{ for some } u\}.$$

The relation σ is reflexive because $aaa \in \Delta_{12} \subseteq \rho$ for all $a \in A$. It is clearly symmetric and therefore trivial. Clearly $\sigma = \iota_2$ and $\sigma = A^2$ correspond to ρ strict and ρ strong. ■

We consider strong trilia. We say that a strong trilium is a *shuffle* if for all $x, y \in A$

$$xyu,\ xuy,\ uyx \in \rho$$

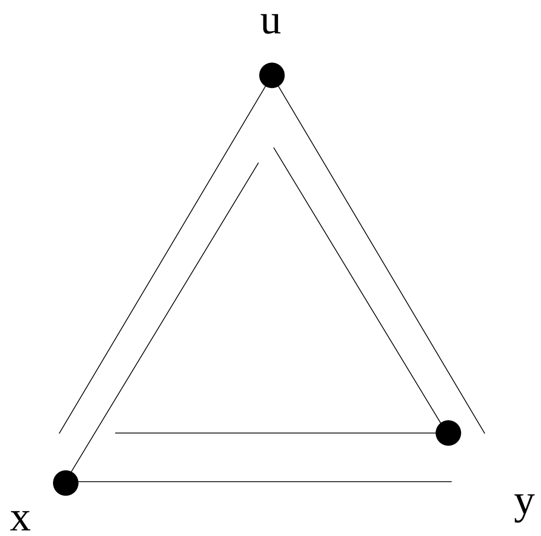
(see Fig. 11) holds for some u. We have:

6.4 Lemma
A strong trilium is either a shuffle or satisfies

$$xyz, xzy, zyx \in \rho \Rightarrow x = y = z. \tag{6.2}$$

Proof. Set

$$\sigma := \{xy : xyu, xuy, uyx \in \rho \text{ for some } u\}.$$

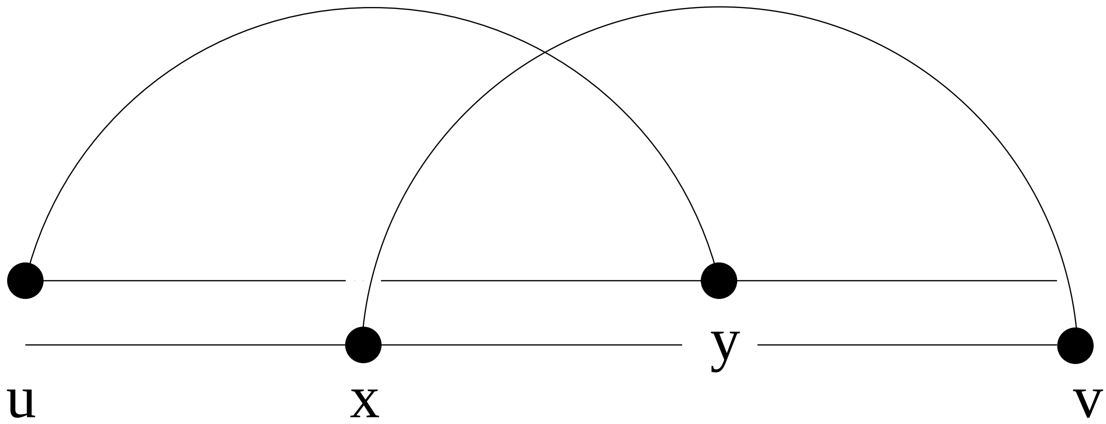


- Fig 11 -

Clearly σ is reflexive and symmetric. Since ρ is repellent, it follows that $\sigma = \iota_2$ or $\sigma = A^2$, whence ρ satisfies (6.2) or is a shuffle. ■

We say that a strong trilium is *taut* if it satisfies (6.2) and to arbitrary $x, y \in A$ there are $u, v \in A$ so that

$$xyu, xvy, uyx, xyv \in \rho \tag{6.3}$$



- Fig 12 -

6.5 Lemma
Let ρ be a trilium. Then

(i) *Either ρ is taut or for all $x, y, u, v \in A$*

$$xyu, xvy, uyx, xvy \in \rho \Rightarrow x = y = u = v, \tag{6.4}$$

(ii) *If ρ is strong and $[\rho]$ contains no reflexive diagraph other than ι_2 then ρ is taut.*

Proof. (i) It suffices to consider

$$\sigma := \{xy : xyu, xvy, uyx, xyv \in \rho \text{ for some } u, v\}\,.$$

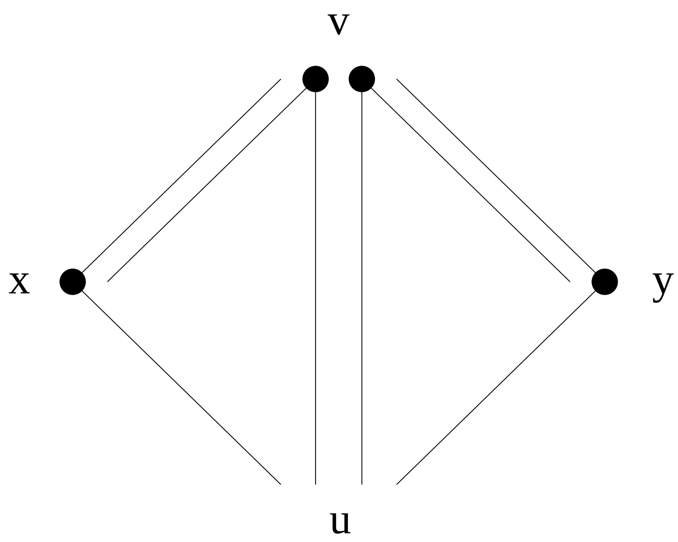
(ii) Suppose to the contrary that ρ is not taut. Set

$$\lambda := \{xy : xyu, xuy \in \rho\}\,.$$

Clearly λ is reflexive. Taking into account that ρ is strong we see that to arbitrary $u, v \in A,\ u \neq v$ there exists $w \in A$ such that $wuv, wvu \in \rho$. Thus $wu \in \lambda$. Moreover, $w \neq u$ (since otherwise $u = v$). Thus $\lambda \supset \iota_2$. Suppose $xy, yx \in \lambda$. Then there are $u, v \in A$ so that

$$xyu, xuy, yxv, yvx \in \rho,$$

hence by (i) we get $x = y$. Thus $\lambda \in [\rho]$ is antisymmetric in contradiction to our assumption. ■



- Fig 13 -

A trilium ρ is of the *first kind* if for all $x, y, u_x, u_y \in A$

$$xu_yu_x, yu_xu_y \in \rho \Rightarrow x = y.$$

The trilium is of the *second kind* if for all $x, y \in A$ there exist $u_x, u_y \in A$ such that

$$xu_yu_x, yu_xu_y \in \rho.$$

6.6 Lemma

Each trilium is either of the first kind or of the second kind.

Proof. Set

$$\sigma := \{xy : xu_yu_x, yu_xu_y \in \rho \text{ for some } u_x, u_y\}.$$

Clearly σ is symmetric and reflexive (chose $u_x = u_y := x$). Since ρ is repellent, $\sigma = \iota_2$ or $\sigma = A^2$ and hence ρ is either of the first or the second kind. ■

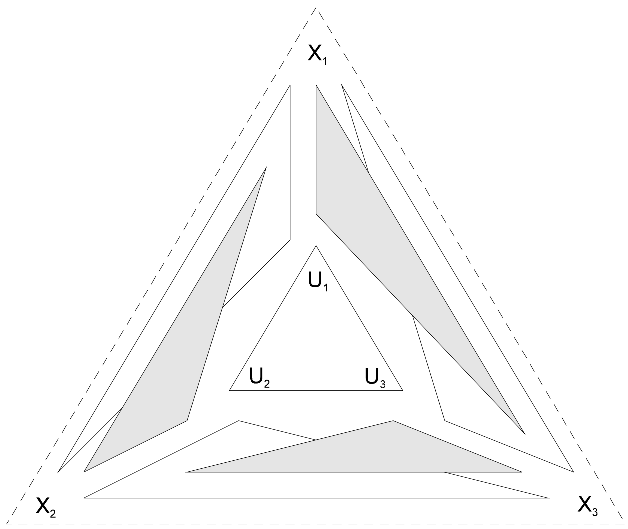
7 Totally reflexive and symmetric relations

7.1 Definitions
Let again T_h denote the set of totally reflexive and totally symmetric h-ary relations on A distinct from A^h and let $T = \bigcup_{h=2}^{\infty} T_h$. Note that $\rho \in T_h$ is completely determined by the family of h-element sets

$$\rho^* = \Big\{\{x_1, ..., x_h\} \in [A]^h \mid (x_1, ..., x_h) \in \rho\Big\}$$

(where, as usual, $[A]^h$ is the set of h-element subsets of A). The reader may find it more convenient to treat $\rho \in T_h$ in this way.
We say that $B \subseteq A$ is *ρ-centered* if there is $c \in A$ such that $\{c\} \cup X \in \rho^*$ for all $X \in [B]^{h-1}$ with $c \notin X$.

We say that $\rho \in T_h$ is *locally central* if each finite $B \subset A$ is ρ-centered (i.e. $[B]^{h-1} \times \{u\} \subseteq \rho$ for some u) and ρ is *homogeneous* if every $\rho-$centered $B \in [A]^h$ belongs to ρ; i.e., $b_1...b_h \in \rho$ whenever for some u we have that $b_1...b_{i-1}ub_{i+1}...b_h \in \rho$ for all $i = 1, ..., h$. Thus $\rho \in T_2$ is homogeneous iff it is transitive; i.e., iff ρ is an equivalence relation on A distinct from A^2. For $h = 3$ see Fig. 14. Note the following absorbing property of T_h.



- Fig 14 -

7.2 Fact *Let $h > 2$ and $\rho \in T_h$. Then all l-ary relations $(1 \leqq l < h)$ from $[\rho]$ are trivial. Each h-ary relation $\sigma \in [\rho]$ satisfies $\sigma \supseteq \rho$ and hence is totally reflexive; moreover $[\sigma] \cap T_h \neq \emptyset$.*

Proof. Let $1 \leq l \leq h$ and σ an l-ary resolvent of ρ. Then $\rho = \lambda \frown_l \rho$ where λ is an h-ary relation on a set V such that $V \supseteq \{1, ..., l\}$. Let $l < h$. By the total reflexivity of ρ each map $f : V \to A$ with $|\operatorname{im} f| \leq l$ belongs to $\operatorname{Hom}(\lambda, \rho)$ and so $\sigma = A^l$. Let $l = h$. For each $a_1...a_h \in \rho$ every map $f : V \to \{a_1, ..., a_k\}$ with $f(i) = a_i$ $(i = 1, ..., h)$ belongs to $\operatorname{Hom}(\lambda, \rho)$.
Thus $a_1...a_h \in \sigma$ and $\rho \subseteq \sigma$. Finally let $h > 2$ and let σ be an h-ary totally reflexive relation. For a permutation p of $\{1, ..., h\}$ set

$$\sigma^{(p)} := \left\{a_{p(1)}...a_{p(h)} : a_1...a_h \in \sigma\right\}.$$

It is easy to see that $\sigma^{(p)} \in [\sigma]$ (in fact, $\operatorname{Pol}\sigma^{(p)} = \operatorname{Pol}\sigma$). The set σ is intersection closed and so $\xi := \bigcap_{p \in S_h} \sigma^{(p)}$ (where S_h denotes the symmetric group of all permutations of $\{1, ..., h\}$) belongs to $[\sigma]$. Obviously $\xi \supseteq \iota_h$ and ξ is totally symmetric; consequently $\xi \in [\sigma] \cap T_h$. This proves the fact. ■

7.3 Definition

Denote by S_h the set of all h-ary totally symmetric relations on A and set $S := \bigcup\limits_{h=2}^{\infty} S_h$. A *functor* is a set $\{(\ )_j : j < \omega\}$ of maps from $T \left(:= \bigcup\limits_{h=2}^{\infty} T_h \right)$ into S such that for all $h \geq 2$, $\sigma \in T_h$ and $j \geq 0$: (7.1)

(i) $(\sigma)_j \in [\sigma] \cap S_{h+j}$, (ii) $\sigma \subseteq (\sigma)_0$, (iii) $(\sigma)_j = A^{h+j} \Rightarrow \iota_{h+j+1} \subseteq (\sigma)_{j+1}$ (7.1)

We abbreviate $(\sigma)_0$ by (σ). Notice that by (ii) clearly $(\sigma) \in T_h \cup \left\{A^h\right\}$. Similarly if $(\sigma)_j = A^{h+j}$ then from (iii) we obtain $(\sigma)_{j+1} \in T_{h+j+1} \cup \left\{A^{h+j+1}\right\}$.

7.4 Example

The following is our basic example of a functor. For $h \geq 2$, $\sigma \in T_h$ and $j \geq 0$ let $(\sigma)_j$ consist of all $x_1...x_{h+j} \in A^{h+j}$ for which there is an $x_{h+j+1} \in A$ such that $x_{i_1}...x_{i_h} \in \sigma$ whenever $h + j + 1 \in \{i_1, ..., i_h\} \subseteq \{1, ..., h + j + 1\}$. In particular,

$$(\sigma) = (\sigma)_0 := \{x_1...x_h : x_1...x_{i-1} u x_{i+1}...x_h \in \sigma \ \ (i = 1, ..., n) \ \text{ for some } u\} \tag{7.2}$$

(i.e. $(\sigma)^*$ consists of all σ-centered $X \in [A]^h$).

We verify (7.1). First (i) and (ii) follow from the definitions. For (iii) suppose $(\sigma)_j = A^p$ where $p := h + j$. Let $x_1, ..., x_p \in A$ be arbitrary. Then $x_1...x_p \in A^p = (\sigma)_j$; whence there is $x_{p+1} \in A$ such that $x_{{}_1}...x_{{}_h} \in \sigma$ whenever $p + 1 \in \{i_1, ..., i_h\} \subset \{1, ..., p + 1\}$. Put $y_i := x_i$ $(i = 1, ..., p)$ and $y_{p+1} := x_p$. To prove $y_1...y_{p+1} \in (\sigma)_{j+1}$ put $y_{p+2} := x_{p+1}$. Let $p + 2 \in \{i_1, ..., i_h\} \subseteq \{1, ..., p + 2\}$. If $\{p, p + 1\} \subseteq \{i_1, ..., i_h\}$ then automatically we have $y_{i_1}...y_{i_h} \in \sigma$ due to the total reflexivity of σ.
If $|\{p, p + 1\} \cap \{i_1, ..., i_h\}| \leqq 1$ then in view of $y_{p+1} = x_p$ and $y_{p+2} = x_{p+1}$ the h-tuple $y_{i_1}...y_{i_h}$ is one of h-tuples $x_{j_1}...x_{j_h}$.

7.5 Definitions
Let $\{(\ )_j : j < w\}$ be a functor and let $\rho \in T_h$. By a transfinite construction we establish a sequence $\langle \rho_0, \rho_1, ...\rangle$ of h_ξ-ary relations ρ_ξ on A (where ξ is an ordinal) satisfying the following condition: if ξ is a nonzero limit ordinal and ρ_ς has been constructed for all $\varsigma < \xi$ then there exists $\theta < \xi$ such that $h_\varsigma = h_\theta$ for all $\theta < \varsigma < \xi$.

Let $\rho_0 := \rho$. Suppose that ς is a nonzero ordinal for which the sequence $\langle \theta_\xi : \xi < \varsigma\rangle$ has been constructed. Denote by θ the least ordinal such that $h_\theta = h_\xi$ for all $\theta \leq \xi < \varsigma$ (if ς is isolated possibly $\theta = \varsigma - 1$). Set $\sigma := \bigcup_{\theta\leq\xi<\varsigma} \rho_\xi$ and $h := h_\theta$. Recall that (σ stands for $(\sigma)_0$.

1. if $\sigma \subset (\sigma) \subset A^h$ set $\rho_\varsigma := (\sigma)$,
2. if $\sigma = A^h$ stop,
3. if $\sigma \subset A^h = (\sigma)$ while for some $l > 0$
$$(\sigma)_l \subset A^{h+l}, \tag{7.3}$$
set $\rho_\varsigma := (\sigma)_j$ where j is the least integer ℓ satisfying (7.3),
4. if $\sigma \subset (\sigma)$ while $(\sigma)_j = A^{h+j}$ for all $j \geq 0$, stop.
5. if $\sigma \subset (\sigma) \subset A^h$, stop.

7.6 Example
Let $\left\{(\ )_j : j < w\right\}$ be the functor from Example 7.4 and let $h = 2$. It is easy to see that $(\rho) = (\rho)_0 = \rho^2$ $(= \rho \circ \rho)$ and so $\rho_1 = \rho^2$, $\rho_2 = \rho^4$ etc. The construction stops if some ρ_i $\left(= \rho^{2^i}\right)$ is an equivalence relation on A. Suppose $(\rho_i) = A^2$. By its definition $(\rho_i)^*_l$ consists of all $(l+2)$-element subsets of A having a joint neighbor in ρ_i $(l \geq 0)$. Thus ρ_{i+1} is the first $(\rho_i)_j \subset A^{j+2}$ provided such a j exists, else the construction stops. If $\rho_i = \rho^{2^i} \subset A^2$ for all $i < w$ then set $\sigma := \bigcup_{i<w} \rho_i$ and continue.

We show some basic properties of the construction.

7.7 Lemma
If $h > 1$ and $\rho \in T_h$ then

(i) *The construction of the sequence $\langle \rho_0, \rho_1, ...\rangle$ stops at steps* 2, 4, 5 *or at the least ordinal ς such that $\{h_\xi : \xi < \varsigma\}$ is an unbounded subset of* $\mathbb{N}$,

(ii) $\rho_\xi \in [\rho] \cap T_{h_\xi}$ *for all $\xi < \varsigma$, and*

(iii) *For all* $\theta < \xi < \varsigma$ *we have that* a) $h_\theta \leq h_\xi$, b) $\operatorname{Pol}\rho_\theta \subseteq \operatorname{Pol}\rho_\xi$ *and* c) $\rho_\theta \subset \rho_\xi$ *provided* $h_\theta = h_\xi$.

Proof. An easy transfinite induction (based on Definition 7.3 and the fact that $[\ ]$ is closed under both arbitrary intersections and directed unions proves (ii) and (iii-a) and (iii-c) as well as $\operatorname{Pol}\rho_\theta \subseteq \operatorname{Pol}\rho_\xi$. We show that $\operatorname{Pol}\rho_\theta \not\supseteq \operatorname{Pol}\rho_\xi$. Indeed if $h_\theta < h_\xi$ then by Fact 7.2 every h_θ-ary $\sigma \in [\rho_\xi]$ is trivial while if $h_\theta = h_\xi$ then every h_ξ-ary $\sigma \in [\rho_\xi]$ contains $\rho_\xi \supset \rho_\theta$. In both cases $\operatorname{Pol}\rho_\theta \not\supseteq \operatorname{Pol}\rho_\xi$. This proves (iii-b).
To prove (i) suppose there is an ordinal τ such that the subset

$$\{h_\xi : \xi < \tau,\ \rho_\xi \text{ has been constructed}\}$$

of $\mathbb{N}$ is unbounded (i.e. infinite). Denote by ς the least ordinal with this property. It is easy to verify that then the construction stops at ς. Thus assume that there exists an ordinal θ such that $h_\xi = h' := h_\theta$ for all $\xi > \theta$ for which ρ_ξ has been constructed. By (iii-c) for each such ξ the sequence $\langle \rho_\varsigma : \theta < \varsigma < \xi \rangle$ is a strictly increasing sequence in the ordered set $(T_{h'}, \subseteq)$. It follows that $R := \{\rho_\varsigma : \varsigma > \theta,\ \rho_\varsigma \text{ constructed}\}$ is a subset of $T_{h'}$. The chain $(R, \subseteq)$ is union-closed and so by Zorn's lemma it has a maximal element σ. If $\sigma \in T_{h'}$ then $(\sigma) = \sigma$ and the construction stops at step 5. Thus let $\sigma = A^h$. Then the construction stops at step 2. ■

We need the following statements (Towers were defined in §1.5).
A tower $< p_\xi \cdot \xi < \zeta >$ where each p_ξ is an h_ξ-ary relations is *arity increasing (constant)* if $\xi < \xi' < \zeta$ implies $h_\xi < h_{\xi'} (h_\xi = h_{\xi'})$

7.8 Proposition
For $i = 1, 2, ...$ *let* ρ_i *be an* h_i*-ary relation on* A. *If* (i) $h_1 < h_2 < ...$, (ii) $\iota_{h_i} \subseteq \rho_i \subset A^{h_i}$ *for* $i = 1, 2, ...$ *and* (iii) $\operatorname{Pol}\rho_1 \subseteq \operatorname{Pol}\rho_2 \subseteq ...$ then $\langle \rho_1, \rho_2, ... \rangle$ *is an increasing arity tower.*

Proof. First we show that $\operatorname{Pol}\rho_i \subset \operatorname{Pol}\rho_{i+1}$ for all $i > 0$. First suppose $\iota_{h_i} < p$.Choose

$$a_1 ... a_{h_i} \in \rho_i \backslash \iota_{h_i},\ b_1 ... b_{h_i} \in A^{h_i} \backslash \rho_i \tag{7.4}$$

and define $f \in \mathcal{O}_A^{(1)}$ by setting $f(a_i) := b_i$ for all $i = 1, ..., h_i$ and $f(x) := b_i$ otherwise. Clearly $f \notin \operatorname{Pol}\rho_i$ by (7.4) while $f \in \operatorname{Pol}\rho_{i+1}$ due to $|\operatorname{im} f| = |\{b_1, ..., b_{h_i}\}| = h_i < h_{i+1}$.
Next let $\iota_{h_1} = p_1$. As in the finite case one can show that Pol_{z_k} consists of the clone generated by $\mathcal{O}^{[1]}$ and all $f \in \mathcal{O}$ with $\lim f| \leq h_i$. From $\iota_{h_{i+1}} \subseteq p_{i+1}$ and $h_i < h_{i+1}$ it follows that any $f \in \mathcal{O}^{(2)}$ with $\lim f| = h_{i+1}$ belongs to

$\operatorname{Pol} p_{i+1} \setminus \operatorname{Pol}\ p_1$.Set $C := \bigcup_{i=0}^{\infty} \operatorname{Pol} \rho_i$. To show that $\operatorname{Loc} C = \mathcal{O}_A$ let $f \in \mathcal{O}_A^{(n)}$ be arbitrary and let $F \subset A$ be finite. Set $G := \operatorname{im}(f \upharpoonright F)$ (where $f \upharpoonright F$ is the restriction of f to F) and denote by i the least integer such that $h_i > |G|$. Fix $g \in G$ and define $f^* \in \mathcal{O}_A^{(n)}$ by

$$f^*(a_1, ..., a_n) = \begin{cases} f(a_1, ..., a_n) & \text{if } a_1, ..., a_n \in F, \\ g & \text{otherwise.} \end{cases} \tag{7.5}$$

Clearly $\operatorname{im} f^* = G$ and $f^* \upharpoonright F = f \upharpoonright F$. Next $f^* \in \operatorname{Pol} \rho_i \subseteq C$ due to $\iota_{h_i} \subseteq \rho_i$. Thus $f \in \operatorname{Loc} C$, hence $\mathcal{O}_A \subseteq \operatorname{Loc} C \subseteq \mathcal{O}_A$ shows $\operatorname{Loc} C = \mathcal{O}_A$. ■

7.9 Proposition
Let $h > 1$, let ς be *an ordinal and let $\rho_\xi \in T_h$ for all $\xi < \zeta$.* If

$$\xi < \varsigma \Rightarrow \rho_\theta \subset \rho_\xi,\ \operatorname{Pol} \rho_\theta \subset \operatorname{Pol} \rho_\xi, \tag{7.6}$$

$$\bigcup_{\xi < \varsigma} \rho_\xi = A^h, \tag{7.7}$$

then $\langle \rho_\xi : \xi < \varsigma \rangle$ is a constant arity tower.

Proof. We proceed as in the proof of the preceding proposition. By (7.6) and (7.7) there exists the least ordinal ξ such that $G^h \subseteq \rho_\xi$. Clearly $f^* \in \operatorname{Pol} \rho_\xi$ and $\operatorname{Loc} C = \mathcal{O}_A$ follows in the same way as in the above mentioned proof. ■

The homogenous and locally central relations were defined in 7.1

7.10 Proposition
Let $h > 1$ and $\rho \in T_h$. *Then ρ is dominated by* (i) *a homogeneous relation,* (ii) *by a locally central relation* (iii) *by a constant arity tower or* (iv) *by an increasing arity tower.*

Proof. Let $\left\{ (\quad)_j : j < \omega \right\}$ be the functor from Example 7.4 and let $\langle \rho_0, \rho_1 ... \rangle$ be the corresponding sequence defined in 7.5. According to Lemma 7.6 it stops at steps 2,4,5 or at the least ordinal ς such that the set $\{h_\xi : \xi < \varsigma\}$ is unbounded. We consider separately the four cases.

1. Suppose that for some ordinal ς the construction stops at step 2. Then from Lemma 7.7 (iii) and Proposition 7.9 we obtain that $\langle \rho_\xi : \xi < \varsigma \rangle$ is a constant arity tower and (iii) holds.

2. Suppose that for some ordinal ς the construction stops at step 4. For an h-ary relation σ and $j \geq 0$ the validity of $(\sigma)_j = A^{h+j}$ means that every $(h+j)$-element subset of A is σ^*-centered $(j = 0, 1, ...)$; whence σ is locally central and we have (ii).

3. Suppose that the construction stops at step 5. Then $\sigma = (\sigma)$ and therefore σ is homogeneous.

4. Suppose that the construction stops at the least ordinal ς such that $H := \{h_\xi : \xi < \varsigma\}$ is unbounded. Write H as $\{h'_i : i < \omega\}$ where $h'_1 < h'_2 < ...$. For each $i < \omega$ denote by ξ_i the least ordinal such that $h_{\xi_i} = h'_i$. Now Lemma 7.7 and Proposition 7.8 show that $\langle \rho_{\xi_1}, \rho_{\xi_2}, ... \rangle$ is an increasing arity tower and (iv) holds. ■

Remark. Proposition 7.9 shows the existence of many towers. If ρ is locally central then $\mathrm{Pol}\,\rho$ is locally maximal [R-Sc 82]. Thus in the remainder of this paper we can concentrate on the set H_h of all homogenous relations from T_h. Recall from Example 7.4 that H_2 is the set of proper equivalence relations on A. Since for such relations ρ the clone $\mathrm{Pol}\,\rho$ is locally maximal, we assume $h > 2$.

7.11 Definition
We introduce another functor $\{(\ \ )_j : j < \omega\}$. Let $h \geq 3$, $\sigma \in T_h$, $j \geq 0$ and $p := h + j$. The relation $(\sigma)_j$ consists of all $x_1 ... x_p \in A^p$ such that some $u_1, ..., u_p \in A$ satisfy (i) $u_{i_1} ... u_{i_h} \in \sigma$ whenever $1 \leq i_1 < ... < i_h \leq p$ and (ii) for all $i = 1, ..., p$ and $1 \leq j_1 < ... < j_{h-2} \leq p$

$$x_i u_i x_{j_1} ... x_{j_{h-2}} \in \sigma. \tag{7.8}$$

In particular, $(\sigma) := (\sigma)_0$ consists of all $x_1 ... x_h \in A^h$ such that some $u_1 ... u_h \in \sigma$ satisfies for all $1 \leq i, j \leq h$ with $i \neq j$

$$x_1 ... x_{j-1} u_i x_{j+1} ... x_h \in \sigma \tag{7.9}$$

(see Fig. 14 for $h = 3$). Informally, we can view u_i as a bodyguard of the master x_i ($i = 1, ..., h$). Every set of h bodyguards forms a coalition in the sense of (i). A master and his bodyguard form a coalition for every choice of $h - 2$ other masters.
Call a relation $\sigma \in T_h$ *strongly homogeneous* if $\sigma = (\sigma)$. Thus σ is strongly homogeneous if $x_1 ... x_h \in \sigma$ whenever there exists $u_1 ... u_h \in \sigma$ satisfying (7.9). Notice that a strongly homogeneous relation σ is homogeneous. Indeed if to $x_1, ..., x_h \in A$ there exists u satisfying $x_1 ... x_{i-1} u x_{i+1} ... x_h \in \sigma$ for all $i = 1, ..., h$ then it suffices to choose $u ... u \in \sigma$.
For $h \geq 3$ a relation of $\sigma \in T_h$ is *protective* if $(\sigma)_j = A^{h+j}$ for all $j \geq 0$.

7.12 Lemma
$\left\{(\ \ )_j : j < \omega\right\}$ *from Definition* 7.10 is *a functor.*

Proof. We verify (7.1). Let $\sigma \in T_h$ and $j \geq 0$. Clearly $(\sigma)_j \in [\sigma] \cap S_{h+j}$. If $a_1...a_h \in \sigma$ then $a_1...a_h \in (\sigma)$ because we can choose $u_1...u_h = a_1...a_h$ and (7.9) holds on account of the total reflexivity. Finally let $p := h + j$, $(\sigma) = A^p$ and $x_1, ..., x_p \in A$. Then $x_1...x_p \in A^p = (\sigma)_j$ and so there exist $u_1, ..., u_p$ satisfying the conditions (i) and (ii) from Definition 7.10. Set $u_{p+1} := u_p$. It can be easily verified that $x_1...x_p x_p \in (\sigma)_{j+1}$. Since $(\sigma)_{j+1}$ is totally symmetric, we obtain (iii). ■

We have an analog of Proposition 7.9.

7.13 Proposition
Every homogeneous relation ρ of arity at least 3 *is dominated by*
(i) *a strongly homogeneous relation,*
(ii) *a protective relation,*
(iii) *a constant arity tower, or*
(iv) *an increasing arity tower.*

Proof. Let $\langle \rho_0, \rho_1, ... \rangle$ be the transfinite sequence corresponding to ρ and the functor from Definition 7.10. The proof is the same as the proof of Proposition 7.9. ■

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