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Hypocoercivity for infinite dimensional non-linear degenerate stochastic differential equations with multiplicative noise

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Abstract

We analyze infinite dimensional Langevin dynamics with multiplicative noise. Such a dynamic is described via a coupled system of an infinite dimensional differential equation with an infinite dimensional non-linear stochastic differential equation with multiplicative noise. The coupled system is defined on the Cartesian product of two real separable Hilbert spaces U and V. The non-linearity of the equation is caused by considering external forces, induced by a potential function $\Phi : U \to (-\infty, \infty]$. Moreover, we allow stochastic perturbations in terms of a multiplicative noise, driven by an infinite dimensional cylindrical Wiener process in V.
First, the essential m-dissipativity of the associated Kolmogorov backwards operator L^Φ on $L^2(\mu^\Phi)$ defined on smooth finitely based functions is established. Moreover, we show that the strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ generated by the closure of L^Φ in $L^2(\mu^\Phi)$ is sub-Markovian and conservative. Here, μ^Φ is the canonical invariant measure with density $e^{-\Phi}$ with respect to an infinite dimensional non-degenerate Gaussian measure on $U \times V$. The main difficulty, besides the non-sectorality of L^Φ, is the coverage of a large class of potentials.
Second, we apply a refinement of the abstract Hilbert space hypocoercivity method, developed by Dolbeault, Mouhot and Schmeiser, to derive the hypocoercivity of $(T_t)_{t\geq 0}$. We take domain issues into account and use the formulation in the Kolmogorov backwards setting worked out by Grothaus and Stilgenbauer. The method enables us to explicitly compute the constants determining the exponential convergence rate to equilibrium of $(T_t)_{t\geq 0}$. To utilize this method, we derive a general Poincaré inequality for measures of type μ^Φ. We also derive the essential m-dissipativity and a second order regularity estimate for a perturbed infinite dimensional Ornstein-Uhlenbeck operator with possibly unbounded diffusion coefficient.
In the third part, we use abstract analytic potential theoretic results to construct a right process that solves the martingale problem for the Kolmogorov backwards generator with respect to the equilibrium measure. Under stronger assumptions, we construct a μ^Φ-invariant Hunt process with infinite life-time and weakly continuous paths, whose transition semigroup is associated with $(T_t)_{t\geq 0}$. This process provides a stochastically and analytically weak solution to the infinite dimensional Langevin dynamics with multiplicative noise. Hypocoercivity of $(T_t)_{t\geq 0}$ and the identification of $(T_t)_{t\geq 0}$ with the transition semigroups of the processes yields exponential ergodicity of the processes.
Finally, we apply our results to degenerate second order in time stochastic reaction-diffusion and Cahn-Hilliard-type equations with multiplicative noise. A discussion of the class of applicable potentials and coefficients governing these equations completes our analysis.

Zusammenfassung

Wir analysieren unendlichdimensionale Langevin Dynamiken mit multiplikativem Rauschen. Eine solche Dynamik wird durch ein gekoppeltes System beschrieben, welches durch eine unendlichdimensionale Differentialgleichung und eine unendlichdimensionale nichtlineare stochastische Differentialgleichung mit multiplikativem Rauschen gegeben ist. Das gekoppelte System ist auf dem kartesischen Produkt zweier reeller separabler Hilberträume U und V definiert. Die Nichtlinearität der Gleichung wird durch die Berücksichtigung externer Kräfte hervorgerufen, die durch die Potentialfunktion $\Phi : U \to (-\infty, \infty]$ induziert werden. Zusätzlich erlauben wir stochastische Störungen in Form eines multiplikativen Rauschens, das von einem unendlichdimensionalen zylindrischen Wiener Process in V getrieben wird.
Zunächst wird die essentielle m-Dissipativität des zugehörigen Kolmogorov Rückwärtsoperators L^Φ, welcher auf dem Raum der glatten Zylinderfunktionen definiert ist, in $L^2(\mu^\Phi)$ etabliert. Außerdem zeigen wir, dass die stark stetige Kontraktionshalbgruppe $(T_t)_{t\geq 0}$, die vom Abschluss von L^Φ in $L^2(\mu^\Phi)$ erzeugt wird, sub-Markovsch und konservativ ist. Hierbei bezeichnet μ^Φ das kanonische invariante Maß mit Dichte $e^{-\Phi}$ bezüglich eines unendlichdimensionalen nicht-entarteten Gaußschen Maßes auf $U \times V$. Die Herausforderung, neben der Nicht-Sektoralität von L^Φ, ist dabei die Betrachtung einer möglichst großen Klasse von Potentialen.
Zweitens wenden wir eine Verfeinerung der von Dolbeault, Mouhot und Schmeiser entwickelten abstrakten Hilbertraum Hypokoerzitivitätsmethode an, um die Hypokoerzitivität von $(T_t)_{t\geq 0}$ herzuleiten. Das heißt wir achten sorgfältig auf Definitionsbereiche und verwenden die von Grothaus und Stilgenbauer ausgearbeitete Formulierung im Kolmogorov Rückwärtsrahmen. Die Methode erlaubt die explizite Bestimmung der Konstanten, die die exponentielle Konvergenzgeschwindigkeit ins Gleichgewicht von $(T_t)_{t\geq 0}$ festlegen. Um diese Methode anzuwenden, beweisen wir eine allgemeine Poincaré Ungleichung für Maße vom Typ μ^Φ. Wir stellen auch die wesentliche m-Dissipativität und eine Regularitätsabschätzungen zweiter Ordnung für einen gestörten unendlichdimensionalen Ornstein-Uhlenbeck Operator mit möglicherweise unbeschränktem Diffusionskoeffizienten bereit.
Im dritten Teil benutzen wir Methoden der analytischen Potentialtheorie, um einen stochastischen Prozess zu konstruieren, der das Martingalproblem für den Kolmogorov Rückwärtsoperator bezüglich des Gleichgewichtsmaßes löst. Unter stärkeren Annahmen konstruieren wir einen μ^Φ-invarianten Hunt Prozess, mit schwach stetigen Pfaden und unendlicher Lebensdauer, dessen Übergangshalbgruppe mit $(T_t)_{t\geq 0}$ assoziiert ist. Dieser Prozess löst die unendlichdimensionale Langevin Dynamik mit multiplikativem Rauschen im stochastisch und analytisch schwachen Sinne. Hypokoerzitivität von $(T_t)_{t\geq 0}$ und die Identifikation von $(T_t)_{t\geq 0}$ mit den Übergangshalbgruppen der Prozesse resultieren in exponentieller Ergodizität der Prozesse.

Schließlich wenden wir unsere Ergebnisse auf entartete stochastische Reaktions-Diffusions und Cahn-Hilliard Gleichungen, zweiter Ordnung in der Zeitvariablen, mit multiplikativem Rauschen an. Eine Diskussion der Klasse zulässiger Potentiale und Koeffizienten, die diese Gleichungen beschreiben, vervollständigt unsere Analyse.

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Contents

1

Introduction

The classical Langevin dynamics

$$\begin{aligned} \mathrm{d}X_t &= Y_t \,\mathrm{d}t \\ \mathrm{d}Y_t &= -\gamma Y_t \,\mathrm{d}t - D\Phi(X_t)\,\mathrm{d}t + \sqrt{\frac{2\gamma}{\beta}}\,\mathrm{d}W_t, \end{aligned} \tag{1.1}$$

describes the evolution of a particle via its position $X_t \in \mathbb{R}^d$ and its velocity $Y_t \in \mathbb{R}^d$ in the d-dimensional euclidean space, $d \in \mathbb{N}$. The velocity of the particle is subjected to friction, whose magnitude is determined by $\gamma \in (0,\infty)$ and to a stochastic force, induced by a Wiener process $(W_t)_{t\geq 0}$ in $\mathbb{R}^d$. The parameter $\beta \in (0,\infty)$ is up to a constant, the inverse temperature. External forces affecting the motion of the particle are described via the gradient $D\Phi$ of a potential $\Phi : \mathbb{R}^d \to \mathbb{R}$.
The Itô stochastic differential equation describing the dynamic can be examined through its associated Kolmogorov backwards operator L_d^{Φ}, acting on $C_c^{\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ as follows

$$L_d^{\Phi} f(x,y) = \frac{\gamma}{\beta}\operatorname{tr}\left[D_2^2 f(x,y)\right] - \gamma\langle y, D_2 f(x,y)\rangle - \langle D\Phi, D_2 f(x,y)\rangle + \langle y, D_1 f(x,y)\rangle.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$ and D_1 and D_2 the gradients with respect to the first and second component, respectively. Establishing the essential m-dissipativity of $(L_d^{\Phi}, C_c^{\infty}(\mathbb{R}^d \times \mathbb{R}^d))$ on $L^2(\mathbb{R}^d \times \mathbb{R}^d, \mu_d^{\Phi})$, where

$$\mu_d^{\Phi} \coloneqq (2\pi)^{-\frac{d}{2}} e^{-\Phi(x) - \frac{1}{2}y^2}\,\mathrm{d}x \otimes \mathrm{d}y,$$

is the canonical invariant measure, provides an associated conservative strongly continuous sub-Markovian semigroup $(T_t)_{t\geq 0}$. Due to the non-sectorality of L_d^{Φ}, this is highly non-trivial. As the quadratic form associated to L_d^{Φ} is not coercive, classical spectral gap methods to provide exponential convergence to equilibrium of $(T_t)_{t\geq 0}$ are not available. However, abstract hypocoercivity methods are applicable. Analytic potential theoretic methods in the context of sub-Markovian resolvents ensure the existence of a stochastic process solving (1.1).
The objective of this thesis is to examine an infinite dimensional version of Equation (1.1), whereas we allow multiplicative noise. Let $W \coloneqq U \times V$ be the Cartesian product of two infinite dimensional real separable Hilbert spaces $(U, (\cdot,\cdot)_U)$ and $(V, (\cdot,\cdot)_V)$, respectively. Then, the infinite dimensional Langevin equation with multiplicative noise is described

on W, by the following infinite dimensional non-linear degenerate stochastic differential equation with multiplicative noise

$$\begin{aligned} \mathrm{d}X_t &= K_{21}Q_2^{-1}Y_t\,\mathrm{d}t \\ \mathrm{d}Y_t &= \sum_{i=1}^{\infty}\partial_{e_i}K_{22}(Y_t)e_i\,\mathrm{d}t - K_{22}(Y_t)Q_2^{-1}Y_t\,\mathrm{d}t - K_{12}Q_1^{-1}X_t\,\mathrm{d}t - K_{12}D\Phi(X_t)\,\mathrm{d}t \\ &\quad + \sqrt{2K_{22}(Y_t)}\,\mathrm{d}W_t. \end{aligned} \tag{1.2}$$

The action of the associated Kolmogorov backwards operator L^Φ on the space of bounded smooth cylinder functions, in the following denoted by $\mathcal{F}C_b^\infty(B_W)$, is given as

$$\begin{aligned} L^\Phi f(u,v) :=\ & \mathrm{tr}\left[K_{22}(v)\circ D_2^2 f(u,v)\right] + \sum_{i=1}^{\infty}(\partial_{e_i}K_{22}(v)D_2 f(u,v), e_i)_V \\ & - (v, Q_2^{-1}K_{22}(v)D_2 f(u,v))_V - (u, Q_1^{-1}K_{21}D_2 f(u,v))_U \\ & - (D\Phi(u), K_{21}D_2 f(u,v))_U + (v, Q_2^{-1}K_{12}D_1 f(u,v))_V. \end{aligned}$$

Above, K_{21} is a bounded linear operator from V to U and K_{12} its adjoint. The diffusion part is determined by the variable coefficient K_{22}, where $K_{22}(v)$ is a bounded symmetric positive linear operator on V for every $v \in V$. The stochastic force is governed by a cylindrical Wiener process $(W_t)_{t\geq 0}$ with values in V. Moreover, $D\Phi$ is the gradient of a potential $\Phi : U \to (-\infty, \infty]$ and Q_1, as well as Q_2, are the covariance operators of two centered non-degenerate Gaussian measures μ_1 and μ_2 on U and V, respectively. The partial derivatives $\partial_{d_i}K_{22}$, $i \in \mathbb{N}$, are taken with respect to the orthonormal Basis $(e_i)_{i\in\mathbb{N}}$, diagonalizing the covariance operator Q_2. Regularity assumptions for Φ and suitable invariance properties for the coefficients ensure that L^Φ is well-defined on $\mathcal{F}C_b^\infty(B_W)$.
The degeneracy of the Equation (1.2) corresponds to the degeneracy of L^Φ in the sense that the second order differential operator in the definition of L^Φ only acts in the second component. Hereafter, $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is also referred to as the infinite dimensional Langevin operator.
To analyze the equation (1.2), we focus on the infinite dimensional Langevin operator. The following enumeration summarizes the major achievements of this thesis and outlines our strategy.

- We establish the essential m-dissipativity of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ on $L^2(W;\mu^\Phi)$, where

  $$\mu^\Phi := e^{-\Phi}\mu_1 \otimes \mu_2.$$

  Consequently, the closure $(L^\Phi, D(L^\Phi))$ of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$.

- We apply abstract hypocoercivity methods to provide and quantify the exponential convergence rate to equilibrium of $(T_t)_{t\geq 0}$.

- We show the existence of a right process with infinite life-time, whose transition semigroup is associated with $(T_t)_{t\geq 0}$. The process solves the martingale problem for $(L^\Phi, D(L^\Phi))$ with respect to the equilibrium measure. Further, we give sufficient conditions, ensuring the existence of a μ^Φ-invariant Hunt process with infinite lifetime and weakly continuous paths, providing a stochastically and analytically weak solution to (1.2). Hypocoercivity of $(T_t)_{t\geq 0}$ translates into L^2-exponential ergodicity of the processes.

- We formulate degenerate second order in time stochastic reaction-diffusion and Cahn-Hilliard equations in the context of infinite dimensional Langevin equations with multiplicative noise. They are analyzed based on the previous results.

Essential m-dissipativity

Under mild regularity assumption on the potential Φ and K_{22}, as well as reasonable block invariance properties of the coefficients, compare Section 5.1, we derive an integration by parts formula with respect to measures of type μ^Φ. This results in the dissipativity of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ on $L^2(W;\mu^\Phi)$. The essential m-dissipativity of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ then follows, in view of the famous Lumer-Phillips theorem, if there is some $\lambda \in (0,\infty)$ such that

$$(\lambda - L^\Phi)(\mathcal{F}C_b^\infty(B_W)) \quad \text{is dense in} \quad L^2(W;\mu^\Phi). \tag{1.3}$$

The difficulty to establish this so-called dense range condition, is governed by the degeneracy of the operator, the infinite dimensionality of the problem and the regularity properties of the potential and the coefficients.
We use two different approaches to establish (1.3). In the first approach, we provide a first order L^2 regularity estimate for the solution f of $\lambda f - Lf = g$, $g \in \mathcal{F}C_b^\infty(B_W)$, where $L := L^0$. The existence of sufficient regular solutions to this equation is due to [Ale23]. Afterwards, we employ a perturbation argument to establish (1.3), where we assume that the gradient of Φ is bounded. Actually, we only need existence and boundedness of the gradient in a weaker sense described in Assumption $\mathrm{Bd}_\theta(\Phi)$. These strong essential m-dissipativity results have already been published in [EG22] and [BEG23], whereby the second mentioned reference deals with multiplicative noise.
For the second approach, we assume that $V \ni v \mapsto K_{22}(v)e_i \in V$ is two times continuously differentiable with bounded derivatives up to order two for all $i \in \mathbb{N}$. The potential Φ comes with an approximating double sequence $(\Phi_n^m)_{n,m\in\mathbb{N}}$ and a constant $\lambda \in (0,\infty)$ independent of $m, n \in \mathbb{N}$, such that for each $g \in \mathcal{F}C_b^\infty(B_W)$ there exists a function $f_{n,m} \in \mathcal{F}C_b^3(B_W)$ with

$$\lambda f_{n,m} - L^{\Phi_n^m} f_{n,m} = g.$$

By means of Assumption $\mathrm{App}(\Phi)$, we then establish an $L^4(W;\mu^{\Phi_n^m})$ first order regularity estimate for $f_{n,m}$, independent of $m, n \in \mathbb{N}$, which allows us to show (1.3). The strategy to derive such $L^4(W;\mu^{\Phi_n^m})$ first order regularity estimates is inspired by the considerations in [DL05], where m-dissipativity for degenerate elliptic operators corresponding to finite dimensional degenerate stochastic differential equations with additive noise has been established.
Other results, concerning the essential m-dissipativity of such degenerate Kolmogorov backwards operators, are, to our knowledge, only available in finite dimensional situations, compare e.g. [DL05; GS16; BG23]. However, concerning the essential m-dissipativity and

even the essential self-adjointness of (perturbed) infinite dimensional Ornstein-Uhlenbeck operators, there are strong results, compare e.g. [DT00; DA14; LD15; LP20; BF22]. The essential m-dissipativity for generators associated to (singular) dissipative stochastic equations in Hilbert space was derived in [DR02; Big22].
The essential m-dissipativity of the infinite dimensional Langevin operator yields the existence of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ generated by the closure $(L^\Phi, D(L^\Phi))$ of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$. Consequently, for each $u_0 \in D(L^\Phi)$, the function $[0,\infty) \ni t \mapsto u(t) := T_t u_0 \in D(L^\Phi)$ is the unique classical solution to the abstract Cauchy Problem

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) = L^\Phi u(t), \quad u(0) = u_0,$$

compare [Are+01, Theorem 3.1.12]. We highlight that our approaches to establish essential m-dissipativity of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ are applicable, if the variable diffusion coefficient K_{22} and the potential Φ are not C^∞-smooth. In this sense, our results complement those of [Bog+15], where existence and uniqueness (for C^∞-smooth coefficients) of solutions for a large class of highly degenerate Fokker-Planck-Kolmogorov equations for probability measures on infinite dimensional spaces has been established.
Essential m-dissipativity of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ not only ensures the existence and the uniqueness of the solution to the abstract Cauchy problem but also plays a crucial role in the context of the abstract hypocoercivity methods that we introduce next.

Hypocoercivity

In [Vil06], Villani developed hypocoercivity methods to provide and quantify convergence rates to equilibrium of non-coercive and, in this sense, degenerate diffusive equations. Inspired by the ideas of Villani and influenced by the methods from Hérau in [Hér05], Dolbeault, Mouhot and Schmeiser developed an abstract hypocoercivity concept, compare [DMS15]. They studied exponential convergence to equilibrium of non-coercive evolution equations in a general Hilbert space setting, by means of entropy methods. The core of their idea is the construction of an entropy functional, which is equivalent to the underlying Hilbert space norm and for which the operator, governing the evolution equation, is coercive. Although their results were fundamental and opened the door to studying a wide range of degenerate evolution equations, the authors failed to address domain issues that commonly arise when dealing with unbounded linear operators. Grothaus and Stilgenbauer's significant contribution, in [GS14] and [GS16], was to incorporate these concepts into a Kolmogorov backwards setting while also taking domain issues into account.
It is the rigorous method of Grothaus and Stilgenbauer, in the following called the abstract hypocoercivity method, we use to establish hypocoercivity, with explicitly computable constants determining the exponential speed of convergence to equilibrium, for the infinite dimensional Langevin operator.
We contribute by formulating assumptions on the coefficients and the potential, determining L^Φ, under which the abstract hypocoercivity method is applicable. To check the sufficiency of these assumptions, we derive a general Poincaré inequality for measures of type μ^Φ and a second order regularity estimate for infinite dimensional (perturbed) Ornstein-Uhlenbeck operators with possibly unbounded diffusion coefficient. We cover situations in which Φ is not convex and the gradient of Φ merely exists in a suitable Sobolev space, compare

Chapter 6. The results we present in this context are based on the already published articles [EG23] and [BEG23], where hypocoercivity for infinite dimensional Langevin dynamics with additive and multiplicative noise, respectively, has been established. We highlight that the results from [EG23] were utilized to demonstrate exponential convergence to equilibrium of the infinite dimensional Boomerang Sampler, compare [DB23].
Other approaches to provide explicit exponential convergence rates for infinite dimensional degenerate dynamics can be found in [Zim17] and [Wan17]. Using coupling methods, the author of [Zim17] derived explicit contraction rates for degenerate and infinite dimensional diffusions in an L^1 Wasserstein distance. In [Wan17], the author established L^2-L^4 hypercontractivity (stronger notion than hypocoercivity) for stochastic Hamiltonian systems. The result is obtained by means of a dimension free Harnack inequality and coupling methods. However, both dynamics considered in [Zim17] and [Wan17] are less general in terms of the allowed coefficients describing the dynamic and are limited to additive noise. Moreover, the assumptions on the non-linearity in [Zim17] and [Wan17] translate to Lipschitz continuity of $D\Phi$, which we do not need for our approach.
In finite dimensions, the available literature is significantly more extensive. Using generalized Dirichlet forms and martingale techniques, the ergodicity and the rate of convergence to equilibrium of finite dimensional Langevin dynamics with weakly differentiable and singular potentials were studied in [GS15]. Singular but C^∞-smooth potentials were treated also by Lyapunov techniques, see [Cam+21a], [Cam+21b] and [BGH21]. To study finite dimensional Langevin dynamics with multiplicative noise, a Lyapunov function approach was used in [Lim01]. Hypocoercivity of Langevin dynamics on abstract smooth manifolds was established in [GM22]. Probabilistic coupling methods were applied in [EGZ19], to derive quantitative contraction rates for finite dimensional Langevin dynamics in L^1 Wasserstein distance. In the context of sub-exponential convergence rates to equilibrium of finite dimensional Langevin dynamics, the aforementioned results have been generalized in [GW19] and with multiplicative noise in [BG23].

The associated process

We establish that $(T_t)_{t\geq 0}$ is sub-Markovian and conservative. The analytic potential theoretic results, described by Beznea, Boboc and Röckner in [BBR06b], guarantee existence of a right process whose transition semigroup is associated with $(T_t)_{t\geq 0}$. The process provides a martingale solution for the infinite dimensional Langevin operator L^Φ with respect to the equilibrium measure. This approach is applicable without imposing any further conditions on the potential and the coefficients. However, the state space of the process is not necessarily W, instead a reasonable larger Lusin topological space. The identification of $(T_t)_{t\geq 0}$ and the transition semigroup enables us to derive L^2-exponential ergodicity of the process, provided $(T_t)_{t\geq 0}$ is hypocoercive.
By equipping the state space W with the weak topology and in presence of the assumptions described in Chapter 7, we are able to apply the abstract resolvent methods from [BBR06a] to construct a μ^Φ-invariant Hunt process $\mathbf{M}$ with weakly continuous paths and whose transition semigroup is associated with $(T_t)_{t\geq 0}$. The existence of a suitable core and a μ^Φ-nest of weakly compact sets is essential for this approach. By calculating the quadratic covariation of an huge class of martingales, induced by $\mathbf{M}$ and L^Φ by means of the martingale problem, we construct a cylindrical Wiener process with values in V. Afterwards, we show

that **M** provides a stochastically and analytically weak solution to (1.2). Parts of these results have already been published in [BEG23] and [EG23].
For finite dimensional Langevin equations, similar approaches were used e.g. in [CG08; Con11; BG22], where the associated processes were constructed by using the theory of generalized Dirichlet forms, compare [Sta99] and [Tru00; Tru03].

Applications

We apply the aforementioned results in the context of stochastic reaction-diffusion and Cahn-Hilliard equations. Our analysis is based on [DA14], where Lunardi and Da Prato studied maximal Sobolev regularity and m-dissipativity for second order elliptic partial differential equations in infinite dimensions to analyze stochastic reaction-diffusion and Cahn-Hilliard equations. In addition, we point out [ES09], where the existence of invariant measures and the m-dissipativity in an L^1-setting for non-degenerate stochastic Cahn-Hilliard type equations were discussed.
We translate the classic non-degenerate reaction-diffusion and Cahn-Hilliard type equations into our framework of infinite-dimensional Langevin equations with multiplicative noise. For both U and V, in the context of equation (1.2), we choose $L^2((0,1);\mathrm{d}\xi)$ for the reaction-diffusion equation and the dual space of the Sobolev space $\{x \in W^{1,2}(0,1) \mid \int_0^1 x(\xi)\,\mathrm{d}\xi = 0\}$ for the Cahn-Hilliard equation. In principle, we consider potentials of type

$$\Phi : U \to (-\infty, \infty], \quad \text{with} \quad \Phi(u) := \int_0^1 \phi(u(\xi))\,\mathrm{d}\xi, \quad u \in U,$$

where we assume that ϕ is continuously differentiable with at most polynomial growth. Perturbations of Φ, by bounded functions with bounded first and second order derivatives, are possible. Depending on the chosen approach to establish essential m-dissipativity of the associated generators, we assume boundedness of ϕ', compare Section 8.1 and Section 8.2 or more smoothness and structure of ϕ, compare Section 8.3. Moreover, the operators K_{22}, K_{21}, K_{12}, Q_1 and Q_2 are determined by suitable powers of minus the second order derivative with Dirichlet boundary condition in the reaction-diffusion and by powers of the fourth order derivative with zero boundary condition for the first and third order derivative in the Cahn-Hilliard setting. The assumptions to obtain the essential m-dissipativity of the corresponding Langevin operators, hypocoercivity of the semigroups and associated martingale respectively stochastically and analytically weak solutions with weakly continuous paths, are translated into inequalities in terms of the powers determining the coefficient operators. These examples emphasize the strength of our results, as they are more general than the degenerate semi-linear infinite dimensional stochastic differential equations discussed in [Wan17].

1.1 Outline

In Chapter 2, we discuss basic functional analytic and probabilistic notions and results, including strongly continuous contraction semigroups and their generators. Further, we consider strongly continuous sub-Markovian semigroups and resolvents, as well as their stochastic counterparts. We include basic potential theoretic notions and state related process construction theorems. In Chapter 3, we construct Sobolev spaces with respect to infinite dimensional Gaussian measures (with densities) by means of a corresponding integration by parts formula. Classical (perturbed) infinite dimensional Ornstein-Uhlenbeck semigroups are considered and an important Poincaré inequality for measures of type μ^Φ is derived. Furthermore, we include a brief introduction into the theory of (cylindrical) Wiener processes in Hilbert spaces and corresponding stochastic integration. Then, we recall the abstract Hilbert space hypocoercivity method from Stilgenbauer and Grothaus in Chapter 4. In Chapter 5, we show the essential m-dissipativity of the infinite dimensional Langevin operator, using the different approaches presented above. Chapter 6 deals with the application of the method from Chapter 4 and establishes the hypocoercivity of the semigroup generated by the infinite dimensional Langevin operator. Therefore, Chapter 6 includes the analysis of infinite dimensional Ornstein Uhlenbeck operators (perturbed by the gradient of a potential) in terms of essential self-adjointness results and second order regularity estimates. Using the general process construction theorems stated in Section 2.3.3, we first construct a right process solving the martingale problem for L^Φ with respect to the equilibrium measure. By imposing additional assumptions, we construct a μ^Φ-invariant Hunt process $\mathbf{M}$ with weakly continuous paths and infinite life-time, providing a stochastically and analytically weak solution to (1.2) in Chapter 7. Via the identification of the semigroup generated by L^Φ and the transition semigroups of the processes, we derive an L^2-exponential ergodicity result for the processes. Finally, in Chapter 8, we focus on degenerate second order in time stochastic reaction-diffusion and Cahn-Hilliard type equations with multiplicative noise. We emphasize how the results from above can be applied.

1.2 Notation

The natural, rational, real and complex numbers are denoted by $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. For each complex number z, $\Re(z)$ and $\Im(z)$ denote its real and imaginary part, respectively. Moreover, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each element $x \in \mathbb{R}^d$, $d \in \mathbb{N}$, we denote by $|x|$ the euclidean norm of x, which is induced by the euclidean inner product, in the following denoted by $\langle \cdot, \cdot \rangle$.

The set of all linear bounded operators, mapping from a normed space X to a normed space Y, is denoted by $\mathcal{L}(X;Y)$. If $X = Y$, we simply write $\mathcal{L}(X)$ for $\mathcal{L}(X;X)$. The space of bounded linear operators from X to Y is again a normed space, by equipping it with the operator norm $\|T\|_{\mathcal{L}(X;Y)} := \sup_{\|x\|_X \le 1} \|Tx\|_Y$, $T \in \mathcal{L}(X;Y)$. If $\mathcal{D}$ is a linear subspace of X and $L : \mathcal{D} \to Y$ is linear, we say that $(L, \mathcal{D})$ is a linear operator from X to Y. For $X = Y$, we abbreviate and say $(L, \mathcal{D})$ is a linear operator on X. The kernel of a linear operator $(L, \mathcal{D})$ is denoted by $\ker(L)$.

Assume X is $\mathbb{K}$-vector space with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $E \subseteq X$. Then, $\operatorname{span}\{E\}$ denotes the $\mathbb{K}$-vector space of all linear combinations of elements from E. If X is equipped with an inner product, $E^\perp$ is defined as the set of all elements in X orthogonal to all elements from E.

For a non-empty set E and a subset $F \subseteq \{f : E \to \mathbb{R}\}$, we define $F^+ := \{f^+ \mid f \in F\}$, where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$ denote the positive and negative part of a real valued function f, respectively. Moreover, we set $F_b := \{f \in F \mid f \text{ bounded}\}$. The signum of a real valued function $f : E \to \mathbb{R}$ is defined by

$$\text{sign} f(x) := \begin{cases} \frac{f(x)}{|f(x)|} & \text{if} \quad f(x) \neq 0 \\ 0 & \text{else.} \end{cases}$$

For each bounded function $f : E \to \mathbb{R}$, we set $\|f\|_\infty := \sup_{x\in E}|f(x)|$. This notation is generalized to the case where $\mathbb{R}$ is replaced by a subset of a normed space $(X, \|\cdot\|_X)$.

Suppose $(E, \mathcal{T})$ is a topological space, then the corresponding Borel σ-algebra is denoted by $\mathscr{B}_\mathcal{T}(E)$. We omit the subscript $\mathcal{T}$, if the topology considered on E is clear from the context. If $(\tilde{E}, \tilde{\mathcal{T}})$ is another topological space, we denote by $C(E; \tilde{E})$ the set of continuous maps from $(E, \mathcal{T})$ to $(\tilde{E}, \tilde{\mathcal{T}})$. If $\tilde{E} = \mathbb{R}$, we sometimes use the abbreviation $C(E)$. If not explicitly stated otherwise, $\mathbb{R}^d$, $d \in \mathbb{N}$, is always equipped with the topology generated by the open sets with respect to the euclidean norm.

Let $E \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be an open set and $k \in \mathbb{N} \cup \{\infty\}$. Then, $C^k(E)$ denotes the spaces of k-times differentiable real-valued functions on E with values in $\mathbb{R}$. $C_c^k(E)$ is defined as the subset of those functions in $C^k(E)$ having compact support in E. The support of a function $f : E \to \mathbb{R}$ is denoted by $\text{supp}(f)$ and defined as the closure of $\{x \in E \mid f(x) \neq 0\}$ in $\mathbb{R}^d$. For $1 \leq i, j \leq d$ and a sufficient regular function $f : E \to \mathbb{R}$, $\partial_i f$ denotes the partial derivative of f in the i-th component. Moreover, we set $\partial_{ij} f = \partial_i \partial_j f$ and $\partial_i^2 f = \partial_{ii} f$. For higher order derivatives, we make use of the multi-index notation and set $\partial^\alpha f = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d} f$ for each multiindex $\alpha \in \mathbb{N}_0^d$.

Suppose $(E, \mathcal{F}, \mu)$ is a measure space and F a collection of functions from E into another measurable space $(\tilde{E}, \tilde{\mathcal{F}})$. Then, $\sigma(F)$ denotes the σ-algebra generated by F. For a measurable function $f : (E, \mathcal{F}) \to (\tilde{E}, \tilde{\mathcal{F}})$, we denote by $\mu \circ f^{-1}$ the image measure of μ under f. Typically, the Lebesgue measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$, $n \in \mathbb{N}$, is denoted by $\mathrm{d}x$.
Let $p \in [1, \infty]$. If $p < \infty$, then $L^p(E; \mu)$ denotes the space of equivalence classes of real valued p-integrable functions with respect to the measure μ. For $p = \infty$, $L^\infty(E; \mu)$ is defined as the space of equivalence classes of real valued μ-essentially bounded functions. The corresponding norms are denoted by $\|\cdot\|_{L^p(\mu)}$. For $\tau \in (0, \infty)$, we set

$$\ell^\tau(\mathbb{N}) := \left\{ (a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \mid \sum_{n=1}^\infty |a_n|^\tau < \infty \right\}.$$

We define $\|(a_n)_{n\in\mathbb{N}}\|_{\ell^\tau} := (\sum_{n=1}^\infty |a_n|^\tau)^{\frac{1}{\tau}}$ for all $(a_n)_{n\in\mathbb{N}} \in \ell^\tau(\mathbb{N})$. Then, $(\ell^\tau(\mathbb{N}), \|\cdot\|_{\ell^\tau})$ is a Banach space, if $\tau \in [1, \infty)$.

1.3 Publications

Parts of the work presented in this thesis have been published in the following articles.

[BEG23] Alexander Bertram, Benedikt Eisenhuth, and Martin Grothaus. *Hypocoercivity for infinite-dimensional non-linear degenerate stochastic differential equations with multiplicative noise*. 2023. eprint: https://arxiv.org/abs/2306.13402.

[EG22] Benedikt Eisenhuth and Martin Grothaus. "Essential m-dissipativity for Possibly Degenerate Generators of Infinite-dimensional Diffusion Processes". In: *Integral Equations and Operator Theory* 94.3 (July 2022). ISSN: 1420-8989. DOI: 10.1007/s00020-022-02707-2.

[EG23] Benedikt Eisenhuth and Martin Grothaus. "Hypocoercivity for non-linear infinite-dimensional degenerate stochastic differential equations". In: *Stochastics and Partial Differential Equations: Analysis and Computations* 12.2 (June 2023), pp. 984–1020. ISSN: 2194-041X. DOI: 10.1007/s40072-023-00299-5.

To be precise, Section 5.1.1, Chapter 6, Chapter 7, Section 8.1 and Section 8.2 are based on [EG22], [EG23] and [BEG23].

2

Functional analytic and probabilistic background

In this chapter we develop and describe the functional analytic and probabilistic background needed in this thesis. The definitions and results are formulated in a way that allows to apply them as easily as possible in the course of this thesis.

2.1 Basics

2.1.1 Linear operators

The definitions and results in this section are rather basic and well known, therefore stated without proof. We refer the interested reader to the textbooks [RS81] and [Rud91].
Below, $(X, (\cdot,\cdot)_X)$ and $(Y, (\cdot,\cdot)_Y)$ are two Hilbert spaces, both over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The norms induced by $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$ are denoted by $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.

Definition 2.1. Let $(L, D(L))$ be a linear operator from $(X, (\cdot,\cdot)_X)$ to $(Y, (\cdot,\cdot)_Y)$.

(i) Assume that $(L, D(L))$ is densely defined, then the unique linear operator $(L^*, D(L^*))$ from $(Y, (\cdot,\cdot)_Y)$ to $(X, (\cdot,\cdot)_X)$ is defined via

$$\begin{aligned} D(L^*) &:= \{y \in Y \mid \text{ there is } z_y \in X \text{ such that } (Lx, y)_Y = (x, z_y)_X \text{ for all } x \in D(L)\} \\ L^* y &:= z_y. \end{aligned}$$

$(L^*, D(L^*))$ is called the adjoint of $(L, D(L))$.

(ii) Suppose $X = Y$ and $(L, D(L))$ is densely defined. If $(L^*, D(L^*))$ is an extension of $(L, D(L))$, i.e. $D(L) \subseteq D(L^*)$ with $Lx = L^*x$ for all $x \in D(L)$, we say that $(L, D(L))$ is symmetric. $(L, D(L))$ is called antisymmetric, if it is extended by $(-L^*, D(L^*))$. A symmetric operator $(L, D(L))$ with $D(L^*) \subseteq D(L)$ is called self-adjoint.

(iii) Suppose $X = Y$. $(L, D(L))$ is said to be positive semidefinite if $(Lx, x)_X \in \mathbb{R}$ and $(Lx, x)_X \geq 0$ for all $x \in D(L)$. The operator is called positive, if $(Lx, x)_X > 0$ for all $x \in D(L) \setminus \{0\}$. Moreover, $(L, D(L))$ is said to be negative (semidefinite) if $(-L, D(L))$ is positive (semidefinite).

(iv) $(L, D(L))$ is called closed if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x, y \in D(L)$ with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} Lx_n = y$, it follows $y = Lx$.

(v) We say that $(L, D(L))$ is closable if it has a closed extension. Every closable operator has a smallest closed extension, which we denote by $(\overline{L}, D(\overline{L}))$. Equivalently, $(L, D(L))$ is closable if for each sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} Lx_n = y$ for some $y \in Y$, it follows $y = 0$.

Lemma 2.2. *Suppose $T \in \mathcal{L}(Y; X)$ and $(L, D(L))$ is a densely defined linear operator from $(X, (\cdot,\cdot)_X)$ to $(Y, (\cdot,\cdot)_Y)$. Then it holds*

(i) $(L^*, D(L^*))$ is closed. If $D(L^*)$ is dense in Y, then $(L, D(L))$ is closable with $(\overline{L}, D(\overline{L})) = (L^{**}, D(L^{**}))$.

(ii) $T^* \in \mathcal{L}(X; Y)$ with $\|T\|_{\mathcal{L}(Y;X)} = \|T^*\|_{\mathcal{L}(X;Y)}$.

(iii) If $(L, D(L))$ is closed, then $D(L^*)$ is dense in Y and by (i) we directly get $(L, D(L)) = (L^{**}, D(L^{**}))$.

(iv) If $(L, D(L))$ is closed, then also $(LT, D(LT))$ with domain

$$D(LT) := \{y \in Y \mid Ty \in D(L)\}.$$

(v) $(TL, D(L))$ is not necessarily closed, however

$$((TL)^*, D((TL)^*)) = (L^*T^*, D(L^*T^*)).$$

In the following definition we introduce important subsets of $\mathcal{L}(X; Y)$, where we additionally assume that $(X, (\cdot,\cdot)_X)$ is a real separable Hilbert space.

Definition 2.3. Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of $(X, (\cdot,\cdot)_X)$. Define

$$\begin{aligned}
\mathcal{L}^+(X) &:= \{T \in \mathcal{L}(X) \mid T = T^* \quad \text{and} \quad (Tx, x)_X \geq 0 \quad \text{for all} \quad x \in X\}, \\
\mathcal{L}^+_{>0}(X) &:= \{T \in \mathcal{L}(X) \mid T = T^* \quad \text{and} \quad (Tx, x)_X > 0 \quad \text{for all} \quad x \in X \setminus \{0\}\}, \\
\mathcal{L}^+_1(X) &:= \left\{T \in \mathcal{L}^+(X) \mid \operatorname{tr}[T] := \sum_{i=1}^{\infty} (Te_i, e_i)_X < \infty\right\} \quad \text{and} \\
\mathcal{L}_2(X; Y) &:= \left\{T \in \mathcal{L}(X; Y) \mid \|T\|^2_{\mathcal{L}_2(X;Y)} := \sum_{i=1}^{\infty} \|Te_i\|^2_Y < \infty\right\}.
\end{aligned}$$

Hence, $\mathcal{L}^+(X)$ is the set of all bounded symmetric positive semidefinite operators on X and $\mathcal{L}^+_{>0}(X)$ denotes the subset of all positive operators. $\mathcal{L}^+_1(X)$ is the set of all trace class operators on X and $\mathcal{L}_2(X; Y)$ is the set of all Hilbert-Schmidt operators on X with values in Y. It is easy to see that the definition of $\mathcal{L}^+_1(X)$ and $\mathcal{L}_2(X)$ is independent of the chosen orthonormal basis. Moreover, $\mathcal{L}_2(X; Y)$, equipped with the inner product $(\cdot,\cdot)_{\mathcal{L}_2(X;Y)}$ defined by

$$(S, T)_{\mathcal{L}_2(X;Y)} := \sum_{i=1}^{\infty} (Se_i, Te_i)_Y, \quad S, T \in \mathcal{L}_2(X; Y),$$

is a real Hilbert space and $\|\cdot\|_{\mathcal{L}_2(X;Y)}$ is induced by $(\cdot,\cdot)_{\mathcal{L}_2(X;Y)}$. In the following, we use the abbreviation $\mathcal{L}_2(X)$ for $\mathcal{L}_2(X; X)$. Finally, note that $\mathcal{L}^+_1(X)$ and $\mathcal{L}_2(X)$ are subsets of the space of compact operators from X to X.

Remark 2.4. *Suppose $T \in \mathcal{L}(X)$ and that there is an orthonormal basis of eigenvectors $(e_i)_{i\in\mathbb{N}}$ of T with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. It is easy to see that $(\lambda_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{N})$, if and only if $T \in \mathcal{L}_2(X)$.*
Moreover, if $T \in \mathcal{L}^+(X)$, then $T \in \mathcal{L}_1^+(X)$ if and only if $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N})$ and $\lambda_i \geq 0$ for all $i \in \mathbb{N}$.

Definition 2.5. Let T be an injective operator in $\mathcal{L}_1^+(X)$. By the spectral theorem for symmetric compact operators, we know that there exists an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of eigenvectors of T with corresponding (positive) eigenvalues $(\lambda_i)_{i\in\mathbb{N}} \in \ell^1(\mathbb{N})$. For $\tau \in \mathbb{R}$ we define

$$D(T^\tau) := \left\{ x \in X \mid \sum_{i=1}^{\infty} \lambda_i^{2\tau} (x, e_i)_X^2 < \infty \right\} \quad \text{and} \quad T^\tau x := \sum_{i=1}^{\infty} \lambda_i^\tau (x, e_i)_X, \quad x \in D(T^\tau).$$

Remark 2.6. *In the setting of Definition 2.5, it is obvious that $D(T^\tau) = X$ for all $\tau \in [0, \infty)$. Moreover, $D(T^\tau)$ is closed for all $\tau \in \mathbb{R}$ and*

$$\operatorname{span}\{e_1, e_2, ...\} \subseteq D(T^\tau) \quad \textit{with} \quad T^\tau e_i = \lambda_i^\tau e_i, \quad i \in \mathbb{N}.$$

For our further considerations, it is important to mention that $(T^\tau, D(T^\tau))$ is self-adjoint for every $\tau \in \mathbb{R}$.

2.1.2 Derivatives

In this section $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are real normed vector spaces. In addition, we fix an open set $U \subseteq X$ and a function $f : U \to Y$. All the results, concerning Gâteaux and Fréchet differentiability are contained in [AP95, Chapter 1].

Definition 2.7. (i) We call f Gâteaux differentiable at $x \in U$, if there is a linear operator $T \in \mathcal{L}(X; Y)$ such that

$$\partial_v f(x) := \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h} = Tv \quad \text{for all} \quad v \in X. \tag{2.1}$$

In this case we define the Gâteaux derivative $Df(x) \in \mathcal{L}(X; Y)$ of f in x by

$$Df(x)(v) := \partial_v f(x) = Tv, \quad v \in X.$$

(ii) f is called Fréchet differentiable at $x \in U$, if the convergence in (2.1) is uniform with respect to $v \in X$ with $\|v\|_X \leq 1$. This is equivalent to the existence of $T \in \mathcal{L}(X; Y)$, such that

$$f(x + v) = f(x) + Tv + r(v), \quad \text{for all} \quad v \in X \quad \text{with} \quad \lim_{\|v\|_X \to 0} \frac{r(v)}{\|v\|_X} = 0. \tag{2.2}$$

In this case we define the Fréchet derivative $df(x) \in \mathcal{L}(X; Y)$ of f in x by

$$df(x)(v) := Tv, \quad v \in X.$$

(iii) The function f is called Gâteaux (Fréchet) differentiable on U, if f is Gâteaux (Fréchet) differentiable at every $x \in U$. If f is Gâteaux (Fréchet) differentiable on U, we define the Gâteaux and Fréchet derivative of f by

$$Df : U \to \mathcal{L}(X;Y),\ x \mapsto Df(x) \quad \text{and} \quad df : U \to \mathcal{L}(X;Y),\ x \mapsto df(x).$$

In the following theorem we summarize some important results about Gâteaux and Fréchet differentiable functions.

Theorem 2.8. *(i) Let f be Gâteaux differentiable and $x_0, x_1 \in U$ be fixed and assume that $I := \{x_0 + \lambda x_1 \mid \lambda \in [0,1]\} \subseteq U$. Then*

$$\|f(x_0 + x_1) - f(x_0)\|_Y \leq \sup_{\xi \in I} \|Df(\xi)\|_{\mathcal{L}(X;Y)} \|x_1\|_X.$$

(ii) If f is Gâteaux differentiable and $Df : U \to \mathcal{L}(X;Y)$ is continuous, then f is Fréchet differentiable. In this case we call f continuously differentiable.

Of course the Gâteaux and Fréchet derivative are linear objects and there is a natural generalization of the classical chain rule for the composition for appropriate Fréchet differentiable functions. Next, we introduce n-times Gâteaux and Fréchet differentiable functions.

Definition 2.9. Let $f : U \to Y$ be Gâteaux (Fréchet) differentiable. If Df (df) : $U \to \mathcal{L}(X;Y)$ is Gâteaux (Fréchet) differentiable, we call f two times Gâteaux (Fréchet) differentiable and denote the second order Gâteaux (Fréchet) derivative by

$$D^2 f\ (d^2 f) : U \to \mathcal{L}(X; \mathcal{L}(X;Y)).$$

Inductively, this construction generalizes to higher order Gâteaux (Fréchet) derivatives. The space of (bounded) n-times Fréchet differentiable functions, $n \in \mathbb{N}$, from U to Y with continuous (and bounded) derivatives up to order n, is denoted by $C^n(U;Y)$ $(C_b^n(U;Y))$. Since for $f \in C^1(U;Y)$ the Gâteaux and Fréchet derivative coincide, we sometimes just call f continuously differentiable and Df its derivative.

Remark 2.10. *Assume $(X, (\cdot,\cdot)_X)$ is a real separable Hilbert space and $f : U \to \mathbb{R}$ is Gâteaux differentiable. For $u \in U$ the Riesz representation theorem allows us to identify $Df(u) \in \mathcal{L}(X;\mathbb{R})$ with the gradient of $\nabla f(u) \in X$, i.e. with the unique element such that*

$$Df(u)(v) = (\nabla f(u), v)_X \quad \textit{for all} \quad v \in X.$$

Analogously, for a two times Gâteaux differentiable function $f : U \to \mathbb{R}$, we identify $D^2 f(u) \in \mathcal{L}(X; \mathcal{L}(X;\mathbb{R}))$ with the unique element $\nabla^2 f(u) \in \mathcal{L}(X)$ such that

$$D^2 f(u)(v)(w) = (\nabla^2 f(u)(v), w)_X \quad \textit{for all} \quad v, w \in X.$$

We end this section with the introduction of the Moreau-Yosida approximation, which provides a useful approximation scheme. We quickly state some results about subdifferential functions before. Suppose $(X, (\cdot,\cdot)_X)$ is a real separable Hilbert space and $\Phi : X \to \mathbb{R} \cup \{\infty\}$

is convex, bounded from below, lower semi-continuous and not identically to ∞, then for each $x \in X$, the set

$$\partial\Phi(x) := \{y \in X \mid \text{for all } z \in X \text{ it holds } (z-x, y)_X + \Phi(x) \leq \Phi(z)\},$$

denotes the subdifferential of Φ in x. For each $x \in X$, where $\Phi(x) = \infty$ we set $D_0\Phi(x) := \infty$. For all $x \in X$ with $\Phi(x) \neq \infty$ the set $\partial\Phi(x)$ is closed and convex, compare [BC17, Proposition 16.4]. In particular, for such x it is reasonable to define $D_0\Phi(x)$ as the element in $\partial\Phi(x)$ with minimal norm if $\partial\Phi(x) \neq \emptyset$ and ∞ otherwise.

Example 2.11. Let $(X, (\cdot,\cdot)_X)$ be a real separable Hilbert space and suppose $\Phi : X \to \mathbb{R} \cup \{\infty\}$ is not identical to ∞, convex, bounded from below and lower semicontinuous. For such functions, the so-called Moreau-Yosida approximation Φ_t, $t > 0$, is defined by

$$\Phi_t : X \to \mathbb{R}, \quad \Phi_t(y) = \inf_{x \in X} \left\{ \Phi(x) + \frac{\|y-x\|_X^2}{2t} \right\}.$$

One can show that for all $t > 0$, Φ_t is convex and Fréchet differentiable with

(i) $-\infty < \inf_{y \in X} \Phi(y) \leq \Phi_t(x) \leq \Phi(x)$ for all $x \in X$.

(ii) $\lim_{t \to 0} \Phi_t(x) = \Phi(x)$ for all $x \in X$.

(iii) $D\Phi_t$ is Lipschitz continuous and for all $x \in X$ with $\partial\Phi(x) \neq \emptyset$, $\|D\Phi_t(x)\|_X$ converges monotonically to $\|D_0\Phi(x)\|_X$ with

$$\|D\Phi_t(x) - D_0\Phi(x)\|_X^2 \leq \|D_0\Phi(x)\|_X^2 - \|D\Phi_t(x)\|_X^2.$$

A proof of these statements, except the convergence result of in Item (iii), is given in [BC17]. The statement in Item (iii), is shown in [Bré73, Chapter 2].

In Section 3.2, where we discuss weaker notions of differentiability, we come back to this example.

2.2 Operator Semigroups

This section introduces the concepts and provides the results we need for our applications in the context of strongly continuous (contraction) semigroups and their corresponding resolvents and generators. We state the famous semigroup generating theorems of Hille-Yosida and Lumer-Phillips and focus on essential m-dissipative operators. These fundamental results can be found in almost every introduction book into the topic of strongly continuous semigroups, such as e.g. [EN00] and [Paz83].

Moreover, we introduce the concept of strongly continuous sub-Markovian semigroups, as they are the link to the probabilistic interpretation of the dynamic described by the semigroup. Unless stated otherwise, all the results are taken from [Ebe99].

The last subsection deals with basic potential theoretic notions and results in an $L^p(E;\mu)$ setting, which can be found e.g. in [MR92] (for $p = 2$) and the article [BBR06a]. These notions and results are necessary to formulate Theorem 2.70, which is an applicable method to construct a μ-standard right process (compare Definition 2.66) associated to a generator of a sub-Markovian strongly continuous contraction semigroup.

Throughout this section, $(X, \|\cdot\|_X)$ denotes a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $X' := \mathcal{L}(X;\mathbb{K})$ its continuous dual space.

2.2.1 Strongly continuous (contraction) semigroups and their generators

Definition 2.12. A family of linear operators $(T_t)_{t\geq 0}$ in $\mathcal{L}(X)$ satisfying the conditions

(S1) $T_0 = \mathrm{Id}$,

(S2) $T_{s+t} = T_s T_t$ for $s, t \geq 0$ and

(S3) $\lim_{t\to 0} \|T_t x - x\|_X = 0$ for all $x \in X$,

is called a strongly continuous semigroup (s.c.s.) of bounded linear operators on X. If additionally

(S4) $\|T_t\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$,

then $(T_t)_{t\geq 0}$ is called a strongly continuous contraction semigroup (s.c.c.s.).

Example 2.13. For a linear operator $L \in \mathcal{L}(X)$ it is easy to see that the mapping $[0,\infty) \ni t \mapsto T_t := e^{Lt} := \sum_{k=0}^{\infty} \frac{(Lt)^k}{k!} \in \mathcal{L}(X)$ defines a s.c.s. and that

$$\lim_{t\to 0} \frac{T_t x - x}{t} = Lx.$$

One can even show that the limit above exists with respect to the operator topology induced by $\|\cdot\|_{\mathcal{L}(X)}$.
Inspired by this example we define the generator of a general s.c.s..

Definition 2.14. Let $(T_t)_{t\geq 0}$ be a s.c.s. on X. The linear operator $(L, D(L))$ defined by

$$Lx := \lim_{t\to 0} \frac{T_t x - x}{t}, \quad x \in D(L) := \left\{ x \in X \mid \lim_{t\to 0} \frac{T_t x - x}{t} \text{ exists in } X \right\},$$

is called the generator of $(T_t)_{t\geq 0}$. Moreover, we say that $(T_t)_{t\geq 0}$ is generated by $(L, D(L))$.

The next lemma shows that a s.c.s. is uniquely determined by its generator. Furthermore, we collect some important facts about s.c.s..

Lemma 2.15. *Let $(T_t)_{t\geq 0}$ be a s.c.s. on X with generator $(L, D(L))$. Then the following statements are valid.*

(i) For all $x \in X$ and $t \in (0,\infty)$ the Riemann integral $\int_0^t T_s x \,\mathrm{d}s$ is an element of $D(L)$ with $L\int_0^t T_s x \,\mathrm{d}s = T_t x - x$. Moreover, $D(L)$ is dense in X.

(ii) For all $x \in D(L)$ and $t \in [0,\infty)$ it holds $T_t A x = A T_t x$ as well as $T_t x - x = \int_0^t T_s L x \,\mathrm{d}s$.

(iii) If $(S_t)_{t\geq 0}$ is another s.c.s. with generator $(L, D(L))$, then $T_t = S_t$ for all $t \in [0,\infty)$.

(iv) If $(T_t)_{t\geq 0}$ is a s.c.c.s., then $(0,\infty) \subseteq \rho(L)$ and for all $\alpha \in (0,\infty)$

$$(\alpha - L)^{-1} = \int_0^\infty e^{-\alpha t} T_t \,\mathrm{d}t \quad \textit{and} \quad \|\alpha(\alpha - L)^{-1}\|_{\mathcal{L}(X)} \leq 1. \tag{2.3}$$

In particular, $(L, D(L))$ is a closed operator ($\rho(L) \neq \emptyset$).

It is natural to ask under which condition a densely defined closed operator $(L, D(L))$ is the generator of a s.c.s.. The answer is given in the famous Hille-Yosida theorem. As we focus only on s.c.c.s. in the upcoming considerations we don't state the theorem in its full generality.

Theorem 2.16 (Hille-Yosida)**.** *A linear operator $(L, D(L))$ on X is the generator of a s.c.c.s. $(T_t)_{t\geq 0}$ on X if and only if*

(G1) $(L, D(L))$ *is closed,*

(G2) $(L, D(L))$ *is densely defined and*

(G3) $(0,\infty) \subseteq \rho(L)$ *and* $\|\alpha(\alpha - L)^{-1}\|_{\mathcal{L}(X)} \leq 1$ *for all* $\alpha \in (0,\infty)$.

Note that in this case, the bounded linear operator $(\alpha - L)^{-1}$ is given via the Laplace-transform of $(T_t)_{t\geq 0}$ defined in (2.3).

Next, we introduce the so-called strongly continuous contraction resolvents. They provide a new approach to characterize s.c.c.s. and their generators.

Definition 2.17. A family $(R_\alpha)_{\alpha>0}$ of bounded linear operators on X is called a strongly continuous contraction resolvent (s.c.c.r.), if

(R1) $\lim_{\alpha\to\infty} \alpha R_\alpha x = x$ for all $x \in X$,

(R2) $\|\alpha R_\alpha\|_{\mathcal{L}(X)} \leq 1$ for all $\alpha \in (0,\infty)$ and

(R3) $R_\alpha - R_\beta = (\beta - \alpha) R_\alpha R_\beta$ for all $\alpha, \beta \in (0,\infty)$.

Lemma 2.18. *Let $(R_\alpha)_{\alpha>0}$ be a s.c.c.r. on X, then there is a unique linear operator $(L, D(L))$ on X such that $(0,\infty) \subseteq \rho(L)$ and $R_\alpha = (\alpha - L)^{-1}$ for all $\alpha \in (0,\infty)$. $(L, D(L))$ is closed, densely defined and is called the generator of $(R_\alpha)_{\alpha>0}$.*
On the other hand, let $(L, D(L))$ be a densely defined operator with $(0,\infty) \subseteq \rho(L)$. Set $R_\alpha := (\alpha - L)^{-1}$ for each $\alpha \in (0,\infty)$. If $\|\alpha R_\alpha\|_{\mathcal{L}(X)} \leq 1$ for all $\alpha \in (0,\infty)$, then $(R_\alpha)_{\alpha>0}$ is a s.c.c.r..

Corollary 2.19. *A densely defined linear operator $(L, D(L))$ on X generates a s.c.c.s. $(T_t)_{t\geq 0}$ on X if and only if it generates a s.c.c.r. $(R_\alpha)_{\alpha>0}$ on X and in that case (2.3) holds true.*

2.2.2 Essential m-dissipativity

The Hille-Yosida theorem characterizes generators of s.c.c.s. completely, nevertheless, it is rather difficult to apply in concrete situations. In this section we introduce the concept of essential m-dissipativity and state the Lumer-Phillips theorem. Both are central tools for our analysis of infinite dimensional degenerate Langevin operators.

Definition 2.20. Let $x \in X$. By the Hahn-Banach theorem the duality set $F(x) \subseteq X'$ defined by

$$F(x) := \left\{ x' \in X' \mid \; x'(x) = \|x\|_X^2 = \|x'\|_{X'} \right\}$$

is nonempty.

Example 2.21. If $(X, (\cdot,\cdot)_X)$ is a Hilbert space and $x \in X$, then $F(x) = \{(\cdot, x)_X\}$. Indeed, for $x = 0$ this is trivial. So suppose $x \in X \setminus \{0\}$ and without loss of generality assume $\|x\|_X = 1$. Obviously, $(\cdot, x)_X \in F(x)$. Now let $\tilde{x}' \in F(x)$, which can be uniquely identified using the Riesz isomorphism, with some element $\tilde{x}$, via $\tilde{x}' = (\cdot, \tilde{x})_X$. Then, by the Cauchy-Schwarz inequality

$$1 = (x, \tilde{x})_X \leq \|x\|_X \|\tilde{x}\|_X \leq 1$$

and thus equality in the Cauchy-Schwarz inequality holds. Therefore, $x = \lambda\tilde{x}$ for some $\lambda \in \mathbb{K}$. But since $1 = (x, \tilde{x})_X = \lambda(\tilde{x}, \tilde{x})_X = \lambda$ we get $x = \tilde{x}$ and the statement is shown.

Definition 2.22. A linear operator $(L, D(L))$ is called dissipative, if for each $x \in D(L)$ there is some $x' \in F(x)$ with

$$\Re(x'(Lx)) \leq 0.$$

This is equivalent to

$$\|(\alpha - L)x\|_X \geq \alpha\|x\|_X, \quad \text{for all} \quad x \in D(L).$$

Remark 2.23. *If X is a Hilbert space with inner product $(\cdot,\cdot)_X$, we can use Example 2.21 to get an easy characterization of dissipative operators. Indeed, $(L, D(L))$ is dissipative if and only if $\Re((Lx, x)_X) \leq 0$ for all $x \in D(L)$.*

Lemma 2.24. *Let $(L, D(L))$ be a dissipative linear operator on X.*

(i) If $(L, D(L))$ is densely defined, then it is closable and the closure $(\overline{L}, D(\overline{L}))$ is dissipative as well. Furthermore, $\overline{(\alpha - L)(D(L))} = (\alpha - \overline{L})(D(\overline{L}))$ for all $\alpha \in (0, \infty)$.

(ii) If $(\alpha_0 - L)(D(L)) = X$ for some $\alpha_0 \in (0, \infty)$, then $(L, D(L))$ does not posses a proper dissipative extension. Moreover, $(0, \infty) \subseteq \rho(L)$ and $\|\alpha(\alpha - L)^{-1}\|_{\mathcal{L}(X)} \leq 1$. This implies that $(L, D(L))$ is closed and $(\alpha - L)(D(L)) = X$ for all $\alpha \in (0, \infty)$.

The maximality described above is summarized in the next definition.

Definition 2.25. Let $(L, D(L))$ be a densely linear operator on X.

(i) $(L, D(L))$ is called m-dissipative, if it dissipative and $(\alpha - L)(D(L)) = X$ for one (hence all) $\alpha \in (0, \infty)$.

(ii) $(L, D(L))$ is called essentially m-dissipative, if it is dissipative and $(\alpha - L)(D(L))$ is dense in X for one (hence all) $\alpha \in (0, \infty)$.

The tools and terminology we collected above is enough to state the Lumer-Phillips theorem.

Theorem 2.26 (Lumer-Phillips)**.** *Let $(L, D(L))$ be a linear operator on the Banach space X. Then $(L, D(L))$ is the generator of a s.c.c.s. on X if and only if it is densely defined and m-dissipative. In that case, it follows that $\Re(x'(Lx)) \leq 0$ for all $x \in D(L)$ and all $x' \in F(x)$.*

Corollary 2.27. *Let $(L, D(L))$ be an essentially m-dissipative operator on X, then its closure generates a s.c.c.s. on X.*

Example 2.28. (i) Assume that $(B, D(B))$ is a dissipative and self-adjoint operator on some Hilbert space $(X, (\cdot,\cdot)_X)$. It is well known, that for each $\alpha \in (0,\infty)$

$$X = \overline{(\alpha - B)(D(B))} \oplus \ker(\alpha - B^*).$$

Using that $(B, D(B)) = (B^*, D(B^*))$, the fact that self-adjoint operators are closed and that $\alpha - B$ is injective for dissipative $(B, D(B))$, as well as Item (i) from Lemma 2.24, we obtain $X = \overline{(\alpha - B)(D(B))} = (\alpha - B)(D(B))$.

(ii) Set $D(\Delta) := W^{1,2}_0(0,1) \cap W^{2,2}(0,1)$, where $W^{1,2}_0(0,1) \subseteq L^2((0,1), \mathrm{d}\xi)$ is the first order Sobolev space with Dirichlet boundary conditions and $W^{2,2}(0,1) \subseteq L^2((0,1), \mathrm{d}\xi)$ is the second order Sobolev space on the unit interval $(0,1)$, respectively. The linear operator $(\Delta, D(\Delta))$, defined by $\Delta f = f'' \in L^2((0,1), \mathrm{d}\xi)$, $f \in D(\Delta)$, is dissipative by the integration by parts formula and self-adjoint. Consequently, $(\Delta, D(\Delta))$ is the generator of a strongly continuous contraction semigroup.

(iii) In Section 3.2.3 we discuss the so-called infinite dimensional (perturbed) Ornstein Uhlenbeck semigroups.

We end this section with two useful results, dealing with self-adjoint generators.

Lemma 2.29. *Suppose that $(T_t)_{t\geq 0}$ is a s.c.c.s. with generator $(L, D(L))$ on a Hilbert space $(X, (\cdot,\cdot)_X)$. Then T_t is self-adjoint for all $t \in [0,\infty)$ if and only if $(L, D(L))$ is self-adjoint.*

Theorem 2.30. *Let $(L, D(L))$ be a densely defined, symmetric and negative semidefinite operator on a Hilbert space $(X, (\cdot,\cdot)_X)$. Then $(L, D(L))$ is self-adjoint if and only if it is m-dissipative.*

2.2.3 Sub-Markovian semigroups and resolvents

We fix a a measure space $(E, \mathcal{F}, \mu)$ and $p \in [1,\infty)$. For simplicity we assume that μ is a probability measure, even though most of the results in this section are also valid for σ-finite measures. If we consider elements $f, g \in L^p(E;\mu)$, i.e. equivalence classes of functions, we write $f \leq g$ if there are corresponding representatives fulfilling this inequality. Analogously, we define the relations $\geq, <, >$ and $=$ on $L^p(E;\mu)$. By f^+, f^- and $|f|$ we denote the equivalence class of the positive part, negative part and the absolute value of a representative of f, respectively.

Definition 2.31. Let $(L, D(L))$ be a closed densely defined linear operator on $L^p(E;\mu)$ and $(T, \mathcal{D})$ be a linear operator on $L^p(E;\mu)$.

(i) $(L, D(L))$ is called a Dirichlet operator if

$$\int_E Lf((f-1)^+)^{p-1}\,\mathrm{d}\mu \leq 0$$

for all $f \in D(L)$. Here we use the convention $0^0 = 0$.

(ii) $(T, \mathcal{D})$ is called positive preserving if for all $f \in \mathcal{D}$ with $0 \leq f$ it holds $0 \leq Tf$.

(iii) $(T,\mathcal{D})$ is called sub-Markovian if for all $f \in \mathcal{D}$ with $f \leq 1$ it holds $Tf \leq 1$.

(iv) A s.c.c.s. $(T_t)_{t\geq 0}$ (s.c.c.r. $(R_\alpha)_{\alpha>0}$) is called positive preserving or sub-Markovian if T_t (αR_α) is positive preserving or sub-Markovian for all $t \in [0,\infty)$ $(\alpha \in (0,\infty))$, respectively.

Lemma 2.32. *(i) If $(T, L^p(E;\mu))$ is positive preserving, then $T \in \mathcal{L}(L^p(E;\mu))$.*

(ii) If $(T,\mathcal{D})$ is sub-Markovian, then $(T,\mathcal{D})$ is positive preserving.

For a positive preserving operator $(T, L^1(E;\mu))$, a Cauchy-Schwarz type inequality holds true. Even though this is a well-known fact, we could not find a proof in the literature. Therefore, it is included in the next lemma.

Lemma 2.33. *Assume $(T, L^1(E;\mu))$ is positive preserving. Then for all $f, g \in L^2(E;\mu)$ it holds*

$$T(fg) \leq (Tf^2)^{\frac{1}{2}}(Tg^2)^{\frac{1}{2}}.$$

In particular, if $T1 = 1$ we obtain

$$T(f) \leq (T|f|^2)^{\frac{1}{2}}.$$

Proof. For $f, g \in L^2(E;\mu)$ we get μ-a.e. for each $\lambda \in \mathbb{R}$ (first for $\lambda \in \mathbb{Q}$ and then by density for all $\lambda \in \mathbb{R}$)

$$0 \leq T((f-\lambda g)^2) = Tf^2 - 2\lambda T(fg) + \lambda^2 Tg^2.$$

As this is a quadratic polynomial in $\lambda \in \mathbb{R}$ with at most one root, it holds μ-a.e.

$$(-2T(fg))^2 - 4Tg^2Tf^2 \leq 0.$$

This ends the proof. □

Lemma 2.34. *Let $(L, D(L))$ be the generator of a s.c.c.s. $(T_t)_{t\geq 0}$ with corresponding s.c.c.r. $(R_\alpha)_{\alpha>0}$ on $L^p(E;\mu)$. Then the following statements are equivalent*

(i) $(T_t)_{t\geq 0}$ is sub-Markovian.

(ii) $(R_\alpha)_{\alpha>0}$ is sub-Markovian.

(iii) $(L, D(L))$ is a Dirichlet operator.

Next, we deal with μ-invariance as well as conservative operators and semigroups. As previously mentioned in the introduction of Section 2.2 these concepts have a translation into the context of associated stochastic processes, compare Section 2.3.1.

Definition 2.35. We say that the measure μ is invariant for the linear operator $(L,\mathcal{D})$ on $L^p(E;\mu)$, if

$$\int_E Lf \,\mathrm{d}\mu = 0 \quad \text{for all non-negative} \quad f \in \mathcal{D}, \tag{2.4}$$

and sub-invariant if (2.4) holds with "$\leq$".

Definition 2.36. Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on $L^p(E;\mu)$.

(i) The measure μ is called invariant for $(T_t)_{t\geq 0}$, if

$$\int_E T_t f \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu \quad \text{for all} \quad f \in L^p(E;\mu).$$

(ii) If $T_t 1 = 1$ holds for all $t \in [0,\infty)$, then $(T_t)_{t\geq 0}$ is said to be conservative.

Lemma 2.37. *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on $L^p(E;\mu)$ with generator $(L, D(L))$.*

(i) $(T_t)_{t\geq 0}$ is conservative/sub-Markovian if and only if $(T_t^*)_{t\geq 0}$ is conservative/sub-Markovian.

(ii) $(T_t)_{t\geq 0}$ is conservative if and only if $1 \in D(L)$ with $L1 = 0$. In this case we also call $(L, D(L))$ conservative.

(iii) Let $\mathcal{D}$ be a core for $(L, D(L))$. Then μ is invariant for $(L, D(L))$ if and only if it is invariant for $(L, \mathcal{D})$.

(iv) Let $(T_t)_{t\geq 0}$ be a s.c.c.s. and sub-Markovian. Then conservativity and μ invariance are equivalent.

Below we introduce the concept of abstract diffusion operators. Lemma 2.39 shows that this is a useful concept to establish sub-Markovianity of s.c.c.s..

Definition 2.38. Let $(L, \mathcal{D})$ be a densely defined linear operator on $L^p(E;\mu)$. The Carré du champ operator of $(L, \mathcal{D})$ is the bilinear operator $\Gamma : \mathcal{D} \times \mathcal{D} \to L^0(E;\mu)$ defined by

$$\Gamma(f,g) := \frac{1}{2}\left(L(fg) - fL(g) - gL(f)\right)$$

Here $L^0(E;\mu)$ refers to the space of all μ-classes of functions on E. $(L, \mathcal{D})$ is called an abstract diffusion operator, if and only if

(i) For any $m \in \mathbb{N}$, $f_1, \dots, f_m \in \mathcal{D}$ and $\varphi \in C^\infty(\mathbb{R}^m;\mathbb{R})$ with $\varphi(0) = 0$, it holds that $\varphi(f_1, \dots, f_m) \in \mathcal{D}$ and

$$L\varphi(f_1,\dots,f_m) = \sum_{k=1}^m \partial_k \varphi(f_1,\dots,f_m) L(f_k) + \sum_{k,l=1}^m \partial_l \partial_k \varphi(f_1,\dots,f_m)\Gamma(f_k, f_l).$$

(ii) $\Gamma(f,f) \geq 0$ for all $f \in \mathcal{D}$.

Lemma 2.39. *Let $(L, \mathcal{D})$ be an abstract diffusion operator on $L^p(E;\mu)$.*

(i) Suppose μ is sub-invariant for $(L-\alpha, \mathcal{D})$, then $(L, \mathcal{D})$ is dissipative. If additionally the closure $(L, D(L))$ of $(L, \mathcal{D})$ generates a s.c.c.s. $(T_t)_{t\geq 0}$, then $(T_t)_{t\geq 0}$ is sub-Markovian and $(L, D(L))$ is a Dirichlet operator.

(ii) *Suppose $(L, \mathcal{D})$ is a densely defined and essentially m-dissipative operator on $L^p(E;\mu)$ and such that μ is sub-invariant for $(L-\alpha, \mathcal{D})$. Then $(L, \mathcal{D})$ is essentially m-dissipative on $L^q(E;\mu)$, for all $q \in [1,p]$.*

Denote by $(L^{(p)}, D(L^{(p)}))$ and $(L^{(q)}, D(L^{(q)}))$ the closures of $(L, \mathcal{D})$ in $L^p(E;\mu)$ and $L^q(E;\mu)$, respectively. Further, let $(T_t^{(p)})_{t\geq 0}$ and $(R_\alpha^{L^{(p)}})_{\alpha>0}$, as well as $(T_t^{(q)})_{t\geq 0}$ and $(R_\alpha^{L^{(q)}})_{\alpha>0}$ be the corresponding strongly continuous contraction semigroups and resolvents, then $L^{(p)}f = L^{(q)}f$ for all $f \in D(L^{(p)})$ and $T_t^{(p)}f = T_t^{(q)}f$, as well as $R_\alpha^{L^{(p)}}f = R_\alpha^{L^{(q)}}f$ for all $f \in L^p(E;\mu)$, $t \in [0,\infty)$ and $\alpha \in (0,\infty)$.

Proof. (i) This is [Ebe99, Lemma 1.8] and [Ebe99, Lemma 1.9].

(ii) By the Hölder inequality we have $\|f\|_{L^q(\mu)} \leq \|f\|_{L^p(\mu)}$ for all $f \in L^p(E;\mu)$. Therefore, $\mathcal{D} \subseteq L^q(E;\mu)$ as well as

$$L^p(E;\mu) = \overline{(\alpha - L)(\mathcal{D})}^{L^p(\mu)} \subseteq \overline{(\alpha - L)(\mathcal{D})}^{L^q(\mu)},$$

for each $\alpha \in (0,\infty)$. Since $\overline{L^p(E;\mu)}^{L^q(\mu)} = L^q(E;\mu)$, the dense range condition follows.

Similar we can show that $\mathcal{D}$ is dense in $L^q(E;\mu)$. Moreover, $(L, \mathcal{D})$ is an abstract diffusion operator on $L^q(E;\mu)$. By means of Item (i) we conclude that $(L, \mathcal{D})$ is dissipative on $L^q(E;\mu)$. Hence, we know that $(L, \mathcal{D})$ is essentially m-dissipative on $L^q(E;\mu)$.

Since $L^{(p)}f = L^{(q)}f$ for all $f \in \mathcal{D} \subseteq L^p(E;\mu)$, we obtain $L^{(p)}f = L^{(q)}f$ for all $f \in D(L^{(p)})$ by the density of $\mathcal{D}$ in $D(L^{(p)})$ with respect to the $L^{(p)}$ graph norm. Using that semigroups and resolvents are uniquely determined by their generators, the proof is finished.

□

2.2.4 Analytic potential theory of sub-Markovian resolvents

In this section $(E, \mathcal{T})$ denotes a Lusin space, i.e., $(E, \mathcal{T})$ is a topological Hausdorff space such that E carries a finer topology $\mathcal{T}'$, such that $(E, \mathcal{T}')$ is a polish space. We always consider E with the Borel σ-algebra $\mathscr{B}_\mathcal{T}(E)$ induced by $\mathcal{T}$. Further, let μ be a probability measure on $(E, \mathscr{B}_\mathcal{T}(E))$. We fix $p \in [1,\infty)$ and assume that $(T_t)_{t\geq 0}$ is a strongly continuous sub-Markovian semigroup on $L^p(E;\mu)$ with generator $(L, D(L))$ and corresponding strongly continuous sub-Markovian resolvent $(R_\alpha)_{\alpha>0}$.

Definition 2.40. Let $\alpha \in (0,\infty)$ be given. An element $f \in L^p(E;\mu)$ is called α-excessive (with respect to $(R_\alpha)_{\alpha>0}$) if $\beta R_{\beta+\alpha} f \leq f$ for all $\beta > 0$. By $\mathcal{F}_\alpha$ we denote the set of all α-excessive elements.

The proof of the following properties can be found in [Non20, Proposition 2.33.]. Some of them are also contained in [MR92, Chapter III] for the special case $p = 2$.

Proposition 2.41. *Let $f \in L^p(E;\mu)$ and $\alpha \in (0,\infty)$. Then the following statements hold true.*

(i) f is α-excessive if and only if $e^{-\alpha t}T_t f \leq f$ for all $t \in [0, \infty)$.

(ii) If f is α-excessive, then $f \geq 0$ and if additionally $f > 0$ also $R_\alpha f > 0$ for all $\alpha \in (0, \infty)$.

(iii) If $f \in D(L)$, then f is α-excessive if and only if $(\alpha - L)f \geq 0$. Moreover, for $f \in D(L) \cap \mathcal{F}_\alpha$ there exists $g \geq 0$ such that $f = R_\alpha g$.

(iv) If f and g are α-excessive, then also $\min\{f, g\}$.

(v) If $f \geq 0$, then $R_\alpha f$ is α-excessive.

(vi) If $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}_\alpha$ is an increasing sequence such that $\sup_{n\in\mathbb{N}} f_n \in L^p(E;\mu)$, then also $\sup_{n\in\mathbb{N}} f_n \in \mathcal{F}_\alpha$.

Definition 2.42. For an element $f \in L^p(E;\mu)$ we define the level set $\mathcal{L}_f$ of f by

$$\mathcal{L}_f := \{g \in L^p(E;\mu) \mid g \geq f\}.$$

Lemma 2.43. *Let $f \in L^p(E;\mu)$ and $\alpha \in (0, \infty)$. Assume that $\mathcal{L}_f \cap \mathcal{F}_\alpha \neq \emptyset$. Then, there exists an element $B_\alpha \in \mathcal{L}_f \cap \mathcal{F}_\alpha$ such that $B_\alpha f \leq g$ for all $g \in \mathcal{L}_f \cap \mathcal{F}_\alpha$. $B_\alpha f$ is unique and is called the α-reduced element of f (with respect to $(R_\alpha)_{\alpha>0}$).*

Definition 2.44. Let $(E_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed sets in $(E, \mathcal{T})$.

(i) $(E_n)_{n\in\mathbb{N}}$ is said to be a μ-nest (with respect to $(R_\alpha)_{\alpha>0}$) if $\lim_{n\to\infty} B_1(\mathbb{1}_{E_n^c} f) = 0$ in $L^p(E;\mu)$ for all $f \in D(L) \cap \mathcal{F}_1$.

(ii) A function $f : E \to \mathbb{R}$ is said to be μ-quasi continuous, if there exists a μ-nest such that $f\,|_{E_n}$ is $\mathcal{T}$-continuous for all $n \in \mathbb{N}$.

(iii) A set F is called μ-exceptional if $F \subseteq \bigcap_{n\in\mathbb{N}}(E \setminus E_n)$ for some μ-nest $(E_n)_{n\in\mathbb{N}}$. A property of points in E holds μ-quasi everywhere (abbreviated μ-q.e.) if it holds outside some μ-exceptional set.

The definitions above play a prominent role in Section 2.3.3. Moreover, note that every μ-exceptional set is clearly μ-negligible.
To conclude the section we state a very useful result to verify that an increasing sequence of closed sets in $(E, \mathcal{T})$ is a μ-nest. For its proof, we refer to the explanations in [BBR06b, Section 3], because the result is stated without proof in [BBR06a]. In the special case $p = 1$ there is also a proof in [Non20, Proposition 2.38.], using techniques from the analytic potential theory of generalized Dirichlet forms.

Proposition 2.45. *[BBR06a, Remark 2.2] Suppose that $(E_n)_{n\in\mathbb{N}}$ is an increasing sequence of closed sets in $(E, \mathcal{T})$ and $f_0 \in L^p(E;\mu)$ with $f_0 > 0$. Then, $(E_n)_{n\in\mathbb{N}}$ is a μ-nest with respect to $(R_\alpha)_{\alpha>0}$ if and only if $\lim_{n\to\infty} B_1(\mathbb{1}_{E_n^c} R_1 f_0) = 0$ in $L^p(E;\mu)$.*

The proposition above is applied in Chapter 7.

2.3 Stochastic processes

We start this section by introducing probability laws on the space of càdlàg and continuous paths with values in a Polish space F. Basic concepts such as (infinite) life-time, associated sub-Markovian s.c.c.s., invariance of a measure μ and martingale problems corresponding to a linear operator $(L, \mathcal{D})$ on $L^p(F;\mu)$ are discussed. All the results in this first subsection are well-known, but sometimes reformulated to make them better accessible for our particular applications. We refer to [EK86] and [Con11] for a more detailed analysis of such laws.
Afterwards, different types of Markov processes (compare Definition 2.56 and Definition 2.66) with a general topological Hausdorff space as state space are defined. Moreover, we investigate connections to the probability laws considered before. Most of the presented results can be found in any textbook about time continuous Markov processes, e.g. [MR92], [Sha90], [EK86] and [KB68].
In the last part of this section we state general process construction theorems. They are the central tool to construct solutions for the infinite dimensional Langevin equation, compare Chapter 7.
We assume that the reader is familiar with aspects of martingale theory as presented e.g. in [KS98, Chapter 1].

Without further mentioning, we use the following construction for a topological space $(E, \mathcal{T})$. Let $\Delta \notin E$ be an isolated point of E. In the context of stochastic processes, Δ is also referred to as a cemetery.
Define $E_\Delta := E \cup \{\Delta\}$ and $\mathcal{T}_\Delta := \mathcal{T} \cup \{A \cup \{\Delta\} \mid A \in \mathcal{T}\}$, then $(E_\Delta, \mathcal{T}_\Delta)$ is a topological space and $\mathscr{B}_{\mathcal{T}_\Delta}(E_\Delta) = \mathscr{B}_{\mathcal{T}}(E) \cup \{A \cup \{\Delta\} \mid A \in \mathscr{B}_{\mathcal{T}}(E)\}$. Moreover, every measure μ on $(E, \mathscr{B}_{\mathcal{T}}(E))$ is extended to a measure on $(E_\Delta, \mathscr{B}_{\mathcal{T}_\Delta}(E_\Delta))$ by setting $\mu(\{\Delta\}) = 0$. Similarly, every function $f : E_\Delta \to \mathbb{R}$ is extended to E_Δ via $f(\{\Delta\}) = 0$. Finally, note that every function $f : E \to \mathbb{R}$ is $\mathscr{B}_{\mathcal{T}}(E)/\mathscr{B}(\mathbb{R})$ measurable if and only if its extension to E_Δ is $\mathscr{B}_{\mathcal{T}_\Delta}(E_\Delta)/\mathscr{B}(\mathbb{R})$ measurable.

2.3.1 Probability laws, path spaces and the martingale problem

In this section, we assume that E is a Polish space and that its Borel σ-algebra is generated by the continuous real-valued functions on E, i.e. $\mathscr{B}(E) = \sigma(C(E;\mathbb{R}))$. It is easy to see that E_Δ is again a Polish space. Let $F \in \{E, E_\Delta\}$, then the set $D([0,\infty);F)$ denotes the space of càdlàg paths, i.e.

$$D([0,\infty);F) := \{[0,\infty) \ni t \mapsto Z_t \in F \text{ is right cont. with left limits for all } t \in (0,\infty)\}.$$

Generic paths in $D([0,\infty);F)$ are denoted by $(Z_t)_{t\geq 0}$. In [EK86, Theorem 3.5.6] it is explained that $D([0,\infty);E_\Delta)$, equipped with the Skorokhod topology, is again a Polish space. We denote the corresponding Borel σ-algebra by $\mathscr{B}^D$. The set $C([0,\infty);E_\Delta)$ of continuous paths is closed in $D([0,\infty);E_\Delta)$ with respect to Skorokhod topology. Even though the topology of uniform convergence on compact sets on $D([0,\infty);E_\Delta)$ is weaker than the Skorokhod topology, they coincide on $C([0,\infty);E_\Delta)$. The corresponding σ-algebra on $C([0,\infty);E_\Delta)$ is denoted by $\mathscr{B}^C$. Moreover, $D([0,\infty);E) \in \mathscr{B}^D$ (identified with the subset of $D([0,\infty);E_\Delta)$ of paths which stay away from Δ on compact intervals), as well as $C([0,\infty);E) \in \mathscr{B}^C$ (seen as the subset of $C([0,\infty);E_\Delta)$ that do not hit Δ). Therefore, every measure $\mathbb{P}$ on $(D([0,\infty);E_\Delta),\mathscr{B}^D)$ can be restricted to $C([0,\infty);E)$, $C([0,\infty);E_\Delta)$ or $D([0,\infty);E)$.

Definition 2.46. The set of zombie paths is denoted by $\mathcal{Z}$ and is defined as the complement of

$$\{(Z_t)_{t\geq 0} \in D([0,\infty);E_\Delta) \mid \ Z_t = \Delta \text{ for any } t\in[0,\infty) \text{ implies } Z_s = \Delta \text{ for } s \geq t\}\,.$$

A probability measure $\mathbb{P}$ on $(D([0,\infty);E_\Delta),\mathscr{B}^D)$ or another path space is called a probability law if $\mathcal{Z}$ is a subset of a $\mathbb{P}$-null set. Finally, the life-time $\zeta : D([0,\infty);E_\Delta) \to [0,\infty]$ is defined via

$$\zeta\left((Z_t)_{t\geq 0}\right) := \inf\left\{t\in[0,\infty) \mid \ Z_t = \Delta\right\}.$$

Remark 2.47. *(i) Suppose $\mathbb{P}$ is the image measure of a probability measure P, defined on a measurable space $(\Omega,\mathcal{A})$, under a measurable map $\Phi : \Omega \to D([0,\infty);E_\Delta)$ with $\Phi(\Omega)\cap\mathcal{Z} = \emptyset$, then $\mathbb{P}$ is probability law, compare [Con11, Remark 2.1.2].*

(ii) Let $\mathbb{P}$ be a probability law on $(D([0,\infty);E_\Delta),\mathscr{B}^D)$. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. For $(Z_t)_{t\geq 0}\in D([0,\infty);E_\Delta)$ and $s\in[0,\infty)$ we identify Z_s with the projection of the path $(Z_t)_{t\geq 0}$ to its state at time s. Without further mentioning, we use this identification also in the course of this chapter. In particular, we can write

$$\mathbb{E}[f(Z_s)] = \int_{D([0,\infty);E_\Delta)} f(Z_s)\,\mathbb{P}(\mathrm{d}(Z_t)_{t\geq 0}),$$

for each sufficient regular $f : E_\Delta \to \mathbb{R}$.

In the following μ is a probability measure on $(E,\mathscr{B}(E))$. Moreover, if h is a probability density function with respect to μ, we denote by $h\mu$ the associated probability measure on $(E,\mathscr{B}(E))$ defined by

$$h\mu(A) := \int_A h\,\mathrm{d}\mu, \quad A\in\mathscr{B}(E).$$

At this point we want to mention, that in all statements below, it would be possible to consider a σ-finite measure μ and a corresponding probability density function h. But to avoid technical difficulties and since we only consider probability measures for our applications, we limit ourselves to the case that $\mu(E) = 1$.

Definition 2.48. Let $\mathbb{P}$ be a probability law on $(D([0,\infty);E_\Delta),\mathscr{B}^D)$.

(i) The initial distribution of $\mathbb{P}$ is defined as the image measure of $\mathbb{P}$ under Z_0.

(ii) Suppose $h\mu$ is the initial distribution of $\mathbb{P}$, where $h\in L^1(E;\mu)^+$ is a probability density with respect to μ. A sub-Markovian strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ is said to be associated with $\mathbb{P}$, if for all $f_1,\dots,f_k\in L^\infty(E;\mu)^+$, $0\leq t_1 < \dots < t_k < \infty$, $k\in\mathbb{N}$ it holds

$$\mathbb{E}\left[\Pi_{1\leq i\leq k} f_i(Z_{t_i})\right] = (h, T_{t_1}(f_1 T_{t_2-t_1}(f_2\dots T_{t_{k-1}-t_{k-2}}(f_{k-1}T_{t_k-t_{k-1}}f_k))))_{L^2(E;\mu)}. \tag{2.5}$$

Note that the right-hand side of (2.5) is indeed well defined, since the sub-Markovian semigroup $(T_t)_{t\geq 0}$ leaves $L^\infty(E;\mu)^+$ invariant.

Definition 2.49. Let $\mathbb{P}$ be a probability law on $(D([0,\infty);E_\Delta),\mathscr{B}^D)$ with initial distribution μ. Further, let $(L,\mathcal{D})$ be a linear operator on $L^p(E;\mu)$. Then $\mathbb{P}$ is said to solve the martingale problem for $(L,\mathcal{D})$ if

(i) for every $f\in\mathcal{D}$ and $t\in[0,\infty)$ it holds $\int_0^\infty |Lf(Z_s)|\,\mathrm{d}s<\infty$ $\mathbb{P}$-a.s. and $f(Z_t)$, as well as $Lf(Z_t)$ are $\mathbb{P}$-a.s. well-defined on $(D([0,\infty);E_\Delta),\mathscr{B}^D)$, i.e. independent of the chosen μ-version of f and Lf, respectively.

(ii) for all $f\in\mathcal{D}$ and $t\in[0,\infty)$, the random variable $M_t^{[f],L}$ defined by

$$M_t^{[f],L}:=f(Z_t)-f(Z_0)-\int_0^t Lf(Z_s)\,\mathrm{d}s$$

is $\mathbb{P}$-integrable and the corresponding process $(M_t^{[f],L})_{t\geq 0}$ is an $(\mathcal{F}_t^0)_{t\geq 0}$-martingale, where $\mathcal{F}_t^0=\sigma(Z_s\mid s\in[0,t])$.

If f is continuous, we can replace $\mathcal{F}_t^0$ by $\mathcal{F}_t^+:=\bigcap_{s>t}\mathcal{F}_s^0$ and its $\mathbb{P}$-completion, compare [Con11, Remark 2.1.7].

Lemma 2.50. *Suppose $p\in[1,\infty)$. Moreover, let $h\in L^{\frac{p}{p-1}}(E;\mu)^+$ be a probability density with respect to μ and $\mathbb{P}$ be a probability law on $(D([0,\infty);E_\Delta),\mathscr{B}^D)$ with initial distribution $h\mu$. If $\mathbb{P}$ is associated with some sub-Markovian strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ with generator $(L,D(L))$ on $L^p(E;\mu)$, then $\mathbb{P}$ solves the martingale problem for $(L,D(L))$. If additionally $f\in D(L)$ with $f^2\in D(L)$ and $Lf\in L^{2p}(E;\mu)$, then*

$$N_t^{[f],L}:=\left(M_t^{[f],L}\right)^2-\int_0^t L(f^2)(Z_s)-(2fLf)(Z_s)\,\mathrm{d}s\quad t\in[0,\infty),$$

also defines an $(\mathcal{F}_t^0)_{t\geq 0}$-martingale.

Proof. As $L^{\frac{p}{p-1}}(E;\mu)^+\subseteq L^1(E;\mu)^+$ this is exactly [Con11, Lemma 2.1.8] for $r=p$. □

Since we assume that μ is a probability measure, we can always choose $h=1$ (i.e. initial distribution μ) in the lemma above.
Next, we state a result, that establishes continuity properties of a process, which is associated to the semigroup generated by the closure of an abstract diffusion operator on $L^2(E;\mu)$. Its proof is based on [DR02, Theorem 6.3] and worked out in [Con11, Lemma 2.1.10 and Corollary 2.1.11.].

Proposition 2.51. *Let $(L,\mathcal{D})$ be an abstract diffusion operator on $L^2(E;\mu)$ and assume that $\mathcal{D}\subseteq C_b(E;\mathbb{R})$. Moreover, suppose $(L,\mathcal{D})$ has an extension which generates a sub-Markovian s.c.c.s. $(T_t)_{t\geq 0}$ for which μ is invariant. Let $\mathbb{P}$ be a probability law on $(D([0,\infty);E_\Delta),\mathscr{B}^D)$ associated with $(T_t)_{t\geq 0}$ with initial distribution μ, then $(f(Z_t))_{t\geq 0}$ is $\mathbb{P}$-a.s. continuous for every $f\in\mathcal{D}$.*
If additionally there is a countable subset $\tilde{\mathcal{D}}\subseteq\mathcal{D}$ separating the points of E, then $(Z_t)_{t\geq 0}$ is $\mathbb{P}$-a.s. continuous on $[0,\zeta)$ and if E is locally compact we further have $\mathbb{P}(C([0,\infty);E_\Delta))=1$.

Definition 2.52. Let $\mathbb{P}$ be a probability law on $(D([0,\infty); E_\Delta), \mathscr{B}^D)$ with initial distribution μ.

(i) $\mathbb{P}$ is said to be conservative, if $\zeta = \infty$ holds $\mathbb{P}$-a.s..

(ii) The measure μ is called invariant for $\mathbb{P}$, if the image measure of $\mathbb{P}$ under Z_t is equal to μ for all $t \in [0,\infty)$.

The lemma below shows that there is a one to one correspondence between conservativity and μ-invariance of the law and its associated semigroup.

Lemma 2.53. *Let $\mathbb{P}$ be a probability law on $(D([0,\infty); E_\Delta), \mathscr{B}^D)$ with initial distribution μ and assume it is associated with some sub-Markovian strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^p(E;\mu)$. Then*

(i) $\mathbb{P}$ is conservative if and only if $(T_t)_{t\geq 0}$ is conservative.

(ii) μ is invariant for $\mathbb{P}$ if and only if μ is invariant for $(T_t)_{t\geq 0}$.

Proof. The proof of both items is almost the same as in [Con11, Lemma 2.1.14], we only have to replace $(C([0,\infty); E_\Delta), \mathscr{B}^C)$ with $(D([0,\infty); E_\Delta), \mathscr{B}^D)$. □

2.3.2 Markov processes and transition semigroups

Throughout this section, let E be a topological Hausdorff space and suppose its Borel σ-algebra is generated by the continuous real valued functions on E. Therefore,

$$\mathscr{B}(E) = \sigma(C_b(E;\mathbb{R})) = \sigma(C_b(E;\mathbb{R})^+) \quad \text{and} \quad \mathscr{B}(E_\Delta) = \sigma(C_b(E_\Delta;\mathbb{R})) = \sigma(C_b(E_\Delta;\mathbb{R})^+).$$

Definition 2.54. The collection $\mathbf{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ is called a (time-homogeneous) Markov process with state space E, life-time ξ and corresponding filtration $(\mathcal{M}_t)_{t\geq 0}$ if

(M1) $Z_t : \Omega \to E_\Delta$ is $\mathcal{M}_t/\mathscr{B}(E_\Delta)$ measurable for all $t \in [0,\infty)$ and $Z_t(\omega) = \Delta$ if and only if $t \geq \xi(\omega)$ for all $\omega \in \Omega$. Here, the life-time is defined via

$$\xi : \Omega \to [0,\infty], \quad \xi(\omega) := \inf\{t \in [0,\infty) \mid Z_t(\omega) = \Delta\}.$$

Moreover, for each $\omega \in \Omega$ the map $[0,\infty) \ni t \mapsto Z_t(\omega) \in E_\Delta$ is called a path of $\mathbf{M}$.

(M2) For all $t \in [0,\infty)$ there is a map $\theta_t : \Omega \to \Omega$ such that $Z_s \circ \theta_t = Z_{s+t}$ for all $s \in [0,\infty)$.

(M3) $(P_z)_{z\in E_\Delta}$ is a family of probability measures on $(\Omega, \mathcal{M})$ such that $z \mapsto P_z(B)$ is $\mathscr{B}(E_\Delta)^*/\mathscr{B}(\mathbb{R})$ measurable for all $B \in \mathcal{M}$ and $\mathscr{B}(E_\Delta)/\mathscr{B}(\mathbb{R})$ measurable for all $B \in \sigma(Z_t \mid t \in [0,\infty))$. In addition, $P_\Delta(Z_0 = \Delta) = 1$.

(M4) $\mathbf{M}$ fulfills the Markov property, i.e., for all $B \in \mathscr{B}(E_\Delta)$, $s,t \in [0,\infty)$ and $x \in E_\Delta$ it holds P_z-almost surely

$$P_z(Z_{t+s} \in B \mid \mathcal{M}_t) = P_{Z_t}(Z_s \in B).$$

In (M3), $\mathscr{B}(E_\Delta)^*$ is defined as the σ-algebra of universally measurable sets, i.e.

$$\mathscr{B}(E_\Delta)^* := \bigcap_{\nu \in \mathcal{P}(E_\Delta)} \mathscr{B}(E_\Delta)^\nu,$$

where $\mathcal{P}(E_\Delta)$ denotes the set of all probability measures on $(E_\Delta, \mathscr{B}(E_\Delta))$ and $\mathscr{B}(E_\Delta)^\nu$ the ν-completion of $\mathscr{B}(E_\Delta)$.
For a measure $\nu \in \mathcal{P}(E_\Delta)$, we define the measure P_ν on $(\Omega, \mathcal{M})$ via

$$P_\nu(B) := \int_{E_\Delta} P_z(B)\, \nu(\mathrm{d}z) \quad B \in \mathcal{M}.$$

Above, ν also denotes its unique extension to $\mathscr{B}(E_\Delta)^*$. In literature, P_ν is also called the equilibrium measure.

Remark 2.55. *Suppose $\boldsymbol{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ is a Markov process with state space E and $\nu \in \mathcal{P}(E_\Delta)$.*

(i) If the paths of $\boldsymbol{M}$ are regular, e.g. $(Z_t(\omega))_{t\geq 0} \in D([0,\infty); E_\Delta)$ for P_ν-a.e. $\omega \in \Omega$ then

$$\Lambda := \{\omega \in \Omega \mid \ (Z_t(\omega))_{t\geq 0} \in D([0,\infty); E_\Delta)\} \in \mathcal{M}^{P_\nu}$$

and for an arbitrary element $z \in E$,

$$\Phi : (\Omega, \mathcal{M}^{P_\nu}) \to (D([0,\infty); E_\Delta), \mathscr{B}^D), \quad \Phi(\omega) := \begin{cases} (Z_t(\omega))_{t\geq 0} & \text{if} \quad \omega \in \Lambda \\ z & \text{else} \end{cases}$$

is measurable. Therefore, the image measure of P_ν under Φ defined on $(\Omega, \mathcal{M}^{P_\nu})$ is a probability law in the sense of Definition 2.46. Compare also Remark 2.47.

(ii) $\boldsymbol{M}$ can be equivalently defined by replacing (M4) with one of the two items below

(M4') For all $z \in E_\Delta$, $s,t \in (0,\infty)$, $f \in \mathscr{B}(E_\Delta;\mathbb{R})^+$ and every non-negative $\mathcal{M}_s$-measurable function G we have

$$E_z[f(Z_{t+s})G] = E_z[p_t f(Z_s)G], \quad \text{where} \quad p_t f(\tilde{z}) := \mathbb{E}_{\tilde{z}}[f(Z_t)], \quad \tilde{z} \in E_\Delta.$$

(M4") For all $z \in E_\Delta$, $s,t \in (0,\infty)$, $f \in \mathscr{B}(E_\Delta;\mathbb{R})^+$ we have

$$E_z[f(Z_{t+s}) \mid \mathcal{M}_s] = E_{Z_s}[f(Z_t)].$$

Suppose $\mathbf{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ is a Markov process. In the definition of right processes below, we introduce the so-called strong Markov property. It is, as the name suggests, a strengthened version of the Markov property considered in Item (M4). To state the strong Markov property we introduce for any $(\mathcal{M}_t)_{t\geq 0}$-stopping time τ (i.e. $\tau : \Omega \to [0,\infty]$ and $\{\tau \leq t\} \in \mathcal{M}_t$ for all $t \in [0,\infty)$) the σ-algebra of τ past

$$\mathcal{M}_\tau := \{A \in \mathcal{M} \mid A \cup \{\tau \leq t\} \in \mathcal{M}_t \,\text{for all}\, t \in [0,\infty)\}.$$

Furthermore, we define

$$Z_\tau : \Omega \to E_\Delta, \quad Z_\tau(\omega) := \begin{cases} Z_{\tau(\omega)}(\omega) & \text{if} \quad \tau(\omega) < \infty \\ \Delta & \text{if} \quad \tau(\omega) = \infty. \end{cases}$$

Definition 2.56. A Markov process $\mathbf{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ with state space E and life-time ξ is called a right process if

(M5) $P_z(Z_0 = z) = 1$ for all $z \in E_\Delta$.

(M6) $[0,\infty) \ni t \mapsto Z_t(\omega) \in E_\Delta$ is right-continuous for all $\omega \in \Omega$.

(M7) The filtration $(\mathcal{M}_t)_{t\geq 0}$ is right-continuous. Moreover, $\mathbf{M}$ fulfills the strong Markov property, i.e. for all $\nu \in \mathcal{P}(E_\Delta)$ and any $(\mathcal{M}_t)_{t\geq 0}$-stopping time τ, it holds P_ν-almost surely

$$P_z(Z_{\tau+s} \in B \mid \mathcal{M}_\tau) = P_{Z_\tau}(Z_s \in B),$$

for all $B \in \mathscr{B}(E_\Delta)$ and $s \in [0,\infty)$.

Remark 2.57. *Let $\boldsymbol{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ be a Markov process. For $t \in [0,\infty)$ we set $\mathcal{F}_t^0 = \sigma(Z_s \mid s \in [0,t])$ and $\mathcal{F}_\infty^0 = \sigma(Z_s \mid s \in [0,\infty))$. The natural filtration $(\mathcal{F}_t)_{t\geq 0}$ is defined by*

$$\mathcal{F}_t := \bigcup_{\nu\in\mathcal{P}(E_\Delta)} (\mathcal{F}_t^0)^{P_\nu|_{\mathcal{F}_\infty^0}} \quad t \in [0,\infty).$$

Further, we define $\mathcal{F} := \bigcap_{\nu\in\mathcal{P}(E_\Delta)} (\mathcal{F}_\infty^0)^{P_\nu|_{\mathcal{F}_\infty^0}}$. It can be checked that if $\boldsymbol{M}$ is a right process, then also $\boldsymbol{M}' := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$, where P_z is identified with the unique extension of $P_z \mid_{\mathcal{F}_\infty^0}$ to $\mathcal{F}$. Therefore it is no restriction to only consider right processes such that $\mathcal{M} = \mathcal{F}$ and $\mathcal{M}_t = \mathcal{F}_t$.

Convention 2.58. *Every right process $\boldsymbol{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ is automatically considered with respect to its natural filtration and denoted in the following by $\boldsymbol{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$.*

Definition 2.59. Let $(E, \mathscr{B})$ be a measurable space.

(i) A mapping $\pi : E \times \mathscr{B} \to [0,\infty)$ is called a kernel on $(E, \mathscr{B})$, if $\pi(\cdot, B)$ is $\mathscr{B}$-measurable for every $B \in \mathscr{B}$ and $\pi(z, \cdot)$ is a measure on $(E, \mathscr{B})$ for every $z \in E$. Such a kernel is called sub-Markovian, if $\pi(z, E) \leq 1$ for all $z \in Z$. For $f \in \mathscr{B}(E;\mathbb{R})^+$ we define

$$\pi f(z) := \int_E f(y)\pi(z, dy) \quad z \in E.$$

This definition can easily be extended to $\mathscr{B}$-measurable functions f for which πf^+ and πf^- exists.

(ii) A family $(p_t)_{t\geq 0}$ of sub-Markovian kernels on $(E, \mathscr{B})$ is said to be a sub-Markovian semigroup of kernels on $(E, \mathscr{B})$, if for $t, s \geq 0$ it holds $p_{t+s}f = p_t p_s f$ for all non-negative $\mathscr{B}$-measurable functions f on E. The family $(p_t)_{t\geq 0}$ is said to be measurable, if $(t, z) \mapsto p_t f(z)$ is measurable for all bounded $\mathscr{B}$-measurable functions f.

(iii) A family $(r_\alpha)_{\alpha>0}$ of non-negative kernels on $(E, \mathscr{B})$ is said to be a sub-Markovian resolvent of kernels on (E, B), if $(\alpha r_\alpha)_{\alpha>0}$ is a sub-Markovian family and for all $\alpha, \beta > 0$ and bounded $\mathscr{B}$-measurable functions f it holds $r_\alpha - r_\beta = (\beta - \alpha) r_\alpha r_\beta$.

Example 2.60. Let $(E, \mathscr{B}(E))$ be as introduced in the beginning of this section and assume $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ is a right process with state space E. Such processes naturally induce a measurable sub-Markovian family of kernels $(p_t)_{t\geq 0}$ on $(E, \mathscr{B}(E))$ via

$$p_t(z, B) := P_z(Z_t \in B), \quad t \in [0, \infty), z \in E \text{ and } B \in \mathscr{B}(E).$$

One can show that for each $f \in \mathscr{B}(E; \mathbb{R})$ it holds

$$p_t f(z) = E_z[f(Z_t)], \quad z \in E, \tag{2.6}$$

whenever the integral on the right-hand side in (2.6) exists. In analogy to Section 2.2.1, we can define the corresponding sub-Markovian resolvent of kernels on $(E, \mathscr{B}(E))$ by

$$r_\alpha f(z) := \int_0^\infty e^{-\alpha t} p_t f(z)\,\mathrm{d}t \quad z \in E \text{ and } \alpha \in (0, \infty). \tag{2.7}$$

Note that the definition in (2.7) makes sense for $f \in \mathscr{B}(E; \mathbb{R})^+$ or $f \in \mathscr{B}_b(E; \mathbb{R})$ as $t \mapsto p_t f(z)$ is measurable for each $z \in E$.
It is also possible to construct a corresponding measurable sub-Markovian family of kernels $(p_t^\Delta)_{t\geq 0}$ on $(E_\Delta, \mathscr{B}(E_\Delta))$ such that $p_t^\Delta 1 = 1$ and $p_t^\Delta\,|_E = p_t$. In the following, we omit to distinguish between these.

Definition 2.61. Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ be a right process with state space E.

(i) The objects defined in (2.6) and (2.7) are called the transition semigroup and transition resolvent of $\mathbf{M}$, respectively.

(ii) Let $(T_t)_{t\geq 0}$ be a sub-Markovian s.c.c.s. on $L^p(E; \mu)$ with corresponding s.c.c.r. denoted by $(R_\alpha)_{\alpha>0}$, where $p \in [1, \infty)$ and μ is a probability measure on $(E, \mathscr{B}(E))$. Then M is said to be associated with $(T_t)_{t\geq 0}$ $((R_\alpha)_{\alpha>0})$ if for all $t \in [0, \infty)$ $(\alpha \in (0, \infty))$ and all $f \in L^p(E; \mu) \cap L^\infty(E; \mu)$ with μ-version $\hat{f} \in \mathscr{B}_b(E; \mathbb{R})$, the function $p_t \hat{f}$ $(r_\alpha \hat{f})$ is a μ-version of $T_t f$ $(G_\alpha f)$.

(ii) $f \in \mathscr{B}(E; \mathbb{R})^+$ is called α-excessive (with respect to $(r_\alpha)_{\alpha>0}$) if $\beta r_{\beta+\alpha} u \leq u$ for all $\beta > 0$.

Remark 2.62. *Let $\alpha > 0$ be given. By the same reasoning as in Proposition 2.41, we see that $r_\alpha f$ is α-excessive with respect to $(r_\alpha)_{\alpha>0}$, if $f \in \mathscr{B}(E; \mathbb{R})^+$.*

In [BBR06a] the transition semigroup is used to define when a Markov process $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ is a right process. They replace item (M7) with

(M7') The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous. Moreover, for all $\alpha > 0$, for every function f, which is α-excessive with respect to $(r_\alpha)_{\alpha>0}$ and each probability measure $\nu \in \mathcal{P}(E)$, the function $t \mapsto f(Z_t)$ is right continuous on $[0, \infty)$ P_ν-a.s.

The next proposition shows that these two definitions coincide. The proof is a combination and adaption of [KB68, Section I Proposition 8.2] and [KB68, Section I Theorem 8.11]. We make this effort since we want to apply results from [BBR06a] to construct right processes. Hence, we have to be certain that we are dealing with the same objects.

Proposition 2.63. *Let* $\boldsymbol{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ *be a Markov process with state space* E *fulfilling Items (M5), (M6) and (M7'). Then* $\boldsymbol{M}$ *also fulfills (M7).*

Proof. Let τ be an $(\mathcal{F}_t)_{t\geq 0}$ stopping time and define for $n \in \mathbb{N}$

$$\tau_n := \begin{cases} \frac{k+1}{2^n}, & \frac{k}{2^n} \leq \tau < \frac{k+1}{2^n} \text{ for some } k \in \mathbb{N} \\ \infty & \tau = \infty. \end{cases}$$

One can easily check that $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of $(\mathcal{F}_t)_{t\geq 0}$ stopping times converging pointwisely from above to τ. Moreover, $\tau_n > \tau$ on $\{\tau < \infty\}$ for all $n \in \mathbb{N}$. For $z \in E_\Delta$, $\alpha \in (0, \infty)$ and $f \in C_b(E_\Delta; \mathbb{R})^+$ we can calculate using (M7')

$$\begin{aligned} \int_0^\infty e^{-\alpha t} E_z \left[f(Z_{\tau+t}) \right] \mathrm{d}t &= \lim_{n\to\infty} \int_0^\infty e^{-\alpha t} E_z \left[f(Z_{\tau_n+t}) \right] \mathrm{d}t \\ &= \lim_{n\to\infty} \int_0^\infty e^{-\alpha t} \sum_{k=1}^\infty E_z \left[f(Z_{k2^{-n}+t}) \mathbb{1}_{\{\tau_n = k2^{-n}\}} \right] \mathrm{d}t \\ &= \lim_{n\to\infty} \int_0^\infty e^{-\alpha t} \sum_{k=1}^\infty E_z \left[p_t f(Z_{k2^{-n}}) \mathbb{1}_{\{\tau_n = k2^{-n}\}} \right] \mathrm{d}t \\ &= \lim_{n\to\infty} \int_0^\infty e^{-\alpha t} E_z \left[p_t f(Z_{\tau_n}) \right] \mathrm{d}t \\ &= \lim_{n\to\infty} E_z \left[r_\alpha f(Z_{\tau_n}) \right] \\ &= E_z \left[r_\alpha f(Z_\tau) \right] \\ &= \int_0^\infty e^{-\alpha t} E_z \left[p_t f(Z_\tau) \right] \mathrm{d}t. \end{aligned}$$

Above, we also used the theorem of dominated convergence and right continuity of $t \mapsto f(Z_{\tau+t})$ and $t \mapsto r_\alpha f(Z_t)$ on $[0, \infty)$ P_{δ_z}-a.s., which follows by (M6) and by (M7') together with Remark 2.62. Moreover, we used (M4') to get from the third to fourth equality. Further note, that by (M6) the functions $t \mapsto E_z \left[f(Z_{\tau+t}) \right]$ and $t \mapsto E_z \left[p_t f(Z_\tau) \right]$ are right continuous. The equation above shows that they have the same Laplace transform, which implies their equality. Using that $f = f^+ - f^-$ we obtain for all $f \in C_b(E_\Delta; \mathbb{R})$,

$$E_z \left[f(Z_{\tau+t}) \right] = E_z \left[p_t f(Z_\tau) \right], \quad \text{for all} \quad t \in [0, \infty). \tag{2.8}$$

Define

$$\mathcal{H} := \{ f \in \mathscr{B}_b(E_\Delta; \mathbb{R}) \mid (2.8) \text{ holds true} \} \subseteq \mathscr{B}_b(E_\Delta; \mathbb{R}).$$

Then, $\mathcal{H}$ is vector space of functions which is closed under bounded increasing limits and containing the algebra $C_b(E_\Delta; \mathbb{R})$. Therefore, by a monotone class argument, compare e.g. [Wil91, Theorem 3.14], we know that $\mathcal{H}$ contains all bounded $\sigma(C_b(E_\Delta; \mathbb{R}))/\mathscr{B}(\mathbb{R})$ measurable functions. Recalling that we assume that $\mathscr{B}(E_\Delta)$ is generated by $C_b(E_\Delta; \mathbb{R})$, we obtain $\mathbb{1}_B \in \mathcal{H}$ for all $B \in \sigma(C_b(E_\Delta; \mathbb{R})) = \sigma(C(E_\Delta; \mathbb{R})) = \mathscr{B}(E_\Delta)$. Thus,

$$P_z(Z_{\tau+t} \in B) = E_z \left[\mathbb{1}_B(Z_{\tau+t}) \right] = E_z \left[p_t \mathbb{1}_B(Z_\tau) \right] = E_z \left[P_{Z_\tau}(Z_t \in B) \right]$$

for all $B \in \mathscr{B}(E_\Delta)$.
Finally, we can show that this implies the strong Markov property in (M7). Indeed, let $\Lambda \in \mathcal{F}_\tau$ be given and set

$$\tau_\Lambda := \begin{cases} \tau & \text{on} \quad \Lambda \\ \infty & \text{on} \quad \Lambda^c \end{cases}$$

One can verify that τ_Λ is again a stopping time. Moreover, for all $z \in E$, $s \in [0, \infty)$ and $B \in \mathscr{B}(E_\Delta)$ we calculate

$$\begin{aligned} P_z(Z_{\tau_\Lambda + s} \in B) &= P_z(\{Z_{\tau+s} \in B\} \cap \Lambda) + P_z(\{\Delta \in B\} \cap \Lambda^c) \\ &= P_z(\{Z_{\tau+s} \in B\} \cap \Lambda) + P_z(\{\Delta \in B\}) P_z(\Lambda^c) \quad \text{and} \\ E_z\left[P_{Z_{\tau_\Lambda}}(Z_s \in B)\right] &= E_z\left[P_{Z_\tau}(Z_s \in B) \mathbb{1}_\Lambda\right] + E_z\left[P_\Delta(Z_s \in B) \mathbb{1}_{\Lambda^c}\right] \\ &= E_z\left[P_{Z_\tau}(Z_s \in B) \mathbb{1}_\Lambda\right] + P_z(\{\Delta \in B\}) P_z(\Lambda^c). \end{aligned}$$

Since the left-hand sides of the equations above coincide by the considerations in the first part we obtain

$$E_z(\mathbb{1}_{\{Z_{\tau+s} \in B\}} \mathbb{1}_\Lambda) = P_z(\{Z_{\tau+s} \in B\} \cap \Lambda) = E_z\left[P_{Z_\tau}(Z_s \in B) \mathbb{1}_\Lambda\right]$$

and therefore the strong Markov property from (M7). □

In the preceding section we considered solutions to the martingale problem in terms of a probability law $\mathbb{P}$ defined on $(D([0, \infty); E_\Delta, \mathscr{B}^D)$. We generalize this concept by defining solutions of the martingale problem for $(L, \mathcal{D})$ on $L^p(E; \mu)$ in terms of a right process. For a connection of the two solution concepts we refer to Lemma 2.67, below.

Definition 2.64. Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$ be a right process with state space E and $\mu \in \mathcal{P}(E)$. Further, let $(L, \mathcal{D})$ be a linear operator on $L^p(E; \mu)$, $p \in [1, \infty)$. Then $\mathbf{M}$ is said to solve the martingale problem for $(L, \mathcal{D})$ with respect to P_μ, if

(i) For every $f \in \mathcal{D}$ and $t \in [0, \infty)$ it holds $\int_0^\infty |Lf(Z_s)| \, \mathrm{d}s < \infty$ P_μ-a.s. and $f(Z_t)$ as well as $Lf(Z_t)$ are P_μ-a.s. well-defined, i.e. independent of the chosen μ-version of f and Lf, respectively.

(ii) For all $f \in \mathcal{D}$ and $t \in [0, \infty)$, the random variable $M_t^{[f],L}$ defined by

$$M_t^{[f],L} := f(Z_t) - f(Z_0) - \int_0^t Lf(Z_s) \, \mathrm{d}s$$

is P_μ-integrable and the corresponding process $(M_t^{[f],L})_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$-martingale.

Remark 2.65. *Suppose $\mu \in \mathcal{P}(E)$ and $\boldsymbol{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (Z_t)_{t \geq 0}, (P_z)_{z \in E_\Delta})$ is a right process with state space E which is associated with a conservative sub-Markovian strongly continuous contraction semigroup $(T_t)_{t \geq 0}$ on $L^p(E; \mu)$. Then the process $\boldsymbol{M}$ has infinite life-time P_μ-a.s.. Indeed, using the right continuity of $\boldsymbol{M}$ we obtain for every $z \in E_\Delta$ and $t \in [0, \infty)$*

$$P_z(t < \xi) = P_z(Z_t \in E) = p_t \mathbb{1}_E(z).$$

Since $T_t\mathbb{1}_E = 1$ is a μ-version of $p_t\mathbb{1}_E$ we can derive

$$P_\mu(t < \xi) = \int_E P_z(Z_t \in E)\, \mu(\mathrm{d}z) = \int_E p_t\mathbb{1}_E(z)\, \mu(\mathrm{d}z) = \int_E T_t\mathbb{1}_E(z)\, \mu(\mathrm{d}z) = 1.$$

The claim follows as $P_\mu(\xi = \infty) = \bigcap_{t\in\mathbb{N}} P_\mu(t < \xi)$. The result we just derived can be considered as a generalization of the statement from Lemma 2.53 Item (i).

Definition 2.66. Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ be a right process with state space E and life-time ξ.

(i) Let μ be a probability measure on $(E_\Delta, \mathscr{B}(E_\Delta))$. $\mathbf{M}$ is called μ-standard if the following two additional items hold

(M8) $\mathbf{M}$ possesses left limits in E P_μ-a.s., i.e.

$$Z_{t-} := \lim_{s\uparrow t} Z_s \quad \text{exists in } E \text{ for all } t \in (0,\zeta)\ P_\mu\text{-}a.s..$$

(M9) If τ, τ_n are $(\mathcal{F}_t)_{t\geq 0}$-stopping times for all $n \in \mathbb{N}$ and such that $\tau_n \uparrow \tau$ as $n \to \infty$, then $(Z_{\tau_n})_{n\in\mathbb{N}}$ converges to Z_τ P_μ-a.s. on $\{\tau < \xi\}$.

(ii) $\mathbf{M}$ is called a Hunt process if (M8) and (M9) hold with ξ replaced by ∞ and E_Δ by E.

(iii) $\mathbf{M}$ is called a diffusion process if it is a Hunt process and its paths are P_μ-a.s. in $C([0,\infty); E)$.

Lemma 2.67. *Let $\mu \in \mathcal{P}(E)$ and $\boldsymbol{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Z_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ be a right process with state space E, associated with a sub-Markovian s.c.c.s. $(T_t)_{t\geq 0}$ with generator $(L, D(L))$ and assume (M8) holds. Moreover, let $\mathbb{P}_\mu$ be the image measure of P_μ under the map,*

$$\Phi : (\Omega, \mathcal{M}^{P_\mu}) \to (D([0,\infty); E_\Delta), \mathscr{B}^D), \quad \Phi(\omega) := \begin{cases} (Z_t(\omega))_{t\geq 0} & \text{if} \quad \omega \in \Lambda \\ z & \text{else} \end{cases},$$

where $\Lambda \in \mathcal{M}^{P_\mu}$ is defined in Remark 2.55 and $z \in E$ is arbitrary. Then,

(i) $\mathbb{P}_\mu$ is associated with $(T_t)_{t\geq 0}$.

(ii) $\boldsymbol{M}$ solves the Martingale problem for $(L, D(L))$ with respect to P_μ if and only if $\mathbb{P}_\mu$ does.

(iii) $\boldsymbol{M}$ has infinite life-time P_μ-a.e. if and only if $\mathbb{P}_\mu$ has it in the sense of Definition 2.46.

Proof. The proof of Item (i) is given in [Con11, Remark 2.2.9], compare also [Ale23, Lemma 2.3.36]. Item (ii) and (iii) can be directly verified using the definitions of the involved objects. □

2.3.3 Process construction theorems

Before we state the first process construction theorem, we need to discuss different notions of Lusin spaces which are used in literature. To start this discussion we assume that $(E, \mathcal{T})$ is a topological space and μ is a probability measure on $(E, \mathscr{B}_{\mathcal{T}}(E))$. Again, note that the results below are also valid for σ-finite measures.

Definition 2.68. The space $(E, \mathcal{T})$ is called

(i) a Lusin space if there is a stronger topology $\tilde{\mathcal{T}}$ such that $(E, \tilde{\mathcal{T}})$ is a Polish space.

(ii) a Lusin topological space if it is homeomorphic to a Borel measurable subset of a compact metric space.

(iii) a Lusin measurable space if there is some Lusin topological space $(F, \hat{\mathcal{T}})$ such that the measurable space $(E, \mathscr{B}_{\mathcal{T}}(E))$ is measurably isomorphic to $(F, \mathscr{B}_{\hat{\mathcal{T}}}(F))$.

Clearly, every Lusin topological space is a Lusin space and every Lusin space is a Lusin measurable space.
Starting with a sub-Markovian strongly continuous resolvent on some $L^p(E;\mu)$ space, where $(E, \mathcal{T})$ is merely a Lusin measurable space, we can enlarge E to ensure that there is an associated right process. This statement is formulated in the next theorem.

Theorem 2.69. *[BBR06b, Theorem 2.2] Let $p \in [1, \infty)$ and $(E, \mathcal{T})$ be a Lusin measurable space. Assume that $(R_\alpha)_{\alpha>0}$ is a sub-Markovian strongly continuous contraction resolvent on $L^p(E;\mu)$. Then there exists a Lusin topological space $(E_1, \mathcal{T}_1)$ with $E \subseteq E_1$, $E \in \mathscr{B}_{\mathcal{T}_1}(E_1)$, $\mathscr{B}_{\mathcal{T}}(E) = \mathscr{B}_{\mathcal{T}_1}(E_1) |_E$ and a right process with state space E_1 such that its resolvent, regarded on $L^p(E_1; \overline{\mu})$, coincides with $(R_\alpha)_{\alpha>0}$, where $\overline{\mu}$ is the measure on $(E_1, \mathscr{B}_{\mathcal{T}_1}(E_1))$ extending μ by zero on $E_1 \setminus E$.*

For the second process construction theorem, we assume that $(E, \mathcal{T})$ is a Lusin space and $(R_\alpha)_{\alpha>0}$ is a sub-Markovian strongly continuous contraction resolvent on $L^p(E;\mu)$. Under additional assumption we see that there is an associate right process with state space E equipped with a weaker (metrizable) topology, having cádlág paths in the original topology $\mathcal{T}$, P_μ-a.e.. To apply the theorem we need to check that the generator of $(R_\alpha)_{\alpha>0}$ has a nice core and we have access to a μ-nest of compact sets in $(E, \mathcal{T})$.

Theorem 2.70. *[BBR06a, Theorem 1.1] Let $p \in [1, \infty)$ and assume that $(R_\alpha)_{\alpha>0}$ is a sub-Markovian strongly continuous contraction resolvent on $L^p(E;\mu)$ with corresponding generator $(L, D(L))$. Further, suppose that the following statements hold true.*

(I) There exists a μ-nest of compact sets in $(E, \mathcal{T})$.

(II) There exists a countable $\mathbb{Q}$-algebra $\mathcal{A} \subseteq D(L) \cap C_b(E;\mathbb{R})$ such that $\mathcal{A}$ is a core for $(L, D(L))$ separating the points of E.

Denote by $\mathcal{T}_0$ the (metrizable Lusin) topology on E generated by $\mathcal{A}$. Then,

(i) there exists a μ-standard right process with state space E (endowed with the topology $\mathcal{T}_0$) whose resolvent $(r_\alpha)_{\alpha>0}$ regarded on $L^p(E;\mu)$ coincides with $(R_\alpha)_{\alpha>0}$.

(ii) the process is cádlág in the topology $\mathcal{T}$, P_μ-a.e..

(iii) every element from $D(L)$ has a μ-quasi continuous version (with respect to the topology $\mathcal{T}_0$).

Remark 2.71. *In Chapter 7 we apply Theorem 2.70 in the situation where E is a real separable Hilbert space and $\mathcal{T}$ denotes the weak topology. This is indeed a Lusin space and therefore also a Lusin measurable space, as the identity map from E equipped with the strong topology to $(E, \mathcal{T})$ is a continuous one to one map.*
Note that the collection of compact sets in $(E, \mathcal{T})$ is much bigger than the one in E with the strong topology. In particular, by the famous Banach-Alaoglu theorem, the closed balls with respect to the strong topology are weakly compact. Hence, a sequence of centered balls with increasing radius are at least a potential candidate to check Item (II) of Theorem 2.70. Indeed, in Proposition 7.3, we verify that this sequence is a nest of weakly compact sets for the s.c.c.r. associated to the infinite dimensional Langevin operator. In the end, we obtain an associated process with weakly cádlág and by further analysis weakly continuous paths, compare Proposition 7.5.

Even though not used in this thesis, we want to mention the recent results from [BCR23]. In particular we want to highlight [BCR23, Theorem 3.6], which contains a sufficient condition under which a right process with state space E (equipped with a topology $\mathcal{T}$ such that $(E, \mathcal{T})$ is a Lusin topological space) and cádlág paths, has $\mathcal{T}$-continuous paths. Moreover, [BCR23, Corollary 4.10] deals with a Lusin measurable space $(E, \mathcal{T})$ and a sub-Markovian s.c.c.r. on $L^p(E;\mu)$. There, the authors extend the results from Theorem 2.69. They state an extra condition such that there exists a Lusin topological space $(E_1, \mathcal{T}_1)$ with $E \subseteq E_1$, $E \in \mathscr{B}_{\mathcal{T}_1}(E_1)$, $\mathscr{B}_{\mathcal{T}}(E) = \mathscr{B}_{\mathcal{T}_1}(E_1) \mid_E$ and a right process with state space E_1, which is a diffusion μ-q.e. ([BCR23, Definition 2.6]) and such that its resolvent, regarded on $L^p(E_1; \overline{\mu})$, coincides with $(R_\alpha)_{\alpha>0}$. Again, $\overline{\mu}$ denotes the measure on $(E_1, \mathscr{B}_{\mathcal{T}_1}(E_1))$ extending μ by zero on $E_1 \setminus E$.

3

Infinite dimensional stochastic analysis

This chapter includes a presentation of useful concepts and fundamental results from (infinite dimensional) Stochastic analysis which are necessary for the applications we have in mind. Following the textbooks [PR07; Da 06; DZ14], we first introduce Gaussian measures on Hilbert spaces, (cylindrical) Wiener processes and corresponding stochastic integrals. In addition, we study measure theoretic and topological properties of L^p-spaces with respect to Gaussian measures.
After the introduction of Sobolev spaces and a corresponding integration by parts formula with respect to Gaussian measures on Hilbert spaces in Section 3.2, we summarize results about infinite dimensional (perturbed) Ornstein-Uhlenbeck operators and corresponding Poincaré inequalities, in Section 3.2.3 and Section 3.2.4, respectively. We also consider measures having a density with respect to a Gaussian measure. The content presented is based on the books [DT00; DZ02; Da 06; DZ14] and extended by results from the articles [DA14; Big22; BF22].
Even though most of the results in this section are well known, we add, reformulate and extend some results. We want to highlight Theorem 3.32, Proposition 3.51 and Lemma 3.61

3.1 Gaussian analysis

Let X be an infinite dimensional real separable Hilbert space with inner product $(\cdot,\cdot)_X$. Further, assume that $m \in X$ and Q is an injective linear operator in $\mathcal{L}_1^+(X)$. By [Da 06, Theorem 1.12] we know that there exists a unique probability measure denoted by $N(m,Q)$ on $(X, \mathscr{B}(X))$ such that its characteristic function is given via

$$X \ni x \mapsto e^{i(m,x)_X - \frac{1}{2}(Qx,x)_X} \in \mathbb{C}\,.$$

This measure is called the infinite dimensional non-degenerate Gaussian measure with mean $m \in X$ and covariance operator $Q \in \mathcal{L}_1^+(X)$. Obviously, this also works if X is finite dimensional. In this case we use the same notation and Q can be identified with a symmetric positive matrix, which is then called the covariance matrix. For a detailed introduction into the concept of infinite dimensional Gaussian measures, we refer to [Da 06, Chapter 1] or [PR07, Section 2.1].
For the rest of this section we fix a non-degenerate infinite dimensional Gaussian measure $\mu := N(0,Q)$. Since Q is symmetric and of trace class, there is an orthonormal basis $B_X = (e_n)_{n\in\mathbb{N}}$ of X consisting of eigenvectors of Q with corresponding positive eigenvalues

$(\lambda_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$. Without loss of generality, we assume that the eigenvalues are decreasing to zero. In this situation the following definition is useful.

Definition 3.1. For each $n \in \mathbb{N}$, define $X_n := \operatorname{span}\{e_1, \dots, e_n\}$ and denote the orthogonal projection from X to X_n by P_n^X, with the corresponding coordinate map $p_n^X : X \to \mathbb{R}^n$. This means for all $x \in X$

$$P_n^X x := \sum_{k=1}^{n} (x, e_k)_X e_k \in X_n \quad \text{and} \quad p_n^X x := ((x, e_1)_X, \dots, (x, e_n)_X) \in \mathbb{R}^n .$$

By $\overline{p}_n^X : \mathbb{R}^n \to X_n$ we denote the canonical embedding of $\mathbb{R}^n$ into X_n, i.e.

$$\overline{p}_n^X y := \sum_{k=1}^{n} y_k e_k \in X_n, \quad y \in \mathbb{R}^n .$$

If the underlying real separable Hilbert space and the corresponding orthonormal basis is clear from the context, we also write P_n, p_n and $\overline{p}_n$ to avoid an overload of notation. Let $k \in \mathbb{N} \cup \{\infty\}$ and $C \in \left\{C_b^k(\mathbb{R}^n), C_c^k(\mathbb{R}^n)\right\}$. Then we define for each $n \in \mathbb{N}$

$$\mathcal{F}C(B_X, n) := \{\varphi \circ p_n^X : X \to \mathbb{R} \mid \text{ for some } \varphi \in C\} \quad \text{and} \quad \mathcal{F}C(B_X) := \bigcup_{n\in\mathbb{N}} \mathcal{F}C(B_X, n).$$

3.1.1 Gaussian measures on Hilbert spaces

For the sake of a better understanding, we start by stating a different characterization of infinite dimensional non-degenerate Gaussian measures, which will be useful to construct an infinite dimensional Wiener process. If not stated otherwise, the results are taken from [Da 06, Chapter 1].

Proposition 3.2 ([PR07, Proposition 2.1.6])**.** *Let Y be an X-valued random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the image measure $\mathbb{P} \circ Y^{-1}$ of Y under $\mathbb{P}$ on $(X, \mathscr{B}(X))$ is equal to $N(m, Q)$ if and only if*

$$Y = \sum_{k\in\mathbb{N}} \sqrt{\lambda_k} \beta_k e_k + m \quad (\textit{as an object in } L^2(\Omega; \mathcal{F}; \mathbb{P}; X)),$$

where $(\beta_k)_{k\in\mathbb{N}}$ is a sequence of independent real valued random variables with $\mathbb{P} \circ \beta_k^{-1} = N(0, 1)$.

We continue with an easy to derive but central result for our further analysis.

Lemma 3.3. *Given $n \in \mathbb{N}$ and $l_1, ..., l_n \in X$. Denote by μ^n the image measure of μ under the map*

$$X \ni x \mapsto \big((x, l_1)_X, ...(x, l_n)_X\big) \in \mathbb{R}^n .$$

Then, μ^n is a finite dimensional centered Gaussian measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ with covariance matrix $((Ql_i, l_j)_X)_{ij=1,\dots,n}$.
In particular, if $l_i = e_i$ for all $i = 1, \dots, n$, we have $\mu^n = N(0, Q_n)$ with covariance

matrix $Q_n := \operatorname{diag}(\lambda_1, ..., \lambda_n)$. *Note that a finite dimensional centered Gaussian measure on* $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ *with covariance matrix* Q_n *is uniquely determined by its density with respect to the* n*-dimensional Lebesgue measure* $\mathrm{d}x$. *We then have*

$$\mu^n(A) = \frac{1}{\sqrt{(2\pi)^n \det(Q_n)}} \int_A e^{-\frac{1}{2}\left\langle Q_n^{-1}x, x\right\rangle} \, \mathrm{d}x \quad \textit{for all} \quad A \in \mathscr{B}(\mathbb{R}^n).$$

Below we state Fernique's theorem, which tells us that infinite dimensional Gaussian measures have exponential tails. It also shows that Gaussian measures have moments of all orders. The moments up to order four can be calculated by the last lemma in this section.

Proposition 3.4. *Given* $s \in \mathbb{R}$. *It holds*

$$\int_X e^{s\|x\|_X^2} \, \mu(\mathrm{d}x) = \begin{cases} \left(\prod_{k=1}^{\infty}(1 - 2s\lambda_k)\right)^{-\frac{1}{2}} & \textit{for} \quad s < \frac{1}{2\lambda_1} \\ \infty & \textit{else.} \end{cases}$$

Lemma 3.5. *For* $l_1, l_2, l_3, l_4 \in X$ *set* $q_{ij} = (Ql_i, l_j)_X$, $i, j \in \{1, 2, 3, 4\}$. *Then it holds*

$$\int_X (x, l_1)_X \, \mu(\mathrm{d}x) = 0, \qquad \int_X (x, l_1)_X (x, l_2)_X \, \mu(\mathrm{d}x) = q_{12},$$
$$\int_X (x, l_1)_X (x, l_2)_X (x, l_3)_X \, \mu(\mathrm{d}x) = 0 \quad \textit{and}$$
$$\int_X (x, l_1)_X (x, l_2)_X (x, l_3)_X (x, l_4)_X \, \mu(\mathrm{d}x) = q_{12}q_{34} + q_{13}q_{24} + q_{14}q_{23}.$$

Moreover, for all $s \in [0, \infty)$, $X \ni x \mapsto \|x\|_X^s \in \mathbb{R}$ *is* μ*-integrable and particularly*

$$\int_X \|x\|_X^2 \, \mathrm{d}\mu = \int_X \sum_{n \in \mathbb{N}} (x, e_n)_X^2 \, \mathrm{d}\mu = \sum_{n \in \mathbb{N}} (Qe_n, e_n)_X = \sum_{n \in \mathbb{N}} \lambda_n < \infty$$

due to monotone convergence and the fact that Q *is trace class.*

3.1.2 The Wiener process and the stochastic integral

Fix a finite time horizon $T \in (0, \infty)$ and define

$$\mathcal{I} := \begin{cases} N & \text{if} \quad \dim(X) = N \\ \mathbb{N} & \text{if} \quad \dim(X) = \infty. \end{cases}$$

Further, let $Q \in \mathcal{L}_1^+(X)$ be injective with corresponding orthonormal basis $B_X = (e_n)_{n \in \mathcal{I}}$ of eigenvectors and positive eigenvalues $(\lambda_n)_{n \in \mathcal{I}} \in \ell^1(\mathcal{I})$. Without loss of generality we assume that the eigenvalues are decreasing. Besides Lemma 3.15, which will be useful to construct solutions to the infinite dimensional Langevin equation via corresponding martingale solutions (compare Proposition 7.10), every result below is included in [PR07, Chapter 2]. This section should be understood as a summary of the tools we need for further analysis.

Definition 3.6. An X-valued stochastic process $W := (W_t)_{t \in [0,T]}$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a (standard) Q-Wiener process if

(i) $W_0 = 0$,

(ii) W has $\mathbb{P}$-a.s. continuous trajectories and the increments of B are independent, i.e. the random variables
$$W_{t_1}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$$
are independent for all $0 \leq t_1 < \cdots < t_n \leq T$, $n \in \mathbb{N}$.

(iii) For all $0 \leq s \leq t \leq T$ we have:
$$\mathbb{P} \circ (W_t - W_S)^{-1} = N(0, (t-s)Q).$$

Suppose $(\mathcal{F}_t)_{t\in[0,T]}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. Then, a Q-Wiener process B is called Q-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ if

(iii) W_t is adapted to $\mathcal{F}_t$, $t \in [0, T]$.

(iv) $W_t - W_s$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t \leq T$.

Proposition 3.7. *An X-valued stochastic process $(W_t)_{t\in[0,T]}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is a Q-Wiener process if and only if*
$$W_t = \sum_{k\in\mathcal{I}} \sqrt{\lambda_k}\beta_k(t)e_k, \quad t \in [0, T],$$
where $(\beta_k)_{k\in\mathcal{I}}$ is a sequence of independent real valued (standard) Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. One can show that the series above converges in $L^2(\Omega; \mathcal{F}; \mathbb{P}; C^0([0, T]; X))$ and thus always has a $\mathbb{P}$-a.s. continuous modification.
Moreover, $(W_t)_{t\in[0,T]}$ is a Q-Wiener process with respect to the normal filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ (normal in the sense that it is right-continuous and $\mathcal{F}_0$ contains all $A \in \mathcal{A}$ with $\mathbb{P}(A) = 0$) defined by
$$\mathcal{F}_t := \bigcap_{s\in(t,T]} \sigma(W_r \mid r \in [0, s])^{\mathbb{P}},$$
where $\sigma(W_r \mid r \in [0, s])^{\mathbb{P}}$ denotes the completion of $\sigma(W_r \mid r \in [0, s])$ w.r.t $\mathbb{P}$.

Definition 3.8. Let $(Y, (\cdot, \cdot)_Y)$ be another real separable Hilbert space. In the following we set $X_0 := Q^{\frac{1}{2}}X$ and equip it with the inner product
$$(x, y)_{X_0} := (Q^{-\frac{1}{2}}x, Q^{-\frac{1}{2}}y)_X, \quad x, y \in X_0.$$
Then $(X_0, (\cdot, \cdot)_{X_0})$ is a real separable Hilbert space with orthonormal basis $(Q^{\frac{1}{2}}e_i)_{i\in\mathcal{I}}$. In literature, the Hilbert space $(X_0, (\cdot, \cdot)_{X_0})$ is called the Cameron Martin space. Define $\mathcal{L}_2^0 := \mathcal{L}_2(X_0; Y)$ as the Hilbert space of Hilbert-Schmidt operators from X_0 to Y.

Remark 3.9. *If dim$(X) < \infty$, $X_0 = X$ and $\mathcal{L}_2^0 = \mathcal{L}(X_0; Y)$.*

For the rest of this section we a fix probability space $(\Omega, \mathcal{F}, , \mathbb{P})$. Additionally, we define $\Omega_T := [0, T] \times \Omega$ and assume that $(W_t)_{t\in[0,T]}$ is Q-Wiener process with respect to the normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ introduced in Proposition 3.7.
For the sake of completeness, we give a quick summary on how to construct the stochastic integral with respect to an infinite dimensional Q-Wiener process.

Definition 3.10. An $\mathcal{L}(X;Y)$-valued process $(\xi_t)_{t\in[0,T]}$ is said to be elementary if there exist $0 < t_0 \cdots < t_k = T$, $k \in \mathbb{N}$, such that

$$\xi_t = \sum_{m=0}^{k-1} \xi_{(m)} \mathbb{1}_{(t_m,t_{m-1}]}(t), \quad t \in [0,T],$$

where

(i) $\xi_{(m)} : \Omega \to \mathcal{L}(X;Y)$ is $\mathcal{F}_{t_m}$-$\mathscr{B}(\mathcal{L}(X;Y))$ measurable, $0 \le m \le k-1$.

(ii) $\xi_{(m)}$ takes only a finite number of values in $\mathcal{L}(X;Y)$, $0 \le m \le k-1$.

The space of all $\mathcal{L}(X;Y)$-valued elementary processes is denoted by $\mathcal{E}$. Moreover, we denote by $\mathcal{M}_T^2$ the space of all Y-valued square integrable continuous martingales $M = (M_t)_{t\in[0,T]}$. One can show that the normed space $(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2})$ is a Banach space, where

$$\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} \left(\mathbb{E}(\|M_t\|_Y^2)\right)^{\frac{1}{2}} \quad \text{for all} \quad M \in \mathcal{M}_T^2.$$

Definition 3.11. Define the σ-algebra

$$\mathcal{A}_T := \sigma(\{\xi : \Omega_T \to \mathbb{R} \mid \xi \text{ is left-continuous and } (\mathcal{F}_t)_{t\in[0,T]}\text{-adapted}\}),$$

and suppose $(Z, (\cdot,\cdot)_Z)$ is another real separable Hilbert space. A random variable $\xi : \Omega_T \to Z$ is called predictable if it is $\mathcal{A}_T/\mathscr{B}(Z)$-measurable.
For a predictable process $\xi : \Omega_T \to \mathcal{L}_2^0$, we define

$$\|\xi\|_T := \left(\mathbb{E}\left(\int_0^T \|\xi_s \circ Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(X;Y)} \mathrm{d}s\right)\right)^{\frac{1}{2}} \in [0,\infty].$$

Finally, we set

$$\mathcal{N}_T^2 := \left\{\xi : \Omega_T \to \mathcal{L}_2^0 \mid \xi \text{ is predictable and } \|\xi\|_T < \infty\right\} = L^2(\Omega_T; \mathcal{A}_T; \mathrm{d}t \otimes \mathbb{P}; \mathcal{L}_2^0).$$

Remark 3.12. *Note that $\|\cdot\|_T$ is not a norm on $\mathcal{E}$. Indeed, $\|\xi - \eta\|_T = 0$ for two elementary processes ξ and η, merely implies that $\xi = \eta$ $\mathrm{d}t \otimes \mathbb{P}$-a.e. on $Q^{\frac{1}{2}}(X)$. So in the considerations below we work with equivalence classes with respect to $\|\cdot\|_T$, but without changing the notation.*

With these definitions at hand we can show that the map

$$\mathrm{Int} : (\mathcal{E}, \|\cdot\|_T) \to (\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2}), \quad \xi \mapsto \mathrm{Int}(\xi), \quad \text{where}$$

$$\mathrm{Int}(\xi)_t := \int_0^t \xi_s \, \mathrm{d}W_s := \sum_{m=0}^{k-1} \xi_{(m)}(W_{t_{m+1}\wedge t} - W_{t_m \wedge t}), \quad t \in [0,T],$$

is well-defined and linear. Furthermore, it is isometric, i.e. fulfilling the so-called Itô-Isometrie

$$\|\mathrm{Int}(\xi)\|_{\mathcal{M}_T^2} = \|\xi\|_T.$$

We call Int the stochastic integral on $\mathcal{E}$.

Remark 3.13. *(i) One can show that the abstract completion of $\mathcal{E}$ with respect to $\|\cdot\|_T$ can be identified with $\mathcal{N}_T^2$ and that the space of elementary functions with values in $\mathcal{L}(X;Y)_0 := \{T_{|X_0} \mid T \in \mathcal{L}(X;Y)\} \subseteq \mathcal{L}_2^0$ is dense in $\mathcal{N}_T^2$ with respect to $\|\cdot\|_T$.*

Hence, we can extend the stochastic integral on $\mathcal{E}$ to an isometric transformation from $(\mathcal{N}_T^2, \|\cdot\|_T)$ to $(\mathcal{M}_T^2, \|\cdot\|_{\mathcal{M}_T^2})$. We again denote this extension by Int.

(ii) Using a localization argument one can extend the definition of the stochastic integral for processes in

$$\left\{\xi : \Omega_T \to \mathcal{L}_2^0 \mid \xi \text{ is predictable and } \mathbb{P}\left(\int_0^T \|\xi_s \circ Q^{\frac{1}{2}}\|^2_{\mathcal{L}_2(X;Y)} \mathrm{d}s\right) < \infty\right\}.$$

Since this is not necessary for our further considerations, we omit this extension.

Lemma 3.14. *Let $\xi \in \mathcal{N}_T^2$ and $L \in \mathcal{L}(X;Y)$. Then the process $(L(\xi_t))_{t\in[0,T]}$ is in $\mathcal{N}_T^2$ and for all $t \in [0,T]$*

$$L\left(\int_0^t \xi_s \,\mathrm{d}W_s\right) = \int_0^t L(\xi_s)\,\mathrm{d}W_s, \quad \mathbb{P}\text{-a.s.}$$

Before we move on and generalize the results to the case, where Q is not trace class, we derive a result which allows us to reduce certain infinite dimensional stochastic integrals to finite dimensional ones. As already mentioned, we use this result in Chapter 7.

Lemma 3.15. *Fix $N \in \mathbb{N}$. Suppose $\dim(X) = \infty$ and $\xi : \Omega_T \to \mathcal{L}_2(X_0;\mathbb{R})$ is an element of $\mathcal{N}_T^2$ and such that $\xi_s(\omega)(x) = 0$ for all $(s,\omega) \in \Omega_T$ and all $x \in \operatorname{span}\{e_1,\dots,e_N\}^\perp$. Then it holds*

$$\int_0^t \xi_s \,\mathrm{d}W_s = \int_0^t \xi_s^{[N]} \,\mathrm{d}W_s^{[N]} \quad \text{for all} \quad t \in [0,T],$$

where $\xi_s^{[N]} = (\xi_s(\sqrt{\lambda_1}e_1),\dots,\xi_s(\sqrt{\lambda_N}e_N))$ and $W^{[N]} = (\beta_1,\dots,\beta_N)^T$.

Proof. Let $(\xi^{(n)})_{n\in\mathbb{N}}$ be an approximating sequence of elementary functions with values in $\mathcal{L}(X;Y)_0$ of ξ in $\mathcal{N}_T^2$ with respect to $\|\cdot\|_T$. In formulas,

$$\xi_t^{(n)} = \sum_{m=0}^{k^{(n)}-1} \xi_{(m)}^{(n)} \mathbb{1}_{(t_m^{(n)},t_{m-1}^{(n)}]}(t), \quad t \in [0,T],$$

where

(i) $\xi_{(m)}^{(n)} : \Omega \to \mathcal{L}(X;\mathbb{R})_0$ is $\mathcal{F}_{t_m^{(n)}}$-$\mathscr{B}(\mathcal{L}(X;\mathbb{R}))$ measurable, $0 \le m \le k^{(n)}-1$.

(ii) $\xi_{(m)}^{(n)}$ takes only a finite number of values in $\mathcal{L}(X;\mathbb{R})_0$, $0 \le m \le k^{(n)}-1$.

Now define $\tilde{\xi}^{(n)}$ by

$$\tilde{\xi}_t^{(n)} := \sum_{m=0}^{k^{(n)}-1} (\xi_{(m)}^{(n)} \circ P_N^X)\mathbb{1}_{(t_m^{(n)},t_{m-1}^{(n)}]}(t), \quad t \in [0,T].$$

Since $\xi_s(\omega)(x) = 0$ for all $(s,\omega) \in \Omega_T$ and $x \in \text{span}\{e_1, \ldots, e_N\}^\perp$, it is easy to see that $(\tilde{\xi}^{(n)})_{n\in\mathbb{N}}$ is approximating ξ in $\mathcal{N}_T^2$ with respect to $\|\cdot\|_T$ and again a sequence of elementary functions with values in $\mathcal{L}(X;Y)_0$. Moreover, by definition it holds for each $t \in [0,T]$

$$\begin{aligned}\int_0^t \tilde{\xi}^{(n)} \,\mathrm{d}W_s &= \sum_{m=0}^{k^{(n)}-1} \xi_{(m)}^{(n)}(P_N^X(W_{t_{m+1}^{(n)}\wedge t} - W_{t_m^{(n)}\wedge t})) \\ &= \sum_{m=0}^{k^{(n)}-1}\sum_{j=1}^{N} \xi_{(m)}^{(n)}(\sqrt{\lambda_j}e_j)\left(\beta_j(t_{m+1}^{(n)}\wedge t) - \beta_j(t_m^{(n)}\wedge t)\right) \\ &= \int_0^t \xi_s^{[N],(n)}\,\mathrm{d}W_s^{[N]},\end{aligned}$$

where $\xi_s^{[N],(n)} = (\xi_s^{(n)}(\sqrt{\lambda_1}e_1), \ldots, \xi_s^{(n)}(\sqrt{\lambda_N}e_N))$. Obviously, $\xi^{[N],(n)}$ is an elementary process with values in $\mathcal{L}(\mathbb{R}^n;\mathbb{R})$, where $\mathbb{R}^n$ is equipped with the classical inner product $\langle\cdot,\cdot\rangle$. Then we calculate, denoting by a_i the canonical i-th unit vector of $\mathbb{R}^n$

$$\begin{aligned}\|\xi^{[N]} - \xi^{[N],(n)}\|_T^2 &= \mathbb{E}\left(\int_0^T \|\xi_s^{[N]} - \xi_s^{[N],(n)}\|_{\mathcal{L}_2^0(\mathbb{R}^n;\mathbb{R})}^2\,\mathrm{d}s\right) \\ &= \mathbb{E}\left(\int_0^T \sum_{i=1}^N \left(\xi_s^{[N]} - \xi_s^{[N],(n)}a_i\right)^2\,\mathrm{d}s\right) \\ &= \mathbb{E}\left(\int_0^T \sum_{i=1}^N \left(\xi_s(\sqrt{\lambda_i}e_i) - \xi_s^{(n)}(\sqrt{\lambda_i}e_i)\right)^2\,\mathrm{d}s\right) \\ &= \mathbb{E}\left(\int_0^T \sum_{i=1}^\infty \left(\xi_s(Q^{\frac{1}{2}}e_i) - \tilde{\xi}_s^{(n)}(Q^{\frac{1}{2}}e_i)\right)^2\,\mathrm{d}s\right) \\ &= \|\xi - \tilde{\xi}^{(n)}\|_T^2.\end{aligned}$$

Hence, $\int_0^t \tilde{\xi}^{(n)}\,\mathrm{d}W_s$ converges to $\int_0^t \xi\,\mathrm{d}W_s$ and $\int_0^t \xi_s^{[N],(n)}\,\mathrm{d}W_s^{[N]}$ to $\int_0^t \xi_s^{[N]}\,\mathrm{d}W_s^{[N]}$ as $n \to \infty$. This finishes the proof. □

For the rest of this section we consider the case where Q is not necessarily a trace class operator. Hence, we only consider the case when $\dim(X) = \infty$. We fix another real separable Hilbert space $(X_1, (\cdot,\cdot)_{X_1})$ and a Hilbert-Schmidt embedding

$$J : (X_0, (\cdot,\cdot)_{X_0}) \to (X_1, (\cdot,\cdot)_{X_1}).$$

Note that such a Hilbert space with corresponding embedding always exists. Now we are able to generalize our construction from above.

Proposition 3.16. *Let $(Q^{\frac{1}{2}}e_k)_{k\in\mathbb{N}}$ be the canonical orthonormal basis of the Hilbert space $(X_0, (\cdot,\cdot)_{X_0})$ and $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of independent real valued (standard) Brownian motions on $(\Omega,\mathcal{F},\mathbb{P})$. Set $\overline{Q} := JJ^*$. Then $\overline{Q} \in \mathcal{L}_1^+(X_1)$ and the series*

$$W_t = \sum_{k\in\mathbb{N}} \beta_k(t)JQ^{\frac{1}{2}}e_k, \quad t \in [0,T],$$

converges in $\mathcal{M}_T^2$ and defines a $\overline{Q}$-Wiener process on X_1. Moreover, it holds $\overline{Q}^{\frac{1}{2}}(X_1) = J(X_0)$ and $J : X_0 \to \overline{Q}(X_1)$ is isometric, i.e.

$$\|x_0\|_{X_0} = \|Jx_0\|_{\overline{Q}^{\frac{1}{2}}(X_1)} \quad \textit{for all} \quad x_0 \in X_0.$$

We call this process a Q-cylindrical Wiener process.

Remark 3.17. *Basically, a Q-cylindrical Wiener process is a $\overline{Q}$-Wiener process on X_1.*

For a given Q-cylindrical Wiener process $W = (W_t)_{t\in[0,T]}$ as constructed above, one can show that

$$\xi \in \mathcal{L}_2^0 = \mathcal{L}(Q^{\frac{1}{2}}(X); Y) \Leftrightarrow \xi \circ J^{-1} \in \mathcal{L}(\overline{Q}^{\frac{1}{2}}(X_1); Y).$$

Therefore, it is reasonable to define the stochastic integral on $\mathcal{N}_T^2$ with respect to the Q-cylindrical Wiener process by

$$\int_0^t \xi_s \, \mathrm{d}W_s := \int_0^t \xi_s \circ J^{-1} \, \mathrm{d}W_s, \quad t \in [0, T]. \tag{3.1}$$

Remark 3.18.

(i) The stochastic integral with respect to a Q-cylindrical Wiener process does not depend on $(X_1, (\cdot,\cdot)_{X_1})$ and J.

(ii) If Q is already in $\mathcal{L}_1^+(X)$, then it can also be considered as a Q-cylindrical Wiener process by choosing $X_1 = X$ and

$$J = Id : (X_0), (\cdot,\cdot)_{X_0}) \to (X, (\cdot,\cdot)_X).$$

3.2 Sobolev spaces with respect to Gaussian measures on Hilbert spaces

Again, we fix a real separable Hilbert space $(X, (\cdot,\cdot)_X)$ and on it a non-degenerate infinite dimensional Gaussian measure $\mu := N(0, Q)$, with corresponding orthonormal basis $B_X = (e_n)_{n\in\mathbb{N}}$ of eigenvectors of Q. The positive eigenvalues are denoted by $(\lambda_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$, where we assume w.l.o.g., that the eigenvalues are decreasing to zero.

3.2.1 Dense subsets of L^p

In the next chapter, we introduce and investigate the infinite dimensional Langevin operator. As its domain, we consider a space closely related to $\mathcal{F}C_b^\infty(B_X)$. In order to verify that this domain is dense in $L^2(X; \mu)$ and can be modified to a core fulfilling Item (II) from Theorem 2.70, the considerations in this section are necessary. Note that the first part deals merely with finite dimensional results, which can be found in [Ale23, Section 2.4] and most textbooks containing results about convolutions and smoothing, compare e.g. [Eva10]. We fix $d \in \mathbb{N}$ and set $B_r(0) := \{x \in \mathbb{R}^d \mid |x| < r\}$ for each $r \in (0, \infty)$.

Definition 3.19. A sequence $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^1(\mathbb{R}^d; \mathrm{d}x)$ is said to be an approximate identity, if $\varphi_\varepsilon \geq 0$, $\|\varphi_\varepsilon\|_{L^1(\mathrm{d}x)} = 1$ for all $\varepsilon \in (0,\infty)$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^d \setminus B_r(0)} \varphi_\varepsilon(x)\, \mathrm{d}x = 0 \quad \text{for all} \quad r \in (0,\infty).$$

If additionally $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq C_c^\infty(\mathbb{R}^d)$ and $\operatorname{supp}(\varphi_\varepsilon) \subseteq B_\varepsilon(0)$ then $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq L^1(\mathbb{R}^d; \mathrm{d}x)$ is called a standard approximate identity.

Remark 3.20. *There exists a standard approximate identity. Indeed, define $\tilde{\varphi} \in C_c^\infty(\mathbb{R}^d)$ via*

$$\tilde{\varphi}(x) := \begin{cases} \exp\left(-\frac{1}{1-4|x|^2}\right) & \text{if} \quad |x| < \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

Denote by φ the normalization of $\tilde{\varphi}$, i.e. $\varphi := \|\tilde{\varphi}\|^{-1}_{L^1(\mathrm{d}x)}\tilde{\varphi}$ and set $\varphi_\varepsilon(x) := \varepsilon^{-d}\varphi(\frac{x}{\varepsilon})$ for all $x \in \mathbb{R}^d$ and $\varepsilon \in (0,\infty)$. Then it is easy to check that $(\varphi_\varepsilon)_{\varepsilon>0} \subseteq C_c^\infty(\mathbb{R}^d)$ is a standard approximate identity. In literature, φ is called a mollifier.

Lemma 3.21. *Let $p \in [1,\infty]$ and $f \in L^p(\mathbb{R}^d; \mathrm{d}x)$ be given. The convolution of f with $\varphi \in L^1(\mathbb{R}^d; \mathrm{d}x)$ is defined via*

$$(\varphi * f)(x) := \int_{\mathbb{R}^d} \varphi(x-y) f(y)\, \mathrm{d}y, \quad x \in \mathbb{R}^d.$$

The following statements hold true

(i) $\varphi * f \in L^p(\mathbb{R}^d; \mathrm{d}x)$ *with* $\|\varphi * f\|_{L^p(\mathrm{d}x)} \leq \|\varphi\|_{L^1(\mathrm{d}x)} \|f\|_{L^p(\mathrm{d}x)}$.

(ii) Let $k \in \mathbb{N}$ and $s \in \mathbb{N}^d$ with $|s| \leq k$ be given. If $\varphi \in C_c^k(\mathbb{R}^d)$, then $\varphi * f \in C^k(\mathbb{R}^d)$ and $\partial^s(\varphi * f) = (\partial^s \varphi) * f$. Moreover, suppose $f \in C^k(\mathbb{R}^d)$ and $\partial^s f \in L^p(\mathbb{R}^d, \mathrm{d}x)$, then $\partial^s(\varphi * f) = \varphi * (\partial^s f)$.

(iii) $\operatorname{supp}(\varphi * f) \subseteq \overline{\{x + y \mid x \in \operatorname{supp}(f),\ y \in \operatorname{supp}(\varphi)\}}$. In particular, $\varphi * f \in C_c^\infty(\mathbb{R}^d)$ if $\varphi \in C_c^\infty(\mathbb{R}^d)$ and f has compact support.

(iv) If $p < \infty$ and $(\varphi_\varepsilon)_{\varepsilon>0}$ is a standard approximate identity, then $\varphi_\varepsilon * f \in C^\infty(\mathbb{R}^d)$, $\operatorname{supp}(\varphi_\varepsilon * f) \subseteq \overline{B_\varepsilon(\operatorname{supp}(f))}$ and

$$\lim_{\varepsilon\to 0} \|\varphi_\varepsilon * f - f\|_{L^p(\mathrm{d}x)} = 0.$$

Lemma 3.22. *Let $U \subseteq \mathbb{R}^d$ be an open set and $K \subseteq \mathbb{R}^d$ be compact. Then, for all $\varepsilon \in (0,\infty)$ with $B_\varepsilon(K) \subseteq U$, there is a smooth, so-called cut-off function $\eta \in C_c^\infty(\mathbb{R}^d)$ for K with*

$$0 \leq \eta \leq 1, \quad \eta = 1 \quad \text{on} \quad K \quad \text{and} \quad |\partial^s \eta(x)| \leq C_{d,s}\varepsilon^{-s} \quad \text{for all} \quad s \in \mathbb{N}^d \quad \text{and} \quad x \in \mathbb{R}^d.$$

Indeed, the function $\eta := \varphi_{\frac{\varepsilon}{4}} * \mathbb{1}_{B_{\frac{\varepsilon}{2}}(0)}$, where $(\varphi_\varepsilon)_{\varepsilon>0}$ is a standard approximate identity, has the required properties.

Corollary 3.23. *Let $n \in \mathbb{N}$. There is some $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$, $\varphi = 1$ on $B_1(0) = \{x \in \mathbb{R}^d \mid |x| < 1\}$ and $\varphi = 0$ outside $B_2(0)$, a constant $c \in (0,\infty)$, independent of n, such that*

$$|\partial_i \varphi_n(x)| \le \frac{c}{n}, \quad |\partial_i\partial_j \varphi_n(x)| \le \frac{c}{n^2} \quad \text{for all} \quad x \in \mathbb{R}^d, \quad 1 \le i,j \le n,$$

where we define

$$\varphi_n(x) = \varphi(\frac{x}{n}) \quad \text{for each} \quad x \in \mathbb{R}^d.$$

In particular, $0 \le \varphi_n \le 1$ and $\varphi_n = 1$ on $B_n(0)$ for all $n \in \mathbb{N}$. Moreover, $\varphi_n \to 1$ pointwisely on $\mathbb{R}^d$ and $D\varphi_n, D^2\varphi \to 0$ as $n \to \infty$, with respect to $\|\cdot\|_\infty$.

Corollary 3.24. *Let either $(\varphi_\varepsilon)_{\varepsilon>0}$ be a standard approximate identity and $\psi \in C(\mathbb{R})$ or let $(\varphi_\varepsilon)_{\varepsilon>0}$ be an approximate identity and $\psi \in C_b(\mathbb{R})$. Then for each $K \subseteq \mathbb{R}^d$ with $\overline{K}$ compact it holds*

$$\lim_{\varepsilon \to 0} \sup_{x \in K} |(\varphi_\varepsilon * \psi)(x) - \psi(x)| = 0.$$

Although the following result is well-known, we give a proof, as we are not able to find a reference. In addition, its proof is the basis for an analogue result in infinite dimensions.

Proposition 3.25. *Let $p \in [1,\infty)$ be given. Then there is a countable set $\mathcal{C}_{\mathbb{R}^d} \subseteq C_c^\infty(\mathbb{R}^d)$, which is dense in $(C_c(\mathbb{R}^d), \|\cdot\|_\infty)$ and $(L^p(\mathbb{R}^d;\mathrm{d}x), \|\cdot\|_{L^p(\mathrm{d}x)})$. In particular $(C_c(\mathbb{R}^d), \|\cdot\|_\infty)$ and $(L^p(\mathbb{R}^d;\mathrm{d}x), \|\cdot\|_{L^p(\mathrm{d}x)})$ are separable.*

Proof. For each $n \in \mathbb{N}$, define

$$C_{c,n}(\mathbb{R}^d) := \left\{ f \in C_c(\mathbb{R}^d) \mid \operatorname{supp}(f) \subseteq B_n(0) \right\}.$$

Obviously, $C_c(\mathbb{R}^d) = \bigcup_{n\in\mathbb{N}} C_{c,n}(\mathbb{R}^d)$. Hence, if for each $n \in \mathbb{N}$ there is a countable set $\mathcal{C}_{\mathbb{R}^d,n} \subseteq C_c^\infty(\mathbb{R}^d)$, which is dense in $(C_{c,n}(\mathbb{R}^d), \|\cdot\|_\infty)$, the first part of the claim is shown. Consequently, we fix $n \in \mathbb{N}$. By the Stone-Weierstraß theorem we know that the Banach space $(C(\overline{B_n(0)}), \|\cdot\|_\infty)$ is separable. By identifying $C_{c,n}(\mathbb{R}^d)$ with a subspace of $C(\overline{B_n(0)})$, the existence of a countable set $\mathcal{C}^0_{\mathbb{R}^d,n} \subseteq C_{c,n}(\mathbb{R}^d)$, which is dense in $(C_{c,n}(\mathbb{R}^d), \|\cdot\|_\infty)$, is clear. Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a standard approximate identity and define

$$\mathcal{C}_{\mathbb{R}^d,n} := \left\{ \varphi_{\frac{1}{m}} * \psi \mid m \in \mathbb{N} \quad \text{and} \quad \psi \in \mathcal{C}^0_{\mathbb{R}^d,n} \right\} \cap C_{c,n}(\mathbb{R}^d).$$

By Lemma 3.21 and Corollary 3.24, we can conclude that $\mathcal{C}_{\mathbb{R}^d,n} \subseteq C_c^\infty(\mathbb{R}^d)$ is dense in $(C_{c,n}(\mathbb{R}^d), \|\cdot\|_\infty)$. Therefore, the countable set $\mathcal{C}_{\mathbb{R}^d} := \bigcup_{n\in\mathbb{N}} \mathcal{C}_{\mathbb{R}^d,n} \subseteq C_c^\infty(\mathbb{R}^d)$ is dense in $(C_c(\mathbb{R}^d), \|\cdot\|_\infty)$.
Now we show that the closure of $\mathcal{C}_{\mathbb{R}^d}$ is dense in $(L^p(\mathbb{R}^d;\mathrm{d}x), \|\cdot\|_{L^p(\mathrm{d}x)})$. It is easy to see that every $f \in L^p(\mathbb{R}^d;\mathrm{d}x)$ can be approximated in $(L^p(\mathbb{R}^d;\mathrm{d}x), \|\cdot\|_{L^p(\mathrm{d}x)})$ by a function $\tilde{f}$ with compact support. Moreover, by Item (iii) and (iv) from Lemma 3.21, we see that $\tilde{f}$ can be approximated by a function $\tilde{\tilde{f}} \in C_c^\infty(\mathbb{R}^d)$ with respect to $\|\cdot\|_{L^p(\mathrm{d}x)}$. Therefore,

there is some $n \in \mathbb{N}$ such that $\tilde{\tilde{f}} \in C_{c,n}(\mathbb{R}^d)$. Using the first part of the proof, we find a sequence $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{C}_{\mathbb{R}^d,n}$ converging to $\tilde{\tilde{f}}$ as $k \to \infty$ with respect to $\|\cdot\|_\infty$. This yields,

$$\int_{\mathbb{R}^d} (\tilde{\tilde{f}} - f_k)^p \,\mathrm{d}x = \int_{\overline{B_n(0)}} (\tilde{\tilde{f}} - f_k)^p \,\mathrm{d}x \leq \mathrm{d}x(\overline{B_n(0)})\|\tilde{\tilde{f}} - f_k\|_\infty^p \to 0 \quad \text{as} \quad k \to \infty.$$

Consequently the proof is finish. □

Corollary 3.26. *Given $k \in \mathbb{N}$. Then, there exists a countable set in $C_c^\infty(\mathbb{R}^d)$ which is dense in $C_c^k(\mathbb{R}^d)$ equipped with the norm*

$$\|\psi\|_{C^k(\mathbb{R}^d)} := \sum_{s\in\mathbb{N}^d,\ |s|\leq k} \|\partial^s \psi\|_\infty, \quad \psi \in C_c^\infty(\mathbb{R}^d).$$

Proof. Use the arguments from Proposition 3.25 together with Item (ii) from Lemma 3.21. □

For the rest of this section, we go back to an infinite dimensional setting. First, we explain that various well-known function spaces generate the Borel σ-algebra on X.

Lemma 3.27. *Let $k \in \mathbb{N}\cup\{\infty\}$. Denote by $\mathcal{T}$ the weak topology on $(X, (\cdot,\cdot)_X)$ and by $C_\mathcal{T}(X;\mathbb{R})$ the space of weakly continuous functions from X to $\mathbb{R}$. Then,*

$$\sigma(X') = \mathscr{B}(X) = \sigma(\mathcal{F}C_b^k(B_X)) = \sigma(C_\mathcal{T}(X;\mathbb{R})) = \sigma(C(X;\mathbb{R})).$$

Proof. Since every weakly open set is open w.r.t norm topology, we obtain $\sigma(X') \subseteq \mathscr{B}(X)$. The other inclusion follows as $\mathscr{B}(X)$ is generated by the closed balls in $(X, \|\cdot\|_X)$ and

$$\{x \in X \mid \|x-a\|_X \leq r\} = \bigcap_{n\in\mathbb{N}} \left\{ x \in X \mid \sum_{i=1}^n (x-a, e_i)_X^2 \leq r^2 \right\} \in \sigma(X'),$$

for each $a \in X$ and $r \in [0,\infty)$. As $\mathcal{F}C_b^k(B_X) \subseteq C_\mathcal{T}(X;\mathbb{R}) \subseteq C(X;\mathbb{R})$ and preimages of open sets under continuous maps are open we get

$$\sigma(\mathcal{F}C_b^k(B_X)) \subseteq \sigma(C_\mathcal{T}(X;\mathbb{R})) \subseteq \sigma(C(X;\mathbb{R})) \subseteq \mathscr{B}(X) = \sigma(X').$$

To conclude the statement it is enough to show that $\sigma(X') \subseteq \sigma(\mathcal{F}C_b^k(B_X))$. Therefore, let $(\cdot, a)_X \in X'$ be given. It is easy to see that there is a sequence $(\varphi_m)_{m\in\mathbb{N}} \subseteq C_b^k(\mathbb{R})$ converging pointwise to $\mathbb{R} \ni x \mapsto x \in \mathbb{R}$, e.g. use Corollary 3.23. Hence, for each $i \in \mathbb{N}$ we have

$$(x, e_i)_X = \lim_{m\to\infty} \varphi_m((x,e_i)_X) \quad \text{for all} \quad x \in X.$$

As $\varphi_m((\cdot,e_i)_X) \in \mathcal{F}C_b^k(B_X)$ and the limit of measurable functions is measurable, we see that $(\cdot, e_i)_X$ is $\sigma(\mathcal{F}C_b^k(B_X))$ measurable for every $i \in \mathbb{N}$. Since also $(x,a)_X = \sum_{i=1}^\infty (x,e_i)_X(a,e_i)_X$ for every $x \in X$, we obtain $\sigma(\mathcal{F}C_b^k(B_X))$ measurability of $(\cdot,a)_X$. □

We proof the infinite dimensional version of Proposition 3.25, below.

Lemma 3.28. *Suppose $p \in [1,\infty)$, then there is a countable set in $\mathcal{F}C_c^\infty(B_X)$, which is dense in $(L^p(X;\mu), \|\cdot\|_{L^p(\mu)})$. In particular, $(L^p(X;\mu), \|\cdot\|_{L^p(\mu)})$ is separable and $\mathcal{F}C_b^k(B_X)$ is dense in $(L^p(X;\mu), \|\cdot\|_{L^p(\mu)})$.*

Proof. It is well known that $C_b(X)$ is dense in $(L^p(X;\mathrm{d}\mu), \|\cdot\|_{L^p(\mu)})$, compare [DA14, Exercise 9.1] or [DZ02, Proposition 1.2.5]. In view of the theorem of dominated convergence, each $f \in C_b(X)$ can be approximated by $(f \circ \overline{p}_n^X \circ p_n^X)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b(B_X)$, with respect to $\|\cdot\|_{L^p(\mu)}$. So we need to find a countable set in $\mathcal{F}C_c^\infty(\mathbb{R}^d)$ which is dense in

$$\left\{g \circ p_n^X \in \mathcal{F}C_b(B_X) \mid n \in \mathbb{N},\ g \in C_b(\mathbb{R}^n)\right\},$$

with respect to $\|\cdot\|_{L^p(\mu)}$. Recall $\mathcal{C}_{\mathbb{R}^n}$ from Proposition 3.25, then we claim that the countable set

$$\left\{\psi \circ p_n^X \in \mathcal{F}C_c^\infty(B_X) \mid n \in \mathbb{N},\ \psi \in \mathcal{C}_{\mathbb{R}^n}\right\},$$

has the required properties. Let $\varepsilon \in (0,\infty)$ be given. By means of Proposition 3.25, for every $g \in C_b(\mathbb{R}^n)$, there is some $\psi \in \mathcal{C}_{\mathbb{R}^n} \subseteq C_c^\infty(\mathbb{R}^n)$ such that

$$\|g-\psi\|_{L^p(\mathrm{d}x)}^p \leq \varepsilon c_n \quad \text{with} \quad c_n := \sqrt{(2\pi)^n \Pi_{i=1}^n \lambda_i}.$$

By Lemma 3.3 we can conclude

$$\begin{aligned}&\|g \circ p_n^X - \psi \circ p_n^X\|_{L^p(\mu)}^p = \|g-\psi\|_{L^p(\mu^n)}^p = c_n^{-1} \int_{\mathbb{R}^n} (g(x)-\psi(x))^p e^{-\frac{1}{2}\left\langle Q_n^{-1}x,x\right\rangle}\,\mathrm{d}x\\ &\leq c_n^{-1}\|g-\psi\|_{L^p(\mathrm{d}x)}^p \leq \varepsilon.\end{aligned}$$

□

Remark 3.29. *Denote by $\mathcal{A}$ the smallest countable $\mathbb{Q}$-algebra containing the countable subset of $\mathcal{F}C_c^\infty(B_X)$ constructed in Lemma 3.28. $\mathcal{A}$ is dense in $(L^p(X;\mu), \|\cdot\|_{L^p(\mu)})$ and $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{F}C_b^\infty(B_X))$. Using the density of $\mathcal{A}$ in $(L^p(X;\mu), \|\cdot\|_{L^p(\mu)})$ and a similar argument as in Lemma 3.27, we obtain also the converse inclusion and therefore $\sigma(\mathcal{A}) = \sigma(\mathcal{F}C_b^\infty(B_X)) = \mathscr{B}(X)$. Choosing $\mathcal{A}$ for the application of Theorem 2.70 we consequently get $L^p(E;\mathscr{B}(X);\mu) = L^p(E;\sigma(\mathcal{A});\mu)$. Consequently, we do not have to deal with different L^p spaces.*

3.2.2 Weak derivatives and the integration by parts formula

As explained in the beginning of this chapter, the results below are closely related to [DZ02, Chapter 9 and 10], [Da 06, Chapter 10 and 11], as well as [DA14, Section 2] and [LD15, Section 2]. Note that the last mentioned reference also deals with (Gaussian) Sobolev spaces on convex subsets of real separable Hilbert spaces. [DZ02; PR07] work with

$$\mathcal{E}(X) := \operatorname{span}\{\{\Re(e^{i(h,\cdot)_X}), \Im(e^{i(h,\cdot)_X}) \mid h \in X\}\}$$

instead of $\mathcal{F}C_b^\infty(B_X)$. For the applications we have in mind, we need to extend the definitions and results from the above references. We highlight the differences and explain where the extensions take place.

We start with a remark, telling us how the Gâteaux derivative of a sufficient regular function looks like.

Remark 3.30. *It holds that* $Df(x) = \sum_{i\in\mathbb{N}} \partial_{e_i} f(x) e_i$ *for all Gâteaux differentiable functions* $f : X \to \mathbb{R}$ *and* $x \in X$*. Let* $n \in \mathbb{N}$ *be given. If* $f = \varphi \circ p_n$ *for some* $\varphi \in C_b^1(\mathbb{R}^n)$*, then the chain rule implies* $Df(x) = \sum_{i=1}^n \partial_i \varphi(p_n(x)) e_i \in X_n$ *for all* $x \in X$*.*

Lemma 3.31. *For* $f, g \in \mathcal{F}C_b^1(B_X)$ *and* $i \in \mathbb{N}$*, it holds the integration by parts formula*

$$\int_X \partial_{e_i} f g \, \mu = -\int_X f \partial_{e_i} g \, \mathrm{d}\mu + \int_X (x, Q_1^{-1} e_i)_X f g \, \mathrm{d}\mu.$$

Proof. This follows by Lemma 3.3 and the classical integration by parts formula in $\mathbb{R}^n$. □

Theorem 3.32. *Let* $p \in (1, \infty)$ *be given. Moreover, let* $(A, D(A))$ *with* $\mathrm{span}\{e_1, e_2, ...\} \subseteq D(A)$ *and* $\mathrm{span}\{e_1, e_2, ...\} \subseteq D(A^*)$ *be a linear operators on* X*. Assume that for each* $n \in \mathbb{N}$ *there is some* $m \in \mathbb{N}$ *with*

$$A^*(\mathrm{span}\{e_1, ..., e_n\}) \subseteq \mathrm{span}\{e_1, ..., e_m\}.$$

Then the operators

$$AD : \mathcal{F}C_b^1(B_X) \to L^p(X; \mu; X) \quad \text{and}$$
$$(AD, AD^2) : \mathcal{F}C_b^2(B_X) \to L^p(X; \mu; X) \times L^p(X; \mu; \mathcal{L}_2(X))$$

are closable in $L^p(X; \mu)$*. Here and in the following,* $L^p(X; \mu; X)$ *and* $L^p(X; \mu; \mathcal{L}_2(X))$ *are defined in the Bochner-Lebesgue sense.*

Proof. Suppose $(f_j)_{j\in\mathbb{N}}$ is a sequence in $\mathcal{F}C_b^1(B_X)$ converging to 0 in $L^p(X; \mu)$ and such that $ADf_j \to F$ in $L^p(X; \mu; X)$ as $j \to \infty$. Our goal is to show that $F = 0$. Let $k \in \mathbb{N}$ be given. Using the invariance properties of $(A^*, D(A^*))$ we know that there is some $m \in \mathbb{N}$ only depending on k such that

$$(ADf_j, e_k)_X = (Df_j, A^* e_k)_X = \sum_{i=1}^m (Ae_i, e_k)_X \partial_{e_i} f_j.$$

For an arbitrary $g \in \mathcal{F}C_b^1(B_X)$ we obtain by the integration by parts formula 3.31

$$\int_X (ADf_j, e_k)_X g \mathrm{d}\mu = -\sum_{i=1}^m (Ae_i, e_k)_X \int_X f_j \left(\partial_{e_i} g - (x, Q_1^{-1} e_i)_X g\right) \mathrm{d}\mu.$$

Observe that g and $\partial_{e_i} g - (x, Q_1^{-1} e_i)_X g$ are in $L^{\frac{p}{p-1}}(X; \mu)$, by Lemma 3.5. Therefore, by taking the limit $j \to \infty$

$$\int_X (F, e_k)_X g \, \mathrm{d}\mu = 0.$$

By the density of $\mathcal{F}C_b^1(B_X)$ in $L^p(X; \mu)$, we conclude $(F, e_k)_X = 0$ for all $k \in \mathbb{N}$. Consequently $F = 0$, as desired.
To show that the second operator is closable, we proceed similarly. Indeed, let $(f_j)_{j\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(B_X)$ converge to 0 in $L^p(X; \mu)$ and be such that $ADf_j \to F$ in $L^p(X; \mu; X)$ and $AD^2 f_j \to G$ in $L^p(X; \mu; \mathcal{L}_2(X))$, as $j \to \infty$. As above, $F = 0$. Now for $k, l \in \mathbb{N}$, using the

invariance property of $(A^*, D(A^*))$, there is some $m \in \mathbb{N}$ only dependent on $l \in \mathbb{N}$ such that

$$(AD^2 f_j e_k, e_l)_X = \sum_{i=1}^m (A^* e_l, e_i)_X \partial_{e_k} \partial_{e_i} f_j.$$

For arbitrary $g \in \mathcal{F}C_b^2(B_X)$, we obtain by the integration by parts formula

$$\begin{aligned}\int_X (AD^2 f_j e_k, e_l)_X g \,\mathrm{d}\mu &= -\sum_{i=1}^m (A^* e_l, e_i)_X \int_X \partial_{e_i} f_j (\partial_{e_k} g - (x, Q_1^{-1} e_k)_X g) \,\mathrm{d}\mu \\ &= -\int_X (e_l, ADf_j)_X (\partial_{e_k} g - (x, Q_1^{-1} e_k)_X g) \,\mathrm{d}\mu.\end{aligned}$$

Arguing as in the first part, we observe $(Ge_k, e_l)_X = 0$ in $L^p(X;\mu)$ for all $k, l \in \mathbb{N}$, implying $G = 0$ in $L^p(X;\mu;\mathcal{L}_2(X))$. □

Theorem 3.32 is an extension of [DA14, Lemma 2.3], where only the case $A = Q^\theta$ for $\theta \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ is considered. Moreover, note that for $A = Q^{\frac{1}{2}}$ the corresponding Sobolev space coincides with the usual Sobolev space of Malliavin calculus.

Definition 3.33. For an operator $(A, D(A))$, as in Theorem 3.32 and $p \in (1,\infty)$, we denote by $W_A^{1,p}(X;\mu)$ and $W_A^{2,p}(X;\mu)$ the domain of the closure of

$$\begin{aligned} AD &: \mathcal{F}C_b^1(B_X) \to L^p(X;\mu;X) \quad \text{and} \\ (AD, AD^2) &: \mathcal{F}C_b^2(B_X) \to L^p(X;\mu;X) \times L^p(X;\mu;\mathcal{L}_2(X)) \end{aligned}$$

in $L^p(X;\mu)$, respectively. If $A = \mathrm{Id}$, we simply write $W^{1,p}(X;\mu)$ and $W^{2,p}(X;\mu)$. We equip these spaces with the corresponding Graph norms.

Definition 3.34. Let $\theta \in \mathbb{R}$, $p \in (1,\infty)$ and a linear operator $(C, D(C))$ be given. Assume, further that $\mathrm{span}\{e_1, e_2, ...\} \subseteq D(C)$ and $\mathrm{span}\{e_1, e_2, ...\} \subseteq D(C^*)$ and that for each $n \in \mathbb{N}$ there is some $m \in \mathbb{N}$ with

$$C^*(\mathrm{span}\{e_1, ..., e_n\}) \subseteq \mathrm{span}\{e_1, ..., e_m\}.$$

By Theorem 3.32, it is reasonable to consider the infinite dimensional Gaussian Sobolev spaces $W_{Q^\theta C}^{1,p}(X;\mu)$ and $W_{Q^\theta}^{2,p}(X;\mu)$. Moreover, for $f \in W_{Q^\theta}^{1,p}(X;\mu)$ and $n \in \mathbb{N}$ we set

$$\partial_{e_i} f := (Q^\theta Df, e_i)_X \frac{1}{\lambda_i^\theta} \in L^p(X;\mu) \quad \text{and} \quad P_n Df := \sum_{i=1}^n \partial_{e_i} f e_i \in L^p(X;\mu).$$

Note that $\partial_{e_i} f_m$ converges to $\partial_{e_i} f$ in $L^p(X;\mu)$ if $(f_m)_{m\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(B_X)$ is a sequence converging to f in $W_{Q^\theta}^{1,p}(X;\mu)$.

The natural question: "When do weak and classical derivatives coincide?" is discussed in the next proposition, which is taken from [Da 06, Section 10.1.1].

Proposition 3.35. *Assume $f \in C^1(X;\mathbb{R})$ and that there are constants $\kappa \in (0,\infty)$ and $0 < \varepsilon < \frac{1}{2\inf_{k\in\mathbb{N}} \lambda_k^{-1}}$ such that*

$$|f(x)| + \|Df(x)\|_X \leq \kappa e^{\varepsilon \|x\|_X^2} \quad \text{for all} \quad x \in X.$$

Then, $f \in W^{1,2}(X;\mu)$ and the classical and the weak derivative coincide for μ-a.e. $x \in X$.

The following lemmas extend the chain and product rule from [Da 06, Proposition 10.8] and [Da 06, Proposition 10.9], where only $A = \mathrm{Id}$ is considered.

Lemma 3.36. *Let $p \in (1,\infty)$ and $(A, D(A))$ be as in Theorem 3.32 and suppose $f, g \in W_A^{1,p}(X;\mu)$. If g and ADg are bounded, then $fg \in W_A^{1,p}(X;\mu)$ with*

$$AD(fg) = AD(f)g + fAD(g).$$

Proof. Suppose that $f \in \mathcal{F}C_b^1(B_X)$. Since g is in $W_A^{1,p}(X;\mu)$, there is a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(B_X)$ with

$$\lim_{n\to\infty} g_n = g \quad \text{in} \quad L^p(X;\mu) \quad \text{and} \quad \lim_{n\to\infty} ADg_n = ADg \quad \text{in} \quad L^p(X;\mu;X).$$

By the classical product rule we have

$$AD(fg_n) = AD(f)g_n + fAD(g_n).$$

Since f and ADf are bounded, we obtain

$$\begin{aligned} \|fg_n - fg\|_{L^p(\mu)} &\leq \|f\|_\infty \|g_n - g\|_{L^p(\mu)} \to 0 \quad \text{as} \quad n\to\infty \quad \text{and} \\ \|AD(fg_n) - AD(f)g + fAD(g)\|_{L^p(\mu)} &\leq \|ADf\|_\infty \|g_n - g\|_{L^p(\mu)} \\ &\quad + \|f\|_\infty \|AD(g_n) - AD(g)\|_{L^p(\mu)} \\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned}$$

By definition we get the desired product rule for $f \in \mathcal{F}C_b^1(B_X)$.
For general $f \in W_A^{1,p}(X;\mu)$ we choose an approximating sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(B_X)$ with

$$\lim_{n\to\infty} f_n = f \quad \text{in} \quad L^p(X;\mu) \quad \text{and} \quad \lim_{n\to\infty} ADf_n = ADf \quad \text{in} \quad L^p(X;\mu;X).$$

By the previous result we know

$$AD(fg_n) = AD(f_n)g + f_nAD(g).$$

Using the boundedness of g and ADg and a similar reasoning as above, we conclude this proof. ☐

Lemma 3.37. *Let $p \in (1,\infty)$ and $(A, D(A))$ be as in Theorem 3.32 and suppose $f \in W_A^{1,p}(X;\mu)$. If $\Psi \in C_b^1(\mathbb{R})$, then $\Psi \circ f \in W_A^{1,p}(X;\mu)$ with*

$$AD(\Psi \circ f) = (\Psi' \circ f)ADf.$$

Proof. Choose an approximating sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(B_X)$ with

$$\lim_{n\to\infty} f_n = f \quad \text{in} \quad L^p(X;\mu) \quad \text{and} \quad \lim_{n\to\infty} ADf_n = ADf \quad \text{in} \quad L^p(X;\mu;X).$$

Dropping to a subsubsequence (without changing the notation) the above convergence results hold pointwisely μ-a.e.. Note that

$$AD(\Psi \circ f_n) = (\Psi' \circ f_n)ADf_n.$$

Therefore, using the continuity of Ψ, we obtain μ-a.e. pointwise convergence of $\Psi \circ f_n$ and $AD(\Psi \circ f_n)$ to $\Psi \circ f$ and $(\Psi' \circ f)ADf$, respectively. In view of the boundedness of Ψ and Ψ' we conclude by the theorem of dominated convergence that

$$\lim_{n\to\infty} \Psi \circ f_n = \Psi \circ f \quad \text{in} \quad L^p(X;\mu)$$

and also

$$\begin{aligned}\int_X |(\Psi' \circ f_n)ADf_n - (\Psi' \circ f)ADf|^p \,\mathrm{d}\mu &\leq 2^p \|\Psi'\|_\infty^p \int_X |ADf_n - ADf|^p \,\mathrm{d}\mu \\ &\quad + 2^p \int_X |\Psi' \circ f_n - \Psi' \circ f|^p |ADf|^p \,\mathrm{d}\mu \\ &\to 0 \quad \text{as } n \to \infty,\end{aligned}$$

the final result. □

The subsequent lemma is a tool to define Sobolev spaces for measures having exponential type densities with respect to an infinite dimensional Gaussian measure.

Lemma 3.38. *Let $p \in (1,\infty)$ and $\theta \in \mathbb{R}$. Then the following statements are true*

(i) For each $f \in W^{1,p}_{Q^\theta}(X;\mu)$ it holds $|f| \in W^{1,p}_{Q^\theta}(X;\mu)$ with $Q^\theta|f| = Q^\theta Df\, sign(f)$ and $\mathbb{1}_{\{f=0\}} Q^\theta Df = 0$, μ-a.e..

(ii) Suppose $\Phi : X \to (-\infty,\infty]$ is bounded from below and such that $\Phi \in W^{1,p}_{Q^\theta}(X;\mu)$. Then $e^{-\Phi} \in W^{1,p}_{Q^\theta}(X;\mu)$ with

$$Q^\theta D(e^{-\Phi}) = -e^{-\Phi} Q^\theta D\Phi.$$

Proof.

(i) Approximate $|f|$ by the functions $f_n := \sqrt{f^2 + \frac{1}{n}}$, $n \in \mathbb{N}$. By means of the chain rule from Lemma 3.37, we get $Q^\theta Df_n = -\frac{f}{\sqrt{f^2+\frac{1}{n}}} Q^\theta Df$. An application of the theorem of dominated convergence finishes the proof.

For the second part, let $f \in \mathcal{F}C_b^1(B_X)$. It is enough to show that for each $i \in \mathbb{N}$ and $g \in \mathcal{F}C_b^1(B_X)$ we have

$$\int_{\{f=0\}} \partial_{e_i} fg \,\mathrm{d}\mu = 0.$$

This implies $\mathbb{1}_{\{f=0\}}\partial_{e_i} f = 0$, μ-a.e., by the density of $\mathcal{F}C_b^1(B_X)$ in $L^p(X;\mu)$ and therefore $\mathbb{1}_{\{f=0\}}Q^\theta Df = \sum_{i=1}^\infty \lambda_i^\theta e_i \mathbb{1}_{\{f=0\}}\partial_{e_i} f = 0$, μ-a.e..

By Lemma 3.22, there is a function $\eta \in C_c^\infty(\mathbb{R})$ with $\eta(0) = 1$, support in $[-1,1]$ and values in $[0,1]$. Define the sequence $(\eta_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R})$ by $\eta_n(x) := \eta(nx)$, $x \in \mathbb{R}$. Then, the function $\eta_n \circ f$ converges pointwise to $\mathbb{1}_{\{f=0\}}$ as $n \to \infty$ and we have $\eta_n \circ f \in \mathcal{F}C_b^1(B_X)$ with $\partial_{e_i}(\eta_n \circ f) = (\eta_n' \circ f)\partial_{e_i} f$. By means of the integration by parts formula from Lemma 3.31, we calculate

$$\begin{aligned}\int_X \partial_{e_i} f g(\eta_n \circ f)\,\mathrm{d}\mu = -\int_X f(\partial_{e_i} g)(\eta_n \circ f)\,\mathrm{d}\mu - \int_X fg(\eta_n' \circ f)\partial_{e_i} f\,\mathrm{d}\mu \\ + \int_X fg(\eta_n \circ f)(x, Q^{-1}e_i)_X\,\mathrm{d}\mu.\end{aligned}$$

Note that $\eta_n \circ f$ is bounded by 1 and $|f\eta_n' \circ f| = |nf\eta'(nf)| \le \|\eta'\|_\infty$, by the support property of η. Moreover, $nf\eta'(nf)$ converges pointwisely to 0 as $n \to \infty$. Hence, by the theorem of dominated convergence

$$\int_{\{f=0\}} \partial_{e_i} f g\,\mathrm{d}\mu = -\int_{\{f=0\}} f(\partial_{e_i} g)\,\mathrm{d}\mu + \int_{\{f=0\}} fg(x, Q^{-1}e_i)_X\,\mathrm{d}\mu = 0.$$

For general $f \in W^{1,p}_{Q^\theta}(X;\mu)$, recall the definition of $\partial_{e_i} f$ from Definition 3.34 and note that $\mathbb{1}_{\{f=0\}}Q^\theta Df = \sum_{i=1}^\infty \lambda_i^\theta e_i \mathbb{1}_{\{f=0\}}\partial_{e_i} f$, μ-a.e.. Since there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(B_X)$ such that f_n and $\partial_{e_i} f_n$ converges pointwisely μ-a.e. to f and $\partial_{e_i} f$, respectively, the claim follows.

(ii) Without loss of generality we assume that $\Phi \ge 0$. Define the function $\Psi_n \in C_b^1(\mathbb{R})$ by

$$\Psi_n(x) := e^{-\sqrt{x^2+\frac{1}{n}}} \quad \text{for} \quad x \in \mathbb{R}\,.$$

One can calculate for every $x \in \mathbb{R}$

$$\Psi_n'(x) = -\frac{x}{\sqrt{x^2+\frac{1}{n}}}e^{-\sqrt{x^2+\frac{1}{n}}} \to -\operatorname{sign}(x)e^{-|x|}, \quad \text{as} \quad n \to \infty.$$

By Lemma 3.37, we know that $\Psi_n \circ \Phi \in W^{1,p}_{Q^\theta}(X;\mu)$ with

$$Q^\theta D(\Psi_n \circ \Phi) = (\Psi_n' \circ \Phi)Q^\theta D\Phi.$$

It is easy to see that the functions Ψ_n and Ψ_n' are bounded independent of $n \in \mathbb{N}$. Hence, $\Psi_n \circ \Phi \to e^{-\Phi}$ as $n \to \infty$ in $L^p(X;\mu)$ and $\Psi_n' \circ \Phi \to -\operatorname{sign}(\Phi)e^{-\Phi}$ as $n \to \infty$ in $L^{\frac{p}{p-1}}(X;\mu)$. This yields $e^{-\Phi} \in W^{1,p}_{Q^\theta}(X;\mu)$ with

$$Q^\theta D(e^{-\Phi}) = -e^{-\Phi}\operatorname{sign}(\Phi)Q^\theta D\Phi = -e^{-\Phi}Q^\theta D\Phi,$$

where we used $\Phi \ge 0$ and Item (i) for the last equality.

□

The lemma below, provides a useful connection between Lipschitz continuous functions, functions in $W^{1,2}(X;\mu)$ and Gâteaux differentiable functions.

Lemma 3.39. *Denote by $Lip(X)$ the space of Lipschitz continuous function from X to $\mathbb{R}$. It holds $Lip(X) \subseteq W^{1,2}(X;\mu)$. Moreover, every Lipschitz continuous function is Gâteaux differentiable μ-a.e..*

Proof. The first statement is [Da 06, Proposition 10.11], while the second is a consequence of [Phe78, Theorem 6]. □

Next, we extend the integration by parts formula for infinite dimensional Gaussian measures to a bigger class of functions. For $\theta \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ the statement was already mentioned in [DA14], but without giving a detailed proof. This is done for general $\theta \in \mathbb{R}$, below.

Proposition 3.40. *Let $\theta \in \mathbb{R}$, $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} \leq 1$ and $f \in W^{1,p}_{Q^\theta}(X;\mu)$, $g \in W^{1,q}_{Q^\theta}(X;\mu)$. Then*

$$\int_X \partial_{e_i} f g \,\mathrm{d}\mu = -\int_X f \partial_{e_i} g \,\mathrm{d}\mu + \int_X (x, Q^{-1}e_i)_X f g \,\mathrm{d}\mu.$$

Proof. First suppose $g \in \mathcal{F}C_b^1(B_X)$. Since $\partial_{e_i} g$ and $(x, Q^{-1}e_i)_X g$ are in $L^s(X;\mu)$ for all $s \in [1,\infty)$, the claim is valid for all $f \in W^{1,p}_{Q^\theta}(X;\mu)$ with $p \in (1,\infty)$ by an approximation argument.

For $p, q \in (1,\infty)$ with $\frac{1}{p} + \frac{1}{q} \leq 1$, either p or q has to be bigger or equal than two. Without loss of generality we assume, $p \geq 2$ and $f \in W^{1,p}_{Q^\theta}(X;\mu)$. For $g \in W^{1,q}_{Q^\theta}(X;\mu)$ we find a corresponding approximating sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^1(B_X)$ in $W^{1,q}_{Q^\theta}(X;\mu)$. Then, it holds for all $n \in \mathbb{N}$

$$\int_X \partial_{e_i} f g_n \,\mathrm{d}\mu = -\int_X f \partial_{e_i} g_n \,\mathrm{d}\mu + \int_X (x, Q^{-1}e_i)_X f g_n \,\mathrm{d}\mu.$$

Therefore, it is enough to show that $\partial_{e_i} f, f, (x, Q^{-1}e_i)_X f \in L^p(X;\mu)$. For $\partial_{e_i} f$ and f this is obviously valid, while for $(x, Q^{-1}e_i)_X f$ we argue as follows.

First, let $p > 2$. We start by deriving an auxiliary result. A direct calculation shows that $\mathbb{R} \ni x \mapsto \psi(x) := |x|_X^{p-2} x$ is in $C^1(\mathbb{R})$ with $\psi'(x) = (p-1)|x|^{p-2}$. Consider a sequence of cut-off functions $(\varphi_n)_{n\in\mathbb{N}}$ on $\mathbb{R}$ provided by Corollary 3.23 and set $\psi_n(x) := \varphi_n((x, Q^{-1}e_i)_X)\psi((x, Q^{-1}e_i)_X)$. Then $\psi_n \in \mathcal{F}C_b^1(B_X)$. In view of the arguments in the first part, we get for each $h \in \mathcal{F}C_b^1(B_X)$

$$\begin{aligned}
\int_X |h|^p \varphi_n((x, Q^{-1}e_i))|(x, Q^{-1}e_i)|_X^p \,\mathrm{d}\mu &= \int_X (x, Q^{-1}e_i)_X |h|^p \psi_n \,\mathrm{d}\mu \\
&= \int_X p|h|^{p-1}\mathrm{sign}(h)\partial_{e_i} h \psi_n + |h|^p \partial_{e_i}\psi_n \,\mathrm{d}\mu \\
&= \int_X p|h|^{p-1}\mathrm{sign}(h)\partial_{e_i} h \psi_n \\
&\quad + |h|^p \frac{1}{\lambda_i}\varphi_n'((x, Q^{-1}e_i)_X)\psi((x, Q^{-1}e_i)_X) \\
&\quad + |h|^p \varphi_n((x, Q^{-1}e_i)_X)\frac{1}{\lambda_i}\psi'((x, Q^{-1}e_i)_X) \,\mathrm{d}\mu.
\end{aligned}$$

By means of the properties of $(\varphi_n)_{n\in\mathbb{N}}$ and the theorem of dominated convergence, we can conclude for $n \to \infty$

$$\begin{aligned}\int_X |h|^p |(x, Q^{-1}e_i)_X|^p \,\mathrm{d}\mu = \int_X &|(x, Q^{-1}e_i)_X|^{p-2}(x, Q^{-1}e_i)_X p|h|^{p-1}\mathrm{sign}(h)\partial_{e_i} h \\ &+ |h|^p (p-1)\frac{1}{\lambda_i}|(x, Q^{-1}e_i)_X|^{p-2}\,\mathrm{d}\mu.\end{aligned} \tag{3.2}$$

Using Equation (3.2) and Youngs inequality for products, we see that for each $h \in \mathcal{F}C_b^1(B_X)$ and $t \in (0,\infty)$

$$\begin{aligned}&\int_X |h(x, Q^{-1}e_i)_X|^p \,\mathrm{d}\mu \\ &= \int_X p|(x, Q^{-1}e_i)_X|^{p-2}(x, Q^{-1}e_i)_X |h|^{p-1}\mathrm{sign}(h)\partial_{e_i} h + (p-1)\frac{1}{\lambda_i}|(x, Q^{-1}e_i)_X|^{p-2}|h|^p \,\mathrm{d}\mu \\ &\leq p\int_X |h(x, Q^{-1}e_i)_X|^{p-1}|\partial_{e_i} h| + \frac{1}{\lambda_i}|h(x, Q^{-1}e_i)_X|^{p-2}|h|^2\,\mathrm{d}\mu \\ &\leq (p-1)t^{\frac{p}{p-1}}\int_X |h(x, Q^{-1}e_i)_X|^p\,\mathrm{d}\mu + t^{-p}\int_X |\partial_{e_i} h|^p\,\mathrm{d}\mu \\ &\quad + \frac{1}{\lambda_i}(p-2)t^{\frac{p}{p-2}}\int_X |h(x, Q^{-1}e_i)_X|^p\,\mathrm{d}\mu + 2\frac{1}{\lambda_i}t^{-\frac{p}{2}}\int_X |h|^p\,\mathrm{d}\mu.\end{aligned}$$

So by choosing t small enough, we find a constant $C \in (0,\infty)$ only depending on p, t and $i \in \mathbb{N}$ such that

$$\int_X |h(x, Q^{-1}e_i)_X|^p\,\mathrm{d}\mu \leq C\left(\int_X |h|^p + |\partial_{e_i} h|^p\,\mathrm{d}\mu\right). \tag{3.3}$$

Now we can extend (3.3) to functions $h \in W^{1,p}_{Q^\theta}(X;\mu)$ by using a $\mathcal{F}C_b^1(B_X)$ approximation of h in $W^{1,p}_{Q^\theta}(X;\mu)$. For $p = 2$, we refer to [DZ02, Proposition 9.2.8]. This completes our proof. □

Below, we provide a useful criterion to determine if the pointwise limit of a μ-a.e. convergent sequence in $W^{1,p}_{Q^\theta}(X;\mu)$, $\theta \in [0,\infty)$ and $p \in (1,\infty)$, is again in $W^{1,p}_{Q^\theta}(X;\mu)$.

Remark 3.41. *A Banach space $(Y, \|\cdot\|_Y)$ has the so-called Banach-Saks property, if every bounded sequence $(y_n)_{n\in\mathbb{N}} \subseteq Y$ has a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that its Cesàro mean converges in Y, i.e. $\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N y_{n_k}$ exists.*
Let $\theta \in [0,\infty)$ and $p \in (1,\infty)$ be given. Similar to [Bog98, Lemma 5.4.4], we can establish that $W^{1,p}_{Q^\theta}(X;\mu)$ has the Banach-Saks property and for each bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $W^{1,p}_{Q^\theta}(X;\mu)$ with $\lim_{n\to\infty} f_n = f$, μ-a.e., it holds $f \in W^{1,p}_{Q^\theta}(X;\mu)$.

We continue the analysis of Sobolev spaces with respect to infinite dimensional Gaussian measures with a very useful approximation result. Recall the Moreau-Yosida approximation Φ_t, $t > 0$ for $\Phi : X \to (-\infty,\infty]$ and $D_0\Phi$ from Example 2.11.

Lemma 3.42. *[LD15, Lemma 2.2] Suppose $\Phi : X \to (-\infty,\infty]$ is as in Example 2.11, i.e. convex, bounded from below, lower-semicontinuous and not identically to ∞. If $x \mapsto \|D_0\Phi\|_X \in L^{p_1}(X;\mu)$ for some $p_1 \in (1,\infty)$, then for each $1 \leq p_0 < p_1$*

(i) $D\Phi = D_0\Phi$, *μ-a.e..*

(ii) $\Phi \in W^{1,p_0}_{Q^\theta}(X;\mu)$ *and* $\lim_{t\to 0}\Phi_t = \Phi$ *in* $W^{1,p_0}_{Q^\theta}(X;\mu)$ *for all* $\theta \in [0,\infty)$.

Note that similar results hold for the Moreau-Yosida approximation along the Cameron-Martin space, studied in [GF16, Section 3] and [GF18, Section 4]. We omit its introduction, since we don't need it for our further applications.

Even though Lemma 3.42 is only applicable if Φ is convex, it is very useful, as $D\Phi_t$ is Lipschitz continuous. This allows us to construct even more regular approximations in order to derive the Poincaré inequality from Corollary 3.60.
In the last part of this section, we generalize some of the above constructions and results to the case where the infinite dimensional Gaussian measure is additionally equipped with a density. Without further mentioning we consider potential functions $\Phi : X \to (-\infty,\infty]$, as described in the subsequent definition.

Definition 3.43. Suppose $\Phi : X \to (-\infty,\infty]$ is measurable, bounded from below and such that $\int_X e^{-\Phi}\,\mathrm{d}\mu > 0$. For such Φ, we consider the measure $\mu^\Phi := \frac{1}{\int_X e^{-\Phi}\,\mathrm{d}\mu} e^{-\Phi}\mu$ and set $\mu^0 := \mu$, as well as

$$\mu^\Phi(f) := \int_X f\,\mathrm{d}\mu^\Phi \quad \text{for,} \quad f \in L^1(X;\mu^\Phi).$$

Lemma 3.44. *Suppose* $p \in [1,\infty)$. *Then,* $L^p(X;\mu) \subseteq L^p(X;\mu^\Phi)$ *and the space of smooth cylinder functions* $\mathcal{F}C_b^\infty(B_X)$ *is dense in* $L^p(X;\mu^\Phi)$.

Proof. The inclusion of spaces follow, as for each $f \in L^p(X;\mu)$ it holds

$$\|f\|^p_{L^p(\mu^\Phi)} \le \frac{e^{\inf_{x\in X} -\Phi(x)}}{\mu(e^{-\Phi})}\|f\|^p_{L^p(\mu)}. \tag{3.4}$$

The density of $\mathcal{F}C_b^\infty(B_X)$ in $L^p(X;\mu^\Phi)$ follows by [DA14, Lemma 2.2]. □

Definition 3.45. Let $p \in (1,\infty)$ and Φ as in Definition 3.43 be given. Moreover, let $(A, D(A))$ with $\operatorname{span}\{e_1,e_2,...\} \subseteq D(A)$ be a linear operator on X. If the operator

$$AD : \mathcal{F}C_b^1(B_X) \to L^p(X;\mu^\Phi;X)$$

is closable in $L^p(X;\mu^\Phi)$, we set $W^{1,p}_A(X;\mu^\Phi) := D(\overline{AD})$. If additionally, the operator

$$(AD, AD^2) : \mathcal{F}C_b^2(B_X) \to L^p(X;\mu^\Phi;X) \times L^p(X;\mu;\mathcal{L}_2(X))$$

is closable in $L^2(X;\mu^\Phi)$, we set $W^{2,p}_A(X;\mu^\Phi) := D(\overline{(AD,AD^2)})$. Both, $W^{1,p}_A(X;\mu^\Phi)$ and $W^{2,p}_A(X;\mu^\Phi)$, are Banach spaces, if we equip them with the corresponding graph norms.

Lemma 3.46. *Let* $p \in (1,\infty)$ *and* $(A, D(A))$ *be as in Definition 3.45 and suppose* $f,g \in W^{1,p}_A(X;\mu^\Phi)$.

(i) If g *and* ADg *are bounded, then* $fg \in W^{1,p}_A(X;\mu^\Phi)$ *with*

$$AD(fg) = AD(f)g + fAD(g).$$

(ii) If $\Psi \in C_b^1(\mathbb{R})$, then $\Psi \circ f \in W_A^{1,p}(X;\mu^\Phi)$ with

$$AD(\Psi \circ f) = (\Psi' \circ f)ADf.$$

Proof. Use the same arguments as in Lemma 3.36 and Lemma 3.37. □

As in [DA14, Chapter 2.2], we can extend the integration by parts formula for measures of type μ^Φ.

Lemma 3.47. *Let $\theta \in \mathbb{R}$ and $\Phi \in W_{Q^\theta}^{1,2}(X;\mu)$. Then, for $f, g \in \mathcal{F}C_b^1(B_X)$ and $i \in \mathbb{N}$, it holds the integration by parts formula*

$$\int_X \partial_{e_i} f g \, \mu^\Phi = -\int_X f \partial_{e_i} g \, \mathrm{d}\mu^\Phi + \int_X (x, Q_1^{-1} e_i)_X f g \, \mathrm{d}\mu^\Phi + \int_X \partial_{e_i} \Phi f g \, \mathrm{d}\mu^\Phi. \tag{3.5}$$

Proof. Since $\Phi \in W_{Q^\theta}^{1,2}(X;\mu)$, we obtain by Lemma 3.38 that $e^{-\Phi} \in W_{Q^\theta}^{1,2}(X;\mu)$ with $Q^\theta D(e^{-\Phi}) = -e^{-\Phi} Q^\theta D\Phi$. By Lemma 3.36 we know that $ge^{-\Phi} \in W_{Q^\theta}^{1,2}(X;\mu)$ for each $g \in \mathcal{F}C_b^1(B_X)$, with

$$Q^\theta D(ge^{-\Phi}) = e^{-\Phi} Q^\theta D(g) + g Q^\theta D(e^{-\Phi}) = e^{-\Phi} Q^\theta D(g) - ge^{-\Phi} Q^\theta D\Phi.$$

Hence, the claim follows by Proposition 3.40. □

Proposition 3.48. *Let $\theta \in \mathbb{R}$ and $p \in [2,\infty)$ be given. Further, assume that $\Phi \in W_{Q^\theta}^{1,2}(X;\mu)$. Then the following statements hold.*

(i) The operators

$$Q^\theta D : \mathcal{F}C_b^1(B_X) \to L^p(X;\mu^\Phi;X) \quad \textit{and}$$
$$(Q^\theta D, Q^\theta D^2) : \mathcal{F}C_b^2(B_X) \to L^p(X;\mu^\Phi;X) \times L^p(X;\mu;\mathcal{L}_2(X))$$

are closable in $L^p(X;\mu^\Phi)$. Therefore, it is reasonable to consider the Sobolev spaces $W_{Q^\theta}^{1,p}(X;\mu^\Phi)$ and $W_{Q^\theta}^{2,p}(X;\mu^\Phi)$. Again we use the abbreviations $W^{1,p}(X;\mu^\Phi) := W_{Id}^{1,p}(X;\mu^\Phi)$ and $W^{2,p}(X;\mu^\Phi) := W_{Id}^{2,p}(X;\mu^\Phi)$.

(ii) It holds $W_{Q^\theta}^{1,p}(X;\mu) \subseteq W_{Q^\theta}^{1,p}(X;\mu^\Phi)$. Moreover, for each $\bar{\theta} \in \mathbb{R}$ with $\theta \leq \bar{\theta}$ we have

$$W_{Q^\theta}^{1,p}(X;\mu^\Phi) \subseteq W_{Q^{\bar{\theta}}}^{1,p}(X;\mu^\Phi).$$

(iii) For each $q \in [2,\infty)$ with $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}$ the integration by parts formula (3.5) is valid for $f \in W_{Q^\theta}^{1,p}(X;\mu^\Phi)$ and $g \in W_{Q^\theta}^{1,q}(X;\mu^\Phi)$.

Proof. (i) This follows as in [DA14, Lemma 2.3], where only the case $\theta \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$ was considered. For the sake of completeness, we give the proof here. Compare also Theorem 3.32. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}C_b^1(B_X)$ converge to 0 in $L^p(X;\mu^\Phi)$ and be such that

$Q^\theta Df_n \to F$ in $L^p(X;\mu^\Phi;X)$ as $n\to\infty$. Let $k\in\mathbb{N}$ be given. Since $(e_i)_{i\in\mathbb{N}}$ is a basis of eigenvectors of Q, we see

$$(Q^\theta Df_n, e_k)_X = \lambda_k^\theta \partial_{e_k} f_n.$$

For an arbitrary $g\in\mathcal{F}C_b^1(B_X)$, we obtain by the integration by parts formula 3.31

$$\int_X (Q^\theta Df_n, e_k)_X g \mathrm{d}\mu^\Phi = -\lambda_k^\theta \int_X f_n(\partial_{e_k} g - (x, Q_1^{-1}e_k)_X g - \partial_{e_k}\Phi g)\,\mathrm{d}\mu^\Phi.$$

Observe that $L^2(X;\mu^\Phi)\subseteq L^{\frac{p}{p-1}}(X;\mu^\Phi)$, since $p\geq 2$. Therefore, $g\in L^{\frac{p}{p-1}}(X;\mu^\Phi)$ and $\partial_{e_k} g - (\cdot, Q_1^{-1}e_k)_X g - \partial_{e_k}\Phi g \in L^{\frac{p}{p-1}}(X;\mu^\Phi)$. This implies, by taking the limit $n\to\infty$, that

$$\int_X (F, e_k)_X g\,\mathrm{d}\mu^\Phi = 0.$$

By the density of $\mathcal{F}C_b^1(B_X)$ in $L^p(X;\mu^\Phi)$, we conclude $(F,e_k)_X = 0$ for all $k\in\mathbb{N}$. Hence, $F=0$. The proof for the second order Sobolev space follows as in Theorem 3.32.

(ii) $W^{1,p}_{Q^\theta}(X;\mu)\subseteq W^{1,p}_{Q^\theta}(X;\mu^\Phi)$ follows directly, using Inequality (3.4) from Lemma 3.44. Suppose $\bar\theta\in\mathbb{R}$ with $\theta\leq\bar\theta$. Since $Q\in\mathcal{L}_1^+(X)$ is non-degenerate, we know that $\lambda_i\to 0$ for $i\to\infty$ with $\lambda_i>0$. In particular, there is some $k\in\mathbb{N}$ such that $\lambda_i\in(0,1)$ for all $i\in\mathbb{N}$ with $i>k$. Since $x^\theta\leq x^{\bar\theta}$ for all $x\in(0,1)$, we obtain for all $f\in\mathcal{F}C_b^1(B_X)$ and $c_{\theta,\bar\theta} := \max_{1\leq i\leq k}\lambda_i^{2(\bar\theta-\theta)}+1$:

$$\begin{aligned}\|Q^{\bar\theta}Df\|_X^2 &= \sum_{i=1}^k (\partial_{e_i}f)^2\lambda_i^{2\bar\theta} + \sum_{i=k+1}^\infty (\partial_{e_i}f)^2\lambda_i^{2\bar\theta}\\ &\leq \max_{1\leq i\leq k}\lambda_i^{2(\bar\theta-\theta)}\sum_{i=1}^k(\partial_{e_i}f)^2\lambda_i^{2\theta} + \sum_{i=k+1}^\infty(\partial_{e_i}f)^2\lambda_i^{2\theta}\\ &\leq c_{\theta,\bar\theta}\|Q^\theta Df\|_X^2.\end{aligned}$$

The claim follows by definition of the involved Sobolev spaces.

(iii) Since $(\cdot, Q_1^{-1}e_i)_X$ and $\partial_{e_i}\Phi$ are in $L^2(W;\mu^\Phi)$ for all $i\in\mathbb{N}$, this follows by an approximation argument, compare Proposition 3.40.

□

Before we consider (infinite dimensional) Ornstein-Uhlenbeck semigroups in the next section, we introduce an important class of Potential functions Φ. This class plays an important role in Section 8.1 and Section 8.3. For that, assume

$$(X,(\cdot,\cdot)_X) = (L^2((0,1);\mathrm{d}\xi),(\cdot,\cdot)_{L^2(\mathrm{d}\xi)}),$$

where $\mathrm{d}\xi$ denotes the classical Lebesgue measure on $((0,1),\mathscr{B}(0,1))$. In addition, we fix a continuous differentiable function $\phi:\mathbb{R}\to\mathbb{R}$, which is bounded from below and such that its derivative grows at most of order $b\in[0,\infty)$, i.e. there exists $a\in(0,\infty)$ such that

$$|\phi'(x)|\le a(1+|x|^b)\quad\text{for all}\quad x\in\mathbb{R}\,.$$

Using the mean value theorem, it is easy to check that there exists some $\tilde a\in(0,\infty)$ such that

$$|\phi(x)|\le \tilde a(1+|x|^{b+1})\quad\text{for all}\quad x\in\mathbb{R}\,.$$

Therefore, ϕ grows at most of order $b+1$. For such ϕ it is reasonable to define

$$\Phi:X\to(-\infty,\infty],\ x\mapsto\Phi(x):=\begin{cases}\int_0^1\phi(x(\xi))\,\mathrm{d}\xi & \text{if } x\in L^{b+1}((0,1);\mathrm{d}\xi)\\ \infty & \text{else.}\end{cases}$$

Remark 3.49. *Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to some element $x\in X$. Since ϕ is bounded from below, the same applies to Φ. Hence, $\inf_{n\in\mathbb{N}}\Phi(x_n)\in(-\infty,\infty]$ and there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, such that $\lim_{k\to\infty}\Phi(x_{n_k})=\inf_{n\in\mathbb{N}}\Phi(x_n)$. We also find a subsubsequence $(x_{n_{k_i}})_{i\in\mathbb{N}}$ converging to x pointwisely $\mathrm{d}\xi$-a.e.. Suppose $\Phi(x)\neq\infty$. Then using Fatous lemma and the continuity of ϕ, we can conclude*

$$\liminf_{n\to\infty}\Phi(x_n)\ge\inf_{n\in\mathbb{N}}\Phi(x_n)=\liminf_{i\to\infty}\Phi(x_{n_{k_i}})\ge\int_0^1\liminf_{i\to\infty}\phi(x_{n_{k_i}}(\xi))\,\mathrm{d}\xi=\Phi(x).$$

If $\Phi(x)=\infty$ also $\liminf_{n\to\infty}\Phi(x_n)=\infty$. In summary, Φ is lower semicontinuous.

It is well known that $B_X=(e_k)_{k\in\mathbb{N}}=(\sqrt{2}\sin(k\pi\cdot))_{k\in\mathbb{N}}$ is an orthonormal basis of X. Recall the corresponding orthogonal projection P_n and define

$$\Phi_n:=\Phi\circ P_n:X\to(-\infty,\infty).$$

Before we show that $\Phi\in W^{1,p}(X;\mu)$ for all $p\in[1,\infty)$, we state general results about the integrability and approximation of the function

$$X\times(0,1)\ni(x,\xi)\mapsto x(\xi)\in\mathbb{R}$$

in $L^q(X\times(0,1);\mu\otimes\mathrm{d}\xi)$, $q\in[2,\infty)$.

Lemma 3.50. *[DA14, Lemma 5.1] For all $p\in[1,\infty)$, there is a constant $C_p\in(0,\infty)$ such that*

$$\int_X\int_0^1|P_nx(\xi)|^p\,\mathrm{d}\xi\,\mu(\mathrm{d}x)\le C_p\left(\sum_{k=1}^n\frac{1}{(\pi k)^2}\right)^{\frac{p}{2}}$$

and the sequence $((x,\xi)\mapsto P_nx(\xi))_{n\in\mathbb{N}}$ converges in $L^p(X\times(0,1);\mu\otimes\mathrm{d}\xi)$. If $p\ge2$, then it also holds

$$\int_{L^p(0,1)}\int_0^1|x(\xi)|^p\,\mathrm{d}\xi\,\mu(\mathrm{d}x)<\infty,\quad \mu(L^p(0,1))=1$$

and the sequence $((x,\xi)\mapsto P_nx(\xi))_{n\in\mathbb{N}}$ converges to $(x,\xi)\mapsto x(\xi)$ in $L^p(X\times(0,1);\mu\otimes\mathrm{d}\xi)$. Further, the map

$$L^p((0,1);\mathrm{d}\xi)\ni x\mapsto\|x\|_{L^p(0,1)}\in\mathbb{R}$$

is in $L^q(X;\mu)$ for every $q\in[1,\infty)$. In particular, $\Phi\in L^q(X;\mu)$ for every $q\in[1,\infty)$.

Proposition 3.51. *For each $p \in [1, \infty)$, we have $\lim_{n\to\infty} \Phi_n = \Phi$ in $L^p(X;\mu)$. If $p > 1$ it holds,*

$$\Phi \in W^{1,p}(X;\mu) \quad \textit{with} \quad D\Phi(x) = \phi' \circ x \quad \textit{for } \mu\textit{-a.e.} \quad x \in X.$$

In particular, $D\Phi \in L^\infty(X;\mu)$, if $b = 0$ (i.e. ϕ has bounded derivative).

Proof. By Lemma 3.50 and the growth condition on ϕ, it holds $\Phi_n \in L^p(X;\mu)$. Moreover, $\Phi_n \in C^1(X;\mathbb{R})$ with $D\Phi_n(x) = \phi'(P_n x) \in L^p((0,1);\mathrm{d}\xi)$, as it is the composition of the smooth function $X \ni x \mapsto P_n x \in C^0([0,1];\mathbb{R})$ and the $C^1(C^0([0,1];\mathbb{R});\mathbb{R})$ function $C^0([0,1];\mathbb{R}) \ni y \mapsto \int_0^1 \phi(y(\xi))\,\mathrm{d}\xi$. An application of Proposition 3.35 shows that $\Phi_n \in W^{1,2}(X;\mu)$.
Using the Hölder inequality, the mean value theorem and Lemma 3.50, we find $A \in (0,\infty)$ such that

$$\begin{aligned}
\int_X |\Phi_n - \Phi|^p \,\mathrm{d}\mu &\le \int_X \int_0^1 (\phi(P_n x(\xi)) - \phi(x(\xi))^p \,\mathrm{d}\xi\, \mu(\mathrm{d}x) \\
&\le a^p \int_X \int_0^1 \left(1 + (|P_n x(\xi)| + |x(\xi)|)^b\right)^p |P_n x(\xi) - x(\xi)|^p \,\mathrm{d}\xi\, \mu(\mathrm{d}x) \\
&\le a^p \int_X \left\| \left(1 + (|P_n x| + |x|)^b\right)^p \right\|_X \left\| (P_n x - x)^p \right\|_X \mu(\mathrm{d}x) \\
&\le A \left(\int_X \left\| (P_n x - x)^p \right\|_X^2 \mu(\mathrm{d}x) \right)^{\frac{1}{2}} \\
&= A \left(\int_X \int_0^1 \left| (P_n x(\xi) - x(\xi)) \right|^{2p} \mathrm{d}\xi\, \mu(\mathrm{d}x) \right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by Lemma 3.50 we have $\lim_{n\to\infty} \Phi_n = \Phi$ in $L^p(X;\mu)$.
Observe that for each $j \in \mathbb{N}$ with $j \le n$, we can estimate

$$\begin{aligned}
&\int_X |\partial_{e_j} \Phi_n(x) - (\phi'(x), e_j)_X|^p \,\mu(\mathrm{d}x) = \int_X |(\phi'(P_n x) - \phi'(x), e_j)_X|^p \,\mu(\mathrm{d}x) \\
&\le \int_X \int_0^1 |\phi'(P_n x(\xi)) - \phi'(x(\xi)) e_j(\xi)|^p \,\mathrm{d}\xi\, \mu(\mathrm{d}x).
\end{aligned}$$

By Lemma 3.50, we know that

$$(x,\xi) \mapsto \left(\sqrt{2}a \left(2 + |P_n x(\xi)|^b + |x(\xi)|^b\right)\right)^p$$

converges in $L^1(X \times (0,1); \mu \otimes \mathrm{d}\xi)$ as $n \to \infty$. Therefore, [Bré83, Theorem IV.9] provides a function $g \in L^1(X \times (0,1); \mu \otimes \mathrm{d}\xi)$ such that for some subsequence, for $\mu \otimes \mathrm{d}\xi$-a.e. $(x,\xi) \in X \times (0,1)$ and for all $k \in \mathbb{N}$

$$|\phi'(P_{n_k} x(\xi)) - \phi'(x(\xi)) e_j(\xi)|^p \le \left(\sqrt{2}a \left(2 + |P_{n_k} x(\xi)|^b + |x(\xi)|^b\right)\right)^p \le g(x,\xi).$$

Since for a subsequence $\lim_{i\to\infty} P_{n_{k_i}} x(\xi) = x(\xi)$ for $\mu \otimes \mathrm{d}\xi$-a.e. $(x,\xi) \in X \times (0,1)$ and ϕ' is continuous, we can apply the theorem of dominated convergence to show that there exists a subsequence $(\Phi_{n(j)_k})_{k\in\mathbb{N}}$ of $(\Phi_n)_{n\in\mathbb{N}}$ such that $(\partial_{e_j} \Phi_{n(j)_k})_{k\in\mathbb{N}}$ converges pointwisely μ-a.e. and in $L^p(X;\mu)$ to $(\phi'(\cdot), e_j)_X$.

To continue, we first establish that $(D\Phi_{n(j)_k})_{k\in\mathbb{N}}$ is bounded in $L^p(X;\mu;X)$. This follows since

$$\int_X \|D\Phi_n(x)\|_X^p \,\mu(\mathrm{d}x) = \int_X \left(\int_0^1 |\phi'(P_n x)|^2 \,\mathrm{d}\xi\right)^{\frac{p}{2}} \mu(\mathrm{d}x)$$
$$\leq \int_X \int_0^1 \left(a\left(1+|P_n x(\xi)|^b\right)\right)^p \,\mathrm{d}\xi\,\mu(\mathrm{d}x)$$

and the right-hand side is bounded independent of $n\in\mathbb{N}$ by Lemma 3.50. Since the sequence $(\Phi_n)_{n\in\mathbb{N}}\subseteq W^{1,2}(X;\mu)$ is bounded in $L^p(X;\mu)$, we get boundedness of $(\Phi_{n(j)_k})_{k\in\mathbb{N}}$ in $W^{1,p}(X;\mu)$. As $W^{1,p}(X;\mu)$ has the Banach-Saks property for every $p\in(1,\infty)$, see Remark 3.41, we know that there exits a subsequence $(\Phi_{n(j)_{k_i}})_{i\in\mathbb{N}}$ of $(\Phi_{n(j)_k})_{k\in\mathbb{N}}$ and $\Psi\in W^{1,p}(X;\mu)$ such that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \Phi_{n(j)_{k_i}} = \Psi \quad\text{in}\quad W^{1,p}(X;\mu).$$

Using $\lim_{n\to\infty}\Phi_n=\Phi$ in $L^p(X;\mu)$, we see

$$\Psi = \lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^N \Phi_{n(j)_{k_i}} = \Phi \quad\text{in}\quad L^p(X;\mu)$$

and therefore Ψ does not depend on the subsequence we were starting with. In particular, the above argumentation shows that $\Phi\in W^{1,p}(X;\mu)$ with $D\Phi = D\Psi$ in $L^p(X;\mu;X)$. Moreover, we know that there is a subsequence $(N_m)_{m\in\mathbb{N}}$ such that for μ-a.e. $x\in X$

$$\lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m} D\Phi_{n(j)_{k_i}}(x) = D\Phi(x).$$

We finally conclude that for μ-a.e. $x\in X$ and for all $j\in\mathbb{N}$

$$(D\Phi(x),e_j)_X = \lim_{m\to\infty}\frac{1}{N_m}\sum_{i=1}^{N_m}\partial_{e_j}\Phi_{n(j)_{k_i}}(x) = (\phi'(x),e_j)_X$$

and therefore

$$D\Phi(x) = \sum_{j=1}^\infty (\phi'(x),e_j)_X e_j = \phi'(x) \quad\text{for } \mu\text{-a.e. } x\in X,$$

as $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of X. □

Remark 3.52. *Suppose ϕ and Φ are as described above. There are two more natural situations in which we derive similar results as in Proposition 3.51.*

(i) Assume that ϕ and therefore also Φ is convex. Then, the Moreau-Yosida approximation $(\Phi_t)_{t>0}$ from Lemma 3.42 converges to Φ in $W^{1,p}(X;\mu)$ for all $p\in[1,\infty)$ and $D\Phi(x)=\phi'(x)$ for all $x\in L^{2b}(X;\mu)$, compare [DA14, Proposition 5.1]. We do not give the proof here, but it relies on Lemma 3.42 and the facts that for each $x\in L^{2b}(X;\mu)$ we have $\partial\Phi(x)=\{\phi'(x)\}$, as well as $x\mapsto\|\phi'(x)\|_X\in L^p(X;\mu)$ for all $p\in[1,\infty)$.

(ii) *Assume that ϕ'' exists, is continuous and grows at most of order $\tilde{b} \in [0, \infty)$. Then, using similar arguments as in the beginning of the proof of Proposition 3.51, we find $\tilde{A} \in (0, \infty)$, such that for all $p \in [1, \infty)$*

$$\int_X \left\| D\Phi_n(x) - \phi'(x) \right\|_X^p \mu(\mathrm{d}x) \leq \int_X \int_0^1 \left(\phi'(P_n x(\xi)) - \phi'(x(\xi)\right)^p \mathrm{d}\xi \, \mu(\mathrm{d}x)$$
$$\leq \tilde{A} \left(\int_X \left\| (P_n x - x)^p \right\|_X^2 \mu(\mathrm{d}x) \right)^{\frac{1}{2}}.$$

Therefore, by Lemma 3.50, we know that $(D\Phi_n)_{n\in\mathbb{N}}$ converges to $\phi'(\cdot)$ in $L^p(X;\mu;X)$. As $\lim_{n\to\infty} \Phi_n = \Phi$ in $L^p((0,1);\mathrm{d}\xi)$, we get $\lim_{n\to\infty} \Phi_n = \Phi$ in $W^{1,p}(X;\mu)$ with $D\Phi(x) = \phi'(x)$ for μ-a.e. $x \in X$.

Both, the approximation from Proposition 3.51 and the one from Remark 3.52 Item (i), play an important role for our applications. Note that the first one does not demand the convexity of ϕ, while the second yields one with Lipschitz continuous derivatives.

3.2.3 Infinite dimensional (perturbed) Ornstein-Uhlenbeck semigroups

The first part of this section starts with a review about known results concerning infinite dimensional Ornstein-Uhlenbeck semigroups on $L^p(X;\mu)$. Here, μ denotes a centered infinite dimensional Gaussian measure defined in Equation (3.10) below. The semigroups are defined via Mehlers formula on the space of bounded Borel measurable functions and extended to all of $L^p(X;\mu)$, in the case that μ can be identified as the corresponding invariant measure. We focus on the relevant results for our applications, i.e. the smoothing property of the semigroup and conditions under which the Ornstein-Uhlenbeck semigroups are strongly continuous on $L^p(X;\mu)$. Moreover, we give an explicit representation of the corresponding generator (infinite dimensional Ornstein-Uhlenbeck operator), on the core of smooth bounded cylinder functions. All mentioned results are taken from [DZ02, Chapter 10] and [Da 06, Chapter 8].
In the second part of this section, we state general essential m-dissipativity results for infinite dimensional Ornstein-Uhlenbeck operators perturbed by the gradient of a sufficient regular potential. This is important, since variants of such infinite dimensional Ornstein-Uhlenbeck operators naturally appear as we study the longtime behavior of the semigroup associated to the infinite dimensional Langevin operator, compare Chapter 6.
We consider the following setting

(i) $(B, D(B))$ is the generator of a strongly continuous semigroup $(e^{tB})_{t\geq 0}$ on X.

(ii) $C \in \mathcal{L}^+(X)$ and for all $t \in [0,\infty)$, we have $C_t \in \mathcal{L}_1^+(X)$, where

$$C_t x := 2 \int_0^t e^{sB} C e^{sB^*} x \, \mathrm{d}s, \quad x \in X.$$

Then, for all $t \in (0, \infty)$, it is reasonable to consider the Gaussian measure $\mu_t := N(0, C_t)$ on $(X, \mathscr{B}(X))$. Setting $\mu_0 := \delta_0$, we can introduce the family of operators $(S_t)_{t\geq 0}$, defined on $\mathscr{B}_b(X;\mathbb{R})$ by

$$S_t f(x) := \int_X f(e^{tB}x + y) \, \mu_t(\mathrm{d}y), \quad x \in X. \tag{3.6}$$

The formula above is also known as Mehlers formula. One can show that $S_{t+s} = S_t S_s$ for all $s, t \in [0, \infty)$. Moreover, under certain conditions, the semigroup $(S_t)_{t\geq 0}$ enjoys strong smoothing properties. This is part of the next proposition.

Proposition 3.53. *The following statements are equivalent.*

(i) $e^{tB}(X) \subseteq C_t^{\frac{1}{2}}(X)$ *for all* $t \in (0, \infty)$.

(ii) $S_t(\mathscr{B}_b(X;\mathbb{R})) \subseteq C_b^\infty(X;\mathbb{R})$ *for all* $t \in (0, \infty)$.

Remark 3.54.

(i) One can show that Item (i) of Proposition 3.53 is satisfied if C has a continuous inverse.

(ii) The linear operator $\Gamma(t) := C_t^{-\frac{1}{2}} e^{tB}$, *where* $C_t^{-\frac{1}{2}}$ *denotes the pseudo-inverse of* $C_t^{\frac{1}{2}}$ *(for* $x \in X$ *the pseudo-inverse* $C_t^{-\frac{1}{2}}$ *applied to* x *denotes the element* y *with minimal norm such that* $C_t^{\frac{1}{2}} y = x$*), is closable and therefore extendable to a bounded linear operator in* $\mathcal{L}(X)$*, compare [Da 06, Section 8.3.1]. With this newly introduced operator one can show that for* $f \in \mathscr{B}_b(X;\mathbb{R})$

$$S_t f(x) = \int_X f(y) e^{-\frac{1}{2}\|\Gamma(t)x\|_X^2 + (\Gamma(t)x, C_t^{-\frac{1}{2}} y)_X} \mu_t(\mathrm{d}y), \quad x \in X. \tag{3.7}$$

Lemma 3.55. *Assume that one of the items in Proposition 3.53 is valid and let* $f \in C^1(X;\mathbb{R})$ *be convex with Lipschitz continuous derivative. Then, for all* $t \in (0, \infty)$, $S_t f \in C^\infty(X;\mathbb{R})$ *is convex and* $DS_t f$ *is Lipschitz continuous. Moreover,* $DS_t f$ *has bounded derivatives of all orders and we have for all* $x, h \in X$

$$(DS_t f(x), h)_X = \int_X (\Gamma(t)x, C_t^{-\frac{1}{2}} y)_X f(e^{tB} x + y) \, \mu_t(\mathrm{d}y) \tag{3.8}$$

$$= \int_X (Df(e^{tB} x + y), e^{tB} h)_X \, \mu_t(\mathrm{d}y). \tag{3.9}$$

Proof. A function $f \in C^1(X;\mathbb{R})$ with Lipschitz continuous derivative has at most quadratic growth. In particular, we can define $S_t f$ as in (3.6). $S_t f$ can also be represented by the alternative formula (3.7) from Remark 3.54. Therefore, using formula (3.7) and an iterative argument, we derive that $S_t f \in C^\infty(X;\mathbb{R})$ and (3.8) is valid, compare also [Da 06, Theorem 8.16]. Convexity of $S_t f$ is inherited by the convexity of f.
Formula (3.9) follows directly, using the Mehler representation formula. From here, we can calculate denoting by L_{Df} the Lipschitz constant of Df, that for each $x_1, x_2, h \in X$ with $\|h\|_X \leq 1$

$$\begin{aligned} |(DS_t f(x_1) - DS_t f(x_2), h)_X| &\leq \int_X (Df(e^{tB} x_1 + y) - Df(e^{tB} x_2 + y), e^{tB} h)_X \, \mu_t(\mathrm{d}y) \\ &\leq L_{Df} \|e^{tB}(x_1 - x_2)\|_X \|e^{tB} h\|_X \\ &\leq L_{Df} \|e^{tB}\|_{\mathcal{L}(X)}^2 \|x_1 - x_2\|_X. \end{aligned}$$

Consequently, $DS_t f$ is Lipschitz continuous with bounded gradient. An iterative argument generalizes this to higher orders. □

To extend the semigroup $(S_t)_{t\geq 0}$ to an L^p-space, we assume that

$$\sup_{t\in(0,\infty)} \operatorname{tr}[C_t] = 2\int_0^\infty \operatorname{tr}[e^{sB}Ce^{sB^*}]\,\mathrm{d}s < \infty,$$

which implies that

$$C_\infty x := 2\int_0^\infty e^{sB}Ce^{sB^*}x\,\mathrm{d}s, \quad x \in X$$

defines an operator in $\mathcal{L}_1^+(X)$. It is reasonable to define the Gaussian measure

$$\mu := N(0, C_\infty). \tag{3.10}$$

We state a condition implying the existence of $C_\infty \in \mathcal{L}_1^+(X)$ and of an unique invariant measure for $(S_t)_{t\geq 0}$, below. In this case, the invariant measure μ is given by $N(0, C_\infty)$.

Proposition 3.56. *Assume that there are constants $M, \omega \in (0,\infty)$ such that*

$$\|e^{tB}\|_{\mathcal{L}(X)} \leq Me^{-\omega t}, \quad t \in [0,\infty).$$

Then the following statements hold true.

(i) μ is the unique invariant measure for the semigroup $(S_t)_{t\geq 0}$, which can be extended to a sub-Markovian strongly continuous contraction semigroup on $L^p(X;\mu)$ (this extension is again denoted by $(S_t)_{t\geq 0}$). The semigroup and its extension are called the Ornstein-Uhlenbeck semigroup.

(ii) $\mathcal{F}C_b^\infty(B_X)$ is a core for the generator of the semigroup $(S_t)_{t\geq 0}$. Denote the generator by $(N, D(N))$, then for $f \in \mathcal{F}C_b^\infty(B_X)$ it holds

$$Nf(x) = \operatorname{tr}[CD^2 f(x)] + (x, B^* Df(x))_X, \quad x \in X.$$

The operator $(N, D(N)))$ is also called the (infinite dimensional) Ornstein-Uhlenbeck operator (on $L^p(X;\mu)$).

(iii) If $p = 2$, then $S_t^* = S_t$ for all $t \in [0,\infty)$, if and only if $e^{tB}C = Ce^{tB^*}$ for all $t \in [0,\infty)$. In this case, $(N, \mathcal{F}C_b^\infty(B_X))$ is an essentially self-adjoint operator on $L^2(X;\mu)$ with self-adjoint closure $(N, D(N))$.

Proof. (i) This is a combination of [Da 06, Theorem 8.20] and [Da 06, Proposition 8.21].

(ii) Use the same arguments as in [Da 06, Theorem 8.21] and replace the exponential functions by the finitely based bounded smooth cylinder functions.

(iii) Apply [DZ02, Proposition] for the symmetry result. The statement for $(N, D(N))$ follows by Lemma 2.29 and Theorem 2.30.

□

The example below describes a particularly easy situation where all the results from Proposition 3.56 are applicable. It is a useful tool to use the smoothing from Lemma 3.55.

Example 3.57. Let $(B, D(B))$ be a self-adjoint operator $(X, (\cdot,\cdot)_X)$ and such that B^{-1} is of trace class. Let $C = \mathrm{Id}$ and suppose there is some $\omega \in (0,\infty)$ such that $(Bx, x)_X \leq -\omega \|x\|_X^2$ for all $x \in D(B)$. We can therefore consider the strongly continuous contraction semigroup $(e^{tB})_{t\geq 0}$. Furthermore, for each $t \in [0,\infty)$, we can calculate

$$C_t = B^{-1}(e^{2tB} - \mathrm{Id}) \quad \text{and} \quad \|e^{tB}\|_{\mathcal{L}(X)} \leq e^{-\omega t}.$$

Since $C = \mathrm{Id}$ has a continuous inverse, we know that Item (i) from Proposition 3.53 is valid. The assumption from Proposition 3.56 is also satisfied and it holds

$$\mu_t = N(0, B^{-1}(e^{2tB} - \mathrm{Id})) \quad \text{for every } t \in [0,\infty) \text{ and} \quad \mu = N(0, -B^{-1}).$$

Lastly, by the transformation formula for Gaussian measures from [Da 06, Proposition 1.18], we have

$$S_t f(x) = \int_X f(e^{tB}x + y)\,\mu_t(\mathrm{d}y) = \int_X f(e^{tB}x + \sqrt{\mathrm{Id} - e^{2tB}}y)\,\mu(\mathrm{d}y). \tag{3.11}$$

At this point we note that for f, as in Lemma 3.55, the Lipschitz constant of $DS_t f$, $t \in (0,\infty)$ does not depend on t.

For the the next proposition we fix a (sufficient regular) function $\Phi : X \to (-\infty, \infty]$. We perturb the Ornstein-Uhlenbeck operator $(N, D(N))$ in $L^2(X;\mu)$ by $f \mapsto (CD\Phi, Df)_X$. To be more precise, we analyze the perturbed operator $(N^\Phi, \mathcal{F}C_b^\infty(X))$ defined by

$$N^\Phi f(x) := \mathrm{tr}[CD^2 f(x)] + (x, B^* Df(x))_X + (CD\Phi(x), Df(x))_X, \quad x \in X.$$

Even though our general situation in Chapter 6 demands that unbounded linear diffusion coefficients C are allowed, we want to give an overview over typical and well studied situations in which $(N^\Phi, \mathcal{F}C_b^\infty(B_X))$ is an essential self-adjoint operator on $L^2(X;\mu^\Phi)$. Note that such perturbation of Ornstein-Uhlenbeck operators were also studied in [DA14] and for Neumann problems in [LD15].

Proposition 3.58. *Let $(B, D(B))$ be a self-adjoint operator and suppose there is some $\omega \in (0,\infty)$ such that $(Bx, x)_X \leq -\omega\|x\|_X^2$ for all $x \in D(B)$. Further, assume that one of the following items is valid.*

(i) $C = Id$ and B^{-1} is of trace class. Additionally, $e^{-\Phi}, e^{-\frac{1}{2}\Phi} \in W^{1,2}(X;\mu)$ and $\|D\Phi\|_X \in L^4(X;\mu^\Phi)$.

(ii) $C = Id$ and B^{-1} is of trace class. In addition, $e^{-\Phi}, e^{-\frac{1}{2}\Phi} \in W^{1,2}(X;\mu)$, Φ is convex, non-negative, lower semicontinuous and $\|D\Phi\|_X \in L^r(X;\mu^\Phi)$ for some $r \in (2,\infty)$.

(iii) $C = (-B)^{-\varepsilon}$ for some $\varepsilon \in (0,1)$ and the operator $(-B)^{-(1+\varepsilon)}$ is of trace class. Moreover, $e^{-\Phi} \in L^p(X;\mu)$ for all $p \in [1,\infty)$ and $\Phi \in W^{1,4}_{C^{\frac{1}{2}}}(X;\mu^\Phi)$.

Then, $(N^\Phi, \mathcal{F}C_b^\infty(B_X))$ is essentially self-adjoint on $L^2(X;\mu^\Phi)$.

Proof. The proofs can be found in [DZ02, Theorem 12.2.1], [DZ02, Theorem 12.3.2] and [DT00, Theorem 3.2], respectively. □

We end this section by noting that such results can also be derived, where the perturbation $D\Phi$ is replaced by a suitable vector field $F : D(F) \subseteq X \to X$. For that, compare e.g. [DR02] and the recent articles [BF22], [Pri21].

3.2.4 Poincaré inequalities

Basic Poincaré inequalities for infinite dimensional Gaussian measures μ and measures of type μ^Φ where Φ is as in Item (ii) of Proposition 3.58, can be found e.g. in [DZ02, Chapter 10 and 12]. Below, we derive a generalized Poincaré inequality designed for the application in Chapter 6. We use a double approximation defined in terms of the Moreau-Yosida approximation and a smoothing Ornstein-Uhlenbeck semigroup. Such approximations have already been used to establish the essential m-dissipativity of operators associated to singular dissipative stochastic equations in Hilbert spaces, compare e.g. [DR02].
The proof of our Poincaré inequality is based on the following result, which is a special case of [AFP19, Proposition 4.5], where also Poincaré inequalities on convex subsets of X are established. In [AFP19], the authors use pointwise gradient estimates for semigroups associated to perturbed Ornstein-Uhlenbeck operators.

Proposition 3.59. *[AFP19, Proposition 4.5] Let $\Phi : X \to \mathbb{R}$ be convex and suppose $\Phi \in C^2(X;\mathbb{R}) \cap W^{1,p}_{Q^{\frac{1}{2}}}(X;\mu)$ for all $p \in [1,\infty)$. Then, for all $f \in \mathcal{F}C_b^\infty(B_X)$ it holds*

$$\lambda_1 \int_X (QDf, Df)_X \mathrm{d}\mu^\Phi \geq \int_X (f - \mu^\Phi(f))^2 \mathrm{d}\mu^\Phi.$$

Corollary 3.60. *Suppose $\Phi : X \to (-\infty,\infty]$ is convex, bounded from below, lower semicontinuous and not identically to ∞. Then, for all $f \in \mathcal{F}C_b^\infty(B_X)$ it holds*

$$\lambda_1 \int_X (QDf, Df)_X \mathrm{d}\mu^\Phi \geq \int_X (f - \mu^\Phi(f))^2 \mathrm{d}\mu^\Phi.$$

Proof. As mentioned above, the idea of the proof is to approximate Φ. Afterwards, we apply the Poincaré inequality from Proposition 3.59.
Denote by $(\Phi_\alpha)_{\alpha>0}$ the Moreau-Yosida approximation of Φ. By Example 2.11, we know that Φ_α is convex and differentiable with Lipschitz continuous derivative. Furthermore, for all $x \in X$, $\lim_{\alpha\to 0} \Phi_\alpha(x) = \Phi(x)$. To apply the Poincaré inequality from Proposition 3.59 Φ_α is not regular enough. Therefore, let $\beta > 0$ and define the function $\Phi_{\alpha,\beta} := S_\beta \Phi_\alpha$, where $(S_\beta)_{\beta\geq 0}$ is the Ornstein-Uhlenbeck semigroup considered in Example 3.57. In formulas

$$\Phi_{\alpha,\beta}(x) = \int_X \Phi_\alpha(e^{\beta B}x + \sqrt{\mathrm{Id} - e^{2\beta B}}y)\, N(0,-B^{-1})(\mathrm{d}y),$$

where we choose the representation from Equation (3.11). As discussed in Lemma 3.55, it holds for every $\alpha, \beta \in (0,\infty)$

(i) $\Phi_{\alpha,\beta}$ is convex and has derivatives of all orders.

(ii) $D\Phi_{\alpha,\beta}$ is Lipschitz continuous (with Lipschitz constant independent of β) and has bounded derivatives of all orders.

In particular, Proposition 3.59 is applicable. We get for all $f \in \mathcal{F}C_b^\infty(B_X)$

$$\lambda_1 \int_X (QDf, Df)_X \mathrm{d}\mu^{\Phi_{\alpha,\beta}} \geq \int_X (f - \mu^{\Phi_{\alpha,\beta}}(f))^2 \mathrm{d}\mu^{\Phi_{\alpha,\beta}}. \tag{3.12}$$

Since the derivative of Φ_α is Lipschitz continuous, one can show that Φ_α has at most quadratic growth. Hence, there exits a constant $c \in (0,\infty)$ such that for all $x, y \in X$

$$\begin{aligned}|\Phi_\alpha(e^{\beta B}x + \sqrt{\mathrm{Id} - e^{2\beta B}}y)| &\leq c\left(1 + \|e^{\beta B}x + \sqrt{\mathrm{Id} - e^{2\beta B}}y)\|_X^2\right)\\ &\leq 2c\left(1 + \|e^{\beta B}x\|_X^2 + \|\mathrm{Id} - e^{2\beta B}\|_{\mathcal{L}(X)}\|y\|_X^2\right)\\ &\leq 2c\left(1 + \|x\|_X^2 + 2\|y\|_X^2\right).\end{aligned}$$

Above, we also used that $\|e^{\beta B}\|_{\mathcal{L}(\mathcal{X})} \leq 1$ for all $\beta \in [0,\infty)$. Since, for all $x, y \in X$, $\lim_{\beta\to 0}\Phi_\alpha(e^{\beta B}x + \sqrt{\mathrm{Id} - e^{2\beta B}}y) = \Phi_\alpha(x)$, we obtain $\lim_{\beta\to 0}\Phi_{\alpha,\beta}(x) = \Phi_\alpha(x)$ by the theorem of dominated convergence. This yields $\lim_{\alpha\to 0}\lim_{\beta\to 0}\Phi_\alpha(x) = \Phi(x)$ for all $x \in X$. As $-\infty < \inf_{\bar{x}\in X}\Phi(\bar{x}) \leq \Phi_\alpha(x) \leq \Phi(x)$ for all $x \in X$, it is easy to see that $-\infty < \inf_{\bar{x}\in X}\Phi(\bar{x}) \leq \Phi_{\alpha,\beta}(x)$ for all $x \in X$. In particular $e^{-\Phi_\alpha}$ and $e^{-\Phi_{\alpha,\beta}}$ are bounded independent of α, β. An iterative application of the theorem of dominated convergence shows that

$$\lim_{\alpha\to 0}\lim_{\beta\to 0}\mu^{\Phi_{\alpha,\beta}}(g) = \mu^\Phi(g) \quad \text{for all} \quad g \in L^1(X;\mu).$$

Consequently, taking the limits $\beta \to 0$ and $\alpha \to 0$ in Inequality (3.12) yields the claim. □

The subsequent lemma shows that the Poincaré inequality is stable under additive perturbations with bounded oscillation. Consequently, we can also consider potentials which are not necessary convex.

Lemma 3.61. *Suppose $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 : X \to (-\infty,\infty]$ is as in Corollary 3.60 and $\Phi_2 : X \to \mathbb{R}$ is measurable with $\|\Phi_2\|_{osc} := \sup_{x\in X}\Phi_2(x) - \inf_{x\in X}\Phi_2(x) < \infty$. Then*

$$\lambda_1 e^{\|\Phi_2\|_{osc}}\int_X (QDf, Df)_X \,\mathrm{d}\mu^\Phi \geq \int_X \left(f - \mu^\Phi(f)\right)^2 \mathrm{d}\mu^\Phi \quad \textit{for all} \quad f \in \mathcal{F}C_b^\infty(B_X).$$

Proof. For $\Phi_2 = 0$ the claim is already valid by Corollary 3.60. Using that

$$\int_X \left(f - \mu^\Phi(f)\right)^2 \mathrm{d}\mu^\Phi \leq \int_X (f - c)^2 \,\mathrm{d}\mu^\Phi$$

for all $c \in \mathbb{R}$ we can estimate

$$\begin{aligned}\int_X \left(f - \mu^\Phi(f)\right)^2 \mathrm{d}\mu^\Phi &\leq \int_X \left(f - \mu^{\Phi_1}(f)\right)^2 \mathrm{d}\mu^\Phi\\ &\leq e^{-\inf_{x\in X}\Phi_2(x)}\frac{\int_X e^{-\Phi_1}\,\mathrm{d}\mu}{\int_X e^{-\Phi}\,\mathrm{d}\mu}\int_X \left(f - \mu^{\Phi_1}(f)\right)^2 \mathrm{d}\mu^{\Phi_1}\\ &\leq \lambda_1 e^{-\inf_{x\in X}\Phi_2(x)}\frac{\int_X e^{-\Phi_1}\,\mathrm{d}\mu}{\int_X e^{-\Phi}\,\mathrm{d}\mu}\int_X (QDf, Df)_X \,\mathrm{d}\mu^{\Phi_1}\\ &\leq \lambda_1 e^{\|\Phi_2\|_{osc}}\int_X (QDf, Df)_X \,\mathrm{d}\mu^\Phi.\end{aligned}$$

□

Remark 3.62. *Note that the Poincaré inequality from Corollary 3.60 above is valid without assuming that $\Phi \in W^{1,2}(X;\mu)$.*

4

The abstract hypocoercivity framework and method

We include here the slight reformulation from [Ale23, Section 2.2] of the abstract Hilbert space hypocoercivity method presented in [GS14] and further extended in [GS16] by Grothaus and Stilgenbauer. Historically, a variant of this method was first developed by Dolbeault, Mouhot and Schmeiser [DMS09], on an algebraic level, in the context of hypocoercivity for kinetic equations with linear relaxation term. Algebraic in the sense that the authors did not consider domain issues of the involved unbounded operators. This gap was filled in [GS14] by simultaneously making the method more applicable. Indeed, Grothaus and Stilgenbauer explained rigorously that it is sufficient to check the data and hypocoercivity assumption (compare below) on a suitable core for the operator describing the dynamic. Especially in our context of hypocoercivity for infinite dimensional Langevin dynamics, it is essential to work with a core on which we know that the involved operators are well-defined and explicitly given to verify the data and hypocoercivity assumptions. This chapter does not contain any new results. Based on [GS16, Theorem 1.1], we only add a more explicit calculation of the constants determining the speed of convergence, compare Theorem 4.5.

Let H be a separable Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$, which has an orthogonal decomposition $H = H_1 \oplus H_2$ with corresponding orthogonal projections $P : H \to H_1$, $(\mathrm{Id} - P) : H \to H_2$. Let $(L, D(L))$ further be a densely defined linear operator that generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on H. We assume that L has the structure described by the data assumptions **D1**-**D3** formulated in the assumption below.

Assumption (D1). $L = S - A$ on $\mathcal{D}$, where $(S, \mathcal{D})$ is symmetric, $(A, \mathcal{D})$ is antisymmetric and $\mathcal{D} \subseteq D(L)$ is a core for $(L, D(L))$.

Then both $(S, \mathcal{D})$ and $(A, \mathcal{D})$ are closable and we denote their closures by $(S, D(S))$ and $(A, D(A))$, respectively. These two operators are linked to the decomposition of H in the following way.

Assumption (D2). $H_1 \subseteq D(S)$ and $S = 0$ on H_1.

Assumption (D3). $P(\mathcal{D}) \subseteq D(A)$, $AP(\mathcal{D}) \subseteq D((AP)^*)$ and $PAP = 0$ on $\mathcal{D}$. Here, $(AP)^*$ is the adjoint of the densely defined closed operator $(AP, D(AP))$ with

$$D(AP) = \{x \in H \mid Px \in D(A)\}.$$

Definition 4.1. We define the operator $(G, D(G))$ by

$$G := -(AP)^*AP, \qquad D(G) := \{x \in D(AP) \mid APx \in D((AP)^*)\}.$$

Remark 4.2. *Due to von Neumann's theorem ([Ped89, Theorem 5.1.9]), $(G, D(G))$ is self-adjoint and $\mathrm{Id} - G : D(G) \to H$ is bijective with bounded inverse. Since G is dissipative, it generates a strongly continuous contraction semigroup on H.*
Due to Assumption ***D3****, we have $\mathcal{D} \subseteq D(G)$. If additionally, $AP(\mathcal{D}) \subseteq D(A)$, then $G = PA^2P$ on $\mathcal{D}$.*

This allows us to define the following operator, which is bounded with operator norm less than 1, again due to [Ped89, Theorem 5.1.9].

Definition 4.3. Define the operator $(B, D(B))$ as

$$B := (\mathrm{Id} - G)^{-1}(AP)^*, \qquad D(B) := D((AP)^*).$$

Due to boundedness, it extends uniquely to a bounded operator $B : H \to H$.

We continue this section with the formulation of three hypocoercivity assumptions.

Assumption (H1). *Boundedness of auxiliary operators.* The operators $(BS, \mathcal{D})$ and $(BA(I - P), \mathcal{D})$ are bounded and there exist constants $c_1, c_2 < \infty$ such that

$$\|BSx\| \le c_1\|(\mathrm{Id} - P)x\| \quad \text{and} \quad \|BA(\mathrm{Id} - P)x\| \le c_2\|(\mathrm{Id} - P)x\|$$

hold for all $x \in \mathcal{D}$.

Assumption (H2). *Microscopic coercivity.* There exists some $\Lambda_m > 0$ such that

$$-(Sx, x) \ge \Lambda_m\|(\mathrm{Id} - P)x\|^2 \qquad \text{for all } x \in \mathcal{D}.$$

Assumption (H3). *Macroscopic coercivity.* There is some $\Lambda_M > 0$ such that

$$\|APx\|^2 \ge \Lambda_M\|Px\|^2 \qquad \text{for all } x \in D(G). \tag{4.1}$$

Remark 4.4. *If $(G, \mathcal{D})$ is already essentially self-adjoint, then Assumption* ***H3*** *is satisfied if (4.1) holds for all $x \in \mathcal{D}$. For the proof, compare [GS14, Corollary 2.13].*

Next, we formulate the central hypocoercivity theorem. The techniques on how to explicitly compute the constant determining the speed of convergence are worked out in [GS16, Theorem 1.1], for the particular case of classical Langevin dynamics on $\mathbb{R}^n$, $n \in \mathbb{N}$ and for finite dimensional Langevin dynamics with multiplicative noise in [Ale23, Theorem 4.2.10]. The same techniques are applied, below, but in the abstract level of the general hypocoercivity method. Chapter 6 is devoted to the application of this theorem in the context of infinite dimensional Langevin dynamics with multiplicative noise.

Theorem 4.5. *Assume that the data assumptions* ***D1-D3*** *and the hypocoercivity assumptions* ***H1-H3*** *are satisfied. Then for each $\theta_1 \in (1, \infty)$ there exist $\theta_2 \in (0, \infty)$ such that for each $g \in H$ we have*

$$\|T_t g - (g, 1)\| \le \theta_1 e^{-\theta_2 t}\|g - (g, 1)\| \quad \textit{for all} \quad t \ge 0,$$

where $(T_t)_{t\geq 0}$ denotes the s.c.c.s. generated by $(L, D(L))$. The constant θ_2 is explicitly computable in terms of $\Lambda_m, \Lambda_M, c_1$ and c_2 and given as

$$\frac{1}{4}\frac{\theta_1 - 1}{\theta_1}\frac{\min\{\Lambda_m, c_1\}}{(1+c_1+c_2)\Big(1+\frac{1+\Lambda_M}{2\Lambda_M}(1+c_1+c_2)\Big)+\frac{1}{2}\frac{\Lambda_M}{1+\Lambda_M}}\frac{\Lambda_M}{1+\Lambda_M}.$$

Proof. In view of [GS14, Theorem 2.18], we first choose $\delta > 0$ such that

$$\frac{\Lambda_M}{1+\Lambda_M} - (1+c_1+c_2)\frac{\delta}{2} > 0$$

and then $\varepsilon > 0$ small enough such that

$$\Lambda_m - \varepsilon(1+c_1+c_2)\left(1+\frac{1}{2\delta}\right) > 0,$$

as well. This particular choice ensures that

$$\min\left\{\frac{\Lambda_M}{1+\Lambda_M} - (1+c_1+c_2)\frac{\delta}{2}\,,\,\Lambda_m - \varepsilon(1+c_1+c_2)\left(1+\frac{1}{2\delta}\right)\right\} > 0.$$

Now chose $\kappa > 0$ smaller or equal than the minimum above. Again, by [GS14, Theorem 2.18], we obtain

$$\|T_t g - (g,1)\| \leq \kappa_1 e^{-\kappa_2 t}\|g-(g,1)\| \quad \text{for all} \quad t \geq 0,$$

for

$$\kappa_1 = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \quad \text{and} \quad \kappa_2 = \frac{\kappa}{1+\varepsilon}.$$

To explicitly compute θ_1 and θ_2 as promised in the assertion, we have to specify $\delta > 0$, $\varepsilon > 0$ and the corresponding $\kappa > 0$. We use the strategy from [GS16, Theorem 1.1]. Without loss of generality assume $\Lambda_m \leq c_1$, since otherwise we replace Λ_m with $\min\{\Lambda_m, c_1\}$ in **H2**. We set

$$\delta = \frac{\Lambda_M}{1+\Lambda_M}\frac{1}{1+c_1+c_2}.$$

Moreover, we define

$$r_{\Lambda_M,c_1} = (1+c_1+c_2)\left(1+\frac{1+\Lambda_M}{2\Lambda_M}(1+c_1+c_2)\right) \quad \text{and} \quad s_{\Lambda_M} = \frac{1}{2}\frac{\Lambda_M}{1+\Lambda_M}.$$

For arbitrary $v \in (0,\infty)$, we choose $\varepsilon = \frac{v}{1+v}\frac{\Lambda_m}{r_{\Lambda_M,c_1}+s_{\Lambda_M}}$. As $\Lambda_m \leq c_1$ one can check that $\varepsilon \in (0,1)$. Since $\varepsilon(r_{\Lambda_M,c_1}+s_{\Lambda_M}) = \frac{v}{1+v}\Lambda_m < \Lambda_m$, we get

$$\Lambda_m - \varepsilon r_{\Lambda_M,c_1} \geq \varepsilon s_{\Lambda_M} = \frac{v}{1+v}\frac{\Lambda_m}{r_{\Lambda_M,c_1}+s_{\Lambda_M}} s_{\Lambda_M}.$$

In particular, $\kappa = \frac{v}{1+v}\frac{\Lambda_m}{r_{\Lambda_M,c_1}+s_{\Lambda_M}}s_{\Lambda_M}$ is a valid choice. The convergence rate in terms of κ_1 and κ_2 is given by

$$\kappa_1 = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} = \sqrt{\frac{1+v+\frac{\Lambda_m}{r_{\Lambda_M,c_1}+s_{\Lambda_M}}v}{1+v-\frac{\Lambda_m}{r_{\Lambda_M,c_1}+s_{\Lambda_M}}v}} \leq \sqrt{1+2v+v^2} = 1+v \quad \text{and}$$
$$\kappa_2 = \frac{\kappa}{1+\varepsilon} > \frac{1}{2}\kappa.$$

Therefore, choosing $\theta_1 = 1+v$ and $\theta_2 = \frac{1}{2}\kappa$ yields the claimed rate of convergence. □

Especially the hypocoercivity condition **H1** can be hard to verify, therefore we state [BG23, Lemma 3.1], which is a useful tool for its verification.

Lemma 4.6. *Let $\mathcal{D}$ be a core for $(G, D(G))$. Let $(T, D(T))$ be a linear operator with $\mathcal{D} \subseteq D(T)$ and assume $AP(\mathcal{D}) \subseteq D(T^*)$. Then*

$$(\mathrm{Id}-G)(\mathcal{D}) \subseteq D((BT)^*) \quad \text{with} \quad (BT)^*(\mathrm{Id}-G)x = T^*APx, \quad x \in \mathcal{D}.$$

If there exists some $C < \infty$ such that

$$\|(BT)^*y\| \leq C\|y\| \qquad \text{for all } y = (\mathrm{Id}-G)x, \quad x \in \mathcal{D}, \tag{4.2}$$

*then $(BT, D(T))$ is bounded and its closure $(\overline{BT})$ is a continuous operator on H with $\|\overline{BT}\|_{\mathcal{L}(H)} = \|(BT)^*\|_{\mathcal{L}(H)} \leq C$. In particular, if $(S, D(S))$ and $(A, D(A))$ satisfy these assumptions with constant C_S and C_A, respectively, then **H1** is satisfied with $c_1 = C_S$ and $c_2 = C_A$.*

5

Essential m-dissipativity of infinite dimensional Langevin operators

In this chapter we start the analysis of infinite dimensional degenerate Langevin dynamics with multiplicative noise, in terms of the associate infinite dimensional Langevin operators. These operators describe a class of non-sectorial infinite dimensional second-order differential operators with variable diffusion coefficient. Besides the non-sectorality of such operators, the difficulty of the problem is determined by the regularity of the considered potentials describing an external force.
The chapters main results are Theorem 5.23 and Theorem 5.27. By imposing different assumptions regarding the regularity of the coefficient operators and of the potential, the results establish the essential m-dissipativity of the infinite dimensional Langevin operator L^{Φ}, compare Definition 5.7, on the core $\mathcal{F}C_b^{\infty}(B_W)$, defined in Definition 5.2 as the space of finitely based smooth and bounded cylinder functions. The existence of a nice core is crucial for the construction of a stochastic process describing the Langevin dynamic, compare Chapter 7 and for employing the general abstract Hilbert space hypocoercivity method, as outlined in Chapter 6.

The results from Section 5.1.1, where potentials with bounded gradient in a suitable Sobolev space are considered, have been published before in [BEG23]. We point out that the applied techniques were already developed in [EG22], where a similar situation was considered, but without multiplicative noise, i.e. constant diffusion operator K_{22}.

5.1 The infinite dimensional Langevin operator

Let $(U, (\cdot,\cdot)_U)$ and $(V, (\cdot,\cdot)_V)$ be two real separable Hilbert spaces. Moreover, we fix two centered non-degenerate Gaussian measures μ_1 and μ_2 on $(U, \mathscr{B}(U))$ and $(V, \mathscr{B}(V))$, respectively. Let Q_i denote the covariance operator of μ_i, $i = 1, 2$ with corresponding basis of eigenvectors $B_U = (d_k)_{k\in\mathbb{N}}$ and $B_V = (e_k)_{k\in\mathbb{N}}$ and positive eigenvalues $(\lambda_{1,k})_{k\in\mathbb{N}}$ and $(\lambda_{2,k})_{k\in\mathbb{N}}$, respectively. Without loss of generality, we assume that $(\lambda_{1,k})_{k\in\mathbb{N}}$ and $(\lambda_{2,k})_{k\in\mathbb{N}}$ are decreasing to zero. The corresponding projections to the induced subspaces, coordinate maps and embeddings are denoted by P_n^U, p_n^U, $\overline{p}_n^U$ and P_n^V, p_n^V, $\overline{p}_n^V$, respectively.
In addition, we fix a potential $\Phi : U \to (-\infty, \infty]$ and assume for the rest of this section the following assumption.

Assumption 5.1. $\Phi : U \to (-\infty, \infty]$ is bounded from below by zero and there is $\theta \in [0, \infty)$ such that $\Phi \in W^{1,2}_{Q_1^\theta}(U; \mu_1)$. Φ is normalized, i.e. $\int_U e^{-\Phi} \, d\mu_1 = 1$.

All results below are also valid if we replace bounded from below by zero with bounded from below. Through the application of a suitable scaling, $\int_U e^{-\Phi} \, d\mu_1 = 1$ holds without loss of generality.

Definition 5.2. Set $W := U \times V$ and denote by $(\cdot, \cdot)_W$ the canonical inner product on W defined by

$$((u_1, v_1), (u_2, v_2))_W := (u_1, u_2)_U + (v_1, v_2)_V, \quad \text{for all} \quad (u_1, v_1), (u_2, v_2) \in W.$$

Then, $(W, (\cdot, \cdot)_W)$ is a real separable Hilbert space. Furthermore, we define the measure $\mu_1^\Phi := e^{-\Phi}\mu_1$ on $\mathscr{B}(U)$ and set

$$\mu^\Phi := \mu_1^\Phi \otimes \mu_2$$

be the product measure on the Borel σ-algebra $\mathscr{B}(W) = \mathscr{B}(U) \otimes \mathscr{B}(V)$. We set $\mu := \mu^0 = \mu_1 \otimes \mu_2$. Due to [Da 06, Theorem 1.12], μ is a centered Gaussian measure with covariance operator Q defined by

$$Q : W \to W, \quad (u, v) \mapsto (Q_1 u, Q_2 v).$$

Let B_W be an ordered enumeration of the set

$$\{(d_n, 0) \mid n \in \mathbb{N}\} \cup \{(0, e_n) \mid n \in \mathbb{N}\} \subseteq W.$$

Then, B_W is an orthonormal basis of eigenvectors of Q. In analogy to Definition 3.1, we define for each $n \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$ and $C \in \left\{C_b^k(\mathbb{R}^n \times \mathbb{R}^n), C_c^k(\mathbb{R}^n \times \mathbb{R}^n)\right\}$ the spaces of finitely based cylinder functions with respect to B_W by

$$\mathcal{F}C(B_W, n) := \left\{f = \varphi \circ (p_n^U, p_n^V) \text{ for some } \varphi \in C\right\} \quad \text{and} \quad \mathcal{F}C(B_W) := \bigcup_{n \in \mathbb{N}} \mathcal{F}C(B_W, n).$$

Further, we introduce $\mu^n := \mu_1^n \otimes \mu_2^n$ on $\mathscr{B}(\mathbb{R}^n \times \mathbb{R}^n)$, with μ_i^n being a centered Gaussian measure on $\mathscr{B}(\mathbb{R}^n)$ with diagonal covariance matrix $Q_{i,n} := \operatorname{diag}(\lambda_{i,1} \ldots, \lambda_{i,n})$.

Definition 5.3. Let $n \in \mathbb{N}$ and $\varphi \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ be given. By $\partial_{i,1}\varphi$ and $\partial_{i,2}\varphi$, $1 \leq i \leq n$, we denote the i-th partial derivative of φ in the first and second component, respectively. We generalize this notation to gradients, e.g. $D_1\varphi$ denotes the gradient of φ with respect to the first component.
For Gâteaux differentiable $f : W \to \mathbb{R}$, compare Definition 2.7 and all $w = (u, v) \in W$ we define

$$D_1 f(w) := \sum_{n \in \mathbb{N}} (Df(w), (d_n, 0))_W d_n \in U, \quad \partial_{d_i} f(w) := (D_1 f(w), d_i)_U \quad \text{and}$$
$$D_2 f(w) := \sum_{n \in \mathbb{N}} (Df(w), (0, e_n))_W e_n \in V, \quad \partial_{e_i} f(w) := (D_2 f(w), e_i)_V.$$

Higher order (partial) derivatives are defined analogously, compare Definition 2.9.

Remark 5.4. *(i) Let $n \in \mathbb{N}$ and $f = \varphi \circ (p_n^U, p_n^V) \in \mathcal{F}C_b^1(B_W)$. Similar to Remark 3.30, we compute for all $(u,v) \in W$*

$$D_1 f(u,v) = \sum_{n\in\mathbb{N}} \partial_{i,1}\varphi(p_n^U u, p_n^V v) d_i \quad \text{and} \quad D_2 f(u,v) = \sum_{n\in\mathbb{N}} \partial_{i,2}\varphi(p_n^U u, p_n^V v) e_i.$$

(ii) Using similar arguments as in Lemma 3.28 and Lemma 3.44, we know that for each $p \in [1,\infty)$ there is a countable subset of $\mathcal{F}C_c^\infty(B_W)$, which is dense in $L^p(W;\mu^\Phi)$ with respect to $\|\cdot\|_{L^p(\mu^\Phi)}$). In particular, $(L^p(W;\mu^\Phi), \|\cdot\|_{L^p(\mu^\Phi)})$ is separable and for every $k \in \mathbb{N}$, $\mathcal{F}C_c^k(B_W)$ and $\mathcal{F}C_b^k(B_W)$ are dense in $(L^p(W;\mu^\Phi), \|\cdot\|_{L^p(\mu^\Phi)})$.

In the next definition, we fix the coefficient operators determining the infinite dimensional Langevin operator. We directly include the invariance and growth condition needed for our further considerations.

Definition 5.5. We fix $K_{12} \in \mathcal{L}(U;V)$ and set $K_{21} := K_{12}^* \in \mathcal{L}(V;U)$. Moreover, suppose $K_{22} : V \to \mathcal{L}_{>0}^+(V)$ and $v \mapsto K_{22}(v)e_i \in C^1(V;V)$ for all $i \in \mathbb{N}$. Further, assume that there is a strictly increasing sequence $(m_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that for each $n \le m_k$ and $v \in V$, it holds that

$$K_{12}(U_n) \subseteq V_{m_k},\ K_{21}(V_n) \subseteq U_{m_k},\ K_{22}(v)(V_n) \subseteq V_{m_k} \text{ and } K_{22}(v)|_{V_n} = K_{22}(P_{m_k}^V v)|_{V_n}.$$

Moreover, suppose that for each $k \in \mathbb{N}$, there is a constant $M_k \in (0,\infty)$ such that

$$\sup_{v\in V_{m_k}} \|K_{22}(v)\|_{\mathcal{L}(V_{m_k})} \le M_k \quad \text{and}$$
$$\|\partial_{e_i} K_{22}(v)\|_{\mathcal{L}(V_{m_k})} \le M_k(1 + \|v\|_{V_{m_k}}) \quad \text{for all } v \in V_{m_k}, 1 \le i \le m_k.$$

Above, for each $v \in V_{m_k}$ the linear operator $\partial_{e_i} K_{22}(v) : V_{m_k} \to V_{m_k}$ is defined by $\partial_{e_i} K_{22}(v)\tilde{v} := \sum_{j=1}^{m_k} \partial_{e_i} K_{22}(v) e_j (\tilde{v}, e_j)_V$, $\tilde{v} \in V_{m_k}$.
In the following, we set $m^K(n) := \min_{k\in\mathbb{N}}\{m_k : m_k \ge n\}$.

Roughly speaking, the invariance properties K_{12}, K_{21} and K_{22} imply that they have a block invariance structure, where the size of the blocks is described by the increasing sequence $(m_k)_{k\in\mathbb{N}}$.

Remark 5.6. *Suppose $f = \varphi \circ (p_n^U, p_n^V) \in \mathcal{F}C_b^\infty(B_W, n)$ and by trivially extending φ if necessary that $m^K(n) = n$. Then, by invariance properties of the coefficients, we compute*

$$(Q_1^\theta D\Phi, Q_1^{-\theta} K_{21} D_2 f)_U = \sum_{i=1}^n \lambda_{1,i}^{-\theta} (Q_1^\theta D\Phi, d_i)_U (d_i, K_{21} D_2 f)_U = \sum_{i=1}^n \partial_{d_i}\Phi (d_i, K_{21} D_2 f)_U.$$

Therefore, the interpretation of $(D\Phi, K_{21}D_2 f)_U$ as $(Q_1^\theta D\Phi, Q_1^{-\theta} K_{21} D_2 f)_U$ is reasonable even though we do not know if $\Phi \in W^{1,2}(U;\mu_1)$. For the following consideration we define for all $f \in \mathcal{F}C_b^\infty(B_W)$

$$(D\Phi, K_{21} D_2 f)_U := (Q_1^\theta D\Phi, Q_1^{-\theta} K_{21} D_2 f)_U \quad \text{for all} \quad f \in \mathcal{F}C_b^\infty(B_W).$$

We are now able to define the infinite dimensional Langevin operator on $\mathcal{F}C_b^\infty(B_W)$.

Definition 5.7. The differential operators $(S, \mathcal{F}C_b^\infty(B_W))$ and $(A^\Phi, \mathcal{F}C_b^\infty(B_W))$ are defined on $L^2(W;\mu^\Phi)$ by

$$\begin{aligned} Sf(u,v) := \operatorname{tr}\left[K_{22}(v)\circ D_2^2 f(u,v)\right] &+ \sum_{j=1}^\infty (\partial_{e_j} K_{22}(v) D_2 f(u,v), e_j)_V \\ &- (v, Q_2^{-1} K_{22}(v) D_2 f(u,v))_V \end{aligned}$$

and

$$\begin{aligned} A^\Phi f(u,v) :=& (u, Q_1^{-1} K_{21} D_2 f(u,v))_U + (D\Phi(u), K_{21} D_2 f(u,v))_U \\ &- (v, Q_2^{-1} K_{12} D_1 f(u,v))_V, \end{aligned}$$

respectively, for all $(u,v)\in W$. The infinite dimensional Langevin operator denoted by $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is defined via

$$L^\Phi := S - A^\Phi.$$

For notational convenience we set $A := A^0$ and $L := L^0$.

Remark 5.8. *The invariance assumptions made on K_{12}, K_{21} and K_{22} ensure that S and A^Φ and therefore also L^Φ are well-defined on $\mathcal{F}C_b^\infty(B_W)$. Indeed, let $n\in\mathbb{N}$ and suppose $f=\varphi\circ(p_n^U, p_n^V)\in \mathcal{F}C_b^\infty(B_W,n)$. By trivially extending φ if necessary, we can assume $m^K(n)=n$. Then, for all $(u,v)\in W$ we get by Remark 5.4 Item (i)*

$$Q_1^{-1}K_{21}D_2 f(u,v)\in U_n, \quad Q_2^{-1}K_{12}D_1 f(u,v)\in V_n \quad \text{and} \quad Q_2^{-1}K_{22}(v)D_2 f(u,v)\in V_n.$$

Moreover, these maps are uniformly bounded in $(u,v)\in W$ due to uniform boundedness of $K_{22}: V_n\to\mathcal{L}(V_n)$ and the fact that all derivatives of f are bounded. By the observation that all sums appearing in the definition of Sf and $A^\Phi f$ are finite, the fact that $\partial_{d_i}\Phi\in L^2(U;\mu_1^\Phi)$, as well as $\|\cdot\|_U, \|\cdot\|_V\in L^2(W;\mu^\Phi)$ by Lemma 3.5, it follows that

$$Sf(u,v) = Sf(P_n^U u, P_n^V v), \quad A^\Phi f(u,v) = Af(P_n^U u, P_n^V) + (D\Phi(u), K_{21}D_2 f(P_n^U u, P_n^V))_U$$

and Sf, $A^\Phi f\in L^2(W;\mu^\Phi)$. Therefore, $Lf\in L^2(W;\mu^\Phi)$ is finitely based and we have

$$\begin{aligned} L^\Phi f(u,v) &= Sf(P_n^U u, P_n^V v) - Af(P_n^U u, P_n^V) - (D\Phi(u), K_{21}D_2 f(P_n^U u, P_n^V))_U \\ &= Lf(P_n^U u, P_n^V) - (D\Phi(u), K_{21}D_2 f(P_n^U u, P_n^V))_U. \end{aligned}$$

It is also possible to consider $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ on $L^p(W;\mu^\Phi)$ for $p\in[1,2]$.

As the abbreviation should suggest, we show below, among other things, that S is symmetric and A^Φ is antisymmetric.

Lemma 5.9. *The linear operator $(S, \mathcal{F}C_b^\infty(B_W))$ is symmetric and negative semi-definite, whereas $(A^\Phi, \mathcal{F}C_b^\infty(B_W))$ is antisymmetric on $L^2(W;\mu^\Phi)$. Therefore, $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is dissipative on $L^2(W;\mu^\Phi)$.*
Denote by $(S, D(S))$, $(A^\Phi, D(A^\Phi))$ and $(L^\Phi, D(L^\Phi))$ the closures of the respective operators. Then, for all $f,g\in\mathcal{F}C_b^\infty(B_W)$ it holds

$$-\int_W L^\Phi f g \,\mathrm{d}\mu^\Phi = \int_W (D_2 f, K_{22}D_2 g)_V - (D_1 f, K_{21} D_2 g)_U + (D_2 f, K_{12} D_1 g)_V \,\mathrm{d}\mu^\Phi.$$

Proof. Let $f, g \in \mathcal{F}C_b^\infty(B_W)$. As in Remark 5.8, we assume $f, g \in \mathcal{F}C_b^\infty(B_W, n)$ for some $n \in \mathbb{N}$ with $n = m^K(n)$. For any $(u, v) \in W$, it holds that

$$Q_1^{-1} K_{21} D_2 f(u, v) = \sum_{k=1}^{n} \partial_{e_k} f(u, v) Q_1^{-1} K_{21} e_k = \sum_{k,\ell=1}^{n} \partial_{e_k} f(u, v)(K_{21} e_k, d_\ell)_U Q_1^{-1} d_\ell.$$

Using Item (iii) from Proposition 3.48, we obtain

$$\int_W \left((u, Q_1^{-1} d_\ell)_U + \partial_{d_l}\Phi\right) \partial_{e_k} f g \,\mathrm{d}\mu^\Phi = \int_W (g \partial_{d_l} \partial_{e_k} f + \partial_{e_k} f \partial_{d_l} g) \,\mathrm{d}\mu^\Phi,$$

which shows that

$$\begin{aligned}&\left((u, Q_1^{-1} K_{21} D_2 f)_U + (D\Phi, K_{21} D_2 f)_U, g\right)_{L^2(\mu^\Phi)}\\ &= \int_W (K_{21} D_2 f, D_1 g)_U \,\mathrm{d}\mu^\Phi + \sum_{k,\ell=1}^{n} (K_{21} e_k, d_\ell)_U (g, \partial_{d_l} \partial_{e_k} f)_{L^2(\mu^\Phi)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left((v, Q_2^{-1} K_{12} D_1 f)_V, g\right)_{L^2(\mu^\Phi)} &= \int_W (K_{12} D_1 f, D_2 g)_V \,\mathrm{d}\mu^\Phi\\ &\quad + \sum_{k,\ell=1}^{n} (K_{12} d_\ell, e_k)_V (g, \partial_{e_k} \partial_{d_l} f)_{L^2(\mu^\Phi)}.\end{aligned}$$

The property $K_{12}^* = K_{21}$ implies that

$$(A^\Phi f, g)_{L^2(\mu^\Phi)} = \int_W (D_2 f, K_{12} D_1 g)_V - (D_1 f, K_{21} D_2 g)_U \,\mathrm{d}\mu^\Phi.$$

In particular, $(A^\Phi f, f)_{L^2(\mu^\Phi)} = 0$. Now we consider the operator S. As before, we have

$$Q_2^{-1} K_{22}(v) D_2 f(u, v) = \sum_{i,j=1}^{n} \partial_{e_i} f(u, v)(K_{22}(v) e_i, e_j)_V Q_2^{-1} e_j$$

for all $(u, v) \in W$. Due to the assumptions on K_{22}, the maps

$$(u, v) \mapsto \partial_{e_i} f(u, v)(K_{22}(v) e_i, e_j)_V$$

are finitely based. The application of the integration by parts formula is possible and yields

$$\begin{aligned}\int_W (v, Q_2^{-1} e_j) \partial_{e_i} f (K_{22} e_i, e_j)_V g \,\mathrm{d}\mu^\Phi &= \int_W \partial_{e_j} \partial_{e_i} f (K_{22} e_i, e_j)_V g \,\mathrm{d}\mu^\Phi\\ &\quad + \int_W \partial_{e_i} f (\partial_{e_j} K_{22} e_i, e_j)_V g \,\mathrm{d}\mu^\Phi\\ &\quad + \int_W \partial_{e_i} f (K_{22} e_i, e_j)_V \partial_{e_j} g \,\mathrm{d}\mu^\Phi.\end{aligned}$$

Summing over i and j, results in

$$\sum_{i,j=1}^{n} \int_W \partial_{e_i} f (K_{22}e_i, e_j)_V \partial_{e_j} g \,\mathrm{d}\mu^\Phi = \int_W (K_{22}D_2 f, D_2 g)_V \,\mathrm{d}\mu^\Phi$$

and

$$\sum_{i,j=1}^{n} \int_W \partial_{e_j}\partial_{e_i} f (K_{22}e_i, e_j)_V g \,\mathrm{d}\mu^\Phi = \int_W \operatorname{tr}\left[K_{22}D_2^2 f\right] g \,\mathrm{d}\mu^\Phi,$$

due to pointwise symmetry of K_{22}. We obtain

$$(Sf, g)_{L^2(\mu^\Phi)} = -\sum_{i,j=1}^{n} \int_W \partial_{e_i} f (K_{22}e_i, e_j)_V \partial_{e_j} g \,\mathrm{d}\mu^\Phi = -\int_W (D_2 f, K_{22}D_2 g)_V \,\mathrm{d}\mu^\Phi.$$

Hence, S is symmetric and negative semi-definite since K_{22} is positive semi-definite. In particular, all three operators are dissipative on $L^2(W;\mu)$ and therefore closable. □

We explicitly calculate the Carré du champ operator of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ and verify that the infinite dimensional Langevin operator is an abstract diffusion operator, below. Hence, if $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ generates a strongly continuous semigroup, we immediately know that it is sub-Markovian, see Lemma 2.39.

Corollary 5.10. *The measure μ^Φ is invariant for the symmetric operator $(S, \mathcal{F}C_b^\infty(B_W))$ and the antisymmetric operator $(A^\Phi, \mathcal{F}C_b^\infty(B_W))$, therefore also for $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$. Moreover, the infinite dimensional Langevin operator $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is an abstract diffusion operator on $L^p(W;\mu^\Phi)$ for all $p \in [1,2]$ and the corresponding Carré du champ operator is given by*

$$\Gamma(f,g) = (K_{22}D_2 f, D_2 g)_V \quad \text{for all} \quad f, g \in \mathcal{F}C_b^\infty(B_W). \tag{5.1}$$

Proof. As mentioned at the end of Remark 5.8, it is possible to consider $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ on $L^p(W;\mu^\Phi)$, for all $p \in [1,2]$. The first part of the statement directly follows by Lemma 5.9. To calculate the Carré du champ operator, let $f, g \in \mathcal{F}C_b^\infty(B_W)$ be given. Obviously, their product fg is in $\mathcal{F}C_b^\infty(B_W)$ and by the classical product rule for differentiable functions, we obtain (note that all appearing infinite sums below are finite)

$$\begin{aligned}
L^\Phi(fg) &= \sum_{i,j=1}^{\infty} (K_{22}e_i, e_j)_V \partial_{e_i}\partial_{e_j}(fg) + \sum_{j=1}^{\infty} (\partial_{e_j} K_{22} D_2(fg), e_j)_V \\
&\quad - (v, Q_2^{-1} K_{22} D_2(fg))_V - (u, Q_1^{-1} K_{21} D_2(fg))_U \\
&\quad - (D\Phi, K_{21} D_2(fg))_U + (v, Q_2^{-1} K_{12} D_1(fg))_V \\
&= \sum_{i,j=1}^{\infty} (K_{22}e_i, e_j)_V \partial_{e_i} f \partial_{e_j} g + \sum_{i,j=1}^{\infty} (K_{22}e_i, e_j)_V \partial_{e_i} g \partial_{e_j} f + f L^\Phi g + g L^\Phi f \\
&= 2\,(K_{22}D_2 f, D_2 g)_V + f L^\Phi g + g L^\Phi f.
\end{aligned}$$

Therefore, (5.1) holds and since $K_{22}(v) \in \mathcal{L}^+(V)$ for all $v \in V$, we conclude Item (ii) from Definition 2.38 is fulfilled. To show the second item from Definition 2.38, let $m, n \in \mathbb{N}$,

$f_1, \dots, f_m \in \mathcal{F}C_b^\infty(B_W)$ and $\varphi \in C^\infty(\mathbb{R}^m)$ with $\varphi(0) = 0$ be given. Since the composition of a $C^\infty(\mathbb{R}^m)$ function with a vector $(\psi_1, \dots, \psi_m) \in (C_b^\infty(\mathbb{R}^n))^m$ is in $C_b^\infty(\mathbb{R}^n)$, we obtain $\varphi(f_1, \dots, f_m) \in \mathcal{F}C_b^\infty(B_W)$. Finally,

$$L^\Phi \varphi(f_1, \dots, f_m) = \sum_{k=1}^m \partial_k \varphi(f_1, \dots, f_m) L(f_k) + \sum_{k,l=1}^m \partial_l \partial_k \varphi(f_1, \dots, f_m) \left(K_{22} D_2 f_k, D_2 f_l\right)_V,$$

follows similar as above, by the classical chain rule. □

5.1.1 Essential m-dissipativity of infinite dimensional Langevin operators for potentials with bounded gradient

In the first part of this section, we assume $\Phi = 0$ and prove that $(L, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative on $L^2(W;\mu)$. Since the dissipativity of $(L, \mathcal{F}C_b^\infty(B_W))$ is part of Lemma 5.9, it remains to show that $(\mathrm{Id} - L)(\mathcal{F}C_b^\infty(B_W))$ is dense in $L^2(W;\mu)$. As $\mathcal{F}C_b^\infty(B_W)$ is dense in $L^2(W;\mu)$, it suffices to approximate all such functions. The main idea is to interpret L for all m_k, $k \in \mathbb{N}$, as an operator on the finite dimensional subspace determined by $\mathcal{F}C_b^\infty(B_W, m_k)$, which is possible due to Remark 5.8. In that case, we apply the recent finite dimensional m-dissipativity result from [BG23, Thm. 1.1], compare also [Ale23, Chapter 5]. Actually, Proposition 5.14 and Theorem 5.15 are already proved in [Ale23, Chapter 5], however to draw the full picture we give their proofs.
In the second part, we derive first order regularity results for the solution of the resolvent equation. With them, we establish the density of $(\mathrm{Id} - L^\Phi)(\mathcal{F}C_b^\infty(B_W))$ in $L^2(W;\mu^\Phi)$, if $D\Phi$ is bounded in the sense of Assumption $\mathrm{Bd}_\theta(\Phi)$.

To reduce and analyze the problem in a finite dimensional setting, we need the following definition.

Definition 5.11. Fix $n \in \mathbb{N}$ such that $n = m^K(n)$. We define for all $y \in \mathbb{R}^n$

$$K_{12,n} := ((K_{12} d_i, e_j)_V)_{ij}, \quad K_{21,n} := (K_{12,n})^*, \quad K_{22,n}(y) := \left((K_{22}(\overline{p}_n^V y) e_i, e_j)_V \right)_{ij},$$

and denote the entry of $K_{22,n}$ at position i,j by $k_{ij,n}$.
Moreover, we define the operators S_n, A_n and L_n on the Hilbert space $L^2(\mathbb{R}^n \times \mathbb{R}^n; \mu^n)$ with domain $C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$\begin{aligned}
S_n f(x,y) &:= \mathrm{tr}[K_{22,n} D_2^2 f](x,y) + \sum_{i,j=1}^n \partial_j k_{ij,n}(y) \partial_{e_i} f(x,y) - \langle K_{22,n}(y) Q_{2,n}^{-1} y, D_2 f(x,y) \rangle, \\
A_n f(x,y) &:= \langle K_{12,n} Q_{1,n}^{-1} x, D_2 f(x,y) \rangle - \langle K_{21,n} Q_{2,n}^{-1} y, D_1 f(x,y) \rangle \quad \text{and} \\
L_n f &:= (S_n - A_n) f.
\end{aligned}$$

Recall that $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denotes the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. These definitions coincide with the structure of operators considered in [BG23], with the choices $\Theta(x) = \frac{1}{2}\langle x, Q_{1,n}^{-1} x \rangle$ and $\Psi(y) = \frac{1}{2}\langle y, Q_{2,n}^{-1} y \rangle$.

Remark 5.12. *Let $f = \varphi \circ (p_n^U, p_n^V) \in \mathcal{F}C_b^\infty(B_W, n)$ for some $n \in \mathbb{N}$ with $n = m^K(n)$. Then, by Remark 5.8, we immediately see that $Sf(u,v) = S_n \varphi(p_n^U u, p_n^V v)$ for all $(u,v) \in W$ and analogous statements hold for A and L.*

We subsequently state sufficient assumptions under which [BG23, Thm. 1.1] is applicable for $(L_n, C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ on $L^2(\mathbb{R}^n \times \mathbb{R}^n; \mu^n)$. This results in essential m-dissipativity of the infinite dimensional Langevin operator with $\Phi = 0$.

Assumption (K0). Assume that there is some positive operator $K_{22}^0 \in \mathcal{L}^+(V)$, which leaves each V_{m_k} for all $k \in \mathbb{N}$ invariant and such that

$$(v, K_{22}(\tilde{v})v)_V \geq (v, K_{22}^0 v)_V \quad \text{ for all } v, \tilde{v} \in V.$$

Above $(m_k)_{k\in\mathbb{N}}$ is the sequence from Definition 5.5,

Assumption (K1). For each $n \in \mathbb{N}$, let $k(n)$ be such that $m_{k(n)} = m^K(n)$. Assume that there are sequences $(\beta_k)_{k\in\mathbb{N}}$ in $[0,1)$ and $(N_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ such that for all $n \in \mathbb{N}$.

$$|(\partial_{e_i} K_{22}(v)e_n, e_j)_V| \leq N_{k(n)} \left(1 + \|v\|_{V_{m^K(n)}}^{\beta_{k(n)}}\right)$$

for all $v \in V_{m^K(n)}$, $1 \leq i \leq m^K(n)$ and $1 \leq j \leq n$.
For $n \in \mathbb{N}$, set $N^K(n) := 2\max\{N_{k(j)} : 1 \leq j \leq n\}$ and $\beta^K(n) := \max\{\beta_{k(j)} : 1 \leq j \leq n\}$.

Remark 5.13. *Assume that $K_{22}(v)$ leaves V_n invariant for all $n \in \mathbb{N}$ and $v \in V$. Using the strengthened invariance properties of K_{22}, it follows quickly that $K_{22}(v)$ is diagonal, i.e. $K_{22}(v)e_i = \lambda_{22,i}(v)e_i$ for some positive continuous differentiable $\lambda_{22,i} : V \to \mathbb{R}$. In that case, Assumption **K0** means that each $\lambda_{22,i}$ is bounded from below by a positive constant $\lambda_i^0 \in \mathbb{R}$ and Assumption **K1** reduces to*

$$|\partial_{e_i}\lambda_{22,n}(v)| = |\partial_{e_i}\lambda_{22,n}(P_{m^K(n)}^V v)| \leq N_{k(n)}(1 + \|P_{m^K(n)}^V v\|_V^{\beta_{k(n)}})$$

for all $1 \leq i \leq m^K(n)$ and $n \in \mathbb{N}$.

Proposition 5.14. *Let $n \in \mathbb{N}$ such that $n = m^K(n)$ and let K_{22} satisfy Assumption **K0** and **K1**. Then, $(L_n, C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ is essentially m-dissipative on $L^2(\mathbb{R}^n \times \mathbb{R}^n; \mu^n)$.*

Proof. Define $K_{22,n}^0$ analogously to $K_{22,n}$ for K_{22}^0. Since K_{22}^0 is positive, all eigenvalues $\lambda_1^0, \dots, \lambda_n^0$ of $K_{22,n}^0$ are positive and therefore $c_n := \min_{i\in\{1,\dots,n\}} \lambda_i > 0$. Then for all $y, \tilde{y} \in \mathbb{R}^n$, we estimate

$$\langle y, K_{22,n}(\tilde{y})y\rangle = (\overline{p}_n^V y, K_{22}(\overline{p}_n^V \tilde{y})\overline{p}_n^V y)_V \geq (\overline{p}_n^V y, K_{22}^0 \overline{p}_n^V y)_V = \langle y, K_{22,n}^0 y\rangle \geq c_n |y|^2.$$

Assumption (Σ1) from [BG23] therefore holds true with $c_\Sigma := c_n^{-1}$. Due to the definition of K_{22}, all entries of $K_{22,n}$ are bounded and differentiable, hence, by means of Assumption **K1** locally Lipschitz. Consequently, also Assumption (Σ2) from [BG23] is valid. Now assume that $j \leq i$ and let $k \in \{1, \dots, n\}$. Then

$$\begin{aligned}
|\partial_k k_{ij,n}(y)| &= |\partial_k (K_{22}(\overline{p}_n^V y)e_i, e_j)_V| = |(\partial_{e_k} K_{22}(\overline{p}_n^V y)e_i, e_j)_V| \\
&\leq N_{k(i)} \left(1 + \|\overline{p}_n^V y\|_{V_n}^{\beta_n}\right) \\
&\leq 2N_{k(i)} \left(1 + \|\overline{p}_n^V y\|_{V_n}^{\beta^K(n)}\right) \leq N^K(n) \left(1 + \|\overline{p}_n^V y\|_{V_n}^{\beta^K(n)}\right)
\end{aligned}$$

by Assumption **K1**, so $K_{22,n}$ satisfies (Σ3) from [BG23] with constants $M = N^K(n)$ and $\beta = \beta^K(n)$.
Assumption (Ψ1)-(Ψ3) and (Θ1)-(Θ2) from [BG23] for Ψ and Θ, as chosen in Definition 5.11, are immediate. Moreover, Θ satisfies the growth condition (Θ2) for $N = \lambda_{1,n}^{-1}$ and $\gamma = 1 < (\beta^K(n))^{-1}$. Indeed, for any $x \in \mathbb{R}^n$, it holds that

$$|\nabla\Theta(x)|^2 = \sum_{i=1}^{n} \frac{1}{\lambda_{1,i}^2} x_i^2 \leq \frac{1}{\lambda_{1,n}^2}|x|^2$$

since $Q_{1,n} = \operatorname{diag}(\lambda_{1,1}, \dots, \lambda_{1,n})$, where $(\lambda_{1,i})_{i\in\mathbb{N}}$ is the decreasing sequence of eigenvalues of Q_1.
All in all, we justified that [BG23, Thm. 1.1] is applicable. Therefore, $(L_n, C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ is essentially m-dissipative on $L^2(\mathbb{R}^n \times \mathbb{R}^n; \mu^n)$. Since $C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ extends the domain $C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $(L_n, C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ is dissipative on $L^2(\mathbb{R}^n \times \mathbb{R}^n; \mu^n)$, due to Lemma 5.9, the claim follows. □

The application of [BG23, Thm. 1.1] also implies that the strongly continuous contraction semigroup, generated by $(L_n, C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n))$, is sub-Markovian, conservative and possesses μ^n as an invariant measure.
We generalize the results above to our infinite dimensional setting.

Theorem 5.15. *Let K_{22} satisfy Assumption **K0** and **K1**. Then $(L, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative on $L^2(W; \mu)$. Furthermore, the strongly continuous contraction semigroup $(T_t)_{t\geq 0}$, generated by $(L, D(L))$, is sub-Markovian and conservative.*

Proof. To verify that $(L, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative on $L^2(W; \mu)$, it remains to show that $(\operatorname{Id} - L)(\mathcal{F}C_b^\infty(B_W))$ is dense in $L^2(W; \mu)$, since Lemma 5.9 established dissipativity of $(L, \mathcal{F}C_b^\infty(B_W))$ already. Let $g \in \mathcal{F}C_b^\infty(B_W)$, then there is some $n \in \mathbb{N}$ such that $g \in \mathcal{F}C_b^\infty(B_W, n)$. As before, we extend g trivially to $\mathcal{F}C_b^\infty(B_W, m^K(n))$, so that we can assume $n = m^K(n)$. Let $\varphi_g \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $g(u, v) = \varphi_g(p_n^U u, p_n^V v)$ for all $(u, v) \in W$ and let $\varepsilon > 0$. Then,

$$\begin{aligned}
\|(\operatorname{Id} - L)f - g\|_{L^2(\mu)}^2 &= \int_W \left((\operatorname{Id} - L)f(P_n^U u, P_n^V v) - g(P_n^U u, P_n^V v)\right)^2 \mu(\mathrm{d}(u, v)) \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^n} ((\operatorname{Id} - L_n)\varphi_f(x, y) - \varphi_g(x, y))^2 \mu^n(\mathrm{d}(x, y)) \\
&= \|(\operatorname{Id} - L_n)\varphi_f - \varphi_g\|_{L^2(\mu^n)}^2
\end{aligned}$$

for all $f \in \mathcal{F}C_b^\infty(B_W, n)$ with corresponding $\varphi_f \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Due to Proposition 5.14, there is some $\psi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|(\operatorname{Id} - L_n)\psi - \varphi_g\|_{L^2(\mu^n)} < \varepsilon.$$

Setting $f_\psi(u, v) := \psi(p_n^U u, p_n^V v)$ yields $f_\psi \in \mathcal{F}C_b^\infty(B_W, n)$ with

$$\|(\operatorname{Id} - L)f_\psi - g\|_{L^2(\mu)} < \varepsilon.$$

By Remark 5.4, we know that $\mathcal{F}C_b^\infty(B_W)$ is dense in $L^2(W;\mu)$, therefore density of $(\mathrm{Id}-L)(\mathcal{F}C_b^\infty(B_W))$ in $L^2(W;\mu)$ follows as well. Lemma 5.9 tells us that $L1=0$ and $\mu(Lf)=0$ for all $f\in\mathcal{F}C_b^\infty(B_W)$. The former implies $T_t1=1$ in $L^2(W;\mu)$ for all $t\geq 0$, while the latter shows that μ is invariant for L and consequently for $(T_t)_{t\geq 0}$. By Corollary 5.10, we also know that $(L,\mathcal{F}C_b^\infty(B_W))$ is an abstract diffusion operator, which implies together with Lemma 2.39 that $(T_t)_{t\geq 0}$ is sub-Markovian. □

Remark 5.16. *In the setting of Theorem 5.15, denote by $(R_\lambda^L)_{\lambda>0}$ the resolvent associated to $(T_t)_{t\geq 0}$ and $(L,D(L))$. Then, by Lemma 2.34, $(R_\lambda^L)_{\lambda>0}$ is sub-Markovian and $(L,D(L))$ is a Dirichlet-operator.*

By Definition 5.5, we know that $K_{22}(v)$ is a symmetric and positive operator for all $v\in V$ with $K_{22}(v)(V_{m_k})\subseteq V_{m_k}$ for all $k\in\mathbb{N}$. Hence, for each $v\in V$ there is a unique positive symmetric linear operator $K_{22}^{\frac{1}{2}}(v)$ with $K_{22}^{\frac{1}{2}}(v)K_{22}^{\frac{1}{2}}(v)=K_{22}(v)$, compare [PR07, Proposition 2.3.4.]. It is easy to see that $K_{22}^{\frac{1}{2}}$ shares the same invariance properties as K_{22}. Next we study the regularity of the map $v\mapsto K_{22}^{\frac{1}{2}}(v)e_k$, $k\in\mathbb{N}$. This is important for Lemma 5.19 and Remark 5.28.

Lemma 5.17. *Let $l\in\mathbb{N}$ be given and suppose $v\mapsto K_{22}(v)e_k\in C_b^l(V;V)$ for all $k\in\mathbb{N}$. Then for each $i,j\in\mathbb{N}$ it holds $(K_{22}^{\frac{1}{2}}e_i,e_j)_V\in C^l(V;\mathbb{R})$. Moreover, if Assumption **K0** holds true, then $(K_{22}^{\frac{1}{2}}e_i,e_j)_V\in C_b^l(V;\mathbb{R})$.*

Proof. There exists $k\in\mathbb{N}$ such that $i,j\in\{1,\cdots,m_k\}$. For simplicity set $n:=m_k$. Recall $K_{22,n}$ and $K_{22,n}^0$ from Proposition 5.14, then $K_{22,n}^{\frac{1}{2}}$ corresponds to the matrix representation of $K_{22}^{\frac{1}{2}}$ and for each $v\in V$ it holds $(K_{22}^{\frac{1}{2}}(v)e_i,e_j)_V=(K_{22}^{\frac{1}{2}}(p_n(v)))_{ij}$. For the first claim it is therefore enough to show that $\mathbb{R}^n\ni y\mapsto K_{22,n}^{\frac{1}{2}}(y)\in\mathcal{L}^+(\mathbb{R}^n)$ is in $C^l(\mathbb{R}^n;\mathcal{L}^+(\mathbb{R}^n))$. By [DN18, Theorem 1.1] the map $\mathcal{L}^+_{>0}(\mathbb{R}^n)\ni A\mapsto\varphi(A):=A^{\frac{1}{2}}\in\mathcal{L}^+_{>0}(\mathbb{R}^n)$ is Frèchet differentiable of any order. Consequently, the first statement follows by the chain rule.
If **K0** holds true then for each $y\in\mathbb{R}^n$ the minimal eigenvalue of $K_{22,n}^{\frac{1}{2}}(y)$ is bounded from below by the minimal eigenvalue of $K_{22,n}^0$, compare also the proof of Proposition 5.14. Therefore, the second statement follows by [DN18, Theorem 1.1] which tells us for each $m\in\mathbb{N}$ there exists a constant C_m such that $\|D^m\varphi(A)\|\leq C_m\lambda_{\min}(A)^{-m-\frac{1}{2}}$ for all $A\in\mathcal{L}^+_{>0}(\mathbb{R}^n)$. Here, $\lambda_{\min}(A)$ denotes the minimal eigenvalue of $A\in\mathcal{L}^+_{>0}(\mathbb{R}^n)$. □

Remark 5.18. *For each fixed $i,j\in\mathbb{N}$ the boundedness and invariance properties of Definition 5.5 imply that $(K_{22}(v)e_i,e_j)_V$ is uniformly bounded in $v\in V$. Hence, a closer look in the proof of Lemma 5.17 shows that $(K_{22}^{\frac{1}{2}}e_i,e_j)_V$ is Frèchet differentiable with bounded derivative, if Assumption **K0** holds true.*

Recall that the potential $\Phi:U\to(-\infty,\infty]$ fulfills Assumption 5.1. In particular, Φ is measurable, bounded from below by zero and normalized. Our next goal is to show essential m-dissipativity of $(L^\Phi,\mathcal{F}C_b^\infty(B_W))$ on $L^2(W;\mu^\Phi)$. To achieve this, we derive regularity estimates and generalize the construction of Sobolev spaces with respect to the measures of type μ^Φ and differential operators with variable coefficients. We need the following assumption.

Assumption $(\mathrm{Bd}_\theta(\Phi))$**.** There is $\theta \in [0,\infty)$ such that,

$(\mathrm{Bd}_\theta(\Phi 1))$ $\Phi \in W^{1,2}_{Q_1^\theta}(U;\mu_1)$.

$(\mathrm{Bd}_\theta(\Phi 2))$ there exists $c_\theta \in (0,\infty)$ such that $(Q_1^{-\theta}K_{21}v, Q_1^{-\theta}K_{21}v)_U \le c_\theta(K_{22}(\tilde{v})v,v)_V$ for all $\tilde{v}\in V$ and $v \in V_n$, $n\in\mathbb{N}$.

$(\mathrm{Bd}_\theta(\Phi 3))$ $Q_1^\theta D\Phi$ is in $L^\infty(U;\mu_1)$.

Item $(\mathrm{Bd}_\theta(\Phi 2))$ and $(\mathrm{Bd}_\theta(\Phi 3))$ from Assumption $\mathrm{Bd}_\theta(\Phi)$ are contrary to each other in the sense that the first is easy to verify if θ is small, while the second is easier for large θ. Therefore, it is important to mention that the constant θ from Assumption 5.1 might differ from the one from Assumption $\mathrm{Bd}_\theta(\Phi)$.

Lemma 5.19. *Let $p\in[2,\infty)$ and assume that Item $Bd_\theta(\Phi 1)$ from Assumption $Bd_\theta(\Phi)$ and Assumption **K0** are valid. Then, the operators*

$$K_{21}D_2 : \mathcal{F}C_b^1(B_W)\to L^p(W;\mu^\Phi;W), \quad Q_1^{-\theta}K_{21}D_2 : \mathcal{F}C_b^1(B_W)\to L^p(W;\mu^\Phi;W) \quad \text{and}$$
$$K_{22}^{\frac{1}{2}}D_2 : \mathcal{F}C_b^1(B_W)\to L^p(W;\mu^\Phi;W)$$

are closable. We denote their closures by $K_{21}D_2$, $Q_1^{-\theta}K_{21}D_2$ and $K_{22}^{\frac{1}{2}}D_2$ and the corresponding domains by $W^{1,p}_{K_{21}}(W;\mu^\Phi)$, $W^{1,p}_{Q_1^{-\theta}K_{21}}(W;\mu^\Phi)$ and $W^{1,p}_{K_{22}^{\frac{1}{2}}}(W;\mu^\Phi)$. By equipping $W^{1,p}_{K_{21}}(W;\mu^\Phi)$, $W^{1,p}_{Q_1^{-\theta}K_{21}}(W;\mu^\Phi)$ and $W^{1,p}_{K_{22}^{\frac{1}{2}}}(W;\mu^\Phi)$ with the corresponding graph norms, we obtain Banach spaces.

Proof. The closability of the first two operators follows as in Proposition 3.48.
As K_{22} is not constant, we cannot directly use Proposition 3.48, but apply similar arguments. Indeed, let $(f_n)_{n\in\mathbb{N}}\subseteq \mathcal{F}C_b^1(B_W)$ converge to 0 in $L^p(W;\mu^\Phi)$ and be such that $K_{22}^{\frac{1}{2}}D_2 f_n \to F$ in $L^p(W;\mu^\Phi;W)$, as $n\to\infty$. Let $k\in\mathbb{N}$ be given. Using the invariance properties of $K_{22}^{\frac{1}{2}}$ we know that there is some $m\in\mathbb{N}$ independent of $n\in\mathbb{N}$ such that

$$(K_{22}^{\frac{1}{2}}D_2 f_n, e_k)_V = (D_2 f_n, K_{22}^{\frac{1}{2}}e_k)_V = \sum_{i=1}^m (K_{22}^{\frac{1}{2}}e_i,e_k)_V \partial_{e_i} f_n.$$

For an arbitrary $g\in\mathcal{F}C_b^1(B_W)$, we obtain by the integration by parts formula from Proposition 3.48 Item (iii)

$$\int_W (K_{22}^{\frac{1}{2}}D_2 f_n, e_k)_V g \,\mathrm{d}\mu^\Phi = \sum_{i=1}^m \int_W (K_{22}^{\frac{1}{2}}e_i,e_k)_V \partial_{e_i} f_n g \,\mathrm{d}\mu^\Phi$$
$$= -\sum_{i=1}^m \int_W f_n (K_{22}^{\frac{1}{2}}e_i,e_k)_V\partial_{e_i} g + f_n \partial_{e_i}(K_{22}^{\frac{1}{2}}e_i,e_k)_V g - f_n(v,Q_2^{-1}e_i)_V (K_{22}^{\frac{1}{2}}e_i,e_k)_V g\,\mathrm{d}\mu^\Phi.$$

The formula from Proposition 3.40 is indeed applicable, since $K_{22}^{\frac{1}{2}}$ has the appropriate growth and invariance properties, see Remark 5.18. The fact that $(K_{22}^{\frac{1}{2}}e_i,e_k)_V\partial_{e_i}g +$

$\partial_{e_i}(K_{22}^{\frac{1}{2}}e_i, e_k)_V g - (v, Q_2^{-1}e_i)_V(K_{22}^{\frac{1}{2}}e_i, e_k)_V g \in L^{\frac{p}{p-1}}(W;\mu^\Phi)$, implies for $n \to \infty$

$$\int_W (F, e_k)_V g \, \mathrm{d}\mu^\Phi = 0.$$

As $\mathcal{F}C_b^1(B_W)$ is dense in $L^{\frac{p}{p-1}}(W;\mu^\Phi)$, the proof is concluded. □

Lemma 5.20. *Assume that Item $Bd_\theta(\Phi 1)$ from Assumption $Bd_\theta(\Phi)$ and Assumption* ***K0*** *hold true. For $f \in D(L^\Phi)$ and $\lambda \in (0,\infty)$, set*

$$g := \lambda f - L^\Phi f.$$

Then, $f \in W^{1,2}_{K_{22}^{\frac{1}{2}}}(W;\mu^\Phi)$ and the following equation is valid

$$\int_W \lambda f^2 + \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \, \mathrm{d}\mu^\Phi = \int_W fg \, \mathrm{d}\mu^\Phi. \tag{5.2}$$

In particular,

$$\int_W \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \, \mathrm{d}\mu^\Phi \le \frac{1}{2}\int_W f^2 + (Lf)^2 \, \mathrm{d}\mu^\Phi \quad \textit{and} \tag{5.3}$$

$$\int_W \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \, \mathrm{d}\mu^\Phi \le \frac{1}{4\lambda}\int_W g^2 \, \mathrm{d}\mu^\Phi. \tag{5.4}$$

For $f \in \mathcal{F}C_b^\infty(B_W)$ the inequalities above are also valid without assuming Assumption ***K0****.*

Proof. Assume $f \in \mathcal{F}C_b^\infty(B_W)$ and $g = \lambda f - L^\Phi f$. Next, we multiply $g = \lambda f - L^\Phi f$ with f, integrate over W with respect to r.t. μ^Φ and use Lemma 5.9 to obtain the first identity. Rearranging the terms, we obtain

$$\int_W \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \, \mathrm{d}\mu^\Phi = \int_W f(g-\lambda f) \, \mathrm{d}\mu = -\int_W f L^\Phi f \, \mathrm{d}\mu \le \frac{1}{2}\int_W f^2 + (L^\Phi f)^2 \, \mathrm{d}\mu^\Phi.$$

By completing the square, we have

$$\int_W \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \, \mathrm{d}\mu^\Phi = -\int_W \lambda f^2 - fg \, \mathrm{d}\mu^\Phi \le \frac{1}{4\lambda}\int_W g^2 \, \mathrm{d}\mu^\Phi.$$

$\mathcal{F}C_b^\infty(B_W)$ is dense in the L^Φ graph norm. Hence, $K_{22}^{\frac{1}{2}} D_2 f$ exists for $f \in D(L^\Phi)$ as the limit in $L^2(W;\mu^\Phi)$ of $K_{22}^{\frac{1}{2}} D_2 f_n$, where $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^\infty(B_W)$ is the the approximating sequence of f w.r.t L^Φ graph norm. Particularly, $K_{22}^{\frac{1}{2}} D_2 f$ coincides with the application of the closure of the differential operator $K_{22}^{\frac{1}{2}} D_2 : \mathcal{F}C_b^\infty(B_W) \to L^2(W;\mu^\Phi)$, compare Lemma 5.19, to f. Consequently, $D(L^\Phi) \subseteq W^{1,2}_{K_{22}^{\frac{1}{2}}}(W;\mu^\Phi)$ and the (in)equalities above are also valid for $f \in D(L^\Phi)$. □

Lemma 5.21. *Suppose Assumption $Bd_\theta(\Phi)$ and Assumption* ***K0*** *are valid, then*

$$D(L^\Phi) \subseteq W^{1,2}_{K_{22}^{\frac{1}{2}}}(W;\mu^\Phi) \subseteq W^{1,2}_{Q_1^{-\theta}K_{21}}(W;\mu^\Phi) \subseteq W^{1,2}_{K_{21}}(W;\mu^\Phi).$$

Moreover, for all $f \in D(L^\Phi)$ it holds

$$\begin{aligned} \frac{1}{\lambda_{1,1}^{2\theta}} \int_W \|K_{21}D_2 f\|_U^2 \,\mathrm{d}\mu^\Phi &\leq \int_W \|Q_1^{-\theta}K_{21}D_2 f\|_U^2 \,\mathrm{d}\mu^\Phi \\ &\leq \int_W c_\theta \|K_{22}^{\frac{1}{2}} D_2 f\|_V \,\mathrm{d}\mu^\Phi \\ &\leq \frac{c_\theta}{2} \int_W f^2 + (Lf)^2 \,\mathrm{d}\mu^\Phi. \end{aligned} \tag{5.5}$$

Proof. For every $f \in D(L^\Phi)$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^\infty(B_W)$ converging to f with respect to the L^Φ graph norm. Hence, we can estimate for every $n, m \in \mathbb{N}$

$$\begin{aligned} \frac{1}{\lambda_{1,1}^{2\theta}} \int_W \|K_{21}D_2(f_n - f_m)\|_U^2 \,\mathrm{d}\mu^\Phi &\leq \int_W \|Q_1^{-\theta}K_{21}D_2(f_n - f_m)\|_U^2 \,\mathrm{d}\mu^\Phi \\ &\leq \int_W c_\theta \|K_{22}^{\frac{1}{2}} D_2(f_n - f_m)\|_V \,\mathrm{d}\mu^\Phi \\ &\leq \frac{c_\theta}{2} \int_W (f_n - f_m)^2 + (L^\Phi(f_n - f_m))^2 \,\mathrm{d}\mu^\Phi, \end{aligned}$$

where we use Assumption $\mathrm{Bd}_\theta(\Phi)$ and Inequality (5.3). Consequently, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,2}_{K_{21}}(W;\mu^\Phi)$, $W^{1,2}_{Q_1^{-\theta}K_{21}}(W;\mu^\Phi)$ and $W^{1,2}_{K_{22}^{\frac{1}{2}}}(W;\mu^\Phi)$. Since these spaces are complete, we know that f is in all of these Sobolev spaces. The asserted chain of inclusions follows by the chain of inequalities in the estimation above. □

Remark 5.22. *Let Assumption $Bd_\theta(\Phi)$ and Assumption* ***K0*** *are valid, then all the statements and inequalities from Lemma 5.21 hold also for $\Phi = 0$. Particularly, the map*

$$D(L) \ni f \mapsto (Q_1^\theta D\Phi, Q_1^{-\theta}K_{21}D_2 f)_U \in L^2(W;\mu)$$

is well-defined. Since $L^2(W;\mu) \subseteq L^2(W;\mu^\Phi)$, an interpretation as a map to $L^2(W;\mu^\Phi)$ is also reasonable. Now, let $(f_n)_{n\in\mathbb{N}}$ be a sequence converging to some $f \in D(L)$ with respect to the L graph norm. Recalling Remark 5.6 we estimate for all $n \in \mathbb{N}$

$$\begin{aligned} &\int_W \left((D\Phi, K_{21}D_2 f_n)_U - (Q_1^\theta D\Phi, Q_1^{-\theta}K_{21}D_2 f)_U\right)^2 \mathrm{d}\mu^\Phi \\ &= \int_W (Q_1^\theta D\Phi, Q_1^{-\theta}K_{21}D_2(f_n - f))_U^2 \,\mathrm{d}\mu^\Phi \\ &\leq \frac{c_\theta}{2} \|Q_1^\theta D\Phi\|^2_{L^\infty(\mu_1)} \int_W (f_n - f)^2 + (Lf_n - Lf)^2 \,\mathrm{d}\mu. \end{aligned}$$

This implies that the $L^2(W;\mu^\Phi)$ limit of the sequence $((D\Phi, K_{21}D_2 f_n)_U)_{n\in\mathbb{N}}$ exists, is equal to $(Q_1^\theta D\Phi, Q_1^{-\theta}K_{21}D_2 f)_U$ and coincides for all sequences approximating f in L graph norm. Hence, it is reasonable to write $(D\Phi, K_{21}D_2 f)_U$ instead of $(Q_1^\theta D\Phi, Q_1^{-\theta}K_{21}D_2 f)_U$ for all $f \in D(L)$.

We are finally ready to prove the central essential m-dissipativity result for the infinite-dimensional Langevin operator L^Φ in this section.

Theorem 5.23. *Let K_{22} satisfy Assumption* ***K0*** *and* ***K1****. Moreover, suppose Assumption $Bd_\theta(\Phi)$ is valid. Then $D(L) \subseteq D(L^\Phi)$ with*

$$L^\Phi f = Lf - (D\Phi, K_{21}D_2 f)_U, \quad f \in D(L).$$

Furthermore, the infinite dimensional Langevin operator $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative on $L^2(W;\mu^\Phi)$. Additionally, the strongly continuous semigroup $(T_t)_{t\geq 0}$ is sub-Markovian and conservative. $(L^\Phi, D(L^\Phi))$ consequently is a Dirichlet-operator and the corresponding resolvent $(R_\lambda^{L^\Phi})_{\lambda>0}$ is sub-Markovian.

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^\infty(B_W)$ be a sequence converging to $f \in D(L)$ with respect to the L graph norm. Since Φ is bounded from below, it is easy to check that $(f_n)_{n\in\mathbb{N}}$ converges to f in $L^2(W;\mu^\Phi)$. We estimate, using the interpretation discussed in Lemma 5.21 and inequality (5.5) (for $\Phi = 0$ compare also Remark 5.22)

$$\begin{aligned}
&\int_W (L^\Phi f_n - Lf + (D\Phi, K_{21}D_2 f)_U)^2 \,\mathrm{d}\mu^\Phi \\
&\leq 2\int_W (Lf_n - Lf)^2 \,\mathrm{d}\mu^\Phi + 2\int_W (Q_1^\theta D\Phi, Q_1^{-\theta}K_{21}D_2(f_n - f))_U^2 \,\mathrm{d}\mu^\Phi \\
&\leq 2\int_W (Lf_n - Lf)^2 \,\mathrm{d}\mu + c_\theta \|Q_1^\theta D\Phi\|_{L^\infty(\mu_1)}^2 \int_W (f_n - f)^2 + (Lf_n - Lf)^2 \,\mathrm{d}\mu.
\end{aligned}$$

Hence, the sequence $(L^\Phi f_n)_{n\in\mathbb{N}}$ converges to $Lf - (D\Phi, K_{21}D_2 f)_U$ in $L^2(W;\mu^\Phi)$. As the operator $(L^\Phi, D(L^\Phi))$ is closed, we get $D(L) \subseteq D(L^\Phi)$ and for all $f \in D(L)$

$$L^\Phi f = Lf - (D\Phi, K_{21}D_2 f)_U.$$

By Lemma 5.9, we already know that $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is dissipative. In view of the Lumer-Phillips theorem, the dense range condition is left to show. For $f \in L^2(W;\mu)$ and $\lambda \in (0,\infty)$ set

$$T_\lambda f = -(D\Phi, K_{21}D_2 R_\lambda^L f)_U.$$

We estimate using the Assumption $\mathrm{Bd}_\theta(\Phi)$, Inequality (5.4) and (5.5)

$$\begin{aligned}
\int_W (T_\lambda f)^2 \,\mathrm{d}\mu &= \int_W (D\Phi, K_{21}D_2 R_\lambda^L f)_U^2 \,\mathrm{d}\mu \\
&\leq \|Q_1^\theta D\Phi\|_{L^\infty(\mu_1)}^2 \int_W (Q_1^{-\theta}K_{21}D_2 R_\lambda^L f, Q_1^{-\theta}K_{21}D_2 R_\lambda^L f)_U \,\mathrm{d}\mu \\
&\leq \|Q_1^\theta D\Phi\|_{L^\infty(\mu_1)}^2 \int_W c_\theta (K_{22}D_2 R_\lambda^L f, D_2 R_\lambda^L f)_U \,\mathrm{d}\mu \\
&\leq \|Q_1^\theta D\Phi\|_{L^\infty(\mu_1)}^2 \frac{c_\theta}{4\lambda} \int_W f^2 \,\mathrm{d}\mu.
\end{aligned}$$

This yields that $T_\lambda : L^2(W;\mu) \to L^2(W;\mu)$ is well-defined. Moreover, if

$$\|Q_1^\theta D\Phi\|_{L^\infty(\mu_1)}^2 \frac{c_\theta}{4\lambda} < 1, \tag{5.6}$$

the Neumann-Series theorem implies $(\mathrm{Id} - T_\lambda)^{-1} \in \mathcal{L}(L^2(W;\mu))$.
Now fix $\lambda \in (0,\infty)$ such that (5.6) holds. For all $g \in L^2(W;\mu)$, we then find $f \in L^2(W;\mu)$ with $f - T_\lambda f = g$ in $L^2(W;\mu)$. Furthermore, there is $h \in D(L)$ with $(\lambda - L)h = f$ and therefore,

$$(\lambda - L^\Phi)h = (\lambda - L)h + (D\Phi, K_{21}D_2 h)_U = f + (D\Phi, K_{21}D_2 R^L_\lambda f)_U = f - T_\lambda f = g.$$

This implies that $L^2(W;\mu) \subseteq (\lambda - L^\Phi)(D(L))$. Since $L^2(W;\mu)$ is dense in $L^2(W;\mu^\Phi)$ and $D(L) \subseteq D(L^\Phi)$, the dense range condition is shown. Sub-Markovianity, conservativity and μ^Φ-invariance follows as in Theorem 5.15. □

5.1.2 Essential m-dissipativity of infinite dimensional Langevin operators for potentials with possibly unbounded gradient

In the previous section, we showed that $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative, if the potential Φ fulfills Assumption $\mathrm{Bd}_\theta(\Phi)$. In particular, we assumed boundedness of $Q_1^\theta D\Phi$ for some $\theta \in [0,\infty)$. To relax this strong boundedness assumption, we approximate Φ by a sequence $(\Phi_n^m)_{n,m\in\mathbb{N}}$ and suppose that there exists a constant $\lambda \in (0,\infty)$ independent of $m,n \in \mathbb{N}$ such that for each $g \in \mathcal{F}C_b^\infty(B_W)$, there exists a sequence $(f_{n,m})_{n,m\in\mathbb{N}} \subseteq \mathcal{F}C_b^3(B_W)$ with

$$\lambda f_{n,m} - L^{\Phi_n^m} f_{n,m} = g. \tag{5.7}$$

By imposing the properties stated in Assumption App(Φ), we show that $f_{n,m}$ fulfills an $L^4(W;\mu^{\Phi_n^m})$ first order regularity estimate independent of $m,n \in \mathbb{N}$. Roughly speaking, the $L^4(W;\mu^{\Phi_n^m})$ regularity estimate is the key to apply a similar strategy as in Theorem 5.23, but with a more involved version of the Hölder inequality. To prove this important $L^4(W;\mu^{\Phi_n^m})$ inequality, we generalize the arguments from [DT00],[DZ02, Chapter 12.3] and [DL05], where non-degenerate infinite dimensional and degenerate finite dimensional operators, both without variable diffusion coefficient, of this type have been studied.

Assumption (App(Φ))**.** There exists a sequence $(\Phi_n^m)_{n,m\in\mathbb{N}}$ such that,

(App(Φ1)) for each fixed $m \in \mathbb{N}$, $\Phi_n^m(u) = \Phi_n^m \circ P_n^U u$ for all $n \in \mathbb{N}$ and $u \in U$.

(App(Φ2)) $v \mapsto K_{22}(v)e_i \in C_b^2(V;V)$ for all $i \in \mathbb{N}$. Further, for all $m,n \in \mathbb{N}$, $\Phi_n^m \in C^3(U;\mathbb{R})$ and $D\Phi_n^m$ has bounded derivatives up to the second order.

(App(Φ3)) there exists $\lambda \in (0,\infty)$ such that for every $n,m \in \mathbb{N}$ and $g \in \mathcal{F}C_b^\infty(B_W)$, there is a solution $f_{n,m} \in \mathcal{F}C_b^3(B_W)$ of Equation (5.7) with $\|f_{n,m}\|_\infty \le \frac{1}{\lambda}\|g\|_\infty$

(App(Φ4)) there are $\alpha,\beta,\gamma \in [0,\infty)$ and $\kappa \in (1,\infty)$, all independent of $m,n \in \mathbb{N}$ such that

$$\begin{aligned}\int_W \Big\|\sum_{i=1}^\infty \lambda_{1,i}^\alpha K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}^2 \Phi_n^m\Big\|_V^2 + \Big\|\sum_{i=1}^\infty K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi_n^m\,(u, Q_1^{\alpha-1}e_i)_U\Big\|_V^2 \\ + \Big\|\sum_{i=1}^\infty K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi_n^m\,\lambda_{1,i}^\alpha \partial_{d_i}\Phi_n^m\Big\|_V^2 \,\mathrm{d}\mu^{\Phi_n^m} \le \kappa^2\end{aligned} \tag{5.8}$$

$$\sum_{i=1}^\infty \int_W \Big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi_n^m\Big\|_U^2 \,\mathrm{d}\mu^{\Phi_n^m} \le \kappa. \tag{5.9}$$

Additionally, it holds for all $f \in \mathcal{F}C_b^\infty(B_W)$

$$\left\|Q_1^{\frac{\alpha}{2}-1} K_{21} D_2 f\right\|_V \leq \kappa \left\|K_{22}^{\frac{1}{2}} D_2 f\right\|_V \tag{5.10}$$

$$\left\|Q_2^{\frac{\beta}{2}-1} K_{22} D_2 f\right\|_V + \left(\sum_{i=1}^\infty \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{-\frac{1}{2}} \partial_{e_i} K_{22} D_2 f\right\|_V^2\right)^{\frac{1}{2}} \leq \kappa \left\|K_{22}^{\frac{1}{2}} D_2 f\right\|_V \tag{5.11}$$

$$\left\|Q_2^{\frac{\beta}{2}-1} K_{12} D_1 f\right\|_V \leq \kappa \left\|Q_1^{\frac{\alpha}{2}} D_1 f\right\|_U \tag{5.12}$$

$$\left(\sum_{i=1}^\infty \left(\int_W \lambda_{2,i}^{4\gamma} (\partial_{e_i}^2 f)^2 \,\mathrm{d}\mu^{\Phi_n^m}\right)^{\frac{1}{2}}\right)^2 \leq \kappa \int_W \sum_{i=1}^\infty \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{e_i} f\right\|_V^2 \,\mathrm{d}\mu^{\Phi_n^m} \tag{5.13}$$

$$\sum_{i=1}^\infty \lambda_{2,i}^{2\gamma-1} \leq \kappa^{\frac{1}{2}}. \tag{5.14}$$

(App(Φ5)) there is some constant $p^* \in (4,\infty)$ such that

$$\lim_{n\to\infty} \lim_{m\to\infty} \int_U \left\|Q_2^{-\gamma} K_{12}(D\Phi_n^m - P_n D\Phi)\right\|_V^{p^*} \,\mathrm{d}\mu_1^{\Phi_n^m} = 0. \tag{5.15}$$

For $q^* := \frac{2p^*}{p^*-4}$ there are constants $c_1, c_2 \in \mathbb{R}$ and $c_3 < \frac{1}{2\lambda_{1,1}}$ such that for all $m, n \in \mathbb{N}$

$$c_1 \leq \Phi_n^m(u) \quad \text{and} \tag{5.16}$$

$$(q^*-1)\Phi_n^m(u) \leq c_2 + c_3\|u\|_U^2 + q^*\Phi(u) \quad \text{for all} \quad u \in U. \tag{5.17}$$

For the next three lemmas, we consider Assumption App(Φ) as valid. In particular, there exists $\lambda \in (0,\infty)$ such that for $g \in \mathcal{F}C_b^\infty(B_W)$ there is a function $f_{n,m} \in \mathcal{F}C_b^3(B_W)$ with

$$\lambda f_{n,m} - L^{\Phi_n^m} f_{n,m} = g \quad \text{and} \quad \|f_{n,m}\|_\infty \leq \frac{1}{\lambda}\|g\|_\infty. \tag{5.18}$$

We next establish the existence of a constant c, independent of $n, m \in \mathbb{N}$ such that

$$\int_W \left\|Q_2^\gamma D_2 f_{n,m}\right\|_V^4 \,\mathrm{d}\mu^{\Phi_n^m} \leq c,$$

where γ is the parameter from App(Φ4).

Lemma 5.24. *There is a constant $a := a(\lambda, g, \kappa) \in (0,\infty)$, independent of $m, n \in \mathbb{N}$ with*

$$\int_W \left\|Q_1^{\frac{\alpha}{2}} D_1 f_{n,m}\right\|_U^2 \,\mathrm{d}\mu^{\Phi_n^m} \leq a.$$

Proof. To avoid an overload of notation, we fix n, m and substitute Φ_n^m with Φ and $f_{n,m}$ by f in the following proof. Differentiating (5.18) with respect to d_i yields

$$\lambda \partial_{d_i} f - L^\Phi \partial_{d_i} f + (d_i, Q_1^{-1} K_{21} D_2 f)_U + (D\partial_{d_i}\Phi, K_{21} D_2 f)_U = \partial_{d_i} g. \tag{5.19}$$

Multiply the equation above with $\lambda_{1,i}^{\alpha}\partial_{d_i} f$, integrate over W with respect to μ^Φ and sum over all $i \in \mathbb{N}$ to obtain

$$\begin{aligned}\int_W \lambda \big\|Q_1^{\frac{\alpha}{2}} D_1 f\big\|_U^2 + \sum_{i=1}^{\infty} \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{d_i} f\big\|_V^2 + \big(Q_1^{\alpha} D_1 f, Q_1^{-1} K_{21} D_2 f\big)_U \ \mathrm{d}\mu^\Phi \\ + \int_W \sum_{i=1}^{\infty} \lambda_{1,i}^{\alpha} \partial_{d_i} f \,(D\partial_{d_i}\Phi, K_{21} D_2 f)_U \ \mathrm{d}\mu^\Phi = \int_W (Q_1^{\frac{\alpha}{2}} D_1 f, Q_1^{\frac{\alpha}{2}} D_1 g)_U \ \mathrm{d}\mu^\Phi, \qquad (5.20)\end{aligned}$$

where we also used Lemma 5.9 and in particular

$$-\big(L^\Phi \partial_{d_i} f, \partial_{d_i} f\big)_{L^2(\mu^\Phi)} = \int_W (K_{22} D_2 \partial_{d_i} f, D_2 \partial_{d_i} f)_V \ \mathrm{d}\mu^\Phi.$$

Note that

$$\sum_{i=1}^{\infty} \lambda_{1,i}^{\alpha} \partial_{d_i} f \,(D\partial_{d_i}\Phi, K_{21} D_2 f)_U = \sum_{i,j=1}^{\infty} \lambda_{1,i}^{\alpha} \partial_{d_i} f \partial_{d_i} \partial_{d_j} \Phi \,(d_j, K_{21} D_2 f)_U\,.$$

Using the integration by parts formula from Lemma 3.47, we compute

$$\begin{aligned}&\int_W \lambda_{1,i}^{\alpha} \partial_{d_i} f \partial_{d_i} \partial_{d_j} \Phi \,(d_j, K_{21} D_2 f)_U \ \mathrm{d}\mu^\Phi \\ &= - \int_W \lambda_{1,i}^{\alpha} f \partial_{d_i} \partial_{d_i} \partial_{d_j} \Phi \,(d_j, K_{21} D_2 f)_U \ \mathrm{d}\mu^\Phi - \int_W \lambda_{1,i}^{\alpha} f \partial_{d_i} \partial_{d_j} \Phi \,(d_j, K_{21} D_2 \partial_{d_i} f)_U \ \mathrm{d}\mu^\Phi \\ &\quad + \int_W \lambda_{1,i}^{\alpha-1} (u, d_i)_U f \partial_{d_i} \partial_{d_j} \Phi \,(d_j, K_{21} D_2 f)_U \ \mathrm{d}\mu^\Phi \\ &\quad + \int_W \lambda_{1,i}^{\alpha} \partial_{d_i} \Phi f \partial_{d_i} \partial_{d_j} \Phi \,(d_j, K_{21} D_2 f)_U \ \mathrm{d}\mu^\Phi \\ &=: -I_{ij}^1 - I_{ij}^2 + I_{ij}^3 + I_{ij}^4\end{aligned}$$

The next step on our agenda is to estimate $I_{i,j}^1, I_{i,j}^2, I_{i,j}^3$ and $I_{i,j}^4$ separately. Indeed, using Inequality (5.8) from Assumption App(Φ) and the Inequalities (5.18) and (5.4), we estimate

$$\begin{aligned}\Big|\sum_{i,j=1}^{\infty} I_{ij}^1\Big| &= \Big|\int_W \big(\sum_{i=1}^{\infty} \lambda_{1,i}^{\alpha} D\partial_{d_i}^2 \Phi, K_{21} D_2 f\big)_U f \ \mathrm{d}\mu^\Phi\Big| \\ &\leq \|f\|_\infty \left(\int_W \Big\|\sum_{i=1}^{\infty} \lambda_{1,i}^{\alpha} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}^2 \Phi\Big\|_V^2 \ \mathrm{d}\mu^\Phi\right)^{\frac{1}{2}} \left(\int_W \big\|K_{22}^{\frac{1}{2}} D_2 f\big\|_V^2 \ \mathrm{d}\mu^\Phi\right)^{\frac{1}{2}} \\ &\leq \frac{\|g\|_\infty}{\lambda} \kappa \frac{1}{2\sqrt{\lambda}} \|g\|_{L^2(\mu^\Phi)} =: a_1.\end{aligned}$$

Using Inequality (5.9) and Youngs inequality for $\delta \in (0,\infty)$, yields

$$\begin{aligned}
\Big|\sum_{i,j=1}^{\infty} I_{ij}^2\Big| &= \Big|\int_W \sum_{i=1}^{\infty} \big(\lambda_{1,i}^{\alpha} D\partial_{d_i}\Phi, K_{21}D_2\partial_{d_i} f\big)_U f \,\mathrm{d}\mu^{\Phi}\Big| \\
&\leq \|f\|_\infty \sum_{i=1}^{\infty}\int_W \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi\big\|_V \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2\partial_{d_i} f\big\|_V \,\mathrm{d}\mu^{\Phi} \\
&\leq \|f\|_\infty \sum_{i=1}^{\infty}\int_W \frac{1}{2\delta}\big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi\big\|_V^2 + \frac{\delta}{2}\big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2\partial_{d_i} g\big\|_V^2 \,\mathrm{d}\mu^{\Phi} \\
&\leq \frac{\|g\|_\infty}{\lambda}\frac{1}{2\delta}\kappa + \frac{\|g\|_\infty}{\lambda}\frac{\delta}{2}\sum_{i=1}^{\infty}\int_W \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2\partial_{d_i} f\big\|_V^2 \,\mathrm{d}\mu^{\Phi}.
\end{aligned}$$

Taking Inequality (5.8) into account, we estimate

$$\begin{aligned}
\Big|\sum_{i,j=1}^{\infty} I_{ij}^3\Big| &= \Big|\int_W \sum_{i=1}^{\infty} (D\partial_{d_i}\Phi, K_{21}D_2 f)_U\,(u, Q_1^{\alpha-1}e_i)_U f \,\mathrm{d}\mu^{\Phi}\Big| \\
&\leq \|f\|_\infty \int_W \Big\|\sum_{i=1}^{\infty} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi\,(u, Q_1^{\alpha-1}e_i)_U\Big\|_V \big\|K_{22}^{\frac{1}{2}} D_2 f\big\|_V \,\mathrm{d}\mu^{\Phi} \\
&\leq \|f\|_\infty \left(\int_W \Big\|\sum_{i=1}^{\infty} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi\,(u, Q_1^{\alpha-1}e_i)_U\Big\|_V^2 \,\mathrm{d}\mu^{\Phi} \int_W \big\|K_{22}^{\frac{1}{2}} D_2 f\big\|_V^2 \,\mathrm{d}\mu^{\Phi}\right)^{\frac{1}{2}} \\
&\leq \frac{\|g\|_\infty}{\lambda}\kappa\frac{1}{2\sqrt{\lambda}}\|g\|_{L^2(\mu^{\Phi})} = a_1.
\end{aligned}$$

Again, by (5.8), we have

$$\begin{aligned}
\Big|\sum_{i,j=1}^{\infty} I_{ij}^4\Big| &= \Big|\int_W \sum_{i=1}^{\infty} (D\partial_{d_i}\Phi, K_{21}D_2 f)_U\,\lambda_{1,i}^{\alpha}\partial_{d_i}\Phi f \,\mathrm{d}\mu^{\Phi}\Big| \\
&\leq \|f\|_\infty \int_W \Big\|\sum_{i=1}^{\infty} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi\,\lambda_{1,i}^{\alpha}\partial_{d_i}\Phi\Big\|_V \big\|K_{22}^{\frac{1}{2}} D_2 f\big\|_V \,\mathrm{d}\mu^{\Phi} \\
&\leq \frac{\|g\|_\infty}{\lambda}\kappa\frac{1}{2\sqrt{\lambda}}\|g\|_{L^2(\mu^{\Phi})} = a_1.
\end{aligned}$$

A combination of the above (in)equalities implies

$$\sum_{i,j=1}^{\infty} I_{ij}^1 + I_{ij}^2 - I_{ij}^3 - I_{ij}^4 \leq \frac{\|g\|_\infty}{\lambda}\frac{1}{2\delta}\kappa + \frac{\|g\|_\infty}{\lambda}\frac{\delta}{2}\sum_{i=1}^{\infty}\int_W \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2\partial_{d_i} f\big\|_V^2 \,\mathrm{d}\mu^{\Phi} + 3a_1. \quad (5.21)$$

If we plug Inequality (5.21) into Equation (5.20) and apply Youngs inequality two times for $\varepsilon \in (0,\infty)$, we derive the following inequality

$$\begin{aligned}
&\int_W \lambda \big\|Q_1^{\frac{\alpha}{2}} D_1 f\big\|_U^2 + \sum_{i=1}^\infty \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{d_i} f\big\|_V^2 \,\mathrm{d}\mu^\Phi \\
&= -\int_W \big(Q_1^{\alpha} D_1 f, Q_1^{-1} K_{21} D_2 f\big)_U + \big(Q_1^{\frac{\alpha}{2}} D_1 f, Q_1^{\frac{\alpha}{2}} D_1 g\big)_U \,\mathrm{d}\mu^\Phi + \sum_{i,j=1}^\infty I_{ij}^1 + I_{ij}^2 - I_{ij}^3 - I_{ij}^4 \\
&\le \frac{\varepsilon}{2}\int_W \big\|Q_1^{\frac{\alpha}{2}} D_1 f\big\|_U^2 \,\mathrm{d}\mu^\Phi + \frac{1}{2\varepsilon}\int_W \big\|Q_1^{\frac{\alpha}{2}-1} K_{21} D_2 f\big\|_U^2 \,\mathrm{d}\mu^\Phi \\
&\quad + \frac{\varepsilon}{2}\int_W \big\|Q_1^{\frac{\alpha}{2}} D_1 f\big\|_U^2 \,\mathrm{d}\mu^\Phi + \frac{1}{2\varepsilon}\int_W \big\|Q_1^{\frac{\alpha}{2}} D_1 g\big\|_U^2 \,\mathrm{d}\mu^\Phi \\
&\quad + \frac{\|g\|_\infty}{\lambda}\frac{1}{2\delta}\kappa + \frac{\|g\|_\infty}{\lambda}\frac{\delta}{2}\sum_{i=1}^\infty \int_W \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{d_i} f\big\|_V^2 \,\mathrm{d}\mu^\Phi + 3a_1.
\end{aligned}$$

Using that $\big\|Q_1^{\frac{\alpha}{2}-1} K_{21} D_2 f\big\|_V \le \kappa \big\|K_{22}^{\frac{1}{2}} D_2 f\big\|_V$ by Inequality (5.10) from the Assumption App(Φ), we obtain after rearranging the terms

$$\begin{aligned}
&(\lambda-\varepsilon)\int_W \big\|Q_1^{\frac{\alpha}{2}} D_1 f\big\|_U^2 \,\mathrm{d}\mu^\Phi + \left(1 - \frac{\|g\|_\infty}{\lambda}\frac{\delta}{2}\right)\int_W \sum_{i=1}^\infty \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{d_i} f\big\|_V^2 \,\mathrm{d}\mu^\Phi \\
&\le \frac{1}{2\varepsilon}\kappa \int_W \big\|K_{22}^{\frac{1}{2}} D_2 f\big\|_V^2 \,\mathrm{d}\mu^\Phi + \frac{1}{2\varepsilon}\int_W \big\|Q_1^{\frac{\alpha}{2}} D_1 g\big\|_U^2 \,\mathrm{d}\mu^\Phi + \frac{\|g\|_\infty}{\lambda}\frac{1}{2\varepsilon}\kappa + 3a_1 \\
&\le \frac{1}{2\varepsilon}\kappa \frac{1}{4\lambda}\|g\|_{L^2(\mu^\Phi)}^2 + \frac{1}{2\varepsilon}\int_W \big\|Q_1^{\frac{\alpha}{2}} D_1 g\big\|_U^2 \,\mathrm{d}\mu^\Phi + \frac{\|g\|_\infty}{\lambda}\frac{1}{2\delta}\kappa + 3a_1.
\end{aligned}$$

For $\delta := \frac{\lambda}{\|g\|_\infty}$ and $\varepsilon := \frac{\lambda}{2}$ this implies

$$\begin{aligned}
&\frac{\lambda}{2}\int_W \big\|Q_1^{\frac{\alpha}{2}} D_1 f\big\|_U^2 \,\mathrm{d}\mu^\Phi + \frac{1}{2}\int_W \sum_{i=1}^\infty \big\|\lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{d_i} f\big\|_V^2 \,\mathrm{d}\mu^\Phi \\
&\le \frac{\kappa}{4\lambda^2}\|g\|_{L^2(\mu^\Phi)}^2 + \frac{1}{\lambda}\int_W \big\|Q_1^{\frac{\alpha}{2}} D_1 g\big\|_U^2 \,\mathrm{d}\mu^\Phi + \frac{\|g\|_\infty^2}{2\lambda^2}\kappa + 3a_1 =: a_2.
\end{aligned}$$

Finally, setting $a := a(\lambda, g, \kappa) := \frac{2a_2}{\lambda}$ concludes the proof. □

Lemma 5.25. *There is a constant $b := b(\lambda, g, \kappa) \in (0,\infty)$, independent of $m,n \in \mathbb{N}$ such that*

$$\int_W \lambda \big\|Q_2^{\frac{\beta}{2}} D_2 f\big\|_V^2 + \sum_{i=1}^\infty \big\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{e_i} f\big\|_V^2 \,\mathrm{d}\mu^\Phi \le b.$$

Proof. Let n, m be fixed and substitute Φ_n^m and $f_{n,m}$ with Φ and f, respectively. Differentiating (5.18) with respect to e_i yields

$$\begin{aligned}
&\lambda \partial_{e_i} f - L^\Phi \partial_{e_i} f - \Big(\mathrm{tr}[\partial_{e_i} K_{22} D_2^2 f] + \sum_{j=1}^\infty \big(\partial_{e_j}\partial_{e_i} K_{22} D_2 f, e_j\big)_V - (v, Q_2^{-1}\partial_{e_i} K_{22} D_2 f)_V\Big) \\
&+ (e_i, Q_2^{-1} K_{22} D_2 f)_V - (e_i, Q_2^{-1} K_{12} D_1 f) = \partial_{e_i} g. \qquad (5.22)
\end{aligned}$$

We define the operator S_i on $\mathcal{F}C_b^3(B_W)$ by

$$S_i h := \mathrm{tr}[\partial_{e_i} K_{22} D_2^2 h] + \sum_{j=1}^\infty \left(\partial_{e_j}\partial_{e_i} K_{22} D_2 f, e_j\right)_V - (v, Q_2^{-1}\partial_{e_i} K_{22} D_2 h)_V.$$

S_i is an operator of Ornstein-Uhlenbeck type with variable diffusion operator $\partial_{e_i} K_{22}$. Hence, an integration by parts formula, similar to the one from Lemma 5.9, is applicable. We then calculate

$$-\left(L^\Phi \partial_{e_i} f, \partial_{e_i} f\right)_{L^2(\mu^\Phi)} = \int_W (K_{22} D_2 \partial_{e_i} f, D_2 \partial_{e_i} f)_V \ \mathrm{d}\mu^\Phi \quad \text{and}$$
$$-(S_i f, \partial_{e_i} f)_{L^2(\mu^\Phi)} = \int_W (\partial_{e_i} K_{22} D_2 f, D_2 \partial_{e_i} f)_V \ \mathrm{d}\mu^\Phi.$$

Using the equations above, we continue by multiplying Equation (5.22) with $\lambda_{2,i}^\beta \partial_{e_i} f$, by integrating over W with respect to μ^Φ and by summing over all $i \in \mathbb{N}$, to obtain

$$\begin{aligned}&\int_W \lambda \left\|Q_2^{\frac{\beta}{2}} D_2 f\right\|_V^2 + \sum_{i=1}^\infty \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac12} D_2 \partial_{e_i} f\right\|_V^2 + \sum_{i=1}^\infty \lambda_{2,i}^\beta \left(\partial_{e_i} K_{22} D_2 f, D_2 \partial_{e_i} f\right)_V \\ &+ (Q_2^\beta D_2 f, Q_2^{-1} K_{22} D_2 f)_V - (Q_2^\beta D_2 f, Q_2^{-1} K_{12} D_1 f)_V \ \mathrm{d}\mu^\Phi = \int_W \left(Q_2^{\frac{\beta}{2}} D_2 f, Q_2^{\frac{\beta}{2}} D_2 g\right)_V \mathrm{d}\mu^\Phi.\end{aligned}$$

In view of Inequality (5.11) and (5.12) from assumption App(Φ) and Youngs inequality for $\delta \in (0,\infty)$, we estimate

$$\begin{aligned}&\int_W \lambda \left\|Q_2^{\frac{\beta}{2}} D_2 f\right\|_V^2 + \sum_{i=1}^\infty \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac12} D_2 \partial_{e_i} f\right\|_V^2 \ \mathrm{d}\mu^\Phi \\ &\le \int_W \left\|Q_2^{\frac{\beta}{2}} D_2 f\right\|_V \left\|Q_2^{\frac{\beta}{2}} D_2 g\right\|_V + \sum_{i=1}^\infty \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{-\frac12} \partial_{e_i} K_{22} D_2 f\right\|_V \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac12} D_2 \partial_{e_i} f\right\|_V \\ &\quad + \left\|Q_2^{\frac{\beta}{2}} D_2 f\right\|_V \left\|Q_2^{\frac{\beta}{2}-1} K_{22} D_2 f\right\|_V + \left\|Q_2^{\frac{\beta}{2}} D_2 f\right\|_V \left\|Q_2^{\frac{\beta}{2}-1} K_{12} D_1 f\right\|_V \ \mathrm{d}\mu^\Phi \\ &\le \int_W \left\|Q_2^{\frac{\beta}{2}} D_2 f\right\|_V \kappa \left(\left\|Q_2^{\frac{\beta}{2}} D_2 g\right\|_V + \left\|K_{22}^{\frac12} D_2 f\right\|_V + \left\|Q_1^{\frac{\alpha}{2}} D_1 f\right\|_U \right) \ \mathrm{d}\mu^\Phi \\ &\quad + \frac12 \int_W \sum_{i=1}^\infty \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac12} D_2 \partial_{e_i} f\right\|_V^2 \ \mathrm{d}\mu^\Phi + \frac12 \int_W \sum_{i=1}^\infty \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{-\frac12} \partial_{e_i} K_{22} D_2 f\right\|_V^2 \ \mathrm{d}\mu^\Phi \\ &\le \frac{\delta}{2} \int_W \left\|Q_2^{\frac{\beta}{2}} D_2 f\right\|_V^2 \ \mathrm{d}\mu^\Phi + \frac{\kappa^2}{2\delta} \int_W \left(\left\|Q_2^{\frac{\beta}{2}} D_2 g\right\|_V + \left\|K_{22}^{\frac12} D_2 f\right\|_V + \left\|Q_1^{\frac{\alpha}{2}} D_1 f\right\|_U \right)^2 \ \mathrm{d}\mu^\Phi \\ &\quad + \frac12 \int_W \sum_{i=1}^\infty \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac12} D_2 \partial_{e_i} f\right\|_V^2 \ \mathrm{d}\mu^\Phi + \frac12 \int_W \kappa^2 \left\|K_{22}^{\frac12} D_2 f\right\|_V^2 \ \mathrm{d}\mu^\Phi.\end{aligned}$$

The choice of $\delta = \lambda$ leads to

$$\begin{aligned}
&\int_W \lambda \|Q_2^{\frac{\beta}{2}} D_2 f\|_V^2 + \sum_{i=1}^{\infty} \|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{e_i} f\|_V^2 \, \mathrm{d}\mu^\Phi \\
&\leq \kappa^2 \int_W \frac{1}{\lambda} \Big(\|Q_2^{\frac{\beta}{2}} D_2 g\|_V + \|K_{22}^{\frac{1}{2}} D_2 f\|_V + \|Q_1^{\frac{\alpha}{2}} D_1 f\|_U \Big)^2 + \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \, \mathrm{d}\mu^\Phi \\
&\leq \kappa^2 \int_W \frac{4}{\lambda} \Big(\|Q_2^{\frac{\beta}{2}} D_2 g\|_V^2 + \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 + \|Q_1^{\frac{\alpha}{2}} D_1 f\|_U^2 \Big) + \|K_{22}^{\frac{1}{2}} D_2 f\|_V^2 \, \mathrm{d}\mu^\Phi.
\end{aligned}$$

By Lemma 5.24 and Inequality (5.4), we conclude

$$\begin{aligned}
&\int_W \lambda \|Q_2^{\frac{\beta}{2}} D_2 f\|_V^2 + \sum_{i=1}^{\infty} \|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{e_i} f\|_V^2 \, \mathrm{d}\mu^\Phi \\
&\leq \kappa^2 \int_W \frac{4}{\lambda} \Big(\|Q_2^{\frac{\beta}{2}} D_2 g\|_V^2 + \frac{1}{4\lambda} g^2 + a(\lambda, g, \kappa) \Big) + \frac{1}{4\lambda} g^2 \, \mathrm{d}\mu^\Phi =: b.
\end{aligned}$$

□

Lemma 5.26. *There is a constant $c := c(\lambda, g, \kappa) \in (0, \infty)$, independent of $m, n \in \mathbb{N}$ such that*

$$\int_W \|Q_2^\gamma D_2 f_{n,m}\|_V^4 \, \mathrm{d}\mu^{\Phi_n^m} \leq c.$$

Proof. Again, we fix n, m and replace Φ_n^m with Φ and $f_{n,m}$ with g. For $i \in \mathbb{N}$ set

$$p_i := \int_W \lambda_{2,i}^{4\gamma} (\partial_{e_i} f)^4 \, \mathrm{d}\mu^\Phi, \quad h_i := \left(\int_W \lambda_{2,i}^{4\gamma} (\partial_{e_i}^2 f)^2 \, \mathrm{d}\mu^\Phi \right)^{\frac{1}{2}} \quad \text{and} \quad \tilde{h}_i := \sqrt[4]{3} \lambda_{2,i}^{\gamma - \frac{1}{2}}.$$

Using the integration by parts formula from Lemma 3.47, we obtain, by an application of the Cauchy-Schwarz and Hölder Inequality ($p = \frac{4}{3}$ and $q = 4$),

$$\begin{aligned}
p_i &= \int_W \lambda_{2,i}^{4\gamma} (\partial_{e_i} f)^3 \partial_{e_i} f \, \mathrm{d}\mu^\Phi = \int_W -\lambda_{2,i}^{4\gamma} 3 (\partial_{e_i} f)^2 \partial_{e_i}^2 f f + \lambda_{2,i}^{4\gamma} (\partial_{e_i} f)^3 f (v, Q_2^{-1} e_i)_V \, \mathrm{d}\mu^\Phi \\
&\leq 3 \frac{\|g\|_\infty}{\lambda} \left(\int_W \lambda_{2,i}^{4\gamma} (\partial_{e_i} f)^4 \, \mathrm{d}\mu^\Phi \right)^{\frac{1}{2}} \left(\int_W \lambda_{2,i}^{4\gamma} (\partial_{e_i}^2 f)^2 \, \mathrm{d}\mu^\Phi \right)^{\frac{1}{2}} \\
&\quad + \frac{\|g\|_\infty}{\lambda} \left(\int_W \lambda_{2,i}^{\frac{16}{3}\gamma} (\partial_{e_i} f)^4 \, \mathrm{d}\mu^\Phi \right)^{\frac{3}{4}} \left(\int_W (v, Q_2^{-1} e_i)_V^4 \, \mathrm{d}\mu^\Phi \right)^{\frac{1}{4}} \\
&= 3 \frac{\|g\|_\infty}{\lambda} p_i^{\frac{1}{2}} \left(\int_W \lambda_{2,i}^{4\gamma} (\partial_{e_i}^2 f)^2 \, \mathrm{d}\mu^\Phi \right)^{\frac{1}{2}} + \frac{\|g\|_\infty}{\lambda} (\lambda_{2,i}^{\frac{4}{3}\gamma})^{\frac{3}{4}} p_i^{\frac{3}{4}} \lambda_{2,i}^{-1} \left(\int_W (v, e_i)_V^4 \, \mathrm{d}\mu^\Phi \right)^{\frac{1}{4}} \\
&= 3 \frac{\|g\|_\infty}{\lambda} p_i^{\frac{1}{2}} h_i + \frac{\|g\|_\infty}{\lambda} p_i^{\frac{3}{4}} \tilde{h}_i.
\end{aligned}$$

In the last equality above, we used that $\int_W (v, e_i)_V^4 \, \mathrm{d}\mu^\Phi = \int_V (v, e_i)_V^4 \, \mathrm{d}\mu_2 = 3\lambda_{2,i}^2$, by Lemma 3.5. Dividing by $p_i^{\frac{1}{2}}$ and using that $xy \leq \frac{1}{2}(x^2 + y^2)$ for all $x, y \in \mathbb{R}$, we get

$$p_i^{\frac{1}{2}} \leq 3 \frac{\|g\|_\infty}{\lambda} h_i + \frac{\|g\|_\infty}{\lambda} p_i^{\frac{1}{4}} \tilde{h}_i \leq 3 \frac{\|g\|_\infty}{\lambda} h_i + \frac{1}{2} p_i^{\frac{1}{2}} + \frac{1}{2} \frac{\|g\|_\infty^2}{\lambda^2} \tilde{h}_i^2.$$

Set $A := \max\left\{6\frac{\|g\|_\infty}{\lambda}, \frac{\|g\|_\infty^2}{\lambda^2}\right\}$. Then it holds

$$p_i^{\frac{1}{2}} \leq A\left(h_i + \tilde{h}_i^2\right).$$

Lastly, we conclude with the Inequalities (5.13) and (5.14) from Assumption App(Φ) and Lemma 5.25

$$\int_W \|Q_2^\gamma D_2 f\|_V^4 \,\mathrm{d}\mu^\Phi = \int_W \left(\sum_{i=1}^\infty \lambda_{2,i}^{2\gamma}(\partial_{e_i} f)^2\right)^2 \mathrm{d}\mu^\Phi$$
$$= \sum_{i,j=1}^\infty \int_W \lambda_{2,i}^{2\gamma}(\partial_{e_i} f)^2 \lambda_{2,j}^{2\gamma}(\partial_{e_j} f)^2 \,\mathrm{d}\mu^\Phi \leq \left(\sum_{i=1}^\infty p_i^{\frac{1}{2}}\right)^2 \leq A^2\left(\sum_{i=1}^\infty h_i + \tilde{h}_i^2\right)^2$$
$$\leq 2A^2\left(\sum_{i=1}^\infty h_i\right)^2 + 2A^2\left(\sum_{i=1}^\infty \tilde{h}_i^2\right)^2 \leq 2A^2\kappa(b+3) =: c.$$

□

As we have our desired $L^4(W;\mu^{\Phi_m^n})$ regularity estimate at hand, we are able to show that the infinite dimensional Langevin operator $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative on $L^2(W;\mu^\Phi)$. In Section 8.3 we apply this result in a degenerate stochastic reaction-diffusion setting.

Theorem 5.27. *Let Assumption App(Φ) be valid. Then, $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipative on $L^2(W;\mu^\Phi)$. The corresponding strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ is sub-Markovian and conservative.*

Proof. By Lemma 5.9, we already know that $(L, \mathcal{F}C_b^\infty(B_W))$ is dissipative on $L^2(W;\mu^\Phi)$ and therefore closable in $L^2(W;\mu^\Phi)$, with closure denoted by $(L^\Phi, D(L^\Phi))$. To apply the Lumer-Phillips theorem, it is left to show that $(\lambda - L^\Phi)(D(L^\Phi))$ is dense in $L^2(W;\mu^\Phi)$ for some $\lambda \in (0,\infty)$. Since $\mathcal{F}C_b^\infty(B_W)$ is dense in $L^2(W;\mu^\Phi)$, it is enough to show that there exits $\lambda \in (0,\infty)$ such that $(\lambda - L^\Phi)(D(L^\Phi))$ contains $\mathcal{F}C_b^\infty(B_W)$. So take $\lambda \in (0,\infty)$ from Item App(Φ3) of Assumption App(Φ). Further, let $g \in \mathcal{F}C_b^\infty(B_W)$ and $m, n \in \mathbb{N}$, where we assume without loss of generality that $m^K(n) = n$ and $m^K(m) = m$. In view of Item App(Φ3) from Assumption App(Φ), there exists $f_{n,m} \in \mathcal{F}C_b^3(B_W, n)$ with

$$g = \lambda f_{n,m} - L^{\Phi_n^m} f_{n,m} = \lambda f_{n,m} - L f_{n,m} + (K_{12} D\Phi_n^m, D_2 f_{n,m})_V$$
$$= \lambda f_{n,m} - L^\Phi f_{n,m} + (K_{12}(D\Phi_n^m - P_n D\Phi), D_2 f_{n,m})_V.$$

If $(K_{12}(D\Phi_n^m - P_n D\Phi), D_2 f_{n,m})_V$ gets arbitrary small in $L^2(W;\mu^\Phi)$ for big n, m, the dense range condition is shown.

It holds, by an application of the generalized Hölder inequality ($\frac{2}{p^*}+\frac{1}{q^*}+\frac{1}{2}=1$),

$$\begin{aligned}
&\int_W (K_{12}(D\Phi_n^m - P_nD\Phi), D_2 f_{n,m})_V^2 \,\mathrm{d}\mu^\Phi \\
&\leq \mu_1(e^{-\Phi_n^m}) \int_W \|Q_2^{-\gamma}K_{12}(D\Phi_n^m - P_nD\Phi)\|_V^2 \|Q_2^\gamma D_2 f_{n,m}\|_V^2 e^{\Phi_n^m-\Phi}\,\mathrm{d}\mu^{\Phi_n^m} \\
&\leq \mu_1(e^{-\Phi_n^m}) \left(\int_U \|Q_2^{-\gamma}K_{12}(D\Phi_n^m - P_nD\Phi)\|_V^{p^*}\,\mathrm{d}\mu_1^{\Phi_n^m}\right)^{\frac{2}{p^*}} \\
&\quad\times \left(\int_W \|Q_2^\gamma D_2 f_{n,m}\|_V^4\,\mathrm{d}\mu^{\Phi_n^m}\right)^{\frac{1}{2}} \left(\int_W e^{q^*(\Phi_n^m-\Phi)}\,\mathrm{d}\mu^{\Phi_n^m}\right)^{\frac{1}{q^*}}.
\end{aligned}$$

By Item App(Φ5) from Assumption App(Φ), we know that

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_U \|Q_2^{-\gamma}K_{12}(D\Phi_n^m - P_nD\Phi)\|_V^{p^*}\,\mathrm{d}\mu_1^{\Phi_n^m} = 0.$$

To conclude the dense range condition, it is enough to bound

$$\mu_1(e^{-\Phi_n^m}) \left(\int_W \|Q_2^\gamma D_2 f_{n,m}\|_V^4\,\mathrm{d}\mu^{\Phi_n^m}\right)^{\frac{1}{2}} \left(\int_W e^{q^*(\Phi_n^m-\Phi)}\,\mathrm{d}\mu^{\Phi_n^m}\right)^{\frac{1}{q^*}}$$

independent of $m, n \in \mathbb{N}$. To verify this, we argue as follows. Lemma 5.26 implies

$$\int_W \|Q_2^\gamma D_2 f_{n,m}\|_V^4\,\mathrm{d}\mu^{\Phi_n^m} \leq c(\lambda, g, \kappa)$$

for all n, m. Moreover, by Inequality (5.16), we get $\mu_1(e^{-\Phi_n^m}) \leq e^{-c_1}$ independent of m, n. Finally, using Inequality (5.16) from Assumption App(Φ), we get

$$\begin{aligned}
\mu_1(e^{-\Phi_n^m}) \left(\int_W e^{q^*(\Phi_n^m-\Phi)}\,\mathrm{d}\mu^{\Phi_n^m}\right)^{\frac{1}{q^*}} &= \mu_1(e^{-\Phi_n^m})^{1-\frac{1}{q^*}} \left(\int_U e^{(q^*-1)\Phi_n^m - q^*\Phi}\,\mathrm{d}\mu_1\right)^{\frac{1}{q^*}} \\
&\leq (e^{-c_1})^{1-\frac{1}{q^*}} \left(\int_U e^{c_2+c_3\|u\|_U^2}\,\mathrm{d}\mu_1\right)^{\frac{1}{q^*}}.
\end{aligned}$$

The estimate in the last inequality above is valid as $q^* > 1$ and consequently $1-\frac{1}{q^*} > 0$. By means of Proposition 3.4, we know that the right-hand side of the inequality above is bounded. This concludes the proof.
The rest of the statement follows as in Theorem 5.15. □

Especially Item App(Φ3) from Assumption App(Φ), which is necessary to apply Theorem 5.27 is difficult to verify, as the existence of a solution in $\mathcal{F}C_b^3(B_W)$ for equation (5.7) is not trivial. A promising approach to find such solutions is to adapt the strategy from [DL05, Prop. 2.2], where the authors assume $U = V = \mathbb{R}^d$ for some $d \in \mathbb{N}$, $Q_2 = K_{12} = K_{22} = \mathrm{Id}$, Q_1 is a symmetric positive matrix and the potential can be approximated in L^4 by a sequence of C^4 functions whose gradients have bounded derivatives up to order three, compare [DL05, Hypothesis 2.1]. Upon closer examination of the proof of [DL05, Prop. 2.2], a gap in the chain of reasoning becomes apparent. However, we present the strategy from [DL05, Prop. 2.2] in the remark below, by also drawing attention to the gap. In Section 8.3, where we examine concrete examples, Item App(Φ3) from Assumption App(Φ) is considered as a conjecture whose validity is assumed.

Remark 5.28. *Suppose $v \mapsto K_{22}(v)e_i \in C_b^4(V;V)$ for all $i \in \mathbb{N}$ and assume that the sequence $(\Phi_n^m)_{m,n\in\mathbb{N}}$ from Assumption App(Φ) is in $C^4(U;\mathbb{R})$. Moreover, suppose $D\Phi_n^m$ and $V \ni v \mapsto K_{22}(P_n(v))Q_2^{-1}P_n(v) \in V$ have bounded derivatives up to order three for all $m, n \in \mathbb{N}$. Lastly, let Assumption **K0** be valid.*
Given $\lambda \in (0,\infty)$ and $g \in \mathcal{F}C_b^\infty(B_W)$. By a trivial extension procedure, there is some $n \in \{m_k, m_{k+1}...\}$ and $\psi \in C_b^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $g = \psi(p_n^U, p_n^V) \in \mathcal{F}C_b^\infty(B_W)$. Set $\overline{\Phi_n^m} := \Phi_n^m \circ \overline{p}_n^U$ and define for $\varphi \in C_b^2(\mathbb{R}^n \times \mathbb{R}^n)$ and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$

$$L^{\overline{\Phi_n^m}}\varphi(x,y) := L_n\varphi(x,y) - \langle D\overline{\Phi_n^m}(x), K_{21,n}D_2\varphi(x,y)\rangle,$$

where L_n and $K_{21,n}$ are given as in Definition 5.11.
Recall the matrix valued map $K_{22,n}$ considered in Definition 5.11 and define the maps $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n)$ and $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ via

$$a\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{2K_{22,n}(y)} \end{pmatrix} \quad \text{and}$$

$$b\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} K_{21,n}Q_{2,n}^{-1}y \\ -K_{12,n}Q_{1,n}^{-1}x + \sum_{j=1}^n \partial_j K_{22,n}(y)p_n^V e_j - K_{22,n}(y)Q_{2,n}^{-1}y - K_{12,n}D\overline{\Phi_n^m}(x) \end{pmatrix}.$$

Using the Itô formula, we see that the operator $L^{\overline{\Phi_m^n}}$ corresponds to the finite dimensional stochastic differential equation given by

$$\mathrm{d}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} = b\begin{pmatrix} X_t \\ Y_t \end{pmatrix} + a\begin{pmatrix} X_t \\ Y_t \end{pmatrix}\mathrm{d}W_t, \quad \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{5.23}$$

*with $(W_t)_{t\geq 0}$ being a finite dimensional standard Wiener process on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $\mathbb{R}^n \times \mathbb{R}^n$. By means of Assumption **K0** and Lemma 5.17 we obtain $a \in C_b^4(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n))$. Taking the assumptions on the potential and the coefficients into account, it follows that both a and b are Lipschitz continuous. Therefore, the finite dimensional stochastic differential equation (5.23) has a unique global solution in terms of a time-homogeneous Markov process $(X_t(x,y), Y_t(x,y))_{t\geq 0}$ with $(X_0(x,y), Y_0(x,y)) = (x,y)$, compare [GS72, Chapter 3 Paragraph 15].*
As $v \mapsto K_{22}(v)e_i \in C_b^4(V;V)$ and $D\Phi_m^n$, as well as $V \ni v \mapsto K_{22}(P_n(v))Q_2^{-1}P_n(v) \in V$ have bounded derivatives up to order three, we know that $b \in C^3(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^n)$ and b has bounded derivatives up to order three. By a dependence upon initial data result, compare e.g. [GS72, Theorem 1, p. 61], we conclude that for all $t \in (0,\infty)$, the map

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto (X_t(x,y), Y_t(x,y)) \in \mathbb{R}^n \times \mathbb{R}^n$$

is in $C^3(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^n)$ with bounded derivatives up to the third order. Consequently, the associated transition semigroup $(p_t)_{t\geq 0}$ leaves $C_b^3(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R})$ invariant. In view of

[GS72, Chapter 2 Paragraph 8], we even get $\sup_{t\in(0,1]}\|p_t\psi\|_{C^3} < \infty$ *and therefore by the semigroup property of* $(p_t)_{t\geq 0}$ *that* $\|p_t\psi\|_{C^3} \leq M_{m,n}e^{\omega_{m,n}t}$ *for all* $t \in (0,\infty)$*, where* $M_{m,n}$ *and* $\omega_{m,n}$ *are positive constants. Let* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ *and define*

$$\varphi(x,y) := \int_0^\infty e^{-\lambda t}\mathbb{E}[\psi(X_t(x,y), Y_t(x,y))]\,\mathrm{d}t = \int_0^\infty e^{-\lambda t} p_t\psi(x,y)\,\mathrm{d}t, \tag{5.24}$$

i.e. φ *is the transition resolvent corresponding to* $(p_t)_{t\geq 0}$*. For* $\lambda > \omega_{m,n}$ *we can verify that* $\varphi \in C_b^3(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R})$ *and* $\lambda\varphi - L^{\overline{\Phi_n^m}}\varphi = \psi$*. In this case, the function* $f \in \mathcal{F}C_b^3(B_W)$ *defined via* $f(u,v) := \varphi(P_n^U u, P_n V v)$*,* $(u,v) \in W$ *fulfills*

$$\lambda f(u,v) - L^{\Phi_n^m} f(u,v) = \lambda\varphi(p_n^U u, p_n^V v) - L^{\overline{\Phi_n^m}}\varphi(p_n^U u, p_n^V v) = \psi(p_n^U u, p_n^V v) = g(u,v).$$

Using the representation from Equation (5.24), also the inequality $\|f\|_\infty \leq \frac{1}{\lambda}\|g\|_\infty$ *is directly derived. However, we do not know if the smoothing properties holds true for a fixed* $\lambda \in (0,\infty)$ *independent of* m,n*. But this is crucial since the lower bound* $\omega_{m,n}$ *for which the transition resolvent associated to* $(p_t)_{t>0}$ *leaves* $C_b^3(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R})$ *invariant might explode for large* m,n*. This problem also occurs if we only consider a finite dimensional setting (in this case* $\omega_{m,n}$ *only depends on* m*) and additive noise as in [DL05, Prop. 2.2] or in the non-degenerate but infinite dimensional setting in [DT00] and [DZ02, Chapter 12.3].*
By a classical Lumer-Phillips argument we know that $\lambda\varphi - L^{\overline{\Phi_n^m}}\varphi = \psi$ *has a unique solution* $\varphi \in D(L^{\overline{\Phi_n^m}})$*. Here,* $D(L^{\overline{\Phi_n^m}})$ *denotes the domain of the closure of the dissipative operator* $(L^{\overline{\Phi_n^m}}, C_b^2(\mathbb{R}^n \times \mathbb{R}^n,\mathbb{R}))$ *on* $L^2(\mathbb{R}^n \times \mathbb{R}^n;\mu^{\overline{\Phi_n^m}})$*, where* $\mu^{\overline{\Phi_n^m}} := \frac{1}{\mu_1^n(e^{-\overline{\Phi_n^m}})}\mu_1^n \otimes \mu_2^n$*. But this is not enough as general elements from* $D(L^{\overline{\Phi_n^m}})$ *are not regular enough to derive the* L^4*-regularity estimates from Lemma 5.26. For further regularity results for elements in* $D(L^{\overline{\Phi_n^m}})$ *and Schauder-type estimates for the transition semigroup, which are nevertheless not sufficient for our application, we refer to [Lun97; Pri06; PLA07; Sai07; Cer01].*
The methods from [DR02] and [DZ02, Chapter 12.3] are also not applicable in our situation as the degeneracy of our problem results in a non-dissipative non-linearity.

6

Hypocoercivity for infinite dimensional Langevin dynamics

The presentation in this chapter is based on the already published articles [EG23] and [BEG23].
We consider the setting as described in Section 5.1, where we require that $\Phi : U \to (-\infty, \infty]$ is bounded from below and there is $\theta \in [0, \infty)$ such that $\Phi \in W^{1,2}_{Q_1^\theta}(U; \mu_1)$. For simplicity, we again assume without loss of generality that Φ is bounded from below by zero and normalized.
We begin with a situation where the infinite dimensional Langevin operator L^Φ is essentially m-dissipative and has a nice core with corresponding decomposition into a symmetric part S and antisymmetric part A^Φ, respectively. Therefore, we either assume Assumptions **K0**, **K1** and $\mathrm{Bd}_\theta(\Phi)$ to apply Theorem 5.23 or Assumption $\mathrm{App}(\Phi)$ to apply Theorem 5.27.

We additionally assume the following assumption throughout this chapter.

Assumption (K2). The operator $K_{21}K_{12} = K_{12}^* K_{12}$ is positive on U.

Starting from here, our goal is to show that the strongly continuous sub-Markovian semigroup $(T_t)_{t\geq 0}$, generated by $(L^\Phi, D(L^\Phi))$ is hypocoercive, where the exponential speed of convergence to the equilibrium is determined by explicitly computable constants. To achieve this, we use the abstract hypocoercivity framework described in Chapter 4. The strength of our results is emphasized in Chapter 8, where we consider examples of degenerate semi-linear infinite dimensional stochastic differential equations beyond framework discussed in [Wan17].

6.1 Essential self-adjointness and second order regularity for Ornstein-Uhlenbeck operators with possibly unbounded diffusion coefficients

In this section we deal with a new variant of (perturbed) infinite dimensional Ornstein-Uhlenbeck operators, which we already considered in Section 3.2.3. The difference between Section 3.2.3 and this section lies in the fact that we allow possibly unbounded diffusion coefficients. Such Ornstein-Uhlenbeck operators appear naturally, as we apply the abstract Hilbert space hypocoercivity method from Chapter 4. Indeed, the operator $G = P(A^\Phi)^2 P$

is of this type, compare Proposition 6.11. Here, P denotes orthogonal projection defined in Definition 6.10.
The aim of the following explanations is, firstly, to give conditions under which operators of type G are essentially self-adjoint and, secondly, to derive corresponding first and second order regularity estimates.
Similar first and second order regularity results were also derived e.g. in [DA14; LD15] and [DT00], but only if the diffusion operator is the identity or the negative power of a strictly positive self-adjoint operator.

Definition 6.1. The operators $(C, D(C))$ and $(Q_1^{-1}C, D(Q_1^{-1}C))$ on U are defined by

$$C := K_{21}Q_2^{-1}K_{12} \quad \text{with} \qquad D(C) := \{u \in U \mid K_{12}u \in D(Q_2^{-1})\} \quad \text{and}$$
$$Q_1^{-1}C := Q_1^{-1}K_{21}Q_2^{-1}K_{12} \quad \text{with} \qquad D(Q_1^{-1}C) := \{u \in D(C) \mid Cu \in D(Q_1^{-1})\},$$

respectively. Moreover, we define the infinite dimensional Ornstein-Uhlenbeck operator $(N^\Phi, \mathcal{F}C_b^\infty(B_U))$ (perturbed by the gradient of Φ) by

$$N^\Phi : \mathcal{F}C_b^\infty(B_U) \to L^2(U; \mu_1^\Phi), \;\; f \mapsto N^\Phi f := \mathrm{tr}[CD^2 f] - (u, Q_1^{-1}CDf)_U - (D\Phi, CDf)_U.$$

We use the abbreviation $N := N^0$.

In the introduction of this section, we considered $G = PA^\Phi P$ as the object of interest. We derive, in Proposition 6.11, that N^Φ and G are related via $Gf = N^\Phi Pf$ for every $f \in \mathcal{F}C_b^\infty(B_W)$. Hence, the results we establish for N^Φ can be translated into ones for G, compare again Proposition 6.11. For the following arguments, it is more convenient to study N^Φ without the orthogonal projection P.
We continue with the collection of a few properties of the newly defined operators.

Remark 6.2. *(i) $(C, D(C))$ is symmetric and positive on U. Indeed, let $u_1, u_2 \in D(C)$. Then,*

$$(Cu_1, u_2)_U = (Q_2^{-1}K_{12}u_1, K_{12}u_2)_V = (K_{12}u_1, Q_2^{-1}K_{12}u_2)_V = (u_1, Cu_2)_U,$$

*due to symmetry of Q_2^{-1} and the definition of K_{21}. Since Q_2^{-1} is positive, positivity of C follows immediately by Assumption **K2**.*

(ii) For all $n \in \mathbb{N}$, there is m_k with $n \leq m_k$ such that $(C, D(C))$ maps U_n into U_{m_k}. Recall that $(m_k)_{k\in\mathbb{N}}$ is the sequence from Definition 5.5. This follows, as there is some m_k with $n \leq m_k$ such that K_{12} maps U_n to V_{m_k}, K_{21} maps V_n to U_{m_k}, and Q_2^{-1} leaves V_{m_k} invariant.

(iii) Due to Item (i) and (ii), we have $\mathrm{span}\{d_1, d_2, \dots\} \in D(C)$ and we can define a positive symmetric operator $(C^{\frac{1}{2}}, \mathrm{span}\{d_1, d_2, \dots\})$ such that $C^{\frac{1}{2}}C^{\frac{1}{2}}v = Cv$ for all $v \in \mathrm{span}\{d_1, d_2, \dots\}$. For the construction of $(C^{\frac{1}{2}}, \mathrm{span}\{d_1, d_2, \dots\})$ we restrict C to a positive bounded linear operator on $\mathrm{span}\{d_1, d_2, \dots, d_{m_k}\}$ for each $k \in \mathbb{N}$ and use that every positive bounded linear operator has a unique positive square root.

Proposition 6.3. *The operator $(N^\Phi, \mathcal{F}C_b^\infty(B_U))$ is symmetric on $L^2(U; \mu_1^\Phi)$ and therefore closable with closure denoted by $(N^\Phi, D(N^\Phi))$. Moreover, the following statements hold true.*

(i) $D(N^{\Phi}) \subseteq W^{1,2}_{C^{\frac{1}{2}}}(U,\mu_1)$ *and for every* $\lambda \in (0,\infty)$, $f \in D(N^{\Phi})$ *and* $g := \lambda f - N^{\Phi} f$ *we have*

$$\int_U \|C^{\frac{1}{2}} Df\|_U^2 \,\mathrm{d}\mu_1^{\Phi} \leq \frac{1}{4\lambda} \int_U g^2 \,\mathrm{d}\mu_1^{\Phi}. \tag{6.1}$$

(ii) *For all* $f, g \in D(N^{\Phi})$, *it holds*

$$(N^{\Phi} f, g)_{L^2(\mu_1^{\Phi})} = -\int_U (C^{\frac{1}{2}} Df, C^{\frac{1}{2}} Dg)_U \,\mathrm{d}\mu_1^{\Phi}. \tag{6.2}$$

(iii) *Suppose* $\Phi = 0$, *then* $(N, D(N))$ *is essentially self-adjoint. In addition, for* $\lambda \in (0,\infty)$, $f \in \mathcal{F}C_b^{\infty}(B_U)$ *and* $g := \lambda f - Nf$, *the following second order regularity estimate is valid*

$$\int_U \operatorname{tr}[(CD^2 f)^2] + \|Q_1^{-\frac{1}{2}} CDf\|_U^2 \,\mathrm{d}\mu_1 = \int_U (Nf)^2 \,\mathrm{d}\mu_1 \leq 4 \int_U g^2 \,\mathrm{d}\mu_1. \tag{6.3}$$

In particular, $D(N) \subseteq W^{1,2}_{Q_1^{-\frac{1}{2}} C}(U;\mu_1) \subseteq W^{1,2}_C(U;\mu_1)$. *Further,*

$$\int_U \|Q_1^{-\frac{1}{2}} CDf\|_U^2 \,\mathrm{d}\mu_1 \leq 4 \int_U g^2 \,\mathrm{d}\mu_1, \tag{6.4}$$

for each $f \in D(N)$ *and* $g := \lambda f - Nf$.

Proof. The symmetry of $(N, \mathcal{F}C_b^{\infty}(B_U))$ on $L^2(U;\mu_1^{\Phi})$ follows by the integration by parts formula from Lemma 3.47 and the properties of $(C, D(C))$, discussed in Remark 6.2. By means of the same integration by parts formula, we get for all $f, g \in \mathcal{F}C_b^{\infty}(B_U)$

$$(N^{\Phi} f, g)_{L^2(\mu_1^{\Phi})} = -\int_U (CDf, Dg)_U \,\mathrm{d}\mu_1^{\Phi} = -\int_U (C^{\frac{1}{2}} Df, C^{\frac{1}{2}} Dg)_U \,\mathrm{d}\mu_1^{\Phi}.$$

Hence, Item (i) follows by the same reasoning as in Lemma 5.20.
Item (ii) is obtained by approximating $f, g \in D(N^{\Phi})$ by sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{F}C_b^{\infty}(B_U)$ with respect to the N^{Φ} graph norm.
For the rest of the proof, let $\Phi = 0$. We already know that $(N, \mathcal{F}C_b^{\infty}(B_U))$ is symmetric on $L^2(U;\mu_1)$. Hence, essential m-dissipativity implies essential self-adjointness. The first statement of Item (iii) can be shown analogously to Theorem 5.15, since the matrices in $\mathbb{R}^{n\times n}$, induced by $Q_1^{-1}C$, are constant with positive eigenvalues, which allows the usage of Proposition 3.56 (applied in a finite dimensional setting). The second order regularity estimate for $f \in \mathcal{F}C_b^{\infty}(B_U)$ and $g = \lambda f - Nf$ follows by the same reasoning as in Lemma 6.7, where we also consider the case $\Phi \neq 0$. We omit the proof here to avoid a repetition of arguments.
Since Equation (6.3) implies Equation (6.4) for all $f \in \mathcal{F}C_b^{\infty}(B_U)$ and $g = \lambda f - Nf$, we can finish the proof by approximating $f \in D(N)$ with a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^{\infty}(B_U)$ with respect to the N graph norm. Note that $\operatorname{tr}[(CD^2 f)^2] \geq 0$ for all $f \in \mathcal{F}C_b^{\infty}(B_U)$.

□

Assumption (SA(Φ))**.** Assume either

(SA(Φ1)) $\Phi \in W^{1,2}_{Q_1^{\frac{1}{2}}}(U;\mu_1)$ with $\|Q_1^{\frac{1}{2}} D\Phi\|_{L^\infty(\mu_1)} < \frac{1}{2}$.

(SA(Φ2)) $C \in \mathcal{L}(U)$ and $\Phi \in W^{1,2}_{C^{\frac{1}{2}}}(U;\mu_1)$ with $\|C^{\frac{1}{2}} D\Phi\|_{L^\infty(\mu_1)} < \infty$.

(SA(Φ3)) Proposition 3.58 is applicable for the coefficients $C \in \mathcal{L}(U)$ and $(B, D(B))$ defined as the closure of $(-Q_1^{-1}C, \operatorname{span}\{d_1, d_2, \cdots\})$.

Below, we show that Assumption SA(Φ) is sufficient to establish essential self-adjointness of $(N^\Phi, \mathcal{F}C_b^\infty(B_U))$. This is of particular interest for the verification of the macroscopic hypocoercivity assumption **H3**, compare Remark 4.4. The assumption is designed to be applicable for the examples studied in Chapter 8, but any other assumption, implying self-adjointness of $(N^\Phi, \mathcal{F}C_b^\infty(B_U))$ is reasonable. In particular, we are able to consider situations where boundedness of $C^{\frac{1}{2}} D\Phi$ or $Q_1^{\frac{1}{2}} D\Phi$ is not required. This is especially useful when using Theorem 5.27 to show essentially m-dissipativity of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ on $L^2(U;\mu^\Phi)$, where also potentials with unbounded gradient are applicable, compare Section 8.3.

Remark 6.4. *Suppose $f \in D(N)$ and one of the first two items from Assumption SA(Φ) is valid. By Proposition 6.3, we know that $f \in W^{1,2}_{Q_1^{-\frac{1}{2}}C}(U;\mu_1) \cap W^{1,2}_{C^{\frac{1}{2}}}(U;\mu_1)$. Using similar arguments as in Remark 5.22, we interpret $D(N) \ni f \mapsto (D\Phi, CDf)_U \in L^2(U;\mu_1^\Phi)$ either as $D(N) \ni f \mapsto (Q_1^{\frac{1}{2}} D\Phi, Q_1^{-\frac{1}{2}} CDf)_U \in L^2(U;\mu_1^\Phi)$ if SA(Φ1) holds true or as $D(N) \ni f \mapsto (C^{\frac{1}{2}} D\Phi, C^{\frac{1}{2}} Df)_U \in L^2(U;\mu_1^\Phi)$ if SA(Φ2) is valid.*

Theorem 6.5. *Let Assumption SA(Φ) be satisfied. Then $(N^\Phi, \mathcal{F}C_b^\infty(B_U))$ is essentially self-adjoint on $L^2(U;\mu_1^\Phi)$. The resolvent in $\lambda \in (0,\infty)$ of the corresponding closure $(N^\Phi, D(N^\Phi))$ is denoted by $R_\lambda^{N^\Phi}$.*
If either SA(Φ1) or SA(Φ2) from Assumption SA(Φ) is valid, then additionally $D(N) \subseteq D(N^\Phi)$ with

$$N^\Phi f = Nf - (D\Phi, CDf)_U \in L^2(U;\mu_1^\Phi)$$

for all $f \in D(N)$.

Proof. Remember that $(N^\Phi, \mathcal{F}C_b^\infty(B_U))$ is symmetric on $L^2(U;\mu_1^\Phi)$. Consequently, to conclude the first part, it is enough to show the essential m-dissipativity of $(N^\Phi, \mathcal{F}C_b^\infty(B_U))$ on $L^2(U;\mu_1^\Phi)$. We start by assuming that Item SA(Φ1) of Assumption SA(Φ) is valid. For $f \in L^2(U;\mu_1)$ set

$$Tf = -(Q_1^{\frac{1}{2}} D\Phi, Q_1^{-\frac{1}{2}} CDR_1^N f)_U.$$

Since $D(N) \subseteq W^{1,2}_{Q_1^{-\frac{1}{2}}C}(U;\mu_1)$, the definition above is reasonable. Using the Cauchy-Schwarz inequality, Inequality (6.4) and the assumption on Φ, we observe

$$\|Tf\|^2_{L^2(\mu_1)} \le \|Q_1^{\frac{1}{2}} D\Phi\|^2_{L^\infty(\mu_1)} \int_U \|Q_1^{-\frac{1}{2}} CDR_1^N f\|_U^2 \,\mathrm{d}\mu_1 < \|f\|^2_{L^2(\mu_1)}.$$

Consequently, the linear operator $T : L^2(U;\mu_1) \to L^2(U;\mu_1)$ is well-defined with operator norm less than one. By the Neumann-Series theorem, we obtain that $(\mathrm{Id} - T)^{-1}$ exists in $\mathcal{L}(L^2(U;\mu_1))$. For a given $g \in L^2(U;\mu_1)$ we particularly find $f \in L^2(U;\mu_1)$ with $f - Tf = g$ in $L^2(U;\mu_1)$. Since $(N, D(N))$ is m-dissipative, there is $h \in D(N)$ with $(\mathrm{Id} - N)h = f$. This yields

$$(\mathrm{Id} - N)h + (Q_1^{\frac{1}{2}} D\Phi, Q_1^{-\frac{1}{2}} CDh)_U = f + (Q_1^{\frac{1}{2}} D\Phi, Q_1^{\frac{1}{2}} CDR_1^N f)_U = f - Tf = g.$$

By means of the Lumer-Phillips, we conclude the first case, if $D(N) \subseteq D(N^\Phi)$ with $N^\Phi f = Nf - (D\Phi, CDf)_U$ for all $f \in D(N)$. Indeed, this implies

$$\mathcal{F}C_b^\infty(B_U) \subseteq L^2(U;\mu_1) \subseteq (\mathrm{Id} - N^\Phi)(D(N)) \subseteq (\mathrm{Id} - N^\Phi)(D(N^\Phi))$$

and therefore the dense range condition. Note that $\mathcal{F}C_b^\infty(B_U)$ is dense in $L^2(U;\mu_1^\Phi)$. So let $f \in D(N)$ be given. There is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^\infty(B_U)$ such that $f_n \to f$ and $Nf_n \to Nf$ in $L^2(U;\mu_1)$. As $|e^{-\Phi}| \leq 1$, it is easy to see that $f_n \to f$ in $L^2(U;\mu_1^\Phi)$. In view of $\|Q_1^{\frac{1}{2}} D\Phi\|_{L^\infty(\mu_1)} \leq \frac{1}{2}$ and Inequality (6.3), we estimate

$$\begin{aligned}
&\|Nf - (Q_1^{\frac{1}{2}} D\Phi, Q_1^{-\frac{1}{2}} CDf)_U - N^\Phi f_n\|_{L^2(\mu_1^\Phi)} \\
&\leq \|N(f - f_n)\|_{L^2(\mu_1)} + \frac{1}{2}\|Q_1^{-\frac{1}{2}} CD(f - f_n)\|_{L^2(\mu_1)} \\
&\leq \|N(f - f_n)\|_{L^2(\mu_1)} + \|(f - f_n) + N(f - f_n)\|_{L^2(\mu_1)}.
\end{aligned}$$

Therefore, $N^\Phi f_n \to Nf - (Q_1^{\frac{1}{2}} D\Phi, Q_1^{-\frac{1}{2}} CDf)_U$ in $L^2(U;\mu_1^\Phi)$. Since $(N^\Phi, D(N^\Phi))$ is a closed operator, we obtain $D(N) \subseteq D(N^\Phi)$ with $N^\Phi f = Nf - (Q_1^{\frac{1}{2}} D\Phi, Q_1^{-\frac{1}{2}} CDf)_U$ for all $f \in D(N)$ as desired. Remember the interpretation from Remark 6.4.
In the case that Item SA(Φ2) of Assumption SA(Φ) holds true, we proceed as in Theorem 5.23 and use Inequality (6.1).
By means of Proposition 3.58, essential m-dissipativity also follows in presence of Item SA(Φ3). □

The next assumption enables us to generalize the regularity estimates from Proposition 6.3 to the case where N is perturbed by the gradient of the potential Φ.

Assumption (Reg(Φ))**.** $\Phi = \Phi_1 + \Phi_2 : U \to (-\infty, \infty]$.

Reg(Φ_1) There exists a sequence $(\Phi_{1,n,m})_{m,n\in\mathbb{N}}$ of convex functions from U to $\mathbb{R}$ such that for μ_1-almost all $u \in U$ and for all $m, n \in \mathbb{N}$

(i) $-\infty < \inf_{\tilde{u}\in U} \Phi_1(\tilde{u}) \leq \Phi_{1,n,m}(u)$,

(ii) $\lim_{n\to\infty} \lim_{m\to\infty} \Phi_{1,n,m}(u) = \Phi_1(u)$,

(iii) $\Phi_{1,n,m}$ is differentiable with Lipschitz continuous derivative and

$$\lim_{n\to\infty} \lim_{m\to\infty} \|(Q_1^\theta D\Phi_{1,n,m} - Q_1^\theta D\Phi_1, d_k)_U\|_{L^2(\mu_1)} = 0 \text{ for all } k \in \mathbb{N}.$$

Reg(Φ_2) Φ_2 is bounded and two times continuously Fréchet differentiable with bounded first order derivative and second order derivative in $L^1(U;\mu_1^\Phi)$. Moreover, there is a constant $c_{\Phi_2}\in[0,\infty)$ such that for all $n\in\mathbb{N}$

$$(D^2\Phi_2(\tilde u)Cu,Cu)_U \geq -c_{\Phi_2}(Cu,u)_U \quad \text{for all} \quad \tilde u\in U \quad \text{and} \quad u\in U_n.$$

Remark 6.6. *(i) Suppose Item (i) and (ii) of Item Reg(Φ_1) hold true. Then, $e^{-\Phi_{1,n,m}}$ and $e^{-\lim_{m\to\infty}\Phi_{1,n,m}}$ are bounded by $e^{-\inf_{\tilde u\in U}\Phi_1(\tilde u)}$. Since we also have pointwise convergence, we obtain by the theorem of dominated convergence*

$$\lim_{n\to\infty}\lim_{m\to\infty}\mu_1^{\Phi_{1,n,m}}(f)=\mu_1^{\Phi_1}(f) \quad \textit{for all} \quad f\in L^1(U;\mu_1^{\Phi_1}).$$

(ii) Note that Item Reg(Φ_2) above is satisfied, if $D^2\Phi_2$ is bounded and $C\in\mathcal{L}(U)$.

The proof of the second order regularity estimate below is similar to [EG22, Theorem 2], where only convex potentials were considered.

Lemma 6.7. *Assume that Assumption Reg(Φ) holds true. Then for every function $f\in\mathcal{F}C_b^\infty(B_U)$ and $g:=f-N^\Phi f$ we have the following second order regularity estimate*

$$\int_U \operatorname{tr}[(CD^2f)^2]+\|Q_1^{-\frac{1}{2}}CDf\|_U^2\,\mathrm{d}\mu_1^\Phi \leq \left(4+\frac{c_{\Phi_2}}{4}\right)\int_U g^2\,\mathrm{d}\mu_1^\Phi.$$

Proof. For each $m,n\in\mathbb{N}$ define $\Phi_{n,m}:=\Phi_{1,n,m}+\Phi_2$ where $(\Phi_{1,n,m})_{n,m\in\mathbb{N}}$ is the approximation of Φ_1, provided by Assumption Reg(Φ). Further, let $f\in\mathcal{F}C_b^\infty(B_U)$ and set

$$g_{n,m}:=f-N^{\Phi_{n,m}}f.$$

Taking derivatives of the equation above, with respect to the d_k, gives

$$\partial_{d_k}f-N^{\Phi_{n,m}}\partial_{d_k}f+(d_k,Q_1^{-1}CDf)_U+\sum_{i=1}^\infty(d_i,CDf)_U\partial_{d_k}\partial_{d_i}\Phi_{n,m}=\partial_{d_k}g_{n,m}. \tag{6.5}$$

The infinite sum in Equation (6.5) above is a finite one. Moreover, $\partial_{d_k}\partial_{d_i}\Phi_{n,m}$ exists μ_1-a.e., since the Lipschitz continuous function $\partial_{d_i}\Phi_{n,m}:U\to\mathbb{R}$ is Gateaux differentiable μ_1-a.e. by Lemma 3.39. We multiply Equation (6.5) with $\partial_{d_l}f(d_k,Cd_l)_U$. Summing over all $k,l\in\mathbb{N}$ shows that the first and third term, as well as the right-hand side in Equation (6.5), is equal to $(CDf,Df)_U$, $\|Q_1^{-\frac{1}{2}}CDf\|_U^2$ and $(Dg_{n,m},CDf)_U$, respectively. For the second term we calculate

$$\begin{aligned}\sum_{k,l=1}^\infty(d_k,Cd_l)_U\int_U -N^{\Phi_{n,m}}\partial_{d_k}f\partial_{d_l}f\,\mathrm{d}\mu_1^{\Phi_{n,m}} &= \sum_{k,l=1}^\infty(d_k,Cd_l)_U\int_U(CD\partial_{d_k}f,D\partial_{d_l}f)_U\,\mathrm{d}\mu_1^{\Phi_{n,m}}\\ &= \int_U \operatorname{tr}[(CD^2f)^2]\,\mathrm{d}\mu_1^{\Phi_{n,m}}.\end{aligned}$$

Additionally, we have

$$\sum_{k,l,i=1}^\infty (d_i,CDf)_U\partial_{d_l}f(d_k,Cd_l)_U\partial_k\partial_i\Phi_{n,m}=(D^2\Phi_{n,m}CDf,CDf)_U.$$

Putting the results we just derived into Equation (6.5) and rearranging the terms, establishes the following equation

$$\begin{aligned}&\int_U \|Q_1^{-\frac{1}{2}}CDf\|_U^2 + \mathrm{tr}[(CD^2f)^2] + (D^2\Phi_{n,m}CDf, CDf)_U \,\mathrm{d}\mu_1^{\Phi_{n,m}}\\ &= \int_U (D(-N^{\Phi_{n,m}}f), CDf)_U \,\mathrm{d}\mu_1^{\Phi_{n,m}}.\end{aligned}$$

By means of Lemma 3.39 and Lemma 3.36, we know that $N^{\Phi_{n,m}}f \in W^{1,2}(U;\mu_1)$. Together with the fact that $\partial_{d_j} f e^{-\Phi_{n,m}} \in W^{1,2}(U;\mu_1)$ for all $j \in \mathbb{N}$ and the integration by parts formula from Proposition 3.40, we obtain

$$\begin{aligned}&\int_U \left(D(-N^{\Phi_{n,m}}f), CDf\right)_U \;\mathrm{d}\mu_1^{\Phi_{n,m}}\\ &= \frac{1}{\mu_1(e^{-\Phi_{n,m}})}\sum_{i,j=1}^{\infty}(Cd_i,d_j)_U \int_U -\partial_{d_i}N^{\Phi_{n,m}}f\partial_{d_j}fe^{-\Phi_{n,m}} \;\mathrm{d}\mu_1\\ &= \sum_{i,j=1}^{\infty}(Cd_i,d_j)_U \int_U N^{\Phi_{n,m}}f\left(\partial_{d_i}\left(\partial_{d_j}fe^{-\Phi_{n,m}}\right) - (u,Q_1^{-1}d_i)_U\partial_{d_j}f\right) \;\mathrm{d}\mu_1^{\Phi_{n,m}}\\ &= \int_U (N^{\Phi_{n,m}}f)^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}}.\end{aligned}$$

Since the resolvent $R_1^{N^{\Phi_{n,m}}}$ is a contraction, we estimate

$$\int_U f^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}} = \int_U (R_1^{N^{\Phi_{n,m}}}g_{n,m})^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}} \le \int_U g_{n,m}^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}}.$$

This implies

$$\begin{aligned}&\int_U \mathrm{tr}[(CD^2f)^2] + \|Q_1^{-\frac{1}{2}}CDf\|_U^2 + (D^2\Phi_{n,m}CDf, CDf)_U \,\mathrm{d}\mu_1^{\Phi_{n,m}}\\ &= \int_U (N^{\Phi_{n,m}}f)^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}} = \int_U (f-g_{n,m})^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}} \le 4\int_U g_{n,m}^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}}.\end{aligned}$$

Note that $\partial_{d_i}\partial_{d_j}\Phi_{1,n,m}$ and $\partial_{d_i}\partial_{d_j}\Phi_2$ exist in $L^1(\mu_1^{\Phi})$, since $D\Phi_{1,n,m}$ is Lipschitz continuous and $D^2\Phi_2$ is μ_1^{Φ} integrable by Assumption Reg(Φ). Using that $\Phi_{1,n,m}$ is convex, Item Reg(Φ_2) from Assumption Reg(Φ) and Inequality (6.1), we get

$$\int_U \mathrm{tr}[(CD^2f)^2] + \|Q_1^{-\frac{1}{2}}CDf\|_U^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}} \le \left(4 + \frac{c_{\Phi_2}}{4}\right)\int_U g_{n,m}^2 \,\mathrm{d}\mu_1^{\Phi_{n,m}}. \tag{6.6}$$

By Remark 6.6, the left-hand side of the inequality above converges to

$$\int_U \mathrm{tr}[(CD^2f)^2] + \|Q_1^{-\frac{1}{2}}CDf\|_U^2 \,\mathrm{d}\mu_1^{\Phi}.$$

Now we observe

$$\begin{aligned}
\left|\mu_1^{\Phi_{n,m}}\left(g_{n,m}^2\right)-\mu_1^{\Phi}\left(g^2\right)\right| &\le \left|\mu_1^{\Phi_{n,m}}\left((g_{n,m}-g)^2\right)\right|+2\left|\mu_1^{\Phi_{n,m}}\left((g_{n,m}-g)g\right)\right| \\
&\quad+\left|\mu_1^{\Phi_{n,m}}\left(g^2\right)-\mu_1^{\Phi}\left(g^2\right)\right| \\
&\le \left|\mu_1^{\Phi_{n,m}}\left((g_{n,m}-g)^2\right)\right|+2\left|\mu_1^{\Phi_{n,m}}\left((g_{n,m}-g)^2\right)\mu_1^{\Phi_{n,m}}\left(g^2\right)\right|^{\frac{1}{2}} \\
&\quad+\left|\mu_1^{\Phi_{n,m}}\left(g^2\right)-\mu_1^{\Phi}\left(g^2\right)\right| \\
&\le \mu_1(e^{-\Phi_{n,m}})^{-1}e^{-inf_{\tilde{u}\in U}\Phi(\tilde{u})}\|g_{n,m}-g\|_{L^2(\mu_1)}^2 \\
&\quad+2\left(\mu_1(e^{-\Phi_{n,m}})^{-1}e^{-inf_{\tilde{u}\in U}\Phi(\tilde{u})}\mu_1^{\Phi_{n,m}}\left(g^2\right)\right)^{\frac{1}{2}}\|g_{n,m}-g\|_{L^2(\mu_1)} \\
&\quad+\left|(\mu_1^{\Phi_{n,m}}(g^2)-\mu_1^{\Phi}(g^2)\right|.
\end{aligned}$$

By Remark 6.6, we follow that

$$\lim_{n\to\infty}\lim_{m\to\infty}\mu_1(e^{-\Phi_{n,m}})^{-1}=\mu_1(e^{-\Phi})^{-1}\quad\text{and}\quad\lim_{n\to\infty}\lim_{m\to\infty}\left|(\mu_1^{\Phi_{n,m}}(g^2)-\mu_1^{\Phi}(g^2)\right|=0.$$

In view of Item Reg(Φ_1) from Assumption Reg(Φ), we know that

$$\begin{aligned}
&\limsup_{n\to\infty}\limsup_{m\to\infty}\|g_{n,m}-g\|_{L^2(\mu_1)} \\
&=\limsup_{n\to\infty}\limsup_{m\to\infty}\|(Q_1^{\theta}D\Phi_{1,n,m}-Q_1^{\theta}D\Phi_1,Q^{-\theta}CDf)_U\|_{L^2(\mu_1)}=0,
\end{aligned}$$

using that $Q_1^{-\theta}CDf=\sum_{i=1}^{N}(Q_1^{-\theta}CDf,d_i)_U d_i\in U_N$ for some $N\in\mathbb{N}$ and the boundedness of $(Q_1^{-\theta}CDf,d_i)_U$. We conclude that

$$\limsup_{n\to\infty}\limsup_{m\to\infty}\left|\mu_1^{\Phi_{n,m}}\left(g_{n,m}^2\right)-\mu_1^{\Phi}\left(g^2\right)\right|=0.$$

Hence, a successive application of the limes superior, first for m and then for n, in Inequality (6.6) finishes the proof. □

6.2 Application of the abstract Hilbert space hypocoercivity method

Here, we derive sufficient conditions under which the abstract Hilbert space hypocoercivity method, described in Chapter 4, is applicable.
We start by restricting the setting to the Hilbert space

$$H:=\left\{f\in L^2(W;\mu^{\Phi})\mid\mu^{\Phi}(f)=0\right\}$$

and operator domain $\mathcal{D}:=\mathcal{F}C_b^{\infty}(B_W)\cap H$. As the essential m-dissipativity of L^{Φ} holds on $\mathcal{F}C_b^{\infty}(B_W)\subseteq L^2(W;\mu^{\Phi})$, we first need to justify the corresponding result in the restricted setting. This is essential to verify the Assumption **D1** from Chapter 4.

Remark 6.8. *In Corollary 5.10, we established that μ^{Φ} is an invariant measure for the s.c.c.s. $(T_t)_{t\ge0}$ generated by $(L^{\Phi},D(L^{\Phi}))$. The measure μ^{Φ} is also invariant for the operators*

$(A^\Phi, \mathcal{F}C_b^\infty(B_W))$, $(S, \mathcal{F}C_b^\infty(B_W))$ $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$. Hence, $T_t(H)$, $A^\Phi(\mathcal{D}), S^\Phi(\mathcal{D})$ and $L^\Phi(\mathcal{D})$ are contained in H and it is therefore possible to restrict $(T_t)_{t\geq 0}$ to a s.c.c.s. on H, which we denote in the following by $(T_t^0)_{t\geq 0}$. Moreover, we consider $(A^\Phi, \mathcal{D})$, $(S, \mathcal{D})$ and $(L^\Phi, \mathcal{D})$ as operators on H.

Proposition 6.9. *$\mathcal{D}$ is dense in H and the operator $(L^\Phi, \mathcal{D})$ is essentially m-dissipative on H. Therefore, its closure, denoted by $(L_0^\Phi, D(L_0^\Phi))$, generates a s.c.c.s., which is equal to $(T_t^0)_{t\geq 0}$.*

Proof. Let $f \in H$ be given. By Remark 5.4, we know that there exists a sequence $(\tilde{f}_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}C_b^\infty(B_W)$ converging to f in $L^2(W; \mu^\Phi)$. For each $n \in \mathbb{N}$, define $f_n := \tilde{f}_n - \mu^\Phi(\tilde{f}_n) \in \mathcal{D}$. Since $\lim_{n\to\infty} \mu^\Phi(\tilde{f}_n) = \mu^\Phi(f) = 0$, we established that $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}$ converges to f in $L^2(W; \mu^\Phi)$. Consequently, $\mathcal{D}$ is dense in H.
To continue, recall that $(L^\Phi, \mathcal{D})$ is well-defined as an operator on H, see Remark 6.8 above. Dissipativity of $(L^\Phi, \mathcal{D})$ is inherited from $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ and the dense range condition can be verified as follows. For each $f \in H$ there is a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^2(W; \mu^\Phi)$ such that $(\mathrm{Id} - L^\Phi) f_n \to f$ in $L^2(W; \mu^\Phi)$. In particular, $\mu^\Phi(f_n) \to \mu^\Phi(f) = 0$. By setting $g_n := f_n - \mu^\Phi(f_n)$, it follows that $(\mathrm{Id} - L^\Phi) g_n \to f$, since L^Φ acts trivially on constants. Since $g_n \in \mathcal{D}$ for all $n \in \mathbb{N}$, it follows that $(L^\Phi, \mathcal{D})$ is essentially m-dissipative and its closure $(L_0^\Phi, D(L_0^\Phi))$ is the generator of the s.c.c.s. $(T_t^0)_{t\geq 0}$. A direct calculation shows that the generator of $(T_t^0)_{t\geq 0}$ is equal to $(L_0^\Phi, D(L_0^\Phi))$, i.e. also the last claim follows. □

Definition 6.10. Let $H = H_1 \oplus H_2$, where H_1 is provided by the orthogonal projection

$$P : H \to H_1, \quad f \mapsto Pf := \int_V f(\cdot, v)\, \mu_2(\mathrm{d}v).$$

For any $f \in \mathcal{D}$, we can interpret Pf as an element of $\mathcal{F}C_b^\infty(B_U)$, which is then denoted by f_P. Further, let $(S_0, D(S_0))$ and $(A_0^\Phi, D(A_0^\Phi))$ be the closures in H of $(S, \mathcal{D})$ and $(A^\Phi, \mathcal{D})$, respectively.

The following proposition is essential to verify the data assumptions in our infinite dimensional framework.

Proposition 6.11. *Let Assumption SA(Φ) be satisfied. Then*

(i) $H_1 \subseteq D(S_0)$ *with* $S_0 P = 0$.

(ii) $P(\mathcal{D}) \subseteq D(A_0^\Phi)$ *and* $A_0^\Phi P f = -(v, Q_2^{-1} K_{12} D_1 f_P)_V$ *for all* $f \in \mathcal{D}$.

(iii) $P A_0^\Phi P f = 0$ *for all* $f \in \mathcal{D}$.

(iv) $A_0^\Phi P(\mathcal{D}) \subseteq D(A_0^\Phi)$ *and*

$$Gf := P(A_0^\Phi)^2 P f = \mathrm{tr}[C D_1^2 f_P] - (u, Q_1^{-1} C D_1 f_P)_U - (D\Phi(u), C D_1 f_P)_U$$

for all $f \in \mathcal{D}$. *Moreover,* $(G, \mathcal{D})$ *is essentially self-adjoint on* H.

*The data assumptions **D1**-**D3** are in particular satisfied and $\mathcal{D}$ is a core for the operator $(G, D(G))$, defined in Definition 4.1.*

Proof. (i) Let $g \in H_1$ be given, i.e. $g = Pf$ for some $f \in H$. Choose a sequence $(f_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}$ converging to f in H. Obviously, $(Pf_k)_{k\in\mathbb{N}} \subseteq \mathcal{D}$ converges to Pf in H. For all $k \in \mathbb{N}$, it holds $S_0 P f_k = 0$, since Pf_k only depends on the first variable. Using the fact that $(S_0, D(S_0))$ is a closed operator, we conclude that $g = Pf \in D(S_0)$ and $S_0 P = 0$.

(ii) As $P(\mathcal{D}) \subseteq \mathcal{D}$, it immediately follows that $P(\mathcal{D}) \subseteq D(A_0^\Phi)$. The remaining part of Item *(ii)* follows by a direct calculation using that $D_2 P f = 0$ for all $f \in \mathcal{D}$.

(iii) Recalling that the Gaussian measure μ_2 is centered, we calculate for all $f \in \mathcal{D}$ and $(u, v) \in W$ using Item (ii)

$$PA_0^\Phi Pf(u,v) = -\int_V (\tilde{v}, Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mu_2(\mathrm{d}\tilde{v}) = 0.$$

(iv) To obtain $A_0^\Phi P(\mathcal{D}) \subset D(A_0^\Phi)$, consider a function $f \in \mathcal{D}$ and a sequence of cut-off functions $(\varphi_m)_{m\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ provided by Corollary 3.23. Here, $n \in \mathbb{N}$ corresponds to the n such that $f \in \mathcal{F}C_b^\infty(B_W, n) \cap H$. Define $\varphi_m^n := \varphi_m \circ p_n^V$ and the sequence $(g_m)_{m\in\mathbb{N}}$ by

$$g_m : W \to \mathbb{R} \quad (u,v) \mapsto g_m(u,v) := \varphi_m^n(v) A_0^\Phi Pf(u,v) - \mu^\Phi(\varphi_m^n A_0^\Phi Pf).$$

Hence, the lack of boundedness of $A_0^\Phi Pf$ in the second variable is compensated with the sequence of cut-off functions. We obtain $(g_m)_{m\in\mathbb{N}} \subseteq \mathcal{D}$. Using the product rule, we calculate for all $m \in \mathbb{N}$

$$\begin{aligned} A_0^\Phi g_m = &(u, Q_1^{-1} K_{21} D_2(A_0^\Phi Pf))_U \varphi_m^n + (D\Phi, K_{21} D_2(A_0^\Phi Pf))_U \varphi_m^n \\ &- (v, Q_2^{-1} K_{12} D_1(A_0^\Phi Pf))_V \varphi_m^n \\ &+ \big((D\Phi, K_{21} D_2 \varphi_m^n)_U + (u, Q_1^{-1} K_{21} D_2 \varphi_m^n)_U\big) A_0^\Phi Pf. \end{aligned}$$

The theorem of dominated convergence implies that $(A_0^\Phi g_m)_{m\in\mathbb{N}}$ converges to

$$\begin{aligned} (u,v) \mapsto &(u, Q_1^{-1} K_{21} D_2(A_0^\Phi Pf))_U + (D\Phi, K_{21} D_2(A_0^\Phi Pf))_U \\ &- (v, Q_2^{-1} K_{12} D_1(A_0^\Phi Pf))_V \end{aligned}$$

in H as $m \to \infty$. The sequence $(g_m)_{m\in\mathbb{N}}$ converges to $A_0^\Phi Pf$ in H as $m \to \infty$. Since $(A_0^\Phi, D(A_0^\Phi))$ is a closed operator, we conclude $A_0^\Phi P(\mathcal{D}) \subset D(A_0^\Phi)$ and the function defined right above equals $(A_0^\Phi)^2 Pf$.

To finish the proof of Item (iv), we calculate $D_1(A_0^\Phi Pf)$ and $D_2(A_0^\Phi Pf)$. Due to the structure of $A_0^\Phi Pf$, it holds

$$D_2(A_0^\Phi Pf) = -Q_2^{-1} K_{12} D_1 f_P.$$

As $A_0^\Phi Pf$ only depends on the first n-directions in the first variable, we see that

$$D_1(A_0^\Phi Pf)(u,v) = -\sum_{i,j=1}^n (v, Q_2^{-1} K_{12} d_j)_V \partial_{d_i} \partial_{d_j} f_P(u) d_i \quad \text{for all} \quad (u,v) \in W.$$

All in all, this yields

$$(A_0^\Phi)^2 Pf(u,v) = \sum_{i,j=1}^n (v, Q_2^{-1}K_{12}d_i)_V (v, Q_2^{-1}K_{12}d_j)_V \partial_{d_i}\partial_{d_j} f_P(u) \\ - (u, Q_1^{-1}K_{21}Q_2^{-1}K_{12}D_1 f_P(u))_U - (D\Phi(u), K_{21}Q_2^{-1}K_{12}D_1 f_P(u))_U. \tag{6.7}$$

Using Lemma 3.5, we obtain

$$\int_V (v, Q_2^{-1}K_{12}d_i)_V (v, Q_2^{-1}K_{12}d_j)_V \, \mu_2(\mathrm{d}v) = (K_{12}d_i, Q_2^{-1}K_{12}d_j)_V = (Cd_i, d_j)_U$$

for all $1 \le i, j \le n$. This implies,

$$\begin{aligned} &P(A_0^\Phi)^2 Pf(u,v) \\ &= \sum_{i,j=1}^n \partial_{d_i}\partial_{d_j} f_P(u)(Cd_i, d_j)_U - (u, Q_1^{-1}CD_1 f_P(u))_U - (D\Phi(u)CD_1 f_P(u))_U \\ &= \mathrm{tr}[CD_1^2 f_P(u)] - (u, Q_1^{-1}CD_1 f_P(u))_U - (D\Phi(u), CD_1 f_P(u))_U. \end{aligned}$$

Therefore, we have the desired representation of $(G, \mathcal{D})$. It is easy to see that $(G, \mathcal{D})$ is symmetric and consequently dissipative on H. As densely defined symmetric operators on a Hilbert space are essentially self-adjoint if and only if they are essentially m-dissipative, it is left to show that $(\mathrm{Id} - G)(\mathcal{D})$ is dense in H. We prove this by showing that

$$((\mathrm{Id} - G)h, g)_H = 0 \quad \text{for all} \quad h \in \mathcal{D}, \tag{6.8}$$

implies $g = 0$ in H. Suppose $g \in H$ and the statement (6.8) is true. Let $n \in \mathbb{N}$ and $f \in \mathcal{F}C_b^\infty(B_U, n)$ be given. We choose a sequence of cut-off functions $(\varphi_m)_{m\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ provided by Corollary 3.23. Then, the sequence $(f_m)_{m\in\mathbb{N}}$ defined by

$$f_m : W \to \mathbb{R} \quad (u,v) \mapsto f_m(u,v) = f(u)\varphi_m(P_n(v)) - \mu^\Phi(f(\varphi_m \circ P_n))$$

is in $\mathcal{D}$ and for all $m \in \mathbb{N}$ it holds

$$\begin{aligned} 0 &= ((\mathrm{Id} - G)f_m, g)_H \\ &= (f_m, g)_H - \mu_2(\varphi_m \circ P_n) \int_W N^\Phi f(u) g(u,v) \, \mu^\Phi(\mathrm{d}(u,v)) \\ &\to (f, g_P)_{L^2(\mu_1^\Phi)} - (N^\Phi f, g_P)_{L^2(\mu_1^\Phi)} \quad \text{as} \quad m \to \infty, \end{aligned}$$

by the theorem of dominated convergence. Therefore,

$$0 = ((\mathrm{Id} - N^\Phi)f, g_P)_{L^2(\mu_1^\Phi)} = 0 \quad \text{for all} \quad f \in \mathcal{F}C_b^\infty(B_U, n).$$

Since $n \in \mathbb{N}$ was arbitrary, it holds

$$0 = ((\mathrm{Id} - N^\Phi)f, g_P)_{L^2(\mu_1^\Phi)} = 0 \quad \text{for all} \quad f \in \mathcal{F}C_b^\infty(B_U).$$

With the help of Theorem 6.5, we obtain that $(\mathrm{Id} - N^\Phi)(\mathcal{F}C_b^\infty(B_U))$ is dense in $L^2(U;\mu_1^\Phi)$. Hence, $g_P = 0$ in $L^2(U;\mu_1^\Phi)$ and thus for all $f \in \mathcal{D}$

$$(f,g)_H = (Gf,g)_H = (N^\Phi f_P, g_P)_{L^2(\mu_1^\Phi)} = 0.$$

Consequently, $g = 0$ by the density of $\mathcal{D}$ in H.

□

By means of Section 6.1, we define the bounded operator B on H as in Definition 4.3, which acts as $(\mathrm{Id} - G)^{-1}(A_0^\Phi P^*)$ on $D(A_0^\Phi P^*)$. We start verifying the hypocoercivity assumptions. We begin with boundedness of the auxiliary operators $BA_0^\Phi(I-P)$ and BS_0.

Proposition 6.12. *Let Assumptions SA(Φ) and Reg(Φ) be satisfied. Then, the operator* $(BA_0^\Phi(\mathrm{Id} - P), \mathcal{D})$ *is bounded and the second inequality in* ***H1*** *holds with* $c_2 = \sqrt{8 + \frac{c_{\Phi 2}}{4}}$.

Proof. The application of Lemma 4.6 is desired. To achieve this, let $f \in \mathcal{D}$ and $g = (\mathrm{Id} - G)f$ be given. By trivially extending f, we assume without loss of generality that $f \in \mathcal{F}C_b^\infty(B_W, n)$ with $n = m^K(n)$. It holds by the first part of Lemma 4.6

$$(BA_0^\Phi)^* g = -(A_0^\Phi)^2 Pf.$$

Using Formula (6.7), we calculate

$$\begin{aligned}
\|(A_0^\Phi)^2 Pf\|_H^2 = &\int_W \left(\sum_{i,j=1}^n (v, Q_2^{-1}K_{12}d_i)_V (v, Q_2^{-1}K_{12}d_j)_V \partial_{d_i}\partial_{d_j} f_P \right)^2 \, \mathrm{d}\mu^\Phi \\
&- 2\sum_{i,j=1}^n \int_V (v, Q_2^{-1}K_{12}d_i)_V (v, Q_2^{-1}K_{12}d_j)_V \, \mathrm{d}\mu_2 \\
&\times \int_U \partial_{d_i}\partial_{d_j} f_P \Big((u, Q_1^{-1}CD_1 f_P)_U + (D\Phi, CD_1 f_P)_U \Big) \, \mathrm{d}\mu_1^\Phi \\
&+ \int_U \Big((u, Q_1^{-1}CD_1 f_P)_U + (D\Phi, CD_1 f_P)_U \Big)^2 \, \mathrm{d}\mu_1^\Phi.
\end{aligned}$$

Moreover, by Lemma 3.5, it holds for all $i,j,k,l \in \mathbb{N}$

$$\int_V (v, Q_2^{-1}K_{12}d_i)_V (v, Q_2^{-1}K_{12}d_j)_V \, \mathrm{d}\mu_2 = c_{ij} \tag{6.9}$$

$$\begin{aligned}
&\int_V (v, Q_2^{-1}K_{12}d_i)_V (v, Q_2^{-1}K_{12}d_j)_V (v, Q_2^{-1}K_{12}d_k)_V (v, Q_2^{-1}K_{12}d_l)_V \, \mathrm{d}\mu_2 \\
&= c_{ij}c_{kl} + c_{ik}c_{jl} + c_{il}c_{jk},
\end{aligned} \tag{6.10}$$

where $c_{ij} := (Cd_i, d_j)_U$. Using Equation (6.9) and (6.10), we arrive at

$$\begin{aligned}
\|(A_0^\Phi)^2 Pf\|_H^2 &= \int_U \mathrm{tr}[CD_1^2 f_P]^2 - 2\mathrm{tr}[CD_1^2 f_P]\Big((u, Q_1^{-1}CD_1 f_P)_U + (D\Phi(u), CD_1 f_P)_U \Big) \\
&\quad + \Big((u, Q_1^{-1}CD_1 f_P)_U + (D\Phi(u), CD_1 f_P)_U \Big)^2 + 2\mathrm{tr}[(CD^2 f_P)^2] \, \mathrm{d}\mu_1^\Phi \\
&= \int_U (N^\Phi f_P)^2 \, \mu_1^\Phi(\mathrm{d}u) + 2\int_U \mathrm{tr}[(CD_1^2 f_P)^2] \, \mathrm{d}\mu_1^\Phi.
\end{aligned}$$

Since $Pg = Pf - PGf = Pf - N^\Phi Pf$, we get by the regularity estimates from Lemma 6.7 and the contraction property of $R_1^{N^\Phi}$

$$\begin{aligned}\|(A_0^\Phi)^2 Pf\|_H^2 &= \int_U (N^\Phi Pf)^2 \,\mathrm{d}\mu_1^\Phi + 2\int_U \operatorname{tr}[(CD^2Pf)^2]\,\mathrm{d}\mu_1^\Phi \\ &\le 2\int_U (Pf)^2 + (Pg)^2 \mathrm{d}\mu_1^\Phi + \left(4 + \frac{c_{\Phi_2}}{4}\right)\int_U (Pg)^2\,\mathrm{d}\mu_1^\Phi \\ &\le \left(8 + \frac{c_{\Phi_2}}{4}\right)\|Pg\|_H^2 \le \left(8 + \frac{c_{\Phi_2}}{4}\right)\|g\|_H^2.\end{aligned}$$

The claim follows by the second part of Lemma 4.6. □

We state a new assumption ensuring boundedness of BS_0, below. To formulate the assumptions it is useful to introduce $V_\infty := \operatorname{span}\{e_1, e_2, \dots\}$

Assumption (K3). Assume that $K_{22}(v) = K_1 + K_2(v)$, where $K_1 \in \mathcal{L}(V)$ and $K_2 : V \to \mathcal{L}(V)$. In addition, assume that K_1 and K_2 share the same invariance properties as K_{22}. Further, let the following hold

(i) There is some $C_1 \in (0,\infty)$ such that

$$\|Q_2^{-\frac{1}{2}} K_1^* Q_2^{-1} K_1 Q_2^{-\frac{1}{2}}\|_{\mathcal{L}(V_\infty)} \le C_1.$$

(ii) There exists a measurable function $\overline{C}_2 : V \to [0,\infty)$ such that for μ_2-a.e. $v \in V$

$$\|(Q_2^{-\frac{1}{2}} K_2(v)^* Q_2^{-2} K_2(v) Q_2^{-\frac{1}{2}})\|_{\mathcal{L}(V_\infty)} \le \overline{C}_2(v) \quad \text{and} \quad C_2 := \int_V \overline{C}_2(v)\|v\|_V^2\,\mathrm{d}\mu_2 < \infty.$$

(iii) For all $v \in V$, the sequence $(\alpha_n^{22}(v))_{n\in\mathbb{N}}$ defined by

$$\alpha_n^{22}(v) := \sum_{k=1}^{\infty} (Q_2^{-\frac{1}{2}} \partial_{e_k} K_{22}(v) e_k, e_n)_V$$

is in $\ell^2(\mathbb{N})$ and

$$M_{22} := \int_V \|(\alpha_n^{22}(v))_{n\in\mathbb{N}}\|_{\ell^2}^2\,\mu_2(\mathrm{d}v) < \infty.$$

Remark 6.13. *Recall K_{22}^0 from Assumption **K0**, then $K_{22}(v) = K_{22}^0 + K_{22}(v) - K_{22}^0$ is a possible decomposition of K_{22} as assumed in Section 6.2. In general, K_1 has not to be positive definite, which is assumed for K_{22}^0.*

Proposition 6.14. *Let Assumption **K3** be valid. Then, $(BS_0, \mathcal{D})$ is a bounded operator on H and the first inequality in Assumption **H1** is satisfied for*

$$c_1 := \frac{1}{2}\left(\sqrt{C_1} + \sqrt{C_2} + \sqrt{M_{22}}\right).$$

Proof. We prove this via Lemma 4.6. So let $f \in \mathcal{D}$ and $h \in D(S_0)$ be arbitrary. By definition of $D(S_0)$, there is a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that $h_n \to h$ and $S_0 h_n \to S_0 h$ in H as $n \to \infty$. Fix some $n \in \mathbb{N}$, then

$$\begin{aligned}(S_0 h_n, A_0^\Phi P f)_H &= \int_W (D_2 h_n(u,v), K_{22}(v) D_2 A_0^\Phi P f(u,v))_V \, \mu^\Phi(\mathrm{d}(u,v)) \\ &= -\int_U \int_V (D_2 h_n(u,v), K_{22}(v) Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mu^\Phi(\mathrm{d}(u,v)) \\ &= -\sum_{k=1}^\infty \int_U \int_V \partial_{e_k} h_n(u,v) (K_{22}(v) e_k, Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mu^\Phi(\mathrm{d}(u,v)).\end{aligned} \tag{6.11}$$

Above, the first equality follows from the representation of S in Lemma 5.9, the second equality follows from Proposition 6.11 (ii) and the last line is due to symmetry of $K_{22}(v)$ for any $v \in V$. Applying integration by parts formula (see Proposition 3.40), we obtain

$$\begin{aligned}(S_0 h_n, A_0^\Phi P f)_H = &\sum_{k=1}^\infty \int_U \int_V h_n(u,v) (\partial_{e_k} K_{22}(v) e_k, Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mu^\Phi(\mathrm{d}(u,v)) \\ &- \sum_{k=1}^\infty \int_U \int_V (v, Q_2^{-1} e_k)_V h_n(u,v) (e_k, K_{22}(v) Q_2^{-1} K_{12} D_1 f_P(u))_V \, \mu^\Phi(\mathrm{d}(u,v)) \\ = &(h_n, Tf)_H,\end{aligned}$$

where $T : \mathcal{D} \to L^2(W; \mu^\Phi)$ is defined by

$$Tf(u,v) := \sum_{k=1}^\infty (\partial_{e_k} K_{22}(v) e_k, Q_2^{-1} K_{12} D_1 f_P(u))_V - (v, Q_2^{-1} K_{22}(v) Q_2^{-1} K_{12} D_1 f_P(u))_V.$$

For each $f \in \mathcal{D}$, Tf is in $L^2(W; \mu^\Phi)$, since all appearing sums are finite, $\|\cdot\|_V \in L^2(V; \mu_2)$, as well as the growth properties of K_{22} and $\partial_{e_k} K_{22}$. Since $1 \in \mathcal{F}C_b^\infty(B_W)$, it follows analogously to Equation (6.11) that $\mu^\Phi(Tf) = (1, Tf)_H = (S1, A_0^\Phi P f)_H = 0$, so $Tf \in H$. Now letting $n \to \infty$, we see that $A_0^\Phi P f \in D(S_0^*)$ with $S_0^* A_0^\Phi P f = Tf$. Hence, an application of Lemma 4.6 is possible, if there is some $C_T < \infty$ such that for every $g := (\mathrm{Id} - G) f$ with $f \in \mathcal{D}$, we can estimate

$$\|(BS_0)^* g\|_H = \|S_0^* A_0^\Phi P f\|_H = \|Tf\|_H \le C_T \|g\|_H. \tag{6.12}$$

To find such a constant C_T, let $f \in \mathcal{D}$ and fix $k \in \mathbb{N}$ such that $Q_2^{-\frac{1}{2}} K_{12} D_1 f_P \in V_{m_k}$. Due to Assumption **K3** Item (i), we can estimate for each $(u,v) \in W$

$$\begin{aligned}&(K_1 Q_2^{-1} K_{12} D_1 f_P(u), Q_2^{-1} K_1 Q_2^{-1} K_{12} D_1 f_P(u))_V \\ = &\sum_{i,j=1}^{m_k} (Q_2^{-\frac{1}{2}} K_1^* Q_2^{-1} K_1 Q_2^{-\frac{1}{2}} e_i, e_j)_V (Q_2^{-\frac{1}{2}} K_{12} D_1 f_P(u), e_i)_V (Q_2^{-\frac{1}{2}} K_{12} D_1 f_P(u), e_j)_V \\ \le &C_1 \|Q_2^{-\frac{1}{2}} K_{12} D_1 f_P(u)\|_V^2 = C_1 (C D_1 f_P(u), D_1 f_P(u))_U.\end{aligned}$$

We obtain

$$\begin{aligned}
\|(v, Q_2^{-1} K_1 Q_2^{-1} K_{12} D_1 f_P)_V\|_H^2 &= \int_U (K_1 Q_2^{-1} K_{12} D_1 f_P, Q_2^{-1} K_1 Q_2^{-1} K_{12} D_1 f_P)_U \,\mathrm{d}\mu_1^\Phi \\
&\le C_1 \int_U (C D_1 f_P, D_1 f_P)_V \,\mathrm{d}\mu_1^\Phi \le \frac{C_1}{4} \int_U ((\mathrm{Id} - N) f_P)^2 \,\mathrm{d}\mu_1^\Phi \\
&= \frac{C_1}{4} \int_U \left(\int_V (\mathrm{Id} - G) f \,\mathrm{d}\mu_2 \right)^2 \mathrm{d}\mu_1^\Phi \le \frac{C_1}{4} \|g\|_H^2,
\end{aligned}$$

where we applied the estimate above, Lemma 3.5 and Inequality (6.1). On the other hand, by Item (ii) from Assumption **K3**, it holds by similar arguments as above

$$\|Q_2^{-1} K_2(v) Q_2^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} K_{12} D_1 f_P(u)\|_V^2 \le \overline{C}_2(v) (C D_1 f_P, D_1 f_P)_U.$$

This yields,

$$\begin{aligned}
\|(v, Q_2^{-1} K_2 Q_2^{-1} K_{12} D_1 f_P)_V\|_H^2 &\le \int_W \|v\|_V^2 \|Q_2^{-1} K_2(v) Q_2^{-\frac{1}{2}} Q_2^{-\frac{1}{2}} K_{12} D_1 f_P(u)\|_V^2 \,\mathrm{d}\mu^\Phi((u,v)) \\
&\le \int_W \overline{C}_2(v) \, \|v\|_V^2 (C D_1 f_P(u), D_1 f_P(u))_U \; \mu^\Phi(\mathrm{d}(u,v)) \\
&= \int_V \overline{C}_2(v) \, \|v\|_V^2 \, \mu_2(\mathrm{d}v) \int_U (C D_1 f_P(u), D_1 f_P(u))_U \, \mu_1^\Phi(\mathrm{d}u) \\
&\le \frac{C_2}{4} \|g\|_H^2.
\end{aligned}$$

Hence, the second summand of Tf can be bounded relatively to g. To deal with the first summand, we estimate, by means of Item (iii) from Assumption **K3**,

$$\begin{aligned}
&\sum_{k=1}^\infty (\partial_{e_k} K_{22}(v) e_k, Q_2^{-1} K_{12} D_1 f_P(u))_V \\
&= \sum_{k=1}^\infty \sum_{j=1}^\infty (Q_2^{-\frac{1}{2}} \partial_{e_k} K_{22}(v) e_k, e_j)_V (e_j, Q_2^{-\frac{1}{2}} K_{12} D_1 f_P(u))_V \\
&= \sum_{j=1}^\infty \alpha_j^{22}(v) (e_j, Q_2^{-\frac{1}{2}} K_{12} D_1 f_P(u))_V \le \|(\alpha_j^{22}(v))_{j\in\mathbb{N}}\|_{\ell^2} (D_1 f_P(u), C D_1 f_P(u))_U^{\frac{1}{2}}.
\end{aligned}$$

The right-hand side factorizes into an u- and v-dependent component. So an integration over W with respect to μ^Φ yields a product of integrals over U and V with respect to μ_1^Φ and μ_2, respectively. We obtain

$$\int_W \left(\sum_{k=1}^\infty (\partial_{e_k} K_{22} e_k, Q_2^{-1} K_{12} D_1 f_P)_V \right)^2 \mathrm{d}\mu^\Phi \le M_{22} \int_U (D_1 f_P, C D_1 f_P)_U \,\mathrm{d}\mu_1^\Phi \le \frac{M_{22}}{4} \|g\|_H^2.$$

This shows that the first summand of Tf can be bounded relative to g. Overall, we conclude

$$\|Tf\|_H \le \frac{1}{2} \left(\sqrt{C_1} + \sqrt{C_2} + \sqrt{M_{22}} \right) \|g\|_H.$$

Consequently, Inequality (6.12) holds for $C_T := c_1$. In view of of Lemma 4.6, this proves that c_1 is an upper bound for the operator BS_0. □

Remark 6.15. *In the proof of Proposition 6.14, we have always used* $(CD_1 f_P, D_1 f_P)_U$ *as a bounding term, in order to apply the first inequality from Proposition 6.3. By involving* Q_1 *into the assumption* ***K3****, we can use* $(Q_1^{-1} CDf_P, CDf_P)_U$ *instead as a bound, by means of Inequality (6.4). In that case, an imitation of the proof of Proposition 6.14 enables us to bound all terms relative to* g*. This modified assumption is stated below.*

Assumption (K3*). Assume that $K_{22}(v) = K_1 + K_2$, where $K_1 \in \mathcal{L}(V)$ and $K_2 : V \to \mathcal{L}(V)$. Moreover, assume that K_1 and K_2 share the same invariance properties as K_{22}. Further, let the following hold

(i) There is some $C_1 \in (0, \infty)$ such that

$$\|Q_1^{\frac{1}{2}} K_{21}^{-1} K_1^* Q_2^{-1} K_1 K_{12}^{-1} Q_1^{\frac{1}{2}}\|_{\mathcal{L}(V_\infty)} \le C_1.$$

(ii) There exists a measurable function $\overline{C}_2 : V \to [0, \infty)$ such that for all $k \in \mathbb{N}$ and μ_2-a.e. $v \in V$

$$\|Q_2^{-1} K_2(v)^* K_{21}^{-1} Q_1 K_{12}^{-1} K_2(v) Q_2^{-1}\|_{\mathcal{L}(V_\infty)} \le \overline{C}_2(v) \quad \text{and}$$
$$C_2 := \int_V \overline{C}_2(v) \|v\|_V^2 \, \mathrm{d}\mu_2 < \infty.$$

(iii) Assume that the sequence $(\alpha^{22}(v)_n)_{n\in\mathbb{N}}$ defined by

$$\alpha_n^{22}(v) := \sum_{k=1}^{\infty} (Q_1^{\frac{1}{2}} K_{12}^{-1} \partial_{e_k} K_{22}(v) e_k, d_n)_V$$

is an element of $\ell^2(\mathbb{N})$ and that $M_{22} := \int_V \|(\alpha_n^{22}(v))_{n\in\mathbb{N}}\|_{\ell^2}^2 \, \mu_2(\mathrm{d}v) < \infty$.

For the sake of completeness, we state the proposition corresponding to Assumption **K3***. Its validity has already been discussed in Remark 6.15.

Proposition 6.16. *Let Assumption* ***K3**** *be valid. Then* $(BS_0, \mathcal{D})$ *is a bounded operator on* H *and the first inequality in Assumption* ***H1*** *is satisfied for*

$$c_1 := \frac{1}{2}\big(\sqrt{C_1} + \sqrt{C_2} + \sqrt{M_{22}}\big),$$

where the constants C_1*,* C_2 *and* M_{22} *are from Assumption* ***K3****.

As the first hypocoercivity assumption is proven, we are left to verify the microscopic coercivity assumptions **H2** and the macroscopic assumptions **H3**. For this, we assume modified Poincaré type inequalities based on K_{22} and $(C, D(C))$.

Assumption (K4). Assume that there is some $c_S \in (0, \infty)$ such that

$$\int_V (K_{22} D_2 f, D_2 f)_V \, \mathrm{d}\mu_2 \ge c_S \int_V (f - \mu_2(f))^2 \, \mathrm{d}\mu_2 \quad \text{for all} \quad f \in \mathcal{F}C_b^\infty(B_V).$$

Assumption (K5). Assume that there is some $c_A \in (0,\infty)$ such that

$$\int_U (CD_1 f, D_1 f)_V \, \mathrm{d}\mu_1^\Phi \geq c_A \int_U \left(f - \mu_1^\Phi(f)\right)^2 \, \mathrm{d}\mu_1^\Phi \quad \text{for all} \quad f \in \mathcal{F}C_b^\infty(B_U).$$

Remark 6.17. *(i) Recall K_{22}^0 from Assumption **K0** and assume that there is a sequence of eigenvalues $(\lambda_k^0)_{k\in\mathbb{N}}$ of K_{22}^0 with respect to the basis B_V. Let $\lambda_{2,i}$ denote the i-th eigenvalue of Q_2, then, due to Lemma 3.61, we have for all $f \in \mathcal{F}C_b^\infty(B_V)$*

$$\int_V (Q_2 D_2 f, D_2 f)_V \, \mathrm{d}\mu_2 \geq \frac{1}{\lambda_{2,1}} \int_V \left(f - \mu_2(f)\right)^2 \, \mathrm{d}\mu_2.$$

*So, if there is some $\omega_{22} \in (0,\infty)$ such that $\lambda_k^0 \geq \omega_{22}\lambda_{2,k}$ for each $k \in \mathbb{N}$, then Assumption **K4** holds with $c_S = \frac{\omega_{22}}{\lambda_{2,1}}$.*

*(ii) Similarly, if $\Phi = \Phi_1 + \Phi_2$ is as described in Assumption Reg(Φ) and there is some $\omega_{12} \in (0,\infty)$ such that $(Cd_k, d_k) \geq \omega_{12}\lambda_{1,k}$ for all $k \in \mathbb{N}$, then Assumption **K5** holds with $c_A = \frac{\omega_{12}}{\lambda_{1,1} e^{\|\Phi_2\|_{osc}}}$. Indeed this follows by approximating Φ_1 with the sequence $(\Phi_{1,n,m})_{n,m\in\mathbb{N}}$ provided by Reg(Φ), then applying Lemma 3.61 and finally using the approximation properties of $(\Phi_{1,n,m})_{n,m\in\mathbb{N}}$ which are due to Item (i) and (ii) from Item Reg(Φ_1).*

Under these conditions, we verify the macroscopic and microscopic coercivity.

Lemma 6.18. *Let Assumption **K4** be valid, then S_0 satisfies Assumption **H2** with $\Lambda_m = c_S$.*

Proof. Let $f \in \mathcal{D}$ and set $f_u := f(u,\cdot) - Pf(u) \in \mathcal{F}C_b^\infty(B_V)$ for any $u \in U$. Then $\mu_2(f_u) = 0$ and $D_2 f_u(v) = D_2 f(u,v)$ for all $u \in U$, $v \in V$. By means of Assumption **K4** and Lemma 5.9, it holds that

$$\begin{aligned} c_S \|(\mathrm{Id} - P)f\|_H^2 = c_S \int_U \int_V f_u^2 \, \mathrm{d}\mu_2 \, \mathrm{d}\mu_1^\Phi &\leq \int_U \int_V (K_{22} D_2 f_u, D_2 f_u)_V \, \mathrm{d}\mu_2 \, \mathrm{d}\mu_1^\Phi \\ &= \int_W (K_{22} D_2 f, D_2 f)_V \, \mathrm{d}\mu^\Phi = -(S_0 f, f)_H. \end{aligned}$$

□

Remark 6.19. *In view of the proof of Lemma 6.18, we can interpret the Poincaré type inequality from Assumption **K4** as a Poincaré inequality on the orthogonal complement of ker(S_0).*

Lemma 6.20. *Assume that Assumption **K5** holds true. Then A_0^Φ satisfies Assumption **H3** with $\Lambda_M = c_A$.*

Proof. Let $f \in \mathcal{D}$, then $f_P \in \mathcal{F}C_b^\infty(B_U)$ with $\mu_1^\Phi(f_P) = 0$. Using Assumption **K5**, Lemma 3.5 and finally Proposition 6.11 (ii), we estimate

$$\begin{aligned} c_A \|Pf\|_H^2 = c_A \int_U f_P^2 \, \mathrm{d}\mu_1^\Phi &\leq \int_U (Q_2^{-1} K_{12} D_1 f_P, K_{12} D_1 f_P)_V \, \mathrm{d}\mu_1^\Phi \\ &= \int_U \int_V (v, Q_2^{-1} K_{12} D_1 f_P)_V^2 \, \mathrm{d}\mu_2 \, \mathrm{d}\mu_1^\Phi = \|A_0^\Phi Pf\|_H^2. \end{aligned}$$

The claim follows by Proposition 6.11 and Remark 4.4. □

The main result of this section is now immediate.

Theorem 6.21. *Assume that either the assumptions* ***K0-K5****, (with either* ***K3*** *or* ***K3****)*, *SA*(Φ)*, *Reg*(Φ) *and* $Bd_\theta(\Phi)$*, or assumptions* ***K2-K5****, (with either* ***K3*** *or* ***K3****)*, *SA*(Φ)*, *Reg*(Φ) *and* *App*(Φ) *hold true. Then, the semigroup* $(T_t)_{t\geq 0}$ *on* $L^2(W;\mu^\Phi)$ *generated by the closure* $(L^\Phi, D(L^\Phi))$ *of* $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$*, is hypocoercive in the sense that for each* $\theta_1 \in (1,\infty)$*, there is some* $\theta_2 \in (0,\infty)$ *such that*

$$\left\|T_t f - \mu^\Phi(f)\right\|_{L^2(\mu^\Phi)} \leq \theta_1 \mathrm{e}^{-\theta_2 t} \left\|f - \mu^\Phi(f)\right\|_{L^2(\mu^\Phi)}$$

for all $f \in L^2(W;\mu^\Phi)$ *and all* $t \geq 0$*. For* $\theta_1 \in (1,\infty)$*, the constant* θ_2 *determining the speed of convergence can be explicitly computed in terms of* c_S*,* c_A*,* c_{Φ_2} *and* c_1 *as*

$$\theta_2 = \frac{1}{2}\frac{\theta_1 - 1}{\theta_1} \frac{\min\{c_S, c_1\}}{\left(1 + c_1 + \sqrt{8 + \frac{c_{\Phi_2}}{4}}\right)\left(1 + \frac{1+c_A}{2c_A}(1 + c_1 + \sqrt{8 + \frac{c_{\Phi_2}}{4}})\right) + \frac{1}{2}\frac{c_A}{1+c_A}} \frac{c_A}{1 + c_A}.$$

Proof. By the prior considerations, Theorem 4.5 can be applied to $(T_t^0)_{t\geq 0}$ to yield

$$\|T_t^0 f\|_H \leq \theta_1 \mathrm{e}^{-\theta_2 t} \|f\|_H \quad \text{for all } f \in H,\ t \geq 0$$

for θ_1, θ_2, as described in the assertion. By conservativity and μ^Φ-invariance of $(T_t)_{t\geq 0}$ (see Theorem 5.15), this implies

$$\left\|T_t f - \mu^\Phi(f)\right\|_{L^2(\mu^\Phi)} = \|T_t(f - \mu^\Phi(f))\|_H \leq \theta_1 \mathrm{e}^{-\theta_2 t} \|f - \mu^\Phi(f)\|_{L^2(\mu^\Phi)},$$

for all $f \in L^2(W;\mu^\Phi)$ and $t \geq 0$. □

7

The associated stochastic process

The results and statements in this chapter are a generalization of the ones already published in [EG23], where only additive noise was considered. We also extend the results from [BEG23]. Indeed, the results below are valid for situations where the gradient of the potential is not bounded, compare Section 8.3.

Assume that we are in the setting described in Section 5.1. Hence, U and V are two real separable Hilbert spaces, $W = U \times V$, $\Phi : U \to (-\infty, \infty]$ is normalized, bounded from below by zero and there is $\theta \in [0, \infty)$ such that $\Phi \in W^{1,2}_{Q_1^\theta}(U; \mu_1)$. Furthermore, $K_{12} \in \mathcal{L}(U; V)$, $K_{21} = K_{12}^* \in \mathcal{L}(V; U)$, $K_{22}(v) \in \mathcal{L}^+(V)$ for all $v \in V$ and Q_1 and Q_2 are the covariance operators of two infinite dimensional non-degenerate Gaussian measures μ_1 and μ_2, respectively.

As in Chapter 6, we start with the situation where the infinite dimensional Langevin operator $(L^\Phi D(L^\Phi))$ is m-dissipative. This holds, if we either assume Assumptions **K0**, **K1** and $\mathrm{Bd}_\theta(\Phi)$ to apply Theorem 5.23, or Assumption $\mathrm{App}(\Phi)$ to a apply Theorem 5.27. In both situations the strongly continuous contraction semigroup (resolvent) $(T_t)_{t\geq 0}$ $((R_\alpha^{L^\Phi})_{\alpha>0})$, generated by $(L^\Phi D(L^\Phi))$, is sub-Markovian. Moreover, $(L^\Phi D(L^\Phi))$ is conservative and has μ^Φ as an invariant measure.

Equip W with the classical strong topology. A direct application of Theorem 2.69 shows the existence of a Lusin topological space $(W_1, \mathcal{T}_1)$ with $W \subseteq W_1$ and $W \in \mathscr{B}_{\mathcal{T}_1}(W_1)$ and a right process

$$\bar{\mathbf{M}} = (\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geq 0}, (\bar{X}_t, \bar{Y}_t)_{t\geq 0}, (\bar{P}_w)_{w\in W_1})$$

with state space W_1 such that its resolvent, regarded on $L^2(W_1, \bar{\mu}^\Phi)$, coincides with $(R_\alpha^{L^\Phi})_{\alpha>0}$. Recall that $\bar{\mu}^\Phi$ is the measure on $(W_1, \mathscr{B}_{\mathcal{T}_1}(W_1))$ extending μ^Φ by zero on $W_1 \setminus W$. Note that $(R_\alpha^{L^\Phi})_{\alpha>0}$ and $(L^\Phi, D(L^\Phi))$ can also be considered on $L^2(W_1; \bar{\mu}^\Phi)$, since $\bar{\mu}^\Phi(W_1 \setminus W) = 0$. Remember the equilibirum measure $\bar{P}_{\bar{\mu}^\Phi} := \int_{W_1} \bar{P}_w \, \bar{\mu}^\Phi(\mathrm{d}w)$.

In the next proposition we establish that $\bar{\mathbf{M}}$ solves the martingale problem for $(L^\Phi, D(L^\Phi))$ considered on $L^2(W_1; \bar{\mu}^\Phi)$ with respect to $\bar{P}_{\mu^\Phi}$.

Proposition 7.1. *The right process $\bar{\mathbf{M}}$ with state space W_1 solves the martingale problem for $(L^\Phi, D(L^\Phi))$ considered on $L^2(W_1; \bar{\mu}^\Phi)$ with respect to $\bar{P}_{\mu^\Phi}$, in the sense of Definition 2.64.*

Proof. We use the exact same arguments as in [BBR06a, Proposition 1.4] to show that $\bar{\mathbf{M}}$ solves the martingale problem for $(L^\Phi, D(L^\Phi))$ with respect to $\bar{P}_{\mu^\Phi}$, in the sense of Definition 2.64. The assertion in [BBR06a, Proposition 1.4] is stated for μ^Φ-standard right processes but the argumentation works analogously for right processes.

□

For the following considerations we consider $(L^\Phi, D(L^\Phi))$ on $L^2(W;\mu^\Phi)$. Before we continue to construct a more regular process, we provide another core for $(L^\Phi, D(L^\Phi))$. This core is essential to apply Theorem 2.70.

Lemma 7.2. *There exists a countable* $\mathbb{Q}$*-algebra* $\mathcal{A} \subseteq \mathcal{F}C_c^\infty(B_W)$*, which is core for* $(L^\Phi, D(L^\Phi))$ *and also separates the points of* W*.*

Proof. Suppose $f = \psi(p_n^U, p_n^V) \in \mathcal{F}C_b^\infty(B_W)$ for some $n \in \mathbb{N}$ and let $(\varphi_m^n)_{m\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ be the sequence from Corollary 3.23. For each $m, n \in \mathbb{N}$ define the function $g_m^n := \varphi_m^n(p_n^U, p_n^V) \in \mathcal{F}C_c^\infty(B_W)$. By the theorem of dominated convergence, it is easy to see that the sequence $(fg_m^n)_{m\in\mathbb{N}} \subseteq \mathcal{F}C_c^\infty(B_W)$ converges to f in $L^2(W;\mu^\Phi)$ as $m \to \infty$. Moreover, we can calculate

$$L^\Phi(fg_m^n) = fL^\Phi g_m^n + 2(K_{22}D_2 f, D_2 g_m^n)_V + L^\Phi f g_m^n.$$

As g_m^n converges pointwisely to 1 as $m \to \infty$ and has bounded derivatives up to order two independent of m, n, $L^\Phi(fg_m^n) \to L^\Phi f$ as $m \to \infty$ in $L^2(W;\mu^\Phi)$. Hence, $(fg_m^n)_{m\in\mathbb{N}}$ converges to f w.r.t the L^Φ graph norm. This implies that $\mathcal{F}C_c^\infty(B_W)$ is a core for $(L^\Phi, D(L^\Phi))$.
Now suppose $f = \psi(p_n^U, p_n^V) \in \mathcal{F}C_c^\infty(B_W)$, where we assume without loss of generality $n = m^K(n)$. By Corollary 3.26, ψ can be approximated, with respect to $\|\cdot\|_{C^2(\mathbb{R}^n)}$, by a sequence $(\psi_m)_{m\in\mathbb{N}}$, which is contained in a countable dense set $\mathcal{C}_n$ in $C_c^\infty(\mathbb{R}^n)$. Obviously, $g_m^n := \psi_m(p_n^U, p_n^V) \to f$ in $L^2(W;\mu^\Phi)$ as $m \to \infty$. Moreover,

$$\begin{aligned}
&\int_W \left(L^\Phi f - L^\Phi g_m^n\right)^2 \,\mathrm{d}\mu^\Phi \\
&= \int_W \Big(\mathrm{tr}\left[K_{22}(P_n^V v) \circ D_2^2(f - g_m^n)\right] + \sum_{j=1}^n (D_2(f - g_m^n), \partial_{e_j} K_{22}(P_n^V v)e_j)_V \\
&\quad - (K_{22}(P_n^V v)Q_2^{-1}P_n^V v, D_2(f - g_m^n))_V - (K_{12}Q_1^{-1}P_n^U u, D_2(f - g_m^n))_V \\
&\quad - (K_{12}P_n^U D\Phi, D_2(f - g_m^n))_V + (K_{21}Q_2^{-1}P_n^V v, D_1(f - g_m^n))_U \Big)^2 \,\mathrm{d}\mu^\Phi \\
&\le \|\psi - \psi_m\|_{C^2(\mathbb{R}^n)}^2 2^6 \int_W \mathrm{tr}\left[K_{22}(P_n^V v)\right]^2 + \left(\sum_{j=1}^n \|\partial_{e_j} K_{22}(P_n^V v)e_j\|_V\right)^2 \\
&\quad + \|K_{22}(P_n^V v)Q_2^{-1}P_n^V v\|_V^2 + \|K_{12}Q_1^{-1}P_n^U u\|_V + \|K_{12}P_n^U D\Phi\|_V^2 \\
&\quad + \|K_{21}Q_2^{-1}P_n^V v\|_U^2 \,\mathrm{d}\mu^\Phi \to 0, \quad \text{as} \quad m \to \infty.
\end{aligned}$$

Consequently, the countable set $\mathcal{C} := \bigcup_{n\in\mathbb{N}} \{\psi \circ (p_n^U, p_n^V) \mid \psi \in \mathcal{C}_n\}$ is a core for $(L^\Phi, D(L^\Phi))$. Define $\mathcal{A} \subseteq \mathcal{F}C_b^\infty(B_W)$ as the smallest $\mathbb{Q}$-algebra containing $\mathcal{C}$. then $\mathcal{A}$ is countable and a core for $(L^\Phi, D(L^\Phi))$.
Since for each $(u_1, v_1), (u_2, v_2) \in W$, $(u_1, v_1) \neq (u_2, v_2)$ there is some $n \in \mathbb{N}$ such that $(p_n^U u_1, p_n^V v_1) \neq (p_n^U u_2, p_n^V v_2)$, it is easy to show that $\mathcal{A}$ separates the points of W. □

For the rest of this chapter, let $\mathcal{T}$ denote the weak topology on W. Recall that a combination of Lemma 3.27 with Remark 3.29 tells us that the Borel sigma algebra, with respect to the strong and weak topology on W, coincide and are equal to the sigma algebra generated by $\mathcal{A}$ from Lemma 7.2. Next, we establish that

$$F_k := \{w \in W : \|w\|_W \leq k\}$$

defines a μ^Φ-nest of $\mathcal{T}$-compact sets. Our strategy is based on [BBR06a, Section 5], where $(F_k)_{k\in\mathbb{N}}$ was used to construct martingale solutions for generators associated to dissipative stochastic differential equations on Hilbert spaces. We start with the introduction of a new assumption.

Assumption (K6).

(i) There exists $\rho \in L^1(W;\mu^\Phi)$ such that for each $n \in \mathbb{N}$, ρ_n defined via $\rho_n(u,v) := \rho(P_n^U u, P_n^V v)$ is in $L^1(W;\mu^\Phi)$. Moreover, the sequence $(\rho_{m^K(n)})_{n\in\mathbb{N}}$ converges to ρ in $L^1(W;\mu^\Phi)$ as $n \to \infty$ and

$$\begin{aligned}&(P^V_{m^K(n)}v, Q_2^{-1}K_{22}(v)P^V_{m^K(n)}v)_V + (D\Phi(u), K_{21}P^V_{m^K(n)}v)_U \\ &+ (P^U_{m^K(n)}u, Q_1^{-1}K_{21}P^V_{m^K(n)}v)_U - (Q_2^{-1}K_{12}P^U_{m^K(n)}u, P^V_{m^K(n)}v)_V \leq \rho_{m^K(n)}(u,v)\end{aligned}$$

for all $n \in \mathbb{N}$ and $(u,v) \in W$.

(ii) There exist $a, b \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ and $v \in V$

$$\sum_{j=1}^n \|\partial_{e_j} K_{22}(v)e_j\|_V \leq a(1 + \|v\|_V^b).$$

Proposition 7.3. *Let Assumption* ***K6*** *be valid. Then $(F_k)_{k\in\mathbb{N}}$ is μ^Φ-nest of $\mathcal{T}$-compact sets for $(L^\Phi, D(L^\Phi))$.*

Proof. By the Theorem of Banach-Alaoglu F_k is $\mathcal{T}$-compact for all $k \in \mathbb{N}$. It remains to prove that $(F_k)_{k\in\mathbb{N}}$, is a μ^Φ-nest. For notional purposes, we write $N(u,v) := \|(u,v)\|_W^2$ and $N_n(u,v) = N(P_n^U u, P_n^V v)$ for $n \in \mathbb{N}$. We only consider those $n \in \mathbb{N}$ that satisfy $n = m^K(n)$, which provide an increasing sequence. By an approximation argument with a sequence of smooth cut-off functions provided by Corollary 3.23, we see that $N_n \in D(L^\Phi)$, compare also the proof of Lemma 7.2, with

$$\begin{aligned}\frac{1}{2}L^\Phi N_n(u,v) &= \mathrm{tr}[K_{22}(P_n^V v)] + \sum_{j=1}^n (\partial_{e_j} K_{22}(v)e_j, P_n^V v)_V - (P_n^V v, Q_2^{-1}K_{22}(v)P_n^V v)_V \\ &\quad - (D\Phi(u), K_{21}P_n^V v)_U - (P_n^U u, Q_1^{-1}K_{21}P_n^V v)_U + (P_n^V v, Q_2^{-1}K_{12}P_n^U u)_V \\ &\geq \sum_{j=1}^n (\partial_{e_j} K_{22}(v)e_j, P_n^V v)_V - \rho_n(u,v).\end{aligned}$$

Using the second item from Assumption **K6**, we find $a, b \in \mathbb{N}$ with

$$\Big|\sum_{j=1}^n (\partial_{e_j} K_{22}(v)e_j, P_n^V v)_V\Big| \leq a(1 + \|v\|_V^b)\|P_n^V v\|_V =: h_n(v).$$

Note that $(h_n)_{n\in\mathbb{N}}$ converge in $L^1(V;\mu_2)$. In summary, this implies, when setting

$$g_n(u,v) := 2\big(\rho_n(u,v) + h_n(v) + \frac{1}{2}N_n(u,v)\big),$$

that

$$(\mathrm{Id} - L^\Phi)N_n \le g_n \quad \mu^\Phi\text{-a.e.} \tag{7.1}$$

for all $n \in \mathbb{N}$ with $n = m^K(n)$. We proceed as in [BBR06a, Proposition 5.5]. By means of Lemma 2.39, we know that $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is also essentially m-dissipative on $L^1(W;\mu^\Phi)$, with strongly continuous contraction resolvent denoted by $(R_\lambda^{L_1^\Phi})_{\lambda>0}$. On $L^2(W;\mu^\Phi)$, $R_\lambda^{L_1^\Phi}$ and $R_\lambda^{L^\Phi}$ coincide. Consequently, also $R_1^{L^\Phi}$ is sub-Markovian. Applying $R_1^{L_1^\Phi}$ on both sides of Equation (7.1) results in

$$N_n \le R_1^{L_1^\Phi} g_n \quad \mu^\Phi\text{-a.e..}$$

Since $(g_n)_{n\in\mathbb{N}}$ converges to some $g \in L^1(W;\mu^\Phi)$ and $N_n \to N$ in $L^1(W;\mu^\Phi)$ as $n\to\infty$, we get, by taking the limit $n \to \infty$,

$$N \le R_1^{L_1^\Phi} g =: g_* \in L^1(W;\mu^\Phi) \quad \mu^\Phi\text{-a.e..}$$

For each $\lambda \in (0,\infty)$ we obtain,

$$\begin{aligned}\lambda R_{\lambda+1}^{L_1^\Phi} g_*^{\frac{1}{2}} &= \frac{\lambda}{\lambda+1}(\lambda+1)R_{\lambda+1}^{L_1^\Phi} g_*^{\frac{1}{2}} \le \frac{\lambda}{\lambda+1}\left((\lambda+1)R_{\lambda+1}^{L_1^\Phi} g_*\right)^{\frac{1}{2}} \\ &= \left(\frac{\lambda}{\lambda+1}\right)^{\frac{1}{2}}\left(\lambda R_{\lambda+1}^{L_1^\Phi} g_*\right)^{\frac{1}{2}} \le \left(\lambda R_{\lambda+1}^{L_1^\Phi} g_*\right)^{\frac{1}{2}} \le (g_*)^{\frac{1}{2}},\end{aligned}$$

where we use Lemma 2.33 in the fist inequality and Item (v) from Proposition 2.41 in the last. This shows that $g_*^{\frac{1}{2}}$ is a 1-excessive function in $L^2(W;\mu^\Phi)$, dominating the function N. Recall that N has $\mathcal{T}$-compact level sets. Hence, for each $k\in\mathbb{N}$ it holds by definition of the 1-reduced element, see Lemma 2.43,

$$B_1\left(\mathbb{1}_{F_k^c}\right) \le \frac{1}{k} g_*.$$

Therefore, by Proposition 2.45 applied to $f_0 = 1$, the assertion follows. □

Remark 7.4. *To verify the first item of Assumption **K6**, the following considerations are useful. We first give a condition under which*

$$\begin{aligned}&W \ni (u,v) \mapsto (P^U_{m^K(n)}u, Q_1^{-1}K_{21}P^V_{m^K(n)}v)_U \in \mathbb{R} \quad \textit{and} \\ &W \ni (u,v) \mapsto (P^V_{m^K(n)}v, Q_2^{-1}K_{12}P^U_{m^K(n)}u)_U \in \mathbb{R},\end{aligned}$$

define Cauchy sequences in $L^1(W;\mu^\Phi)$. For that, let $n \ge m$, where we assume without loss *of generality $m = m_K(m)$ and $n = m_K(n)$. Then, it holds*

$$\begin{aligned}
&\int_W |(P_n^U u, Q_1^{-1} K_{21} P_n^V v)_U - (P_m^U u, Q_1^{-1} K_{21} P_m^V v)_U| \mu^\Phi(\mathrm{d}(u,v)) \\
&\le \int_W \Big| \sum_{i,j=m+1}^n \lambda_{1,i}^{-1} (u, e_i)_U (v, e_j)_V (d_i, K_{21} e_j)_U \Big| \, \mu(\mathrm{d}(u,v)) \\
&\le \sum_{i,j=m+1}^n \lambda_{1,i}^{-1} |(d_i, K_{21} e_j)_U| \int_U |(u, e_i)_U| \, \mu_1(\mathrm{d}u) \int_V |(v, e_i)_V| \, \mu_2(\mathrm{d}v) \\
&= \frac{2}{\pi} \sum_{i,j=m+1}^n \left(\sqrt{\lambda_{1,i}}\right)^{-1} \sqrt{\lambda_{2,i}} |(d_i, K_{21} e_j)_U| \\
&= \frac{2}{\pi} \sum_{i,j=m+1}^n |(Q_1^{-\frac{1}{2}} d_i, K_{21} Q_2^{\frac{1}{2}} e_j)_U|.
\end{aligned}$$

Since the same calculation applies to $(P_n^V v, Q_2^{-1} K_{12} P_n^U u)_V$, one can check that

$$\sum_{i,j=1}^\infty |(Q_1^{-\frac{1}{2}} d_i, K_{21} Q_2^{\frac{1}{2}} e_j)_U| < \infty \quad \textit{and} \quad \sum_{i,j=1}^\infty |(Q_2^{-\frac{1}{2}} e_i, K_{12} Q_1^{\frac{1}{2}} d_j)_V| < \infty.$$

We secondly verify that $((D\Phi, K_{21} P_n^V v)_U)_{n\in\mathbb{N}}$ converges in $L^1(W; \mu^\Phi)$, if Item $Bd_\theta(\Phi 2)$ from Assumption $Bd_\theta(\Phi)$ is valid. In that case, for each $m, n \in \mathbb{N}$ with $m \le n$ and $\tilde{v} \in V$, we calculate and estimate by means of Lemma 3.5 and $Bd_\theta(\Phi 2)$

$$\begin{aligned}
\int_W |(D\Phi, K_{21}(P_n^V - P_m^V)v)_V| \, \mathrm{d}\mu^\Phi &\le \|Q_1^\theta D\Phi\|_{L^2(\mu_1)} \left(\int_V \|Q_1^{-\theta} K_{21} (P_m^V v - P_n^V v)\|_V^2 \, \mathrm{d}\mu_2 \right)^{\frac{1}{2}} \\
&\le \|Q_1^\theta D\Phi\|_{L^2(\mu_1)} \left(c_\theta \int_V \|K_{22}(\tilde{v})(P_m^V v - P_n^V v)\|_V^2 \, \mathrm{d}\mu_2 \right)^{\frac{1}{2}} \\
&\le \|Q_1^\theta D\Phi\|_{L^2(\mu_1)} \|K_{22}(\tilde{v})\|_{\mathcal{L}(V)} \left(c_\theta \sum_{i=m+1}^n \lambda_{2,i} \right)^{\frac{1}{2}}.
\end{aligned}$$

As Q_2 has finite trace, we get that $((D\Phi, K_{21} P_n^V v)_U)_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent in $L^1(W; \mu^\Phi)$. In the above argumentation, boundedness of $Q_1^\theta D\Phi$ is not involved. Lastly, suppose K_{22} is diagonal, as described in Remark 5.13 and that there is a sequence $(c_k)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$ such that $\lambda_{22,k}(v) \le c_k$ for all $k \in \mathbb{N}$ and $v \in V$. Then,

$$W \ni (u, v) \mapsto (P_{m^K(n)}^V v, Q_2^{-1} K_{22}(v) P_{m^K(n)}^V v)_V \in \mathbb{R}$$

*is a Cauchy sequence in $L^1(W; \mu^\Phi)$, using Lemma 3.5. This consideration is especially useful, when combined with Assumption **K3**.*

For the rest of this section, we assume that Assumption **K6** holds.

Proposition 7.5. *There exists a μ^Φ-invariant Hunt process*

$$\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, Y_t)_{t\geq 0}, (P_w)_{w\in W})$$

with P_{μ^Φ}-a.s. infinite life-time and weakly continuous paths, which is associated with $(T_t)_{t\geq 0}$ ($(R_\alpha)_{\alpha>0}$) and solving the martingale problem for $(L^\Phi, D(L^\Phi))$ with respect to P_{μ^Φ}, in the sense of Definition 2.64. Further, if $f^2 \in D(L^\Phi)$ with $L^\Phi f \in L^4(W;\mu^\Phi)$, then

$$N_t^{[f],L^\Phi} := \left(M_t^{[f],L^\Phi}\right)^2 - \int_0^t L^\Phi(f^2)(X_s, Y_s) - (2fL^\Phi f)(X_s, Y_s)\,\mathrm{d}s, \quad t \geq 0$$

describes an $(\mathcal{F}_t)_{t\geq 0}$-martingale.

Proof. Let $\mathcal{A}$ be the countable core for $(L^\Phi, D(L^\Phi))$, which separates the points of W, constructed in Lemma 7.2. Since $(F_k)_{k\in\mathbb{N}}$ provides a μ^Φ-nest of $\mathcal{T}$-compact sets, we can apply Theorem 2.70 to show that there exists a μ^Φ-standard right process $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, Y_t)_{t\geq 0}, (P_w)_{w\in W})$ with state space W equipped with the topology generated by $\mathcal{A}$, whose paths are càdlàg with respect to the weak topology P_{μ^Φ}-a.e. and which is associated with $(T_t)_{t\geq 0}$ ($(R_\alpha)_{\alpha>0}$). By the same arguments as in Proposition 7.1 we derive that $\mathbf{M}$ solves the martingale problem for $(L^\Phi, D(L^\Phi))$.
As $(L^\Phi, D(L^\Phi))$ is conservative, we are able to apply [Con11, Lemma 2.1.14] to obtain infinite life-time P_{μ^Φ}-a.s.. By Proposition 2.51, the paths of $\mathbf{M}$ are P_{μ^Φ}-a.s. weakly continuous, compare also [BBR06a, Proposition 5.6].
Finally, the statement about $(N_t^{[f],L^\Phi})_{t\geq 0}$ for suitable $f \in D(L^\Phi)$ follows by applying Lemma 2.50 together with Lemma 2.67. □

In [KS98, Chapter 5 Proposition 4.6], it is described how to construct a weak solution to a finite dimensional stochastic differential equation starting from a (local) martingale solution. Even though this result cannot be translated directly into our infinite dimensional setting, the considerations below are inspired by the finite dimensional one. For the definition of quadratic (co)variation and increasing processes we refer to [KS98, Section 1.5]. The following approach is also used in [Ale23], where infinite dimensional Langevin equations with multiplicative noise but with $\Phi = 0$ and stronger invariance assumptions on the coefficients are considered.
First, we evaluate $M_t^{[f],L^\Phi}$ and $N_t^{[f],L^\Phi}$ for a sufficiently rich class of functions f. This class is introduced in the next lemma.

Lemma 7.6. *For any $i \in \mathbb{N}$, define f_i, g_i via*

$$W \ni (u,v) \mapsto f_i(u,v) := (u, d_i)_U \in \mathbb{R} \quad \text{and} \quad W \ni (u,v) \mapsto g_i(u,v) := (v, e_i)_V \in \mathbb{R}\,.$$

Then, for $i,j \in \mathbb{N}$, $f_i, g_i, f_if_j, g_ig_j \in D(L^\Phi)$ and $L^\Phi(f_i^2), L^\Phi(g_ig_j) \in L^4(W;\mu^\Phi)$. Moreover, for all $i,j \in \mathbb{N}$

$$L^{\Phi} f_i = (v, Q_2^{-1} K_{12} d_i)_V, \tag{7.2}$$
$$L^{\Phi}(f_i^2) = 2 f_i L^{\Phi} f_i, \tag{7.3}$$
$$L^{\Phi} g_i = \sum_{k=1}^{\infty} (\partial_{e_k} K_{22} e_i, e_k)_V - (v, Q_2^{-1} K_{22} e_i)_V - (u, Q_1^{-1} K_{21} e_i)_U - (D\Phi, K_{21} e_i)_U, \tag{7.4}$$
$$L^{\Phi}(g_i g_j) = 2(e_i, K_{22} e_j)_V + g_i L^{\Phi} g_j + g_j L^{\Phi} g_i. \tag{7.5}$$

Proof. Note that

$$D_1 f_i = d_i, \quad D_2 f_i = D_1 g_i = 0 \quad \text{and} \quad D_2 g_i = e_i \quad \text{for all} \quad i \in \mathbb{N}\,.$$

The claim follows by using a sequence of smooth cut-off functions for each f_i and g_i, the integrability properties of $Q_1^{\theta} D\Phi$ and the assumptions on the coefficient operators K_{21} and K_{22}, described in Definition 5.5. Compare also the calculations in Corollary 5.10 and Proposition 6.11. □

Remark 7.7. *For each $v \in V$ and all indices $i, k \in \mathbb{N}$, we have $(\partial_{e_k} K_{22}(v) e_i, e_k)_V = (\partial_{e_k} K_{22}(v) e_k, e_i)_V$, using the symmetry properties of K_{22}. Suppose that Item (ii) of Assumption **K6** holds true, then it immediately follows that $\lim_{n\to\infty} \sum_{k=1}^{n} \partial_{e_k} K_{22}(v) e_k \in V$. Hence,*

$$\sum_{k=1}^{\infty} (\partial_{e_k} K_{22}(v) e_i, e_k)_V = \left(\sum_{k=1}^{\infty} \partial_{e_k} K_{22}(v) e_k, e_i \right)_V .$$

Without further mentioning, we use this alternative formula.

Proposition 7.8. *For any $i \in \mathbb{N}$, the real-valued processes $(X_t^i)_{t\geq 0}$ and $(Y_t^i)_{t\geq 0}$ defined by $X_t^i = (X_t, d_i)_U$ and $Y_t^i = (Y_t, e_i)_V$ satisfy $P_{\mu^{\Phi}}$-a.s.*

$$\begin{aligned} X_t^i - X_0^i &= \int_0^t (Y_s, Q_2^{-1} K_{12} d_i)_V \, \mathrm{d}s \qquad \textit{and} \\ Y_t^i - Y_0^i &= \int_0^t \left(\sum_{k=1}^{\infty} \partial_{e_k} K_{22}(Y_s) e_k, e_i \right)_V - (Y_s, Q_2^{-1} K_{22}(Y_s) e_i)_V - (X_s, Q_1^{-1} K_{21} e_i)_U \\ &\quad - (D\Phi(X_s), K_{21} e_i)_U \, \mathrm{d}s + M_t^{[g_i], L^{\Phi}}. \end{aligned} \tag{7.6}$$

Above, $(M_t^{[g_i],L^{\Phi}})_{t\geq 0}$ is a continuous $(\mathcal{F}_t)_{t\geq 0}$-martingale such that for $i, j \in \mathbb{N}$ the quadratic covariation of $M^{[g_i],L^{\Phi}}$ and $M^{[g_j],L^{\Phi}}$ fulfills

$$[M^{[g_i],L^{\Phi}}, M^{[g_j],L^{\Phi}}]_t = 2 \int_0^t (e_i, K_{22}(Y_s) e_j)_V \, \mathrm{d}s \quad \textit{for all} \quad t \in [0, \infty).$$

Proof. We see that $X_t^i = f_i(X_t, Y_t)$ and $Y_t^i = g_i(X_t, Y_t)$, where f_i and g_i are as in Lemma 7.6. Therefore, by Equation (7.2), we directly obtain

$$M_t^{[f_i],L^{\Phi}} = X_t^i - X_0^i - \int_0^t (Y_s, Q_2^{-1} K_{12} d_i)\, \mathrm{d}s,$$

and by Equation (7.3),

$$N_t^{[f_i],L^\Phi} = (M_t^{[f_i],L^\Phi})^2 - \int_0^t L^\Phi(f_i^2)(X_s,Y_s) - (2f_iL^\Phi f_i)(X_s,Y_s)\,\mathrm{d}s = (M_t^{[f_i],L^\Phi})^2.$$

Hence, $[M^{[f_i],L^\Phi}]_t = 0$, i.e. $M^{[f_i],L^\Phi}$ has zero quadratic variation. Consequently, $M_t^{[f_i],L^\Phi} = 0$, P_{μ^Φ}-a.s.. This proves the first Equation in (7.6). To show the second equation, note that

$$\begin{aligned} M_t^{[g_i],L^\Phi} = Y_t^i - Y_0^i - \int_0^t &\left(\sum_{k=1}^\infty \partial_{e_k} K_{22}(Y_s)e_k, e_i\right)_V \\ &- (Y_s, Q_2^{-1}K_{22}(Y_s)e_i)_V - (X_s, Q_1^{-1}K_{21}e_i)_U - (D\Phi(X_s), K_{21}e_i)_U\,\mathrm{d}s, \end{aligned}$$

by Equation (7.4). Moreover, by Equation (7.5),

$$N_t^{[g_i],L^\Phi} = (M_t^{[g_i],L^\Phi})^2 - \int_0^t 2(e_i, K_{22}(Y_s)e_i)_V\,\mathrm{d}s.$$

This implies $[M^{[g_i],L^\Phi}]_t = 2\int_0^t (e_i, K_{22}(Y_s)e_j)_V\,\mathrm{d}s$. For all $i,j\in\mathbb{N}$ we calculate

$$\begin{aligned} &N_t^{[g_i+g_j],L^\Phi} - \left(M_t^{[g_i+g_j],L^\Phi}\right)^2 \\ &= -\int_0^t L^\Phi((g_i+g_j)^2)(X_s,Y_s) - (2(g_i+g_j)L^\Phi(g_i+g_j))(X_s,Y_s)\,\mathrm{d}s \\ &= -2\int_0^t (e_i, K_{22}(Y_s)e_i)_V + 2(e_i, K_{22}(Y_s)e_j)_V + (e_j, K_{22}(Y_s)e_j)_V\,\mathrm{d}s \\ &= -2\int_0^t (e_i+e_j, K_{22}(Y_s)(e_i+e_j))_V\,\mathrm{d}s. \end{aligned}$$

Since $(2\int_0^t (e_i+e_j, K_{22}(Y_s)(e_i+e_j))_V\,\mathrm{d}s)_{t\geq 0}$ is an increasing process, we obtain

$$\begin{aligned} &[M^{[g_i],L^\Phi}, M^{[g_j],L^\Phi}]_t \\ &= \frac{1}{2}\left([M^{[g_i+g_j],L^\Phi}]_t - [M^{[g_i],L^\Phi}]_t - [M^{[g_j],L^\Phi}]_t\right) \\ &= \int_0^t (e_i+e_j, K_{22}(Y_s)(e_i+e_j))_V - (e_i, K_{22}(Y_s)e_i)_V - (e_j, K_{22}(Y_s)e_j)_V\,\mathrm{d}s \\ &= 2\int_0^t (e_i, K_{22}(Y_s)e_j)_V\,\mathrm{d}s. \end{aligned}$$

Hence, the proof is finished. □

Below, we establish that the process **M** provides a stochastically and analytically weak solution for Equation (1.2). For this, we need to construct a suitable cylindrical Wiener process on V, such that we can express the process described by $M_t^V := \sum_{i\in\mathbb{N}} M_t^{[g_i],L^\Phi} e_i$ as a stochastic integral of $\sqrt{K_{22}}$ with respect to to the constructed cylindrical Wiener process. In the following, we set for $k\in\mathbb{N}$

$$M_t^{(k)} := \left(M_t^{[g_1],L^\Phi}, \dots, M_t^{[g_{m^K(k)}],L^\Phi}\right) \quad \text{and} \quad \Sigma_t^{(k)} := \left((K_{22}^{-\frac{1}{2}}(P_{m^K(k)}^V Y_t)e_i, e_j)_V\right)_{i,j=1}^{m^K(k)}.$$

By means of Lévy's characterization of the Wiener process, we show the following lemma.

Lemma 7.9. *For each $k \in \mathbb{N}$, the process $(W_t^{(k)})_{t\geq 0}$ defined by*

$$W_t^{(k)} := \frac{1}{\sqrt{2}} \int_0^t \Sigma_s^{(k)} \, \mathrm{d}M_s^{(k)}$$

is an $m^K(k)$-dimensional Wiener process. Let $\beta^{(k)}$ be the k-th component of $W^{(k)}$, then $(\beta^{(k)})_{k\in\mathbb{N}}$ is an independent sequence of one dimensional Brownian motions.

Proof. First of all, let $k_1, k_2 \in \mathbb{N}$ be given and assume without loss of generality $k_1 \leq k_2$. Due to the block diagonal structure of $\Sigma^{(k_2)}$, which comes from the invariance properties of K_{22}, we know that the j-th component, $j \in \{1, \dots, m^K(k_1)\}$, of $W_t^{(k_1)}$ is equal to the j-th component of $W_t^{(k_2)}$. Hence, $\beta^{(j)}$ does not depend on the $k \in \mathbb{N}$ such that $j \leq k$.
To construct a Wiener process, as described in the assertion, we fix $k \in \mathbb{N}$ with $k = m^K(k)$ and $i, j \in \mathbb{N}$ with $i, j \leq k$.
In view of basic properties of the stochastic integral, compare [KS98, Chapter 3 Proposition 2.19 and Corollary 2.20], we calculate

$$\begin{aligned}
[\beta^{(i)}, \beta^{(j)}]_t &= \frac{1}{2} \int_0^t \sum_{p,q=1}^{k} (\Sigma_s^{(k)})_{ip} (\Sigma_s^{(k)})_{jq} \, \mathrm{d}[M^{[g_p],L^\Phi}, M^{[g_q],L^\Phi}]_s \\
&= \int_0^t \sum_{p,q=1}^{k} (\Sigma_s^{(k)})_{ip} (\Sigma_s^{(k)})_{jq} (K_{22}^{-\frac{1}{2}} (P_k^V Y_t) e_p, e_q)_V \, \mathrm{d}s \\
&= \int_0^t (\Sigma_s^{(k)} K_{22,k}(Y_s) (\Sigma_s^{(k)})^*)_{ij} \, \mathrm{d}s \\
&= \delta_{ij} t.
\end{aligned}$$

By Lévy's characterization, it follows that $(W_t^{(k)})_{t\geq 0}$ is an k-dimensional Wiener process. As $\beta_t^{(k)}$ is the k-th component of $W_t^{(k)}$, we know that $\{\beta^{(1)}, \dots, \beta^{(k)}\}$ is independent for any $k \in \mathbb{N}$. □

Fix some time horizon $T \in (0, \infty)$ and define the process $(W_t)_{t\in[0,T]}$ on V via

$$W_t := \sum_{k=1}^{\infty} \beta_t^{(k)} e_k, \qquad t \in [0, T].$$

This is an Id-cylindrical Wiener process on V, as defined in Proposition 3.16. This follows by choosing $J : V \to V$, $J := Q_2^{\frac{1}{2}}$, since then

$$W_t^{Q_2} := \sum_{k=1}^{\infty} \beta_t^{(k)} J e_k = \sum_{k=1}^{\infty} \beta_t^{(k)} \sqrt{\lambda_k} e_k$$

defines a Q_2-Wiener process on V. Set $V_0 := Q_2^{\frac{1}{2}} V$ and equip it with the inner product

$$(a, b)_{V_0} := (Q_2^{-\frac{1}{2}} a, Q_2^{-\frac{1}{2}} b)_V \qquad \text{for all } a, b \in V_0,$$

which makes V_0 a separable Hilbert space with orthonormal basis $(Q_2^{\frac{1}{2}} e_i)_{i\in\mathbb{N}}$. In accordance to Section 3.1.2, we define $\mathcal{L}_2^0 := \mathcal{L}_2(V_0; V)$ as the Hilbert space of Hilbert-Schmidt operators from V_0 to V.

Assumption (K7). There is a a non-negative function k_{22} in $L^1(V;\mu_2)$ such that for all $v \in V$

$$\operatorname{tr}[K_{22}(v)] \le k_{22}(v).$$

Proposition 7.10. *Suppose Assumption* ***K7*** *holds true. Then, the stochastic process* $\sqrt{K_{22}(Y_t)}J^{-1}$, $t \in [0,T]$ *is* $\mathcal{L}_2^0$*-valued and predictable, i.e.* $\mathcal{A}_T$*-*$\mathcal{B}(\mathcal{L}_2^0)$*-measurable. Moreover, we have*

$$M_t^V := \sum_{i\in\mathbb{N}} M_t^{[g_i],L^\Phi} e_i = \int_0^t \sqrt{2K_{22}(Y_s)}\, \mathrm{d}W_s. \tag{7.7}$$

Proof. For each $v \in V$, we have, by Assumption **K7**,

$$\sum_{i\in\mathbb{N}} (\sqrt{K_{22}(v)}J^{-1}Q_2^{\frac{1}{2}} e_i, \sqrt{K_{22}(v)}J^{-1}Q_2^{\frac{1}{2}} e_i)_V \le k_{22}(v).$$

This implies that $\sqrt{K_{22}(v)}J^{-1} \in \mathcal{L}_2^0$ for any $v \in V$. Furthermore, for all $t \in (0,T)$, we have

$$\begin{aligned}
\|\sqrt{K_{22}(Y_\cdot)}J^{-1}\|_T^2 &= \mathbb{E}_{P_{\mu^\Phi}} \left(\int_0^t \|\sqrt{K_{22}(Y_s)}J^{-1} \circ Q_2^{\frac{1}{2}}\|^2_{\mathcal{L}_2(V)} \,\mathrm{d}s \right) \\
&= \int_\Omega \int_0^t \|\sqrt{K_{22}(Y_s(\omega))}\|^2_{\mathcal{L}_2(V)} \,\mathrm{d}s\, P_{\mu^\Phi}(\mathrm{d}\omega) \\
&\le \int_0^t \int_\Omega k_{22}(Y_s(\omega))\, P_{\mu^\Phi}(\mathrm{d}\omega)\,\mathrm{d}s \\
&= \int_0^t \int_W (T_s k_{22})(v)\, \mu^\Phi(\mathrm{d}(u,v))\,\mathrm{d}s \\
&\le t\|k_{22}\|_{L^1(V,\mu_2)}.
\end{aligned}$$

Recall that $\|\cdot\|_T$ was defined in Section 3.1.2. In addition,

$$A_i := (\sqrt{K_{22}(Y_t)}J^{-1}Q_2^{\frac{1}{2}} e_i, \sqrt{K_{22}(Y_t)}J^{-1}Q_2^{\frac{1}{2}} e_i)_V = (K_{22}(P^V_{m^K(i)}Y_t)e_i, e_i)_V$$

is continuous and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted for any $i \in \mathbb{N}$. Fix some $\varepsilon > 0$ and set

$$B := \left\{ (t,\omega) \in [0,T] \times \Omega \mid \|\sqrt{K_{22}(Y_t(\omega))}J^{-1}\|_{\mathcal{L}_2^0} \le \varepsilon, \right\}$$

as well as $B_k := \{\sum_{i=1}^k A_i \le \varepsilon\} \in \mathcal{A}_T$ for each $k \in \mathbb{N}$. Then, $B = \bigcap_{k\in\mathbb{N}} B_k \in \mathcal{A}_T$ as well. Similarly, all pre-images of closed ε-balls in $\mathcal{L}_2^0$ under $\sqrt{K_{22}(Y_\cdot)}J^{-1}$ are in $\mathcal{A}_T$, so that the process is indeed predictable, since $\mathcal{L}_2^0$ is separable. By the explanations from Remark 3.13, the previous results imply that $\sqrt{K_{22}(Y_t)}J^{-1}$ is integrable with respect to the Q_2-Wiener process $(W_t^{Q_2})_{t\in[0,T]}$, which shows that $\sqrt{K_{22}(Y_t)}$ is integrable with respect to $(W_t)_{t\in[0,T]}$ with

$$\int_0^t \sqrt{K_{22}(Y_s)}\,\mathrm{d}W_s = \int_0^t \sqrt{K_{22}(Y_s)}J^{-1}\,\mathrm{d}W_s^{Q_2}$$

for all $t \in [0,T]$, in the sense of formula (3.1). To verify Equation (7.7), let $i \in \mathbb{N}$ with $i = m^K(i)$ be given. By applying Lemma 3.14 for the operators $(\cdot, e_i)_V : V \to \mathbb{R}$, Lemma 3.15 and [KS98, Chapter 3 Corollary 2.20], we see that

$$\begin{aligned}
\sqrt{2}\left(\int_0^t \sqrt{K_{22}(Y_s)}\,\mathrm{d}W_s, e_i\right)_V &= \sqrt{2}\int_0^t (\sqrt{K_{22}(Y_s)}J^{-1}\cdot, e_i)_V \,\mathrm{d}W_s^{Q_2} \\
&= \sqrt{2}\int_0^t \big((\sqrt{K_{22}(Y_s)}e_1, e_i)_V, \dots (\sqrt{K_{22}(Y_s)}e_i, e_i)_V\big)\,\mathrm{d}W_s^{(i)} \\
&= \int_0^t \big((\sqrt{K_{22}(Y_s)}e_1, e_i)_V, \dots (\sqrt{K_{22}(Y_s)}e_i, e_i)_V\big)\Sigma_s^{(i)}\,\mathrm{d}M_s^{(i)} \\
&= M_t^{[g_i],L^\Phi}.
\end{aligned}$$

Above, we also used the block invariance properties of K_{22}. The claim follows as $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of V. □

Before we finally state the final theorem in this chapter, it is useful to define $D(Q_2^{-1}K_{12})$, $D(Q_1^{-1}K_{21})$ and $D(Q_1^{-\theta}K_{21})$ according to Definition 6.1, as well as

$$D(Q_2^{-1}K_{22}) := \{v \in V \mid\ K_{22}(\tilde{v})(v) \in D(Q_2^{-1}) \text{ for all } \tilde{v} \in V\}.$$

Theorem 7.11. *Suppose Assumptions **K6** and **K7** hold true, then*

$$\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, Y_t)_{t\geq 0}, (P_w)_{w\in W})$$

is a stochastic and analytic weak solution to Equation (1.2), in the sense that there is a cylindrical Wiener process $(W_t)_{t\geq 0}$ on V such that P_{μ^Φ}-a.s., we have for all $j \in \mathbb{N}$,

$$\begin{aligned}
(X_t - X_0, d_j)_U &= \int_0^t (Y_s, Q_2^{-1}K_{12}d_j)_V \,\mathrm{d}s \quad \textit{and} \\
(Y_t - Y_0, e_j)_V &= \int_0^t \left(\sum_{k=1}^\infty \partial_{e_k}K_{22}(Y_s)e_k, e_j\right)_V - (Y_s, Q_2^{-1}K_{22}(Y_s)e_j)_V \\
&\quad - (X_s, Q_1^{-1}K_{21}e_j)_U - (K_{12}D\Phi(X_s), e_j)_V \,\mathrm{d}s \\
&\quad + \left(\int_0^t \sqrt{2K_{22}(Y_s)}\,\mathrm{d}W_s, e_j\right)_V.
\end{aligned} \tag{7.8}$$

Furthermore, we obtain P_{μ^Φ}-a.s. for every element $\nu_1 \in D(Q_2^{-1}K_{12})$ and every element $\nu_2 \in D(Q_1^{-1}K_{21}) \cap D(Q_2^{-1}K_{22}) \cap D(Q_1^{-\theta}K_{21})$,

$$\begin{aligned}
(X_t - X_0, \nu_1)_U &= \int_0^t (Y_s, Q_2^{-1}K_{12}\nu_1)_V \,\mathrm{d}s \quad \textit{and} \\
(Y_t - Y_0, \nu_2)_V &= \int_0^t \left(\sum_{k=1}^\infty \partial_{e_k}K_{22}(Y_s)e_k, \nu_2\right)_V - (Y_s, Q_2^{-1}K_{22}(Y_s)\nu_2)_V \\
&\quad - (X_s, Q_1^{-1}K_{21}\nu_2)_U - (Q_1^\theta D\Phi(X_s), Q_1^{-\theta}K_{21}\nu_2)_V \,\mathrm{d}s \\
&\quad + \left(\int_0^t \sqrt{2K_{22}(Y_s)}\,\mathrm{d}W_s, \nu_2\right)_V.
\end{aligned} \tag{7.9}$$

Proof. The verification of Equation (7.8) directly follows by Proposition 7.10. Recall that all classical integrals appearing in Equation (7.8) are well defined due to Proposition 7.1. For ν_1, ν_2 as in the assertion and all $n \in \mathbb{N}$ with $n = m^K(n)$, it holds $Q_2^{-1} K_{12} P_n^V \nu_1 = P_n^V Q_2^{-1} K_{12} \nu_1, Q_2^{-1} K_{22}(Y_s) P_n^V \nu_2 = P_n^V Q_2^{-1} K_{22}(Y_s) \nu_2, Q_1^{-\theta} K_{21} P_n^V \nu_2 = P_n^V Q_1^{-\theta} K_{21} \nu_2$ and $Q_1^{-1} K_{21} P_n^V \nu_2 = P_n^U Q_1^{-1} K_{21} \nu_2$. By means of Equation (7.8) we have P_{μ^Φ}-a.s. for all $n \in \mathbb{N}$ with $n = m^K(n)$,

$$
\begin{aligned}
(X_t - X_0, P_n^U \nu_1)_U &= \int_0^t (Y_s, P_n^V Q_2^{-1} K_{12} \nu_1)_V \, \mathrm{d}s \\
(Y_t - Y_0, P_n^V \nu_2)_V &= \int_0^t \left(\sum_{k=1}^{\infty} \partial_{e_k} K_{22}(Y_s) e_k, P_n^V \nu_2 \right)_V - (Y_s, P_n^V Q_2^{-1} K_{22}(Y_s) \nu_2)_V \\
&\quad - (X_s, P_n^U Q_1^{-1} K_{21} \nu_2)_U - (Q_1^\theta D\Phi(X_s), P_n^V Q_1^{-\theta} K_{21} \nu_2)_U \, \mathrm{d}s \\
&\quad + \left(\int_0^t \sqrt{2 K_{22}(Y_s)} \, \mathrm{d}W_s, P_n^V \nu_2 \right)_V .
\end{aligned}
$$

Note that $P_n^V Q_2^{-1} K_{22}(Y_s) \nu_2$ converges to $Q_2^{-1} K_{22}(Y_s) \nu_2$ in V. We estimate for all $s \in [0, t]$,

$$
|(Y_s, P_n^V Q_2^{-1} K_{22}(Y_s) \nu_2)_V| \leq \|Y_s\|_V \|Q_2^{-1} K_{22}(Y_s) \nu_2\|_V .
$$

Consequently, we are able to apply the theorem of dominated convergence for $n \to \infty$ with $n = m^K(n)$, to obtain P_{μ^Φ}-a.s.

$$
\lim_{n \to \infty} \int_0^t (Y_s, P_n^V Q_2^{-1} K_{22}(Y_s) \nu_2)_V \, \mathrm{d}s = \int_0^t (Y_s, Q_2^{-1} K_{22}(Y_s) \nu_2)_V \, \mathrm{d}s.
$$

Using that $P_n^U \nu_1$, $P_n^V Q_2^{-1} K_{12} \nu_1$, $P_n^V \nu_2$, $P_n^V Q_1^{-\theta} K_{21} \nu_2$ and $P_n^U Q_1^{-1} K_{21} \nu_2$ converge to ν_1, $Q_2^{-1} K_{12} \nu_1$, ν_2, $Q_1^{-\theta} K_{21} \nu_2$ and $Q_1^{-1} K_{21} \nu_2$ in U and V, respectively, we conclude the proof. □

We end this section with an L^2-exponential ergodicity result for the analytically and stochastically weak solution, provided by Theorem 7.11. A similar result was already established in a manifold setting in [GM22, Corollary 5.2].

Corollary 7.12. *Assume that either the assumptions **K0-K5**, (with either **K3** or **K3***), SA(Φ), Reg(Φ) and Bd$_\theta$(Φ), or assumptions **K2-K5**, (with either **K3** or **K3***), SA(Φ), Reg(Φ) and App(Φ) hold true. Let $\theta_1 \in (1, \infty)$ and $\theta_2 \in (0, \infty)$ be the constants determined by Theorem 6.21. If the Assumptions **K6** and **K7** hold true, then the process*

$$
\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (X_t, Y_t)_{t \geq 0}, (P_w)_{w \in W}),
$$

constructed in Proposition 7.5, is a μ^Φ-invariant Hunt process with infinite life-time and weakly continuous paths solving Equation (7.8). Moreover, for all $t \in (0, \infty)$ and $g \in L^2(W; \mu^\Phi)$, it holds

$$
\left\| \frac{1}{t} \int_0^t g(X_s, Y_s) \mathrm{d}s - \mu^\Phi(g) \right\|_{L^2(P_{\mu^\Phi})} \leq \frac{1}{\sqrt{t}} \sqrt{\frac{2\theta_1}{\theta_2} \left(1 - \frac{1}{t\theta_2} (1 - e^{-t\theta_2}) \right)} \|g - \mu^\Phi(g)\|_{L^2(\mu^\Phi)} .
$$

We call a solution $\mathbf{M}$ with this property L^2-exponentially ergodic, i.e. ergodic with a rate that corresponds to exponential convergence of the corresponding semigroup.

Proof. Besides the ergodicity property, all statements follow by the previous considerations. To show ergodicity, let $t \in (0,\infty)$ and $g \in L^2(W;\mu^\Phi)$ be given. For $f := g - \mu^\Phi(g)$, it holds

$$\begin{aligned}\left\|\frac{1}{t}\int_0^t f(X_s,Y_s)\mathrm{d}s\right\|^2_{L^2(P_{\mu^\Phi})} &= \frac{2}{t^2}\int_0^t\int_0^s (T_{s-u}f,f)_{L^2(\mu^\Phi)}\mathrm{d}u\mathrm{d}s\\ &\leq \frac{2\|f\|^2_{L^2(\mu^\Phi)}}{t^2}\int_0^t\int_0^s \theta_1 e^{-(s-u)\theta_2}\mathrm{d}u\mathrm{d}s\\ &= \frac{1}{t}\frac{2\theta_1}{\theta_2}\left(1-\frac{1}{t\theta_2}(1-e^{-t\theta_2})\right)\|f\|^2_{L^2(\mu^\Phi)}.\end{aligned}$$

To obtain the first equality, we argue as in [GM22, Corollary 5.2]. Afterwards, we use the Cauchy-Schwarz inequality and the hypocoercivity of the semigroup. In the last line we compute the integral. □

Remark 7.13. *We can formulate a similar statement as in Corollary 7.12 in terms of the right process from Proposition 7.1. Indeed for the computations in Corollary 7.12 we only need the Markov property and that the semigroup of $(L^\Phi, D(L^\Phi))$ is associated with the transition semigroup of the process.*

We end this chapter with a remark concerning the optimality of the convergence rate from Corollary 7.12.

Remark 7.14. *From Corollary 7.12 above, we follow, that time average converges to space average in $L^2(P_{\mu^\Phi})$ with rate $t^{-\frac{1}{2}}$. If the spectrum of $(L^\Phi, D(L^\Phi))$ contains a negative eigenvalue $-\kappa$ with corresponding eigenvector g, then this rate is optimal. Indeed, by a similar reasoning as in the calculation above, we then get for all $t \in (0,\infty)$*

$$\left\|\frac{1}{t}\int_0^t g(X_s,Y_s)\mathrm{d}s\right\|_{L^2(P_{\mu^\Phi})} = \frac{1}{\sqrt{t}}\sqrt{\frac{2}{\kappa}\left(1-\frac{1}{t\kappa}(1-e^{-t\kappa})\right)}\|g\|_{L^2(\mu^\Phi)}.$$

Equality above holds, as the application of the Cauchy-Schwarz inequality is not necessary. Moreover, note that $\mu^\Phi(g) = \frac{1}{-\kappa}\mu^\Phi(L_\Phi g) = 0$.

8

Applications

This chapter deals with the application of the results we established above, in the framework of stochastic reaction-diffusion and Cahn-Hilliard type equations. First, we translate these equations into our setting of degenerate second order in time and infinite dimensional stochastic differential equations with multiplicative noise. Afterwards, we show essential m-dissipativity of their associated generators and establish hypocoercivity of the corresponding semigroups. The construction of the associated stochastic processes and the analysis of their long time behavior, by means of the associated hypocoercive semigroups, finishes our analysis. The chapter is separated into three parts. We start with degenerate second order in time stochastic reaction-diffusion equations with multiplicative noise where only potentials with bounded gradient are considered. Then, Cahn-Hilliard equations of this type, allowing potentials with bounded gradient in a suitable infinite dimensional Sobolev space, are studied. The consideration of theses examples is contained in [BEG23]. Lastly, we reconsider reaction-diffusion equations and invoke the results from Section 5.1.2 to allow potentials with unbounded gradient.
Non-degenerate first order stochastic reaction-diffusion and Cahn-Hilliard type equations have been extensively analyzed by many authors. We highlight [DA14, Section 5 and 6] and [ES09; DDT04], where most of the inspiration for our considerations originates. In these articles, the authors were able to treat nonlinearities in terms of potentials and vector fields which grow at most polynomial.

8.1 Degenerate second order in time stochastic reaction-diffusion equations with multiplicative noise (potentials with bounded gradient)

Let $\mathrm{d}\xi$ be the standard Lebesgue measure on $((0,1),\mathscr{B}(0,1))$ and define $U \coloneqq L^2((0,1);\mathrm{d}\xi)$. Moreover, we denote by $W_0^{1,2}(0,1)$ the classical Sobolev space of weakly differentiable functions with zero boundary conditions on $(0,1)$ and by $W^{2,2}(0,1)$ the Sobolev space of two times weakly differentiable functions on $(0,1)$. In the following, we set $W \coloneqq U \times U$ and let $(-\partial_\xi^2, D(\partial_\xi^2))$ be the negative second order derivative with Dirichlet boundary conditions, i.e.

$$D(\partial_\xi^2) \coloneqq W_0^{1,2}(0,1) \cap W^{2,2}(0,1) \subseteq U.$$

It is well known that the inverse of $(-\partial_\xi^2, D(\partial_\xi^2))$ can be extended to a bounded linear operator on U. This extension is denoted by $(-\partial_\xi^2)^{-1}$. Therefore, it is reasonable to define the linear continuous operator

$$Q = (-\partial_\xi^2)^{-1} : (U, \|\cdot\|_U) \to (D(\partial_\xi^2), \|\cdot\|_U) \subseteq (U, \|\cdot\|_U).$$

Q is positive and self-adjoint. Further, $B_U = (d_k)_{k\in\mathbb{N}} = (\sqrt{2}\sin(k\pi\cdot))_{k\in\mathbb{N}}$ is an orthonormal basis of U diagonalizing Q with corresponding eigenvalues $(\lambda_k)_{k\in\mathbb{N}} = ((k\pi)^{-2})_{k\in\mathbb{N}}$.
For parameters $\alpha_1, \alpha_2 \in \mathbb{R}$ with

$$\alpha_1, \alpha_2 > \frac{1}{2},$$

we consider two centered non-degenerate infinite dimensional Gaussian measures μ_1 and μ_2 on $(U, \mathcal{B}(U))$, with covariance operators

$$Q_1 := Q^{\alpha_1} \quad \text{and} \quad Q_2 := Q^{\alpha_2},$$

respectively.
Since $(\lambda_k)_{k\in\mathbb{N}} \in \ell^r(\mathbb{N})$ for $r > \frac{1}{2}$, Q_1 and Q_2 are indeed trace class. By construction, B_U is a basis of eigenvalues of Q_1 and Q_2 with corresponding eigenvalues given by

$$\lambda_{1,k} := \lambda_k^{\alpha_1} \quad \text{and} \quad \lambda_{2,k} := \lambda_k^{\alpha_2}, \quad k \in \mathbb{N},$$

respectively.

8.1.1 Essential m-dissipativity

To determine the coefficient operators K_{12} and K_{21}, we fix $\sigma_1 \in [0, \infty)$ and set $K_{12} := Q^{\sigma_1}$. Since $K_{21} = K_{12}^*$, we also have $K_{21} = Q^{\sigma_1}$. The variable diffusion coefficient operator K_{22} is assumed to be diagonal with respect to B_U and defined by specifying its eigenvalue functions $\lambda_{22,k} : U \to \mathbb{R}$. To do that, let $\sigma_2, \sigma_3 \in [0, \infty)$ and $\beta_k \in (0, 1)$, $\varphi_k \in C_b^1(\mathbb{R}; [0, \infty))$, as well as $\psi_k \in C_b^1(\mathbb{R}^k; [0, \infty))$ for every $k \in \mathbb{N}$.
Define

$$\lambda_{22,k}(v) := \lambda_k^{\alpha_2} + \lambda_k^{\sigma_2} + \gamma_k \left(\varphi_k(|p_k v|^{\beta_k+1}) + \psi_k(p_k v)\right),$$

where

$$\gamma_k := \frac{\lambda_k^{\sigma_3}}{\|\varphi_k\|_{C^1} + \|\psi_k\|_{C^1}}.$$

Remark 8.1. *With appropriate modifications, we could also treat eigenvalue functions of the form* $\lambda_{22,k}(v) := c_1\lambda_k^{\alpha_2} + c_2\lambda_k^{\sigma_2} + c_3\gamma_k \left(\varphi_k(|p_k v|^{\beta_k+1}) + \psi_k(p_k v)\right)$ *for some constants* $c_1, c_2, c_3 \in [0, \infty)$. *To maintain a clear and simple presentation, we do not consider this generalization in the following.*

One easily checks that $\lambda_k^{\alpha_2} \le \lambda_k^{\alpha_2} + \lambda_k^{\sigma_2} \le \lambda_{22,k}(v) = \lambda_{22,k}(P_k v) \le \lambda_k^{\alpha_2} + \lambda_k^{\sigma_2} + \lambda_k^{\sigma_3}$ for every $v \in U$. Moreover, for $i \ge k$, we have $\partial_{d_i}\lambda_{22,k}(v) = 0$ and for $1 \le i \le k$, it holds that

$$\begin{aligned}
|\partial_{d_i}\lambda_{22,k}(v)| &= \gamma_k \left|\varphi_k'(|p_k v|^{\beta_k+1})(\beta_k+1)|p_k v|^{\beta_k-1}(v, d_i)_U + \partial_i\psi(p_k v)\right| \\
&\le \gamma_k(\beta_k+1)(\|\varphi_k'\|_\infty + \|D\psi_k\|_\infty)\left(1 + |p_k v|^{\beta_k}\right) \\
&\le 2\lambda_k^{\sigma_3}\left(1 + \|P_k v\|_U^{\beta_k}\right) \le 2\lambda_k^{\sigma_3}(2 + \|v\|_U)
\end{aligned}$$

for all $v \in U$. Now, we simply set $K_{22}(v)d_k \coloneqq \lambda_{22,k}(v)d_k$, which describes a symmetric positive bounded linear operator on U as required for Definition 5.5. Assumption **K0** holds for $K_{22}^0 = Q^{\alpha_2} = Q_2$ and Assumption **K1** is satisfied for $N_k \coloneqq 2\lambda_k^{\sigma_3}$, compare Remark 5.13.

The nonlinearity in the stochastic reaction diffusion equation comes from a potential Φ, which we construct in the following. We fix a continuous differentiable function $\phi : \mathbb{R} \to \mathbb{R}$, which is bounded from below and with bounded derivative. Further, let $\Phi_2 \in C_b^2(U;\mathbb{R})$. Then, we define

$$\begin{aligned} \Phi_1 : U \to \mathbb{R}, \quad & u \mapsto \Phi_1(u) \coloneqq \int_0^1 \phi(u(\xi))\,\mathrm{d}\xi \quad \text{and} \\ \Phi : U \to \mathbb{R}, \quad & u \mapsto \Phi(u) \coloneqq \Phi_1(u) + \Phi_2(u). \end{aligned}$$

The boundedness of ϕ' implies that ϕ grows at most linear. By Remark 3.49 and Proposition 3.51, we know that Φ_1 is lower semicontinuous, bounded from below and in $L^p(U;\mu_1)$ for all $p \in [1,\infty)$. In addition, Φ is in $W^{1,2}(U;\mu_1)$ with $D\Phi(u) = \phi' \circ u + D\Phi_2(u)$ for all $u \in U$. In particular, we have

$$\|D\Phi\|_{L^\infty(\mu_1)} \leq \sup_{t\in\mathbb{R}} |\phi'(t)| + \|D\Phi_2\|_{L^\infty(\mu_1)} < \infty.$$

The corresponding infinite dimensional Langevin operator reads on $\mathcal{F}C_b^\infty(B_W)$ as

$$\begin{aligned} L^\Phi f = \operatorname{tr}\left[K_{22} \circ D_2^2 f\right] &+ \sum_{j=1}^\infty (\partial_{d_j} K_{22} D_2 f, d_j)_U - (v, (-\partial_\xi^2)^{\alpha_2} K_{22} D_2 f)_U \\ &- (u, (-\partial_\xi^2)^{\alpha_1-\sigma_1} D_2 f)_U + (v, (-\partial_\xi^2)^{\alpha_2-\sigma_1} D_1 f)_U \\ &- (\phi'(u), (-\partial_\xi^2)^{-\sigma_1} D_2 f)_U - (D\Phi_2(u), (-\partial_\xi^2)^{-\sigma_1} D_2 f)_U. \end{aligned}$$

If we assume

$$\sigma_2 \leq 2\sigma_1,$$

we obtain for each $v = \sum_{k=1}^n (v,d_k)_U d_k \in U_n$, $n \in \mathbb{N}$ and $\tilde{v} \in U$,

$$\begin{aligned} (K_{21}v, K_{21}v)_U &= \sum_{k=1}^n \left((k\pi)^{-2}\right)^{2\sigma_1} (v,d_k)_U^2 \leq \sum_{k=1}^n \left((k\pi)^{-2}\right)^{\sigma_2} (v,d_k)_U^2 \\ &\leq \sum_{k=1}^n \lambda_{22,k}(\tilde{v})(v,d_k)_U^2 = (K_{22}(\tilde{v})v, v)_U. \end{aligned}$$

Hence, Assumption $\mathrm{Bd}_\theta(\Phi)$ is valid for $\theta = 0$. Theorem 5.23 is consequently applicable and we obtain essential m-dissipativity on $L^2(W;\mu^\Phi)$ of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$. The semigroup $(T_t)_{t\geq 0}$ generated by $(L^\Phi, D(L^\Phi))$ is sub-Markovian and conservative.

8.1.2 Hypocoercivity

To show that the semigroup generated by the closure of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is hypocoercive, we strengthen our assumptions. Indeed, by means of Theorem 6.21, we need to check **K2**–**K5** with either **K3** or **K3***, as well as Assumption $\mathrm{Reg}(\Phi)$ and $\mathrm{SA}(\Phi)$.

Assumption **K2** is obviously valid. Using the first Item of Remark 6.17 and the fact that $\lambda_{22,k}(v) \geq \lambda_k^{\alpha_2}$ for all $k \in \mathbb{N}$ and $v \in V$, also Assumption **K4** is valid.
To continue, we assume that ϕ is convex (hence also Φ_1) and

$$2\sigma_1 - \alpha_2 \leq \alpha_1.$$

In order to apply Theorem 6.21, we distinguish two cases.

1.Case.

$$\sigma_2 \geq \alpha_2 \quad \text{and} \quad \sigma_3 \geq \frac{3}{2}\alpha_2.$$

This choice of parameters particularly implies that $C = Q^{2\sigma_1-\alpha_2}$ is bounded, as we already assume $\sigma_2 \leq 2\sigma_1$. Therefore, Assumption SA(Φ) holds true. We verify Assumption Reg(Φ) by means of the Moreau-Yosida approximation, using Example 2.11, Lemma 3.42 and Remark 3.52, as well as Item (ii) from Remark 6.6.
Invoking the second Item of Remark 6.17 and the inequality $2\sigma_1 - \alpha_2 \leq \alpha_1$, also Assumption **K5** is valid.
Finally, Item (i) and (ii) from assumption **K3** hold for

$$C_1 = 1 \quad \text{and} \quad C_2(v) = 1 \quad v \in U,$$

by choosing the natural decomposition for K_{22} into K_1 and K_2 induced by its definition. At this point, it is important that $\sigma_2 \geq \alpha_2$ and $\sigma_3 \geq \frac{3}{2}\alpha_2$. For Item (iii), note that

$$\alpha_n^{22}(v) \leq 2(2 + \|v\|_U)\lambda_n^{\sigma_3 - \frac{\alpha_2}{2}},$$

for all $v \in U$, which describes an ℓ^2-sequence, since

$$\sigma_3 - \frac{\alpha_2}{2} \geq \alpha_2 > \frac{1}{2}.$$

Moreover,

$$\int_U \|(\alpha_n^{22}(v))_{n\in\mathbb{N}}\|_{\ell^2}^2 \,\mu_2(\mathrm{d}v) \leq 4\|(\lambda_n^{\sigma_3 - \frac{\alpha_2}{2}})_{n\in\mathbb{N}}\|_{\ell^2}^2 \int_U (2 + \|v\|_U)^2 \,\mu_2(\mathrm{d}v) < \infty.$$

2.Case.

$$\Phi_2 = 0 \text{ and } \Phi \text{ is scaled such that } \|Q_1^{\frac{1}{2}} D\Phi\|_{L^\infty(\mu_1)} < \frac{1}{2}.$$

The verification of Assumption SA(Φ) is immediate by the assumption above, while Reg(Φ) follows again by the means of the Moreau-Yosida approximation as in the first case. Also Assumption **K5** follows as in the first case. To include cases where $(C, D(C))$ might be unbounded, we verify Assumption **K3*** instead of Assumption **K3**. To do that, let

$$-\alpha_2 + 2\sigma_2 - 2\sigma_1 + \alpha_1 \geq 0 \quad \text{and} \quad -2\alpha_2 + 2\sigma_3 - 2\sigma_1 + \alpha_1 \geq 0.$$

Item (i) and (ii) from Assumption **K3*** consequently hold for

$$C_1 = 1 \quad \text{and} \quad C_2(v) = 1,$$

again, by taking the natural decomposition for K_{22} and recalling that $2\sigma_1 - \alpha_2 \leq \alpha_1$. For Item (iii), note that

$$\alpha_n^{22}(v) \leq 2(2 + \|v\|_U)\lambda_n^{\frac{\alpha_1}{2} - \sigma_1 + \sigma_3},$$

for all $v \in U$, which describes an ℓ^2-sequence, since

$$\frac{\alpha_1}{2} - \sigma_1 + \sigma_3 \geq \alpha_2 > \frac{1}{2}.$$

Finally,

$$\int_U \|(\alpha_n^{22}(v))_{n\in\mathbb{N}}\|_{\ell^2}^2 \, \mu_2(\mathrm{d}v) \leq 4\|(\lambda_n^{\frac{\alpha_1}{2} - \sigma_1 + \sigma_3})_{n\in\mathbb{N}}\|_{\ell^2}^2 \int_U (2 + \|v\|_U)^2 \, \mu_2(\mathrm{d}v) < \infty.$$

8.1.3 The process

Below, we assume that $\sigma_2 \leq 2\sigma_1$. Therefore, Theorem 5.23 is applicable and $(T_t)_{t\geq 0}$ generated by $(L^\Phi, D(L^\Phi))$ is sub-Markovian and conservative.
By Proposition 7.1, there exists a right process with the Lusin topological space $(W_1, \mathcal{T}_1)$ as state space, such that its transition semigroup coincide on $L^2(W_1; \bar{\mu}^\Phi)$ with $(T_t)_{t\geq 0}$. Moreover, this process solves the martingale problem for $(L^\Phi, D(L^\Phi))$ under $\bar{P}_{\bar{\mu}^\Phi}$. We emphasize that $K_{22} : U \to \mathcal{L}(U)$ is not finitely based and $\sigma_2 = 0$ is a valid choice. In this case, the variable diffusion matrix K_{22} is not trace class valued.
To show that the there is a μ^Φ-invariant Hunt process

$$\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, Y_t)_{t\geq 0}, (P_w)_{w\in W}),$$

solving the martingale problem for $(L^\Phi, D(L^\Phi))$ under P_{μ^Φ} and with P_{μ^Φ}-a.s. weakly continuous paths and infinite life-time, we invoke Remark 7.4 and assume

$$-\frac{\alpha_1}{2} + \sigma_1 + \frac{\alpha_2}{2} > \frac{1}{2}, \quad -\frac{\alpha_2}{2} + \sigma_1 + \frac{\alpha_1}{2} > \frac{1}{2}, \quad \sigma_2 > \frac{1}{2} \quad \text{and} \quad \sigma_3 > \frac{1}{2},$$

to verify Assumption **K6**. Next, we construct a stochastically and analytically weak solution with weakly continuous paths, in the sense of Theorem 7.11, to the following degenerate second order in time stochastic reaction-diffusion equation

$$\begin{aligned} \mathrm{d}X_t &= (-\partial_\xi^2)^{-\sigma_1+\alpha_2} Y_t \, \mathrm{d}t \\ \mathrm{d}Y_t &= \sum_{i=1}^\infty \partial_{d_i} K_{22}(Y_t) d_i - K_{22}(Y_t)(-\partial_\xi^2)^{\alpha_2} Y_t - (-\partial_\xi^2)^{-\sigma_1+\alpha_1} X_t \\ &\quad - (-\partial_\xi^2)^{-\sigma_1} \phi'(X_t) - (-\partial_\xi^2)^{-\sigma_1} D\Phi_2(X_t) \, \mathrm{d}t + \sqrt{2K_{22}(Y_t)} \, \mathrm{d}W_t. \end{aligned} \tag{8.1}$$

By means of Theorem 7.11, it is left to verify Assumption **K7**. This is redundant, since we already assume $\sigma_2, \sigma_3 > \frac{1}{2}$.

8.1.4 Summary

The table below summarizes the results we established in the previous sections. It includes the combinations of parameters and conditions on the potential such that $(L^{\Phi}, D(L^{\Phi}))$ is m-dissipative on $L^2(W;\mu^{\Phi})$ and the μ^{Φ}-invariant Hunt process $\mathbf{M}$ provides a stochastically and analytically weak solution with $P_{\mu^{\Phi}}$-a.s. weakly continuous paths and infinite life-time for the infinite dimensional stochastic differential equation Equation (8.1). It also tells us when the semigroup $(T_t)_{t\geq 0}$ generated by $(L^{\Phi}, D(L^{\Phi}))$ is hypocoercive.

Table 8.1: degenerate second order in time stochastic reaction-diffusion equation

M-dissipativity and right process solving the Martingale problem (enlarged state space)		
$\sigma_2 \leq 2\sigma_1$ and ϕ' bounded		
μ^{Φ}-invariant Hunt process M with infinite life-time weak sol., weakly cont. paths	$(T_t)_{t\geq 0}$ **hypocoercive**	
$\pm\frac{\alpha_1}{2} + \sigma_1 \mp \frac{\alpha_2}{2} > \frac{1}{2}$	ϕ is convex and $2\sigma_1 - \alpha_2 \leq \alpha_1$	
$\sigma_2, \sigma_3 > \frac{1}{2}$	$\sigma_2 \geq \alpha_2$, $\sigma_3 \geq \frac{3}{2}\alpha_2$	$\Phi_2 = 0$, $\Vert Q_1^{\frac{1}{2}} D\Phi\Vert_{L^{\infty}(\mu_1)} < \frac{1}{2}$
		$-\alpha_2 + 2\sigma_2 - 2\sigma_1 + \alpha_1 \geq 0$
		$-2\alpha_2 + 2\sigma_3 - 2\sigma_1 + \alpha_1 \geq 0$

By means of Corollary 7.12, we can combine the results stated in the table to verify that $\mathbf{M}$ is L^2-exponentially ergodic, compare e.g. the next example. Other situations can be considered by adjusting the parameters accordingly.

Example 8.2. Here we describe two sets of parameters, corresponding to the two cases in Section 8.1.2, such that all of the conditions in the table above are fulfilled. In both cases we assume that ϕ is convex and has bounded derivative.

1. **Case.** Let $\alpha_1, \alpha_2 > \frac{1}{2}$ and set $\sigma_1 = \frac{\alpha_1+\alpha_2}{2}$, $\sigma_2 = \alpha_2$, as well as $\sigma_2 = \frac{3}{2}\alpha_2$.
2. **Case.** Let $\alpha_2 > 3$, $\alpha_1 = 2 + \frac{\alpha_2}{2}$, $\sigma_1 = \frac{\alpha_2}{4}$, $\sigma_2 = \frac{\alpha_2}{2}$, $\sigma_3 = \alpha_2$, $\Phi_2 = 0$ and ϕ is scaled such that $\Vert Q_1^{\frac{1}{2}} D\Phi\Vert_{L^{\infty}(\mu_1)} < \frac{1}{2}$. In this case $(C, D(C))$ is an unbounded operator.

8.2 Degenerate second order in time stochastic Cahn-Hilliard equations with multiplicative noise (potentials with bounded gradient)

In this section, we denote by $W^{1,2}(0,1)$ the classical Sobolev space of weakly differentiable functions and by $\tilde{W}^{1,2}(0,1)$ the functions in $W^{1,2}(0,1)$ with zero mean, i.e.

$$\tilde{W}^{1,2}(0,1) := \left\{ x \in W^{1,2}(0,1) \mid \int_0^1 x(\xi)\,\mathrm{d}\xi = 0 \right\}.$$

$\tilde{W}^{1,2}(0,1)$ becomes a real separable Hilbert space by equipping it with the inner product $(\cdot,\cdot)_{\tilde{W}^{1,2}}$ defined by

$$(x,y)_{\tilde{W}^{1,2}} := \int_0^1 \partial_\xi x(\xi)\partial_\xi y(\xi)\,\mathrm{d}\xi, \quad x,y \in \tilde{W}^{1,2}.$$

Let U be the continuous dual space of $(\tilde{W}^{1,2}(0,1), (\cdot,\cdot)_{\tilde{W}^{1,2}})$, endowed with the canonical dual inner product and norm. Further, set $W = U \times U$ and for $p \in [1,\infty)$,

$$\tilde{L}^p((0,1);\mathrm{d}\xi) = \left\{ x \in L^p((0,1);\mathrm{d}\xi) \mid \int_0^1 x(\xi)\,\mathrm{d}\xi = 0 \right\}.$$

In the following, we consider $\tilde{L}^p((0,1);\mathrm{d}\xi)$ as a subspace of U by identifying an element $x \in \tilde{L}^p((0,1);\mathrm{d}\xi)$ with the continuous linear functional $y \mapsto \int_0^1 x(\xi)y(\xi)\,\mathrm{d}\xi$ in U. We define the map

$$B : \tilde{W}^{1,2}(0,1) \to (\tilde{W}^{1,2}(0,1))', \quad Bx(y) = \int_0^1 \partial_\xi x(\xi)\partial_\xi y(\xi)\,\mathrm{d}\xi, \quad y \in \tilde{W}^{1,2}(0,1).$$

For every $x \in \left\{ x \in W^{2,2}(0,1) \cap \tilde{W}^{1,2}(0,1) \mid \partial_\xi x \in W_0^{1,2}(0,1) \right\}$, i.e. x is two times weakly differentiable with Neumann boundary conditions, we have

$$Bx(y) = -\int_0^1 \partial_\xi^2 x(\xi) y(\xi)\,\mathrm{d}\xi, \quad y \in \tilde{W}^{1,2}(0,1). \tag{8.2}$$

Hence, B can be identified with the extension of minus the second order derivative with Neumann boundary conditions. One can verify that B is isometric and fulfills

$$(z, Bx)_U = (z,x)_{L^2(\mathrm{d}\xi)} \tag{8.3}$$

for all $z \in \tilde{L}^2((0,1);\mathrm{d}\xi)$ and $x \in \tilde{W}^{1,2}(0,1)$. It is well known that the sequence $(e_k)_{k\in\mathbb{N}} = (\sqrt{2}\cos(k\pi\cdot))_{k\in\mathbb{N}}$ is an orthonormal basis of $\tilde{L}^2((0,1);\mathrm{d}\xi)$ with $Be_k = (k\pi)^2 e_k$. Therefore, $(d_k)_{k\in\mathbb{N}}$ defined by $d_k = k\pi e_k$ is an orthonormal basis of U. Now define

$$D(B^2) := \left\{ x \in W^{4,2}(0,1) \cap \tilde{W}^{1,2}(0,1) \mid \partial_\xi x, \partial_\xi^3 x \in W_0^{1,2}(0,1) \right\} \subseteq U$$
$$B^2 x := \partial_\xi^4 x \in U.$$

We can interpret $(B^2, D(B^2))$ as a realization of the fourth order derivative with zero boundary conditions for the first and third order derivative. Moreover, it is easy to show that $(B^2, D(B^2))$ is symmetric with

$$(B^2 x, x)_U \geq \pi^4 (x, x)_U \quad \text{for all} \quad x \in \operatorname{span}\{d_1, d_1, \dots\} \subseteq D(B^2).$$

Therefore, $(B^2)^{-1} \in \mathcal{L}(\operatorname{span}\{d_1, d_1, \dots\}; U)$. Since $(d_k)_{k\in\mathbb{N}}$ is an orthonormal basis of U, we can extend $(B^2)^{-1}$ to a positive self-adjoint operator in $\mathcal{L}(U)$. We denote this extension by $Q \in \mathcal{L}(U)$. As $Qd_k = (\pi k)^{-4} d_k$ for all $k \in \mathbb{N}$, it is evident that the orthonormal basis $B_U := (d_k)_{k\in\mathbb{N}}$ is a basis of eigenvectors of Q, with corresponding eigenvalues $(\lambda_k)_{k\in\mathbb{N}} = ((\pi k)^{-4})_{k\in\mathbb{N}}$. We fix $\alpha_1, \alpha_2 \in \mathbb{R}$ with

$$\alpha_1, \alpha_2 > \frac{1}{4}.$$

Since the sequence of eigenvalues of Q is in $l^r(\mathbb{N})$ for all $r > \frac{1}{4}$, it is reasonable to consider

$$Q_1 = Q^{\alpha_1} \quad \text{and} \quad Q_2 = Q^{\alpha_2}$$

as covariance operators for the infinite dimensional Gaussian measures μ_1 and μ_2, respectively. In analogy to the previous section,

$$\lambda_{1,k} := \lambda_k^{\alpha_1} \qquad \text{and} \quad \lambda_{2,k} := \lambda_k^{\alpha_2}, \quad k \in \mathbb{N}$$

are the eigenvalues of Q_1 and Q_2, respectively.

8.2.1 Essential m-dissipativity

As in the reaction-diffusion setting, we choose $K_{12} = Q^{\sigma_1}$ for some $\sigma_1 \in [0, \infty)$. Since $K_{21} = K_{12}^*$, also $K_{21} = Q^{\sigma_1}$. We assume K_{22} is diagonal with respect to B_U and therefore specified by its eigenvalue functions $\lambda_{22,k} : U \to \mathbb{R}$. Fix $\sigma_2, \sigma_3 \in [0, \infty)$, $\beta_k \in (0, 1)$, $\varphi_k \in C_b^1(\mathbb{R}; [0, \infty))$ and $\psi_k \in C_b^1(\mathbb{R}^k; [0, \infty))$ for all $k \in \mathbb{N}$. Define $v \mapsto \lambda_{22,k}(v)$ and $v \mapsto K_{22}(v)$ as in the previous example such that the requirements from Definition 5.5 are met and note that we could also incorporate the generalization mentioned in Remark 8.1. Then, by the exact same reasoning as in the reaction-diffusion setting, Assumption **K0** holds for $K_{22}^0 = Q^{\alpha_2}$ and Assumption **K1** is satisfied for $N_k := 2\lambda_k^{\sigma_3}$.
We continue our consideration by fixing a function $\phi \in C^1(\mathbb{R})$, which is bounded from below and assume that there are constants $A \in (0, \infty)$ and $b \in [1, \infty)$ such that

$$|\phi'(x)| \leq A(1 + |x|^{b-1}), \quad x \in \mathbb{R}\,.$$

Hence, ϕ and its derivative grow at most of order b and $b - 1$, respectively. For such ϕ we consider potentials $\Phi_1 : U \to (-\infty, \infty]$ defined by

$$\Phi_1(u) = \begin{cases} \int_0^1 \phi(u(\xi))\,\mathrm{d}\xi, & u \in \tilde{L}^b((0,1); \mathrm{d}\xi), \\ \infty, & u \notin \tilde{L}^b((0,1); \mathrm{d}\xi). \end{cases}$$

Before we investigate the Sobolev regularity of Φ_1, we need some auxiliary results from [DA14, Section 6]. Afterwards, we include Lemma 8.4, which contains a refinement of

[DA14, Proposition 6.5], where only the case $\tilde{\theta} = \frac{1}{2}$ is considered. In contrast to the previous example, in general, $\Phi \notin W^{1,2}(U;\mu_1)$. Instead, we show that $W^{1,2}_{Q^{\tilde{\theta}}}(U;\mu_1)$ for all $\tilde{\theta} > \frac{3}{8}$.

Lemma 8.3. *[DA14, Proposition 6.3 and Corollary 6.4] For all $p \in [1,\infty)$ there is a constant $C_p \in (0,\infty)$ such that*

$$\int_U \int_0^1 |P_n u(\xi)|^p \,\mathrm{d}\xi \,\mathrm{d}\mu_1 \leq C_p \left(\sum_{i=1}^n \frac{1}{(\pi k)^2} \right)^{\frac{p}{2}}$$

and $\mu_1(\tilde{L}^p((0,1);\mathrm{d}\xi)) = 1$. Moreover, the sequence $((u,\xi) \mapsto P_n u(\xi))_{n\in\mathbb{N}}$ converges to $(u,\xi) \mapsto u(\xi)$ in $L^p(U \times (0,1); \mu \otimes \mathrm{d}\xi)$.

The statement below is close to the one from Proposition 3.51. The goal is to find a good approximation of Φ_1. This is achieved by taking an appropriate subsequence of the sequence $((\Phi_1)_n)_{n\in\mathbb{N}}$ where the latter is defined for each $n \in \mathbb{N}$ by

$$(\Phi_1)_n(u) = \int_0^1 \phi(P_n u(\xi)) \,\mathrm{d}\xi, \quad u \in U.$$

Even in the case that ϕ is convex, a Moreau-Yosida approximation of Φ_1 is not applicable, as in general $\Phi_1 \notin W^{1,2}(U;\mu_1)$.

Lemma 8.4. *Let $p \in [1,\infty)$, then it holds $\lim_{n\to\infty} \Phi_n = \Phi$ in $L^p(U;\mu_1)$. If $p > 1$, then $\Phi_1 \in W^{1,p}_{Q^{\tilde{\theta}}}(U;\mu_1)$ for all $\tilde{\theta} \in (\frac{3}{8},\infty)$ and for μ_1-a.e. $u \in U$*

$$\partial_{d_k}(\Phi_1)(u) = \int_0^1 \phi'(u(\xi)) d_k(\xi) \,\mathrm{d}\xi.$$

Moreover, if $b = 1$ (i.e. if ϕ' is bounded), we additionally have

$$\|Q^{\tilde{\theta}} D\Phi_1\|_{L^\infty(\mu_1)} < \infty.$$

Proof. As in Proposition 3.51, we can show that $((\Phi_1)_n)_{n\in\mathbb{N}}$ is a sequence of continuously differentiable functions from U to $\mathbb{R}$ converging to Φ_1 in $L^p(U;\mu_1)$ for all $p \in [1,\infty)$. By dropping to a subsequence, we assume without loss of generality that $((\Phi_1)_n)_{n\in\mathbb{N}}$ converges to Φ_1 pointwisely μ_1-a.e..
To verify that $\Phi_1 \in W^{1,2}_{Q^{\tilde{\theta}}}(U;\mu_1)$, we first show that $((\Phi_1)_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}_{Q^{\tilde{\theta}}}(U;\mu_1)$ for all $\tilde{\theta} \in (\frac{3}{8},\infty)$.
Boundedness of $((\Phi_1)_n)_{n\in\mathbb{N}}$ in $L^p(U;\mu_1)$ for all $p \in [1,\infty)$ follows by Lemma 8.3 and the polynomial growth of ϕ. As in [DA14, Proposition 6.5], we obtain

$$\partial_{d_k}(\Phi_1)_n(u) = \begin{cases} \int_0^1 \phi'(P_n u(\xi)) d_k(\xi) \,\mathrm{d}\xi & k \leq n \\ 0 & k > n \end{cases}.$$

Hence, for $k \leq n$

$$\begin{aligned} |\partial_{d_k}(\Phi_1)_n(u)| &= |\int_0^1 \phi'(P_n u(\xi)) d_k(\xi) \,\mathrm{d}\xi| \leq A\sqrt{2}\pi k \int_0^1 (1 + |P_n u(\xi)|^{b-1}) \,\mathrm{d}\xi \\ &= A\sqrt{2}\pi k \|1 + |P_n u|^{b-1}\|_{L^1(\mathrm{d}\xi)}. \end{aligned}$$

This yields $(\Phi)_n \in W^{1,2}_{Q^{\tilde{\theta}}}(U;\mu_1)$ by Proposition 3.35 and

$$\begin{aligned}\|Q^{\tilde{\theta}} D(\Phi_1)_n(u)\|_U^p &= \left(\|Q^{\tilde{\theta}} D(\Phi_1)_n(u)\|_U^2\right)^{\frac{p}{2}} \\ &= \left(\sum_{k=1}^n \frac{1}{(\pi k)^{8\tilde{\theta}}} |\partial_{d_k}(\Phi_1)_n(u)|^2\right)^{\frac{p}{2}} \\ &\leq \left(\sqrt{2}A\|1+|P_n u|^{b-1}\|_{L^1(\mathrm{d}\xi)}\right)^p \left(\sum_{k=1}^n \frac{(\pi k)^2}{(\pi k)^{8\tilde{\theta}}}\right)^{\frac{p}{2}} \\ &= \left(\sqrt{2}A\|1+|P_n u|^{b-1}\|_{L^1(\mathrm{d}\xi)}\right)^p \left(\sum_{k=1}^\infty \frac{1}{(\pi k)^{8\tilde{\theta}-2}}\right)^{\frac{p}{2}}.\end{aligned}$$

By Proposition 8.3, we get boundedness of $(Q^{\tilde{\theta}} D(\Phi_1)_n)_{n\in\mathbb{N}}$ in $L^p(U;\mu_1)$ and therefore boundedness of $((\Phi_1)_n)_{n\in\mathbb{N}}$ in $W^{1,p}_{Q^{\tilde{\theta}}}(U;\mu_1)$, as desired. For $p>1$, we use the Banach-Saks property of $W^{1,p}_{Q^{\tilde{\theta}}}(U;\mu_1)$, compare Remark 3.41, to find a subsequence $(n_k)_{k\in\mathbb{N}}$ such that the Cesaro mean $\psi_N := \frac{1}{N}\sum_{k=1}^N (\Phi_1)_{n_k}$ converges to Φ_1 in $W^{1,p}_{Q^{\tilde{\theta}}}(U;\mu_1)$. Hence, $\Phi_1 \in W^{1,p}_{Q^{\tilde{\theta}}}(U;\mu_1)$. As $((\Phi_1)_n)_{n\in\mathbb{N}}$ converges pointwisely μ_1-a.e. to Φ_1, the same holds true for $(\psi_N)_{N\in\mathbb{N}}$. To show the statement about the partial derivatives, we can argue as in Proposition 3.51, compare also [DA14, Proposition 6.5].
To end the proof, let $b=1$. This implies that $Q^{\tilde{\theta}} D(\Phi_1)_{n_k}$ is bounded in U independent of k. Consequently, the same holds true for $Q^{\tilde{\theta}} D\psi_N$. Since $Q^{\tilde{\theta}} D\psi_N$ converges pointwisely μ_1-a.e. to $Q^{\tilde{\theta}} D\Phi_1$ for a subsequence, we are done. □

Remark 8.5.

(i) As in [DA14, Section 6], one can show that for every $u \in \tilde{L}^{2(b-1)}((0,1);\mathrm{d}\xi)$ we have

$$\partial_{d_k}\Phi_1(u) = \lambda_k^{-\frac{1}{2}}\left(\phi'\circ u - \int_0^1 \phi'(u(\xi))\,\mathrm{d}\xi, d_k\right)_U, \text{ hence also}$$
$$Q^{\frac{1}{2}} D\Phi_1(u) = \phi'\circ u - \int_0^1 \phi'(u(\xi))\,\mathrm{d}\xi.$$

(ii) Let $n\in\mathbb{N}$ with $m^K(n)=n$ and $f\in \mathcal{F}C_b^\infty(B_W,n)$ be given. Using the interpretation from Remark 5.6, we get for $u\in U$ with $\phi'\circ u \in \tilde{W}^{1,2}(0,1)$

$$\begin{aligned}-(\partial_\xi^2\phi'(u),(\partial_\xi^4)^{-\sigma_1}D_2 f(u))_U &= (\phi'(u),(\partial_\xi^4)^{-\sigma_1}D_2 f(u))_{L^2(\mathrm{d}\xi)} \\ &= \sum_{i=1}^n \partial_{d_i}\Phi_1(u)\lambda_i^{\sigma_1}\partial_{d_i} f(u) \\ &= (D\Phi_1(u),(\partial_\xi^4)^{-\sigma_1}D_2 f(u))_U.\end{aligned}$$

Above, we also used equality (8.3) and the identification of B with the extension of minus the second order derivative with Neumann boundary conditions. From *this point on, we write* $-(\partial_\xi^2\phi'(u), (\partial_\xi^4)^{-\sigma_1} D_2 f(u))_U = (D\Phi_1(u), (\partial_\xi^4)^{-\sigma_1} D_2 f(u))_U$ *for* $f \in \mathcal{F}C_b^\infty(B_W)$ *and* $u \in U$*, even though we can only make sense of it, if* $\phi' \circ u \in \tilde{W}^{1,2}(0,1)$.

As in the previous example, we add $\Phi_2 \in C_b^2(U;\mathbb{R})$ and consider potentials of the form

$$\Phi := \Phi_1 + \Phi_2.$$

The infinite dimensional Langevin operator, considered in this Cahn-Hilliard setting, acts on $\mathcal{F}C_b^\infty(B_W)$ as follows,

$$\begin{aligned} L^\Phi f = \operatorname{tr}\left[K_{22} \circ D_2^2 f\right] &+ \sum_{j=1}^\infty (\partial_{d_j} K_{22} D_2 f, d_j)_U - (v, (\partial_\xi^4)^{\alpha_2} K_{22} D_2 f)_U \\ &- (u, (\partial_\xi^4)^{\alpha_1-\sigma_1} D_2 f)_U + (v, (\partial_\xi^4)^{\alpha_2-\sigma_1} D_1 f)_U \\ &+ (\partial_\xi^2 \phi'(u), (\partial_\xi^4)^{-\sigma_1} D_2 f)_U - (D\Phi_2(u), (\partial_\xi^4)^{-\sigma_1} D_2 f)_U. \end{aligned}$$

In order to establish essential m-dissipativity of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$, we assume that ϕ' is bounded. Moreover, we fix $\theta \in (0,\infty)$ with $\theta\alpha_1 \in (\frac{3}{8},\infty)$ and choose the parameters σ_1 and σ_2 such that

$$\sigma_2 \le -2\theta\alpha_1 + 2\sigma_1. \tag{8.4}$$

Assumption $\mathrm{Bd}_\theta(\Phi)$ consequently holds true and Theorem 5.23 can be applied. Therefore, $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ is essentially m-dissipativity on $L^2(W;\mu^\Phi)$. In particular, the associated semigroup $(T_t)_{t\ge0}$ is sub-Markovian and conservative.

8.2.2 Hypocoercivity

In this section we study hypocoercivity, of the semigroup $(T_t)_{t\ge0}$ generated by the closure of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$. To do that we assume that we are in the situation of the previous section, i.e. ϕ' is bounded $\theta \in (0,\infty)$ with $\theta\alpha_1 \in (\frac{3}{8},\infty)$ and $\sigma_2 \le -2\theta\alpha_1 + 2\sigma_1$. We use the same strategy as in Section 8.1.2, where we checked the assumptions from Chapter 6 and particularly from Theorem 6.21.
Note that Assumption **K2** is obviously valid. The next lemma shows that convexity of ϕ is enough to verify Item $\mathrm{Reg}(\Phi_1)$ of Assumption $\mathrm{Reg}(\Phi)$.

Lemma 8.6. *Suppose that* ϕ *is convex, then Item* $\mathrm{Reg}(\Phi_1)$ *of Assumption* $\mathrm{Reg}(\Phi)$ *is satisfied.*

Proof. Recall the sequence $(\psi_N)_{N\in\mathbb{N}}$, given by $\psi_N = \frac{1}{N}\sum_{k=1}^N (\Phi_1)_{n_k}$ from Lemma 8.4, converging to Φ in $W^{1,2}_{Q_1^\theta}(U;\mu_1)$ and pointwisely μ_1-a.e. to Φ_1. Since ϕ is convex, the same holds true for Φ_1 and each member of the sequence $((\Phi_1)_n)_{n\in\mathbb{N}}$. As $(\Phi_1)_n$ is a continuously differentiable function from U to $\mathbb{R}$ and bounded from below for every $n \in \mathbb{N}$, the same is true for $(\psi_N)_{N\in\mathbb{N}}$. Let $(t_M)_{M\in\mathbb{N}} \subseteq (0,\infty)$ be a sequence converging to zero. Denote by $\psi_{(M,N)}$ the Moreau-Yosida approximation of ψ_N, compare Lemma 3.42, of order t_M. Hence, $\psi_{M,N}$ is a convex function from U to $\mathbb{R}$ such that

(i) For all $u \in U$ and $N, M \in \mathbb{N}$, $-\infty < \inf_{u\in U}\Phi(u) \le \inf_{u\in U}\psi_N(u) \le \psi_{M,N}(u) \le \psi_N(u)$, as well as $\lim_{M\to\infty}\psi_{M,N}(u) = \psi_N(u)$.

(ii) $\psi_{M,N}$ is Fréchet-differentiable with Lipschitz continuous gradient.

(iii) $\lim_{M\to\infty}\|(Q_1^\theta D\psi_{M,N} - Q_1^\theta D\psi_N, d_i)_U\|_{L^2(\mu_1)} = 0$ for all $i \in \mathbb{N}$.

Using the approximation properties of $(\psi_N)_{N\in\mathbb{N}}$, we obtain Reg$(\Phi_1)$. □

In analogy to the previous section, we obtain Item Reg(Φ_2), if either $C \in \mathcal{L}(U)$ or $\Phi_2 = 0$. Consequently, Assumption **K4** and **K5** are valid, if

$$2\sigma_1 - \alpha_2 \leq \alpha_1,$$

by Remark 6.17. In this case, we are left to verify Assumption SA(Φ) and either Assumption **K3** or Assumption **K3***, to obtain the final hypocoercivity result. Similar to the reaction-diffusion setting, we distinguish two major cases. One with bounded C, which is implied by assuming $\sigma_2 \geq \alpha_2$ and one with potentially unbounded C but $\Phi_2 = 0$.

1.case.

$$\sigma_2 \geq \alpha_2 \quad \text{and} \quad \sigma_3 \geq \frac{3}{2}\alpha_2.$$

Then, $C = Q^{2\sigma_1-\alpha_2}$ is bounded, as we already assume $\sigma_2 \leq -2\theta\alpha_1 + 2\sigma_1$. Assumption SA(Φ) follows by verifying Item SA(Φ2), if

$$2\sigma_1 - \alpha_2 > \frac{3}{4}.$$

Item (i)-(iii) from Assumption **K3** are valid by the exact same reasoning as in the stochastic reaction diffusion case. Note that the potential Φ, which is the major difference in the two examples, is not involved in Assumption **K3**.

2.case.

$$\alpha_1 > \frac{3}{4},\ \Phi_2 = 0 \text{ and } \Phi \text{ is scaled such that } \|Q_1^{\frac{1}{2}} D\Phi\|_{L^\infty(\mu_1)} < \frac{1}{2}.$$

In this case, Assumption SA(Φ) directly follows. Assumption **K3*** can be verified as in the previous example by demanding

$$-\alpha_2 + 2\sigma_2 - 2\sigma_1 + \alpha_1 \geq 0 \quad \text{and} \quad -2\alpha_2 + 2\sigma_3 - 2\sigma_1 + \alpha_1 \geq 0.$$

8.2.3 The process

Also here we choose the parameters as in Section 8.2.1 to guarantee that $(L^\Phi, D(L^\Phi))$ is m-dissipative. By the same arguments as in Section 8.1.3, there exists a right process with enlarged state space providing a solution to the martingale for $(L^\Phi, D(L^\Phi))$ problem with respect to the equilibrium measure.
To establish existence of a μ^Φ-invariant Hunt process

$$\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, Y_t)_{t\geq 0}, (P_w)_{w\in W}),$$

solving the martingale problem for $(L^\Phi, D(L^\Phi))$ under P_{μ^Φ} and with P_{μ^Φ}-a.s. weakly continuous paths and infinite life-time, we invoke Remark 7.4 and assume

$$-\frac{\alpha_1}{2}+\sigma_1+\frac{\alpha_2}{2}>\frac{1}{4},\quad -\frac{\alpha_2}{2}+\sigma_1+\frac{\alpha_1}{2}>\frac{1}{4}\quad \text{and}\quad \sigma_2,\sigma_3>\frac{1}{4}$$

to verify all items from Assumption **K6**.
Since we already assume that $\sigma_2, \sigma_3 > \frac{1}{4}$, Assumption **K7** holds true and we get a stochastically and analytically weak solution, as explained in Theorem 7.11, for the following degenerate second order in time Cahn-Hilliard type equation

$$\begin{aligned} \mathrm{d}X_t &= (\partial_\xi^4)^{-\sigma_1+\alpha_2} Y_t \,\mathrm{d}t \\ \mathrm{d}Y_t &= \sum_{i=1}^{\infty} \partial_{d_i} K_{22}(Y_t) d_i - K_{22}(Y_t)(\partial_\xi^4)^{\alpha_2} Y_t - (\partial_\xi^4)^{-\sigma_1+\alpha_1} X_t \\ &\quad - (\partial_\xi^4)^{-\sigma_1}\partial_\xi^2 \phi'(X_t) - (\partial_\xi^4)^{-\sigma_1} D\Phi_2(X_t)\mathrm{d}t + \sqrt{2K_{22}(Y_t)}\,\mathrm{d}W_t. \end{aligned} \tag{8.5}$$

8.2.4 Summary

We summarize the results from the previous section in the following table. It has the same structure as the one for the reaction-diffusion setting in Section 8.1.4.

Table 8.2: degenerate second order in time stochastic Cahn-Hilliard equation

M-dissipativity and right process solving the Martingale problem (enlarged state space)		
$\theta > 0$ such that $\theta\alpha_1 > \frac{3}{8}$, $\sigma_2 \leq -2\theta\alpha_1 + 2\sigma_1$ and ϕ' bounded		
μ^Φ-invariant Hunt process M with infinite life-time weak sol., weakly cont. paths	$(T_t)_{t\geq 0}$ **hypocoercive**	
$\pm\frac{\alpha_1}{2} + \sigma_1 \mp \frac{\alpha_2}{2} > \frac{1}{4}$	ϕ is convex and $2\sigma_1 - \alpha_2 \leq \alpha_1$	
$\sigma_2, \sigma_3 > \frac{1}{4}$	$\sigma_2 \geq \alpha_2$, $\sigma_3 \geq \frac{3}{2}\alpha_2$	$\Phi_2 = 0$, $\alpha_1 > \frac{3}{4}$, $\Vert Q_1^{\frac{1}{2}} D\Phi \Vert_{L^\infty(\mu_1)} < \frac{1}{2}$
	$2\sigma_1 - \alpha_2 > \frac{3}{4}$	$-\alpha_2 + 2\sigma_2 - 2\sigma_1 + \alpha_1 \geq 0$
		$-2\alpha_2 + 2\sigma_3 - 2\sigma_1 + \alpha_1 \geq 0$

Example 8.7. As explained in Corollary 7.12 and in the stochastic reaction-diffusion setting in Example 8.2, this example describes two sets of parameters such that all statements in the table above are satisfied and consequently such that **M** is L^2-exponentially ergodic. Many other combinations are possible. In both cases we assume that ϕ is convex with bounded derivative.

1. Case. Let $\alpha_2 > \frac{1}{2}$, $\alpha_1 > \frac{3}{4}$, $\theta = \frac{1}{2}$ and set $\sigma_1 = \frac{\alpha_1+\alpha_2}{2}$, $\sigma_2 = \alpha_2$, as well as $\sigma_3 = \frac{3}{2}\alpha_2$.

2. Case. Let $\alpha_2 > 2$, $\alpha_1 = \frac{4}{3}\alpha_2$, $\theta = \frac{10}{32\alpha_2}$ and set $\sigma_1 = \frac{\alpha_2}{3}$, $\sigma_2 = \frac{\alpha_2}{6}$, $\sigma_3 = \alpha_2$, $\Phi_2 = 0$ and ϕ is scaled such that $\|Q_1^{\frac{1}{2}} D\Phi\|_{L^\infty(\mu_1)} < \frac{1}{2}$. In this case $(C, D(C))$ is an unbounded operator.

8.3 Degenerate second order in time stochastic reaction-diffusion equations with multiplicative noise (potentials with unbounded gradient)

In this section, we again analyze degenerate second order in time stochastic reaction-diffusion equations with multiplicative noise, whereby different to Section 8.1, the gradient of the potential might be unbounded. Instead of the results from Section 5.1.1, we use the techniques described in Section 5.1.2 to show essential m-dissipativity of the corresponding infinite dimensional Langevin operator.
We have to note, that Item App(Φ3) from Assumption App(Φ), which is needed to apply the central essential m-dissipativity result from Section 5.1.2, is not shown in this section. Indeed, as explained in Section 5.1.2 it is considered as a conjecture, whose validity is reasonable by the strategy described in Remark 5.28. To be consistent with this strategy, we derive stronger regularity results for the potential and coefficients than required in Assumption App(Φ).
Nevertheless, our starting point is the same as in the introduction of Section 8.1. So, $U = V = L^2((0,1); \mathrm{d}\xi)$, $B_U = (d_k)_{k\in\mathbb{N}} = (\sqrt{2}\sin(k\pi\cdot))_{k\in\mathbb{N}}$, $(\lambda_k)_{k\in\mathbb{N}} = (k\pi^{-2})_{k\in\mathbb{N}}$, $Q = (-\partial_\xi)^{-1} \in \mathcal{L}(U)$ and $Q_i = Q^{\alpha_i}$ with $\alpha_i > \frac{1}{2}$, $i = 1, 2$.

8.3.1 Essential m-dissipativity

For $\sigma_1 \in [0, \infty)$, we choose $K_{12} = Q^{\sigma_1}$ and since $K_{21} = K_{12}^*$, also $K_{21} = Q^{\sigma_1}$. Moreover, we assume that K_{22} is diagonal with respect to B_U and therefore determined by its eigenvalue functions $\lambda_{22,k} : U \to \mathbb{R}$, $k \in \mathbb{N}$. Let $\sigma_2, \sigma_3 \in [0, \infty)$. For each $k \in \mathbb{N}$, choose $\psi_k \in C_c^4(\mathbb{R}^k; [0, \infty))$ and define

$$\lambda_{22,k}(v) := \lambda_k^{\alpha_2} + \lambda_k^{\sigma_2} + \lambda_k^{\sigma_3} \frac{\psi_k(p_k v)}{\|\psi_k\|_{C^4}}. \tag{8.6}$$

Also in this example it is possible to consider a generalized version of $\lambda_{22,k}$ as described in Remark 8.1.
One can check that

$$\lambda_k^{\alpha_2}, \lambda_k^{\sigma_2} \leq \lambda_{22,k}(v) = \lambda_{22,k}(P_k v) \leq \lambda_k^{\alpha_2} + \lambda_k^{\sigma_2} + \lambda_k^{\sigma_3}$$

for all $k \in \mathbb{N}$ and $v \in U$. For $i \geq k$ and all $v \in U$, we have $\partial_{d_i}\lambda_{22,k}(v) = 0$ and for $1 \leq i \leq k$, it holds that

$$|\partial_{d_i}\lambda_{22,k}(v)| = \left|\lambda_k^{\sigma_3} \frac{\partial_i\psi_k(p_k v)}{\|\psi_k\|_{C^4}}\right| \leq \lambda_k^{\sigma_3}.$$

We simply set $K_{22}(v)d_k := \lambda_{22,k}(v)d_k$, which describes a symmetric positive and bounded linear operator on U, as required for Definition 5.5. Extending the arguments from above to higher order derivatives, we see that $v \mapsto K_{22}(v)d_k \in C_b^4(V;V)$ for all $k \in \mathbb{N}$ and also Assumption **K0** holds true. The compact support property of ψ_k implies that $v \mapsto K_{22}(P_n(v))Q_2^{-1}P_n(v)$ has bounded derivatives up to order three for all $n \in \mathbb{N}$, which is essential to use the arguments from Remark 5.28. Actually, to check Item App(Φ2) from Assumption App(Φ) it is enough to have $v \mapsto K_{22}(v)d_k \in C_b^2(V;V)$ for all $k \in \mathbb{N}$, i.e. it is enough to assume that $\psi_k \in C_b^2(\mathbb{R}^k, [0,\infty))$ for the definition of $\lambda_{22,k}$.
The class of potentials we consider below, is inspired by the considerations in [DL05], where the m-dissipativity of degenerate Langevin operators with additive noise, in a finite dimensional setting, were investigated.

Definition 8.8. Fix $\phi \in C^4(\mathbb{R})$, which is bounded from below by zero. Assume that there are constants $A, \bar{B}, R, m_0 \in (0,\infty)$ and $m_1 \in \mathbb{N}_{\geq 4}$ such that

$$\phi(x) \geq A|x|^{m_0} \quad \text{for all} \quad |x| \geq R \tag{8.7}$$

and

$$|\phi^{(4)}(x)| \leq \bar{B}(1 + |x|^{m_1-4}) \quad \text{for all} \quad x \in \mathbb{R}\,.$$

Using the mean value theorem, there is a constant $B \in (0,\infty)$ such that for all $x \in \mathbb{R}$ and $j \in \{0,1,2,3,4\}$

$$|\phi^{(j)}(x)| \leq B(1 + |x|^{m_1-j}). \tag{8.8}$$

The potential $\Phi : L^2((0,1);\mathrm{d}\xi) \to \mathbb{R}$ is defined in terms of ϕ via

$$\Phi(u) = \begin{cases} \int_0^1 \phi(u(\xi))\,\mathrm{d}\xi, & u \in L^{m_1}((0,1);\mathrm{d}\xi) \\ \infty, & \textit{else} \end{cases}.$$

Let $q \in \mathbb{N}$ be even and $(\alpha_m)_{m\in\mathbb{N}} \subseteq (0,\infty)$ be a monotone sequence converging to zero. For $m \in \mathbb{N}$, we set

$$\Psi_m := \Psi_{m,q} : \mathbb{R} \to \mathbb{R}, \quad \Psi_m(x) := \frac{x}{1+\alpha_m x^q} \quad \text{and} \quad \phi_m := \Psi_m \circ \phi \in C^4(\mathbb{R}).$$

We start investigating $(\phi_m)_{m\in\mathbb{N}}$ by establishing that all derivatives up to order four are polynomial bounded independent of the index m. This helps to approximate Φ, as required in Assumption App(Φ).

Lemma 8.9. *There exists a constant $q \in \mathbb{N}$ only dependent on ϕ such that $\phi_m^{(j)}$ is bounded for all $j \in \{1,2,3,4\}$ and there is a constant $\tilde{B} \in \mathbb{N}$ with*

$$|\phi_m^{(j)}(x)| \leq \tilde{B}(1 + |x|^{j(m_1-1)}) \quad \textit{for all} \quad j \in \{1,2,3,4\} \quad \textit{and} \quad m \in \mathbb{N}\,. \tag{8.9}$$

Proof. We calculate for all $m \in \mathbb{N}$

$$\begin{aligned}
\phi_m' &= \Psi_m'(\phi)\phi', \quad \phi_m'' = \Psi_m''(\phi)(\phi')^2 + \Psi_m'(\phi)\phi'', \\
\phi_m''' &= \Psi_m'''(\phi)(\phi')^3 + 3\Psi_m''(\phi)\phi'\phi'' + \Psi_m'(\phi)\phi''', \\
\phi_m'''' &= \Psi_m''''(\phi)(\phi')^4 + 6\Psi_m'''(\phi)(\phi')^2\phi'' + 3\Psi_m''(\phi)(\phi'')^2 + 4\Psi_m''(\phi)\phi'\phi''' + \Psi_m'(\phi)\phi'''',
\end{aligned}$$

and for all $x \in \mathbb{R}$

$$\Psi_m'(x) = \frac{1-\alpha_m(q-1)x^q}{(1+\alpha_m x^q)^2}, \quad \Psi_m''(x) = \frac{\alpha_m q x^{q-1}\left(-\alpha_m x^q + q(\alpha_m x^q - 1) - 1\right)}{(1+\alpha_m x^q)^3}$$
$$\Psi_m'''(x) = -\frac{x^{q-2}(q(\alpha_m x^q+1)^2 - q^3(\alpha_m x^q(\alpha_m x^q - 4) + 1))}{(\alpha_m x^q + 1)^4},$$
$$\Psi_m''''(x) = \alpha_m q x^{q-3}\frac{\alpha_m^3(q-1)(q+1)(q+2)x^{3q} - \alpha_m^2(11q^3+6q^2+q+6)x^{2q}}{(\alpha_m x^q+1)^5},$$
$$+\alpha_m q x^{q-3}\frac{\alpha_m(q-1)(q(11q+5)+6)x^q - q^3 + 2q^2 + q - 2}{(\alpha_m x^q+1)^5}.$$

Recall that we assume that $q \in \mathbb{N}$ is even. We show the claim exemplary for $j = 1$. The other cases follow similarly, using the calculations from above. First, we verify that Ψ_m' is bounded independent of m. This follows by

$$\left|\frac{1-\alpha_m(q-1)x^q}{(1+\alpha_m x^q)^2}\right| \leq \frac{1}{1+\alpha_m x^q} + q\frac{\alpha_m x^q}{(1+\alpha_m x^q)^2} \leq (q+1)\frac{1}{1+\alpha_m x^q} \leq (q+1).$$

Using Inequality (8.8) and the estimate right above, we obtain Inequality (8.9) for $j = 1$. To show that ϕ_m' is bounded, we proceed as follows. Let $x \in \mathbb{R}$, then, by means of Inequality (8.7), we can estimate

$$|\phi_m'(x)| \leq (q+1)\left|\frac{\phi'(x)}{1+\alpha_m\phi(x)^q}\right| \leq (q+1)\begin{cases}\frac{\bar{B}(1+|x|^{m_1-1})}{1+A^q\alpha_m|x|^{qm_0}} & \text{for} \quad |x| > R\\ \sup_{|x|\leq R}|\phi'(x|) & \text{for} \quad |x| \leq R.\end{cases}$$

Therefore, boundedness of ϕ_m' follows for $q > \frac{m_1-1}{m_0}$. □

For the rest of this section, we assume that $q \in \mathbb{N}$ is as in Lemma 8.9.

Definition 8.10. For $n, m \in \mathbb{N}$, we define $\Phi_n : U \to \mathbb{R}$ and $\Phi_n^m : U \to \mathbb{R}$ by

$$\Phi_n(u) := \int_0^1 \phi(P_n u(\xi))\,\mathrm{d}\xi \quad \text{and} \quad \Phi_n^m(u) := \int_0^1 \phi_m(P_n u(\xi))\,\mathrm{d}\xi.$$

It is evident that $(\Phi_n^m)_{n,m\in\mathbb{N}}$ fulfills App(Φ1) from Assumption App(Φ).

Lemma 8.11. *For all $r \geq 1$, it holds*

$$\lim_{n\to\infty}\Phi_n = \Phi \textit{ in } \quad L^r(U;\mu_1), \ \lim_{n\to\infty}\mu_1(e^{-\Phi_n}) = \mu_1(e^{-\Phi}) \textit{ and } 0 < \inf_{n,m\in\mathbb{N}}\mu_1(e^{-\Phi_n^m}) \leq 1.$$

Moreover, the measures $\mu_1^{\Phi_n^m}$ is uniformly dominated by μ_1, i.e. for all non-negative measurable functions f and $n, m \in \mathbb{N}$ it holds

$$\int_U f\,\mathrm{d}\mu_1^{\Phi_n^m} \leq \frac{1}{\inf_{n,m\in\mathbb{N}}\mu_1(e^{-\Phi_n^m})}\int_U f\,\mathrm{d}\mu_1.$$

Proof. The first claim follows by Proposition 3.51. The second claim immediately follows by the first and the mean value theorem, since $\Phi, \Phi_n \geq 0$ for all $n \in \mathbb{N}$ and the derivative of $[0, \infty) \ni x \mapsto e^{-x} \in \mathbb{R}$ is bounded by 1.
By definition, it holds $0 \leq \phi_m \leq \phi$ and therefore $0 \leq \Phi_n^m \leq \Phi_n$. Hence, $0 \leq \mu_1(e^{-\Phi_n}) \leq \mu_1(e^{-\Phi_n^m}) \leq 1$.
As $\lim_{n\to\infty} \mu_1(e^{-\Phi_n}) = \mu_1(e^{-\Phi}) > 0$, we know that the sequence $(\mu_1(e^{-\Phi_n}))_{n\in\mathbb{N}}$ is bounded from below by a positive constant and therefore the third statement is shown. Finally the last one follows, noting that $e^{-\Phi_n^m} \leq 1$ for all $m, n \in \mathbb{N}$. ☐

Lemma 8.12. *For all $m, n \in \mathbb{N}$, it holds $\Phi_n, \Phi_n^m \in C^4(U; \mathbb{R})$ and for $i, j, k, l \in \{1, ..., n\}$ we have*

$$\partial_{d_i} \Phi_n(u) = \int_0^1 \phi'(P_n u) d_i \,\mathrm{d}\xi, \quad \partial_{d_j}\partial_{d_i} \Phi_n(u) = \int_0^1 \phi''(P_n u) d_i d_j \,\mathrm{d}\xi$$

$$\partial_{d_k}\partial_{d_j}\partial_{d_i} \Phi_n(u) = \int_0^1 \phi'''(P_n u) d_i d_j d_k \,\mathrm{d}\xi, \quad \partial_{d_l}\partial_{d_k}\partial_{d_j}\partial_{d_i} \Phi_n(u) = \int_0^1 \phi''''(P_n u) d_i d_j d_k d_l \,\mathrm{d}\xi,$$

$$\text{and} \quad \partial_{d_i} \Phi_n^m(u) = \int_0^1 \phi_m'(P_n u) d_i \,\mathrm{d}\xi, \quad \partial_{d_j}\partial_{d_i} \Phi_n^m(u) = \int_0^1 \phi_m''(P_n u) d_i d_j \,\mathrm{d}\xi$$

$$\partial_{d_k}\partial_{d_j}\partial_{d_i} \Phi_n^m(u) = \int_0^1 \phi_m'''(P_n u) d_i d_j d_k \,\mathrm{d}\xi \quad \partial_{d_l}\partial_{d_k}\partial_{d_j}\partial_{d_i} \Phi_n^m(u) = \int_0^1 \phi_m''''(P_n u) d_i d_j d_k d_l \,\mathrm{d}\xi.$$

The partial derivatives evaluate to zero if one of the indices exceeds n. Furthermore, we have $D\Phi_n^m \in C_b^3(U; \mathbb{R})$ and consequently we know that Item App(Φ2) from Assumption App(Φ) is valid.

Proof. The calculation of the partial derivatives follows as in Proposition 3.51. The proof of Proposition 3.51 also contains the arguments to show $\Phi_n, \Phi_n^m \in C_b^4(U; \mathbb{R})$. Note that the main ingredients are Inequality (8.9) and Lemma 3.50. ☐

To verify Item App(Φ2) from Assumption App(Φ) it is enough that $\Phi_n^m \in C^3(U; \mathbb{R})$ such that $D\Phi_n^m$ has bounded derivatives up to the second order is enough. However, the stronger regularity statement from 8.12 shows that we are consistent with the strategy described in Remark 5.28.
We are now able to verify that there are constants $\alpha, \beta, \gamma \in [0, \infty)$ such that Item App(Φ4) from Assumption App(Φ) is valid.

Proposition 8.13. *Suppose that $\sigma_3 \geq \frac{\min\{\sigma_2, \alpha_2\}}{2}$, $2\sigma_1 - \min\{\sigma_2, \alpha_2\} \geq \frac{1}{2}$ and*

$$\alpha > \frac{1}{2\alpha_1} + \frac{1}{2} \quad \text{and} \quad \alpha \geq 2\left(\frac{\frac{\min\{\sigma_2,\alpha_2\}}{2} - \sigma_1}{\alpha_1} + 1\right)$$

$$\beta > \frac{1}{2\alpha_2} + \frac{\min\{\sigma_2, \alpha_2\}}{\alpha_2} \quad \text{and} \quad \beta \geq \max\left\{2\left(1 - \frac{\min\{\sigma_2, \alpha_2\}}{2\alpha_2}\right), 2\left(1 - \frac{\sigma_3}{2\alpha_2}\right)\right\}$$

$$\gamma > \frac{1}{4\alpha_2} + \frac{1}{2} \quad \text{and} \quad (4\gamma - \beta) > \frac{1}{2\alpha_2} + \frac{\min\{\sigma_2, \alpha_2\}}{\alpha_2}.$$

Then, Φ_n^m fulfills App(Φ4) from Assumption App(Φ).

Proof. Before we start verifying the inequalities from App(Φ4), we derive an useful integral estimate for products of the partial derivatives of Φ_n^m. So let $i, j, k \in \mathbb{N}$ be given and recall the constants $\tilde{B}$ and m_1 from Inequality (8.9). Using Lemma 8.12 and Inequality (8.9), we estimate

$$\begin{aligned}\int_U \left|\partial_{d_j}\partial_{d_i}^2\Phi_n^m \partial_{d_j}\partial_{d_k}^2\Phi_n^m\right| \,\mathrm{d}\mu_1^{\Phi_n^m} &\le \left(\int_U \left(\partial_{d_j}\partial_{d_i}^2\Phi_n^m\right)^2 \,\mathrm{d}\mu_1^{\Phi_n^m}\right)^{\frac{1}{2}} \left(\int_U \left(\partial_{d_j}\partial_{d_k}^2\Phi_n^m\right)^2 \,\mathrm{d}\mu_1^{\Phi_n^m}\right)^{\frac{1}{2}} \\ &\le \int_U \left(\sqrt{2}^3\tilde{B}\int_0^1 \left(1+|P_n u(\xi)|^{3(m_1-1)}\right) \mathrm{d}\xi\right)^2 \mathrm{d}\mu_1^{\Phi_n^m} \\ &\le 8\tilde{B}^2 \int_U\int_0^1 \left(1+|P_n u(\xi)|^{3(m_1-1)}\right)^2 \mathrm{d}\xi \,\mathrm{d}\mu_1^{\Phi_n^m}.\end{aligned}$$

In a similar way, one can show

$$\begin{aligned}\int_U (\partial_{d_i}\Phi_n^m)^4 \,\mathrm{d}\mu_1^{\Phi_n^m} &\le 16\tilde{B}^4 \int_U\int_0^1 \left(1+|P_n u(\xi)|^{(m_1-1)}\right)^4 \mathrm{d}\xi \,\mathrm{d}\mu_1^{\Phi_n^m}, \\ \int_U (\partial_{d_j}\partial_{d_i}\Phi_n^m)^2 \,\mathrm{d}\mu_1^{\Phi_n^m} &\le 4\tilde{B}^2 \int_U\int_0^1 \left(1+|P_n u(\xi)|^{2(m_1-1)}\right)^2 \mathrm{d}\xi \,\mathrm{d}\mu_1^{\Phi_n^m} \quad \text{and} \\ \int_U \left(\partial_{d_j}\partial_{d_i}\Phi_n^m\right)^4 \mathrm{d}\mu_1^{\Phi_n^m} &\le 16\tilde{B}^4 \int_U\int_0^1 \left(1+|P_n u(\xi)|^{2(m_1-1)}\right)^4 \mathrm{d}\xi \,\mathrm{d}\mu_1^{\Phi_n^m}.\end{aligned}$$

Using the generalized Hölder inequality ($\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=1$) and the estimates above, we estimate

$$\begin{aligned}&\int_U \left|\partial_{d_j}\partial_{d_i}\Phi_n^m \,\partial_{d_i}\Phi_n^m \partial_{d_j}\partial_{d_k}\Phi_n^m \,\partial_{d_k}\Phi_n^m\right| \,\mathrm{d}\mu_1^{\Phi_n^m} \\ &\le 16\tilde{B}^4 \left(\int_U\int_0^1 \left(1+|P_n u(\xi)|^{2(m_1-1)}\right)^4 \mathrm{d}\xi \,\mathrm{d}\mu_1^{\Phi_n^m}\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_U\int_0^1 \left(1+|P_n u(\xi)|^{(m_1-1)}\right)^4 \mathrm{d}\xi \,\mathrm{d}\mu_1^{\Phi_n^m}\right)^{\frac{1}{2}}.\end{aligned}$$

Combing the estimates we just derived, the measure dominance from Lemma 8.11 and the results from Lemma 3.50, we know that there is a constant $C \in (0,\infty)$, independent of i, j, k, such that for all $m, n \in \mathbb{N}$

$$\begin{aligned}&\int_U \left|\partial_{d_j}\partial_{d_i}^2\Phi_n^m \partial_{d_j}\partial_{d_k}^2\Phi_n^m\right| + (\partial_{d_j}\partial_{d_i}\Phi_n^m)^2 \\ &\quad + \left(\partial_{d_j}\partial_{d_i}\Phi_n^m\right)^4 + \left|\partial_{d_j}\partial_{d_i}\Phi_n^m \,\partial_{d_i}\Phi_n^m\partial_{d_j}\partial_{d_k}\Phi_n^m \,\partial_{d_k}\Phi_n^m\right| \mathrm{d}\mu_1^{\Phi_n^m} \le C. \qquad (8.10)\end{aligned}$$

Recalling Inequality (8.6), we are able to estimate, by means of Inequality (8.10),

$$\begin{aligned}
&\int_W \Big\| \sum_{i=1}^{\infty} \lambda_{1,i}^{\alpha} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}^2 \Phi_n^m \Big\|_U^2 \, \mathrm{d}\mu^{\Phi_n^m} \\
&= \sum_{j=1}^{\infty} \int_W \left(\sum_{i=1}^{\infty} \lambda_i^{\alpha\alpha_1} \lambda_{22,j}^{-\frac{1}{2}} \lambda_j^{\sigma_1} \left(D\partial_{d_i}^2 \Phi_n^m, d_j \right)_V \right)^2 \mathrm{d}\mu^{\Phi_n^m} \\
&= \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1} \int_V \lambda_{22,j}^{-1} \, \mathrm{d}\mu_2 \sum_{i,k=1}^{\infty} \lambda_i^{\alpha\alpha_1} \lambda_k^{\alpha\alpha_1} \int_U \partial_{d_j} \partial_{d_i}^2 \Phi_n^m \partial_{d_j} \partial_{d_k}^2 \Phi_n^m \, \mathrm{d}\mu_1^{\Phi_n^m} \\
&\leq C \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1} \int_V \lambda_{22,j}^{-1} \, \mathrm{d}\mu_2 \left(\sum_{i=1}^{\infty} \lambda_i^{\alpha\alpha_1} \right)^2 \\
&\leq C \left(\sum_{i=1}^{\infty} \lambda_i^{\alpha\alpha_1} \right)^2 \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1 - \min\{\sigma_2, \alpha_2\}} =: \kappa_1.
\end{aligned}$$

Note that $\kappa_1 < \infty$, as we assume $2\sigma_1 - \min\{\sigma_2, \alpha_2\} > \frac{1}{2}$ and $\alpha\alpha_1 > \frac{1}{2}$ is implied by $\alpha > \frac{1}{2\alpha_1} + \frac{1}{2}$. Similar arguments yield

$$\begin{aligned}
\sum_{i=1}^{\infty} \int_W \big\| \lambda_{1,i}^{\frac{\alpha}{2}} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i} \Phi_n^m \big\|_V^2 \, \mathrm{d}\mu^{\Phi_n^m} &= \sum_{i,j=1}^{\infty} \lambda_j^{2\sigma_1} \lambda_{1,i}^{\alpha} \int_V \lambda_{22,j}^{-1} \, \mathrm{d}\mu_2 \int_U (\partial_{d_j} \partial_{d_i} \Phi_n^m)^2 \, \mathrm{d}\mu_1^{\Phi_n^m} \\
&\leq C \sum_{i=1}^{\infty} \lambda_i^{\alpha\alpha_1} \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1 - \min\{\sigma_2, \alpha_2\}} =: \kappa_2 < \infty.
\end{aligned}$$

Furthermore, we can derive, using the generalized Hölder inequality and Lemma 3.5

$$\begin{aligned}
&\int_W \Big\| \sum_{i=1}^{\infty} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i} \Phi_n^m \, (u, Q_1^{\alpha-1} d_i)_U \Big\|_V^2 \, \mathrm{d}\mu^{\Phi_n^m} \\
&\leq \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1} \int_V \lambda_{22,j}^{-1} \, \mathrm{d}\mu_2 \sum_{i,k=1}^{\infty} \lambda_i^{\alpha_1(\alpha-1)} \lambda_k^{\alpha_1(\alpha-1)} \int_U \partial_{d_j} \partial_{d_i} \Phi_n^m \, (u, d_i)_U \partial_{d_j} \partial_{d_k} \Phi_n^m \, (u, d_k)_U \mathrm{d}\mu_1^{\Phi_n^m} \\
&\leq C \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1 - \min\{\sigma_2, \alpha_2\}} \sum_{i,k=1}^{\infty} \lambda_i^{\alpha_1(\alpha-1)} \lambda_k^{\alpha_1(\alpha-1)} \left(\int_U (u, d_i)_U^4 \, \mathrm{d}\mu_1^{\Phi_n^m} \right)^{\frac{1}{4}} \left(\int_U (u, d_k)_U^4 \, \mathrm{d}\mu_1^{\Phi_n^m} \right)^{\frac{1}{4}} \\
&\leq \sqrt{3} \tilde{c} C \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1 - \min\{\sigma_2, \alpha_2\}} \left(\sum_{i=1}^{\infty} \lambda_i^{\alpha_1(\alpha - \frac{1}{2})} \right)^2 =: \kappa_3.
\end{aligned}$$

Above the constants $\tilde{c}$, independent of $m, n \in \mathbb{N}$, exists by uniform dominance of the measures $\mu_1^{\Phi_n^m}$ by μ_1, which is due to Lemma 8.11 and the results from Lemma 3.5. Moreover, $\kappa_3 < \infty$, since $\alpha > \frac{1}{2\alpha_1} + \frac{1}{2}$ is equivalent to $\alpha_1(\alpha - \frac{1}{2}) > \frac{1}{2}$. To find $\kappa \in (1, \infty)$

such that the Inequalities (5.8) and (5.9) from App(Φ3) are valid, we continue to estimate

$$\begin{aligned}
&\int_W \Big\| \sum_{i=1}^{\infty} K_{22}^{-\frac{1}{2}} K_{12} D\partial_{d_i}\Phi\, \lambda_{1,i}^{\alpha}\partial_{d_i}\Phi \Big\|_V^2 \,\mathrm{d}\mu^{\Phi_n^m} \\
&\le \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1} \int_V \lambda_{22,j}^{-1}\,\mathrm{d}\mu_2 \sum_{i,k=1}^{\infty} \lambda_i^{\alpha_1\alpha}\lambda_k^{\alpha_1\alpha} \int_U \partial_{d_j}\partial_{d_i}\Phi_n^m\,\partial_{d_i}\Phi_n^m \partial_{d_j}\partial_{d_k}\Phi_n^m\,\partial_{d_k}\Phi_n^m \mathrm{d}\mu_1^{\Phi_n^m} \\
&\le C \sum_{j=1}^{\infty} \lambda_j^{2\sigma_1-\min\{\sigma_2,\alpha_2\}} \left(\sum_{i=1}^{\infty} \lambda_i^{\alpha_1\alpha}\right)^2 = \kappa_1 < \infty.
\end{aligned}$$

Now let $f \in \mathcal{F}C_b^{\infty}(B_U)$ be arbitrary. To show Inequality (5.10) from Assumption App(Φ), let $k \in \mathbb{N}$ and assume $\alpha \ge 2\left(\frac{\frac{\min\{\sigma_2,\alpha_2\}}{2}-\sigma_1}{\alpha_1}+1\right)$ or equivalently $\alpha_1\left(\frac{\alpha}{2}-1\right)+\sigma_1 \ge \frac{\min\{\sigma_2,\alpha_2\}}{2}$ to estimate for all $v \in V$

$$\lambda_k^{\alpha_1(\frac{\alpha}{2}-1)+\sigma_1} \le \lambda_k^{\frac{\min\{\sigma_2,\alpha_2\}}{2}} \le \lambda_{22,k}^{\frac{1}{2}}(v).$$

Using the fact that $(d_k)_{k\in\mathbb{N}}$ is an orthonormal basis, results in

$$\big\|Q_1^{\frac{\alpha}{2}-1} K_{21} D_2 f\big\|_V \le \big\|K_{22}^{\frac{1}{2}} D_2 f\big\|_V.$$

Suppose $\beta \ge \max\left\{2(1-\frac{\min\{\sigma_2,\alpha_2\}}{2\alpha_2}), 2(1-\frac{\sigma_3}{2\alpha_2})\right\}$, which is equivalent to $\frac{\beta}{2} \ge \frac{1}{2}$, $\sigma_2 + \alpha_2(\frac{\beta}{2}-1) \ge \frac{\sigma_2}{2}$ and $\sigma_3 + \alpha_2(\frac{\beta}{2}-1) \ge \frac{\sigma_3}{2}$. We obtain for all $v \in V$

$$\begin{aligned}
\lambda_k^{\alpha_2(\frac{\beta}{2}-1)}\lambda_{22,k}(v) &\le \lambda_k^{\alpha_2\frac{\beta}{2}} + \lambda_k^{\sigma_2+\alpha_2(\frac{\beta}{2}-1)} + \lambda_k^{\sigma_3+\alpha_2(\frac{\beta}{2}-1)} \frac{\psi_k(p_k v)}{\|\psi_k\|_{C^4}} \\
&\le \lambda_k^{\frac{\alpha_2}{2}} + \lambda_k^{\frac{\sigma_2}{2}} + \lambda_k^{\frac{\sigma_3}{2}} \frac{\psi_k(p_k v)}{\|\psi_k\|_{C^4}} \\
&\le \sqrt{3}\lambda_{22,k}^{\frac{1}{2}}(v).
\end{aligned}$$

Hence, as above,

$$\big\|Q_2^{\frac{\beta}{2}-1} K_{22} D_2 f\big\|_V \le \sqrt{3}\big\|K_{22}^{\frac{1}{2}} D_2 f\big\|_V.$$

To establish Inequality (5.11), we estimate for all $v \in V$

$$\begin{aligned}
\sum_{i=1}^{\infty} \big\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{-\frac{1}{2}}(v)\partial_{d_i} K_{22}(v) d_k\big\|_V^2 &= \sum_{i=1}^{\infty} \lambda_{2,i}^{\beta}\lambda_{22,k}^{-1}(v)(\partial_{d_i}\lambda_{22,k}(v))^2 \\
&\le \sum_{i=1}^{\infty} \lambda_i^{\alpha_2\beta-\min\{\sigma_2,\alpha_2\}}\lambda_k^{2\sigma_3} \\
&\le \kappa_4 \lambda_k^{2\sigma_3},
\end{aligned}$$

where $\kappa_4 := \sum_{i=1}^{\infty} \lambda_i^{\alpha_2\beta-\min\{\sigma_2,\alpha_2\}} < \infty$ as $\beta > \frac{1}{2\alpha_2} + \frac{\min\{\sigma_2,\alpha_2\}}{\alpha_2}$ by assumption. Since we assume $\sigma_3 \ge \frac{\min\{\sigma_2,\alpha_2\}}{2}$, we can derive

$$\sum_{i=1}^{\infty} \big\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{-\frac{1}{2}} \partial_{d_i} K_{22} D_2 g\big\|_V^2 \le \kappa_4 \big\|K_{22}^{\frac{1}{2}} D_2 g\big\|_V^2.$$

Inequality (5.12) is valid for $\alpha_2(\frac{\beta}{2}-1)+\sigma_1 \geq \alpha_1\frac{\alpha}{2}$. In order to verify Inequality (5.13), we use that $(4\gamma-\beta) > \frac{1}{2\alpha_2} + \frac{\min\{\sigma_2,\alpha_2\}}{\alpha_2}$ implies $\sum_{i=1}^{\infty} \lambda_i^{\alpha_2(4\gamma-\beta)-\min\{\sigma_2,\alpha_2\}} < \infty$ and the following estimate

$$\begin{aligned}
&\left(\sum_{i=1}^{\infty}\left(\int_W \lambda_{2,i}^{4\gamma}(\partial_{d_i}^2 g)^2 \,\mathrm{d}\mu^{\Phi_n^m}\right)^{\frac{1}{2}}\right)^2 \\
&\leq \sum_{i=1}^{\infty} \lambda_i^{\alpha_2(4\gamma-\beta)-\min\{\sigma_2,\alpha_2\}} \sum_{i=1}^{\infty} \int_W \lambda_{2,i}^{\beta}\lambda_i^{\min\{\sigma_2,\alpha_2\}}(\partial_{d_i}^2 g)^2 \,\mathrm{d}\mu^{\Phi_n^m} \\
&\leq \sum_{i=1}^{\infty} \lambda_i^{\alpha_2(4\gamma-\beta)-\min\{\sigma_2,\alpha_2\}} \int_W \sum_{i=1}^{\infty} \left\|\lambda_{2,i}^{\frac{\beta}{2}} K_{22}^{\frac{1}{2}} D_2 \partial_{d_i} g\right\|_V^2 \,\mathrm{d}\mu^{\Phi_n^m}.
\end{aligned}$$

Finally, if $\gamma > \frac{1}{4\alpha_2} + \frac{1}{2}$ or equivalently $\alpha_2(2\gamma-1) > \frac{1}{2}$, we obtain Inequality (5.14) as

$$\begin{aligned}
&\sum_{i=1}^{\infty} \lambda_{2,i}^{2\gamma-2}\left(\int_W (v,d_i)_V^4 \,\mathrm{d}\mu^{\Phi_n^m}\right)^{\frac{1}{2}} \leq \frac{\sum_{i=1}^{\infty}\lambda_i^{\alpha_2(2\gamma-2)}}{\inf_{n,m\in\mathbb{N}}\mu_1(e^{-\Phi_n^m})}\left(\int_W (v,d_i)_V^4 \,\mathrm{d}\mu\right)^{\frac{1}{2}} \\
&= \frac{\sqrt{3}\sum_{i=1}^{\infty}\lambda_i^{\alpha_2(2\gamma-1)}}{\inf_{n,m\in\mathbb{N}}\mu_1(e^{-\Phi_n^m})} =: \kappa_5 < \infty.
\end{aligned}$$

Hence, we choose $\kappa \in (1,\infty)$, in terms of $\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5$ such that App(Φ4) holds true. □

Proposition 8.14. *Let $\sigma_1 \geq \alpha_2\gamma$ and suppose that $\phi(x) = a_1x^2 + \psi(x)$, $x \in \mathbb{R}$, where $a_1 \in [0,\infty)$ and $\psi \in C^4(\mathbb{R};\mathbb{R}_{\geq 0})$ grows less than quadratic and its fourth order derivative is polynomial bounded, i.e. ϕ is as demanded in Definition 8.8. Then, Φ induced by ϕ fulfills Item App(Φ5) from Assumption App(Φ).*

Proof. Let $p^* \in (4,\infty)$ and q^* be as in App(Φ5). By means of Remark 3.52, we know that $D\Phi_n \to D\Phi$ as $n \to \infty$ in $L^{p^*}(U;\mu_1^{\Phi};U)$. Moreover, we have pointwisely $\lim_{m\to\infty}\phi_m' = \phi'$. Using Inequality (8.9) and Lemma 3.50, we conclude, by the theorem of dominated convergence,

$$\lim_{m\to\infty}\int_U \left\|D\Phi_n^m - D\Phi_n\right\|_U^{p^*} \,\mathrm{d}\mu_1^{\Phi} = \lim_{m\to\infty}\int_U \|\phi_m'(P_n(u)) - \phi'(P_n(u))\|_U^{p^*} \,\mu_1^{\Phi}(\mathrm{d}u) = 0.$$

We obtain the desired convergence (5.15) from Item App(Φ4), as $\sigma_1 \geq \alpha_2\gamma$.
Let $m,n \in \mathbb{N}$ be given, then obviously $0 \leq \Phi_n^m$ and inequality (5.16) holds true. So far, we have not used the special structure of ϕ described in the assertion. We need this to verify Inequality (5.17). As ψ grows less than quadratic, there are constants $a_2 \in [0,\infty)$ and $a_3 \in [0,2)$ such that for all $x \in \mathbb{R}$

$$\psi(x) \leq 1 + a_2|x|^{a_3}.$$

Using Youngs inequality, there exists $a_4 \in (0,\infty)$ with

$$\psi(x) \leq a_4 + \frac{1}{q^*4\lambda_{1,1}}x^2 \quad \text{for all} \quad x \in \mathbb{R}.$$

This implies (5.17) from App(Φ4) by

$$\begin{aligned}\Phi_n^m(u) \le \Phi_n(u) &= \int_0^1 a_1 P_n(u)^2 + \psi(P_n(u))\,\mathrm{d}\xi \\ &\le a_1 \sum_{i,j=1}^n (u,d_i)_U(u,d_j)_U(d_i,d_j)_U + a_4 + \frac{1}{q^* 4\lambda_{1,1}} \sum_{i,j=1}^n (u,d_i)_U(u,d_j)_U(d_i,d_j)_U \\ &= a_1 \sum_{i=1}^n (u,d_i)_U^2 + a_4 + \frac{1}{q^* 4\lambda_{1,1}} \sum_{i=1}^n (u,d_i)_U^2 \\ &\le \Phi(u) + a_4 + \frac{1}{q^* 4\lambda_{1,1}} \|u\|_U^2\end{aligned}$$

and therefore

$$(q^*-1)\Phi_n^m(u) \le q^* a_4 + \frac{1}{4\lambda_{1,1}} \|u\|_U^2 + q^* \Phi(u).$$

This ends the proof. □

Remark 8.15. *The special structure of ϕ, described in Proposition 8.14, is only used to verify Inequality (5.17) from App(Φ5). So different situations, in which this inequality is valid, can be imagined, e.g. by perturbing Φ with a suitable finitely based function. For fixed $k \in \mathbb{N}$ and a function $\tilde{\phi}$ with the properties stated in Definition 8.8 we can incorporate perturbations with potentials of type $\tilde{\Phi}_k$ defined as in Definition 8.10.*

As a consequence of the above considerations, we obtain the following result.

Corollary 8.16. *Suppose Item App(Φ3) from Assumption App(Φ) is valid and we are in the setting of Proposition 8.13 and Proposition 8.14. Then Theorem 5.27 is applicable and consequently essential m-dissipativity of $(L^\Phi, \mathcal{F}C_b^\infty(B_W))$ on $L^2(W;\mu^\Phi)$ is established. Additionally, the semigroup generated by $(L^\Phi, D(L^\Phi))$ is sub-Markovian and conservative. For each $f \in \mathcal{F}C_b^\infty(B_W)$ it holds*

$$\begin{aligned}L^\Phi f = \operatorname{tr}\left[K_{22} \circ D_2^2 f\right] &+ \sum_{j=1}^\infty (\partial_{d_j} K_{22} D_2 f, d_j)_U - (v, (-\partial_\xi^2)^{\alpha_2} K_{22} D_2 f)_U \\ &- (u, (-\partial_\xi^2)^{\alpha_1-\sigma_1} D_2 f)_U + (v, (-\partial_\xi^2)^{\alpha_2-\sigma_1} D_1 f)_U - (\phi'(u), (-\partial_\xi^2)^{-\sigma_1} D_2 f)_U\end{aligned}$$

Assuming that Item App(Φ3) from Assumption App(Φ) holds true, there are several situations, where Corollary 8.16 can be applied. We give an example below.

Example 8.17. First choose σ_2 and σ_3 such that $\sigma_3 \ge \frac{\min\{\sigma_2,\alpha_2\}}{2}$. Since α in Proposition 8.13 can be chosen arbitrary large without imposing restrictions to the other parameters the existence of a suitable α is trivial. For our fixed set of parameters σ_2 and σ_3 we first choose β and then γ large enough so that the inequalities involving β and γ from Proposition 8.13 are fulfilled. Finally, we can choose σ_1 such that the missing inequality $2\sigma_1 - \min\{\sigma_2,\alpha_2\} \ge \frac{1}{2}$ from Proposition 8.13 and the inequality $\sigma_1 \ge \alpha_2\gamma$ from Proposition 8.14 is valid. To apply Corollary 8.16, it is left to choose a potential as described in Proposition 8.14 or even more general as in Remark 8.15.

8.3.2 Hypocoercivity

Assume that

$$\sigma_2 \geq \alpha_2, \quad \sigma_3 \geq \frac{3}{2}\alpha_2$$

and the potential Φ is as described in Proposition 8.14. The assumptions above imply that K_{22} has a similar structure as in Section 8.1. In order to show that the semigroup generated $(L^\Phi, D(L^\Phi))$ is hypocoercive, we additionally assume that ϕ is convex (hence also Φ) and

$$2\sigma_1 - \alpha_2 \leq \frac{\alpha_1}{2}.$$

Assumption **K2** is obviously valid. Moreover, $\lambda_{22,k}(v) \geq \lambda_k^{\alpha_2}$ for all $k \in \mathbb{N}$ and $v \in V$, gives us validity of Assumption **K4**. Assumption Reg(Φ) can be checked as in Section 8.1.2, because the boundedness of $D\Phi$ was not required there. Consequently, Assumption **K5** follows by Item (ii) from Remark 6.17 and the fact that $2\sigma_1 - \alpha_2 \leq \frac{\alpha_1}{2} \leq \alpha_1$. **K3** follows as in the first case of Section 8.1.2, recalling that $\sigma_2 \geq \alpha_2$, $\sigma_3 \geq \frac{3}{2}\alpha_2$ and Φ is not involved. Lastly, we check Assumption SA(Φ). This time, we cannot use the boundedness of $D\Phi$. We instead use Item (iii) of Proposition 3.58 and verify SA(Φ3), which is sufficient to check Assumption SA(Φ).
Indeed, let $(B, D(B))$ be the closure of $(-Q_1^{-1}C, \operatorname{span}\{d_1, d_2, \dots\})$, then $(B, D(B))$ is self-adjoint with

$$(Bu, u)_U = (-Q^{-\alpha_1-\alpha_2+2\sigma_1}u, u)_U \leq -\lambda_1^{-\alpha_1-\alpha_2+2\sigma_1}\|u\|_U^2 \text{ for all } u \in \operatorname{span}\{d_1, d_2, \dots\}. \tag{8.11}$$

In the inequality above, we used $-\alpha_1 - \alpha_2 + 2\sigma_1 < 0$, which is true, as we assume $2\sigma_1 - \alpha_2 \leq \frac{\alpha_1}{2}$. By definition of $(B, D(B))$, Inequality (8.11) also holds for $u \in D(B)$. Moreover, $C = (-B)^{-\varepsilon}$ for $\varepsilon := -\frac{2\sigma_1-\alpha_2}{2\sigma_1-\alpha_2-\alpha_1}$ and $\varepsilon \in (0,1)$ using $2\sigma_1 - \alpha_2 > 0$ and $2\sigma_1 - \alpha_2 \leq \frac{\alpha_1}{2}$. Further, $(-B)^{-(1+\varepsilon)} = Q_1 \in \mathcal{L}_1^+(U)$, $e^{-\Phi} \in L^p(U;\mu_1)$ for all $p \in [1,\infty)$ and finally $\Phi \in W^{1,4}_{C^{\frac{1}{2}}}(U;\mu_1^\Phi)$ by Remark 3.52 and Proposition 3.48. Proposition 3.58 is consequently applicable and Item SA(Φ3) follows.

8.3.3 The process

Suppose that Corollary 8.16 is applicable. Then, as in Section 8.1.3 and Section 8.2.3, there exists a right process with enlarged state space providing a solution to the martingale for $(L^\Phi, D(L^\Phi))$ problem with respect to the equilibrium measure. Next, suppose that

$$\sigma_2, \sigma_3 > \frac{1}{2}, \quad -\frac{\alpha_1}{2} + \sigma_1 + \frac{\alpha_2}{2} > \frac{1}{2}, \quad \frac{\alpha_1}{2} + \sigma_1 - \frac{\alpha_2}{2} > \frac{1}{2} \quad \text{and} \quad 2\sigma_1 \geq \alpha_2. \tag{8.12}$$

This implies that there exists a μ^Φ-invariant Hunt process

$$\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, Y_t)_{t\geq 0}, (P_w)_{w\in W}),$$

solving the martingale problem for $(L^\Phi, D(L^\Phi))$ under P_{μ^Φ} and with P_{μ^Φ}-a.s. weakly continuous paths and infinite life-time.
Indeed, Assumption **K7** holds true, as $\lambda_{22,k}(v) \leq \lambda_k^{\alpha_2} + \lambda_k^{\sigma_2} + \lambda_k^{\sigma_3}$ for all $v \in V$. **K6** is checked by means of Remark 7.4 and the assumption on the parameters described in (8.12), whereas $2\sigma_1 \geq \alpha_2$ implies that $\mathrm{Bd}_\theta(\Phi 2)$ is valid with $\theta = 0$, since we already know that $\lambda_k^{\alpha_2} \leq \lambda_{22,k}(v)$ for each $v \in V$. In summary, Theorem 7.11 is applicable and the existence of $\mathbf{M}$ and we obtain a stochastically and analytically weak solution with weakly continuous paths, in the sense of Theorem 7.11, to the degenerate second order in time stochastic reaction-diffusion equation associated to L^Φ, compare also Section 8.1.3.

8.3.4 Summary

The table below summarizes the results we established in the previous sections. It includes the combinations of parameters and conditions on the potential such that $(L^\Phi, D(L^\Phi))$ is m-dissipative on $L^2(W;\mu^\Phi)$, whereas we use as a standing assumption that the potential is as described in Proposition 8.14 or even more general in Remark 8.15. Moreover, recall that we assume the validity of item App(Φ3) from Assumption App(Φ) as a conjecture.

Table 8.3: degenerate second order in time stochastic reaction-diffusion equation (unbounded gradient of the potential)

M-dissipativity and right process solving the Martingale problem (enlarged state space)	
$\sigma_2 \geq 0$, $\sigma_3 \geq \frac{\min\{\sigma_2,\alpha_2\}}{2}$, then $\alpha,\beta,\gamma \geq 0$ according to Example 8.17	
$2\sigma_1 - \min\{\sigma_2,\alpha_2\} \geq \frac{1}{2}$ and $\sigma_1 \geq \alpha_2\gamma$	
μ^Φ-invariant Hunt process M with infinite life-time weak sol., weakly cont. paths	**$(T_t)_{t\geq 0}$ hypocoercive**
$\pm\frac{\alpha_1}{2} + \sigma_1 \mp \frac{\alpha_2}{2} > \frac{1}{2}$	Φ is convex and as in Remark 8.15
$\sigma_2, \sigma_3 > \frac{1}{2}$	$2\sigma_1 - \alpha_2 \leq \frac{\alpha_1}{2}$
$2\sigma_1 \geq \alpha_2$	$\sigma_2 \geq \alpha_2$, $\sigma_3 \geq \frac{3}{2}\alpha_2$

It is important to mention that the combination of parameters for which $(L^\Phi, D(L^\Phi))$ is m-dissipative on $L^2(W;\mu^\Phi)$ might be adapted if we additionally want to ensure the existence a μ^Φ-invariant Hunt process $\mathbf{M}$ providing a stochastically and analytically weak solution with weakly continuous paths and infinite life-time or hypocoercivity of the associated semigroup.

Example 8.18. By using Corollary 7.12, we can combine the results from the table above to verify that $\mathbf{M}$ is L^2-exponentially ergodic. Such a situation is e.g. given by taking a suitable potential and assuming $\alpha_1 > 1$, $-\alpha_1 + 4\alpha_2 > 2$, $\sigma_1 = \frac{\alpha_1}{4} + \frac{\alpha_2}{2}$, $\sigma_2 \geq \alpha_2$, $\sigma_3 \geq \frac{3}{2}\alpha_2$.

In this case we choose α large, $\beta = \frac{1}{2\alpha_2} + 1 + \frac{\alpha_1 - 1}{16\alpha_2}$ and $\gamma = \frac{1}{4\alpha_2} + \frac{1}{2} + \frac{\alpha_1 - 1}{8\alpha_2}$ to verify the inequalities in Proposition 8.13 and Proposition 8.14.

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