On Center Cycles in Grid Graphs

Horst W. Hamacher *
Anita Schöbel †
‡

December 14, 1998

Abstract

Finding "good" cycles in graphs is a problem of great interest in graph theory as well as in locational analysis. We show that the center and median problems are NP hard in general graphs. This result holds both for the variable cardinality case (i.e. all cycles of the graph are considered) and the fixed cardinality case (i.e. only cycles with a given cardinality p are feasible). Hence it is of interest to investigate special cases where the problem is solvable in polynomial time.

In grid graphs, the variable cardinality case is, for instance, trivially solvable if the shape of the cycle can be chosen freely.

If the shape is fixed to be a rectangle one can analyse rectangles in grid graphs with, in sequence, fixed dimension, fixed cardinality, and variable cardinality. In all cases a complete characterization of the optimal cycles and closed form expressions of the optimal objective values are given, yielding polynomial time algorithms for all cases of center rectangle problems.

Finally, it is shown that center cycles can be chosen as rectangles for small cardinalities such that the center cycle problem in grid graphs is in these cases completely solved.

*Department of Engineering Science, University of Auckland (on sabbatical leave from the University of Kaiserslautern)

†Institut für Techno- und Wirtschaftsmathematik (ITWM), University of Kaiserslautern

‡The work of both authors was partially supported by a grant of the Deutsche Forschungsgemeinschaft (DFG)

Contents

1 Introduction

Let $G = (V, E)$ denote a simple, connected, undirected graph with vertex set V with $|V| = n$, and edge set E. (For the graph theoretic notation and terminology used in this paper, see, for instance, Foulds [7].) An edge between u and v is denoted by $[u, v]$. The **distance** between vertices x and y, denoted by $d(x, y)$, is defined to be the length of a shortest path in G between vertices x and y, expressed as the number of edges.

The **eccentricity of vertex** x, denoted by $e(x)$, is defined to be

$$e(x) = \max_{y \in V} d(x, y),$$

and x is called a **center** of G iff $e(x) = \min_{v \in V} e(v)$.
The **center (vertex) problem** finds a vertex in the graph G minimizing $e(x)$.

In this paper the previous vertex problem is extended to subsets of vertices. For any subset $U \subseteq V$, and any $x \in V$, let

$$d(x, U) = \min_{u \in U} d(x, u).$$

The **eccentricity of U** is defined as

$$e(U) = \max_{x \in V} d(x, U).$$

If the vertex set of any (simple) cycle C in G is denoted by $V(C)$, then $e(C) := e(V(C))$ is called the **eccentricity of cycle** C.

Definition 1 *Corresponding to the vertex center problem we define the*

$$(CC) \quad \textbf{cycle-center problem} \quad : \quad \min \{e(C) \quad : \quad C \textit{ cycle}\}$$

If in addition the number $|E(C)|$ of edges which equals the number $|V(C)|$ of vertices in the cycle is fixed we obtain the

$$(p - CC) \quad \textbf{p-cycle-center problem} \quad : \quad \min \{e(C) \quad : \quad C \textit{ cycle with } |E(C)| = p\}$$

Finding cycle-centers in a graph is a problem which was posed in a graph theoretic context by Buckley and Harary [3]. In location theory this problem is under discussion for a long time. In the covering salesman problem (see Current and Schilling [4]) the aim is to find a tour visiting q of the n nodes of a given network such that the tour lies within a given distance from the non-tour nodes and such that the length of the tour is minimized. In a similar type of problem, the same authors [5] and Akinc and Srikanth [1] are looking for a tour with just one node in each of several disjoint regions minimizing the length of the tour and the access of the customers to the designated node in their region. Applications of

such problems include newspaper delivery (see Jacobsen and Madson [9]) or optimization of postal services (Labbe and Laporte [10]). An overview on the location of extensive facilities in networks (paths, trees, and cycles) is given by Mesa and Boffey [12].
In the plane the problem of finding a cycle close to a given point set has been investigated extensively, too. The classical case is the Traveling Salesman Problem in the plane, which is NP-hard (see Garey and Johnson [8]) or the geometric covering salesman problem in which the maximal distance from the cycle to the point set is restricted (see e.g., Arkin and Hassin [2]). It is also interesting to look for planar cycles with special shape. Here the main emphasis is on the location of circles (i.e. the determination of center point and diameter of the circle). These problems are non-differentiable and non-convex and are therefore mathematically difficult to tackle. Circle location problems have applications in production processes, as, for instance, discussed by Ventura and Yeralan [16, 19]. If the objective is to minimize the maximum distance to the given point set the center cycle problem in the plane is also called *out-of-roundness problem*. Le and Lee [11] show that the center of an optimal circle can be found by investigating the intersection points of the closest and farthest point Voronoi diagram. Heuristic approaches for finding a center circle can be found in Drezner, Steiner and Wesolowsky [6]. The circle location problem and some related problems have been discussed by Witzgall [17]. Späth [14, 15] deals with circle and ellipse location problems minimizing the sum of *squared* distances to a given point set. The location of a circle minimizing the sum of distances to a given point set seems to be an open problem so far. First models of this problem can be found in Schöbel [13].

In this short note we will first establish that CC and p-CC are NP hard in general graphs and that CC is trivially solvable in grid graphs. In the third section we investigate p-CC for the case where the cycle to be found in a given grid graph is a rectangle. By first fixing the dimensions of the rectangle, then its circumference, one finally can solve this problem without fixing any data of the rectangle by identifying optimal rectangles and their objective function values with closed-form expressions. A justification for considering the special case of rectangles is given in Section 4 which concludes the paper.

2 Basic Results

The complexity status of the problems introduced in the previous section is easily determined.

Theorem 2.1 *The problems (CC) and $(p - CC)$ are in general graphs NP-hard.*

Proof: Obviously, $e(C) = 0$ if and only if G contains a Hamiltonian cycle. Hence (CC) is NP-hard. Since any polynomial algorithm for $(p - CC)$ applied to $p = 1, \dots, |V|$ would solve (CC) the same is true for $(p - CC)$.

□

In this note the $p-$cycle-center problem is investigated on grid graphs $G = G_{n_1 \times n_2}$ with node and edge set

$$\begin{aligned} V &:= \{ij : 0 \leq i \leq n_1,\ \ 0 \leq j \leq n_2,\ i,j \ \text{ integer}\} \text{ and} \\ E &:= \{[ij, kl] : \{|i-k|, |j-l|\} = \{0,1\}\}\text{, respectively.} \end{aligned}$$

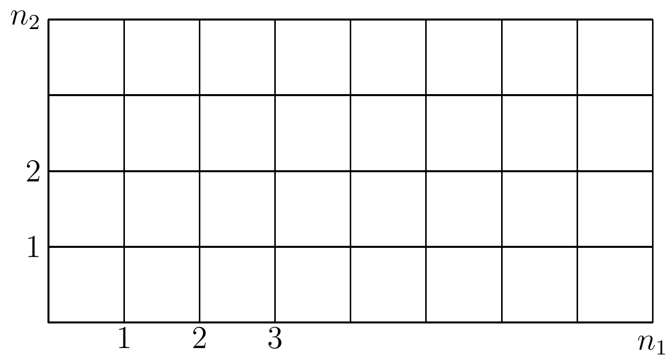

Figure 1: A grid graph of dimension $n_1 \times n_2$. Throughout we assume $n_1 \geq n_2$.

If at least one of n_1, n_2 $(n_1, n_2 > 1)$ is odd, $G_{n_1 \times n_2}$ contains a Hamiltonian cycle, e.g. the one shown in Figure 2.

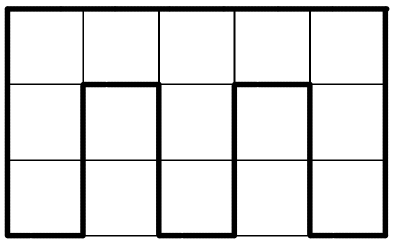

Figure 2: A Hamiltonian cycle in $G_{5\times 3}$.

If, on the other hand, both n_1 and n_2 are even, $G = G_{n_1 \times n_2}$ contains no Hamiltonian cycle. This follows from the fact that G is bipartite (partition G according to the diagonals, see Figure 3) such that all cycles in G have an even number of edges. But G has $(n_1+1)\cdot(n_2+1)$, i.e. an odd number, of nodes, such that G can, indeed, not contain a Hamiltonian cycle. We thus get

Theorem 2.2 *The cycle-center problem (without cardinality constraints) is trivially solvable in grid graphs $G_{n_2 \times n_2}$.*

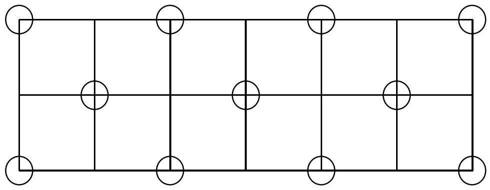

Figure 3: The nodes of $G_{6\times 2}$ can be partitioned into V_1 (circled nodes) and $V_2 = V \setminus V_1$, showing that $G_{6\times 2}$ is bipartite.

Proof: If n_1 or n_2 is odd, we have shown that $G = G_{n_1 \times n_2}$ contains a Hamiltonian cycle C such that $e(C) = 0$ and C solves (CC) .
If n_1 and n_2 are both even, G contains no Hamiltonian cycle, such that $e(C) \geq 1$ for all cycles C in G. But we can easily construct a cycle containing all but one node such that $e(C) = 1$, (see Figure 4). Hence this cycle solves (CC).

□

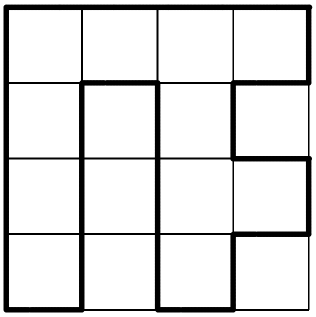

Figure 4: Cycle C in $G_{4\times 4}$ with $e(C) = 1$ and $\sum_{v\in V} d(v, C) = 1$.

The same arguments hold, if we consider the **median-cycle problem** and the **p-median-cycle problem** instead of the corresponding center-cycle problems (CC) and (p-CC). In these problems the distances of all nodes of a given graph to the cycle C are added up instead of just taking the maximum of these distances (see Figure 4).

Corollary 2.3 *The median-cycle and the p-median-cycle problem are both NP-hard in general graphs and trivially solvable in grid graphs.*

It should be noted that in order to solve cycle problems in grid graphs using Theorem 2.2 and Corollary 2.3 it is crucial that one is free in choosing the shape of the cycle. This is in general not the case. In the following section we will therefore discuss the situation that the shape of C is restricted to a rectangle. In addition to being a challenging problem for its only sake this is also a useful intermediate step in solving the general cycle problem (see Section 4).

3 Cycle-Center Problems with Rectangular Cycles

In this section we consider only cycles of rectangular form within $G_{n_1 \times n_2} = G$. Such cycles are completely characterized by choosing one of the corner points, say the lower left corner point $LL = l_1 l_2$ with $0 \leq l_i \leq n_i, i = 1, 2$, and its breadth m_1 and height m_2 with $0 < m_i \leq n_i,\ i = 1, 2$. Consequently, we denote such a cycle by $C(LL, m_1, m_2)$ if we want to emphasize its location and size (see Figure 5)

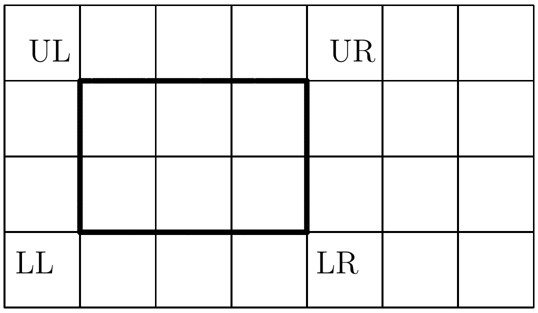

Figure 5: Rectangular cycle $C(LL, m_1, m_2) = C(11, 3, 2)$ with corner points $LL = l_1 l_2 = 11$ (lower left), $LR = 41 = (l_1 + m_1) l_2$ (lower right), $UL = 13 = l_1(l_2 + m_2)$ (upper left), and $UR = 43 = (l_1 + m_1)(l_2 + m_2)$ (upper right).

In this section we will in sequence minimize $e(C(LL, m_1, m_2))$ where

1. m_1 and m_2 are given (**fixed shape**),
2. the cardinality $p = 2(m_1 + m_2)$ is given (**fixed circumference**), and
3. neither m_1, m_2 nor p is fixed.

Using the denotations **int C** $:= \{ij \in V \ :\ l_1 \leq i \leq l_1 + m_1,\ l_2 \leq j \leq l_2 + m_2\}$, **out C** $:= V \setminus int\ C$, $\mathbf{e_{int}}(\mathbf{C}) := \max_{y \in\, int\, C} d(C, y)$, and $\mathbf{e_{out}}(\mathbf{C}) := \max_{y \in\, out\, C} d(C, y)$ we can rewrite the eccentricity of C as

$$e(C) = \max\{e_{int}(C), e_{out}(C)\}. \qquad (1)$$

This reformulation will be used in the following subsections extensively.

3.1 Rectangular cycles with fixed shape

In this subsection m_1 and m_2 are given and we solve $\min_{LL \in V} e(C(LL, m_1, m_2))$. Obviously,

$$e_{int}(C) = \min\left\{\left\lfloor\frac{1}{2}m_1\right\rfloor, \left\lfloor\frac{1}{2}m_2\right\rfloor\right\} \tag{2}$$

for all $C = C(LL, m_1, m_2)$ - independent of LL. (Here and in the following $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. Correspondingly, $\lceil x \rceil$ is the smallest integer larger than or equal to x.)

In order to analyze $e_{out}(C)$ for some given cycle $C = C(LL, m_1, m_2)$ we first observe that for any $y \in$ *out* C there exists one of the corner nodes $x \in \{0, X, Y, Z\}$ with $0 = 00, X = n_1 0, Y = n_1 n_2, Z = 0 n_2$ such that

$$d(C, y) \leq d(C, x) \tag{3}$$

This is illustrated in Figure 6, where the following inequalities hold.

$$\begin{array}{llllll}
\text{For} & y \in & I \cup II \cup VIII, & d(C,y) & \leq & d(C,0) & = & d(LL,0) \\
\text{for} & y \in & II \cup III \cup IV, & d(C,y) & \leq & d(C,X) & = & d(LR,X) \\
\text{for} & y \in & IV \cup V \cup VI, & d(C,y) & \leq & d(C,Y) & = & d(UR,Y) \\
\text{for} & y \in & VI \cup VII \cup VIII, & d(C,y) & \leq & d(C,Z) & = & d(UL,Z)
\end{array}$$

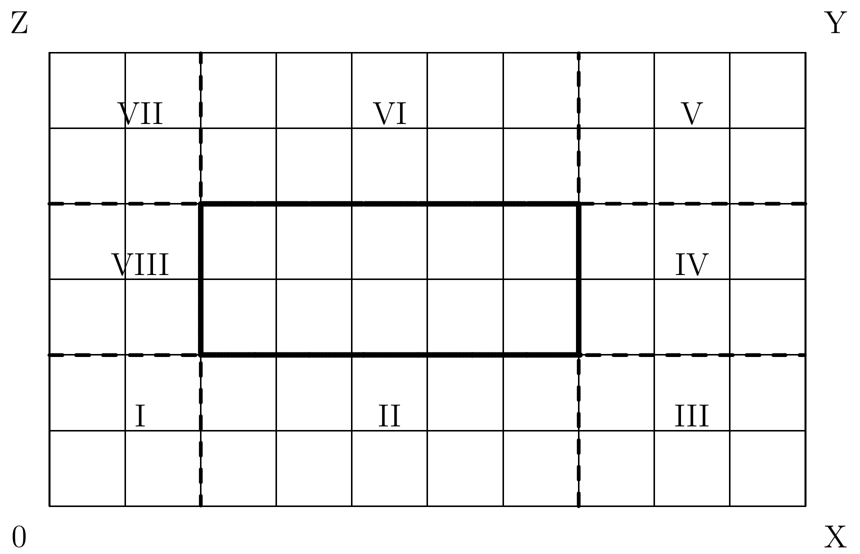

Figure 6: Illustration of Inequality 3 with $C = C(22, 5, 2)$.

Hence

$$\begin{aligned}
e_{out}(C) = \max_{y \in \; out \; C} d(C, y) &= \max_{x \in \{0, X, Y, Z\}} d(C, x) \\
&= \max\{d(LL, 0),\ d(UL, Z),\ d(LR, X),\ d(UR, Y)\}.
\end{aligned}$$

Without loss of generality, we may assume that

$$l_1 \geq n_1 - (l_1 + m_1)$$
$$\text{and} \tag{4}$$
$$l_2 \geq n_2 - (l_2 + m_2)$$

(if this is not the case, denotations can be switched appropriately). Hence

$$\begin{aligned}
d(C,0) &= l_1 + l_2 \geq n_1 - (l_1 + m_1) + l_2 = d(C,X) \\
d(C,0) &= l_1 + l_2 \geq n_1 - (l_1 + m_1) + n_2 - (l_2 + m_2) = d(C,Y) \\
\text{and} & \\
d(C,0) &= l_1 + l_2 \geq l_1 + n_2 - (l_2 + m_2) = d(C,Z).
\end{aligned}$$

such that $e_{out}(C) = d(C,O)$.
Obviously, (4) is equivalent to

$$l_1 \geq \frac{1}{2}(n_1 - m_1)$$
$$\text{and} \tag{5}$$
$$l_2 \geq \frac{1}{2}(n_2 - m_2)$$

such that we choose

$$l_1 = \left\lceil \frac{1}{2}(n_1 - m_1) \right\rceil$$
$$\text{and} \tag{6}$$
$$l_2 = \left\lceil \frac{1}{2}(n_2 - m_2) \right\rceil$$

in order to find $C = C(LL, m_1, m_2)$ with minimal value $e_{out}(C)$.

Summarizing (1), (2) and (6) we obtain the following result which has already been observed by Yamagouchi, Foulds and Lamb [18].

Theorem 3.1 *Given m_1, m_2 with $0 < m_i \leq n_i,\ i = 1, 2$. Then the problem*

$$\min_{LL \in V} e(C(LL, m_1, m_2))$$

has the optimal objective value

$$\max\left\{\min\ \{\left\lfloor \frac{1}{2}m_1 \right\rfloor, \left\lfloor \frac{1}{2}m_2 \right\rfloor\},\ \left\lceil \frac{1}{2}(n_1 - m_1) \right\rceil + \left\lceil \frac{1}{2}(n_2 - m_2) \right\rceil\right\}. \tag{7}$$

An optimal cycle $C(LL, m_1, m_2)$ is defined by its lower left corner point $LL = l_1 l_2$ with

$$l_i = \left\lceil \frac{1}{2}(n_i - m_i) \right\rceil \quad i = 1, 2.$$

If all optimal rectangles are required the wlog assumption (4) is relaxed to find 1, 2 or 4 optimal rectangles (depending on whether $n_1 - m_1$ and $n_2 - m_2$ are even or odd) for the case where $e(C) = e_{out}(C)$.
If $e(C) = e_{int}(C)$ all C with

$$\left\lceil \frac{1}{2}(n_1 - m_1) \right\rceil + \left\lceil \frac{1}{2}(n_2 - m_2) \right\rceil \leq \min \left\{ \left\lfloor \frac{1}{2} m_1 \right\rfloor, \left\lfloor \frac{1}{2} m_2 \right\rfloor \right\}$$

are optimal.

3.2 Rectangular cycles with fixed circumference

Compared with the previous subsection we allow now more freedom in choosing the shape of the rectangular cycle $C = C(LL, m_1, m_2)$. Instead of fixing m_1 and m_2 we fix the circumference $p = 2m_1 + 2m_2$ or equivalently $M = m_1 + m_2$. We thus solve

$$\begin{array}{rl} \min & e(C(LL, m_1, m_2)) \\ \text{such that} & LL \in V \\ & m_1 + m_2 = M \\ & 0 < m_i \leq n_i, \quad i = 1, 2 \\ & m_i \in \mathbb{Z} \end{array} \tag{8}$$

By Theorem 3.1 we can always assume that the lower-left corner point $LL = l_1 l_2$ of $C = C(LL, m_1, m_2)$ satisfies condition (6). Since $m_2 = M - m_1$, Problem (8) only deals with finding an optimal m_1 with respect to a given M.

Proposition 3.2 *Given integers m_1 and M such that $0 < m_1 \leq n_1$ and $m_1 < M < n_1 + n_2$. Then for any rectangle with breadth m_1 and circumference M satisfying condition(6)*

$$e_{out}(C) = \begin{cases} \frac{1}{2}(n_1 + n_2 - M) & \text{if } n_1 + n_2 - M \text{ and } n_1 - m_1 \text{ are even} \\ \frac{1}{2}(n_1 + n_2 - M + 1) & \text{if } n_1 + n_2 - M \text{ is odd} \\ \frac{1}{2}(n_1 + n_2 - M + 2) & \text{if } n_1 + n_2 - M \text{ is even and } n_1 - m_1 \text{ is odd} \end{cases} \tag{9}$$

holds.

Proof: Due to $m_2 = M - m_1$ and $(n_1 - m_1) + (n_2 - M + m_1) = n_1 + n_2 - M$ we can compute $c_{out}(C)$ for any rectangle satisfying condition(6) using the following case analysis.

Case 1 : If $n_1 + n_2 - M$ is odd then exactly one of $n_1 - m_1$ or $n_2 - M + m_1$ is odd, i.e. $e_{out}(C) = \frac{1}{2}(n_1 + n_2 - M + 1)$

Case 2 : If $n_1 + n_2 - M$ is even, then
Case 2a: $n_1 - m_1$ even implies $n_2 - M + m_1$ even and hence $e_{out}(C) = \frac{1}{2}(n_1 + n_2 - M)$
Case 2b: $n_1 - m_1$ odd implies $n_2 - M + m_1$ odd and hence $e_{out}(C) = \frac{1}{2}(n_1 + n_2 - M + 2)$

□

It should be noted that Proposition 3.2 holds for all m_1, m_2 with $M = m_1 + m_2$ such that we can choose m_1 minimizing (see (2))

$$e_{int}(c) = \min\left\{ \left\lfloor \frac{1}{2} m_1 \right\rfloor, \left\lfloor \frac{1}{2}(M - m_1) \right\rfloor \right\}.$$

- For $M \leq n_1$ this is done by constructing a rectangle such that

  $$\begin{array}{rlll} & m_1 = M - 1 & \text{and thus} & m_2 = 1 \\ \text{or} & m_1 = M - 2 & \text{and thus} & m_2 = 2 \end{array}$$

  In the former case $e_{int}(C) = 0$ while in the latter case $e_{int}(C) = 1$. In both cases $e_{int}(C) \leq e_{out}(C)$, and we can choose m_1 such that $e_{out}(C)$ is as small as possible, i.e. $e(C) = \lceil \frac{1}{2}(n_1 + n_2 - M) \rceil$.

- If $M > n_1$ we choose

  $$m_1 = n_1 \text{ and thus } m_2 = M - n_1$$

  to obtain

  $$e_{int}(C) = \left\lfloor \frac{1}{2}(M - n_1) \right\rfloor$$

  (recall the general assumption $n_1 \geq n_2$), and

  $$e_{out}(C) = \left\lceil \frac{1}{2}(n_1 + n_2 - M) \right\rceil$$

  such that in this case

  $$e(C) = \max\left\{ \left\lceil \frac{1}{2}(n_1 + n_2 - M) \right\rceil, \left\lfloor \frac{1}{2}(M - n_1) \right\rfloor \right\}.$$

In summary, we obtain

Theorem 3.3 *An optimal cycle $C = C(LL, m_1, m_2)$ with fixed circumference $p = 2M = 2(m_1 + m_2)$ is given by*

$$(m_1, m_2) \quad = \quad \begin{cases} (n_1, M - n_1) & \text{if} \quad M > n_1 \\ (M-1, 1) & \text{if} \quad M \leq n_1 \quad \text{and} \quad n_1 - M \text{ odd} \\ (M-2, 2) & \text{if} \quad M \leq n_1 \quad \text{and} \quad n_1 - M \text{ even} \end{cases} \tag{10}$$

and

$$LL = l_1 l_2 \quad \text{where} \quad l_i = \left\lceil \frac{1}{2}(n_i - m_i) \right\rceil .$$

Its objective value is

$$e(C) \quad = \quad \begin{cases} \left\lceil \frac{1}{2}(n_1 + n_2 - M) \right\rceil & \text{if} \quad M \leq n_1 \\ \max\left\{ \left\lceil \frac{1}{2}(n_1 + n_2 - M) \right\rceil , \left\lfloor \frac{1}{2}(M - n_1) \right\rfloor \right\} & \text{if}\, M > n_1 . \end{cases} \tag{11}$$

3.3 Rectangular cycles without constraints

In this section we neither fix m_1 and m_2 nor $M = m_1 + m_2$. Thus M is in Problem (8) variable. Using the results of Sections 3.1 and 3.2 we know that $e(C) = \max\{e_{out}(C),\ e_{int}(C)\}$ is computed by (11). A function depending on M is shown in Figure 7

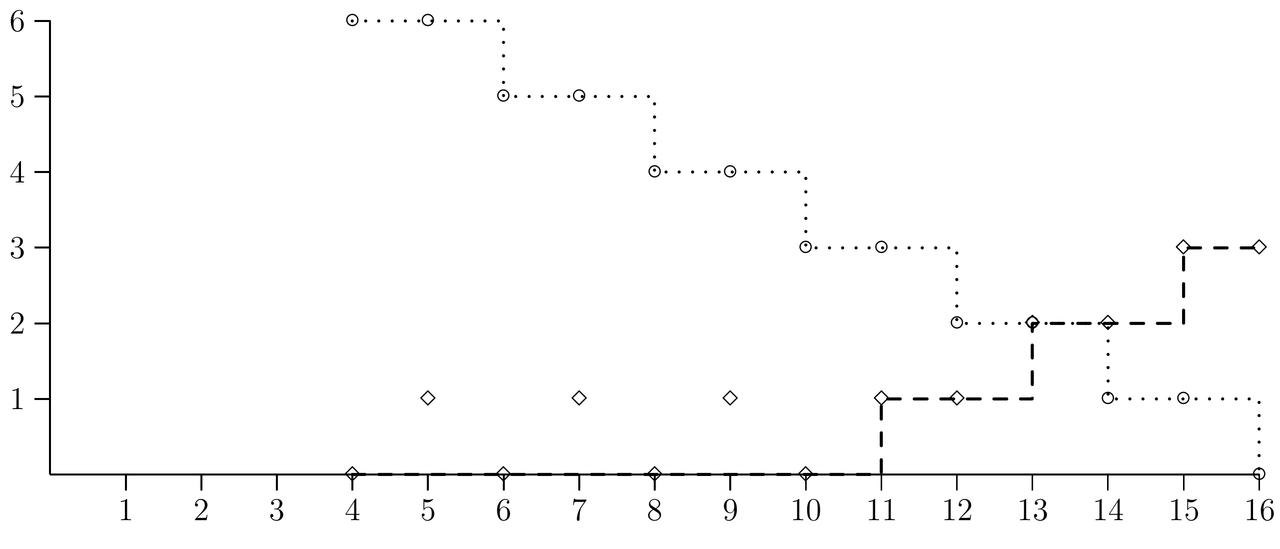

Figure 7: $e(C)$ as upper envelope of $e_{out}(C)$ ($\circ$) and $e_{int}(C)$ ($\diamond$) for a grid graph with $n_1 = 9$ and $n_2 = 7$.

Obviously, its minimum taken over $M \in \{2, \ldots, n_1 + n_2\}$ is attained where e_{out} (C) and e_{int} (C) intersect each other. Depending on n_1 and n_2 the intersecting parts of the two functions is a point or a horizontal piece of length 1 or 2. But in all cases we obtain that the minimum is attained for $M = n_1 + \left\lceil \frac{n_2}{2} \right\rceil$.

We have thus proved:

Theorem 3.4 *The cycle-center problem in grid graphs $G_{n_1 \times n_2}$ with respect to rectangular cycles is obtained by $C = (LL, m_1, m_2)$ given as follows: The circumference of C is $p = 2M^*$ with $M^* = n_1 + \left\lceil \frac{n_2}{2} \right\rceil$, m_1 and m_2 with $m_1 + m_2 = M^*$ are given by (10) and $LL = l_1 l_2$ is given by (6).*

4 Conclusion and Extension

In this note we have shown that the problem of finding optimal center rectangles in grid graphs can be solved by closed form expressions. This is true if the shape of the rectangle is fixed, if its circumference is fixed, or if the best overall rectangle is required. At the beginning of the paper it was established that the center cycle problem in general graphs is NP hard and trivially solvable in grid graphs without fixing the cardinality. In order to resolve the complexity issue for general cycles in grid graphs the special case of optimal rectangles is an important intermediate stage.
The interrelation between the general cycle and rectangle problem in grid graphs is currently not yet fully understood, but the following result indicates that the results for the rectangle problem are useful in solving the more general cycle problem

Theorem 4.1 *Let p be the cardinality of a cycle in the grid graph $G_{n_1 \times n_2}$ such that $2 \leq \frac{p}{2} \leq M^*$ where M^* is taken from Theorem 3.4. Then there is always an optimal center cycle which is a rectangle and which can be calculated according to Theorem 3.3.*

Proof: Suppose C^* is an optimal center cycle which is not a rectangle with cardinality p and objective value $e^* = e(c^*)$. Due to the definition of the eccentricity there exist nodes a_0, a_X, a_Y, a_Z in C^* such that

$$d(a_i, i) \leq e^* \quad i = 0, X, Y, Z$$

Consider the smallest rectangle $R = R(a_0, a_X, a_Y, a_Z)$ containing a_0, a_X, a_Y, a_Z. Clearly,

$$e(R) \leq \max\{d(a_i, i) : i = 0, X, Y, Z\} \leq e^* \quad \text{and} \quad |R| \leq |C^*|$$

Moreover, if R_M denotes the center rectangular cycle with respect to fixed circumference $2M$, we know from Subsection 3.3 that $e(R_M)$ is a non-increasing function in M for $1 \leq M \leq M^*$. Consequently,

$$e(R_{\frac{p}{2}}) \leq e(R) \leq e^* = e(C)$$

such that C^* can be replaced by the rectangle $R_{\frac{p}{2}}$.

□

The question of optimal center cycles in grid graphs with cardinality $p > 2M^*$ is currently under research and will be published shortly together with results on the median problem.

5 Acknowledgment

We would like to thank Professor Les R. Foulds, University of Waikato, for focusing our attention to cycle problems in grid graphs.

References

[1] U. Akinc and K.N. Srikanth. Optimal routing and process scheduling for a mobile service facility. *Networks*, 22:163–183, 1992.

[2] E.M. Arkin and R. Hassin. Approximation algorithms for the geometric covering salesman problem. *Discrete Applied Mathematics*, 55:197–218, 1994.

[3] F. Buckley and F. Harary. *Distance in Graphs.* Addison-Wesley, 1990.

[4] J.R. Current and D.A. Schilling. The covering salesman problem. *Transportation Science*, 23:208–213, 1989.

[5] J.R. Current and D.A. Schilling. The median tour and maximal covering tour problems: Formulations and heuristics. *European Journal of Operational Research*, 73:114–126, 1994.

[6] Z. Drezner, S. Steiner, and G.O. Wesolowsky. On the circle closest to a set of points. Technical report, California State University, Department of Management Science and Information Systems, 1996.

[7] L.R. Foulds. *Graph Theory Applications.* Springer, 1998.

[8] M.R. Garey and D.S. Johnson. *Computers and Intractability — A Guide to the Theory of NP-Completeness.* Freeman, San Francisco, 1979.

[9] S.K. Jacobsen and Madsen O. A comparative study of heuristics for a two-level location-routing problem. *European Journal of Operational Research*, 5:378–387, 1980.

[10] M. Labbe and G. Laporte. Maximizing user convenience and postal service efficiency in post box location. *Belgian Journal of Operations Research Statistics and Computer Science*, 26:21–35, 1986.

[11] V.-B. Le and D.T. Lee. Out-of-roundness problem revisited. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13:217–223, 1991.

[12] J.A. Mesa and T.B. Boffey. A review of extensive facility location in networks. *European Journal of Operational Research*, 95(3):592–603, 1996.

[13] A. Schöbel. *Locating Lines and Hyperplanes — Theory and Algorithms.* Applied Optimization Series. Kluwer, 1999. to appear.

[14] H. Späth. Least squares fitting of ellipses and hyperbolas. *Computational Statistics*, 12(3):329–341, 1997.

[15] H. Späth. Orthogonal distance fitting by circles and ellipses. *Computational Statistics*, 12(3):343–354, 1997.

[16] J.A. Ventura and S. Yeralan. The minmax center estimation problem. *European Journal of Operational Research*, 41:64–72, 1989.

[17] C. Witzgall, S.I. Gass, H.H. Harary, and D.R. Shier. Linear programming techniques for fitting circles and spheres. Talk held at the ISMP'97, Lausanne, august 1997.

[18] T. Yamaguchi, L.R. Foulds, and J. Lamb. Central cycles in graphs. Technical report, preprint, 1998.

[19] S. Yeralan and J.A. Ventura. Computerized roundness inspection. *International Journal of Production Research*, 26:1921–1935, 1988.