Optimal Order Results for a Class of Regularization Methods Using Unbounded Operators

M. Thamban Nair

Department of Mathematics, Indian Institute of Technology Madras
Chennai – 600 036, India.

Abstract

A class of regularization methods using unbounded regularizing operators is considered for obtaining stable approximate solutions for ill–posed operator equations. With an a posteriori as well as an priori parameter choice strategy, it is shown that the method yields optimal order. Error estimates have also been obtained under stronger assumptions on the the generalized solution. The results of the paper unify and simplify many of the results available in the literature. For example, the optimal results of the paper includes, as particular cases for Tikhonov regularization, the main result of Mair (1994) with an a priori parameter choice and a result of Nair (1999) with an a posteriori parameter choice. Thus the observations of Mair (1994) on Tikhonov regularization of ill–posed problems involving finitely and infinitely smoothing operators is applicable to various other regularization procedures as well. Subsequent results on error estimates include, as special cases, an optimal result of Vainikko (1987) and also recent results of Tautenhahn (1996) in the setting Hilbert scales.

1 Introduction and Preliminaries

Many inverse problems in science and engineering have their mathematical formulation as an operator equation

$$Tx = y \tag{1.1}$$

where $T : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y with its range $R(T)$ not closed in Y (cf. [2], [5], [3]). It is well known that if $R(T)$ is not closed, then equation (1.1) or the problem of solving (1.1) is ill-posed (cf. [4]). A prototype of an ill-posed equation is the Fredholm integral equation of the first kind,

$$\int_a^b k(s,t)x(t)\,dt = y(s), \qquad a \le s \le b,$$

with a non-degenerate kernel $k(\cdot,\cdot) \in L^2([a,b] \times [a,b])$ and $X = Y = L^2[a,b]$.

For ill–posed equations one normally looks for a *least residual norm* (LRN) solution, as the solution may not exist. By an LRN solution we mean an element x_0 in the set

$$S_y = \{x \in X : \|Tx_0 - y\| \le \|Tu - y\|, \ \forall u \in X\}.$$

Of course, an LRN solution also may not exist, unless $y \in R(T) + R(T)^\perp$ (c.f. [4]).

If T is not injective, then one has to specify conditions to single out a particular type of LRN solution. In applications one often looks for a unique element $\hat{x} \in S_y \cap D$ such that

$$\|L\hat{x}\| \le \|Lx\|, \quad \forall x \in S_y \cap D,$$

where D is the domain of an unbounded operator $L : D \subseteq X \to Z$ from X to another Hilbert space Z.

Now the question is the existence of such $\hat{x}$ and also its stability under perturbation of the data y. To deal with such issues we adopt the following formalism.

Let

$$L : D(L) \subseteq X \to Z$$

be a densily defined closed linear operator from the Hilbert space X to another Hilbert space Z such that

$$\|Tx\|^2 + \|Lx\|^2 \ge \gamma \|x\|^2, \quad \forall x \in D, \tag{1.2}$$

for some $\gamma > 0$. Observe that, the above condition is satisfied, if for example L is bounded below, which is the case for many of the differential operators. In such case, the range of L is seen to be a closed subspace of Z.

Under the above assumption on L, it is known (see e.g. [12]) that the map

$$(u, v) \mapsto \langle u, v\rangle_0 := \langle Tu, Tv\rangle + \langle Lu, Lv\rangle, \quad \forall (u, v) \in D \times D,$$

is a complete inner product on D with corresponding norm

$$\|x\|_0 = \left(\|Tx\|^2 + \|Lx\|^2\right)^{1/2}, \quad x \in D(L).$$

Let

$$X_0 = D(L) \quad \text{with} \quad \langle \cdot, \cdot \rangle_0,$$

and let

$$T_0 = T|_{X_0},$$

the rstriction of T to the Hilbert space X_0. Note that

$$\|T_0 x\| = \|Tx\| \le \|x\|_0, \quad \forall x \in X_0,$$

so that $T_0 : X_0 \to Y$ is a bounded linear operator with $\|T_0\| \leq 1$. Let $D_0^\dagger$ be the domain of the Moore–Penrose generalized inverse of T_0, i.e.,

$$D_0^\dagger = R(T_0) + R(T)^\perp.$$

It is known (see e.g. [12]) that, for $y \in D_0^\dagger = R(T_0) + R(T_0)^\perp$,

$$\hat{x} := T_0^\dagger y$$

is the unique element in $S_y \cap D$ such that

$$\|L\hat{x}\| \leq \|Lx\|, \quad \forall x \in S_y \cap D.$$

Here the notation $A^\dagger$ is for the Moore–Penrose generalized inverse of A (c.f. [4]).

It can be seen, by using the denseness of D in X, that if range of T is not closed in Y, then range of T_0 is also not closed in Y. Hence the generalized problem of determining $\hat{x} = T_0^\dagger y$ is also ill–posed whenever the range of T is not closed. In such situations regularization methods are employed for obtaining stable approximations for $\hat{x}$.

Recall that, by a regularization we mean a family $\{R_\alpha\}_{\alpha>0}$ of bounded linear operators from X to Y together with a parameter choice strategy

$$\alpha(\cdot,\cdot) : (0,\infty) \times Y \to X$$

such that for $\tilde{y} \in Y$ with $\|y - \tilde{y}\| \leq \delta$, we must have

$$\alpha(\delta, \tilde{y}) \to 0 \quad \text{as} \quad \delta \to 0,$$

and

$$R_{\alpha(\delta,\tilde{y})}\tilde{y} \to \hat{x} \quad \text{as} \quad \delta \to 0.$$

After obtaining such a regularization method, the next concern would be to see whether the error $\|\hat{x} - \tilde{x}_\alpha\|$ is optimal order in the sense of the *best possible maximal error* $E(M,\delta)$ defined by

$$E(M,\delta) = \inf_R \sup \Big\{ \|x - Rv\| : x \in M \text{ and } v \in Y \text{ with } \|Tx - v\| \leq \delta \Big\}.$$

Here M is some preassigned *source–like* set, and the infimimum is taken over all algorithms $R : Y \to X$. If M is a convex and balanced subset, then it is proved in [10] that

$$e(M,\delta) \leq E(M,\delta) \leq 2e(M,\delta),$$

where

$$e(M,\delta) = \sup\{\|x\| : x \in M,\ \|Tx\| \le \delta\}. \tag{1.3}$$

Thus the attempt is to show that

$$\|\hat{x} - \tilde{x}_\alpha\| \le c\, e(M,\delta),$$

for some constant $c > 0$. A standard source–like set which has been considered in the literature is

$$M_\rho = \{x \in D : \|Lx\| \le \rho\}, \quad \rho > 0. \tag{1.4}$$

Clearly, this set is convex and balanced.

The regularization procedure considered in the next section is the same as the one in Hanke [6]. But, we give a different, apparently simpler, motivation for its introduction. Optimality results under the assumption that the generalized solution belongs to M_ρ are proved in Section 3, under apriori and a posteriori parameter choice strategies. Error estimates under additional smoothness assumptions on the generalized solution are derived in Section 4. The results of this section, in particular include the optimal result obtained by Vainikko [16]. These results have been effectively used in unifying the results availablle in the setting of Hilbert scales (c.f. Tautenhahn [14]) and the classical results (c.f. [4], [8], [16]).

2 The Regularization

Our idea is to decompose the *generalized solution* $\hat{x}$ into two parts x_0 and $\hat{x}_0$ with x_0 being stable under perturbation, and then apply a regularization method to approximate $\hat{x}_0$. For this purpose first we assume that

- $R(L)$ is closed in Z.

Let us introduce a few notation. Let L_0 be the operator L considered as from X_0 into Z, i.e.,

$$L_0 : X_0 \to Z, \quad L_0 x = Lx,\ \forall x \in X_0.$$

It is seen, by the definition of $\|\cdot\|_0$, that L_0 is a bounded linear operator with $\|L_0\| \le 1$. More over, since $R(L)$ is closed, $R(L_0)$ is also closed so that the generalized inverse $L_0^\dagger$ of L_0 is continuous. Now let

$$A = T_0 L_0^\dagger.$$

Clearly $A : Z \to Y$ is a continuous linear operator, and it has the property (see [3] for a proof) that

$$A^\dagger = \left(T_0 L_0^\dagger\right)^\dagger = L_0 T_0^\dagger.$$

In particular, for $y \in D_0^\dagger$,
$$A^\dagger y = L_0 \hat{x} = L\hat{x}.$$

By projection theorem, we can write $\hat{x}$ as

$$\hat{x} = x_0 + \hat{x}_0 \quad \text{with} \quad x_0 \in N(L_0), \quad \hat{x}_0 \in N(L_0)^\perp.$$

Since $L_0^\dagger L_0$ is the orthogonal projection onto the space $N(L_0)^\perp$, we have

$$\hat{x}_0 = L_0^\dagger L_0 \hat{x} = L_0^\dagger (A^\dagger y).$$

We show that x_0 is stable under perturbation in y. For this purpose we consider the operator
$$\hat{T}_0 := T_0|_{N(L_0)},$$
the restriction of T_0 to the null space $N(L_0)$ of the operator L_0. Using (1.2) it can be seen that $\hat{T}_0 : N(L_0) \to Y$ is injective and its range is closed in Z. In particular $\hat{T}_0^\dagger$ is continuous.

PROPOSITION 2.1
$$x_0 = \hat{T}_0^\dagger y.$$

Proof.
It is enough to show that
$$\hat{T}_0^* \hat{T}_0 x_0 = \hat{T}_0^* y.$$
For this, let $u \in N(L_0)$. Then we have

$$\langle \hat{T}_0 u, y\rangle = \langle T_0 u, y\rangle = \langle T_0 u, P_0 y\rangle = \langle T_0 u, T_0 \hat{x}\rangle,$$

where $P_0 : Y \to Y$ is the orthogonal projection onto $R(T_0)$. Thus for every $u \in N(L_0)$,
$$\langle \hat{T}_0 u, y\rangle = \langle T_0 u, T_0 x_0 + T_0 \hat{x}_0\rangle.$$
But
$$\langle T_0 u, T_0 \hat{x}_0\rangle = \langle u, \hat{x}_0\rangle_0 = 0,$$
for every $u \in N(L_0)$, since $\hat{x}_0 \in N(L_0)^\perp$. Hence we have

$$\langle \hat{T}_0 u, y\rangle = \langle T_0 u, T_0 x_0\rangle,$$

showing tht
$$\hat{T}_0^* \hat{T}_0 x_0 = \hat{T}_0^* y.$$

$\square$

By the above proposition,

$$\hat{x} = x_0 + \hat{x}_0 = \hat{T}_0^\dagger y + L_o^\dagger(A^\dagger y), \tag{2.1}$$

where $x_0 = \hat{T}_0^\dagger y$ is stable under perturbation in y. The above representation motivates us to use a regularization procedure for approximating $A^\dagger y$, and thereby giving a regularization for the original problem. For this we use a well–known class of regularization method (c.f. e.g., Groetsch [4], Louis [8]), namely

$$R_\alpha = g_\alpha(A^*A)A^*, \quad \alpha > 0,$$

where $\{g_\alpha : \alpha > 0\}$ is a family of piecewise continuous functions defined on $[0, \|A\|^2]$ satisfying the following :

(i) There exists $\beta_0 \geq \frac{1}{2}$, and for every $\beta \in [0, \beta_0]$ there exists $c_\beta > 0$ such that

$$\sup_{0\leq\lambda\leq\|T\|^2} |\lambda^\beta[1 - \lambda g_\alpha(\lambda)]| \leq c_\beta \alpha^\beta.$$

(ii) For every $\mu \in [0, 1]$, there exists $d_\mu > 0$ such that

$$\sup_{0\leq\lambda\leq\|T\|^2} |\lambda^\mu g_\alpha(\lambda)| \leq d_\mu \alpha^{\mu-1}.$$

From these properties of $\{g_\alpha\}_{\alpha>0}$, using spectral theory, we obtain the following inequalities for $0 \leq \beta \leq \beta_0$, $0 \leq \mu \leq 1$, $0 \leq \nu \leq \beta_0 - \frac{1}{2}$ and $0 \leq \omega \leq \frac{1}{2}$:

$$\|(A^*A)^\beta[I - A^*Ag_\alpha(A^*A)]\| \leq \sup_{0\leq\lambda\leq\|T\|^2} |\lambda^\beta[1 - \lambda g_\alpha(\lambda)]| \leq c_\beta \alpha^\beta, \tag{2.2}$$

$$\|A(A^*A)^\nu[I - A^*Ag_\alpha(A^*A)]\| \leq \sup_{0\leq\lambda\leq\|T\|^2} |\lambda^{\nu+\frac{1}{2}}[1 - \lambda g_\alpha(\lambda)]| \leq c_{\nu+\frac{1}{2}} \alpha^{\nu+\frac{1}{2}}, \tag{2.3}$$

$$\|(A^*A)^\mu g_\alpha(A^*A)\| \leq \sup_{0\leq\lambda\leq\|T\|^2} |\lambda^\mu g_\alpha(\lambda)| \leq d_\mu \alpha^{\mu-1}, \tag{2.4}$$

$$\|A(A^*A)^\omega g_\alpha(A^*A)\| \leq \sup_{0\leq\lambda\leq\|T\|^2} |\lambda^{\omega+\frac{1}{2}} g_\alpha(\lambda)| \leq d_{\omega+\frac{1}{2}} \alpha^{\omega-\frac{1}{2}}. \tag{2.5}$$

We assume that the data y is known only approximately, say $\tilde{y}$ with

$$\|y - \tilde{y}\| \leq \delta,$$

for some known error level $\delta > 0$. Then the regularized solution corrsponding to the data y and $\tilde{y}$ are defined as

$$x_\alpha = x_0 + L_0^\dagger g_\alpha(A^*A)A^*y \tag{2.6}$$

and

$$\tilde{x}_\alpha = \tilde{x}_0 + L_0^\dagger g_\alpha(A^*A)A^*\tilde{y} \tag{2.7}$$

respectively, where $x_0 = \hat{T}_0^\dagger y$ and $\tilde{x}_0 = \hat{T}_0^\dagger \tilde{y}$.

Remarks.

The regularized solutions (2.6) and (2.7) can be defined with out R(L) being closed. The assumption that $R(L)$ is closed was made only to guarantee the stability of x_0 under perturbation in y.

It should be mentioned that the above regularization (2.6) has been considered by Hanke [6]. But the approach adopted by us to motivate the definition of the regularization, seems to be much simpler than that in [6]. We looked into two orthogonal components of $\hat{x}$, and motivated the definition of x_α and $\tilde{x}_\alpha$, whereas Hanke [6] considered orthogonal decomposition of $D_0^\dagger$ into three parts, and then show that the element to which x_α converges is of the form (2.1).

3 Order Optimality

Our concern now is to choose the regularization parameter α in such a way that

$$\|\hat{x} - \tilde{x}_\alpha\| \le c\ e(M_\rho, \delta)$$

for some constant $c > 0$, where M_ρ and $e(M_\rho, \delta)$ are as in (1.4) and (1.3) respectively.

From the representations (2.1), (2.6) and (2.7) of $\hat{x}$, x_α and $\tilde{x}_\alpha$, we obtain the relations

$$L(\hat{x} - x_\alpha) = L_0 L_0^\dagger [I - g_\alpha(A^*A)A^*A]A^\dagger y, \tag{3.1}$$

$$L(x_\alpha - \tilde{x}_\alpha) = L_0 L_0^\dagger g_\alpha(A^*A)A^*(y - \tilde{y}), \tag{3.2}$$

$$T(\hat{x} - x_\alpha) = A[I - g_\alpha(A^*A)A^*A]A^\dagger y, \tag{3.3}$$

$$T(x_\alpha - \tilde{x}_\alpha) = AA^* g_\alpha(AA^*)(y - \tilde{y}). \tag{3.4}$$

PROPOSITION 3.1 *Let c_β and d_μ be as in the definition of $\{g_\alpha\}_{\alpha>0}$. Then*

(i) $\|L(\hat{x} - x_\alpha)\| \le c_0 \|L\hat{x}\|$,

(ii) $\|L(x_\alpha - \tilde{x}_\alpha)\| \le d_{1/2}\frac{\delta}{\sqrt{\alpha}}$,

(iii) $\|T(\hat{x} - x_\alpha)\| \le c_{1/2}\sqrt{\alpha}\|L\hat{x}\|$,

(iv) $\|T(x_\alpha - \tilde{x}_\alpha)\| \le d_1 \delta$.

Proof.
Since $L_0 L_0^\dagger : Z \to Z$ is the orthogonal projection onto $R(L_0)$ and since $A^\dagger y = L\hat{x}$, the results follow from (3.1)–(3.4) by making use of (2.2)–(2.5).

□

3.1 Results under an a priori parameter choice

THEOREM 3.2 *If $\hat{x} \in M_\rho$ and $\alpha = \left(\frac{\delta}{\rho}\right)^2$, then*

$$\frac{1}{\kappa}(\hat{x} - \tilde{x}_a) \in M_\rho \quad \textit{and} \quad \|\hat{x} - \tilde{x}_\alpha\| \leq \kappa \; e(M_\rho, \delta),$$

where $\kappa = \max\{c_0 + d_{\frac{1}{2}}, \; c_{\frac{1}{2}} + d_1\}$.

Proof.
If $\hat{x} \in M_\rho$ and $\alpha = (\delta/\rho)^2$, then by Proposition 3.1, we have

$$\|L(\hat{x} - \tilde{x}_\alpha)\| \leq \|L(\hat{x} - x_\alpha)\| + \|L(x_\alpha - \tilde{x}_\alpha)\| \leq c_0\rho + d_{1/2}\rho \leq 2\kappa \; \rho,$$

and

$$\|T(\hat{x} - \tilde{x}_\alpha)\| \leq \|T(\hat{x} - x_\alpha)\| + \|T(x_\alpha - \tilde{x}_\alpha)\| \leq c_{1/2}\delta + d_1 \; \delta \leq 2\kappa \; \delta.$$

Hence the result follows from the definition of M_ρ and $e(M_\rho, \delta)$.

□

3.2 Result under an a posteriori parameter choice

Now we obtain the optimal result under a Morozov's–type discrepancy principle.

Suppose that the regularization parameter is chosen such that the the Morozov's–type *discrepancy principle*

$$\tau_1\delta \leq \|T\tilde{x}_\alpha - \tilde{y}\| \leq \tau_2\delta \tag{3.5}$$

is satisfied for some fixed τ_1, τ_2 with $\tau_1 > c_0$.

Lemma 3.3 *Suppose α is chosen according to (3.5). Then*

$$(\tau_1 - c_0)\delta \leq \|Tx_\alpha - y\| \leq (\tau_2 + c_0)\delta.$$

Proof.
Writing
$$Tx_\alpha - y = [T(x_\alpha - \tilde{x}_\alpha) - (y - \tilde{y})] + (T\tilde{x}_\alpha - \tilde{y}),$$
we have
$$\|Tx_\alpha - y\| \leq \|T(x_\alpha - \tilde{x}_\alpha) - (y - \tilde{y})\| + \|T\tilde{x}_\alpha - \tilde{y}\|$$
and
$$\|Tx_\alpha - y\| \geq \|T\tilde{x}_\alpha - \tilde{y}\| - \|T(x_\alpha - \tilde{x}_\alpha) - (y - \tilde{y})\|.$$
But, by (2.2),
$$\|T(x_\alpha - \tilde{x}_\alpha) - (y - \tilde{y})\| = \|[I - AA^* g_\alpha(AA^*)](y - \tilde{y})\| \leq c_0\delta,$$
so that we get
$$\|Tx_\alpha - y\| \leq (\tau_2 + c_0)\delta$$
and
$$\|Tx_\alpha - y\| \geq (\tau_1 - c_0)\delta.$$
□

THEOREM 3.4 *If $\hat{x} \in M_\rho$, $y = T\hat{x}$ and α is chosen according to (3.5), then*
$$\frac{1}{\tilde{\kappa}}(\hat{x} - \tilde{x}_a) \in M_\rho \quad \textit{and} \quad \|\hat{x} - \tilde{x}_\alpha\| \leq \tilde{\kappa}\; e(M_\rho, \delta),$$
where
$$\tilde{\kappa} = \max\left\{1 + \tau_2, c_0 + \frac{c_{\frac{1}{2}} d_{\frac{1}{2}}}{\tau_1 - c_0}\right\}.$$

Proof.
By Lemma 3.3 we have
$$\|Tx_\alpha - y\| \geq (\tau_1 - c_0)\delta.$$
This, together with the estimate in Proposition 3.1 (iii) gives
$$(\tau_1 - c_0)\delta \leq \|Tx_\alpha - y\| = \|T(x_\alpha - \hat{x})\| \leq c_{\frac{1}{2}}\sqrt{\alpha}\|L\hat{x}\|.$$
Hence
$$\frac{\delta}{\sqrt{\alpha}} \leq \frac{c_{\frac{1}{2}}}{\tau_1 - c_0}\rho.$$

Therefore, by Proposition 3.1,

$$\|L(\hat{x}-\tilde{x}_\alpha)\| \le \|L(\hat{x}-x_\alpha)\| + \|L(x_\alpha-\tilde{x}_\alpha)\| \le \left(c_0 + \frac{c_{1/2}d_{1/2}}{\tau_1 - c_0)}\right)\rho \le \tilde{\kappa}\rho.$$

Also, we have

$$\|T(\hat{x}-\tilde{x}_\alpha)\| = \|y - T\tilde{x}_\alpha\| \le \|y-\tilde{y}\| + \|\tilde{y} - T\tilde{x}_\alpha\| \le (1+\tau_2)\delta \le \tilde{\kappa}\rho.$$

From these relations, the result follows.

□

Remarks.
For the case of Tikhonov regulaization we have $g_\alpha(\lambda) = \frac{1}{\lambda+\alpha}$. For this particular case, optimal result has been obtained by Baumeister [1], Mair [9] and Nair [13]. While results of Baumeister and Mair are based on the a priori parameter choice $\alpha = (\delta/\rho)^2$, the results in Nair is based on Morozov's discrepancy principle. It should be mentioned that many of the important deductions for Tikhonov regularization of ill–posed problems involving finitely and infinitely smoothing operators considered by Mair [9] were based on his optimal result. Theorems 3.2 and 3.4 show that such deductions are valid not only for Tikhonov regularization but also for other regularization methods which include methods such as truncated singular value expansion, semi–iterative methods, pre–conditioned conjugate gradient method (See Hanke [6] for a discussion on these methods).

4 Error Estimates Under Additional Assumptions

In this section, we derive certain error estimates with additional smoothness requirements on the generalized slution $\hat{x}$.

Let β_0, c_β and c_μ be as in the definition of $\{g_\alpha\}_{\alpha>0}$. We assume that

$$L\hat{x} \in R((A^*A)^\nu)$$

for some $\nu \in [0, \beta_0 - \frac{1}{2}]$, so that there exists $\hat{u} \in Z$ such that

$$L\hat{x} = (A^*A)^\nu \hat{u}.$$

For $0 \le \nu \le \beta_0 - \frac{1}{2}$, the quantities

$$\kappa_\nu = \max\{c_\nu + d_{\frac{1}{2}},\ c_{\nu+\frac{1}{2}} + d_1\}$$

and

$$\tilde{\kappa}_\nu = \max\left\{1+\tau_2,\ (\tau_2+c_0)^{\frac{2\nu}{2\nu+1}} c_0^{\frac{1}{2\nu+1}} + d_{1/2}\left(\frac{c_{\nu+\frac{1}{2}}}{\tau_1 - c_0}\right)^{\frac{1}{2\nu+1}}\right\}$$

will appear in our results.

PROPOSITION 4.1 *The following inequalities hold :*

(i) $\|L(\hat{x}-x_\alpha)\| \le c_\nu \|\hat{u}\| \alpha^\nu$,

(ii) $\|L(x_\alpha - \tilde{x}_\alpha)\| \le d_{1/2}\frac{\delta}{\sqrt{\alpha}}$,

(iii) $\|T(\hat{x}-x_\alpha)\| \le c_{\nu+1/2}\|\hat{u}\|\alpha^{\nu+1/2}$,

(iv) $\|T(x_\alpha - \tilde{x}_\alpha)\| \le d_1\delta$.

Proof.
The results are obtained from (3.1)–(3.4) by making use of the relations in (2.2)–(2.5).

□

4.1 Result under an a priori parameter choice

THEOREM 4.2 *Let $\tau > 0$ be such that $\|\hat{u}\| \le \tau$, and let $\alpha = \left(\frac{\delta}{\tau}\right)^{\frac{2}{2\nu+1}}$. Then*

$$\|\hat{x} - \tilde{x}_\alpha\| \le \kappa_\nu \eta\ e\left(M_1, \delta_\nu\right) = \kappa_\nu e\left(M_\eta, \delta\right),$$

where

$$\delta_\nu = \left(\frac{\delta}{\tau}\right)^{\frac{1}{2\nu+1}} \quad \textit{and} \quad \eta = \tau^{\frac{1}{2\nu+1}}\delta^{\frac{2\nu}{2\nu+1}}.$$

Proof.
From the assumptions on $\hat{u}$ and α, we have

$$\|\hat{u}\|\alpha^\nu \le \tau\left(\frac{\delta}{\tau}\right)^{\frac{2\nu}{2\nu+1}} = \tau^{\frac{1}{2\nu+1}}\ \delta^{\frac{2\nu}{2\nu+1}} = \eta$$

and

$$\frac{\delta}{\sqrt{\alpha}} = \delta\left(\frac{\tau}{\delta}\right)^{\frac{1}{2\nu+1}} = \tau^{\frac{1}{2\nu+1}}\delta^{\frac{2\nu}{2\nu+1}} = \eta.$$

Hence by Proposition 4.1 we have

$$\begin{aligned}
\|L(\hat{x}-\tilde{x}_\alpha)\| &\leq \|L(\hat{x}-x_\alpha)\| + \|L(x_\alpha-\tilde{x}_\alpha)\| \\
&\leq c_\nu \|\hat{u}\|\alpha^\nu + d_{\frac{1}{2}}\frac{\delta}{\sqrt{\alpha}} \\
&\leq \kappa_\nu \eta
\end{aligned}$$

and

$$\begin{aligned}
\|T(\hat{x}-\tilde{x}_\alpha)\| &\leq \|T(\hat{x}-x_\alpha)\| + \|T(x_\alpha-\tilde{x}_\alpha)\| \\
&\leq c_{\nu+\frac{1}{2}}\|\hat{u}\|\alpha^{\nu+\frac{1}{2}} + d_1\delta \\
&= \sqrt{\alpha}\left(c_{\nu+\frac{1}{2}}\|\hat{u}\|\alpha^\nu + d_1\frac{\delta}{\sqrt{\alpha}}\right) \\
&\leq \kappa_n \left(\frac{\delta}{\tau}\right)^{\frac{1}{2\nu+1}}\eta \\
&= \kappa_\nu \delta.
\end{aligned}$$

$\square$

4.2 Result under an posteriori parameter choice

We shall make use of the following known result. For its proof one may refer ([13], Lemma 2.3) or ([3], relation (2.46)).

Lemma 4.3 *If B is a bounded self adjoint operator on a Hilbert space H and $0 \leq \mu \leq 1$, then*

$$\|B^\mu x\| \leq \|Bx\|^\mu \|x\|^{1-\mu}, \quad \forall x \in H.$$

THEOREM 4.4 *Suppose $y = T\hat{x}$. Let $\tau > 0$ be such that $\|\hat{u}\| \leq \tau$, and let α be chosen according to (3.5). Then*

$$\|\hat{x}-\tilde{x}_\alpha\| \leq \tilde{\kappa}_\nu \eta e(M_1, \delta_\nu) = \tilde{\kappa}_\nu e\left(M_\eta, \delta\right),$$

where

$$\delta_\nu = \left(\frac{\delta}{\tau}\right)^{\frac{1}{2\nu+1}} \quad \text{and} \quad \eta = \tau^{\frac{1}{2\nu+1}}\delta^{\frac{2\nu}{2\nu+1}}.$$

Proof.

First we obtain an estimate for $\|L(\hat{x} - \tilde{x}_\alpha)\|$ which is different form that in Proposition 3.1. From (3.1), we obtain

$$\begin{aligned}
\|L(\hat{x} - x_\alpha)\| &\leq \|[I - A^*Ag_\alpha(A * A)]L\hat{x}\| \\
&= \|[I - A^*Ag_\alpha(A^*A)](A^*A)^\nu \hat{u}\| \\
&= \|(A^*A)^\nu [I - A^*Ag_\alpha(A^*A)]\hat{u}\|.
\end{aligned}$$

Writing $w = I - A^*Ag_\alpha(A^*A)]\hat{u}$, and using Lemma 4.3, we get

$$\begin{aligned}
\|(A^*A)^\nu w\| &= \left\| \left((A^*A)^{\frac{2\nu+1}{2}} \right)^{\frac{2\nu}{2\nu+1}} w \right\| \\
&\leq \left\| (A^*A)^{\frac{2\nu+1}{2}} w \right\|^{\frac{2\nu}{2\nu+1}} \|w\|^{\frac{1}{2\nu+1}}.
\end{aligned}$$

Now using relation (3.3) and Lemma 3.3, we have

$$\begin{aligned}
\|((A^*A)^{\frac{2\nu+1}{2}})w\| &= \|(A^*A)^{\nu+\frac{1}{2}}[I - A^*Ag_\alpha(A^*A)]\hat{u}\| \\
&= \|(A^*A)^{1/2}[I - A^*Ag_\alpha(A * A)]L\hat{x}\| \\
&= \|A[I - A^*Ag_\alpha(A * A)]L\hat{x}\| \\
&= \|T(\hat{x} - x_\alpha)\| \\
&\leq (\tau_2 + c_0)\delta.
\end{aligned}$$

Also by (2.2) ,

$$\|w\| \leq \|I - A^*Ag_\alpha(A^*A)\| \; \|\hat{u}\| \leq c_0\tau.$$

Hence

$$\|L(\hat{x} - x_\alpha\| \leq [(\tau_2 + c_0)\delta]^{\frac{2\nu}{2\nu+1}} (c_0\tau)^{\frac{1}{2\nu+1}}.$$

Again by Lemma 3.3 and (2.3),

$$\begin{aligned}
(\tau_1 - c_0)\delta &\leq \|T(\hat{x} - x_\alpha)\| \\
&\leq \|A[I - g_\alpha(A^*A)A^*A]L\hat{x}\| \\
&\leq \|A[I - g_\alpha(A^*A)A^*A](A^*A)^\nu \hat{u}\| \\
&\leq \|A[I - g_\alpha(A^*A)A^*A](A^*A)^\nu\| \; \|\hat{u}\| \\
&\leq c_{\nu+\frac{1}{2}} \alpha^{\nu+\frac{1}{2}} \tau.
\end{aligned}$$

From this we get

$$\frac{\delta}{\sqrt{\alpha}} \leq \left(\frac{c_{\nu+\frac{1}{2}}\tau}{\tau_1 - c_0} \right)^{\frac{1}{2\nu+1}} \delta^{\frac{2\nu}{2\nu+1}}.$$

Hence, by Proposition 3.1 (ii),

$$\|L(x_\alpha - \tilde{x}_\alpha)\| \le d_{1/2}\frac{\delta}{\sqrt{\alpha}} \le d_{1/2}\left(\frac{c_{\nu+\frac{1}{2}}\tau}{\tau_1 - c_0}\right)^{\frac{1}{2\nu+1}} \delta^{\frac{2\nu}{2\nu+1}}.$$

Thus

$$\begin{aligned} \|L(\hat{x} - \tilde{x}_\alpha)\| &\le \|L(\hat{x} - x_\alpha)\| + \|L(x_\alpha - \tilde{x}_\alpha) \\ &\le \left\{(\tau_2 + c_0)^{\frac{2\nu}{2\nu+1}} c_0^{\frac{1}{2\nu+1}} + d_{\frac{1}{2}}\left(\frac{c_{\nu+\frac{1}{2}}\tau}{\tau_1 - c_0}\right)^{\frac{1}{2\nu+1}}\right\}\eta \\ &= \tilde{\kappa}_\nu \eta. \end{aligned}$$

Since we already know that

$$\|T(\hat{x} - \tilde{x}_\alpha)\| = \|y - T\tilde{x}_\alpha\| \le \|y - \tilde{y}\| + \|\tilde{y} - T\tilde{x}_\alpha\| \le (1 + \tau_2)\delta,$$

we get the required result.

$\square$

Remarks.

We see that Theorems 3.2 and 3.4 are special cases of Theorems 4.2 and 4.4 respectively, obtained by taking $\nu = 0$. Also we observe that if $L = I$, then $e(M_1, \delta_\nu) \le 1$, so that a result of Vainikko [16] is recovered.

5 Error Estimates Using Hilbert Scales

Suppose the operators T and L are related to a Hilbert scale $\{H_s\}_{s\in\mathbf{R}}$ (cf. [7]) with $H_0 = X$ by

$$\|Tx\| \ge c\|x\|_{-a}, \qquad x \in X, \tag{5.1}$$

and

$$\|Lx\| \ge d\|x\|_b, \qquad x \in H_b \cap D, \tag{5.2}$$

for some $a > 0$, $b \ge 0$, $c > 0$ and $d > 0$.

To obtain our results we shall make use of the *interpolation inequality* (cf. [7])

$$\|x\|_s \le \|x\|_r^\theta \|x\|_t^{1-\theta} \qquad x \in H_t \tag{5.3}$$

where $r \le s \le t$ and $\theta = \frac{t-s}{t-r}$.

As in the previous section we assume that

$$L\hat{x} = (A^*A)^\nu \hat{u}$$

for some $\hat{u} \in Z$ and $\nu \in [0, \beta_0 - \frac{1}{2}]$. Let us denote by $\hat{\kappa}_\nu$, the quantity κ_ν if $\alpha = \left(\frac{\delta}{\rho}\right)^2$, and the quantity $\tilde{\kappa}_\nu$ if α is chosen according to (3.5).

THEOREM 5.1 *Let $\tau > 0$ be such that $\|\hat{u}\| \leq \tau$ and let α be as in Theorem 4.2 or 4.4. Then*

$$\|\hat{x} - \tilde{x}_\alpha\| \leq \hat{\kappa}_\nu \tau_\nu \delta^t,$$

where

$$t = \frac{2\nu}{2\nu+1} + \frac{b}{(a+b)(2\nu+1)},$$

and

$$\tau_\nu = \left(\frac{1}{c}\right)^{\frac{b}{a+b}} \left(\frac{1}{d}\right)^{\frac{a}{a+b}} \tau^{\frac{a}{(2\nu+1)(a+b)}}.$$

Proof.
By taking $r = -a$, $t = b$ and $s = 0$ in the interpolation inequality (5.3), we get

$$\|x\| \leq \|x\|_{-a}^{\frac{b}{a+b}} \|x\|_b^{\frac{a}{a+b}},$$

so that, if $x \in M_r$, then

$$\|x\| \leq \left(\frac{r}{d}\right)^{\frac{a}{a+b}} \left(\frac{\delta}{c}\right)^{\frac{b}{a+b}}.$$

This shows that

$$e\left(M_r, \delta\right) \leq \left(\frac{r}{d}\right)^{\frac{a}{a+b}} \left(\frac{\delta}{c}\right)^{\frac{b}{a+b}}. \tag{5.4}$$

Now the result follows from Theorems 4.2 and 4.4 by taking $r = \eta$ in the estimate (5.4).

□

Remark.
The above result unifies the cases of the known result for $L = I$ (c.f. [16], [8], [4]) and also the result in the setting of Hilbert scales (c.f. [14]).

Acknowlegements. This work is supported by a project grant from NBHM (DAE), Government of India, and it is completed during the author's visit to the Fachbereich Mathematik, Universität Kaiserslautern, Germany. I am gratefull to Professor Eberhard Schock for the invitaion and also for the lively discussions I had with him. Also, the partial travel support received from the International Mathmatical Union is gratefully acknowledged.

References

[1] J. Baumeister, *Stable Solution of Inverse Problems*, Friedr. Viewweg & Sohn, Braunsweig, Wiesbaden, Germany, 1987.

[2] H.W. Engl, *Regularization methods for the stable solution of inverse problems*, Surveys Math. Indust. **3** (1993), 71–143.

[3] H.W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996.

[4] C.W. Groetsch, *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind*, Boston - London - Melbourne: Pitman Pub. 1984.

[5] C.W. Groetsch, *Inverse Problems in Mathematical Sciences*, Braunschweg - Wiesbaden: Vieweg 1993.

[6] M. Hanke, *Regularization with differential operators: An alternative approach*, Numer. Funct. Anal. Optim. **13** (1992), 523–540.

[7] S.G. Krein and P. M. Petunin, *Scales of Banach spaces*, Russian Math. Surveys **21** (1966), 85–160.

[8] A.K. Louis, *Inverse und Schlecht Gestellte Probleme*, B.G.Teubner, Stuttgart, 1989.

[9] B.A. Mair, *Tikhonov regularization for finitely and infinitely smoothing operators*, SIAM J. Math. Anal. **25** (1994), 135–147.

[10] C.A. Micchelli, and T. J. Rivlin, *A survey of optimal recovery.* In: Optimal Estimation in Approximation Theory (eds.: C. A. Micchelli and T. J. Rivlin), Plenum Press, New York, 1977, pp. 1–53.

[11] V.A. Morozov, *Methods for Solving Incorrectly Posed Problems*, New York et al., Springer - Verlag 1986.

[12] M.T.Nair, M. Hegland, M. and R. S. Anderssen, *The trade–off between regularity and stability in Tikhonov regularization*, Mathematics of Computation **66** (1997), 193 – 206.

[13] M.T. Nair, *On Morozov's method for Tikhonov regularization as an optimal order yielding algorithm*, Zeit. Anal. und ihre Anwend. **18** (1999), 37–46.

[14] U. Tautenhahn, *Error estimates for regularization methods in Hilbert scales*, SIAM J. Numer. Anal. **33** (1996), 2120 – 2130.

[15] G.M. Vainikko, *The critical level of discrepancy in regularization mehtods*, U.S.S.R. Comput. Math. and Math. Physics., **23** (1983), 1–9.

[16] G.M. Vainikko, *On the optimality of methods for ill–posed problems*, Zeit. Anal. und ihre Anwend. **6** (1987), 351–362.