Morozov's discrepancy principle for Tikhonov regularization of severely ill-posed problems in finite-dimensional subspaces.

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Abstract

In this paper severely ill-posed problems are studied which are represented in the form of linear operator equations with infinitely smoothing operators but with solutions having only a finite smoothness. It is well known, that the combination of Morozov's discrepancy principle and a finite dimensional version of the ordinary Tikhonov regularization is not always optimal because of its saturation property. Here it is shown, that this combination is always order-optimal in the case of severely ill-posed problems.

1 Introduction

In this paper we consider the problem of finding an approximate solution to a linear ill-posed problem represented in the form of an operator equation

$$Ax = y, \tag{1}$$

where instead of y noisy data y_δ are available with $\|y - y_\delta\| \leq \delta$ and A is a linear compact injective operator between Hilbert spaces X and Y. Usually

(1) is called a severely ill-posed problem if its solution $x_0 = A^{-1}y$ has a finite "smoothness" in some sense, but A is an infinitely smoothing operator. More precisely, x_0 belongs to some subspace V continuously embedded in X, and the singular values of the canonical embedding operator J_V from V into X tend to zero with polynomial rate, while the singular values $\{\sigma_k\}_{k=1}^{\infty}$ of the operator A tend to zero exponentially. Following [5], [10] in such a situation it is natural to assume that

$$x_0 \in M_{p,\rho}^{\log}(A) := \{x : x = \ln^{-p}(A^*A)^{-1}v,\ \|v\| \le \rho\} \tag{2}$$

for some $p > p_0,\ \ \rho > 0$, where the operator function $\ln^{-p}(A^*A)^{-1}$ is well defined via spectral decomposition

$$A^*A = \sum_{k=1}^{\infty} \sigma_\kappa^2 \left(\Psi_k, \cdot\right) \Psi_k$$

of the operator A^*A, i.e.

$$\ln^{-p}\left(A^*A\right)^{-1} v = \sum_{k=1}^{\infty} \ln^{-p} \sigma_k^{-2} \left(\Psi_k, v\right) \Psi_k.$$

Here $(\cdot,\cdot)$ denotes an inner product in X. Moreover, without loss of generality, we assume that $\|A\| \le \theta \le e^{-1/2}$ i.e. $\sigma_k \le \theta \le e^{-1/2},\ k = 1, 2, \dots$.
From [5], [10] it follows, in particular, that the order of the best possible error for identifying x_0 from y_δ under the assumption (2) is $\ln^{-p} \frac{1}{\delta}$. The methods, proposed in [5], [10] for obtaining this optimal error, use the information about the structure of the source set $M_{p,\rho}^{\log}(A)$. For example in [10] a special variant of the method of generalized Tikhonov regularization has been derived which is optimal on the set $M_{p,\rho}^{\log}(A)$. In this method an approximation x_δ for x_0 is determined from the minimization problem

$$\|Ax - y_\delta\|^2 + c\delta^2 \left\|\ln^p \left(A^*A\right)^{-1} x\right\|^2 \to \min\,,$$

where c is some constant. On the other hand, in practice one often does not know the exact value of smoothness index p or some reasonable limits for it. Moreover, it is worth noting that the above variant of Tikhonov regularization is more complicated than ordinary Tikhonov regularization, where the functional

$$I_\alpha(x) = \|Ax - y_\delta\|^2 + \alpha \|x\|^2 , \ \alpha > 0 ,$$

is minimized in X. But the main difficulty in applying the ordinary Tikhonov regularization occurs in the choice of the regularizing parameter α without any a priori smoothness information about the exact solution. Such a posteriori methods of choosing α have been developed for the case of finitely smoothing operators A when (1) is not a severely ill-posed problem, and

$$x_0 \in Range \ (A^*A)^p \ . \tag{3}$$

It is well known, in this case the best possible error of the ordinary Tikhonov regularization is $\mathcal{O}\left(\delta^{2/3}\right)$ and it can not be improved by enlarging the smoothness index p in (3). Occasionally it is referred to as a saturation effect of the ordinary method of Tikhonov regularization. But on accout of the foregoing results [5], [10], the order of the accuracy $\mathcal{O}\left(\delta^{2/3}\right)$ can not be reached for problems (1), (2). Therefore, it is natural to expect that the above mentioned saturation effect will not reveal itself for severely ill-posed problems. In this paper we prove that such is indeed the case. More precisely, we show that the combination of some finite-dimensional version of ordinary Tikhonov regularization with Morozov's discrepancy principle of an a posteriori parameter selection is order optimal for the sets (2) with any $p > p_0$.

1.1 Finite-dimensional approximations

Any numerical realization of the Tikhonov regularization scheme requires to carry out all computations with a finite-dimensional approximation A_n instead of A. Usually, the variation problem $I_\alpha(X) \to \min$ is replaced by the finite-dimensional analogue

$$I_{\alpha,n}(x) := \|A_n x - y_\delta\|^2 + a \|x\|^2 \to \min ,$$

where A_n is some finite-dimensional approximation with $rank\,(A_n) = n$. The computation of the approximation $x_{\alpha,n}^\delta$ for $x_0 = A^{-1}y$ requires in this case to solve the linear operator equation

$$\alpha x + A_n^* A_n x = A_n^* y_\delta \ . \tag{4}$$

It is easy to see that $x_{\alpha,n}^\delta \in Range(A_n^*)$ and can be expressed in the form

$$x_{\alpha,n}^\delta = \sum_{j=1}^{n} x_j \Psi_j \ ,$$

where $\{\Psi_j\}_{j=1}^n$ is some basis of Range (A_n^*) . If

$$A_n = \sum_{i,j=1}^{n} a_{ij} \Phi_i \left(\Psi_j, \cdot\right) \ ,$$

where $\{\Phi_i\}_{i=1}^n$ is a basis of $Range\,(A_n)$, and the matrix $\mathbb{A} = \{a_{ij}\}_{i,j=1}^n$ is known, then (4) is equivalent to the following system of linear algebraic equations for determining $\bar{x} = \{x_j\}_{j=1}^n$:

$$\alpha \bar{x} + \mathbb{A}^T \Phi \mathbb{A} \Psi \bar{x} = \bar{b} \ ,$$

where

$$\bar{b} = \{b_j = \sum_{i=1}^{n} a_{ij} < \Phi_i, y_\delta >\}_{j=1}^n \ ,$$
$$\Psi = \{(\Psi_i, \Psi_j)\}_{i,j=1}^n \ , \ \Phi = \{< \Phi_i, \Phi_j >\}_{i,j=1}^n \ ,$$

and $< \cdot, \cdot >$ denotes an inner product in Y.
Keeping in mind that the singular values of the operator A involved in a severely ill-posed problem (1) tend to zero exponentially it is no restriction of the generality to assume that A_n is chosen in such a way that for some $q \in (0,1)$

$$\|A - A_n\| \le q^{n^\beta} \ , \ \beta > 0. \tag{5}$$

The following examples serve to illustrate this assumption.

Example 1 *Satellite gravity gradiometry problem.*

If we assume a spherical surface of the earth Ω_{r_1} as well as the satellite orbit Ω_{r_2}, $r_2 > r_1$, $\Omega_{r_i} = \{u \in \mathbb{R}^3,\ |u| = r_i\}$, $i = 1, 2$, then one of the problems arising in satellite gradiometry can be formulated as an equation (1) with the operator

$$Ax(u) := \frac{1}{4\pi r_1} \int\limits_{\Omega_{r_1}} \frac{d^2}{dr_2^2} \left(\frac{r_2^2 - r_1^2}{|u - v|^3} \right) x(v) d\Omega_{r_1}(v),\ u \in \Omega_{r_2} \ . \tag{6}$$

For more details we refer the reader to [3], [9]. Let $\{Y_{m,k},\ m = 0, 1, ..., k = 1, 2, ..., 2m + 1\}$ be a set of spherical harmonics L_2-orthonormalized with respect to the unit sphere in R^3. Then, as in [3] we can rewrite A in the form of a singular-value decomposition

$$Ax(u) = \sum_{m=0}^{\infty} \sigma_m \sum_{j=1}^{2m+1} Y_{m,j}^{(2)}(u) \left\langle Y_{m,j}^{(1)}, x \right\rangle \ ,$$

where

$$\sigma_m = \left(\frac{r_1}{r_2} \right)^m (m + 1)\,(m + 2)\, r_2^{-2} \ ,$$

$$Y_{m,j}^{(i)}(w) = \frac{1}{r_i} Y_{m,j} \left(\frac{w}{r_i} \right),\ w \in \Omega_{r_i}\ ,\ i = 1, 2,$$

$$\left\langle Y_{m,j}^{(1)}, x \right\rangle = \int\limits_{\Omega_{r_1}} Y_{m,j}^{(1)}(v) x(v) d\Omega_{r_1}(v) \ .$$

For $n = (m + 1)^2$ consider a finite-dimensional approximation $A_n = AQ_m$, where

$$Q_m x(v) = \sum_{\ell=0}^{m} \sum_{k=1}^{2\ell+1} Y_{\ell,k}^{(1)}(v) \left\langle Y_{\ell,k}^{(1)}, x \right\rangle$$

is the orthogonal projector on the corresponding spherical harmonic space, $rank\,(A_n) = rank(Q_m) = (m+1)^2$ Now, as in [7], one can show that

$$\|A - A_n\| \leq cn \left(\frac{r_1}{r_2}\right)^{\sqrt{n}} ,$$

where c is a constant independent of n. Thus, in the case under consideration the assumption (5) is fulfilled with $\beta = \frac{1}{2}$ and some $q \in \left(\frac{r_1}{r_2}, 1\right)$. By the way, in satellite gradiometry one assumes usually that the exact solution x_0 of (1), (6) is an element of the spherical Sobolev space

$$\mathcal{H}_s := \{f \in L_2(\Omega_{r_1}) : \|f\|_s^2 = \sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} \left(\ell + \frac{1}{2}\right)^{2s} \left|\left\langle Y_{\ell,k}^{(1)}, f\right\rangle\right|^2 < \infty\}$$

for some index $s > 0$. On the other hand, for the singular values $\{\sigma_\ell\}$ of the operator (6) the following relation is valid: $\ln \sigma_\ell^{-2} \asymp \left(\ell + \frac{1}{2}\right)$. Then there are some constants $c_1, c_2 > 0$ such that for any $f \in \mathcal{H}_s$

$$c_1 \|f\|_s \leq \left\|\ln^s (A^*A)^{-1} f\right\| \leq c_2 \|f\|_s \ .$$

It means that any element of $\mathcal{H}_s$ belongs to source set (2) with $p = s$.

Example 2 *Integral equations with analytic kernels.*

Many inverse problem from applications give rise to integral equations of the first kind

$$Ax(t) := \int_0^1 a(t, \tau)\, x(\tau) d\tau = y(t) \tag{7}$$

where the kernel $a(t, \tau)$ is an analytic with respect to t, τ.
A typical example of such a severely ill-posed problem is the Fujita equation having the form (7) with $a(t, \tau) = \frac{c\tau e^{-ct\tau}}{(1-e^{-c\tau})}$, where c is some constant, and occuring in the theory of a sedimentation-diffusion equilibrium in a centrifuge [6],[11]. Other examples of equations (7) with analytic kernels can be found

in [1],[2], where a conditional stability estimates could be proved, provided an a priori smoothness information about the solution was known. Moreover, in [1] Tikhonov regularization for such integral equations was studied, but the corresponding minimization problem involved the norm of the first derivative and the regularizing parameter was equal to δ^2. As a finite-dimensional approximation for the operator A from (7) one can take an integral operator A_n with degenerate kernel

$$a_n(t,\tau) = \sum_{i,j=1}^{n} a(t_i,t_j)\ell_i(t)\ell_i(\tau),$$

where $t_j = \cos^2 \frac{2j-1}{4n}\pi$, $j = 1,2,...,n$, are the zeros of Tschebyscheff polynomial of degree n corresponding to the interval [0,1], and $\ell_j(u)$ are the fundamental polynomials of degree $n-1$ for the pointwise Lagrange interpolation at $\{t_j\}$, i.e. $a_n(t_i,t_j) = a(t_i,t_j)$, $i,j = 1,2,...,n$.
By analogy with the case of one variable functions, the behaviour of an analytic kernel $a(t,\tau)$ can be characterized by the growth of its derivatives in the following way:

$$\left|\frac{\partial^{k+\ell}a(t,\tau)}{\partial t^k \partial \tau^\ell}\right| \leq r_a^{k+\ell}k!\ell!,\ k,\ell = 0,1,2,...,\ t,\tau \in [0,1], \tag{8}$$

where the constant r_a depends on a only. Consider the operators

$$L_{n,1}[f(\cdot,\tau)] := \sum_{i=1}^{n} f(t_i,\tau)\ell_i(t),\ L_{n,2}[f(t,\cdot)] = \sum_{j=1}^{n} f(t,t_j)\ell_j(\tau).$$

Using the well-known estimation of the remainder for the polynomial interpolation carried out on the zeros of the Tschebyscheff polynomial we have

$$|f(t,\tau) - L_{n,1}[f(\cdot,\tau)]| \leq \left(2^{2n-1}n!\right)^{-1} \max_{0\leq t,\tau\leq 1}\left|\frac{\partial^n f(t,\tau)}{\partial t^n}\right|, \tag{9}$$

$$|f(t,\tau) - L_{n,2}[f(t,\cdot)]| \leq \left(2^{2n-1}n!\right)^{-1} \max_{0\leq t,\tau\leq 1}\left|\frac{\partial^n f(t,\tau)}{\partial \tau^n}\right|, \tag{10}$$

Now we observe that

$$\begin{aligned} a(t,\tau) - a_n(t,\tau) &= (a(t,\tau) - L_{n,1}[a(\cdot,\tau)]) + (a(t,\tau) - L_{n,2}[a(t,\cdot)]) \\ &- (a(t,\tau) - L_{n,1}[a(\cdot,\tau)] - L_{n,2}[a(t,\cdot) - L_{n,1}[a(\cdot,\cdot)]]). \end{aligned}$$

Moreover, from (8)-(10) we obtain

$$\max\{|a\,(t,\tau)-L_{n,1}[a(\cdot,\tau)]|\,,\ |a(t,\tau)-L_{n,2}[a(t,\cdot)]|\}\le r_a^n 2^{1-2n},$$

$$\begin{aligned}&|a(t,\tau)-L_{n,1}[a(\cdot,\tau)]-L_{n,2}[a(t,\cdot)-L_{n,1}[a(\cdot,\cdot)]]|\le\\ &\le (2^{2n-1}n!)^{-1}\max_{0\le t,\tau\le 1}\left|\tfrac{\partial^n}{\partial\tau^n}[a(t,\tau)-\sum_{i=0}^{n}a(t_i,\tau)\ell_i(t)]\right|=\\ &=(2^{2n-1}n!)^{-1}\max_{t,\tau}\left|\tfrac{\partial^n a(t,\tau)}{\partial\tau^n}-L_{n,1}\left[\tfrac{\partial^n a(\cdot,\tau)}{\partial\tau^n}\right]\right|\le\\ &\le (2^{2n-1}n!)^{-2}\max_{t,\tau}\left|\tfrac{\partial^n}{\partial t^n}\left[\tfrac{\partial^n a(t,\tau)}{\partial\tau^n}\right]\right|\le r_a^{2n}2^{2-4n}.\end{aligned}$$

Then

$$\|A-A_n\|\le\max_{0\le t,\tau\le 1}|a(t,\tau)-a_n(t,\tau)|\le 4(\tfrac{r_a}{4})^n\left(1+\left(\tfrac{r_a}{4}\right)^n\right).$$

Thus, if $r_a\in(0,4)$ then in the considered case the assumption (5) is fulfilled with $\beta=1$ and some $q\in(\frac{r_a}{4},1)$.

2 A Posteriori parameter choice.

Following [8], we shall consider Morozov's discrepancy principle in a form tailored to the finite-dimensional version of the ordinary Tikhonov regularization.
Let a finite-dimensional approximation A_n be chosen such that

$$\|A-A_n\|\le\delta\rho^{-1} \tag{11}$$

From (5) it follows that for this purpose it will suffice to take rank $(A_n)=n\sim\ln^{\frac{1}{\beta}}\frac{1}{\delta}$.
We will choose the regularization parameter α out of the finite ordered set

$$\Delta_h(\delta)=\{\alpha:\alpha=\alpha_m=\alpha_0 h^m,\ m=0,1,....,\ \alpha\in(\delta^2,\alpha_0),\ h\in(0,1)\}.$$

Namely, we will compute $x^\delta_{\alpha_m,n}=(\alpha_m I+A_n^*A_n)^{-1}A_n^*y_\delta$ by solving

$$\alpha_m x + A_n^* A_n x = A_n^* y_\delta \ , \ m = 0, 1, 2, ...,$$

until

$$\left\| A_n x_{\alpha_m, n}^\delta - y_\delta \right\| \le d_0 \delta \tag{12}$$

where $d_0 \ge \frac{\rho}{\theta} + \frac{9}{4} + \frac{1}{\rho}$ and without loss of generality we assume that $\|y_\delta\| > d_0 \delta$. As we will see in the following this choice strategy insures the best possible order of accuracy $\mathcal{O}\left(\ln^{-p} \frac{1}{\delta}\right)$ on the source set (2) without any information about p.

Lemma 1 *Let* $\|A\| \le \theta < e^{-1/2}$ *and* $x_0 = A^{-1} y \in M_{p,\rho}^{\log}(A)$. *If* $x_\alpha = (\alpha I + A^* A)^{-1} A^* y$ *then for sufficiently small* $\alpha \in (0, e^{-2p})$

$$\|Ax_\alpha - y\| \le \theta^{-1} \rho \sqrt{\alpha} \ln^{-p} \tfrac{1}{\alpha}.$$

Proof. Using the spectral decomposition of the operator $A^* A$ we have

$$\|Ax_\alpha - y\| = \left\{ \sum_{k=1}^{\infty} \left[\tfrac{\alpha \sigma_k}{\alpha + \sigma_k^2} \ln^{-p} \sigma_k^{-2} \right]^2 |(\Psi_k, v)|^2 \right\}^{1/2} , \ \sigma_k \in (0, \theta] \ . \tag{13}$$

Consider the two functions: $g_\alpha(\lambda) = \frac{\lambda}{\alpha + \lambda^2} \ln^{-p} \lambda^{-2}$, $\lambda \in (0, \theta]$ and $g(t) = t \ln^{-p} t^2, t \in [\theta^{-1}, \infty)$. Simple calculations show that $g'(t) = 2(\ln t - p) \ln^{-p-1} t^2$. So, $g(t)$ monotonically decreases in $t \in (1, e^p)$ and increases in $t \in [e^p, \infty)$. Using this simple fact we prove now that for any $\lambda \in (0, \theta]$ and for sufficiently small $\alpha \in (0, e^{-2p})$

$$g_\alpha(\lambda) \le \theta^{-1} \frac{\ln^{-p} \frac{1}{\alpha}}{\sqrt{\alpha}} \tag{14}$$

Indeed, if $\lambda < \sqrt{\alpha}$ then $\ln^{-p} \frac{1}{\lambda^2} < \ln^{-p} \frac{1}{\alpha}$ and

$$g_\alpha(\lambda) \le \tfrac{\lambda}{\alpha} \ln^{-p} \lambda^{-2} < \tfrac{1}{\sqrt{\alpha}} \ln^{-p} \tfrac{1}{\alpha}.$$

Assume now that $\lambda \ge \sqrt{\alpha}$. If $e^{-p} > \theta$ then for $\lambda \in [\sqrt{\alpha}, \theta]$, $\frac{1}{\lambda} \in (e^p, \frac{1}{\sqrt{\alpha}}]$ and

$$g_\alpha(\lambda) \le \tfrac{1}{\lambda}\ln^{-p}\tfrac{1}{\lambda^2} = g(\tfrac{1}{\lambda}) \le g\left(\tfrac{1}{\sqrt{\alpha}}\right) = \tfrac{1}{\sqrt{\alpha}}\ln^{-p}\tfrac{1}{\alpha}$$

For $e^{-p} \le \theta$ and $\lambda \in [\sqrt{\alpha}, \theta]$, $\frac{1}{\lambda} \in [\theta^{-1}, \frac{1}{\sqrt{\alpha}}]$. Then keeping in mind the behaviour of $g(t)$, for sufficiently small α we have

$$\begin{aligned} g_\alpha(\lambda) &\le g\left(\tfrac{1}{\lambda}\right) \le \max\left\{\theta^{-1}\ln^{-p}\theta^{-2},\ \tfrac{1}{\sqrt{\alpha}}\ln^{-p}\tfrac{1}{\alpha}\right\} \\ &\le \max\left\{\theta^{-1}, \tfrac{1}{\sqrt{\alpha}}\ln^{-p}\tfrac{1}{\alpha}\right\} \le \theta^{-1}\tfrac{1}{\sqrt{\alpha}}\ln^{-p}\tfrac{1}{\alpha} \end{aligned}$$

Thus, the inequality (14) is proved. Now from (14) and (13) it follows that

$$\|Ax_\alpha - y\| = \alpha\left\{\sum_{k=1}^{\infty}\left[g_\alpha(\sigma_k)(\Psi_k, v)\right]^2\right\}^{1/2} \le \theta^{-1}\sqrt{\alpha}\ln^{-p}\tfrac{1}{\alpha}\cdot\|v\| \qquad (15)$$

The lemma is proved. ■

Lemma 2 *Assume the condition of Lemma 1. Then there exists an* $\alpha{=}\alpha_k \in \Delta_h(\delta)$ *satisfying the condition (12). Moreover, there exist* $d_1, d_2 > 0$ *such that*

$$d_1\delta \le \|Ax_{a_k} - y\| \le d_2\delta.$$

Proof. First of all we note that

$$\|x_0\| = \left\|\ln^{-p}(A^*A)^{-1}v\right\| \le \rho\sup_{0<\lambda\le\theta}\left|\ln^{-p}\tfrac{1}{\lambda^2}\right| \le \rho.$$

Moreover, for any compact operator B

$$\begin{aligned} &B(\alpha I + B^*B)^{-1} = (\alpha I + BB^*)^{-1}B, \\ &\left\|(\alpha I + B^*B)^{-1}\right\| \le \alpha^{-1},\ \left\|(\alpha I + B^*B)^{-1}B^*\right\| \le \tfrac{1}{2\sqrt{\alpha}}, \\ &\left\|B(\alpha I + B^*B)^{-1}B^*\right\| \le 1 \end{aligned}$$

As in [4], one can represent the residual as

$$Ax_\alpha - y = A_n x^\delta_{\alpha,n} - y_\delta + \sum\nolimits_1 + \sum\nolimits_2, \qquad (16)$$

where

$$\sum\nolimits_1 = \left(A_n\left(\alpha I + A_n^* A_n\right)^{-1} A_n^* - I\right)(y - y_\delta) =$$
$$= \left(\alpha I + A_n A_n^*\right)^{-1}\left(A_n A_n^* - \left(\alpha I + A_n A_n^*\right)\right)(y - y_\delta) =$$
$$= \alpha\left(\alpha I + A_n A_n^*\right)^{-1}(y - y_\delta)\,,$$

$$\left\|\sum\nolimits_1\right\| \leq \alpha\left\|\left(\alpha I + A_n A_n^*\right)^{-1}\right\|\|y - y_\delta\| \leq \delta\,,$$

$$\sum\nolimits_2 = \left(A\left(\alpha I + A^* A\right)^{-1} A^* - A_n\left(\alpha I + A_n^* A_n\right)^{-1} A_n^*\right) y =$$
$$= (AA^*(\alpha I + AA^*)^{-1} - \left(\alpha I + A_n A_n^*\right)^{-1} A_n A_n^*) y =$$
$$= \alpha\left(\alpha I + A_n A_n^*\right)^{-1}\left(AA^* - A_n A_n^*\right)\left(\alpha I + AA^*\right)^{-1} y$$

Now we estimate the norm of $\sum_2$ using the representation

$$\sum\nolimits_2 = I_1 + I_2 + I_3$$

where

$$I_1 = \alpha\left(\alpha I + A_n A_n^*\right)^{-1}\left(A - A_n\right)\left(A^* - A_n^*\right)\left(\alpha I + AA^*\right)^{-1} A x_0,$$

$$\|I_1\| \leq \frac{\|A - A_n\|^2}{2\sqrt{\alpha}}\|x_0\| \leq \frac{\rho}{2\sqrt{\alpha}}\|A - A_n\|^2 \leq \frac{\delta^2}{2\rho\sqrt{\alpha}}$$

$$I_2 = \alpha\left(\alpha I + A_n A_n^*\right)^{-1} A_n\left(A^* - A_n^*\right)\left(\alpha I + AA^*\right)^{-1} A x_0,$$

$$\|I_2\| \leq \frac{\|A^* - A_n^*\|\|x_0\|}{4} \leq \frac{\rho}{4}\|A - A_n\| \leq \frac{\delta}{4}$$

$$I_3 = \alpha\left(\alpha I + A_n A_n^*\right)^{-1}\left(A - A_n\right) A_n^*\left(\alpha I + AA^*\right)^{-1} A x_0 \quad .$$

$$
\begin{aligned}
\|I_3\| &\le \alpha \left\| (\alpha I + A_n A_n^*)^{-1} (A - A_n)(A_n^* - A^*)(\alpha I + AA^*)^{-1} Ax_0 \right\| \\
&+ \alpha \left\| (\alpha I + A_n A_n^*)^{-1} (A - A_n) A^* (\alpha I + AA^*)^{-1} Ax_0 \right\| \\
&\le \tfrac{\|A - A_n\|^2}{2\sqrt{\alpha}} \rho + \rho \|A - A_n\| \le \tfrac{\delta^2}{2\rho\sqrt{\alpha}} + \delta
\end{aligned}
$$

Then

$$
\left\| \sum\nolimits_2 \right\| \le \tfrac{5}{4}\delta + \tfrac{\delta^2}{\rho\sqrt{\alpha}} .
$$

From *Lemma* 1 and (16) it follows that

$$
\begin{aligned}
\left\| A_n x_{\alpha,n}^{\delta} - y_\delta \right\| &\le \|Ax_\alpha - y\| + \tfrac{9}{4}\delta + \tfrac{\delta^2}{\rho\sqrt{\alpha}} \le \\
&\le \theta^{-1} \rho \sqrt{\alpha} \ln^{-p} \tfrac{1}{\alpha} + \tfrac{9}{4}\delta + \tfrac{\delta^2}{\rho\sqrt{\alpha}} ,
\end{aligned}
$$

and, for example, for $\alpha = \delta^2 \ln^{2p} \frac{1}{\delta}$ we have

$$
\left\| A_n x_{\alpha,n}^{\delta} - y_\delta \right\| \le \theta^{-1}\rho\delta + \tfrac{9}{4}\delta + \rho\delta \ln^{-p} \tfrac{1}{\delta} \le d_0 \delta.
$$

Taking into account that $\left\| A_n x_{\alpha,n}^{\delta} - y_\delta \right\|$ monotonically depends on α and, moreover, for sufficiently small δ and $h > \ln^{-2p_o} \frac{1}{\delta}$ the interval $\left(\delta^2, \delta^2 \ln^{2p} \frac{1}{\delta}\right)$ contains at least one element of $\Delta_h(\delta)$ we conclude that there exists an $\alpha = \alpha_k \in \Delta_h(\delta)$ satisfying (12). From (16) for this α_k we have

$$
\begin{aligned}
\|Ax_{\alpha_k} - y\| &\le \left\| A_n x_{\alpha_{k,n}}^{\delta} - y_\delta \right\| + \delta + \tfrac{5}{4}\delta + \tfrac{\delta^2}{\rho\sqrt{\alpha_k}} \le \\
&\le d_0\delta + \tfrac{9}{4}\delta + \tfrac{\delta}{\rho} = d_2\delta
\end{aligned}
$$

On the other hand, from (13) and (16) it follows that

$$
\begin{aligned}
\|Ax_{\alpha_k} - y\| &= \left\| Ax_{h\alpha_{k-1}} - y \right\| \ge h \left\| Ax_{\alpha_{k-1}} - y \right\| \ge \\
&\ge h \left[\left\| A_n x_{\alpha_{k-1,n}}^{\delta} - y_\delta \right\| - \delta - \tfrac{5}{4}\delta - \tfrac{\delta^2}{\rho\sqrt{\alpha_{k-1}}} \right] \ge \\
&\ge h \left[d_0\delta - \tfrac{9}{4}\delta - \tfrac{\delta}{\rho} \right] = d_1\delta.
\end{aligned}
$$

Thus, we obtain the assertion of the lemma for $d_2 = \left(d_o + \frac{9}{4} + \frac{1}{\rho}\right)$, $d_1 = h\left(d_0 - \frac{9}{4}\delta - \frac{1}{\rho}\right)$. ■

Lemma 3 *Assume the conditions of Lemma 1. If* α *is chosen such that*

$$\|Ax_a - y\| \leq d_2\delta \ ,$$

then

$$\|x_0 - x_\alpha\| \leq c \ln^{-p} \tfrac{1}{\delta} \ ,$$

where the constant c *depends on* d_2, p *and* ρ.

To prove this lemma we use the following result by Mair [5].

Theorem [5]. *Let the operators* A *and* B *be such that for all* $x \in$ *Range* (B^*B)

$$\int \varphi\left(\tfrac{1}{\lambda}\right) \lambda d\mu_{x,x}(\lambda) \leq \|Ax\|^2 \ ,$$

where $\mu_{x,x}$ *is the spectral measure of* B^*B *and* $\varphi(s) = s\exp\left(-s^{-\frac{1}{2p}}\right)$. *If for some* $u \in X$ $\|Au\| \leq \varepsilon$ *and* $\|Bu\| \leq 1$ *then*

$$\|u\| \leq \ln^{-p} \tfrac{1}{\varepsilon^2}\left(1 + o(1)\right) \ .$$

Proof of Lemma 3. We put $u = \rho^{-1}\left(x_0 - x_\alpha\right)$. Then using the spectral decomposition of A^*A we have

$$u = \rho^{-1} \sum_{k=1}^{\infty} \tfrac{\alpha}{\alpha + \sigma_k^2} \ln^{-p} \sigma_k^{-2} \left(\Psi_k, v\right) \Psi_k.$$

If

$$B = \sum_{k=1}^{\infty} \tfrac{\alpha + \sigma_k^2}{\alpha} \ln^{p} \sigma_k^{-2} \left(\Psi_k, \cdot\right) \Psi_k$$

then it is easy to see that

$$\|Bu\|^2 = \rho^{-2} \sum_{k=1}^{\infty} \left(\Psi_k, v\right)^2 = \rho^{-2} \|v\|^2 \leq 1.$$

Moreover, for such B

$$\int \varphi\left(\tfrac{1}{\lambda}\right)\lambda d\mu_{x,x}(\lambda) = \sum_{k=1}^{\infty} \varphi\left(\left(\tfrac{\alpha}{\alpha+\sigma_k^2}\ln^{-p}\sigma_k^{-2}\right)^2\right)\left(\tfrac{\alpha+\sigma_k^2}{\alpha}\ln^{p}\sigma_k^{-2}\right)^2(\Psi_k,x)^2 =$$

$$= \sum_{k=1}^{\infty} \exp\left(-\left(\tfrac{\alpha}{\alpha+\sigma_k^2}\ln^{-p}\sigma_k^{-2}\right)^{-\frac{2}{2p}}\right)(\Psi_k,x)^2 =$$

$$= \sum_{k=1}^{\infty} \exp\left(-\left(\tfrac{\alpha+\sigma_k^2}{\alpha}\right)^{\frac{1}{p}}\ln\sigma_k^{-2}\right)(\Psi_k,x)^2 \leq$$

$$\leq \sum_{k=1}^{\infty} \exp\left(-\ln\sigma_k^{-2}\right)(\Psi_k,x)^2 = \|Ax\|^2 \ .$$

Now we are in the position to apply the above mentioned result of Mair [5]. Keeping in mind that

$$\|Au\| = \rho^{-1}\|Ax_0 - Ax_\alpha\| = \rho^{-1}\|Ax_\alpha - y\| \leq d_2\rho^{-1}\delta$$

we conclude that

$$\|u\| = \rho^{-1}\|x_0 - x_\alpha\| \leq \ln^{-p}\left(\tfrac{\rho^2}{d_2^2\delta^2}\right)(1+o(1)) \leq c\ln^{-p}\tfrac{1}{\delta} \ .$$

The lemma is proved.

3 The main result

Theorem. *Let* $\|A\| \leq \theta \leq e^{-1/2}$ *and* $x_0 = A^{-1}y \in M_{p\cdot\rho}^{\log}(A)$. *If* n *and* $\alpha = \alpha_m \in \Delta_h(\delta)$ *are chosen according to (11),(12) then*

$$\left\|x_0 - x_{\alpha_{m,n}}^{\delta}\right\| \leq c_0\ln^{-p}\tfrac{1}{\delta} \ ,$$

where the constant c_0 *depends on* ρ, p, θ, h, d_0.

Proof. First of all we note that

$$\left\|x_0 - x_{\alpha_{m,n}}^{\delta}\right\| \leq \|x_0 - x_{\alpha_m}\| + \left\|x_{\alpha_m} - x_{\alpha_{m,n}}\right\| + \left\|x_{\alpha_{m,n}} - x_{\alpha_{m,n}}^{\delta}\right\|,$$

where $x_{\alpha_{m,n}} = (\alpha_m I + A_n^* A_n)^{-1} A_n^* y$. If $\alpha = \alpha_m$ satisfies (12) then by virtue of Lemma 2

$$d_1\delta \leq \|Ax_{\alpha_m} - y\| \leq d_2\delta \ , \tag{17}$$

and from Lemma 3 we obtain

$$\|x_0 - x_{\alpha_m}\| \leq c \ln^{-p} \tfrac{1}{\delta} \ .$$

Moreover, the same steps as in the proof of Lemma 2 lead to the estimates

$$\begin{aligned} \left\|x_{\alpha_{m,n}} \text{-} x^\delta_{\alpha_{m,n}}\right\| &= \left\|(\alpha_m I + A_n^* A_n)^{-1} A_n^* (y\text{-}y_\delta)\right\| \leq \tfrac{\delta}{2\sqrt{\alpha_m}} \ , \\ \left\|x_{\alpha_m} \text{-} x_{\alpha_{m,n}}\right\| &= \left\|(\alpha_m I + A_n^* A_n)^{-1} \left[\alpha_m (A^* \text{-} A_n^*) + A_n^* (A_n \text{-} A) A^*\right] (\alpha_m I + AA^*)^{-1} Ax_0\right\| \\ &\leq I_1 + I_2 \end{aligned}$$

$$\begin{aligned} I_1 &= \alpha_m \left\|(\alpha_m I + A_n^* A_n)^{-1} (A^* - A_n^*) (\alpha_m I + AA^*)^{-1} Ax_0\right\| \leq \tfrac{\|A - A_n\|}{2\sqrt{\alpha_m}} \rho \leq \tfrac{\delta}{2\sqrt{\alpha_m}} \ , \\ I_2 &= \left\|(\alpha_m I + A_n^* A_n)^{-1} A_n^* (A_n - A) A^* (\alpha_m I + AA^*) Ax_0\right\| \leq \tfrac{\delta}{2\sqrt{\alpha_m}} \ . \end{aligned}$$

Summarizing these estimates we have

$$\left\|x_0 - x^\delta_{\alpha_{m,n}}\right\| \leq c \ln^{-p} \tfrac{1}{\delta} + \tfrac{3}{2} \tfrac{\delta}{\sqrt{\alpha_m}} \tag{18}$$

If the parameter choice strategy (12) gives us α_m such that $\alpha_m > \delta$, for example, then from (18) it follows

$$\left\|x_0 - x^\delta_{\alpha_{m,n}}\right\| \leq c \ln^{-p} \tfrac{1}{\delta} + \tfrac{3}{2}\sqrt{\delta} \leq c_0 \ln^{-p} \tfrac{1}{\delta} \tag{19}$$

On the other hand, if $\alpha_m \leq \delta$ then Lemma 1 and (17) lead to the inequality

$$d_1\delta \leq \|Ax_{\alpha_m} - y\| \leq \theta^{-1} \rho \sqrt{\alpha_m} \ln^{-p} \tfrac{1}{\alpha_m} \leq \theta^{-1} \rho \sqrt{\alpha_m} \ln^{-p} \tfrac{1}{\delta}.$$

It means that

$$\tfrac{\delta}{\sqrt{\alpha_m}} \leq \tfrac{\rho}{\theta d_1} \ln^{-p} \tfrac{1}{\delta} \ ,$$

and (18) again leads to (19). The theorem is proved. ∎

Thus, we have shown that the a posteriori parameter choice strategy (12) insures the best possible order of accuracy on the source set (2).

Remark. Our Theorem describes an asymptotic behavior of the accuracy of ordinary Tikhonov regularization for $\delta \to 0$. In this case it is natural to assume that $\delta^{\frac{1}{2}} < \ln^{-p} \frac{1}{\delta}$. On the other hand, if p is sufficiently large, then one can find an interval $(\delta_0, \delta_1) \subset (0, 1)$ such that for $\delta \in (\delta_0, \delta_1)$ $\ln^{-p} \frac{1}{\delta} < \delta^{\frac{1}{2}}$, that is the accuracy that can be reached for solving (1),(2) is estimated for $\delta \in (\delta_0, \delta_1)$ as $\mathcal{O}\left(\delta^{\frac{1}{2}}\right)$. It means that for noise level δ belonging to (δ_0, δ_1) inverse problem (1),(2) is not in fact severely ill-posed, because it can be solved at least with a power rate of accuracy regarding δ.

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