THE ELECTRON AS A SELFINTERACTING LIGHTLIKE POINTCHARGE: CLASSIFICATION OF LIGHTLIKE CURVES IN SPACETIME UNDER THE GROUP OF SO(1,3) MOTIONS.

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Abstract. Isotropic curves in Minkowski spacetime $\mathbf{M}(1,3)$ are classified with respect to multiplicity of curvature. The natural curve parameter is closely related to one introduced by E. Vessiot. Four sets of Frenet equations are obtained. This multiplicity of four corresponds to the four disconnected parts into which the lightcone of a two dimensional Minkowski space $\mathbf{M}(1,1)$ is decomposed after removal of its apex. This implies four Darboux bivectors and hence, four different equations of motion on Spin(1,3). Comparing with an explicit representation of real isotropic curves by M. Pinl et al., the way to the general solution of the equations of motion is discovered.

1. Introduction

With the aim to develop a relativistic quantum dynamic for several interacting electrons (positrons), David Hestenes [1] conjectured the electron as a selfinteracting pointcharge which moves on a lightlike, i.e., an isotropic curve in spacetime $\mathbf{M}(1,3)$.

Whereas differential geometry of complex isotropic curves has a long tradition [2], [3], real–valued isotropic curves in Minkowski space $\mathbf{M}(1,3)$ have been discussed quite rarely [4], [5].

The purpose of this article therefore is to give a selfcontained treatment of real isotropic curves in spacetime $\mathbf{M}(1,3)$ with no recourse to complex analysis. Real geometric algebra [6] will be made use of throughout.

Differential geometry of curves in $\mathbf{R}^n$ is so widespread [7] because the euclidean structure of $\mathbf{R}^n$ admits an obvious choice of a natural invariant curve parameter, namely, the arclength. Just this quantity vanishes for isotropic curves. The missing natural curve parameter caused the main problem in section 5 of ref. [1].

Section 2 of this article starts with the simple demonstration, that the natural parameter discovered by E. Vessiot [2] for complex curves also applies to non–straight real isotropic curves after a slight modification. Subsequently higher order derivatives of the position vector and various inner products of them are formed. In this way, two basic differential invariants are obtained which, supplemented by the Vessiot parameter, allow to represent all higher order invariants in terms of linear combinations of their derivatives. Squaring outer products of position vector derivatives up to order four, one finds that in $\mathbf{M}(1,3)$ isotropic curves of only double and triple curvature exist. This means, that the first four derivatives of the position vector with respect to the Vessiot parameter either span a 3–space $\mathbf{M}(1,2)$ in case of double curvature, or, the full 4–space $\mathbf{M}(1,3)$ in case of triple curvature.

In section 3, the derivatives of the position vector are related to an orthogonal frame by means of a generalized Gram–Schmidt process. This generalization is needed in order to take account of the pseudo–euclidean structure of $\mathbf{M}(1,3)$. Four linear mappings from the orthogonal frame to the derivatives of the curve position vector are found, which imply four different sets of Frenet equations. With the help of spatial rotations in $\mathbf{M}(1,3)$, this fourfold multiplicity is concentrated on a two–dimensional subspace $\mathbf{M}(1,1)$. The isotropic cone of this subspace, after removal of the apex, just decays into the four disconnected parts on which the four different sets of Frenet equations generate SO(1,1) motions. Guided by this geometrical explanation, two reflection elements of the group O(1,3) are found, which allow to obtain all three other Frenet systems from one by successive applications of them.

Section 4 is devoted to the derivation of equations of motion on one cover of Spin(1,3) from the Frenet equations acting on SO(1,3). Corresponding to the fourfold multiplicity of the Frenet equations, four Darboux bivectors result, which, by means of the two reflections mentioned above, successively may be reduced to one normal form.

Finally, in section 5, the quadrature–free, explicit representation of real isotropic curves by Max Pinl et al. [4] is translated into the highly efficient language of spacetime algebra. After calculating the isotropy group of the curve tangent vector (a 3–parameter Lie group), a closed–form expression is discovered for the general solution spinor of the equations of motion on Spin(1,3). This spinor then provides *all* solutions of the corresponding Monge problem. In general however, it is no longer free of quadratures and depends on the solutions of a linear, homogeneous second order differential equation over the field of complex valued functions composed of a scalar and a pseudoscalar part. The independent variable of these functions is the *real–valued* Vessiot parameter.

2. The natural parameter, differential invariants and comoving frames

Throughout this article the coordinate–free, elegant formulation of vector algebra in spacetime $\mathbf{M}(1,3)$ due to David Hestenes [8] is employed.

Let the position vector of a representative curve point in $\mathbf{M}(1,3)$ be lz, where l is a characteristic length and the dimensionless $z = (z_0 + \vec{z})\gamma_0$. Now, if $\alpha \in \mathbf{R}$ is an arbitrary parameter, a curve $z = z(\alpha)$ is called *lightlike* = *isotropic* if the tangent vector $\dfrac{dz}{d\alpha} = z'(\alpha)$ satisfies the isotropy condition

$$\left(\frac{dz}{d\alpha}\right)^2 = 0. \tag{2.1}$$

Writing $\dfrac{dz}{d\alpha} = n = (n_0 + \vec{n})\gamma_0$, $n^2 = 0$ implies $n = (\pm|\vec{n}| + \vec{n})\gamma_0 = \dfrac{dz}{d\alpha}$. So,

$\frac{d^2 z}{d\alpha^2} = \left(\pm \frac{\vec{n}}{|\vec{n}|} \cdot \frac{d\vec{n}}{d\alpha} + \frac{d\vec{n}}{d\alpha} \right) \gamma_0$, and hence

$$\left(\frac{d^2 z}{d\alpha^2} \right)^2 = - \left(\frac{\vec{n}}{|\vec{n}|} \times \frac{d\vec{n}}{d\alpha} \right)^2 \leq 0. \tag{2.2}$$

The conclusion therefore is: *Except for straight isotropic lines* $\frac{d\vec{n}}{d\alpha} = \mu \vec{n}$, $\mu \in \mathbf{R}$, *the vector* $\frac{d^2 z}{d\alpha^2}$ *always is spacelike*, i.e.,

$$\left(\frac{d^2 z}{d\alpha^2} \right)^2 < 0. \tag{2.3}$$

Thus, except for straight isotropic lines, the quantity $-\left(\frac{d^2 z}{d\alpha^2} \right)^2$ is always positive. Therefore it is ideally suited to take over the rôle played by the squared tangent vector or arclength squared in case of (nonisotropic) curves in euclidean space. Namely, to provide the *definition of a natural (invariant) curve parameter* β *according to* [2]:

$$\left(\frac{d^2 z(\beta)}{d\beta^2} \right)^2 = -1. \tag{2.4}$$

The derivation of the relation between the Vessiot parameter β and an arbitrary parameter α is straightforward. Equation (2.1) implies $\frac{dz}{d\alpha} \cdot \frac{d^2 z}{d\alpha^2} = 0$ and with $\frac{dz}{d\beta} = \frac{d\alpha}{d\beta} \frac{dz}{d\alpha}$, $\frac{d^2 z}{d\beta^2} = \frac{d^2 \alpha}{d\beta^2} \frac{dz}{d\alpha} + \left(\frac{d\alpha}{d\beta} \right)^2 \frac{d^2 z}{d\alpha^2}$ equations (2.1) and (2.4) lead to

$$\left(\frac{d\beta}{d\alpha} \right)^4 = - \left(\frac{d^2 z}{d\alpha^2} \right)^2. \tag{2.5}$$

The invariance of β may better be displayed when writing it in terms of (vector–valued) first and second differentials of z, $dz = d\alpha \frac{dz}{d\alpha}$, $d^2 z = d\alpha^2 \frac{d^2 z}{d\alpha^2}$, i.e.,

$$(d\beta)^4 = -(d^2 z)^2. \tag{2.6}$$

The advantage of employing β as a curve parameter is obvious. All quantities formed of derivatives of the vector z with respect to β then are connected with the curve in a motion– and parameter–invariant manner.

Higher derivatives soon get clumsy in the traditional notation of quotients and primes. Let me therefore meet the convention

$$z_1 = \frac{dz}{d\beta} = z'(\beta) = z', \quad z_{k+1} = z'_k = \frac{dz_k}{d\beta} = \frac{d^{k+1} z}{d\beta^{k+1}}, \quad k \geq 1. \tag{2.7}$$

With this compact notation, equations (2.1) and (2.4) lead to

$$z_1^2 = 0, \quad z_1 \cdot z_2 = 0, \quad z_2^2 = -1. \tag{2.8}$$

Taking successive higher order derivatives of this basic set of equations, as e.g. $z_2 \cdot z_3 = 0$, $z_2 \cdot z_2 + z_1 \cdot z_3 = 0 \quad \Rightarrow \quad z_1 \cdot z_3 = 1$, one finds the following

TABLE I
Scalar differential invariants

order	invariants
2	$z_1^2 = 0$
3	$z_1 \cdot z_2 = 0$
4	$z_2^2 = -1, \quad z_1 \cdot z_3 = 1$
5	$z_2 \cdot z_3 = 0, \quad z_1 \cdot z_4 = 0$
6	$\boxed{z_3^2 = \sigma}\,, \quad z_2 \cdot z_4 = -\sigma, \quad z_1 \cdot z_5 = \sigma$
7	$2z_3 \cdot z_4 = \sigma', \quad z_2 \cdot z_5 = -3z_3 \cdot z_4 = -\frac{3}{2}\sigma', \quad z_1 \cdot z_6 = \frac{5}{2}\sigma', \quad \sigma' = \frac{d\sigma}{d\beta}$
8	$\boxed{z_4^2 = -\tau}\,, \quad z_3 \cdot z_5 = \tau + \frac{\sigma''}{2}, \quad z_2 \cdot z_6 = -\frac{3}{2}\sigma'' - z_3 \cdot z_5 = -2\sigma'' - \tau$ $z_1 \cdot z_7 + z_2 \cdot z_6 = \frac{5}{2}\sigma'', \quad z_1 \cdot z_7 = \frac{9}{2}\sigma'' + \tau$
9	$2z_4 \cdot z_5 = -\tau', \quad z_3 \cdot z_6 = (z_3 \cdot z_4)'' + \frac{3}{2}\tau', \quad z_2 \cdot z_7 = -\frac{5}{2}\tau' - 5(z_3 \cdot z_4)'',$ $z_1 \cdot z_8 = 14(z_3 \cdot z_4)'' + \frac{7}{2}\tau', \quad z_4 \cdot z_5 = -\frac{\tau'}{2}, \quad z_3 \cdot z_6 = \frac{\sigma'''}{2} + \frac{3}{2}\tau',$ $z_2 \cdot z_7 = -\frac{5}{2}(\sigma''' + \tau'), \quad z_1 \cdot z_8 = 7\sigma''' + \frac{7}{2}\tau'$

Inspection of Table I shows that

$$\sigma = z_3^2 = -z_2 \cdot z_4, \quad \tau = -z_4^2 \tag{2.9}$$

may be choosen as lowest order invariants beyond β. Also, as mentioned already in the introduction, all inner products of higher order derivatives are expressible in terms of linear combinations of derivatives of σ and τ with rational coefficients.

In order to gain information about the shape of the curve $z(\beta)$, the square of the trivector $T = z_1 \wedge z_2 \wedge z_3 = ti$, $i = \gamma_0\gamma_1\gamma_2\gamma_3$ now is calculated: $T^2 = T \cdot T = t^2 = -\tilde{T} \cdot T = -(z_3 \wedge z_2 \wedge z_1) \cdot (z_1 \wedge z_2 \wedge z_3) = -(z_1 \cdot z_3)(z_3 \wedge z_2) \cdot (z_1 \wedge z_2) = z_2^2(z_1 \cdot z_3)^2$. Thus, according to Table I,

$$T^2 = (z_1 \wedge z_2 \wedge z_3)^2 = -1, \tag{2.10}$$

which means that the vectors z_1, z_2 and z_3 always span a 3–dimensional subspace of $\mathbf{M}(1,3)$. Isotropic curves in $\mathbf{M}(1,3)$ therefore either are straight or of double curvature at least. A condition for the occurrence of triple curvature is found after evaluating the square of the pseudoscalar $P = z_1 \wedge z_2 \wedge z_3 \wedge z_4 = T \wedge z_4 = -z_4 \wedge T$,

$$P^2 = \tilde{P} \cdot P = (z_4 \wedge z_3 \wedge z_2 \wedge z_1) \cdot (z_1 \wedge z_2 \wedge z_3 \wedge z_4) = -(\tau - \sigma^2) \leq 0. \tag{2.11}$$

Consequently $P = 0$ if and only if $\tau = \sigma^2$, whereas typically $\tau > \sigma^2$. The quantity

$$\kappa = \sqrt{\tau - \sigma^2} \geq 0, \tag{2.12}$$

which is needed subsequently, therefore is welldefined. That $P = 0$ implies $\kappa = 0$, may be seen from

$$z_4 = -z_4 T^2 = -(z_4 \cdot T) \cdot T - (z_4 \wedge T) \cdot T = -(z_4 \cdot T) \cdot T + P \cdot T \tag{2.13}$$

and

$$-(z_4 \cdot T) \cdot T = \frac{\sigma'}{2} z_1 + \sigma z_2. \tag{2.14}$$

In fact, $P = 0$ in (2.13) leads to

$$z_4 = \frac{\sigma'}{2} z_1 + \sigma z_2, \tag{2.15}$$

and hence, according to Table I, $\tau = -z_4^2 = \sigma^2$, or, $\kappa = 0$.

Summing up: Isotropic curves in $\mathbf{M}(1,3)$ are either straight lines or typically, i.e. for $\kappa > 0$, of triple curvature. Only in the atypical, particular case $\kappa = 0$ they degenerate to double curvature. Making the agreement to exclude straight isotropic lines, the vectors z_1, z_2, z_3 and z_4 for $\kappa > 0$ always generate a comoving, non–orthogonal frame for the curve z.

Let me now calculate the reciprocal frame $\{z^1, z^2, z^3, z^4\}$ defined by

$$z^k P^2 = (-1)^{k+1}(z_1 \wedge \ldots \wedge z_{k-1} \wedge z_{k+1} \wedge \ldots z_4) \cdot P, \quad k = 1, 2, 3, 4. \tag{2.16}$$

The result is

$$\begin{aligned} z^1 &= -\sigma z_1 + z_3 + \frac{\sigma'}{2\kappa^2}\left(z_4 - \frac{\sigma'}{2} z_1 - \sigma z_2\right) \\ z^2 &= -z_2 + \frac{\sigma}{\kappa^2}\left(z_4 - \frac{\sigma'}{2} z_1 - \sigma z_2\right) \\ z^3 &= z_1 \\ z^4 &= -\frac{1}{\kappa^2}\left(z_4 - \frac{\sigma'}{2} z_1 - \sigma z_2\right) \end{aligned} \tag{2.17}$$

and satisfies $z^k \cdot z_l = \delta^k_l = \begin{cases} 1 & \text{for} \quad k = l \\ 0 & \text{for} \quad k \neq l \end{cases}$ as it should.

In the exceptional case $\kappa = 0$, equation (2.15) is valid and (2.17) degenerates into

$$\begin{aligned} z^1 &= -\sigma z_1 + z_3 \\ z^2 &= -z_2 \\ z^3 &= z_1, \end{aligned} \tag{2.18}$$

if the definition

$$\frac{1}{\kappa^2}(z_4 - \frac{\sigma'}{2} z_1 - \sigma z_2) = 0$$

is met for $\kappa = 0$.

For $\kappa > 0$, all derivatives of z of order greater than four may be decomposed into the frame $\{z_1, z_2, z_3, z_4\}$. For instance

$$z_4' = z_5 = \sum_{k=1}^{4} z_k (z^k \cdot z_5), \quad \kappa > 0. \tag{2.19}$$

Making use of Table I, one finds

$$z_4' = z_5 = z_1\left(\kappa^2 + \frac{\sigma''}{2} - \frac{\kappa'\sigma'}{2\kappa}\right) + z_2\left(\frac{3}{2}\sigma' - \sigma\frac{\kappa'}{\kappa}\right) + z_3\sigma + z_4\frac{\kappa'}{\kappa}, \quad \kappa > 0, \tag{2.20}$$

which supplemented by

$$z_1' = z_2, \quad z_2' = z_3, \quad z_3' = z_4 \tag{2.21}$$

may be considered as a linear, homogeneous system which determines every isotropic curve of triple curvature in $\mathbf{M}(1,3)$ for given κ and σ. In the exceptional case $\kappa = 0$, equation (2.15) implies the system

$$z_3' = z_4 = \frac{\sigma'}{2}z_1 + \sigma z_2, \quad z_2' = z_3, \quad z_1' = z_2, \quad \kappa = 0, \tag{2.22}$$

which determines every isotropic curve of double curvature in $\mathbf{M}(1,3)$ for given σ. The systems (2.20), (2.21) and (2.22) via generalized Gram–Schmidt orthonormalizations of the frame z_k lead to Frenet formulas and Darboux bivectors for isotropic curves as is shown in the following section.

3. Frenet formulas

As indicated already in the introduction, the Gram–Schmidt orthonormalization process, known for euclidean vector spaces, needs modifications in order to account for the pseudo–euclidean structure of $\mathbf{M}(1,3)$. Let me start this modified orthonormalization with the triple $\{z_1, z_2, z_3\}$ since according to (2.10) it always generates a 3–dimensional frame. In this triple z_2 is a spacelike unit vector, i.e. $z_2^2 = -1$. So, without loss of generality, one may put

$$z_2 = \varepsilon_3 g_3, \quad \varepsilon_3^2 = 1, \quad \varepsilon_3 \in \mathbf{R}. \tag{3.1}$$

The vectors z_1 and z_3 are orthogonal to $g_3 = \varepsilon_3 z_2$, as seen from Table I. Thus, there are two normalizable orthogonal vectors g_0 and g_2, $\quad g_0 \cdot g_2 = 0$, such that $g_0 = \lambda_1 z_1 + \lambda_3 z_3$ and $g_2 = \mu_1 z_1 + \mu_3 z_3$, $\quad \lambda_k, \mu_k \in \mathbf{R}$. Now, $g_0 \wedge g_2 = g_0 g_2 = (\lambda_1\mu_3 - \lambda_3\mu_1) z_1 \wedge z_3 = D z_1 \wedge z_3$, and hence $(g_0 g_2)^2 = D^2(z_1 \wedge z_3)^2 = D^2 = -g_0^2 g_2^2 = +1$, which finally implies (except for an interchange of indices 0 and 2)

$$g_0^2 = 1 = -g_2^2 = -g_3^2, \quad g_0 \cdot g_2 = 0, \quad g_{\substack{0\\2}} \cdot g_3 = 0. \tag{3.2}$$

Note, that because of (3.2), the triple $\{z_1, z_2, z_3\}$ spans a Minkowski space $\mathbf{M}(1,2)$ *for which the signature had to be calculated!* This is one of the peculiarities of the pseudo–euclidean structure which do not occur in standard theory of curves [7]. The determination of linear mappings from z_1, z_3 to g_0, g_2 now is straightforward. One finds

$$\begin{gathered}\varepsilon_0 g_0\sqrt{2\varrho + \sigma} = \varrho z_1 + z_3, \quad \varepsilon_2 g_2\sqrt{2\varrho + \sigma} = -z_1(\varrho + \sigma) + z_3, \\ 2\varrho + \sigma > 0, \quad \varepsilon_\mu^2 = 1, \quad \varepsilon_\mu \in \mathbf{R}, \quad \mu = 0, 2, 3,\end{gathered} \tag{3.3}$$

with the inverse

$$z_1 = \frac{1}{\sqrt{2\varrho+\sigma}}\left(\varepsilon_0 g_0 - \varepsilon_2 g_2\right), \quad z_3 = \frac{1}{\sqrt{2\varrho+\sigma}}\left[\varepsilon_0 g_0(\varrho+\sigma) + \varrho\varepsilon_2 g_2\right]. \tag{3.4}$$

Equations (3.1) and (3.4) represent the triple $\{z_1, z_2, z_3\}$ as the image of $2^3 = 8$ different one–parameter families of linear mappings applied to the orthonormal triad $\{g_0, g_2, g_3\}$. This eightfold multiplicity arises because each of the parameters ε_0, ε_2 and ε_3 independently may take the values ± 1.

The question now is, how many of these different linear mappings can be made equivalent by means of the group SO(1,3) of Lorentz rotations? Or, otherwise asked, how many normal forms of (3.1) and (3.4) remain modulo the group SO(1,3) [9], when the triad $\{g_0, g_2, g_3\}$ is extended (lifted) to the tetrad

$$g_\mu = LUe_\mu\tilde{U}\tilde{L}, \quad \mu = 0,1,2,3, \tag{3.5}$$

where

$$e_\mu = R\gamma_\mu\tilde{R}, \quad \gamma'_\mu = 0, \quad R\tilde{R} = 1, \quad iR = Ri, \tag{3.6}$$

and

$$L = e^{\frac{\lambda}{2}e_2 e_0}, \quad \lambda \in \mathbf{R}, \quad U = e^{\frac{\pi}{4}(1-\varepsilon_3)e_3 e_2}. \tag{3.7}$$

The task of the spinor L is to cancel $\sqrt{2\varrho+\sigma}$ in (3.3) and (3.4), as will be shown later. The spinor U generates a spatial rotation in the e_3, e_2–plane with an angle π for $\varepsilon_3 = -1$ and with the angle 0, i.e., $U = 1$, for $\varepsilon_3 = +1$. For $\mu = 0,2,3$, equations (3.5) therefore read

$$g_0 = Le_0\tilde{L}, \quad g_2 = \varepsilon_3 Le_2\tilde{L}, \quad g_3 = \varepsilon_3 Le_3\tilde{L} = \varepsilon_3 e_3, \tag{3.8}$$

whence equation (3.1) yields

$$z_2 = e_3. \tag{3.9}$$

Inserting (3.8) into (3.4), one finds first of all the vector e_2 multiplied by the signfactor $\varepsilon_3\varepsilon_2$. Since however ε_0 and ε_2 are independent signfactors, no generality is lost in replacing $\varepsilon_3\varepsilon_2$ by ε_2 simply. In this way, one obtains

$$z_1 = L\frac{\varepsilon_0 e_0 - \varepsilon_2 e_2}{\sqrt{2\varrho+\sigma}}\tilde{L}, \quad z_3 = L\frac{\varepsilon_0 e_0(\varrho+\sigma) + \varrho\varepsilon_2 e_2}{\sqrt{2\varrho+\sigma}}\tilde{L}, \tag{3.10}$$

where

$$L = e^{\frac{\lambda}{2}e_2 e_0}, \quad \lambda \in \mathbf{R}, \quad 2\varrho + \sigma > 0. \tag{3.11}$$

In order to arrive at a particularly simple form of (3.10), the freedom of choosing λ appropriately is exploited. The formulas

$$L(\varepsilon_0 e_0 \mp \varepsilon_2 e_2)\tilde{L} = e^{\mp\varepsilon_0\varepsilon_2\lambda}(\varepsilon_0 e_0 \mp \varepsilon_2 e_2) = L\varepsilon_0 e_0\tilde{L} \mp L\varepsilon_2 e_2\tilde{L} \tag{3.12}$$

when inserted for z_1 in (3.10) lead to

$$z_1 = \frac{e^{-\varepsilon_0\varepsilon_2\lambda}}{\sqrt{2\varrho+\sigma}}(\varepsilon_0 e_0 - \varepsilon_2 e_2) = \varepsilon_0 e_0 - \varepsilon_2 e_2 = z_1 \tag{3.13}$$

for the choice

$$e^{\varepsilon_0\varepsilon_2\lambda}\sqrt{2\varrho+\sigma} = 1, \quad 2\varrho+\sigma > 0. \tag{3.14}$$

Adding and substracting equations (3.12), the choice (3.14) implies

$$\begin{gathered} 2L\varepsilon_{\substack{0\\2}} e_{\substack{0\\2}} \tilde{L} = \varepsilon_{\substack{0\\2}} e_{\substack{0\\2}} \left(\frac{1}{\sqrt{}} + \sqrt{}\right) + \varepsilon_{\substack{2\\0}} e_{\substack{2\\0}} \left(\frac{1}{\sqrt{}} - \sqrt{}\right), \\ \sqrt{} = \sqrt{2\varrho+\sigma}, \end{gathered} \tag{3.15}$$

and hence,

$$2z_3 = \varepsilon_0 e_0(1+\sigma) + \varepsilon_2 e_2(1-\sigma). \tag{3.16}$$

So, one may conclude from (3.9) and (3.10)

$$z_1 = \varepsilon_0 e_0 - \varepsilon_2 e_2, \quad z_2 = e_3, \quad 2z_3 = \varepsilon_0 e_0(1+\sigma) + \varepsilon_2 e_2(1-\sigma). \tag{3.17}$$

The process of orthonormalization is complete if z_4 also is decomposed with respect to the vectors e_μ in (3.6)

$$z_4 = \sum_{\mu=0}^{3} \alpha_\mu e_\mu, \quad \alpha_\mu \in \mathbf{R}, \quad e_\mu \cdot e_\nu = \gamma_\mu \cdot \gamma_\nu, \tag{3.18}$$

such that the following conditions from Table I hold

$$z_1 \cdot z_4 = 0, \quad z_2 \cdot z_4 = -\sigma, \quad z_3 \cdot z_4 = \frac{\sigma'}{2}, \quad z_4 \cdot z_4 = -\tau. \tag{3.19}$$

A nonunique result follows from (3.19) since the condition $z_4^2 = -\tau$ is quadratic

$$z_4 = \frac{\sigma'}{2}(\varepsilon_0 e_0 - \varepsilon_2 e_2) + \sigma e_3 + \varepsilon_1 e_1 \kappa, \quad \varepsilon_1 = \pm 1, \quad \kappa = \sqrt{\tau - \sigma^2} \geq 0. \tag{3.20}$$

Again the signfactor ε_1 in (3.20) may be moved to the vector e_2 by rotating the tetrad e_μ in the e_1, e_2–plane with the spinor $e^{\frac{\pi}{4}(1-\varepsilon_1)e_2 e_1}$, whence the signfactor ε_2 of e_2 becomes $\varepsilon_1\varepsilon_2$. Because of the independence of ε_0 and ε_2 in (3.20), the product $\varepsilon_1\varepsilon_2$ finally may be replaced by ε_2 without loss of generality. So, effectively ε_1 in (3.20) may be replaced by $\varepsilon_1 = 1$ and (3.17) together with (3.20) then comprise the result of this generalized Gram–Schmidt orthonormalization process, viz.,

$$\begin{gathered} z_1 = \varepsilon_0 e_0 - \varepsilon_2 e_2, \quad z_2 = e_3, \quad 2z_3 = \varepsilon_0 e_0(1+\sigma) + \varepsilon_2 e_2(1-\sigma) \\ z_4 = \frac{\sigma'}{2}(\varepsilon_0 e_0 - \varepsilon_2 e_2) + \kappa e_1 + \sigma e_3, \quad \varepsilon_0^2 = 1 = \varepsilon_2^2, \quad \varepsilon_{\substack{0\\2}} \in \mathbf{R}. \end{gathered} \tag{3.21}$$

Note, that this expression for z_4 also holds for $\kappa = \sqrt{\tau - \sigma^2} = 0$. In that case, z_4 just satisfies equation (2.15), as seen from (3.21).

Conclusion: There are precisely *four* linear mappings, non–equivalent under SO(1,3), from an orthonormal frame $e_\mu = R\gamma_\mu \tilde{R}$ to the vectors $z_k = \dfrac{d^k z(\beta)}{d\beta^k}$, k =1, 2, 3, 4. For these four mappings the normal form is given by (3.21).

From the trivector

$$z_1 \wedge z_2 \wedge z_3 = -\varepsilon_0 \varepsilon_2 e_0 e_2 e_3 = -\varepsilon_0 \varepsilon_2 e_1 i \tag{3.22}$$

and the pseudoscalar

$$z_1 \wedge z_2 \wedge z_3 \wedge z_4 = -\varepsilon_0 \varepsilon_2 \kappa e_0 e_1 e_2 e_3 = -\varepsilon_0 \varepsilon_2 \kappa i \tag{3.23}$$

one notes that the mapping (3.21): $e_\mu \to z_k$ is injective only for $\kappa > 0$. So, for $\kappa > 0$ equation (3.21) yields the inverse

$$\begin{gathered} \varepsilon_0 e_0 = \frac{1-\sigma}{2} z_1 + z_3, \quad \kappa e_1 = -\frac{\sigma'}{2} z_1 - \sigma z_2 + z_4, \\ \varepsilon_2 e_2 = -\frac{1+\sigma}{2} z_1 + z_3, \quad e_3 = z_2. \end{gathered} \tag{3.24}$$

Frenet equations are obtained from (3.24) by derivation with respect to β, $\varepsilon_0 e_0' = \dfrac{1-\sigma}{2} z_1' + z_3' - \dfrac{\sigma'}{2} z_1$, elimination of the "primes"over z by means of $z_k' = z_{k+1}$ and finally mapping back from the z_k to the e_μ with the help of (3.21). In detail: $\varepsilon_0 e_0' = \dfrac{1-\sigma}{2} z_2 + z_4 - \dfrac{\sigma'}{2} z_1 = \kappa e_1 + \dfrac{1+\sigma}{2} e_3$, or,

$$e_0' = \varepsilon_0 \kappa e_1 + \varepsilon_0 \frac{1+\sigma}{2} e_3. \tag{3.25}$$

In the calculation of e_1' according to $(\kappa e_1)' = \kappa e_1' + \kappa' e_1 = -\frac{\sigma'}{2} z_2 - \sigma z_3 + z_5 - \frac{\sigma''}{2} z_1 - \sigma' z_2$, the fifth derivative z_5 is needed. This however already has been decomposed into the frame $z_k, \quad k = 1, 2, 3, 4$ in (2.20), for which the reciprocal frame (2.17) was needed. So, finally one obtains the set of equations

$$\begin{gathered} e_1' = \varepsilon_0 \kappa e_0 - \varepsilon_2 \kappa e_2, \quad e_2' = \varepsilon_2 \kappa e_1 + \varepsilon_2 \frac{\sigma - 1}{2} e_3, \\ e_3' = \varepsilon_0 \frac{1+\sigma}{2} e_0 + \varepsilon_2 \frac{1-\sigma}{2} e_2, \end{gathered} \tag{3.26}$$

which together with (3.25) constitute a generalization of the euclidean Frenet equations [7] to (non–straight) isotropic curves in $\mathbf{M}(1,3)$. Recall, that because of $\varepsilon_0^2 = 1 = \varepsilon_2^2, \quad \varepsilon_{\substack{0\\2}} \in \mathbf{R}$, equations (3.25) and (3.26) in fact are *four* different sets of Frenet equations, which can not be made equivalent (coincident) by acting on the tetrad e_μ with arbitrary elements of the group SO(1,3)!

A geometrical explanation for the existence of *four* "branches" of Frenet equations is obvious when the dimensions spanned by the vectors e_1 and e_3 are projected away.

The fourfoldness remains untouched when projecting on the $\mathbf{M}(1,1)$ generated by e_0 and e_2, as seen from (3.21). In this $\mathbf{M}(1,1)$ the isotropic cone (lightcone) however consists of two intersecting straight lines and decays into *four* disconnected straight halflines when the intersection point (apex) is removed. It is evident, that the group SO(1,1) can act transitively only along each straight halfline separately. Elements of SO(1,1) can not map the motion along one halfline into the motion along another halfline. Therefore, precisely *four* different sets of Frenet equations are needed.

There is another lesson to be learned from this consideration. On the euclidean plane $\mathbf{R}^2$, which carries the $\mathbf{M}(1,1)$ with its quadratic form, *all four branches of the lightcone are obtained from a single one by successive reflections at the axes e_0 and e_2.* This also holds in $\mathbf{M}(1,3)$, since the dimension is even [10], viz.,

$$\Gamma_\nu e_\mu \Gamma_\nu^{-1} = \begin{cases} -e_\mu & \text{for} \quad \mu = \nu \\ e_\mu & \text{for} \quad \mu \neq \nu \end{cases}, \quad \Gamma_\nu i \Gamma_\nu^{-1} = -i, \tag{3.27}$$

where

$$\Gamma_\nu = e_\nu i, \quad i = e_0 e_1 e_2 e_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3. \tag{3.28}$$

It is easy to see, that by successive applications of the reflections (3.27) with $\nu = 0$ or $\nu = 2$, the Frenet equations (3.25) and (3.26) are reducible to their O(1,3)–equivalent normal form with $\varepsilon_0 = 1 = -\varepsilon_2$

$$\begin{gathered} e_0' = \kappa e_1 + \frac{1+\sigma}{2} e_3, \quad e_1' = \kappa e_0 + \kappa e_2, \quad e_2' = -\kappa e_1 + \frac{1-\sigma}{2} e_3, \\ e_3' = \frac{1+\sigma}{2} e_0 + \frac{\sigma - 1}{2} e_2. \end{gathered} \tag{3.29}$$

Correspondingly, equation (3.21) becomes

$$\begin{gathered} z_1 = e_0 + e_2, \quad z_2 = e_3, \quad 2z_3 = (1+\sigma) e_0 + (\sigma - 1) e_2, \\ z_4 = \frac{\sigma'}{2}(e_0 + e_2) + \kappa e_1 + \sigma e_3. \end{gathered} \tag{3.30}$$

Conclusion: For isotropic curves in even–dimensional Minkowski spaces there is precisely *one* set of Frenet equations with respect to the *full orthogonal group.*

The final remark in this section concerns the exceptional case of double curvature, $\kappa = 0$. Even though the assumption $\kappa > 0$ has been made for the derivation of (3.24), equations (3.29) and (3.30) remain valid in the case $\kappa = 0$. This may be checked by repeating the precedure of this section for $\kappa = 0$ with (2.15) instead of (2.20). The result is again (3.29) and (3.30) with $\kappa = 0$. So, one can work *in general* with (3.29) and (3.30), or on SO(1,3), with (3.21), (3.25) and (3.26).

4. Equations of motion on Spin(1,3)

Equation (3.6) relates the comoving tetrad e_μ to the tetrad γ_μ which is fixed in $\mathbf{M}(1,3)$, i.e., $\gamma_\mu' = \dfrac{d\gamma_\mu}{d\beta} = 0$, by means of the variable unimodular spinor $R = R(\beta)$. Equations of motion for R will now be derived from the condition of unimodularity

$$R\tilde{R} = 1 \ (= \tilde{R}R), \tag{4.1}$$

the mapping from Spin(1,3) on SO(1,3)

$$e_\mu = R\gamma_\mu \tilde{R}, \quad R' = \frac{dR(\beta)}{d\beta}, \tag{4.2}$$

and from the Frenet equations (3.25), (3.26).

Equation (4.1) implies that

$$\Omega = 2R'\tilde{R} = -\tilde{\Omega} \tag{4.3}$$

is a bivector, called *Darboux bivector*, whence (4.2) leads to the Frenet system

$$e'_\mu = \frac{1}{2}(\Omega e_\mu - e_\mu \Omega) = \Omega \cdot e_\mu . \tag{4.4}$$

That conversely a given Frenet system like (3.25) or (3.26) *uniquely* determines a Darboux bivector Ω rests on the *important theorem*, that for every bivector Ω

$$\sum_\mu (\Omega \cdot g_\mu) \wedge g^\mu = 2\Omega, \tag{4.5}$$

where $\{g_\mu\}$ is an *arbitrary* (non–orthogonal) basis in $\mathbf{M}(1,3)$ and $\{g^\mu\}$ the corresponding reciprocal basis defined by

$$g^\mu \cdot g_\mu = \delta^\mu_\nu = \begin{cases} 1 & \text{for} \quad \mu = \nu \\ 0 & \text{for} \quad \mu \neq \nu \end{cases} . \tag{4.6}$$

The proof of (4.5) follows from the formula

$$\begin{aligned} \sum_\mu [(a \wedge b) \cdot g_\mu] \wedge g^\mu &= \sum_\mu g^\mu \wedge [b(a \cdot g_\mu) - a(b \cdot g_\mu)] \\ &= \sum_\mu [(a \cdot g_\mu) g^\mu \wedge b - (b \cdot g_\mu) g^\mu \wedge a] = 2a \wedge b, \end{aligned} \tag{4.7}$$

valid for arbitrary grade 1 vectors a and b. Now, in every finite–dimensional linear vector space, a bivector Ω always is a finite linear combination of outer products of grade 1 vectors. By linearity therefore, equation (4.7) implies the theorem (4.5).

The calculation of the Darboux bivector is straightforward when the tetrad e_μ is choosen in (4.5). In that case equations (4.4) and (4.2) lead to

$$2\Omega = \sum_{\mu=0}^{3} e'_\mu \wedge e^\mu = e'_0 \wedge e_0 - \sum_{k=1}^{3} e'_k \wedge e_k = e'_0 \wedge e_0 + \sum_{k=1}^{3} e_k \wedge e'_k . \tag{4.8}$$

The following Darboux bivector is found from (3.25) and (3.26)

$$\Omega = \varepsilon_0 \left(\kappa e_1 + \frac{1+\sigma}{2} e_3 \right) e_0 + i\varepsilon_2 \left(\frac{1-\sigma}{2} e_1 + \kappa e_3 \right) e_0 , \tag{4.9}$$

with the same fourfold sign multiplicity with respect to ε_0 and ε_2 as the Frenet equations of course. The Darboux bivector is an extremely powerful tool in curve theory because it compactly comprises all structural information which sparsely is distributed in a Frenet system.

As a simple application, let me display the action of the reflections (3.27) on Ω, viz.,

$$\Gamma_0\Omega\Gamma_0^{-1} = -\varepsilon_0\left(\kappa e_1 + \frac{1+\sigma}{2}e_3\right)e_0 + i\varepsilon_2\left(\frac{1-\sigma}{2}e_1 + \kappa e_3\right)e_0 \tag{4.10}$$

$$\Gamma_2\Omega\Gamma_2^{-1} = \varepsilon_0\left(\kappa e_1 + \frac{1+\sigma}{2}e_3\right)e_0 - i\varepsilon_2\left(\frac{1-\sigma}{2} + \kappa e_3\right)e_0. \tag{4.11}$$

One notes again, that by successive applications of Γ_0 and Γ_2, as in (4.10), (4.11), every sign constellation of ε_0 and ε_2 can be achieved.

As it stands in (4.9), Ω depends on the comoving tetrad e_μ. For the purpose of integrating (4.3), it is however more convenient to refer Ω the fixed tetrad γ_μ in (4.2), i.e.,

$$\begin{aligned} &\Omega = R\omega\tilde{R}, \quad \omega = \tilde{R}\Omega R = \varepsilon_0\left(\kappa\vec{\sigma}_1 + \frac{1+\sigma}{2}\vec{\sigma}_3\right) + i\varepsilon_2\left(\frac{1-\sigma}{2}\vec{\sigma}_1 + \kappa\vec{\sigma}_3\right), \\ &\vec{\sigma}_k = \gamma_k\gamma_0, \end{aligned} \tag{4.12}$$

whence

$$2R' = \Omega R = R\omega = 2\frac{dR}{d\beta}. \tag{4.13}$$

Equations (4.12) and (4.13) describe the motion on Spin(1,3) in terms of the unimodular spinor R. When this spinor is calculated, z_1 is found from (3.21) and (4.2)

$$z_1 = \frac{dz}{d\beta} = R(\beta)[\varepsilon_0\gamma_0 - \varepsilon_2\gamma_2]\tilde{R}(\beta). \tag{4.14}$$

The position vector $lz(\beta)$ of a curve point then may be obtained from (4.14) by a quadrature.

5. Closed–form solutions on Spin(1,3)

For complex isotropic curves in $\mathbf{C}^3$, Karl Weierstrass (1866) found an explicit, closed–form expression *free of any quadratures.* Comparable explicit representations of real isotropic curves in pseudometric spaces, to my knowledge, may all be traced back to Max Pinl, a follower of Josef Lense at Vienna. Max Pinl et al. published in ref. [4] a quadrature–free explicit representation for real isotropic curves in $\mathbf{M}(3,1)$. This representation, equation (8) in ref. [4], correspondingly written for the signature $(1,3)$ using a parameter $\alpha \in \mathbf{R}$ *instead of* the Vessiot parameter β (2.4), then takes the form

$$z = z(\alpha) = e^{\frac{\alpha}{2}B}ae^{-\frac{\alpha}{2}B}, \quad B = (1+i)\vec{\sigma}_1, \quad a = (a_0 + \vec{a})\gamma_0 = a(\alpha), \tag{5.1}$$

$$a_0 = f_1 + f_3, \quad \vec{a} = -\vec{\sigma}_1(f + f_2) + \vec{\sigma}_2(f_3 - f_1) + \vec{\sigma}_3(f_2 - f), \quad f = f(\alpha),$$
$$f_k = \frac{d^k f(\alpha)}{d\alpha^k}, \quad k \geq 1. \tag{5.2}$$

The first derivative of z with respect to α is

$$\frac{dz}{d\alpha} = e^{\frac{\alpha}{2}B} \dot{a} e^{-\frac{\alpha}{2}B}, \tag{5.3}$$

where the "dot–derivative" $\dot{a}$ defines a shorthand notation for

$$\dot{a} = \frac{da}{d\alpha} + B \cdot a. \tag{5.4}$$

One finds

$$\dot{a} = (\gamma_0 + \gamma_2)(f_4 - f) = b \tag{5.5}$$

$$\ddot{a} = \dot{b} = (\gamma_0 + \gamma_2)(f_5 - f_1) + (\gamma_1 - \gamma_3)(f_4 - f). \tag{5.6}$$

Note, that according to (5.5) and (5.3), the vectors $\dot{a}$ and $\frac{dz}{d\alpha}$ are isotropic, as they should.

The relation between an arbitrary curve parameter α and the Vessiot parameter β is given by equation (2.5). With

$$\frac{d^2 z}{d\alpha^2} = e^{\frac{\alpha}{2}B} \ddot{a} e^{-\frac{\alpha}{2}B}, \tag{5.7}$$

equation (5.6) and

$$\left(\frac{d^2 z}{d\alpha^2}\right)^2 = (\ddot{a})^2 = -2(f_4 - f)^2, \tag{5.8}$$

the result

$$\left(\frac{d\alpha}{d\beta}\right)^4 2\left[\frac{d^4 f(\alpha)}{d\alpha^4} - f(\alpha)\right]^2 = 1 \tag{5.9}$$

is obtained. So, $z_1 = \dfrac{dz}{d\beta} = \dfrac{dz}{d\alpha} \Big/ \dfrac{d\beta}{d\alpha}$ may be written in the form

$$\frac{dz}{d\beta} = z_1 = \theta e^{\frac{\alpha}{2}B} (\gamma_0 + \gamma_2) e^{-\frac{\alpha}{2}B}, \quad B = (1 + i)\vec{\sigma}_1, \tag{5.10}$$

where

$$\theta = \frac{f_4 - f}{[2(f_4 - f)^2]^{1/4}}, \quad f_4 = \frac{d^4 f(\alpha)}{d\alpha^4}, \quad f = f(\alpha). \tag{5.11}$$

Equation (5.10) for z_1 should be compared with (4.14). One notes that $\theta \neq 0$ is necessary. A zero of θ means that z_1 is situated in the apex of the (lightcone) isotropic cone. Excluding from this article zeroes and sign changes of θ (particle transmutations), it is no loss of generality to assume $\theta > 0$ henceforth, i.e., to put $\varepsilon_0 = 1 = -\varepsilon_2$ in (4.14). The case $\theta < 0$ then follows from $\theta > 0$ with the help of sign–changing reflections (particle conjugations).

For $\theta > 0$, equations (3.11) and (3.12) with e_μ replaced by γ_μ provide the formula

$$\theta(\gamma_0 + \gamma_2) = e^{\lambda}(\gamma_0 + \gamma_2) = e^{\frac{\lambda}{2}\vec{\sigma}_2}(\gamma_0 + \gamma_2)e^{-\frac{\lambda}{2}\vec{\sigma}_2}, \quad \lambda = \ln\theta, \tag{5.12}$$

which allows to change (5.10) into

$$z_1 = e^{\frac{\alpha}{2}B} e^{\frac{\lambda}{2}\vec{\sigma}_2}(\gamma_0 + \gamma_2)e^{-\frac{\lambda}{2}\vec{\sigma}_2} e^{-\frac{\alpha}{2}B}. \tag{5.13}$$

The intention now is to determine the spinor R by comparison of (5.13) with (4.14) for $\varepsilon_0 = 1 = -\varepsilon_2$,

$$z_1 = R(\gamma_0 + \gamma_2)\tilde{R}. \tag{5.14}$$

Apart from being a product of the spinors

$$R_1 = e^{\frac{\alpha}{2}B}, \quad R_2 = e^{\frac{1}{2}\vec{\sigma}_2 \ln\theta}, \tag{5.15}$$

R in general may have a right–hand unimodular factor R_3,

$$R = R_1 R_2 R_3, \tag{5.16}$$

with the property

$$R_3(\gamma_0 + \gamma_2)\tilde{R}_3 = \gamma_0 + \gamma_2, \quad R_3 i = i R_3. \tag{5.17}$$

Equation (5.17) has the general solution

$$R_3 = e^{-\frac{\zeta}{2}N} e^{\frac{i_2\varphi}{2}} e^{\frac{\zeta}{2}N}, \quad i_2 = i\vec{\sigma}_2 = \gamma_1\gamma_3, \tag{5.18}$$

$$N = \vec{\sigma}_1(1 - \vec{\sigma}_2) = \gamma_1(\gamma_0 + \gamma_2) = \vec{\sigma}_1 - i\vec{\sigma}_3, \tag{5.19}$$

$$\zeta = \zeta_1 + i\zeta_2, \quad \zeta_{\frac{1}{2}}, \varphi \in \mathbf{R}. \tag{5.20}$$

Note, that R_3 generates a Lie group with 3 real parameters ζ_1, ζ_2 and φ, the socalled isotropy group of the vector $\gamma_0 + \gamma_2$. The bivector N, (5.19), is nilpotent

$$N^2 = 0, \tag{5.21}$$

and has the projection properties

$$\vec{\sigma}_2 N = N = -N\vec{\sigma}_2. \tag{5.22}$$

Now the following question will be answered affirmatively: Is the representation (5.16) of R sufficiently general that (4.13) implies the bivector ω in the canonical form (4.12)? Note, that ω depends on two arbitrary functions of β, namely $\sigma = \sigma(\beta)$

and $\kappa = \kappa(\beta) = \sqrt{\tau - \sigma^2} \geq 0, \quad \tau = \tau(\beta)$. The calculation of ω in (4.13) proceeds via

$$\omega = 2\tilde{R}R' = 2(\tilde{R}_3 R_3' + \tilde{R}_3\tilde{R}_2 R_2' R_3 + \tilde{R}_3\tilde{R}_2\tilde{R}_1 R_1' R_2 R_3) \tag{5.23}$$

and a little skill in spacetime algebra. The result is

$$\begin{aligned} \omega = \; & i_2\varphi' + N[i\varphi'\zeta + \zeta'(1 - e^{-i\varphi})] + (\ln\theta)'[\vec{\sigma}_2 + N\zeta(1 - e^{-i\varphi})] \\ & + \frac{\alpha'}{2}(1+i)\Bigg\{ \left[\left(\theta + \frac{1}{\theta}\right)\vec{\sigma}_1 + \left(\theta - \frac{1}{\theta}\right) i\vec{\sigma}_3\right] e^{i_2\varphi} \\ & + 2\theta\left[\vec{\sigma}_2\zeta(1 - e^{i\varphi}) + N\zeta^2(1 - \cos\varphi)\right] \Bigg\}, \end{aligned} \tag{5.24}$$

where the primes denote derivatives with respect to β. In (4.12) there are no contributions proportional to $i_2 = i\vec{\sigma}_2$ and $\vec{\sigma}_2$. To eliminate these unwanted terms from (5.24), it is important to exploit the freedom of choosing ζ in (5.20) and (5.18) in such a way, that these terms cancel, viz.,

$$\zeta\theta^2 e^{i\varphi}(e^{i\varphi} - 1)(1+i) = \frac{d}{d\alpha}\left[\log(\theta e^{i\varphi})\right], \quad \theta > 0. \tag{5.25}$$

With (5.25) equation (5.24) becomes

$$\begin{aligned} \omega = \; & N\left[i\varphi'\zeta + \zeta'(1 - e^{-i\varphi}) + \vartheta'\zeta(1 - e^{-i\varphi}) + \alpha'(1+i)e^{\vartheta}\zeta^2(1 - \cos\varphi)\right] \\ & + \alpha'(1+i)\vec{\sigma}_1 e^{\vec{\sigma}_2(\vartheta + i\varphi)}, \end{aligned} \tag{5.26}$$

where instead of $\theta > 0$ now the variable ϑ

$$\theta = e^{\vartheta} \tag{5.27}$$

is used. In terms of ϑ, equation (5.25) becomes

$$\zeta\alpha' e^{2(\vartheta + i\varphi)}(1 - e^{-i\varphi})(1+i) = \vartheta' + i\varphi'. \tag{5.28}$$

Equation (5.26) coincides with the normal form of ω, for $\varepsilon_0 = 1 = -\varepsilon_2$ in equation (4.12), i.e.,

$$\omega = \vec{\sigma}_3 + [\kappa + \frac{i}{2}(\sigma - 1)]N, \quad N = \vec{\sigma}_1(1 - \vec{\sigma}_2). \tag{5.29}$$

This is the clue to the general solution of (4.13). Comparing (5.26) with (5.29), one notes that the solutions found by Max Pinl et al. [4] only cover the particular case

$$\vartheta = -\ln(\alpha'\sqrt{2}), \quad \varphi = -\pi. \tag{5.30}$$

The *general solution* of (4.13) for ω according to (5.29), however is obtained by putting

$$R = i\phi' N + \frac{1}{2}\phi N\vec{\sigma}_1, \quad i\phi = \phi i, \quad R\tilde{R} = 1, \tag{5.31}$$

where $\phi = \phi(\beta)$ may be considered as a spinor–valued auxiliary potential. In the same way a corresponding quantity is introduced, when passing from the first–order Dirac equation to a second–order spinor equation. Insertion of (5.31) in (4.13), where ω is given by (5.29), after some algebra leads to

$$4\phi''(\beta) = [\sigma(\beta) - 2i\kappa(\beta)]\,\phi(\beta). \tag{5.32}$$

This reduces the task of finding the general solution of (4.13) to the problem of solving a linear, homogeneous second order differential equation for the spinor–valued function $\phi(\beta)$. In general, ϕ is a linear combination of the form

$$\phi = \phi_0 + i\phi_7 + \phi_1\vec{\sigma}_1 + \phi_2\vec{\sigma}_2 + \phi_3\vec{\sigma}_3 + i(\phi_4\vec{\sigma}_1 + \phi_5\vec{\sigma}_2 + \phi_6\vec{\sigma}_3), \tag{5.33}$$

which contains all elements of the even subalgebra of spacetime and each of the coefficients $\phi_\nu \in \mathbf{C}(i), \quad \nu = 0, \ldots, 7$ has to solve equation (5.32). This diversity of complex coefficients ϕ_ν however rapidely decreases on two remaining ones, when the quantities ϕN and $\phi' N$ are calculated. Only these enter into (5.31), viz.,

$$\phi N = [\phi_0 + \phi_2 + i(\phi_5 + \phi_7)]N + [\phi_1 + \phi_6 + i(\phi_4 - \phi_3)]\vec{\sigma}_1 N, \tag{5.34}$$

$$\phi' N = [\phi_0 + \phi_2 + i(\phi_5 + \phi_7)]' N + [\phi_1 + \phi_6 + i(\phi_4 - \phi_3)]'\vec{\sigma}_1 N, \tag{5.35}$$

since, because of the linearity of equation (5.32), every $\mathbf{C}(i)$–complex linear combination of solutions ϕ_ν again is a solution. Therefore, only ϕ_0 and ϕ_1 are needed in (5.34), (5.35), whence (5.31) and (5.32) take the form

$$R = i(\phi_0' + \phi_1'\vec{\sigma}_1)N + \frac{1}{2}(\phi_0 + \phi_1\vec{\sigma}_1)N\vec{\sigma}_1, \quad N = \vec{\sigma}_1(1 - \vec{\sigma}_2), \tag{5.36}$$

$$4\phi''_{\substack{0\\1}}(\beta) = [\sigma(\beta) - 2i\kappa(\beta)]\phi_{\substack{0\\1}}(\beta), \quad \phi_{\substack{0\\1}} \in \mathbf{C}(i), \quad \beta \in \mathbf{R}. \tag{5.37}$$

From (5.36) one notes, that the condition of unimodularity, $R\tilde{R} = 1$, implies the Wronski relation

$$R\tilde{R} = 2i(\phi_0\phi_1' - \phi_0'\phi_1) = 1 \tag{5.38}$$

for the complex–valued solutions ϕ_0 and ϕ_1 of (5.37).

Let me recapitulate: *Equations (5.36) – (5.38) provide the general solution of (4.12), (4.13) for the case* $\varepsilon_0 = 1 = -\varepsilon_2$. The further three sign combinations are obtained from this generic case with the help of reflections as in (4.10) and (4.11). For instance, $\varepsilon_0 = -1 = \varepsilon_2$,

$$\begin{aligned}
\omega_{--} &= -\left(\kappa\vec{\sigma}_1 + \frac{1+\sigma}{2}\vec{\sigma}_3\right) - i\left(\frac{1-\sigma}{2}\vec{\sigma}_1 + \kappa\vec{\sigma}_3\right) \\
&= \gamma_0 i\left[\kappa\vec{\sigma}_1 + \frac{1+\sigma}{2}\vec{\sigma}_3 - i\left(\frac{1-\sigma}{2}\vec{\sigma}_1 + \kappa\vec{\sigma}_3\right)\right]\gamma_0 i \\
&= \gamma_0 i\omega_{+-}\gamma_0 i = \gamma_0 i[\vec{\sigma}_3 + (\kappa + \frac{i}{2}(\sigma - 1)N]\gamma_0 i.
\end{aligned} \tag{5.39}$$

So,

$$2R'_{--} = R_{--}\omega_{--} = R_{--}\gamma_0 i\omega_{+-}\gamma_0 i, \tag{5.40}$$

or,

$$2\gamma_0 iR'_{--}\gamma_0 i = \gamma_0 iR_{--}\gamma_0 i\omega_{+-} \tag{5.41}$$

compared with

$$2R'_{+-} = R_{+-}\omega_{+-} \tag{5.42}$$

implies

$$R_{--} = \gamma_0 iR_{+-}\gamma_0 i = \gamma_0 R_{+-}\gamma_0 = R^*_{+-}. \tag{5.43}$$

A corresponding sign change of ε_2 is effectuated with the help of the formula

$$R_{++} = \gamma_2 iR_{+-}(\gamma_2 i)^{-1} = -\gamma_2 iR_{+-}\gamma_2 i = -\gamma_2 R_{+-}\gamma_2. \tag{5.44}$$

This article ends with a discussion of the canonical form of the isotropic tangent vector (4.14), as implied for $\varepsilon_0 = 1 = -\varepsilon_2$ by the generic representation (5.36). Making use of (5.21), the result of this straightforward calculation of the tangent vector $z_1 = v$ is

$$\begin{aligned} v\gamma_0 &= \phi_0\phi_0^* + \phi_1\phi_1^* + \vec{\sigma}_1(\phi_0\phi_1^* + \phi_0^*\phi_1) + \vec{\sigma}_2(\phi_0\phi_0^* - \phi_1\phi_1^*) + \vec{\sigma}_3 i(\phi_0^*\phi_1 - \phi_0\phi_1^*) \\ &= v_0 + \sum_{k=1}^{3}\vec{\sigma}_k v_k, \end{aligned} \tag{5.45}$$

where

$$\phi_\nu^* = \gamma_0\phi_\nu\gamma_0, \quad \nu = 0, 2 \tag{5.46}$$

denotes the complex conjugation on $\mathbf{C}(i)$. For those, who enjoy components and matrices, expressions (5.45) are repeated once more in the following notation

$$v_0 = (\phi_0^* \;\; \phi_1^*)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \tag{5.47}$$

$$v_1 = (\phi_0^* \;\; \phi_1^*)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \tag{5.48}$$

$$v_2 = (\phi_0^* \;\; \phi_1^*)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \tag{5.49}$$

$$v_3 = (\phi_0^* \;\; \phi_1^*)\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}. \tag{5.50}$$

May this article help to convince those, who still believe in Van der Waerden's definition of "spinors for physicists", that spacetime algebra is superior!

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