A QUANTUM/CLASSICAL ENTROPY CONCEPT FOR MEASURING PHASE-SPACE LOCALIZATION

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We present an entropy concept measuring phase-space localization in dynamical systems based on time-averaged phase-space densities. This entropy has a direct classical counterpart; its local scaling with $\ln \hbar^{-1/2}$ is the information dimension of the underlying invariant sets of the classical dynamics. The proposed entropy concept allows to visualize the global phase-space structure of the quantum dynamics and by comparison with the corresponding classical entropy to detect and quantify quantum localization.

1 The entropy concept

Basis quantity is the time-averaged probability density obtained from the time-evolution of an initial coherent state, i.e. $|\psi(t{=}0)\rangle = |p,q\rangle$,

$$\varrho(p,q;p',q') = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t \, |\langle p',q'|\psi(t)\rangle|^2 \, . \tag{1}$$

The entropy

$$S(p,q) = -\int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi\hbar} \, \varrho(p,q;p',q') \, \ln \varrho(p,q;p',q') \tag{2}$$

measures then the spreading of this initial Gaussian in phase space. This entropy has two important properties which quantify it as a suitable measure for quantum localization.

One advantage is that due to the time-averaging process, the influence of inherent quantum fluctuations on the entropy (2) is reduced to order $\hbar$. A comprehensive discussions of the fluctuation properties and the relation to other measures of localization like Wehrl's entropy can be found in [2].

Most important is that the quantum entropy (2) has a direct classical counterpart which can be defined in a entirely analogous manner on basis of time-averaged coarse-grained phase-space distributions. This classical density and thus the classical entropy depends on $\hbar$ as a coarse graining parameter; for details see [3,2]. The purely classical entropy had already been introduced by Núñez et al. [4] as an indicator of chaos in models of celestial dynamics.

The classical entropy $S_c(p,q)$ shows a local scaling behavior with the coarse-graining parameter $\hbar$ like

$$e^{S(p,q)} \sim \hbar^{-D/2}, \tag{3}$$

where D is the information dimension of the invariant set the point (p,q) belongs to.

2 Illustrative example

As an example we study the special case of a harmonically driven rotor ([1] and references therein)

$$H(t) = \frac{J^2}{2} - f \cos\phi \, \cos\omega t \,. \tag{4}$$

For the particular parameter choice of $f = 0.45$ and $\omega = 1$, the phase space shows a typical mixture of regular and chaotic motion (see Fig. 1).

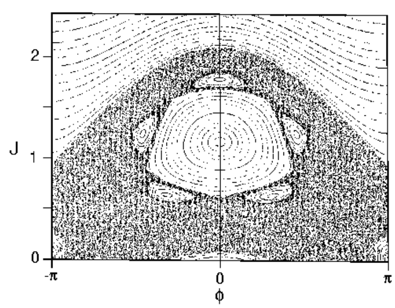

Figure 1: Classical stroboscopic Poincaré section for a driven rotor with $f=0.45$ and $\omega=1$. The lower half plane is omitted because of symmetry.

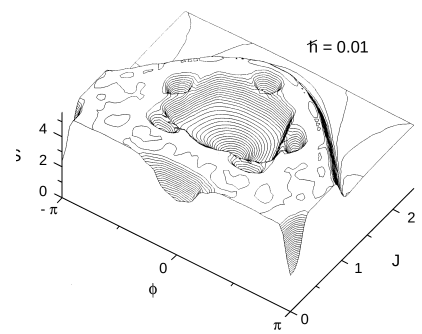

Figure 2: Quantum phase space entropy $S(J,\phi)$ shown as a contour plot over phase-space for $\hbar=0.01$.

The quantum phase space entropy computed for $\hbar=0.01$ is shown in Fig. 2 as contour plot over the (J,ϕ)-plane. One observes a clear image of the global dynamical long-time properties of the quantum system. Wave-packets placed in the high entropy regions spread over extended regions of phase space, those started in the low entropy remain localized on smaller regions. Therefore the chaotic sea appears as a high entropy plateau with the stability islands as valleys.

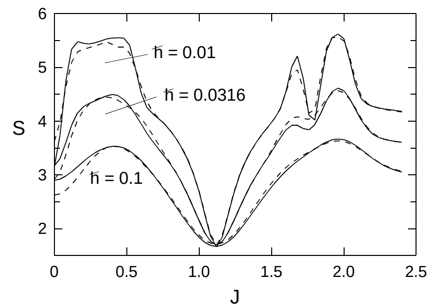

Figure 3: Classical (—–) and quantum (- - -) phase space entropies as a function of J for $\phi=0$ for $\hbar=0.1$, 0.0316, and 0.01.

Figure 3 finally shows the entropy as a function of the angular momentum J at $\phi=0$, i.e. a vertical cut through the plot shown in Figs. 2, but for three different values of $\hbar$ which were chosen equidistant in $\ln\hbar$. One recognizes a different scaling of $S(p,q)$ with $\ln\hbar$ in the different phase-space regions according to the dimension of the underlying invariant set (compare (3)). For periodic phase-space points the entropy approaches a constant which depends on the chosen squeezing parameter and can be estimated as $S\geq 1+\ln n$. where n is the period of the orbit times its multiplicity under the discrete symmetry transformation of the system.

3 Conclusion

We have introduced an entropy concept based on time-averaged phase-space densities which provides a useful *quantitative* measure for quantum localization effects. Its strengths are that it has a direct classical counterpart and that it is less affected from inherent quantum fluctuations than measures used so far. In this way, by comparing the quantum with the classical entropy, localization phenomena can be detected and quantified. Although the entropy as presented here is based on coherent states, the concept can be generalized to any basis set appropriate for the the system and the physical question under consideration.

References

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