The 'Ermakov-Lewis' invariants for Coupled Linear Oscillators

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Abstract

We consider N coupled linear oscillators with time-dependent coefficients. An exact complex amplitude - real phase decomposition of the oscillatory motion is constructed. This decomposition is further used to derive N exact constants of motion which generalise the so-called *Ermakov-Lewis invariant* of a single oscillator. In the Floquet problem of periodic oscillator coefficients we discuss the existence of periodic complex amplitude functions in terms of existing Floquet solutions.

The construction of time-dependent integrals of motion for the parametric harmonic oscillator is currently of interest for the canonical formulation of more general parametric systems [1, 2, 3, 4], their semiclassical quantisation [5, 6] and the theory of coherent and squeezed states [7, 8, 9, 10]. Integrals of one-dimensional motion that are quadratic in position and momentum were rediscovered by Lewis [11, 12, 13], unaware of Ermakov's results [14]. Time-dependent constants of motion, which are linear in momentum and position, have been developed beyond the single degree of freedom [15]. In either case the connection to the abstract symmetries of Noether's theorem has been established [16], and for the quadratic case the relation to Berry's phase is also revealed [17]. Previous studies of the invariants for N-dimensional linear oscillators haven been mainly restricted to the (decoupled) anisotropic case [18]

$$\ddot{x}_i + k_i(t)x_i = 0, \quad i = 1, \ldots, N \tag{1}$$

or to the isotropic oscillator

$$\ddot{x}_i + k(t)x_i = 0, \quad i = 1, \ldots, N \tag{2}$$

(see, e.g., [18, 19, 20, 21]). Here we report the first construction of the (quadratic) Ermakov-Lewis invariant for coupled parametric oscillators.

In the recent applications of the *Ermakov-Lewis invariant* in semiclassical narrow-tube quantisations [5, 22] and time-dependent normal-form transformations [4] and [23], the amplitude-phase analysis [14, 24] of solutions has been an important ingredient. We briefly summarise the basic equations for the single degree of freedom.

In the parametric oscillator equation:

$$\ddot{x} + k(t)x = 0, \tag{3}$$

the amplitude-phase ansatz $x(t) = \rho(t)\exp(i\phi(t))$ with real and positive functions $\rho(t)$ and $\phi(t)$, results in two separated equations:

$$\ddot{\rho} + k(t)\rho = \frac{\Lambda^2}{\rho^3}, \tag{4}$$

and

$$\dot{\phi} = \frac{\Lambda}{\rho^2}, \tag{5}$$

with an arbitrary parameter $\Lambda > 0$. The Ermakov-Lewis invariant (see Ref. [11], and more recent comments in Lichtenberg and Lieberman [25]):

$$\mathcal{L} = \frac{1}{2\Lambda}\left((p\rho(t) - q\dot{\rho}(t))^2 + \Lambda^2 \left(q/\rho(t)\right)^2\right), \tag{6}$$

is a non-trivial combination of the canonical variables $(p, q) = (\dot{x}, x)$, and $\rho(t)$, where $\rho(t)$ is a particular solution of the auxiliary Milne equation (4). For time-periodic coefficients $k(t)$, the Ermakov-Lewis invariant is an explicit expression for the invariant cross section (at fixed times) of the phase-space flow on vortex tubes, provided $\rho(t) > 0$ is a particular *periodic* Milne (amplitude) solution.

Indeed, since we consider an integrable case, the trajectories can be described in terms of an action variable and an angle variable. The Ermakov-Lewis invariant is identical to the action variable, but the corresponding angle variable has a time-dependent time-derivative (i.e. the angular velocity is time-dependent). In the periodic case they parametrize, together with the time variable, the surface of a manifold which has the topology of a cylinder, a so-called vortex tube, and the whole phase space is entirely stratified into such vortex tubes. The vortex tubes can be considered closed with the natural angle identification $0 = 2\pi$, as discussed in [5, 22]. When one is actually mapping a single calculated trajectory in the interval $[0, 2\pi]$ of (p, q, t)-space, this will appear winding along one of the vortex tubes and eventually it fills its surface.
This completes the brief summary.

In the present work we generalise the amplitude-phase idea to coupled equations of classical parametric oscillators, and later use this idea in the construction of the new invariants, which reduce to the Ermakov-Lewis invariants in the uncoupled limit.

The equations of motion for the coupled oscillators are given by

$$\ddot{\mathbf{r}} + \mathbf{k}(t)\mathbf{r} = \mathbf{0}, \tag{7}$$

where we have introduced the N-dimensional column vector

$$\mathbf{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix}, \tag{8}$$

and the *real symmetric* $N \times N$ 'angular velocity' matrix

$$\mathbf{k}(t) = \begin{pmatrix} k_{11}(t) & \cdots & k_{1N}(t) \\ \vdots & \vdots & \vdots \\ k_{N1}(t) & \cdots & k_{NN}(t) \end{pmatrix}. \tag{9}$$

Equation (7) appears in many different branches of theoretical physics: collisions of atoms, molecules and nuclei; scattering of wave components propagating in inhomogeneous media; mechanical oscillations, stability analysis of nonlinear oscillations, etc..

We try to develop an amplitude-phase decomposition which generalises the approach in [5]. We put as basic amplitude-phase solutions:

$$\mathbf{r}(t) = \mathbf{R}_j(t) = \mathbf{u}_j(t) \exp(i\phi_j(t)), \tag{10}$$

where $\mathbf{u}_j(t)$ is a so far unspecified *complex* vector function while the phase $\phi_j(t)$ is assumed to be *real and positive.* The important assumption is the realness of the phase, this seems to rule out the realness of $\mathbf{u}_j(t)$ in the vector case (but not in the scalar case). Similar considerations have been made by Fulling [26, 27] in his search for *approximate* solutions of (7) in a different context.

An independent oscillatory solution given by the complex conjugate of (10), i.e.

$$\mathbf{r}^*(t) = \mathbf{R}_j^*(t) = \mathbf{u}_j^*(t) \exp(-i\phi_j(t)), \tag{11}$$

will also solve the equation (7) with real $\mathbf{k}(t)$. The index refers to our expectation of finding N independent *pairs* of solutions of this form. On substituting the ansatz (10) into (7), we find

$$\ddot{\mathbf{u}}_j - \dot{\phi}_j^2 \mathbf{u}_j + \mathbf{k}(t)\mathbf{u}_j + i\left(\ddot{\phi}_j \mathbf{u}_j + 2\dot{\phi}_j \dot{\mathbf{u}}_j\right) = \mathbf{0}. \tag{12}$$

We have the freedom to introduce an auxiliary condition, since we introduced the phase $\phi_j(t)$ in addition to the complex vector $\mathbf{u}_j$. In analogy with the 1D case we could choose to set the imaginary parts of the equations equal to zero; but this turns out to be too restrictive in general. Instead, by suitable scalar multiplications of equation (12) and its complex conjugate, followed by a subtraction, we find:

$$\ddot{\mathbf{u}}_j \cdot \mathbf{u}_j^* - \ddot{\mathbf{u}}_j^* \cdot \mathbf{u}_j + 2i\,(\ddot{\phi}_j\, u_j^2 + \dot{\phi}_j\, du_j^2/dt) = 0, \tag{13}$$

where $u_j^2 = \mathbf{u}_j \cdot \mathbf{u}_j^*$ is the 'real amplitude' squared. Our choice is to eliminate the bracketed terms, which results in the generalized 'local angular velocity' relation for the amplitude-phase solutions:

$$\dot{\phi}_j = \frac{\Lambda_j}{u_j^2}, \tag{14}$$

where Λ_j is a constant that has been referred to as the angular momentum parameter or the mixing parameter [4]. Provided the norms of the complex amplitudes do not vanish, and $\Lambda_j \neq 0$, the function $\dot{\phi}_j$ has a definite sign. In this work we consider $\Lambda_j > 0$.

¿From (13) we also see that:

$$\dot{\mathbf{u}}_j \cdot \mathbf{u}_j^* - \dot{\mathbf{u}}_j^* \cdot \mathbf{u}_j = \text{const.} \tag{15}$$

A second look at the equation shows that the right hand side of (15) will differ from zero if the initial conditions are not purely real. This can in general be seen as an invariant for the vector-Milne solutions. Lets define this vector-Milne invariant as a real quantity $\mathcal{M}_j$ from the equation

$$\mathcal{M}_j = \left(\dot{\mathbf{u}}_j \cdot \mathbf{u}_j^* - \dot{\mathbf{u}}_j^* \cdot \mathbf{u}_j\right)/(2i) = \mathrm{Im}\left[\dot{\mathbf{u}}_j \cdot \mathbf{u}_j^*\right]. \tag{16}$$

This invariant is always zero in the scalar Milne equation and plays no role in the corresponding parametric oscillator dynamics.

Any solution of the oscillator equation (7) is specified by $2N$ independent integration constants (e.g. the initial position and velocity). We associate with this equation $2N$ complex fundamental solutions $(\mathbf{f}_j, \mathbf{f}_j^*)$, $j = 1, \cdots, N$, that later will be subject to the amplitude-phase decompositions (10) and (11). ¿From the theory of (more general) linear equations [28] with Hermitian symmetry $\mathbf{k}^\dagger(t) = \mathbf{k}(t)$, arbitrary real solutions $\mathbf{r}(t)$ would thus have $2N$ Wronskian constants (associated with any set of complex fundamental solutions $\mathbf{f}_j$, $j = 1, \cdots, N$), namely

$$W_j = \mathbf{p} \cdot \mathbf{f}_j - \mathbf{r} \cdot \dot{\mathbf{f}}_j, \tag{17}$$

with $\mathbf{p} = \dot{\mathbf{r}}$. These constant Wronskians are, together with their complex conjugates, nothing less than the 'linear' time-dependent integrals obtained from Noether's theorem by Castaños et. al. [15]. The values depend of course on various initial conditions of the ingredients. To fit into the Hamiltonian scheme of symplectic phase-space flow, it is important that the fundamental solutions form a complexified version of a symplectic matrix basis, which is always possible to construct (see Lichtenberg and Lieberman [25]). The main requirement for complex fundamental solutions is the proper normalization according to:

$$\dot{\mathbf{f}}_j \cdot \mathbf{f}_j^* - \mathbf{f}_j \cdot \dot{\mathbf{f}}_j^* = 2i, \;\; j = 1, 2, \cdots, N. \tag{18}$$

The main item in the further refinement of the Ermakov-Lewis invariants is the amplitude-phase decomposition of the set of fundamental solutions. The differentiated expressions of our amplitude-phase solutions $\mathbf{R}_j$ are given by:

$$\dot{\mathbf{R}}_j = \left(\dot{\mathbf{u}}_j + i\Lambda_j \frac{\mathbf{u}_j}{u_j^2}\right) \mathrm{e}^{i\phi_i}, \tag{19}$$

and

$$\dot{\mathbf{R}}_j^* = \left(\dot{\mathbf{u}}_j^* - i\Lambda_j \frac{\mathbf{u}_j^*}{u_j^2}\right) \mathrm{e}^{-i\phi_i}. \tag{20}$$

The normalization (18) is generally not satisfied by the basic amplitude-phase functions. The normalization constants n_j are therefore determined from the initial values. We find the condition:

$$n_j^2(\dot{\mathbf{u}}_j(0) \cdot \mathbf{u}_j^*(0) - \dot{\mathbf{u}}_j^*(0) \cdot \mathbf{u}_j(0) + 2i\Lambda_j) = 2i, \;\; j = 1, 2, \cdots, N, \tag{21}$$

i.e., from (15)

$$n_j = \frac{1}{\sqrt{\mathcal{M}_j + \Lambda_j}}. \tag{22}$$

In terms of normalised amplitude-phase solutions, the Wronskian constant W_j is now expressed as

$$W_j = \frac{1}{\sqrt{\mathcal{M}_j + \Lambda_j}} \left(\dot{\mathbf{r}} \cdot \mathbf{u}_j - \mathbf{r} \cdot \dot{\mathbf{u}}_j - i\Lambda_j \left(\frac{\mathbf{r} \cdot \mathbf{u}_j}{u_j^2}\right)\right) \mathrm{e}^{i\phi_j(t)}. \tag{23}$$

Finally we construct the N generalised 'Ermakov-Lewis invariants' according to the prescription

$$\mathcal{L}_j = \frac{1}{2} W_j W_j^* \,. \tag{24}$$

We are thus left with only half of the number of integration constants, i.e. N; one Ermakov-Lewis invariant for each dimension (or normal mode). Formula (24) is the main result of this letter.

In the limit of decoupled oscillators the independent amplitude vectors take the form

$$\mathbf{u}_j(t) = \mathbf{u}_j^*(t) = \begin{pmatrix} \vdots \\ 0 \\ \rho_j(t) \\ 0 \\ \vdots \end{pmatrix}, \tag{25}$$

with the real function $\rho_j(t)$ satisfying Milne's differential equation (4) with $k(t) = k_{jj}(t)$, and we immediately see that the realness of $\rho_j(t)$ implies that $\mathcal{M}_j = 0$. Hence, the general Ermakov-Lewis invariants reduce to the form (6) for each separate oscillator in agreement with previous studies [18].

The present result has a great potential of further developments. The Ermakov- Lewis invariants are currently of interest in canonical formulations of the action-angle type, where the angular velocities realistically follow the true 'geometric' vortex-tube motion [4, 6, 22]. Indeed, some tubes in time-periodic models [6] look dramatically flat and folded so that the winding process is certainly not uniform. Furthermore, the approximations introduced in [23] suggest a theoretical frame for systematic (higher-order) adiabatic descriptions of coupled oscillators, related to Fulling's work [26, 27].

The amplitude phase decomposition of a set of independent solutions is the crucial step in the construction of the Ermakov-Lewis invariants. This decomposition *per se* has an interesting theoretical aspect connected to it. It introduces quantities (amplitude and phase) that can be made considerably less oscillating than the solution itself. However, the decomposition used here turns out not to be unique. In fact, the 'mixing parameters' Λ_j are rather arbitrary, and some choices of them can of course be inadequate for the description of the solution (this will be demonstrated for Floquet solutions below). Still all choices lead to exact representations of the solution. The best choice can sometimes not be definitely determined, but there are cases with particular symmetries that allow criteria for such choices.

For example, in the case of a time-periodic coefficient $\mathbf{k}(t)$, one is interested in finding periodic amplitude vectors $\mathbf{u}_j(t)$. The experience from the research on 1D systems is that the single amplitude component (Milne's solution) should be periodic in order to describe the canonical phase-space vortex flow correctly. For other choices of the amplitude function the invariant tube, corresponding to a given Ermakov-Lewis invariant, would not be periodic with the same cross section area as the phase-space period map. Hence, there is an obvious interest to secure periodic functions $\mathbf{u}_j(t)$ also in this more general system.

One of the difficulties here is that we have to deal with *complex* quantities, another that we need vectors. Let us review the 1D stable parametric oscillations in the complex-amplitude formulation and show the existence of such periodic amplitudes (cf. [29] with real Milne solutions):

Our first assumption is that equation (3) with periodic and real $k(t)$ has two independent Floquet solutions given by

$$\Phi_F(t) = P(t)\mathrm{e}^{i\alpha t} \text{ and } \Phi_F^*(t) = P^*(t)\mathrm{e}^{-i\alpha t}, \tag{26}$$

with a periodic complex function $P(t+T) = P(t)$ and a real characteristic coefficient $\alpha > 0$. Without loss of generality we can always consider one initial condition to be real and positive $P(0) > 0$, since the oscillator equation is linear.

Our second main assumption is that any solution of (3) has its amplitude-phase decomposition according to the complex version presented here, which is exact. The new situation is that the complex amplitude functions satisfy the one-component version of equation (12) rather than Milne's original equation (4).

Hence, we can conclude that the existing Floquet solution $\Phi_F(t)$ gives rise to the following equation:

$$P(t)\mathrm{e}^{i\alpha t} = u(t)\mathrm{e}^{i\phi(t)}, \tag{27}$$

with real and positive phase function satisfying $\phi(0) = 0$. The differentiated equation, with due regard to (14), is then given by

$$\left(\dot{P}(t) + i\alpha P(t)\right)\mathrm{e}^{i\alpha t} = \left(\dot{u}(t) + i\Lambda\frac{u(t)}{|u(t)|^2}\right)\mathrm{e}^{i\phi(t)}. \tag{28}$$

The two relations (27) and (28) allow us to specify the initial conditions for the function $u(t)$ and its derivative. We note that $\dot{u}(0)$ will depend on the actual value used for the 'mixing parameter' Λ.

We now argue that $u(t)$ can be found periodic. From the equality (27) we immediately see that the absolute value of the amplitude $|u(t)|(=|P(t)|)$ is in fact periodic and independent of Λ, but perhaps not the phase of $u(t)$. The parameter Λ thus monitors the amount of the total phase to be explicit in $\phi(t)$ and the rest hidden in $u(t)$. Since anyway $|u(t)|$ is periodic and positive and independent of Λ, we have

$$\phi(nT) = \Lambda \int_0^{nT} |u(t)|^{-2} dt = n\phi(T), \tag{29}$$

so that we can always find a suitable $\Lambda = \Lambda_d$ to satisfy the equation $\phi(T) = \alpha T$. As a result

$$\Lambda_d = \frac{\alpha T}{\int_0^T |P(t)|^{-2} dt} \tag{30}$$

and

$$u(t) = P(t) \mathrm{e}^{i(\alpha t - \phi_d(t))} \tag{31}$$

is a periodic complex valued function, with $\phi_d(t)$ specified by the choice $\Lambda = \Lambda_d$. A closer look at the complex equation for $u(t)$ in one dimension also reveals that $u(t)$ will be real if $k(t)$ is real and $\mathrm{Im}\, u(0) = \mathrm{Im}\, \dot{u}(0) = 0$.

In an equivalent proof we could have started with the equalities:

$$AP(t)\mathrm{e}^{i\alpha t} = u(t)\mathrm{e}^{i\phi(t)}, \tag{32}$$

with any constant A, and

$$A\left(\dot{P}(t) + i\alpha P(t)\right)\mathrm{e}^{i\alpha t} = \left(\dot{u}(t) + i\Lambda \frac{u(t)}{|u(t)|^2}\right)\mathrm{e}^{i\phi(t)}. \tag{33}$$

This leads finally to a different 'mixing parameter' $\Lambda_d(A) = \Lambda_d/A^2$, which is a consequence of the scaling symmetry of the Milne equation (4) which prevails in (12). A generalisation to finding the periodic complex vectors $\mathbf{u}_j(t)$ if independent Floquet solutions of the above type exist is now straight forward.

Our result can be summarised as follows: We have derived time-dependent invariants which are linear as well as quadratic in momenta. Both types of invariants are directly related to the well known Wronskian constants for linear equations, but the Ermakov-Lewis invariant (quadratic in momenta) also uses a non-trivial amplitude-phase decomposition of a fundamental set of solutions. The theory is a new tool for analysing stable coupled oscillators with varying (periodic or not) parameters. We note that amplitude-phase decompositions have a wider applicability than Floquet solutions, since they are valid notions also in non-periodic dependences of time.

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