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ORIGINAL ARTICLE

Cohomological connectivity of perturbations of map-germs

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Abstract
Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a finite map-germ with $n < p$ and Y_δ the image of a small perturbation f_δ. We show that the reduced cohomology of Y_δ is concentrated in a range of degrees determined by the dimension of the instability locus of f. In the case $n \geq p$, we obtain an analogous result, replacing finiteness by $\mathcal{K}$-finiteness and Y_δ by the discriminant $\Delta(f_\delta)$. We also study the monodromy associated to the perturbation f_δ.

KEYWORDS
algebraic geometry, derived categories, perverse sheaves, singularity theory, vanishing cycles

1 | INTRODUCTION

In this paper, we establish bounds for the vanishing cohomology for images and discriminants of map-germs with non-isolated instabilities. As will become clear, these results are parallel to the classical bounds of Kato and Matsumoto on the cohomology of the Milnor fiber of nonisolated hypersurface singularities [15].

It is also our intention to illustrate that perverse sheaves are a powerful tool in the study of singular mappings. For the reader who is unfamiliar with the machinery of perverse sheaves, [3] provides a quick introduction. Here, we have included only the information that is necessary for our applications, and have avoided technical definitions. For those who are already well versed in the topic, some nontrivially perverse sheaves related to the alternating cohomology of multiple points are introduced in Section 5. The perversity of these sheaves was discovered by Houston [13].

Throughout this paper, we will consider homology and cohomology groups with different coefficients, depending on the context.

1.1 | The Milnor fibration

Let $(X, 0)$ be a germ of an analytic space and $g : (X, 0) \to (\mathbb{C}, 0)$ a nonconstant germ of a holomorphic function. We may assume that the representative X of $(X, 0)$ has a closed embedding into some open domain in $\mathbb{C}^N$ and choose a closed ball B_r of sufficientaly small radius $r > 0$ centered at the point $0 \in X$. Consider a punctured open disc D^*, centered at the origin in $\mathbb{C}$ and of radius $\delta > 0$ with δ sufficiently small with respect to r. As was shown in this generality by Lê in [16] (building on the existence of a_g-stratifications from [10] in the same volume), this gives rise to a locally trivial fibration

$$g : B_r \cap g^{-1}(D^*) \to D^*,$$

called the *Milnor fibration*, whose generic fiber

$$M = M_g(0) = g^{-1}(\delta) \cap B_r$$

is known as the *Milnor fiber* of g at $0 \in X$. This construction is well known to be independent of the choices made and the Milnor fibration is well defined up to homotopy equivalence.

Milnor showed [23] that if $X = \mathbb{C}^{n+1}$ and $g^{-1}(0)$ has an isolated singularity at 0, then M is homotopy equivalent to a bouquet of n-dimensional spheres. In particular, the reduced cohomology of M—that is the vanishing cohomology of g at 0—is concentrated in the middle degree. Later, Kato and Matsumoto [15] established their results for functions on $(\mathbb{C}^{n+1}, 0)$ with nonisolated singularities: If the critical locus $\Sigma(g)$ has dimension d, then the Milnor fiber M is at least $(n - d - 1)$-connected. Consequently, the reduced cohomology $\tilde{H}^p(M)$ is concentrated in the range of degrees $n - d \leq p \leq n$. Moreover, if one is using integer coefficients, then for the lowest such degree $p = n - d$, the cohomology is actually free, see, for example, [29, Example 6.0.12]. This concentration of reduced cohomology is what we refer to as *cohomological connectivity*.

Besides the cohomological version of Kato's and Matsumoto's connectivity result, we wish to also study the monodromy transformations. Since our setting will be slightly different from that of the Milnor fibration, we briefly review the general notion of monodromy for a topologically locally trivial fibration with fiber F

$$E \xrightarrow{\pi} D^*$$

over a punctured disk D^*. Let $\exp : S \to D^*$ be the universal cover of D^* by an infinite strip S. Since S is simply connected, we may choose a *global* trivialization of the the fiber product

$$E' := E \times_{D^*} S \cong F \times S$$

and deliberately identify the fiber $F_\delta = \pi^{-1}(\delta)$ over a point $\delta \in D^*$ with any fiber of E' over a point δ' mapping to $\delta \in D$. The chosen global trivialization of E' over S furnishes a notion of parallel transport of the fiber F over D^*, which is unique up to homotopy and independent of the choices made. The monodromy operator is defined to be the homeomorphism

$$h : F \to F$$

obtained from parallel transport along a closed loop in D^* passing counterclockwise around the origin once. It follows that the induced maps on cohomology

$$h^i : H^i(F) \to H^i(F)$$

are well-defined automorphisms.

Let us recall the classical monodromy theorem in the Milnor fibration setting.

Theorem 1.1. *Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a complex analytic germ and*

$$g : (X, 0) \to (\mathbb{C}, 0)$$

a holomorphic function. Denote by M the Milnor fiber of g at the origin and by $h^i : H^i(M) \to H^i(M)$ the i-th monodromy morphism.

(1) The eigenvalues of the monodromy operator are roots of unity.
(2) The size of the Jordan blocks of h^i is bounded by $i + 1$.

For the first statement, see [18]. The second statement is proved in [8, Corollaire 2.10 on p. 52]. See also [2, Theorem 3.1.20] and [17].

1.2 | The fibration associated to an unfolding

Rather than looking at germs of hypersurfaces, we will be looking at singularities that arise from multigerms of mappings

$$f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0).$$

Here, S is a *finite* subset of $\mathbb{C}^n$, mapped to 0 by f. Moreover, we always assume f to be $\mathcal{K}$-finite (which is the same as having finite singularity type, see Definition 2.1 and the lines following it).

Whenever $p > n$, our attention will be directed toward the image of f, denoted by

$$(Y, 0) = (\mathrm{Im} f, 0) \subset (\mathbb{C}^p, 0).$$

For obvious reasons, the analytic space $(Y, 0)$ is sometimes called a *parameterizable singularity*.

Whenever $n \geq p$, $\mathcal{K}$-finite map-germs are surjective so there is no interest in studying their images. The attention is directed toward the *discriminant* instead: The *critical locus* of f is the germ

$$\Sigma(f) = \{x \in \mathbb{C}^n | \mathrm{d}f_x \text{ is not surjective}\}$$

and the discriminant of f is defined to be the image

$$\Delta(f) = f(\Sigma(f)).$$

Observe that the study of discriminants comprises the case of parameterizable singularities in the sense that, for $p > n$, the differential cannot be surjective and hence $\Sigma(f) = \mathbb{C}^n$ and $Y = \Delta(f)$.

Let us summarize how the classical Milnor fibration is replaced by the fibration determined by a one-parameter family in the context of map-germs: A *one-parameter unfolding* $F = (f_t, t)$ of f is a germ

$$F : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, 0),$$

of the form $F(x,t) = (f_t(x), t)$ and such that $f_0 = f$. In the case $n < p$, the projection from the image$Y = \mathrm{Im} F$ to the parameter space gives a fibration, the Milnor fiber of which is Y_δ, the image of a perturbation f_δ. In the case $n \geq p$, we consider the fibration defined on the discriminant $\Delta(F)$ whose Milnor fiber is $\Delta(f_\delta)$.

These fibrations depend not only on the map-germ f, but also on the chosen unfolding F. Special attention is paid to the case where the perturbations f_δ are stable. In this case, the Milnor fiber Y_δ (or $\Delta(f_\delta)$) plays a role closer to that of the classical Milnor fiber of a map-germ $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. In the case $n < p$, the image Y_δ of a stable perturbation is called a *disentanglement* of f. For the precise definition of these terms, see Section 3.1.

Unfoldings play the role of deformations and are still local objects. To define perturbations—the nearby objects—we need a well-chosen representative. It is customary to absorb all associated technicalities in the definition of a *good representative*.

Definition 1.2. Let $V \subset \mathbb{C}^p$ and $T \subset \mathbb{C}$ be open neighborhoods of the origin and

$$F : W \to V \times T$$

a representative of a one-parameter unfolding, defined on an open subset $W \subset \mathbb{C}^n \times \mathbb{C}$. We call F a *good representative* if it is of finite singularity type and satisfies the following conditions:

(1) the family $\pi : \Delta(F) \to T$ is a locally trivial fibration over $T \setminus \{0\}$,
(2) the central fiber $\pi^{-1}(0)$ is contractible, and
(3) the space $\Delta(F)$ retracts onto the central fiber.

For any fixed nonzero value $\delta \in T \setminus \{0\}$ in the parameter space of a good representative, the map

$$f_\delta : W_\delta \to V$$

on $W_\delta := W \cap (\mathbb{C}^n \times \{\delta\})$ is called a *perturbation* of f. With no risk of confusion, we also write f for the representative $f_0 : W_0 \to V$.

Remark 1.3. A good representative of a $\mathcal{K}$-finite germ can be obtained from an arbitrary representative $F : U \times T \to V \times T$ as follows: The discriminant $\Delta(F)$ in $V \times T$ is a closed complex analytic set and the projection $\pi : V \times T \to T$ a holomorphic function on it. The Milnor-Lê fibration asserts that for sufficiently small ball $B_r \subset V$ around the origin and a subsequently chosen disc $D_\delta \subset T$ with $r \gg \delta > 0$, the restriction

$$\pi : \Delta(F) \cap (B_r \times D_\delta) \to D_\delta$$

is a map satisfying the three properties mentioned above. Now the good representative is furnished by choosing $B_r \times D_\delta$ small enough such that the restriction

$$F|_{F^{-1}(B_r \times D_\delta)} : F^{-1}(B_r \times D_\delta) \to B_r \times D_\delta$$

is of finite singularity type, replacing $V \times T$ by $B_r \times D_\delta$, and finally setting $W := F^{-1}(B_r \times D_\delta)$.

Parameterizable singularities and even their disentanglements are usually highly singular. This is due to the fact that the image of a map f may be singular even if f is stable (for a map$f$ to be stable means that it does not admit any nontrivial unfoldings; see the examples below; for the formal definition, see Section 3). Thus, the singular locus of $(Y, 0)$ is not well suited to be the analog of the critical locus of g from the classical Kato–Matsumoto result. Instead, we will be considering

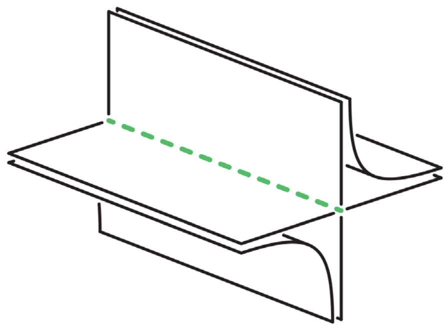
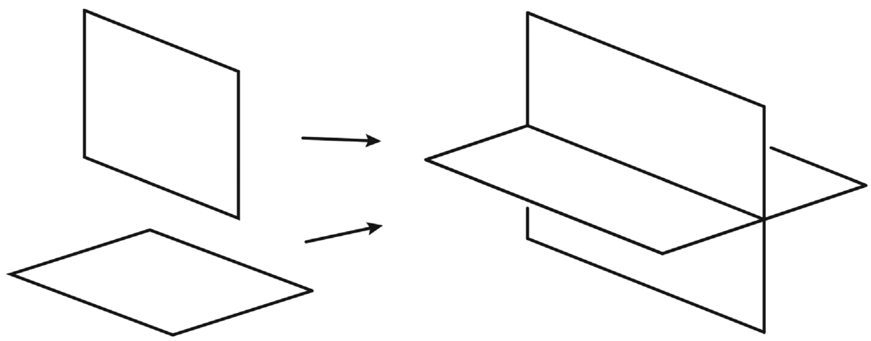


FIGURE 1 The singularity $X = \{xy = 0\}$ and its Milnor fiber $\{xy = \delta\}$. The dashed green line is the singular locus of X.

FIGURE 2 The same singularity as in Figure 1, now regarded as the image of a transverse double point bi-germ.



the *instability locus*

$$\mathrm{Inst}(f) \subseteq \Delta(f),$$

which is an analytic subset of the discriminant of f (see Section 3.1). The bound on the vanishing cohomology of the disentanglement will be given in terms of the dimension d of $\mathrm{Inst}(f)$. We give some examples to illustrate the situation.

Example 1.4. Consider the function

$$g : \mathbb{C}^3 \to \mathbb{C}, \quad (x, y, z) \mapsto xy$$

and the associated hypersurface $X = g^{-1}(0)$. Since z is a dummy coordinate, the Milnor fiber $M_g(0)$ has the homotopy type of a circle. This is consistent with Kato and Matsumoto's theorem, because the critical locus of g, and hence also the singular locus of X, has dimension one, see Figure 1.

The situation changes when we think of $(X, 0)$ as a parameterized singularity given by a bi-germ

$$f : (\mathbb{C}^2, \{p, q\}) \to (\mathbb{C}^3, 0)$$

with the two obvious branches, see Figure 2.

This map-germ is known as a "transverse double point." It is stable, meaning that it cannot be perturbed by any unfolding, up to analytic isomorphism. Therefore, a sufficiently small representative of the image of f coincides with its disentanglement. We see that, unlike the Milnor fiber $M_g(0)$ of g, the disentanglement is a singular space with two smooth branches crossing transversally and it has trivial reduced cohomology.

Example 1.5. We will now consider map-germs f with *isolated instability*, that is, those for which $d = \dim \mathrm{Inst}(f) = 0$. This property turns out to be equivalent to f being *finitely determined* [20]. The families S_k, B_k, and H_k of Mond [24] are examples of such finitely determined germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. Here, we consider the germ

$$(x, y) \mapsto (x^2, y^2, x^3 + y^3 + xy),$$

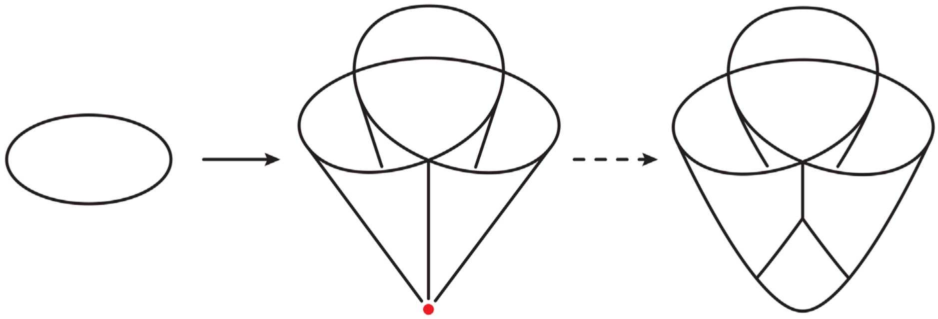


FIGURE 3 A germ f with isolated instability and its perturbation f_δ. The red thick dot at the vertex of the cone represents the instability locus Inst(f).

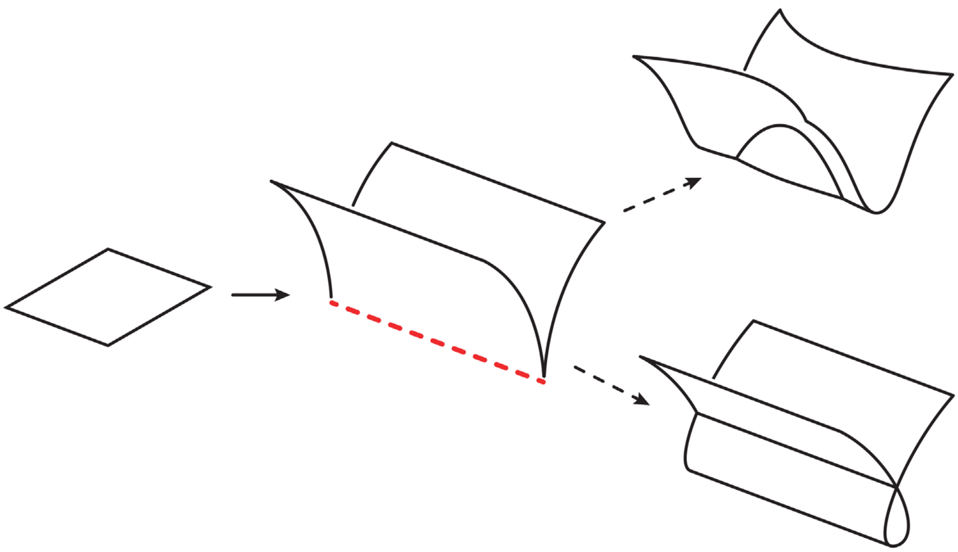


FIGURE 4 The cuspidal edge and two different perturbations. The red dashed line represents the instability locus Inst(f).

depicted in Figure 3.

Mond showed that the disentanglement of a finitely determined germ $(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ has the homotopy type of a bouquet of n-dimensional spheres [25]. Moreover, he showed that the number of spheres is independent of the chosen stabilization. Moreover, all finitely determined map-germs $f:(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ with $n\le 14$ admit perturbations to stable mappings[1] f_δ.

A similar connectivity result holds for discriminants of perturbations of finitely determined map-germs $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$, when $n\ge p$ [1]. The discriminant $\Sigma(f_\delta)$ has the homotopy type of a bouquet of spheres of dimension $p-1$.

Example 1.6. The cuspidal edge $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$, given by

$$(x,y)\mapsto(x,y^2,y^3),$$

parameterizes the cartesian product of a usual cusp and the complex line, see Figure 4.

It is not finitely determined, because it has a line of instabilities. Maps with nonisolated instability locus may admit more than one disentanglement: The cuspidal edge can, for example, be perturbed into a cuspidal node

$$(x,y)\mapsto(x,y^2,y^3-\delta y),$$

which is stable and has the homotopy type of a circle. Another perturbation of f is

$$(x, y) \mapsto (x, y^2, y^3 + \delta y(x^2 - \delta)).$$

The image of this last one has the homotopy type of a bouquet of two-dimensional spheres.

The main point of this paper is to show that despite the fact that different unfoldings may lead to different disentanglements with possibly distinct homotopy types, there is *always* a bound on the degrees of the nontrivial reduced cohomology groups of *any* given disentanglement. As the three examples above suggest, this bound is not related to the dimension of the singular locus of the image (equal to one for all the germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ in our examples), but it is controlled by the dimension of the instability locus (empty, zero-dimensional, and one-dimensional, respectively, in the examples).

Before stating our results, we shall introduce a construction that connects the Milnor fibration to the disentanglement fibration. Indeed, we will show that the study of the Milnor fibration of germs of hypersurfaces in $\mathbb{C}^n$ is equivalent to that of disentanglements of bi-germs of immersions $(\mathbb{C}^n, \{p, q\}) \to (\mathbb{C}^{n+1}, 0)$.

Example 1.7. To any hypersurface $X = V(g) \subseteq (\mathbb{C}^n, 0)$, not necessarily with isolated singularity, we are going to associate a bi-germ of immersions $(\mathbb{C}^n, 0) \sqcup (\mathbb{C}^n, 0) \to \mathbb{C}^{n+1}$. For convenience, label the two copies of $(\mathbb{C}^n, 0)$ as U_1 and U_2, then let $f : U_1 \sqcup U_2 \to (\mathbb{C}^{n+1}, 0)$ be the map of the form

$$x \mapsto \begin{cases} (x, g(x)) & \text{if} \, x \in U_1, \\ (x, 0) & \text{if} \, x \in U_2. \end{cases}$$

A different choice g' of a generator of the ideal $\langle g \rangle$ gives rise to a different map f', but there is a change of coordinates in $(\mathbb{C}^{n+1}, 0)$ turning f into f'. In other words, f and f' are $\mathcal{A}$-equivalent and, consequently, the study of their disentanglements is equivalent.

Conversely, every bi-germ of immersion $(\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ arises—up to $\mathcal{A}$-equivalence—by this construction: Given such a bi-germ, we can take a change of coordinates turning the second branch into $x \mapsto (x, 0)$, and so that the normal vector to the first branch at the origin has a nonzero last coordinate in $\mathbb{C}^{n+1}$. This makes the first branch locally into a graph, as desired. A more direct way to invert the process is as follows: The two branches of a bi-germ f of immersion are two germs $f^{(1)}, f^{(2)} : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. Take a reduced equation $L = 0$ for the image of the first branch. Applying the above construction to the function germ $g = L \circ f^{(2)} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ gives rise to a bi-germ, which is $\mathcal{A}$-equivalent to the original immersion f.

Having explained the construction and its inverse, let us describe the relation between the hypersurface X and the immersion f: First and most obvious, the intersection of the two branches is

$$\operatorname{Im}(f|_{U_1}) \cap \operatorname{Im}(f|_{U_2}) = X \times \{0\}.$$

Moreover, the two branches cross transversally, except on the singularities of X. It is well known that the instabilities of an immersion are located precisely at points where the branches do not intersect in general position (this is a particular case of Theorem 3.3 in [26], taking into account that monogerms of immersions are stable). In particular, it follows that

$$\operatorname{Inst}(f) = \operatorname{Sing} X \times \{0\}.$$

The Milnor fiber of X has the form $M = g^{-1}(\{\delta\}) \cap B_r$, for suitable choices of $r \gg \delta > 0$. The same discussion about singularities and stabilities shows that a stabilization of f is given by the bi-germ $F = (f_t, t)$, with

$$f_t|_{U_1} = f|_{U_1} \quad \text{and} \quad f_t|_{U_2} = (x, t).$$

A stable perturbation of f is given by $f_\delta : B_r \sqcup B_r \to \mathbb{C}^{n+1}$ and the Milnor fiber M of g is recovered as the intersection of the two branches of f_δ.

Our claim that the study of the Milnor fibration of hypersurfaces is the same as the study of disentanglements of bi-germs of immersions also carries over to the vanishing topology in the following sense:

Remark 1.8. Let M be the Milnor fiber of a hypersurface $X = V(g)$ and $Y_\delta = \operatorname{Im} f_\delta$ be the disentanglement of the bi-germ associated to g, as in Example 1.7. Then, there is an isomorphism

$$\tilde{H}^i(M) \cong \tilde{H}^{i+1}(Y_\delta),$$

compatible with the monodromy actions on M and Y_δ. In fact, Y_δ is homotopy equivalent to the suspension of M and we will come back to this isomorphism on Example 5.16 and Remark 5.17 to illustrate the more general approach to the computation of the cohomology of mappings.

2 | MAIN RESULTS

In this paper, we always assume that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is $\mathcal{K}$-finite, a notion, which we briefly recall below. Also, we would like to emphasize that in Sections 3 and 4 we will be using integer coefficients for cohomology, whereas in Sections 5–7 we use rational, resp. complex coefficients.

Definition 2.1. A map f is of *finite singularity type* if the restriction $f|_{\Sigma(f)}$ is a finite map.

For map-germs, this condition is equivalent to *$\mathcal{K}$-finiteness* [26, Proposition 4.3], a condition that arises naturally in the study of contact equivalence of singular map-germs [26, Section 4.4] and it is generally considered mild. The definition of $\mathcal{K}$-finiteness is irrelevant to our work, and the reader may regard $\mathcal{K}$-finiteness as a synonym of being of finite singularity type that can be used for map-germs but not for mappings.

Observe that, whenever $\dim Y > \dim X$, for a map $f : X \to Y$ to be of finite singularity type means just that f is finite

The finite singularity–type condition is helpful in at least three ways: First of all, it entails that for dimensions $p \leq n+1$ the discriminant $\Delta(f)$ is a hypersurface in $\mathbb{C}^p$ (Proposition 3.3). Second, the instability locus of mappings of finite singularity type is analytic (Proposition 3.2). Lastly, $\mathcal{K}$-finiteness of germs ensures the existence of stable unfoldings, which are used in the definition of multiple point spaces in Section 5.

Theorem 2.2. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a $\mathcal{K}$-finite map-germ, with $p \leq n$, and let $d = \dim(\operatorname{Inst}(f))$. The reduced cohomology of the discriminant of any perturbation f_δ satisfies*

$$\tilde{H}^q(\Delta(f_\delta); \mathbb{Z}) = 0, \text{for any} q \notin [p-1-d, p-1].$$

Moreover, $\tilde{H}^{p-1-d}(\Delta(f_\delta); \mathbb{Z})$ is free.

The previous result also holds for disentanglements, as they are particular cases of perturbations.

Theorem 2.3. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a finite map-germ and let $d = \dim(\operatorname{Inst}(f))$. The reduced cohomology of the image of any perturbation f_δ satisfies*

$$\tilde{H}^q(Y_\delta; \mathbb{Z}) = 0, \text{for any} q \notin [n-d, n].$$

Moreover, $\tilde{H}^{n-d}(Y_\delta; \mathbb{Z})$ is free.

The last result is not new, and appeared already in [9, Proposition 2.1]. We include our version for completeness and prove it together with Theorem 2.2, making the connection between both results and between finiteness and $\mathcal{K}$-finiteness clear.

In the case of germs $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, apart from the image Y, it is common to study the *source double point space*$D(f) \subseteq \mathbb{C}^n$. For a map $f : X \to Z$ between complex manifolds, it consists of the set of points $x \in X$ such that the germ $f : (X, f^{-1}(\{f(x)\})) \to (\mathrm{Im}f, f(x))$ is not an isomorphism, that is,

$$D(f) = \{x \in X \mid f^{-1}(\{f(x)\}) \neq \{x\}\} \cup \{x \in X \mid f \text{ is not immersive at } x\}.$$

A point $x \in D(f)$ is called a *double point* of f, even in the case that $f^{-1}(\{f(x)\}) = \{x\}$. Details on the analytic structure of $D(f)$ are found in Section 4. For now, it is enough to know the following: Whenever $f : X \to Z$ is a finite mapping between complex manifolds with $\dim Z = \dim X + 1$ and $\dim(\mathrm{Inst}(f)) < \dim X$, then $D(f)$ is a hypersurface in X.

Theorem 2.4. *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a finite map-germ and let $d = \dim(\mathrm{Inst}(f))$. The reduced cohomology of the source double point space of any perturbation f_δ satisfies*

$$\tilde{H}^q(D(f_\delta); \mathbb{Z}) = 0, \text{for any} q \notin [n-1-d, n-1].$$

Moreover $\tilde{H}^{n-1-d}(Y_\delta; \mathbb{Z})$ is free.

Remark 2.5. Just as the disentanglements and discriminants of Example 1.5, the source double points space $D(f_\delta)$ of a *finitely determined* map-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ has the homotopy type of a bouquet of spheres of dimension $n - 1$. This observation is due to R. Giménez Conejero and J.J. Nuño-Ballesteros [6]. Note that this corresponds to the case of *isolated* instability in our setup, that is, $d = 0$.

Example 2.6. Let us come back to the bi-germ of immersions

$$f : U_1 \sqcup U_2 \to \mathbb{C}^{n+1}$$

associated to a hypersurface $X \subseteq \mathbb{C}^n$, as described in Example 1.7. Since each of the branches $f|_{U_i}$ is injective and immersive, the space $D(f)$ is the preimage by f of the intersection of the the two branches $Y_i = \mathrm{Im}f|_{U_i}$. Therefore, the source double point space consists of two disjoint copies

$$D(f) = X \sqcup X.$$

The same argument shows that the double point space of the stable perturbation of f is

$$D(f_\delta) = M \sqcup M,$$

where M is the Milnor fiber of X. From this point of view, the cohomological version of Kato's and Matsumoto's connectivity result is a particular case of Theorem 2.4. The classical connectivity result due to Milnor appears as a particular case of Remark 2.5.

Corollary 2.7. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a finite map-germ and assume that* $\dim(\mathrm{Inst}(f)) < n - 1$. *Then, the singular locus* $\mathrm{Sing}Y_\delta$ *of the image of any perturbation of f is connected.*

Proof. Theorem 2.4 implies immediately that $D(f_\delta)$ is connected. Moreover, from the set-theoretical description of the source double point space given above, it follows that the singular locus of the image Y of a finite and generically one-to-one map $f : X \to Z$ between complex manifolds is, as a set, the image of the double point space. In other words, finite and generically one-to-one maps satisfy $\mathrm{Sing}Y = f(D(f))$, which clearly implies our claim. Now observe that, for dimensions $\dim Y > \dim X$, stable mappings are generically one-to-one. Therefore, mappings satisfying $\dim(\mathrm{Inst}(f)) < n - 1$ (hence generically stable) are necessarily generically one-to-one as well. □

In order to obtain cohomological connectivity results for germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with arbitrary $p > n + 1$, something stronger than $\mathcal{K}$-finiteness is required: Let $f : U \to V$ be a map between complex manifolds of dimensions $n = \dim U$ and $p = \dim V$, and let $D^k(f)$ be the k-th multiple point space of f (see Section 5 for the definition of multiple points). Then,

(1) we say that f is *dimensionally correct* if for each $k \geq 2$ the multiple point space $D^k(f)$ is empty or has dimension $kn - (k-1)p$,
(2) we say that f is *dimensionally correct for nonnegative dimensions* if for each $k \geq 2$ such that $kn - (k-1)p \geq 0$, the multiple point space $D^k(f)$ is empty or has dimension $kn - (k-1)p$.

The difference between these two notions is less subtle than one may think. For example, dimensional correctness fails for map-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ having quadruple points, ruling out many interesting examples such as

$$f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0), \quad (x, y) \mapsto (x^2, y^3, (x+y)^5).$$

The quadruple points of this germ are nonempty. It is, nevertheless, dimensionally correct for nonnegative dimensions since $\dim D^2(f) = 1$ and $\dim D^3(f) = 0$.

It is clear from the above definitions that being dimensionally correct implies being dimensionally correct for nonnegative dimensions. However, while some assertions (see, e.g., Theorem 2.8 below) can be proved for *arbitrary* deformations of dimensionally correct map-germs, the strictly weaker condition to be dimensionally correct for nonnegative dimensions only allows us to consider deformations, which are *stabilizations*. Whenever applicable, we will therefore state our theorems for either one of the two setups.

Theorem 2.8. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a map-germ, with $p > n + 1$, and let f_δ be a perturbation of f. Assume that either*

(a) f is dimensionally correct, or
(b) f is dimensionally correct for nonnegative dimensions and f_δ is a stable perturbation of f.

Then all possibly nontrivial reduced cohomologies $\tilde{H}^q(Y_\delta, \mathbb{Q})$ are concentrated in the degrees q satisfying

$$q = kn - (k-1)(p-1) - s,$$

for some $1 < k \leq \left\lfloor \frac{p}{p-n} \right\rfloor$ *and* $0 \leq s \leq d = \dim \operatorname{Inst}(f)$.

As the following example shows, Theorem 2.8 does not hold if the dimensionally correct hypothesis is omitted.

Example 2.9. By adding a zero coordinate function to the cuspidal edge of Example 1.6, we obtain the germ $f' : (\mathbb{C}^2, 0) \to (\mathbb{C}^4, 0)$ given by

$$(x, y) \mapsto (x, y^2, y^3, 0).$$

Similarly, adding a zero coordinate to the perturbation of Example 1.6 gives a perturbation f'_δ, given by

$$(x, y) \mapsto (x, y^2, y^3 + \delta y(x^2 - \delta), 0),$$

whose image is, obviously, isomorphic to that of the mentioned example. This image has nontrivial cohomology in degree two. If f' was dimensionally correct, then Theorem 2.8 would allow for nontrivial vanishing cohomology only in degrees 0 and 1. Indeed, this condition on f' is violated since the source double point space $D(f')$ contains the whole line $\{y = 0\}$ (f' is not immersive there) whereas, for germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^4, 0)$, the minimal dimension for double points is $2 \cdot 2 - 1 \cdot 4 = 0$.

Remark 2.10. A first version of Theorems 2.3 and 2.8 was proven in [28], under the extra assumptions that f_δ is stable, that f has corank one and, in the case of $p = n + 1$, that f is a dimensionally correct mono-germ. Two examples of map-germs that only the new version can handle are $(x, y) \mapsto (x^2, y^2, x^3 + y^3 + xy)$ (which has corank two) and $(x, y) \mapsto (x, y^3, y^4)$ (which is not dimensionally correct, since it has a line of triple points, but satisfies the hypotheses of Theorem 2.3).

Finally, let us return to the study of monodromy where we are using cohomology with complex coefficients. Let F be a one-parameter unfolding of a finite map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $p > n$. The projection from the image $\mathcal{Y}$ of a good representative of F to the deformation parameter restricts to a locally trivial fibration

$$\mathcal{Y}^* \to D^*$$

over the punctured disk. Since the disentanglement Y_δ is the fiber of the previous fibration, for each i there is a monodromy automorphism

$$h^i : H^i(Y_\delta, \mathbb{C}) \to H^i(Y_\delta, \mathbb{C}).$$

The fact that f is finite implies that $\mathcal{Y}$ is analytic, and thus it is clear from Theorem 1.1 that the eigenvalues of h^i are roots of unity. The following theorem shows that, in some special cases, we can improve the bound on the size of the Jordan blocks for the monodromy of the image compared to the usual bound provided by Theorem 1.1:

Theorem 2.11. *Let F be a one-parameter unfolding of a map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $p > n$. Assume that either*

(a) f is dimensionally correct, or
(b) f is dimensionally correct for nonnegative dimensions and F a stabilization of f.

Then the following hold.

(1) Let $d = \dim \operatorname{Inst}(f)$ and assume $d < p - n - 1$. Then for each degree,

$$q = kn - (k-1)(p-1) - s \quad \text{with } 0 \le s \le d$$

the size of the Jordan blocks of h^q is at most $kn - (k-1)p + 1 - s$.

(2) Assume $D^3(f) = \emptyset$. Then for every degree i, the size of the Jordan blocks of h^i is at most i.

Remark 2.12. As it turns out, the bound for the size of Jordan blocks for *mono-germs* in the case $D^3(f) = \emptyset$ applies to bi-germs of immersion

$$f : U_1 \sqcup U_2 \to (\mathbb{C}^{n+1}, 0), \text{with } U_i = (\mathbb{C}^n, 0),$$

introduced in Example 1.7 (by inspecting the proof of Theorem 2.11, one sees that the obstruction for multigerms is the nonconnectedness of X and the multiple point spaces of f, which introduces nondesired terms in the spectral sequence used in the proof. It is easy to see that these terms do not become an issue for bi-germs of immersions). Note that, as we explained in Remark 1.8, the cohomology of the disentanglement of f corresponds to the cohomology of the Milnor fiber of the hypersurface defined by a function

$$g : (\mathbb{C}^n, 0) \to \mathbb{C}.$$

Applying the bound from Theorem 1.1 blindly to the image of f gives us that the size of Jordan blocks of the monodromy of the disentanglement is at most $n + 1$. However, our bound is $2n - (n+1) + 1 = n$. This is the correct answer from the classical point of view for singular hypersurfaces when considering the Milnor fibration of g.

2.1 | Some remarks on generality

In writing this paper, we decided to sacrifice some generality for the sake of clarity. The following remarks list some improvements that ended up being left behind:

Remark 2.13. Throughout the text, the instability locus can be replaced by the *topological instability locus*. Topological stability is defined by using homeomorphisms instead of diffeomorphisms in the definition of trivial unfolding. In general, the topological instability locus is smaller, giving rise to sharper cohomological connectivity bounds.

Remark 2.14. Theorem 2.8 is stated for monogerms, but it applies to multigerms as well. The proof however involved representations of the symmetric group, which made the exposition more complicated and seemed off-topic.

Remark 2.15. The bounds on the size of Jordan blocks of Theorem 2.11 can be improved easily in many cases, just taking into account the dimensions of the multiple point spaces and making a more detailed study of which entries in the spectral sequence contribute to which degrees in the monodromy of the image (see Section 5). However, incorporating all these considerations to get a sharper bound leads to a much less readable result. We feel that, having understood the ideas we use, the reader will be able to produce the corresponding bound for each particular situation.

3 | PRELIMINARIES

Here, we include definitions related to stability, properties involving $\mathcal{K}$-finiteness, and the basic notions of the theory of perverse sheaves, which are used to establish our results.

3.1 | Stability and $\mathcal{K}$-finiteness

An unfolding $F : (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\}) \to (\mathbb{C}^p, 0)$ of a germ f is *trivial* if there exist an unfolding Φ of $(\mathrm{id}_{\mathbb{C}^n}, S)$ and an unfolding Ψ of $(\mathrm{id}_{\mathbb{C}^p}, 0)$ such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\}) & \xrightarrow{F} & (\mathbb{C}^p \times \mathbb{C}^r, (0,0)) \\ \downarrow \Phi & & \downarrow \Psi \\ (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\}) & \xrightarrow{f \times \mathrm{id}_{\mathbb{C}^r}} & (\mathbb{C}^p \times \mathbb{C}^r, (0,0)). \end{array}$$

Note that these conditions make Φ and Ψ into germs of biholomorphism.

A germ f is *stable* if every unfolding of f is trivial. A map $f : M \to N$ of finite singularity type is stable at $y \in N$ if the germ of f at $S = \Sigma(f) \cap f^{-1}(y)$ is stable. We say that the map f is stable if it has finite singularity type and it is stable at every $y \in N$.

A one-parameter unfolding F is called a *stabilization* of f if it admits a good representative such that for every nonzero $\delta \in T$ the perturbations f_δ are stable. In this case, f_δ is called a *stable perturbation* of f, and $Y_\delta = \mathrm{Im} f_\delta$ is called a *disentanglement* of Y.

The *instability locus* of $f : M \to N$ is the support

$$\mathrm{Inst}(f) = \mathrm{Supp} \frac{\theta(f)}{T\mathcal{A}_e(f)},$$

where $\theta(f)$ is the sheaf of vector fields along f, and $T\mathcal{A}(f)$ is the extended $\mathcal{A}$-tangent space of f, see [30]. What gives its name to the instability locus is the following result [26, Theorem 3.2]:

Proposition 3.1. *Let $f : M \to N$ be a map of finite singularity type, let $q \in N$ and $S = f^{-1}(q) \cap \Sigma(f)$. If $q \notin \mathrm{Inst}(f)$, then the germ of f at S is stable.*

Introducing the instability locus as the support of a sheaf allows us to justify that it is an analytic space. The proof of the following proposition can be extracted from that of [26, Proposition 4.2]:

Proposition 3.2. *If f is of finite singularity type, then $\theta(f)/(T_e\mathcal{A}(f))$ is a coherent $\mathcal{O}_Y$-module. In particular,* $\mathrm{Inst}(f)$ *is analytic.*

We finish this subsection with a result, which is well known, but whose proof we include for lack of a reference.

Proposition 3.3. *Let $f : X \to Z$ be a map of finite singularity type and assume that the dimensions $n = \dim X$ and $p = \dim Z$ satisfy $p \leq n + 1$. Then, $\Delta(f)$ is of pure dimension $p - 1$.*

Proof. The case $p = n + 1$ is clear, because then $\Sigma(f) = X$, the map is finite and $\Delta(f) = \mathrm{Im} f$. We may thus assume $q \leq n$. Since f is of finite singularity type, $\Delta(f)$ is an analytic space of the same dimension as $\Sigma(f)$. On the one hand, Sard's Theorem implies that $\Delta(f)$ is not all of Z, hence its dimension is at most $p - 1$. On the other hand, $\Sigma(f)$ is defined as the vanishing locus of p-minors of an $n \times p$-matrix with $p \leq n$. The results in [4] imply that the dimension of any component of $\Sigma(f)$, and thus of $\Delta(f)$, is greater than or equal to $n - (n - p + 1) = p - 1$. □

3.2 | Perverse sheaves

In this subsection, we will summarize those parts of the machinery of perverse sheaves on complex analytic spaces, which we shall need to prove our theorems. We mainly follow the standard reference [3] where most of the details can be found. We might also refer to the more recent [22] whenever necessary.

Throughout this section, X stands for a complex analytic variety. For a sheaf $\mathscr{F}$ on X its *sheaf cohomology groups* will be denoted by

$$H^i(X; \mathscr{F}).$$

We write $\mathbb{Z}_X$ for the constant sheaf on X associated to the ring $\mathbb{Z}$. This section will be phrased in terms of sheaves of $\mathbb{Z}_X$-modules, but most of the theory—except maybe particular torsion phenomena—works equally well for other rings or fields like $\mathbb{Q}$ or $\mathbb{C}$. In fact, the latter two cases will be needed for the study of the image computing spectral sequence and the monodromy.

Recall that the cohomology of $\mathbb{Z}_X$ is isomorphic to the singular cohomology with coefficients in $\mathbb{Z}$:

$$H^\bullet(X; \mathbb{Z}_X) \cong H^\bullet_{\mathrm{sing}}(X; \mathbb{Z}).$$

By $C(X)$, we denote the category of complexes of sheaves of abelian groups on$X$. Objects in this category are written as

$$\mathscr{F}^\bullet : \qquad \cdots \to \mathscr{F}^{i-1} \to \mathscr{F}^i \to \mathscr{F}^{i+1} \to \cdots$$

From any such complex of sheaves, we obtain the collection of *cohomology sheaves*, denoted by

$$\mathcal{H}^i(\mathscr{F}^\bullet), \quad i \in \mathbb{Z}.$$

A morphism of complexes of sheaves $\varphi^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ is called a *quasi-isomorphism*, if the induced maps on the cohomology sheaves

$$\varphi : \mathcal{H}^i(\mathscr{F}^\bullet) \to \mathcal{H}^i(\mathscr{G}^\bullet)$$

is an isomorphism for all $i \in \mathbb{Z}$. By $\mathcal{F}^\bullet[d]$ we will denote the *shift by d* of the complex $\mathcal{F}$, which is given by the terms

$$(\mathcal{F}^\bullet[d])^k = \mathcal{F}^{k+d}$$

for every k together with the appropriately shifted differentials from $\mathcal{F}$.

By $D(X)$, we denote the *derived category* of sheaves of $\mathbb{Z}_X$-modules on X, which is obtained from $C(X)$ by *localizing* at the set of quasi-isomorphisms. In particular, this construction entails that two complexes of sheaves $\mathscr{F}^\bullet$ and $\mathscr{G}^\bullet$ are *isomorphic* in $D(X)$ if and only if they are *quasi-isomorphic* in $C(X)$.

A complex of sheaves $\mathcal{F}^\bullet$ is in the *bounded* derived category $D^b(X)$ if its nontrivial cohomology sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ are confined to a bounded range of indices i. For any such complex of sheaves $\mathcal{F}^\bullet \in D^b(X)$, we can define the *support* as

$$\mathrm{Supp}\mathcal{F}^\bullet = \overline{\{x \in X \mid \mathcal{H}^i(\mathcal{F}^\bullet)_x \neq 0 \text{ for some} i \in \mathbb{Z}\}},$$

where we denote by $\mathcal{H}^i(\mathcal{F}^\bullet)_x$ the stalk of the cohomology sheaf $\mathcal{H}^i(\mathcal{F}^\bullet)$ at the point $x \in X$.

A sheaf $\mathcal{F}$ on X can be regarded as a complex of sheaves concentrated in degree 0. This is a fully faithful embedding of the category of sheaves on X into the bounded derived category, cf. [3, Proposition 1.3.3 iii]. This allows us to simplify notation and write $\mathcal{F}$ instead of $\mathcal{F}^\bullet$ for the complex of sheaves associated to a sheaf $\mathcal{F}$.

Under this embedding into the bounded derived category, classical sheaf cohomology reappears as *hypercohomology*. More generally, for a continuous map $f : X \to Y$ of topological spaces and a sheaf $\mathcal{F}$ on X, one has

$$R^i f_* \mathcal{F} \cong \mathcal{H}^i(Rf_* \mathcal{F}),$$

where the left-hand side denotes the i-th derived pushforward of a single sheaf and

$$Rf_* : D^b(X) \to D^b(Y)$$

denotes the derived pushforward in the derived categories, cf. [3, Section 2.3]. For the special case of a projection $p : X \to \{pt\}$ to a point, one obtains

$$H^i(X, \mathcal{F}) = R^i p_* \mathcal{F} = \mathcal{H}^i(Rp_* \mathcal{F}) = \mathbb{H}^i(X, \mathcal{F}),$$

where

$$\mathbb{H}^i(X, -) : D^b(X) \to \mathbb{Z}\text{-Mod}$$

is the i-th *hypercohomology* functor.

Again for a map $f : X \to Y$ and a complex of sheaves $\mathcal{F}$ on X the hypercohomology functors satisfy the following fundamental property:

$$\mathbb{H}^i(Y, Rf_* \mathcal{F}) = \mathbb{H}^i(X, \mathcal{F}),$$

the *Leray spectral sequence*, see [3, Section 2.3.4].

Similar translations of the above identifications between the category of $\mathbb{Z}_X$-modules and the derived category exist for relative cohomology for pairs of spaces; see [3] for details.

We shall say that a complex of sheaves $\mathcal{F} \in D^b(X)$ on a *complex analytic* space X is *constructible* if there exists a complex analytic stratification of X with locally closed *complex analytic* strata $S_\alpha \subset X$ such that

(1) the restriction of $\mathcal{F}$ to each one of the S_α has locally constant cohomology sheaves;
(2) all stalks of the cohomology sheaves are finitely generated $\mathbb{Z}$-modules.

The category of bounded constructible complexes of sheaves on X will be denoted by $D^b_c(X)$.

From now on, we will mostly be interested in a very special type of bounded complexes of constructible sheaves: the *(strongly) perverse sheaves* for the *middle perversity*. See, for example, [3] for a definition of perversity and [22, Definition 2.50] for a definition of a strongly perverse sheaf.[2] Note that a perverse sheaf need not be a single sheaf, but usually denotes a whole complex of sheaves. It is our intention for the reader to accept "being (strongly) perverse" as a good property, which can be used without knowing its details. We proceed by stating the necessary properties and results for the intended usage.

Proposition 3.4. *Let $\mathcal{F}$ be a perverse sheaf on a complex analytic variety X and let d be the dimension of the support of $\mathcal{F}$. Then, for any point $x \in X$, the stalk cohomology groups satisfy*

$$\mathcal{H}^i(\mathcal{F})_x = 0 \, for \, i \notin [-d, 0].$$

If, moreover, $\mathcal{F}$ is strongly perverse, then $\mathcal{H}^{-d}(\mathcal{F})_x$ is free.

Note that, since by definition any perverse sheaf is constructible, its support is always a closed *complex analytic* subset and it makes sense to speak of its (complex) dimension.

For a proof of the classical statement of Proposition 3.4 on perverse sheaves see [3, Remark 5.1.19]. Since the case of strongly perverse sheaves is not explicitly covered in the literature, we briefly sketch the argument here. Before that, we need some generalities about (strongly) perverse sheaves.

According to [22, Definition 2.50], a bounded constructible complex $\mathcal{F}$ is strongly perverse if it is an element of

$$^{p}D^{\leq 0}(X) \cap {}^{p^+}D^{\geq 0}(X).$$

The definition of the subcategories and $^{p}D^{\geq 0}$ is found in [22, Definition 2.16] and involves the so-called *support and cosupport conditions*. The subcategory $^{p^+}D^{\geq 0}$ of $^{p}D^{\geq 0}$ tells the strongly perverse sheaves apart from the merely perverse ones, and allows for the statements about torsion-freeness [22, section 2.3]. For our purposes, the relevant information about these categories is captured by the following results:

(1) For a one-point space $X = \{pt\}$, the category $^{p}D^{\leq n}(X)$ (respectively, $^{p}D^{\geq n}(X)$) consists of the bounded complexes whose cohomology is finitely generated and concentrated in degrees $\leq n$ (resp. $\geq n$). The subcategory $^{p^+}D^{\geq 0}(X)$ consists of those complexes in $^{p}D^{\geq n}(X)$ whose cohomology in degree n is torsion free.
(2) Let $f : X \to Y$ be a morphism whose fiber dimension is bounded by d. Then:
   - f^* maps $^{p}D^{\leq n}(Y)$ into $^{p}D^{\leq n+d}(X)$,
   - $f^!$ maps $^{p^+}D^{\geq n}(Y)$ into $^{p^+}D^{\geq n-d}(X)$.
(3) Let $S \subseteq X$ be an analytic subset and assume that locally S can set-theoretically be cut out by k equations. Let $i : S \hookrightarrow X$ be the inclusion morphism. Then, for any $\mathcal{F} \in {}^{p}D^{\geq n}(X)$, we have

$$i^*\mathcal{F}[k] \in {}^{p}D^{\geq n}(S).$$

   The same holds replacing p by p^+.

Proofs of these statements can be found in [29, Proposition 6.0.2, Lemma 6.0.3] and [22, Example 2.22]. We should emphasize that we are using the notation of [22]. In the notation of [29], the space $^{p^+}D^{\geq 0}(X)$ is written as $^{m}D^{\geq 0}(X)$, where the property "≥" is chosen to be the one from [29, Example 6.0.1.3] and where m stands for the *middle perversity*.

Now our result follows by the following standard argument, similar to that around [29, Equation 6.30]:

Proof of Proposition 3.4. We prove only the statements involving strong perversity since the classical version involving merely perversity is already covered in the literature, see, for example, [3, Remark 5.1.19]. Let $\kappa : S \hookrightarrow X$ be the inclusion of the support of $\mathcal{F}$. Then, $\kappa^*\mathcal{F} \cong \kappa^!$ (as in [29, eq. 6.30]) and therefore

$$\kappa^*\mathcal{F} \in {}^{p}D^{\leq 0}(S) \cap {}^{p^+}D^{\geq 0}(S),$$

in other words, $\kappa^*\mathcal{F}$ is strongly perverse. Now take $x \in S$ and take the inclusion morphism $i_x : \{x\} \hookrightarrow S$. Set theoretically, the space $\{x\}$ can locally be cut out from S by $d = \dim S$ equations. Hence, we obtain

$$i_x^*(\kappa^*\mathcal{F}) \in {}^{p}D^{\leq 0}(\{x\}) \cap {}^{p^+}D^{\geq -d}(\{x\}).$$

Now the statement to prove follows simply because $i_x^*(\kappa^*\mathcal{F}) = \mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x. □

Proposition 3.5. *If X is a locally complete intersection of complex dimension d, then the shifted constant sheaf $\mathbb{Z}_X[d]$ is strongly perverse.*

For the classical statement on mere perversity, see [3, Theorem 5.1.20]. For strong perversity, we refer to [22, Proposition 10.2.54] in conjunction with [22, Example 10.4.29].

Let $g : \mathcal{X} \to \mathbb{C}$ be a holomorphic function defined on a complex analytic variety and set $X = g^{-1}(0)$. Associated to g, there is the *vanishing cycle functor*

$$\phi_g : D_c^b(\mathcal{X}) \to D_c^b(X).$$

For any point $x \in X$ and any $\mathcal{F} \in D^b_c(\mathcal{X})$, the stalk cohomology of $\phi_g\mathscr{F}$ can be computed as follows (see, e.g., [3, p. 106 (4.1)]):

$$\mathcal{H}^i(\phi_g\mathcal{F})_x = \mathbb{H}^{i+1}(B_r, B_r \cap g^{-1}(\{\delta\}); \mathcal{F}), \tag{1}$$

where B_r is a sufficiently small open ball in $\mathcal{X}$ centered at x and $\delta \in \mathbb{C} \setminus \{0\}$ is small enough with respect to r. The last result we need is essential, as it allows to produce new perverse sheaves from old:

Theorem 3.6. *The vanishing cycle functor shifted by one $\phi_g[-1] : D^b_c(\mathcal{X}) \to D^b_c(X)$ takes (strongly) perverse sheaves on $\mathcal{X}$ to (strongly) perverse sheaves on X.*

Again, for the classical statement see [3, Theorem 5.2.21]. Those interested in strong perversity are referred to [22, Theorem 10.4.22] or [29, Theorem 6.0.2].

The ingredients we have just introduced can be put together to give a simple proof of the cohomological connectivity of Milnor fibers. Let g be a suitable representative of a holomorphic map-germ $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ defined on some open subset $\mathcal{X} \subset \mathbb{C}^{n+1}$. Take the strongly perverse sheaf $\mathbb{Z}_{\mathcal{X}}[n+1]$ and apply the shifted vanishing cycle functor associated to g to obtain a strongly perverse sheaf on $X = g^{-1}(0)$. The stalk cohomology at a point $x \in X$ recovers the reduced cohomology of the Milnor fiber of the germ of X at x as follows:

$$\begin{aligned}
\mathcal{H}^i(\phi_g\mathbb{Z}_{\mathcal{X}}[n])_x &= \mathbb{H}^{i+1}(B_r, B_r \cap g^{-1}(\delta); \mathbb{Z}_{\mathcal{X}}[n]) \\
&= H^{i+n+1}(B_r, B_r \cap g^{-1}(\delta); \mathbb{Z}_{\mathcal{X}}) \\
&= H^{i+n+1}_{\text{sing}}(B_r, B_r \cap g^{-1}(\delta); \mathbb{Z}) \\
&= \tilde{H}^{i+n}_{\text{sing}}(B_r \cap g^{-1}(\delta); \mathbb{Z}).
\end{aligned}$$

Since the Milnor fiber of g at a regular point of the function has trivial reduced cohomology, we deduce that the support of $\phi_g\mathbb{Z}_{\mathcal{X}}[n]$ is contained in Sing$X$. Letting $d = \dim(\text{Sing}X)$, Proposition 3.4 gives a bound on the stalk cohomologies at the origin

$$\mathcal{H}^i\left(\phi_g\mathbb{Z}_{\mathcal{X}}[n]\right)_0 \cong \tilde{H}^{i+n}_{\text{sing}}(B_r \cap g^{-1}(\delta); \mathbb{Z}) = 0 \quad \text{for all } i \notin [-d, 0],$$

which turns out to be the desired cohomological connectivity result for the Milnor fiber of g. Moreover, the strong perversity of $\mathbb{Z}_X$ implies that $\mathcal{H}^{-d}\left(\phi_g\mathbb{Z}_{\mathcal{X}}[n]\right)_0$ is free.

As the reader will see, the proofs of our results follow the same pattern: Find an appropriate perverse sheaf and apply the vanishing cycle functor associated to the projection to the parameter space. By virtue of Proposition 3.3, an estimate on the dimension of the support of the sheaf will directly translate to a bound on the nonzero degrees of the reduced cohomology of a nearby object.

The vanishing cycle functor can be employed to compute the reduced cohomology group of the Milnor fiber from the restriction of the vanishing cycle complex to the real link of the singularity. For more results in this direction, see the recent preprint [21].

4 | PROOF OF THEOREMS 2.2, 2.3, AND 2.4

Throughout this section, let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a holomorphic map-germ, $f : U \to V$ an appropriate representative, and $F : U \times T \to V \times T$ an unfolding of f.

Recall that, for $p = n + 1$, finiteness and $\mathcal{K}$-finiteness are the same and $Y = \Delta(f)$. Consequently, Theorem 2.2 and Theorem 2.3 can be considered two instances of the same result for $\mathcal{K}$-finite germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $p \leq n + 1$. For a $\mathcal{K}$-finite germ f, any unfolding F is $\mathcal{K}$-finite and, since $p \leq n + 1$, the discriminant $\Delta(F)$ is a hypersurface in $\mathbb{C}^p \times \mathbb{C}$, by Proposition 3.3. In particular, the shifted constant sheaf $\mathbb{Z}_{\Delta(F)}[p]$is strongly perverse. Projecting on the unfolding parameter gives a family

$$\Delta(F) \xrightarrow{\pi} T,$$

and one checks the fiber over $\delta \in T$ to be $\Delta(f_\delta)$. Applying the associated vanishing cycle functor gives a strongly perverse sheaf $\phi_\pi \mathbb{Z}_{\Delta(F)}[p-1]$ on $\Delta(f)$.

Lemma 4.1. *The support of* $\phi_\pi \mathbb{Z}_{\Delta(F)}[p-1]$ *is contained in* $\mathrm{Inst}(f)$.

Proof. Given a point away from the instability locus $y \in \Delta(f) \setminus \mathrm{Inst}(f)$, $S := f^{-1}(y) \cap \Sigma(f)$ is a finite set because f is $\mathcal{K}$-finite. By Proposition 3.1, the germ $f : (U, S) \to (V, y)$ of f at S is stable and thus the unfolding

$$F : (U \times T, S \times \{0\}) \to (V \times T, (y, 0))$$

is a trivial unfolding. Consequently, there exist unfoldings Φ and Ψ of the identity mappings id_U and id_V, respectively, such that the following diagram commutes:

$$\begin{array}{ccc} (U \times T, S \times \{0\}) & \xrightarrow{F} & (V \times T, (y, 0)) \\ \downarrow \Phi & & \downarrow \Psi \\ (U, S) \times (T, 0) & \xrightarrow{f \times \mathrm{id}_T} & (V, y) \times (T, 0). \end{array}$$

The stalk at y of the sheaf of vanishing cycles of π on $\Delta(F)$ is

$$\begin{aligned} \mathcal{H}^i(\phi_\pi \mathbb{Z}_{\Delta(F)}[p-1])_y &= \mathbb{H}^{i+1}(\Delta(F) \cap B_r, \pi^{-1}(\delta) \cap B_r; \mathbb{Z}_{\Delta(F)}[p-1]) \\ &\cong \mathbb{H}^{i+1}((\Delta(f) \times \mathbb{C}) \cap B_r, \Delta(f) \cap B_r; \mathbb{Z}_{\Delta(f)\times\mathbb{C}}[p-1]) \\ &= H^{i+p}_{\mathrm{sing}}((\Delta(f) \times \mathbb{C}) \cap B_r, \Delta(f) \cap B_r; \mathbb{Z}) \\ &= 0, \end{aligned}$$

which finishes the proof. □

Proof of Theorems 2.2 and 2.3. From the inclusion $\mathrm{supp}(\phi_\pi \mathbb{Z}_{\Delta(F)}[p-1]) \subseteq \mathrm{Inst}(f)$ and the hypothesis that $\dim \mathrm{Inst}(f) \leq d$, applying Proposition 3.4 we obtain that

$$\mathcal{H}^i\left(\phi_\pi \mathbb{Z}_{\Delta(F)}[p-1]\right)_0 = 0 \text{ for } i \notin [-d, 0].$$

Theorems 2.2 and 2.3 follow, because for a good representative the stalk at the origin is precisely

$$\begin{aligned} \mathcal{H}^i(\phi_\pi \mathbb{Z}_{\Delta(F)}[p-1])_0 &= \mathbb{H}^{i+1}(\Delta(F), \Delta(f_\delta); \mathbb{Z}_{\Delta(F)}[p-1]) \\ &= \tilde{H}^{i+p-1}_{\mathrm{sing}}(\Delta(f_\delta); \mathbb{Z}), \end{aligned}$$

where the last equality is due to the fact that $\Delta(F)$ is contractible. □

The proof of Theorem 2.4 is very similar and thus will only be sketched. Before that, we discuss some subtleties of the analytic structure of $D(f)$. In order to avoid pathologies related to unfoldings, the source double point space $D(f)$ is given an analytic structure, which need not be reduced. For our purposes, we do not need to know the details of this construction [19, Definition 2.2], but only the following two properties:

(1) For any unfolding $F = (f_t, t) : U \times T \to V \times T$, the fiber over $\delta \in T$ of the family

$$D(F) \xrightarrow{\pi} T$$

is the complex space $D(f_\delta)$.

(2) If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is finite and generically one-to-one, then the space $D(f)$ is a hypersurface.

The first statement follows from [19, Lemma 4.2]. For the second statement, the proof of [19, Lemma 2.3] shows that $D(f)$ is a Cohen Macaulay subspace of $\mathbb{C}^n$ of codimension one, hence a hypersurface.

Proof of Theorem 2.4. First of all, observe that the statement of Theorem 2.4 is trivial unless the dimension d of the instability locus is smaller than $n-1$. Consequently, we may assume $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ to be a finite map-germ with $\dim(\mathrm{Inst}(f)) < n-1$.

The map f is generically one-to-one, because stable mappings are generically one-to-one, and the conditions of finiteness and $\dim(\mathrm{Inst}(f)) < n-1$ imply that the preimage of the instability locus is nowhere dense.

Any unfolding F of f is also finite and generically one-to-one, hence $D(F)$ is a hypersurface in $\mathbb{C}^{n+1}$ and the sheaf $\mathbb{Z}_{D(F)}[n]$ is strongly perverse. The shifted vanishing cycle functor associated to the projection $\pi : D(F) \to T$ gives a strongly perverse sheaf $\phi_\pi \mathbb{Z}_{D(F)}[n-1]$ on $D(f)$. The same argument used in Lemma 4.1 shows the inclusion

$$\mathrm{Supp}\phi_\pi \mathbb{C}_{D(F)} \subseteq f^{-1}(\mathrm{Inst}(f)),$$

where the dimension of $f^{-1}(\mathrm{Inst}(f))$ is equal to d, because f is finite. Then, the result follows from the computation of the stalk

$$\mathcal{H}^i(\phi_\pi \mathbb{Z}_{D(F)}[n-1])_0 \cong \tilde{H}^{n-1+i}_{\mathrm{sing}}(\Delta(f_\delta); \mathbb{Z}).$$

□

5 | MULTIPLE POINT SPACES

Throughout this section and the next section, the ring of coefficients $\mathbb{Z}$ for the cohomology theories considered will be replaced by $\mathbb{Q}$. This is enforced by the use of the *alternating operator* discussed below. In particular, there will be no difference between classical perversity and strong perversity in this setting. All statements made work equally well replacing $\mathbb{Q}$ by any other field of characteristic zero and we will eventually do so when we switch to complex coefficients for our study of the monodromy in the last section.

For finite germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $p > n+1$, the image Y is no longer a hypersurface, and may even fail to be a complete intersection, as shown by the well-known twisted cubic:

$$(\mathbb{C}^2, 0) \to (\mathbb{C}^4, 0), \quad (s, t) \mapsto (s^3, s^2 t, s t^2, t^3).$$

Since the constant sheaf $\mathbb{Q}_X$ on noncomplete intersections X is not necessarily perverse, we cannot follow the same reasoning as in the proof of Theorems 2.3 and 2.2. Instead, we study the cohomology of the disentanglement via the *image computing spectral sequence* due to Goryunov and Mond [7, Proposition 2.3]:

$$E_1^{p,q} = H^q_{\mathrm{Alt}}(\mathcal{D}^{p+1}(f)) \Rightarrow H^{p+q}(Y).$$

and the perversity of the sheaves

$$\mathrm{Alt} R\varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}[kn - (k-1)p]$$

on the image Y, discovered by Houston [13, Theorem 2.9]. This involves the *strict multiple point spaces* $\mathcal{D}^k(f)$ and their *alternating cohomology*, which we will now discuss.

Unfortunately, there is no common agreement on the definition of multiple point spaces and different notions are in circulation. What we will refer to as the "strict multiple points" $\mathscr{D}^k(f)$ is the definition used by Goryunov, Mond [7], and Houston [11-13]. As Example 5.5 shows, the strict multiple point spaces are badly behaved in deformations: They do not specialize to fibers. To remedy this fact, there is another, more subtle definition of multiple point spaces $D^k(f)$ due to Gaffney [5], which we will describe in Section 5.2. These are what we will refer to as simply the *multiple point spaces* of f. Fortunately, the results about $\mathscr{D}^k(f)$ we want to use can be adapted to the spaces $D^k(f)$ without difficulties.

5.1 | The strict multiple point spaces $\mathscr{D}^k(f)$

Let $f : X \to Z$ be a finite holomorphic map between complex analytic manifolds. For $k \geq 1$, the k-th *strict multiple point space* of f is defined to be the analytic closure of the set of strict multiple points:

$$\mathcal{D}^k(f) = \overline{\{(x_1, \dots, x_k) \in X^k \mid f(x_i) = f(x_j), x_i \neq x_j \text{ for all } i \neq j\}}.$$

Note that $\mathcal{D}^1(f) = X$.

We recall the construction of the alternating complex due to Goryunov and Mond [7]. $\mathcal{D}^k(f)$ are symmetric in the sense that the symmetric group S_k acts on them by permuting the points $x_1, \dots, x_k$. This action preserves the fibers of the maps

$$\varepsilon^k : \mathcal{D}^k(f) \to Z, \quad (x_1, \dots, x_k) \mapsto f(x_1) = \cdots = f(x_k).$$

As a consequence, there is an action of S_k on the pushforward sheaf $\varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}$ on Z. For each $\sigma \in S_k$, we write the associated automorphism as

$$\sigma^* : \varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)} \to \varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}$$

and define the *alternating operator*Alt $: \varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)} \to \varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}$ by the formula

$$\mathrm{Alt} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) \sigma^*.$$

The image of Alt defines a subsheaf, which we denote by $\mathrm{Alt}\varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}$.

It is clear that the map $X^k \to X^{k-1}$, which forgets the j-th coordinate, takes a k-multiple point to a $(k-1)$-multiple point. Hence, we have maps

$$\varepsilon^{k,j} : \mathcal{D}^k(f) \to \mathcal{D}^{k-1}(f).$$

These maps induce morphisms $\varepsilon^{k,j}_* : \varepsilon^{k-1}_* \mathbb{Q}_{\mathcal{D}^{k-1}(f)} \to \varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}$, and one can see that there is a well-defined differential

$$\partial^k : \mathrm{Alt}\varepsilon^{k-1}_* \mathbb{Q}_{\mathcal{D}^{k-1}(f)} \to \mathrm{Alt}\varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)},$$

of the form

$$\partial^k = \sum_{i=1}^{k} (-1)^i \varepsilon^{k,i}_*.$$

One can also check the equality $\partial^{k+1} \circ \partial^k = 0$, so that we obtain a complex of sheaves $\left(\mathrm{Alt}\varepsilon^\bullet_* \mathbb{Q}_{\mathcal{D}^\bullet(f)}, \partial^\bullet\right)$, the *alternating complex.*

Proposition 5.1. *[7, Proposition 2.1] The augmented complex*

$$0 \to \mathbb{Q}_Y \to \mathrm{Alt}\varepsilon^1_* \mathbb{Q}_{\mathcal{D}^1(f)} \to \mathrm{Alt}\varepsilon^2_* \mathbb{Q}_{\mathcal{D}^2(f)} \to \cdots$$

is exact.

Goryunov and Mond have already argued that one has an isomorphism

$$H^i(Y, \mathrm{Alt}\varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}) \cong H^i_{\mathrm{Alt}}(\mathcal{D}^k(f)), \tag{2}$$

where the term on the right-hand side is the *alternating cohomology*

$$\left\{ [c] \in H^i_{\mathrm{sing}}(\mathcal{D}^k(f)) \, : \, \sigma^*[c] = \mathrm{sign}(\sigma) \cdot [c] \text{ for all } \sigma \in S_k \right\},$$

a subspace of the singular cohomology of $\mathcal{D}^k(f)$. These considerations were taken to the derived category in [13]. To translate his statements to our setup, note that, since ε^k is finite, the associated pushforward of sheaves is an exact functor and thus in particular

$$\varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)} = R\varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}$$

as complexes of sheaves on Y in the derived category. The same holds for their respective alternating subsheaves.

The last ingredient that enters our discussion of the alternating complexes is the dimension of the multiple point spaces.

Proposition 5.2. *Let $f : X \to Z$ be a complex analytic mapping between two complex analytic manifolds, with $n = \dim X$ and $p = \dim Z$. If f is stable, then, for all $k \geq 2$, the strict multiple point space $\mathcal{D}^k(f)$ is empty or has dimension $kn - (k-1)p$.*

Definition 5.3. Let $f : X \to Z$ be a complex analytic mapping between two complex analytic manifolds with $n = \dim X$ and $p = \dim Z$. Then, f is called *strictly dimensionally correct* if, for all $k \geq 2$, the strict multiple point space $\mathscr{D}^k(f)$ is either empty, or has dimension $kn - (k-1)p$.

With these notations gathered, we may cite the key result due to Houston, cf. [13, Theorem 2.9], slightly adapted to our setup:

Theorem 5.4. *Suppose $f : X \to Z$ is a finite, strictly dimensionally correct complex analytic map between complex manifolds of dimensions $n = \dim X < \dim Z = p$. Then, $\mathrm{Alt}\varepsilon^k_* \mathbb{Q}_{\mathcal{D}^k(f)}[kn - (k-1)p]$ is a perverse sheaf.*

5.2 | The multiple point spaces $D^k(f)$

As already mentioned, the strict multiple point spaces discussed in the previous section are not well behaved in families. This is illustrated by the following example.

Example 5.5. The cuspidal edge $f : (x, y) \mapsto (x, y^2, y^3)$ of Example 1.6 and the similar germ $f' : (x, y) \mapsto (x, y^2, y^3, 0)$ of Example 2.9 are both strictly dimensionally correct because the maps are injective and therefore the multiple point spaces are empty.

However, the map $F' : (x, y, t) \mapsto (x, y^2, y^3 + ty(x^2 - 1), 0, t)$ from the unfolding in Example 2.9 is *not* strictly dimensionally correct: Its multiple point spaces $\mathcal{D}^1(F)$ have dimension two while the expected dimension is one.

Without the dummy variable, that is, for the map f as in Example 1.6, not only f, but also the unfolding

$$F : (x, y, t) \mapsto (x, y^2, y^3 + t \cdot y(x^2 - t))$$

is strictly dimensionally correct with expected dimension two for the double point spaces. In this example, we encounter another problem: Observe that the strict double points of f are empty while those of F are not. This shows the failure of specialization

$$\mathcal{D}^2(F) \cap \{t = 0\} \neq \mathcal{D}^2(f)$$

and thereby illustrates the necessity to use Gaffney's multiple point spaces.

The pathological behavior exhibited in this example can be avoided by taking a different definition of multiple points due to Gaffney [5]. Our exposition follows [27], where the reader can find proofs and details omitted here.

Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a finite multigerm. Since f is finite, hence $\mathcal{K}$-finite, it admits a stable unfolding [26, Theorem 7.2]

$$F = (f_t, t) : (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^r, (0, 0)),$$

where t is the coordinate of the parameter space $(\mathbb{C}^r, 0)$. It is clear that a point in $\mathcal{D}^k(F)$ has $t_i = t_j$, for all $1 \le i, j \le k$. Therefore, $\mathcal{D}^k(F)$ can be embedded in $(\mathbb{C}^n)^k \times \mathbb{C}^r$, rather than in $(\mathbb{C}^n \times \mathbb{C}^r)^k$. The multiple point spaces of f are defined as

$$D^k(f) := \mathcal{D}^k(F) \cap \{t = 0\}.$$

The space $D^k(f)$ does not depend on the chosen stable unfolding F [27, Lemma 2.3]. Multiple point spaces of finite mappings between complex manifolds are defined by patching the multiple point spaces of their corresponding multigerms.

To compare $\mathscr{D}^k(f)$ to $D^k(f)$, we write the *k-th fat diagonal of X* as

$$\Delta^k = \{(x_1, \dots, x_k) \in X^k \mid x_i = x_j \text{ for some } i \neq j\}.$$

Proposition 5.6. *(Properties of the multiple point spaces) Let $f : X \to Z$ be a finite mapping between complex manifolds.*

(1) The spaces $D^k(f)$ and $\mathcal{D}^k(f)$ satisfy the relation

$$\mathcal{D}^k(f) = \overline{D^k(f) \setminus \Delta^k}.$$

(2) If f is stable, then $D^k(f) = \mathcal{D}^k(f)$. In particular, $\dim D^k(f) = kn - (k-1)p$, by Proposition 5.2.
(3) Unlike $\mathscr{D}^k(f)$, the spaces $D^k(f)$ behave well under deformations in the sense that, for any unfolding $F : X \times T \to Z \times T$ of f, the family $\pi : D^k(F) \to T$ satisfies

$$D^k(f_\delta) = \pi^{-1}(\delta), \textit{ for any } \delta \in T.$$

Just as in the case of $\mathcal{D}^k(f)$, the spaces $D^k(f)$ are S_k-invariant spaces endowed with finite maps $\varepsilon^{k,j} : D^k(f) \to D^{k-1}(f)$, giving rise to a complex

$$\left(\mathrm{Alt}\varepsilon_*^\bullet \mathbb{Q}_{D^\bullet(f)}, \partial^\bullet\right).$$

We wish to replace $\mathcal{D}^k(f)$ by our favorite $D^k(f)$ in the study of $\mathbb{Q}_Y$. While the spaces $D^k(f)$ and $D^k(f)$ and hence also the sheaves $\varepsilon_*^k \mathbb{Q}_{D^k(f)}$ and $\varepsilon_*^k \mathbb{Q}_{D^k(f)}$ may differ, the corresponding *alternating* sheaves do not:

Lemma 5.7. *For any finite map f between complex manifolds, the complexes $\left(\mathrm{Alt}\varepsilon_*^\bullet \mathbb{Q}_{D^\bullet(f)}, \partial^\bullet\right)$ and $\left(\mathrm{Alt}\varepsilon_*^\bullet \mathbb{Q}_{D^\bullet(f)}, \partial^\bullet\right)$ are equal.*

Note that this lemma allows us to replace $\mathrm{Alt}\varepsilon_*^k \mathbb{Q}_{D^k(f)}$ by $\mathrm{Alt}\varepsilon_*^k \mathbb{Q}_{D^k(f)}$ in Proposition 5.1. The idea behind the proof of Lemma 5.7 is not new and appears already in [7], as well as [13] and [12]. Due to the lack of a citeable reference, we give a self-contained argument.

Proof. The inclusion $\mathcal{D}^k(f) \hookrightarrow D^k(f)$ induces a morphism between the corresponding sheaves, so it is enough to show that it induces an isomorphism on all stalks at points $y \in Y$. Fix one such point and let

$$P := \left(\varepsilon^k\right)^{-1}(y) \subset D^k(f)$$

be its preimage under the finite map $\varepsilon^k : D^k(f) \to Y$. Set $P' := \mathcal{D}^k(f) \cap P$ to be the preimage in the subspace $\mathcal{D}^k(f)$. According to Proposition 5.6, every point $x \in P \setminus P'$ has to be contained in the fat diagonal Δ^k. Now the pushforward of $\mathbb{Q}_{D^k(f)}$ along ε^k decomposes as

$$\varepsilon^k_* \mathbb{Q}_{D^k(f)} = \left(\bigoplus_{x \in P'} \mathbb{Q}_{D^k(f),x} \right) \oplus \left(\bigoplus_{x \notin P'} \mathbb{Q}_{D^k(f),x} \right).$$

The alternating operator clearly respects this decomposition since P' is itself an S_k-invariant subspace. Note that the first summand is nothing but the stalk of $\varepsilon^k_* \mathbb{Q}_{D^k(f)}$, by definition of P'. Thus, we may conclude the proof by showing that the second summand has no alternating part.

To see this, let $x = (x_1, \dots, x_k) \in P \setminus P'$ be an arbitrary point and $c \in \mathbb{Q}_{D^k(f),x}$ an element of its associated stalk. Since x has to be a diagonal point, there exists one pair of indices $0 < i < j \le k$ such that $x_i = x_j$. Let $\tau \in S_k$ be the corresponding transposition. Then, we have

$$-\mathrm{Alt} c = \tau * \mathrm{Alt} c = \mathrm{Alt}(\tau * c) = \mathrm{Alt} c,$$

because $\mathrm{Alt} c$ is alternating and τ takes the point x to itself. Thus, $\mathrm{Alt} c = 0$ for every element c of any stalk of $\mathbb{Q}_{D^k(f)}$ at points outside $\mathcal{D}^k(f)$. □

Remark 5.8. Parallel to (2) we have the equality

$$H^i(Y, \mathrm{Alt}\varepsilon^k_* \mathbb{Q}_{D^k(f)}) = H^i_{\mathrm{Alt}}(D^k(f), \mathbb{Q}).$$

This shows indirectly that $\mathcal{D}^k(f)$ and $D^k(f)$ have the same alternating cohomology, thanks to the equality $\left(\mathrm{Alt}\varepsilon^\bullet_* \mathbb{Q}_{D^\bullet(f)}, \partial^\bullet\right) = \left(\mathrm{Alt}\varepsilon^\bullet_* \mathbb{Q}_{\mathcal{D}^\bullet(f)}, \partial^\bullet\right)$ proved in Lemma 5.7.

As in the case of $\mathcal{D}^k(f)$, if $f : X \to Z$ is a mapping between two complex analytic manifolds with $n = \dim X$ and $p = \dim Z$, then every irreducible component of $D^k(f)$ has dimension at least $kn - (k-1)p$. The use of $D^k(f)$ rather than $\mathcal{D}^k(f)$ calls for the following adaptation:

Definition 5.9. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, be a finite map-germ, with $p > n$. We say that f is *dimensionally correct* if, for all $k \ge 2$, the multiple point space $D^k(f)$ is empty or has dimension $kn - (k-1)p$.

It is clear that every unfolding F of a finite map-germ f must also be finite. Moreover, any stable unfolding of F is a stable unfolding of f as well. From this observation, one concludes easily the following result:

Proposition 5.10. *If a map-germ f is dimensionally correct, then every unfolding F of f is dimensionally correct.*

Remark 5.11. The previous assertion does not hold if one replaces the property of being dimensionally correct by that of being strictly dimensionally correct (see Example 5.5). Being dimensionally correct implies being strictly dimensionally correct, but the converse is not true, as shown again by the germ $(x, y) \mapsto (x, y^2, y^3, 0)$.

As we mentioned in the Introduction, dimensional correctness is too restrictive, as it does not allow us to study, for example, map-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ with nonempty quadruple points. To remedy this, we introduce a further notion of dimensional correctness:

Definition 5.12. We say that f is *dimensionally correct for nonnegative dimensions* if, for all $k \ge 2$ such that $kn - (k-1)p \ge 0$, the multiple point space $D^k(f)$ is empty or has dimension $kn - (k-1)p$.

From the fact that stable mappings are dimensionally correct, we conclude the following:

Proposition 5.13. *Let f be dimensionally correct for nonnegative dimensions and let F be a stabilization of f, with stable perturbation f_δ. Then either*

(a) $D^k(f_\delta) = \emptyset$, *or*
(b) $\dim D^k(f_\delta) = kn - (k-1)p$ *and* $D^k(F)$ *has dimension* $kn - (k-1)p + 1$

for every $k \geq 2$.

5.3 | The image computing spectral sequence

Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a finite map-germ with $p > n$ and let F be a one-parameter unfolding of f. We wish to compute the reduced cohomology of the disentanglement Y_δ of a good representative of F. Goryunov and Mond have shown [7] how to do this in terms of the alternating complex. We are basically going to rephrase their argument in the derived category so that we can bring the perversity of the alternating sheaves, Theorem 5.4, into the picture.

In the derived category, Proposition 5.1 and Lemma 5.7 assert that the constant sheaf $\mathbb{Q}_\mathcal{Y}$ on the image $\mathcal{Y}$ of the unfolding F is quasi-isomorphic to the alternating complex $\left(\mathrm{Alt}\varepsilon_*^\bullet \mathbb{Q}_{D^k(F)}, \partial^\bullet\right)$ and we may thus replace $\mathbb{Q}_\mathcal{Y}$ by this complex.

If we let $\pi : \mathcal{Y} \to T$ be the projection onto the unfolding parameter, then the reduced cohomology of the disentanglement Y_δ are the vanishing cycles of π at the origin:

$$\begin{aligned}\tilde{H}^i(Y_\delta) &\cong \mathcal{H}^i(\phi_\pi(\mathbb{Q}_\mathcal{Y}))_0 \\ &\cong \mathcal{H}^i\left(\phi_\pi(\mathrm{Alt}\varepsilon_*^\bullet \mathbb{Q}_{D^\bullet(F)})\right)_0.\end{aligned}$$

The nearby and the vanishing cycles of $\pi : \mathcal{Y} \to T$ are computed using the well-known diagram

$$\begin{array}{ccccccc} Y_0 & \overset{i}{\hookrightarrow} & \mathcal{Y} & \overset{j}{\hookleftarrow} & \mathcal{Y} \setminus Y_0 & \overset{\rho}{\longleftarrow} & Y_\delta \times S \\ \downarrow & & \downarrow \pi & & \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & T & \hookleftarrow & T^* & \underset{\exp}{\longleftarrow} & S \end{array} \tag{3}$$

for a good representative F of a one-parameter family and its image $\mathcal{Y}$. Here, $\exp : S \to T^*$ is the universal cover of the punctured unit disc and $Y_\delta \times S$ the trivialized fiber product over the infinite strip S, cf. [3, chapter 4.2].

Lemma 5.14. *For a good representative F of a one-parameter unfolding of a finite map $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $n < p$, there is a spectral sequence with first page*

$$E_1^{i,j} = \mathcal{H}^i\left(\phi_\pi\left(\mathrm{Alt}\varepsilon_*^{j+1} \mathbb{Q}_{D^{j+1}(F)}\right)\right)_0$$

converging to the reduced cohomology of the disentanglement Y_δ of f

$$\tilde{H}^{i+j}(Y_\delta) = \mathcal{H}^{i+j}(\phi_\pi \mathbb{Q}_\mathcal{Y})_0.$$

Here, $\pi : \mathcal{Y} \to T$ is the projection of the image $\mathcal{Y}$ of F to the deformation parameter.

Proof. The standard procedure to obtain a quasi-isomorphism of the complex $\mathrm{Alt}\varepsilon_*^\bullet \mathbb{Q}_{D^\bullet(F)}$ with a complex of injectives is to construct a double complex of injectives $I^{\bullet,\bullet}$ where each column $I^{\bullet,j}$ resolves the sheaf $\mathrm{Alt}\varepsilon_*^{j+1} \mathbb{Q}_{D^{j+1}(F)}$, cf. [7, Section 2]. Then, the original complex is quasi-isomorphic to the total complex $\mathrm{Tot}(I^{\bullet,\bullet})$ of this double complex $I^{\bullet,\bullet}$.

The *nearby cycles* of π are defined as

$$\psi_\pi\left(\mathrm{Alt}\varepsilon_*^\bullet \mathbb{Q}_{D^\bullet(F)}\right) := i^{-1} R(\rho \circ j)_* (\rho \circ j)^{-1} \left(\mathrm{Alt}\varepsilon_*^\bullet \mathbb{Q}_{D^\bullet(F)}\right).$$

This is a composition of three functors. The first one, $(\rho\circ j)^{-1}$, takes injective sheaves on $\mathcal{Y}$ to injective sheaves on the open subset $\mathcal{Y}\setminus Y_0$, see [14, Proposition 2.4.1], and the third one, i^{-1}, is an exact functor. To compute the nearby cycles of $\mathrm{Alt}\varepsilon_*^\bullet\mathbb{Q}_{D^\bullet(F)}$—which involves the derived pushforward in the middle—we may therefore apply the functor $i^{-1}(\rho\circ j)_*(\rho\circ j)^{-1}$ to $\mathrm{Tot}(I^{\bullet,\bullet})$. Note that applying this functor to a separate column $I^{\bullet,j}$ yields the nearby cycles of the single sheaf $\mathrm{Alt}\varepsilon_*^{j+1}\mathbb{Q}_{D^{j+1}(F)}$ considered as a complex of sheaves concentrated in a single degree.

The *vanishing cycles* $\phi_\pi\mathbb{Q}_\mathcal{Y}$ of π are defined as the mapping cone of the comparison morphism c

$$i^{-1}\mathbb{Q}_\mathcal{Y}\xrightarrow{c}\psi_\pi\mathbb{Q}_\mathcal{Y}\longrightarrow\phi_\pi\mathbb{Q}_\mathcal{Y}\xrightarrow{[+1]}.$$

Following Definition 1.2, the image $\mathcal{Y}$ of a good representative is connected, hence the stalk of $i^{-1}\mathbb{Q}_\mathcal{Y}$ at the origin is simply the vector space $\mathbb{Q}$, concentrated in degree zero. It is easy to check that on this single nontrivial term the comparison morphism c is always an inclusion, which makes the vanishing cycles coincide with the reduced cohomology of the disentanglement, cf. (1).

We may again replace $\mathbb{Q}_\mathcal{Y}$ by the alternating complex $\mathrm{Alt}\varepsilon_*^\bullet\mathbb{Q}_{D^\bullet(F)}$ and subsequently by $\mathrm{Tot}(I^{\bullet,\bullet})$ in the distinguished triangle of the comparison morphism:

$$i^{-1}\mathrm{Tot}(I^{\bullet,\bullet})\xrightarrow{c}i^{-1}(\rho\circ j)_*(\rho\circ j)^{-1}\mathrm{Tot}(I^{\bullet,\bullet})\longrightarrow\phi_\pi\mathrm{Tot}(I^{\bullet,\bullet})\xrightarrow{[+1]}.$$

Now note that the functor $i^{-1}(\rho\circ j)_*(\rho\circ j)^{-1}$ commutes with taking the total complex in a natural way. Thus, we may identify the distinguished triangle of the comparison morphism c with the total complex of the following "comparison morphism of double complexes"

$$i^{-1}I^{\bullet,\bullet}\xrightarrow{\tilde{c}}i^{-1}(\rho\circ j)_*(\rho\circ j)^{-1}I^{\bullet,\bullet}\longrightarrow\mathrm{Cone}(\tilde{c})\xrightarrow{[+1]}$$

which turns the vanishing cycles $\phi_\pi\mathrm{Tot}(I^{\bullet,\bullet})$ into the total complex of the double complex $\mathrm{Cone}(\tilde{c})$.

The spectral sequence in question is now the spectral sequence of this double complex $\mathrm{Cone}(\tilde{c})$. A column-wise inspection of this double complex reveals that we indeed find the vanishing cycles of the alternating sheaves on the first page. □

Remark 5.15. From the fact that the spaces $D^k(F)$ are empty for k big enough, it follows that the spectral sequence collapses at a certain page. Since we chose a field for the coefficients in cohomology, the infinity page determines the cohomology of the total complex. To be precise, there is an isomorphism of vector spaces

$$\mathcal{H}^\ell(\phi_\pi\mathbb{Q}_\mathcal{Y})_0\cong\bigoplus_i E_\infty^{i,\ell-i}.$$

Example 5.16. Here, we come back to the immersion bi-germ $f:(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ associated to a hypersurface $X=V(g)$, introduced in Example 2.6, where we find a particularly simple spectral sequence. First of all, there are no triple or higher multiplicity points. This is because the stable unfolding F of f (or any unfolding of f for that matter) is also a bi-germ of an immersion, hence it cannot map more than two points to one. This gives $\mathcal{D}^k(F)=\emptyset$, and thus $D^k(f)=\emptyset$, for all $k\geq 3$. Consequently, the nonzero terms of the first page $E_1^{\bullet,\bullet}$ are concentrated in the second column, corresponding to the alternating cohomologies $H^i_{\mathrm{Alt}}(D^2(f_\delta))$.

The fact that there are no triple points implies also that the maps $D^2(F)\to D(F)$ and $D^2(f_\delta)\to D(f_\delta)$ are homeomorphisms. Recall also from Example 2.6 that $D(f_\delta)=M_1\sqcup M_2$, where M_1 and M_2 are copies of the Milnor fiber of $X=V(g)$. The homeomorphism $D^2(f_\delta)\to D(f_\delta)$ transforms the action of the generator $\sigma\in S_2$ into the map taking a point $x\in M_1$ to the same point in M_2, and vice versa. In particular, after identifying $D^2(f_\delta)$ with $M_1\sqcup M_2$, we can choose a system of generators of the cohomology $H^i(D^2(f_\delta))$ consisting of some cocycles $c_i$generating the cohomology of M_1 and their corresponding cocycles $\sigma\cdot c_i$, which generate the cohomology of M_2. It now follows immediately that the alternating part is generated by the cocycles $c_i-\sigma\cdot c_i\in H^i_{\mathrm{Alt}}(D^2(f_\delta))$. The linear map extending $c_i\mapsto c_i-\sigma\cdot c_i$ gives an isomorphism

$$H^i(M)\cong H^i_{\mathrm{Alt}}(D^2(f)).$$

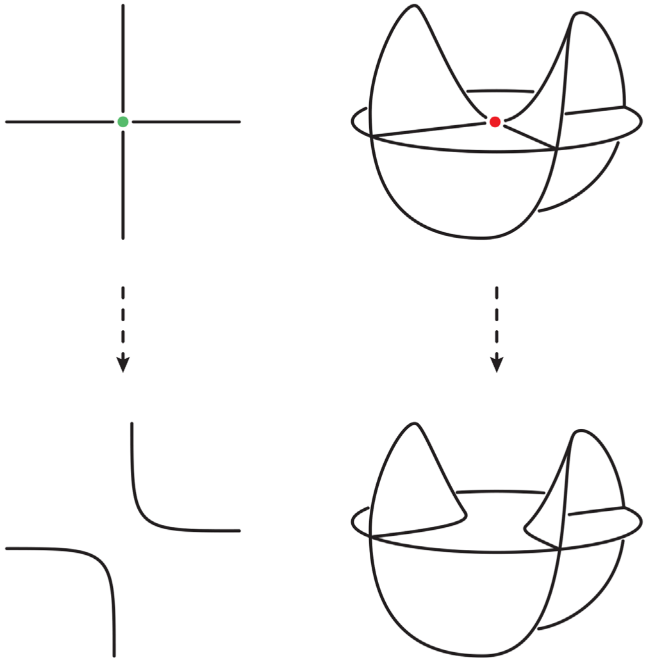


FIGURE 5 The singularity $\{xy = 0\} \subseteq \mathbb{C}^2$ and its Milnor fiber can be realized as the intersection of the branches of a bi-germ and its stable perturbation. The singular locus of $\{xy = 0\}$ corresponds to the instability locus of the bi-germ.

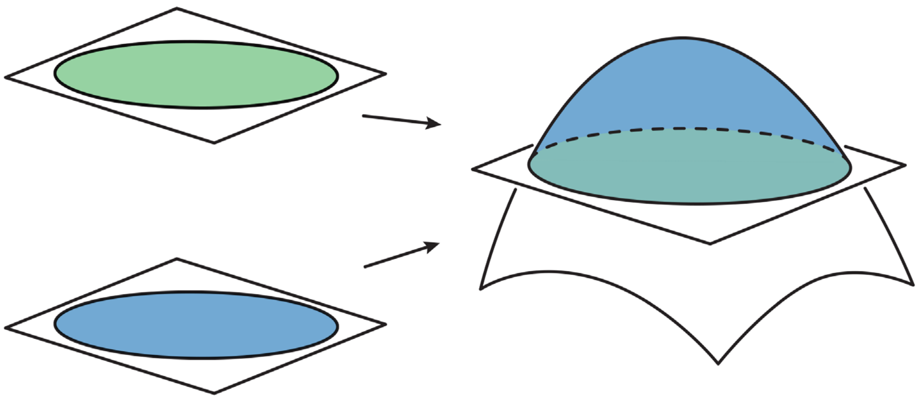


FIGURE 6 The disentanglement of a bi-germ of immersion has the homology of a two-point suspension.

Finally, since $E_1^{\bullet,\bullet}$ contains a single nonzero column, the spectral sequence collapses at page one, that is, $E_\infty^{\bullet,\bullet} = E_1^{\bullet,\bullet}$. Now, the isomorphism $\mathcal{H}^\ell(\phi_\pi \mathbb{Q}_\mathcal{Y})_0 \cong \bigoplus_i E_\infty^{i,\ell-i}$ takes the form

$$\tilde{H}^i(M) \cong \tilde{H}^{i+1}(Y_\delta),$$

as Remark 1.8 claimed.

Remark 5.17. The previous example has a more visual version in homology (Figure 6 depicts the case of $g = x^2 + y^2$ which, after a change of coordinates, is the same as in Figure 5, but with a real picture better suited for the present discussion). We know that $D(f_\delta)$ consists of two copies M_1 and M_2 of M, contained in two copies U_1 and U_2 of an open ball $U \in \mathbb{C}^n$. Now let c be a cycle in M and let c_i be the corresponding copies in U_i (the two circles one the left side in Figure 5). Since U is contractible, there are chains a_i in U_i with boundary $\delta a_i = c_i$ (the green and blue disks). Observe that ∂a_i is supported on M_i, which is the double point space of f_δ, and that f_δ glues a_1 and a_2 along the boundaries ∂a_i. After changing the sign of a_2 if necessary, we observe that $f_*(a_1 + a_2)$ is an $(i+1)$-dimensional cycle on Y_δ (the blue and green cycle on the right side). The desired isomorphism is determined by $c \mapsto f_*(a_1 + a_2)$.

6 | PROOF OF THEOREM 2.8

In this section, we will use the notation of Theorem 2.8 and impose the hypotheses found there, that is, $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a dimensionally correct map-germ with $p > n$ and instability locus $\mathrm{Inst}(f)$ of dimension d, $F : W \to V \times T$ is a good representative of a one-parameter unfolding of f and $\mathcal{Y} = \mathrm{Im}F$ is the image of F. We will continue using rational coefficients for the cohomology theories involved.

We intend to control the stalk cohomology of $\phi_\pi \mathbb{Q}_{\mathcal{Y}}$, that is, the reduced cohomology of the disentanglement, by means of the first page of the spectral sequence in Lemma 5.14. Thus, we have to bound the vanishing cohomology of the alternating sheaves $\mathrm{Alt}\varepsilon_*^{j+1}\mathbb{Q}_{D^{j+1}(F)}$ on the image and we will do so by exploiting the perversity of their respective shifts and estimating the dimension of the support of their vanishing cycles.

Lemma 6.1. *The support of* $\phi_\pi\left(\mathrm{Alt}\varepsilon_*^k\mathbb{Q}_{D^k(F)}\right)$ *is contained in* $\mathrm{Inst}(f)$.

Proof. Take a point away from the instability locus $y \in Y \setminus \mathrm{Inst}(f)$ and let $S = f^{-1}(y)$. Since f is stable at y, the unfolding $(F, S \times \{0\})$ of the germ (f, S) is trivial. Therefore, there exist an unfolding Φ of $(\mathrm{id}_{\mathbb{C}^n}, S)$ and an unfolding Ψ of $(\mathrm{id}_{\mathbb{C}^p}, 0)$, such that

$$(F, S \times \{0\}) = \Psi^{-1}\circ((f, S) \times \mathrm{id}_{(T,0)})\circ\Phi.$$

We choose representatives F, f, Φ, and Ψ satisfying the equality

$$F = \Psi^{-1}\circ(f \times \mathrm{id}_T)\circ\Phi.$$

As one may easily check, this makes the multiple points $D^k(F)$ and $D^k(f \times \mathrm{id}_T)$ isomorphic via $\Phi^{-1}\times \overset{k}{\dots} \times\Phi^{-1}$. Moreover, the multiple points in $D^k(f \times \mathrm{id}_T)$ are of the form $((x_1, t), \dots, (x_k, t))$. Forgetting all but one of the copies of t gives an isomorphism [27, Proposition 3.8]

$$\Omega : D^k(f \times \mathrm{id}_T) \to D^k(f) \times T.$$

In turn, the spaces $\mathcal{Y}$ and $Y \times T$ are isomorphic via Ψ.

There are two geometric considerations where the k-fold product structure of $\Phi^{-1}\times \overset{k}{\dots} \times\Phi^{-1}$ plays a role: First of all, the isomorphisms and the ε^k mappings are compatible in the sense that there is a commutative diagram

$$\begin{array}{ccc} D^k(F) & \xrightarrow{\Omega\circ(\Phi^{-1}\times \overset{k}{\dots} \times\Phi^{-1})} & D^k(f)\times T \\ \downarrow{\scriptstyle \varepsilon^k} & & \downarrow{\scriptstyle \varepsilon^k\times \mathrm{id}_T} \\ \mathcal{Y} & \xrightarrow{\Psi} & Y \times T. \end{array}$$

Second, the map $\Omega\circ(\Phi^{-1}\times \overset{k}{\dots} \times\Phi^{-1})$ is compatible with the action of the symmetric group S_k on both $D^k(F)$ and on $D^k(f) \times T$ (the latter induced by the action of S_k on $D^k(f)$). This makes the construction of the Alt operators on both sides equivalent and the pushforward Ψ_* takes the sheaf $\mathrm{Alt}\varepsilon_*^k\mathbb{Q}_{D^k(F)}$ on $\mathcal{Y}$ to the sheaf $\mathrm{Alt}(\varepsilon^k \times \mathrm{id}_T)_*\mathbb{Q}_{D^k(f)\times T}$ on $Y \times T$.

We can now show that$y$ is not in the support of $\phi_\pi\left(\mathrm{Alt}\varepsilon_*^k\mathbb{Q}_{D^k(F)}\right)$ by checking the vanishing of the stalk cohomology: Take a suitable ball B_r in V around y. Then, for δ sufficiently small we find

$$\begin{aligned} &\mathcal{H}^i\left(\phi_\pi\left(\mathrm{Alt}\varepsilon_*^k\mathbb{Q}_{D^k(F)}\right)\right)_y \\ &= \mathbb{H}^{i+1}\left(\mathcal{Y}\cap B_r, \mathcal{Y}\cap\pi^{-1}(\delta)\cap B_r; \mathrm{Alt}\varepsilon_*^k\mathbb{Q}_{D^k(F)}\right) \\ &\cong \mathbb{H}^{i+1}\left((Y\times T)\cap B_r, (Y\times\{\delta\})\cap B_r; \mathrm{Alt}(\varepsilon^k\times\mathrm{id}_T)_*\mathbb{Q}_{D^k(f)\times T}\right) \\ &= 0. \end{aligned}$$

The vanishing in the last line follows from the fact that due to the product structure clearly $(Y \times T) \cap B_r$ retracts onto the fiber $(Y \times \{\delta\}) \cap B_r$ for $r \gg \delta > 0$ and the restriction of the sheaf $\mathrm{Alt}(\varepsilon^k \times \mathrm{id}_T)_* \mathbb{Q}_{D^k(f) \times T}$ to all $(\{y\} \times T) \cap B_r$ is locally constant. □

Now, we can determine where the nontrivial entries of the first page of the spectral sequence in Lemma 5.14 are concentrated.

Proposition 6.2. *Let F be a dimensionally correct one-parameter unfolding of a germ $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $p > n$. For every integer $k \geq 2$, the nonzero cohomologies*

$$\mathcal{H}^i\left(\phi_\pi\left(\mathrm{Alt}\varepsilon_*^k \mathbb{Q}_{D^k(F)}\right)\right)_0$$

are concentrated in degrees $i \geq 0$ with

$$kn - (k-1)p - d \leq i \leq kn - (k-1)p.$$

In the case $k = 1$, the above vanishing cycles are all zero.

Proof. Since F is dimensionally correct, the sheaves

$$\mathrm{Alt}\varepsilon_*^k \mathbb{Q}_{D^k(F)}[kn - (k-1)p + 1]$$

are perverse, and, by virtue of Theorem 3.6, so are the sheaves

$$\phi_\pi\left(\mathrm{Alt}\varepsilon_*^k \mathbb{Q}_{D^k(F)}\right)[kn - (k-1)p]$$

on Y_0. Since these sheaves are supported on a space of dimension at most d, it follows from Proposition 3.4 that the nontrivial cohomologies of their stalks at the origin are concentrated in degrees i with $-d \leq i \leq 0$.

Shifting everything back by $kn - (k-1)p$, we obtain the desired bounds for $k \geq 2$. Note that, since the alternating cohomology

$$\mathcal{H}^i\left(\phi_\pi\left(\mathrm{Alt}\varepsilon_*^k \mathbb{Q}_{D^k(F)}\right)\right)_0 = \tilde{H}^i_{\mathrm{Alt}}(D^k(f_\delta)) \subset \tilde{H}^i(D^k(f_\delta))$$

is a subspace of the singular reduced cohomology of the disentanglement, there can be no contributions in negative degrees.

For $k = 1$, the disentanglement always provides a product structure

$$(D^1(F), (0,0)) \cong (X \times T, (0,0))$$

and therefore

$$\mathcal{H}^i\left(\phi_\pi\left(\mathrm{Alt}\varepsilon_*^1 \mathbb{Q}_{D^1(F)}\right)\right)_0 = H^i(X \times T, X \times \delta) = 0.$$

This finishes the proof. □

Proof of Theorem 2.8. By Proposition 5.13 in case (a) and Proposition 5.13 in case (b), the assumptions imply that F is a dimensionally correct one-parameter unfolding. Then, the proof follows by the same spectral sequence argument as in [28, Theorem 1.1]: The concentration of cohomology follows immediately from the isomorphism

$$\mathcal{H}^\ell(\phi_\pi \mathbb{Q}_\mathcal{Y})_0 \cong \bigoplus_i E_\infty^{i,\ell-i}$$

from Remark 5.15 and the vanishing of the entries of the first page coming from Proposition 6.2. □

Let us give an example, which illustrates the general situation:

Example 6.3. Let $f : (\mathbb{C}^{16}, 0) \to (\mathbb{C}^{21}, 0)$ be a dimensionally correct map whose instability locus has dimension $d = 2$. According to Proposition 6.2, the possibly nonzero entries in the first page of the spectral sequence $E_1^{i,j}$ from Lemma 5.14 are the following:

$i \uparrow$	0	1	2	3
11		$H^{11}_{\mathrm{Alt}}(D^2(f_\delta))$		
10		$H^{10}_{\mathrm{Alt}}(D^2(f_\delta))$		
9		$H^{9}_{\mathrm{Alt}}(D^2(f_\delta))$		
8				
7				
6			$H^{6}_{\mathrm{Alt}}(D^3(f_\delta))$	
5			$H^{5}_{\mathrm{Alt}}(D^3(f_\delta))$	
4			$H^{4}_{\mathrm{Alt}}(D^3(f_\delta))$	
3				
2				
1				$H^{1}_{\mathrm{Alt}}(D^4(f_\delta))$
0				$H^{0}_{\mathrm{Alt}}(D^4(f_\delta))$
	0	1	2	3 $\longrightarrow j$

In this case, the sequence necessarily collapses on this first page. In general, the differentials between the nonzero entries on the first page can lead to cancellations for the following ones and on the limit page. In either case, the positions of the nonzero entries on the first page give a bound on the nonzero entries of $\bigoplus_i E_\infty^{i,k-i} \cong \mathcal{H}^k(\phi_\pi \mathbb{Q}_\mathcal{Y})_0$. For this particular example, we see that the nontrivial cohomologies of a disentanglement Y_δ are concentrated on the degrees $\ell \in \{3,4\} \cup \{6,7,8\} \cup \{10,11,12\}$.

7 MONODROMY FOR DISENTANGLEMENTS

Our study of the monodromy for disentanglements in Theorem 2.11 will be based on the monodromy on the multiple point spaces and their alternating cohomology. This requires an adaptation of the definitions to fibrations with group actions and, moreover, the use of complex coefficients for all cohomology theories involved.

Note that any stable one-parameter unfolding

$$F : (\mathbb{C}^n, 0) \times (\mathbb{C}, 0) \to (\mathbb{C}^p, 0) \times (\mathbb{C}, 0)$$

of a finite map-germ f with $p > n$ as in Theorem 2.11 gives rise to analytic fibrations

$$\pi \circ \varepsilon^k : D^k(F) \cap (\varepsilon^k)^{-1}(B \times D \setminus \{0\}) \to D \setminus \{0\} \tag{4}$$

for suitable choices of a representative F, a ball $B \subset \mathbb{C}^p$ in the target of f, and a disc $D \subset \mathbb{C}$ in the parameter space. By construction, the symmetric group $\mathfrak{S}_k$ acts fiberwise on the multiple point space $D^k(F) \cap (\varepsilon^k)^{-1}(B \times D)$, for each $k > 0$.

As already noted by K. Houston in [11, Lemma 2.14], there exist *equivariant* local trivializations. We may therefore suppose that the trivialization of the pullback of the fibration (4) to the universal cover $\exp : S \to D \setminus \{0\}$ is $\mathfrak{S}_k$-equivariant and that the parallel transport of the fiber

$$h : D^k(f_\delta) \to D^k(f_\delta)$$

along a closed, counterclockwise loop in D^* around the origin commutes with the $\mathfrak{S}_k$-action on $D^k(f_\delta)$. It is easy to see that this implies that also the maps induced by h on cohomology

$$h^i : H^i(D^k(f_\delta)) \to H^i(D^k(f_\delta))$$

restrict to operators on the alternating cohomology $H^i_{\mathrm{Alt}}(D^k(f_\delta))$. We therefore have a well-defined monodromy action

$$h : \phi_\pi\left(\mathrm{Alt}\varepsilon^k_* \mathbb{C}_{D^k(F)}\right) \to \phi_\pi\left(\mathrm{Alt}\varepsilon^k_* \mathbb{C}_{D^k(F)}\right)$$

on the vanishing cycles of the alternating sheaves. As we shall see now, this gives the vanishing cycles of the alternating sheaves the structure of a $\mathbb{C}[s, s^{-1}]$-module where s is a formal variable standing for the monodromy automorphism h.

To relate monodromy to our usual spectral sequence, it is useful to introduce a point of view about eigenvalues and Jordan blocks slightly different from that of the introduction: Consider the monodromy of a fibration $E \xrightarrow{\pi} D^*$ over the punctured disk with fiber F, and assume that the fibration was obtained as a small representative of an analytic germ $(E, 0) \to (D, 0)$, as in [17]. Consider the one-variable Laurent polynomial ring $R := \mathbb{C}[s, s^{-1}]$. The monodromy action gives each cohomology $H^i(F)$ an R-module structure, by declaring that multiplication by s acts on $H^i(F)$by the monodromy automorphism $h^i : H^i(F) \to H^i(F)$. It is well known that F is a homotopy equivalent to a finite CW-complex, and that each monodromy automorphism h^i is annihilated by some nontrivial polynomial. As a consequence, each $H^i(F)$ is a finitely generated torsion R-module. Note that R is a principal ideal domain so that according to the structure theorem, every finitely generated torsion R-module A can be uniquely decomposed as

$$A \cong \oplus_{j=1}^m R/(s - \lambda_j)^{a_j},$$

where λ_j are nonzero complex numbers and a_j are positive integers. It turns out that the set of eigenvalues of h^i can be recovered as

$$\mathrm{Supp}(A) := \{\lambda_j\}_{1 \le j \le m},$$

and the maximal size of the Jordan blocks is

$$J(A) := \max_{1 \le j \le m} \{a_j\}.$$

We list two simple facts related to finite generated torsion R-modules, the proofs of which we leave to the reader.

Lemma 7.1. *Let $A \xrightarrow{u} B \xrightarrow{v} C$ be a complex of finitely generated torsion R-modules. Then,* $\mathrm{Ker}v/\mathrm{Im}u$ *is also a finitely generated torsion R-module and we have that*

(a) $\mathrm{Supp}(\mathrm{Ker}v/\mathrm{Im}u) \subset \mathrm{Supp}(B)$,
(b) $J(\mathrm{Ker}v/\mathrm{Im}u) \le J(B)$.

Lemma 7.2. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated torsion R-modules. Then, we have that*

(a) $\mathrm{Supp}(B) = \mathrm{Supp}(A) \cup \mathrm{Supp}(C)$,
(b) $\max\{J(A), J(C)\} \le J(B) \le J(A) + J(C)$.

In the case where the short exact sequence splits as a direct sum of R-modules, the first inequality in (b) becomes an equality, that is, $J(B) = \max\{J(A), J(B)\}$.

Proof of Theorem 2.11. The explanation in the beginning of this section shows that the spectral sequence

$$E_1^{i,j} = \mathcal{H}^i\left(\phi_\pi\left(\mathrm{Alt}\varepsilon_* \mathbb{C}_{D^{j+1}(F)}\right)\right)_0 \Rightarrow \tilde{H}^{i+j}(Y_\delta)$$

can be studied in the category of complexes of R-modules, where every entry in the first page E_1 is a torsion R-module. Now let

$$\pi \circ \varepsilon : D^{k+1}(F) \to T$$

be the projection of the multiple point spaces to the parameter space T. Then, the multiple point spaces $D^k(f_\delta)$ of the disentanglement are the Milnor fibers of these fibrations. If either (a) f is dimensionally correct or (b) f is dimensionally correct in nonnegative dimensions and f_δ is a stable perturbation, then by Proposition 5.13 all $D^k(f_\delta)$ are either empty or have dimension

$$\dim D^k(f_\delta) = nk - p(k-1).$$

Observe that only the nonempty $D^k(f_\delta)$ contribute to the spectral sequence and for each nonempty $D^k(f_\delta)$ the total space $D^k(F)$has dimension $\dim D^k(F) = nk - p(k-1) + 1$. We may apply the classical monodromy theorem, Theorem 1.1, to these fibrations: The eigenvalues of the monodromy operator are roots of unity and the size of the Jordan blocks of h^i is bounded from above by $i+1$.

Note that for both cases (2.11) and (2.11), the assumptions imply that the spectral sequence degenerates at the E_1 page and each antidiagonal $\bigoplus_{i+j=k} E_1^{i,j}$ contains at most one nonzero entry. ($D^3(f_\delta) = \emptyset$ implies that$D^k(f_\delta) = \emptyset$ for any $k \geq 3$.) Then the claims follow. □

Remark 7.3. In cases where Theorem 2.11 cannot be applied, one can still use Lemmata 7.1 and 7.2 and the spectral sequence to get bounds on the size of the Jordan blocks in the distinct cohomology groups of the disentanglement. Even though the isomorphism $H^l(Y_\delta) \cong \bigoplus_i E_\infty^{i,l-i}$ (see Remark 5.15) only holds as complex vector spaces, not as $\mathbb{C}[s, s^{-1}]$-modules, the spectral sequence gives $H^l(Y_\delta)$ as an iterated extension of the $E_\infty^{i,l-i}$, which is enough for our purposes.

Remark 7.4. We saw earlier in Remark 5.17 how the study of Milnor fibers of hypersurfaces $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ was equivalent to the study of disentanglements of bi-germs $f : (\mathbb{C}^n, \{p, q\}) \to (\mathbb{C}^{n+1}, 0)$. The above considerations show that this also applies to the monodromy: Since all the ingredients involved in the spectral sequence admit an R-module structure, the isomorphism $\tilde{H}^i(M) \cong \tilde{H}^{i+1}(Y_\delta)$ is an isomorphism of R-modules and therefore the monodromies on both sides are compatible by the considerations in Remark 1.8.

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ENDNOTES

[1] The pairs (n, p) of dimensions where every map-germ $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ admits a stable perturbation are called "Mather's nice dimensions." These comprise all pairs $(n, n+1)$ with $n \leq 14$, cf. [26, Section 5.2.2].

[2] Strongly perverse sheaves have also already been defined in [29], but under a different name and with different notation.

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