FABIAN CHLUMSKY-HARTTMANN

ROBUST MULTI-OBJECTIVE OPTIMIZATION: ANALYSIS AND ALGORITHMIC APPROACHES

ROBUST MULTI-OBJECTIVE OPTIMIZATION: ANALYSIS AND ALGORITHMIC APPROACHES

FABIAN CHLUMSKY-HARTTMANN

Vom Fachbereich Mathematik
der Rheinland-Pfälzischen Technischen Universität Kaiserslautern-Landau
zur Verleihung des akademischen Grades

Doktor der Naturwissenschaften
(*Doctor rerum naturalium, Dr. rer. nat.*)

genehmigte Dissertation

Datum der Disputation: 9. Oktober 2024

Betreuerin:	Prof. Dr. Anita Schöbel
Zweitgutachterin:	Prof. Dr. Gabriele Eichfelder
Promotionskommission:	Prof. Dr. Bernd Simeon, Prof. Dr. Anita Schöbel, Prof. Dr. Sven Krumke

DE-386

ABSTRACT

Many real-world optimization problems not only involve multiple conflicting objective functions, but also a degree of uncertainty in these functions. This thesis considers problems where the outcome of any chosen solution from the decision space is not known precisely. For example, this may be due to measurement errors or unknown future developments.

In order to find robust solutions to uncertain multi-objective optimization problems, several approaches are developed:

First, the problem can primarily be seen as a *robust* optimization problem. Such problems can be solved either by an iterative optimization-pessimization approach or through reformulation. We also show that these methods can be applied to uncertain *multi-objective* problems. Specifically, we show for four different concepts of multi-objective robustness that optimization-pessimization can be used to determine robust solutions and that lower and upper bounds are produced in the process. Convergence conditions are derived. For a particular type of problem, we also show that reformulation is a viable approach.

Alternatively, the problem can be seen as a multi-objective optimization problem, but with the added difficulty that the aim is to find a robust solution. A generalized version of the dichotomic search method from bi-objective optimization is developed, in which, in every iteration, an uncertain single-objective problem has to be solved. Convergence of this method for polyhedral uncertainty sets is shown. On the way, some other results are derived: The dichotomic search is extended from bi-objective linear problems to bi-objective linear minmax problems.

By applying the abovementioned methods, we receive various algorithms that enable finding point-based minmax robust and regret robust efficient solutions. The numerical properties of these algorithms are compared and discussed.

Throughout the thesis, the concepts of (extreme) supported efficiency and nondominance play an essential role. However, different characterizations of (extreme) supported nondominance exist. The relationship between the different definitions is investigated in this thesis for general problems and special problem classes such as discrete, linear, or bi-objective problems. The notion of (extreme) supported nondominatedness/efficiency is extended to multi-objective robust optimization.

ZUSAMMENFASSUNG

Viele reale Optimierungsprobleme beinhalten nicht nur mehrere miteinander in Konflikt stehende Zielfunktionen, sondern auch ein gewisses Maß an Unsicherheit in diesen Funktionen. In dieser Arbeit werden Probleme betrachtet, bei denen der Zielfunktionswert einer gewählten Lösung aus dem Entscheidungsraum nicht genau bekannt ist. Dies kann zum Beispiel auf Messfehler oder auf unbekannte zukünftige Entwicklungen zurückzuführen sein.

Um robuste Lösungen für unsichere Mehrzieloptimierungsprobleme zu finden, werden mehrere Ansätze entwickelt:

Erstens kann das Problem primär als ein *robustes* Optimierungsproblem betrachtet werden. Solche Probleme können entweder durch einen iterativen Ansatz von abwechselnder Optimierung und Pessimierung oder durch Umformulierung gelöst werden. Wir zeigen, dass diese Methoden auch auf unsichere *multikriterielle* Probleme angewendet werden können. Konkret zeigen wir für vier verschiedene Konzepte der Mehrzielrobustheit, dass Optimierung-Pessimierung verwendet werden kann, um robuste Lösungen zu bestimmen und dass dabei untere und obere Schranken erzeugt werden. Konvergenzbedingungen werden abgeleitet. Für einen speziellen Problemtyp wird gezeigt, dass auch Reformulierung einen gangbaren Weg darstellt.

Alternativ kann das Problem als ein multikriterielles Optimierungsproblem betrachtet werden, allerdings mit der zusätzlichen Schwierigkeit, dass eine robuste Lösung gefunden werden soll. Es wird eine verallgemeinerte Version der dichotomischen Suche aus der bikriteriellen Optimierung entwickelt, bei der in jeder Iteration ein unsicheres einkriterielles Problem gelöst werden muss. Die Konvergenz dieser Methode für polyedrische Unsicherheitsmengen wird gezeigt. Auf dem Weg dorthin werden einige weitere Ergebnisse abgeleitet: Die dichotomische Suche wird von linearen bikriteriellen Problemen auf lineare bikriterielle Minmax-Probleme erweitert.

Durch Anwendung der oben beschriebenen Methoden erhalten wir verschiedene Algorithmen, die es ermöglichen, point-based minmax-robuste und regret-robuste effiziente Lösungen zu finden. Die numerischen Eigenschaften dieser Algorithmen werden verglichen und diskutiert.

In der gesamten Arbeit spielen die Konzepte der (extremen) supported efficiency eine wichtige Rolle. In der Literatur existieren jedoch unterschiedliche Charakterisierungen hiervon. Die Beziehung zwischen den verschiedenen Definitionen wird in dieser Arbeit sowohl für allgemeine Probleme als auch für spezielle Problemklassen wie diskrete, lineare oder bikriterielle Probleme untersucht. Der Begriff der (extreme) supported efficiency wird auf die multikriterielle robuste Optimierung erweitert.

ACKNOWLEDGMENTS

First and foremost, I would like to thank Prof. Dr. Anita Schöbel. Despite her busy schedule, she found time for me when I needed her help and supervision. With her support, I was able to complete this thesis.

I would also like to thank Prof. Dr. Gabriele Eichfelder, who agreed to read and evaluate this work.

Vera Grafe and Kathrin Prinz were my proofreaders. They found errors and enabled me to correct them. I am thankful for their help.

My thanks also go to all the members of the optimization research group at the RPTU in Kaiserslautern, with whom I have spent so much time over the last five years. A more pleasant work environment is hardly imaginable and the excellent working atmosphere in the group is responsible for me not losing my nerve and motivation at some point in the last five years. Tim Bergner, Oliver Bachtler, and Tobi Dietz were particularly encouraging to me at various times, perhaps without being aware of it. I would like to thank Philine Schiewe, Reena Urban, Vera Grafe, Moritz Stinzendörfer, Sven Jäger, and Sarah Roth for the great atmosphere in our small sub-working group, in particular during the last two years. I am grateful to the *multi-criteria gang*, Kathrin Prinz and Levin Nemesch in particular, for often short yet frequent exchanges we had over the last year and to Kezang Yuden – *oh no!* – for being such a nice office neighbor.

My parents and my sister Miriam were a great source of encouragement.

Un grazie infinito va a Vera, la più meravigliosa di tutte le giraffe.

CONTENTS

LIST OF FIGURES

LIST OF ALGORITHMS

1 INTRODUCTION

MOTIVATION

Two issues often complicate real-world optimization problems: The presence of multiple objectives and uncertainty.

First, in many cases, a decision maker may not only pursue one goal but has to find the right balance between several–often conflicting–objectives. For example, a driver might expect their navigation system to find the fastest route in order to minimize their travel time. Simultaneously, however, they want to choose a route that minimizes energy consumption or avoids toll costs. These goals rarely go hand in hand. After all, there is an inherent conflict between saving energy and being as fast as possible. For problems like the driver's one, there can be several different solutions, each with a different outcome, that are all good "in some way"–by minimizing travel time, reducing energy consumption, or providing a good mix of all goals. The fact that several different outcomes may all be optimal, yet others that can be improved with respect to one objective without deteriorating the other objectives are clearly undesirable, is the basis of *multi-objective optimization*.

Second, an optimization problem in practice might involve some degree of uncertainty. In reality, neither the travel time nor the energy consumption is known with certainty at the time a route is chosen. A plethora of uncertainties such as traffic and weather conditions impacts thems. Solving an optimization problem by simply ignoring these uncertainties amounts to fair-weather optimization. A better approach is to take all possible realizations of the uncertainty, called scenarios, into account. The expected outcome can be optimized if a probability distribution over all scenarios is known. This is done in *stochastic optimization*. If a solution can be re-planned after traffic or weather conditions are observed, the problem can be solved employing *online optimization*. However, without a known probability distribution and without the possibility of changing a chosen solution, the best approach is to find solutions that are at least acceptable under all scenarios. This is done in *robust optimization*.

In order to deal with problems that are both uncertain and multi-objective, *multi-objective robust optimization* has been studied for over ten years, leading to various models and theoretical results. However, research into methods of actually solving such problems is still in its initial stages. This thesis proposes algorithmic methods to compute robust, efficient solutions to uncertain multi-objective optimization problems.

CONTRIBUTION AND OUTLINE

The remainder of this thesis is organized as follows. In Chapter 2 multi-objective optimization and some of its most important properties as well as robustness concepts for uncertain optimization problems are presented. With this groundwork being laid, we then introduce and illustrate four concepts for *robust multi-objective optimization*, that are investigated later in this thesis. We end the chapter with a review of relevant literature .

In Chapters 3 to 5 we undertake the search for the first type of robust solutions for multi-objective problems, named point-based minmax robust efficient solutions. We start by focusing on the difficulty that is rooted in the uncertainty of the problem in Chapter 3. We review an optimization-pessimization approach for (single-objective) robust optimization and extend it to multi-objective problems, first with uncertainty only in the objectives, then with uncertainty also in the constraints. For the special case of an uncertain *bi-objective* problem we take the opposite approach in Chapter 4: We extend the well-known dichotomic search algorithm for bi-objective problems to robust problems. In Section 5 we combine dichotomic search and optimization-pessimization and receive two different methods for finding point-based minmax robust efficient solutions for bi-objective problems. For a special case with a non-discrete uncertainty set, we additionally develop a reformulation utilizing duality. Results of computational experiments for all the developed algorithmic methods then allow us to compare the performance of these algorithms.

In Chapter 6 we turn two two other robustness concepts for multi-objective optimization: set-based and hull-based minmax robust efficiency and show how the optimization-pessimization approach can employed to determine such solutions. The applicability of previous results for minmax robust efficiency to regret robust efficiency is discussed in Chapter 7.

Finally, in Chapter 8 the notion of (extreme) supported efficiency for multi-objective optimization problems (without uncertainty) is discussed.

We conclude this thesis in Chapter 9 and give directions for further research.

PUBLICATIONS

Parts of this thesis have already been published. Sections 3.1 and 3.2 of Chapter 3 and Chapters 4 and 5 have been published together with Marie Schmidt and Anita Schöbel as a preprint (see [CHSS23]) and are currently under review at the European Journal of Operational Research.

2 PRELIMINARIES AND LITERATURE REVIEW

2.1 MULTI-OBJECTIVE OPTIMIZATION, BUT DETERMINISTIC

The field of multi-objective optimization (other names are: Pareto optimization and multicriteria optimization) is concerned with mathematical optimization problems that involve more than just one objective function. In its most general form we can write a *multi-objective optimization problem* as

$$\min_{x \in \mathcal{X}} \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_p(x) \end{pmatrix}. \tag{MOP}$$

Here we call $\mathcal{X} \subseteq \mathbb{R}^n$ the *feasible set* and $p \in \mathbb{N}$ the *number of objectives*. Accordingly, the scalar-valued functions $g_i \colon \mathcal{X} \to \mathbb{R}$, $i = 1, 2, \dots, p$ are called *objective functions* (or shorter: *objectives*). At times, the vector-valued function $g := (g_1, g_2, \dots, g_p)^\top \colon \mathcal{X} \to \mathbb{R}^p$ is called *objective* (*function*) as well. A significant part of this thesis (Chapters 4 and 5) focuses on the special case where $p = 2$, such problems are called *bi-objective*. If the objective g is a scalar, i.e., $p = 1$, the problem is called *single-objective*.

In the general case, objective functions might be conflicting, such that there is no solution $x \in \mathcal{X}$ that minimizes all objectives functions g_i, $i = 1, 2, \dots, p$, simultaneously. Instead, in order to find a "good" solution, we have to compare vectors. We use the following vector relations between two vectors $y = (y_1, y_2, \dots, y_p)$ and $y' = (y'_1, y'_2, \dots, y'_p) \in \mathbb{R}^p$:

- $y \leqq y' :\iff y_i \leq y'_i$ for all $i = 1, 2, \dots, p$
- $y \lneq y' :\iff y_i \leq y'_i$ for all $i = 1, 2, \dots, p$ and $y_i < y'_i$ for at least one $i = 1, 2, \dots, p$
- $y < y' :\iff y_i < y'_i$ for all $i = 1, 2, \dots, p$.

All three relations are transitive, yet only "$\leqq$" is reflexive and, thus, defining a partial order on $\mathbb{R}^p$. Note that different authors use different notation. Specifically, "$\leq$" is sometimes used in the sense of "$\lneq$" while at other times it signifies "$\leqq$". For the sake of clarity and to avoid any ambiguity, when comparing vectors we never use $\leq$, but $\leqq$ and $\lneq$ instead.

We use $\mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p \colon y \geqq 0\}$ to denote the non-negative orthant, $\mathbb{R}^p_{\gneq} := \{y \in \mathbb{R}^p \colon y \gneq 0\}$ for the non-negative orthant without zero, and $\mathbb{R}^p_{>} := \{y \in \mathbb{R}^p \colon y > 0\}$ for the (strictly) positive orthant. For the closed and open p-simplexes we use the notations $\Delta^p := \{\lambda \in \mathbb{R}^p_{\geqq} \colon \sum_{i=1}^p \lambda_i = 1\}$ and $\Delta^p_{>} := \{\lambda \in \mathbb{R}^p_{>} \colon \sum_{i=1}^p \lambda_i = 1\}$.

For two vectors $y, y' \in \mathbb{R}^p$ we say that y *dominates* y', if $y \lneq y'$. Similarly, if $y < y'$ we say that y *strictly dominates* y', and if $y \leqq y'$ we say that y *weakly dominates* y'.

Intuitively, it is clear that for a solution $x \in \mathcal{X}$ of a multi-objective optimization problem to be desireable its *outcome* $y := g(x)$ should not be dominated by the outcome $y' := g(x')$ of another solution $x' \in \mathcal{X}$. This property is called *efficiency* (or *Pareto optimality*) and is the most important concept in multi-objective optimization.

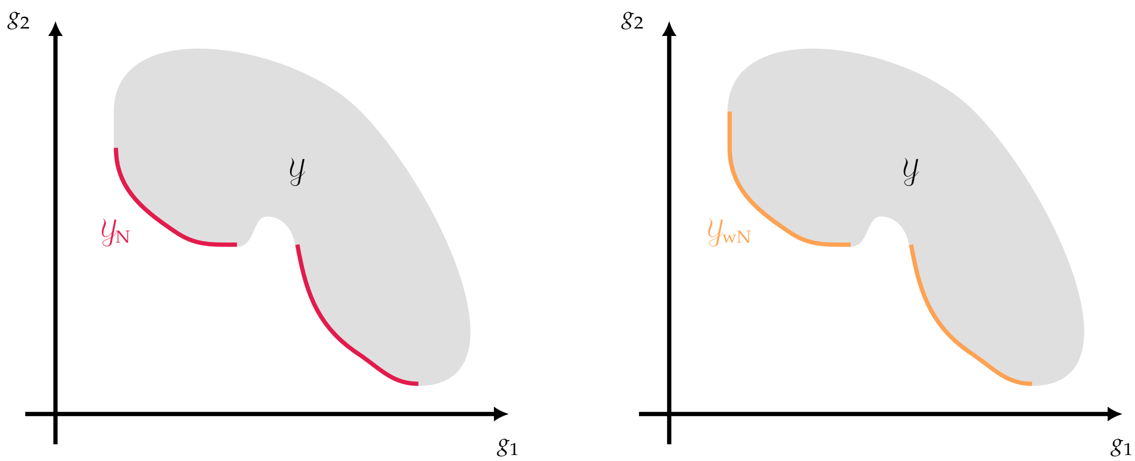
Definition 2.1 (see, e.g., [Ehr05, Definitions 2.1 and 2.24])**.** Given a multi-objective optimization problem (MOP), a solution $x \in \mathcal{X}$ is called

- *efficient* if there is no solution $x' \in \mathcal{X} \setminus \{x\}$ such that $g(x') \lneq g(x)$,
- *strictly efficient* if there is no solution $x' \in \mathcal{X} \setminus \{x\}$ such that $g(x') \leqq g(x)$, and
- *weakly efficient* if there is no solution $x' \in \mathcal{X} \setminus \{x\}$ such that $g(x') < g(x)$.

From the definition it is clear that strict efficiency implies efficiency, and efficiency in turn implies weak efficiency.

We will use $\mathcal{X}_{\mathrm{E}}$ to denote the set of efficient solutions to a given multi-objective problem. Analogously, $\mathcal{X}_{\mathrm{sE}}$ and $\mathcal{X}_{\mathrm{wE}}$ are the set of strictly and weekly efficient solutions, respectively. The image of $\mathcal{X}$ under g will be denoted by $\mathcal{Y}$, i.e., $\mathcal{Y} := g(\mathcal{X}) \subseteq \mathbb{R}^p$.

Each $y \in \mathcal{Y}$ is called *nondominated (in* $\mathcal{Y}$*)*, if there is no $y' \in \mathcal{Y}$ dominating y, and is called *weakly nondominated (in* $\mathcal{Y}$*)*, if there is no $y' \in \mathcal{Y}$ strictly dominating y. Accordingly, (weak) efficiency of a solution x in the feasible space $\mathcal{X}$ corresponds to (weak) non-dominance of its outcome $y = g(x)$ in the image space $\mathcal{Y}$. The set of nondominated outcomes is commonly called the *Pareto frontier* and will be denoted by $\mathcal{Y}_{\mathrm{N}} := g(\mathcal{X}_{\mathrm{E}})$. Similarly, $\mathcal{Y}_{\mathrm{wN}} := g(\mathcal{X}_{\mathrm{wE}})$. Both sets are depicted in Figures 2.1a and 2.1b. Any subset $\mathcal{R} \subseteq \mathcal{X}$ whose image under g is the Pareto frontier $\mathcal{Y}_{\mathrm{N}}$ is called a *representative set*. Clearly, $\mathcal{R} \subseteq \mathcal{X}_{\mathrm{E}}$, but possibly $\mathcal{R} \subsetneq \mathcal{X}_{\mathrm{E}}$.



(a) Nondominated points $\mathcal{Y}_{\mathrm{N}}$ (Pareto frontier) (b) Weakly nondominated points $\mathcal{Y}_{\mathrm{wN}}$

Figure 2.1: Nondominated and weakly nondominated points

We recall some concepts from multi-objective optimization which we will need in this thesis. For a comprehensive and in-depth analysis on the subject of multi-objective optimization, we refer to Ehrgott's book *Multicriteria Optimization* [Ehr05].

WEIGHTED SUM AND SUPPORTED EFFICIENCY. Consider a multi-objective optimization problem (MOP). There are two special types of efficient solutions called *supported efficient* and *extreme supported efficient* solutions. Intuitively, supported efficient solution are those efficient solutions that can be found by solving the *weighted sum scalarization*

$$\min_{x\in\mathcal{X}} \lambda_1 g_1(x) + \lambda_2 g_2(x) + \dots + \lambda_p g_p(x) \tag{MOP(λ)}$$

for "reasonable" weights $\lambda \in \mathbb{R}^p$. However, there exist various definitions of supported efficiency. At this point, we use the following definition.

Definition 2.2. Given a multi-objective optimization problem (MOP), a solution $x \in \mathcal{X}$ is called

- *supported efficient* and its outcome $y = g(x)$ is called *supported nondominated*, if there is no convex combination of nondominated points $y^{(1)}, y^{(2)}, \dots, y^{(n)} \in \mathcal{Y} \setminus \{y\}$, $\lambda \in \Delta^n$, such that $\sum_{i=1}^n \lambda_i y^{(i)} \lneq y$, and
- *extreme supported efficient* and its outcome $y = g(x)$ is called *extreme supported nondominated*, if there is no convex combination of nondominated points $y^{(1)}, y^{(2)}, \dots,$ $y^{(n)} \in \mathcal{Y} \setminus \{y\}$, $\lambda \in \Delta^n$, such that $\sum_{i=1}^n \lambda_i y^{(i)} \leqq y$.

This definition is not widely used in the literature. The first part (concerning *supported efficiency*) is equivalent to a definition by Hamacher, Pedersen and Ruzika (see [HPR07]), the second part (concerning *extreme supported nondominance*) can be found in work of Özpeynirci and Köksalan (see [ÖK10]). The pros and cons of different definitions of (extreme) supported efficiency will be discussed in Chapter 8.

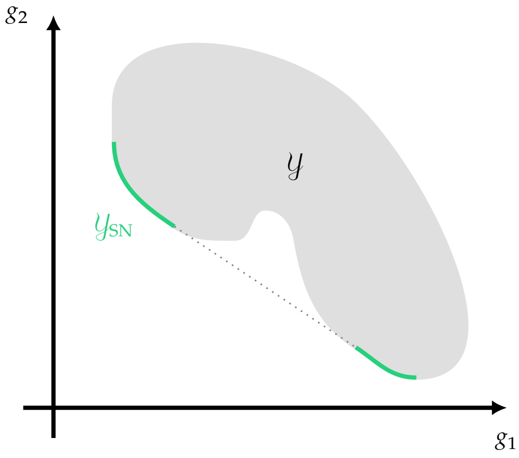
We use $\mathcal{X}_{\text{SN}}$ and $\mathcal{X}_{\text{ESE}}$ to denote the sets of supported efficient and extreme supported efficient solutions and $\mathcal{Y}_{\text{SN}} = g(\mathcal{X}_{\text{SN}})$ and $\mathcal{Y}_{\text{ESN}} = g(\mathcal{X}_{\text{ESE}})$ for the sets of supported nondominated and extreme supported nondominated solutions, respectively. In Figure 2.2a supported nondominated points are illustrated. Supported nondominated points that are not extreme supported nondominated are referred to as *nonextreme supported nondominated* points. The pros and cons of different definitions of (extreme) supported efficiency will be discussed in Chapter 8.

A set whose image under g is equal to the set of extremely supported non-dominated points $\mathcal{Y}_{\text{ESN}}$ is called *representative set for the extreme supported efficient solutions*.

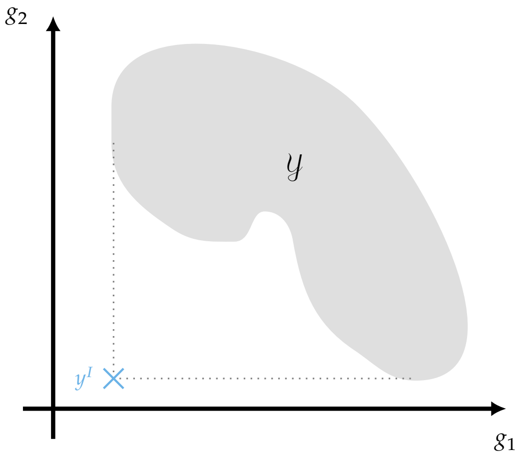
IDEAL POINT AND DOMINATION PROPERTY. In the following we state properties of multi-objective optimization problems (MOP) that are essential to prove some of this thesis' results.

Definition 2.3 (see, e.g., [Ehr05, Definition 2.22]). Consider a multi-objective optimization problem (MOP). If existent, we call

$$y^{\text{I}} := \begin{pmatrix} \min_{x\in\mathcal{X}} g_1(x) \\ \min_{x\in\mathcal{X}} g_2(x) \\ \vdots \\ \min_{x\in\mathcal{X}} g_p(x) \end{pmatrix}$$



(a) Supported nondominated points $\mathcal{Y}_{\text{SN}}$



(b) Ideal point

Figure 2.2: Supported nondominated points and ideal point

its ideal point. Furthermore, we say that that (MOP) has the *ideal point property* if

$$\min_{x \in \mathcal{X}} g_i(x) \text{ exists for all } i = 1, 2, \ldots, p. \tag{2.1}$$

The ideal point is illustrated in Figure 2.2b. Clearly, the outcome set lies in the upper right quadrant of the ideal point, i.e., $\mathcal{Y} \subseteq y^I + \mathbb{R}^p_{\geqq}$. Thus, if the ideal point exists, we can always "shift" $\mathcal{Y}$ by $-y^I$ and assume without loss of generality that $\mathcal{Y}$ lies in the nonnegative orthant $\mathbb{R}^p_{\geqq}$.

Definition 2.4 (Henig [Hen86])**.** A multi-objective optimization problem (MOP) has the *domination property*, if

$$\text{for all } y \in \mathcal{Y} \setminus \mathcal{Y}_{\text{N}} \text{ there exists a point } y' \in \mathcal{Y}_{\text{N}} \text{ with } y' \lneqq y. \tag{2.2}$$

In case a multi-objective optimization problem (MOP) has the domination property, its nondominated set $\mathcal{Y}_{\text{N}}$ is at times called *externally stable* (see, [Ehr05, Definition 2.20]).

The following result is well known.

Lemma 2.5. *Let a multi-objective problem* (MOP) *be given. If $\mathcal{X}$ is finite, or if $\mathcal{X}$ is compact and g is continuous, then both the ideal point property* (2.1) *and the domination property* (2.2) *hold.*

Proof. For (2.1) this is a consequence of Weierstrass' Extreme Value Theorem, for (2.2) we refer to [Hen86] or [Ehr05, Theorem 2.21]. □

2.2 ROBUST OPTIMIZATION, BUT SINGLE-OBJECTIVE

Robust optimization deals with uncertain optimization problems, i.e., problems with some uncertain parameters $\xi \in \mathbb{R}^m$ which depend on measurements, future developments, delays or other uncertainties. Every ξ is called a *scenario* and stems from a known set of all possible scenarios $\mathcal{U} \subseteq \mathbb{R}^m$ which we call *uncertainty set*. A (single-objective) *un-*

certain optimization problem is described by a family of parameterized optimization problems

$$\begin{aligned} &\min h(x,\xi) \\ &\text{s.t. } H_j(x,\xi) \leq 0 \quad \forall j = 1,2,\dots,J \\ &\qquad\quad x \in \mathcal{X} \end{aligned} \tag{P(ξ)}$$

with feasible set $\mathcal{X} \subseteq \mathbb{R}^n$, objective function $h\colon \mathcal{X}\times\mathcal{U} \to \mathbb{R}$ and constraints $H_j\colon \mathcal{X}\times\mathcal{U} \to \mathbb{R}$, $j = 1,2,\dots,J$. We can then write $\mathrm{P}(\mathcal{U}) := \{\mathrm{P}(\xi)\colon \xi \in \mathcal{U}\}$ or

$$\left\{ \begin{array}{ll} \min h(x,\xi) & \\ \text{s.t. } H_j(x,\xi) & \leq 0 \quad \forall j = 1,2,\dots,J \\ \qquad\quad x & \in \mathcal{X} \end{array} \right\}_{\xi\in\mathcal{U}} . \tag{P($\mathcal{U}$)}$$

For the sake of shorter notation, we use $\mathcal{X}_\xi$ to refer to the feasible set of $\mathrm{P}(\xi)$, that is,

$$\mathcal{X}_\xi := \{x \in \mathcal{X}\colon H_j(x,\xi) \leq 0 \ \forall j = 1,2,\dots,J\}.$$

If an optimization problem is uncertain, there usually is no solution that is optimal for all scenarios. Instead one aims to find *robust* (*optimal*) solutions which are feasible and reasonably good for all (or most) scenarios.

Example 2.6. Consider an uncertain optimization problem with three solutions $\mathcal{X} = \{x_1, x_2, x_3\}$ that are all feasible under all scenarios $\mathcal{U} = \{\xi_1, \xi_2, \xi_3\}$, i.e., $\mathcal{X}_\xi = \mathcal{X}$ for all $\xi \in \mathcal{U}$. The objective values, however, depend on the scenario and are depicted in Figure 2.3. We can see that for the first and the second scenario solution x_1 is minimal. However, if the third scenario materializes, x_3 is the optimal solution.

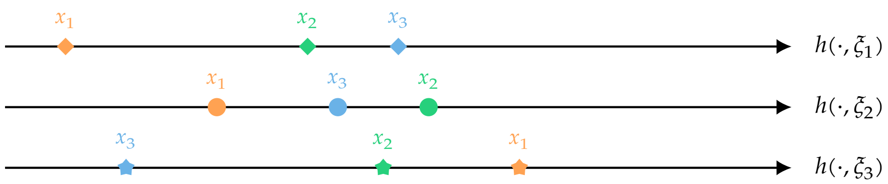


Figure 2.3: An uncertain optimization problem

Many robustness concepts have been defined (see, e.g., Goerigk and Schöbel [GS16] for an overview on different robustness concepts). In the following, we will restate two of the most commonly used concepts, that are relevant for the thesis. For a detailed account on the subject of robust optimization, we refer to the books *Robust Optimization* by Ben-Tal, El Ghaoui and Nemirovski [BTEN09], *Robust discrete optimization and its applications* by Kouvelis and Yu [KY13], and *Robust and Adaptive Optimization* by Bertsimas and Hertog [BH22].

MINMAX ROBUSTNESS (STRICT ROBUSTNESS). A basic, albeit quite conservative robustness concept is *minmax robustness* (sometimes also *strict robustness*). Its idea is to only consider those solutions that are feasible for all scenarios and evaluate each solution in their individual worst-case scenario. This leads to the following definition.

Definition 2.7 (see, e.g., [BTEN09])**.** Given an uncertain optimization problem ($\mathrm{P}(\mathcal{U})$) we call the problem

$$\begin{aligned} \min \sup_{\xi \in \mathcal{U}} & h(x, \xi) \\ \text{s.t. } & H_j(x, \xi) \leq 0 \quad \forall j = 1, 2, \ldots, J \ \forall \xi \in \mathcal{U} \\ & x \in \mathcal{X} \end{aligned} \tag{RC($\mathcal{U}$)}$$

its *robust counterpart*.

A solution $x \in \mathcal{X}$ to the uncertain problem ($\mathrm{P}(\mathcal{U})$) is called *strictly robust feasible* if it is feasible for the robust counterpart ($\mathrm{RC}(\mathcal{U})$), and is called *minmax robust* (*optimal*) or *strictly robust* (*optimal*), if it is an optimal solution to the robust counterpart ($\mathrm{RC}(\mathcal{U})$).

In general, uncertainty can occur in both the objective and the constraints. However, since by introducing a bottleneck variable y the uncertain problem ($\mathrm{P}(\mathcal{U})$) can be reformulated as

$$\left\{ \begin{aligned} \min y & \\ \text{s.t. } h(x, \xi) & \leq y \\ H_j(x, \xi) & \leq 0 \quad \forall j = 1, 2, \ldots, J \\ x & \in \mathcal{X} \\ y & \in \mathbb{R} \end{aligned} \right\}_{\xi \in \mathcal{U}} \tag{$\mathrm{P}^{\mathrm{BN}}(\mathcal{U})$}$$

and the reformulation's robust counterpart

$$\begin{aligned} \min y & \\ \text{s.t. } h(x, \xi) & \leq y \quad \forall \xi \in \mathcal{U} \\ H_j(x, \xi) & \leq 0 \quad \forall j = 1, 2, \ldots, J \ \forall \xi \in \mathcal{U} \\ x & \in \mathcal{X} \\ y & \in \mathbb{R} \end{aligned} \tag{$\mathrm{RC}^{\mathrm{BN}}(\mathcal{U})$}$$

is equivalent to the robust counterpart ($\mathrm{RC}(\mathcal{U})$), it is often assumed without loss of generality that uncertainty only occurs in the constraints (see [BTEN09]). We will see, however, that this simplification is not quite so straightforward in the case of uncertain *multi-objective* problems, where its validity depends on which generalization of minmax robustness is chosen.

In the following we will use

$$\mathcal{X}_{\mathcal{U}} := \bigcap_{\xi \in \mathcal{U}} \mathcal{X}_{\xi} = \{x \in \mathcal{X} \colon H_j(x, \xi) \leq 0 \ \forall j = 1, 2, \ldots, J \ \forall \xi \in \mathcal{U}\}$$

to denote the set of those solutions that are (*strictly*) *robust feasible*, i.e., feasible under all possible scenarios $\xi \in \mathcal{U}$. The robust counterpart (RC($\mathcal{U}$)) can then alternatively be written as

$$\min \left\{ \sup_{\xi \in \mathcal{U}} f(x, \xi) : x \in \mathcal{X}_{\mathcal{U}} \right\}. \qquad \text{(short formulation of RC}(\mathcal{U}))$$

If uncertainty occurs only in the objective, whilst the constraints remain deterministic, we have $\mathcal{X}_{\mathcal{U}} = \mathcal{X}$.

Example 2.6 (Continued). The fourth axis in Figure 2.4 shows the worst-case objective value for all solutions in $\mathcal{X}_{\mathcal{U}} = \mathcal{X} = \{x_1, x_2, x_3\}$. As $\sup_{\xi \in \mathcal{U}} f(x_3)$ is minimal, x_3 is minmax robust optimal.

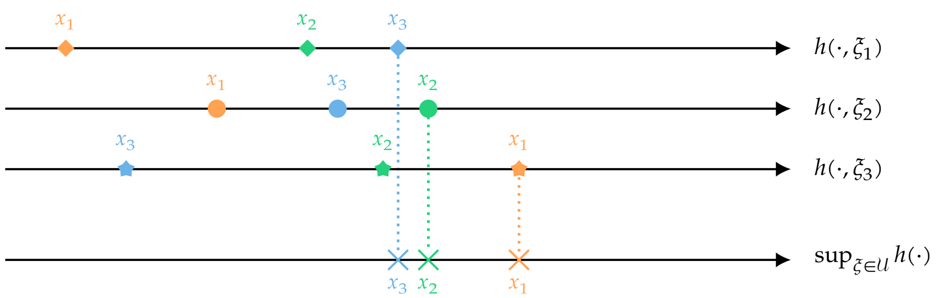


Figure 2.4: Minmax robustness

REGRET ROBUSTNESS. An alternative robustness concept, which is often used in portfolio optimization (see, e.g., [Sim+18]), is called *regret robustness*. The idea is to measure a decision maker's *regret* as the gap between the objective value of their chosen solution and what would have been the optimal solution for the scenario that materializes.

Definition 2.8. A solution $x \in \mathcal{X}$ to a given uncertain problem (P($\mathcal{U}$)) is called *regret robust optimal*, if it is optimal to the *regret robust counterpart*

$$\begin{aligned} \min \sup_{\xi \in \mathcal{U}} \; & (h(x, \xi) - \min_{x^* \in \mathcal{X}} h(x^*, \xi)) \\ \text{s.t. } & H_j(x, \xi) \leq 0 \quad \forall j = 1, 2, \ldots, J, \forall \xi \in \mathcal{U} \\ & x \in \mathcal{X}. \end{aligned} \qquad \text{(RRC}(\mathcal{U}))$$

Here,

$$r(x, \xi) := h(x, \xi) - \min_{x^* \in \mathcal{X}} h(x^*, \xi) \tag{2.3}$$

is the function measuring the decision maker's regret if solution x is chosen and scenario ξ materializes. It is called *regret function*. As in the case of strict robustness, we can write the *robust feasible set* as $\mathcal{X}_{\mathcal{U}}$. After rearranging the objective we get

$$\min \left\{ \sup_{(\xi, x^*) \in \mathcal{U} \times \mathcal{X}} (h(x, \xi) - h(x^*, \xi)) : x \in \mathcal{X}_{\mathcal{U}} \right\}. \qquad \text{(short formulation of RRC}(\mathcal{U}))$$

In this alternative formulation, it is more apparent that the regret robust counterpart still has the structure of a minmax problem – just with a more complicated objective.

Example 2.6 (Continued). For each solution in $\mathcal{X} = \{x_1, x_2, x_3\}$ its worst-case regret $\sup_{\xi \in \mathcal{U}} r(x, \xi)$ is depicted by the length of the arrows in Figure 2.5. Since the worst-case regret is minimal for x_2, solution x_2 is regret robust optimal.

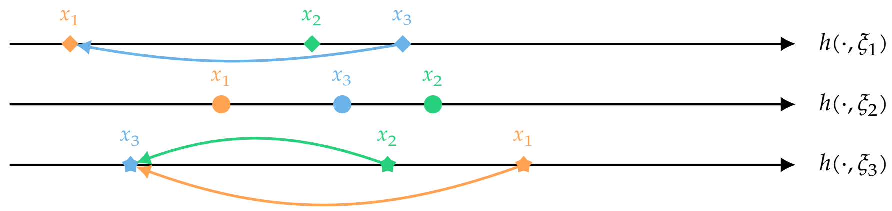


Figure 2.5: Regret robustness

All observation from Example 2.6 are summarized in Table 2.1.

A solution x is...	...if it minimizes	x_1	x_2	x_3
optimal w.r.t. scenario ξ_1	$h(\cdot, \xi_1)$	×		
optimal w.r.t. scenario ξ_2	$h(\cdot, \xi_2)$	×		
optimal w.r.t. scenario ξ_3	$h(\cdot, \xi_3)$			×
minmax robust optimal	$\sup_{\xi \in \mathcal{U}} h(\cdot, \xi)$			×
regret robust optimal	$\sup_{\xi \in \mathcal{U}} (h(\cdot, \xi) - \min_{x^* \in \mathcal{X}} h(x^*, \xi))$		×	

Table 2.1: Observations from Example 2.6

2.3 MULTI-OBJECTIVE ROBUST OPTIMIZATION

Real-world optimization problems often have multiple objective functions *and* involve uncertainty. We consider multi-objective optimization problems which depend on a scenario $\xi \in \mathcal{U} \subseteq \mathbb{R}^m$. A family of such problems

$$\left\{ \min \left\{ \begin{pmatrix} f_1(x, \xi) \\ f_2(x, \xi) \\ \vdots \\ f_p(x, \xi) \end{pmatrix} : \begin{array}{ll} F_j(x, \xi) \leq 0 & \forall j = 1, 2, \ldots, J \\ x \in \mathcal{X} & \end{array} \right\} \right\}_{\xi \in \mathcal{U}} \tag{MOP($\mathcal{U}$)}$$

with feasible set $\mathcal{X} \subseteq \mathbb{R}^n$, uncertainty set $\mathcal{U} \subseteq \mathbb{R}^m$, objectives $f_i \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}, i = 1, 2, \ldots, p$, and constraints $F_j \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}, j = 1, 2, \ldots, J$, is called an *uncertain multi-objective optimization problem*. As for single-objective problems, we write

$$\mathcal{X}_{\mathcal{U}} =: \bigcap_{\xi \in \mathcal{U}} \mathcal{X}_\xi = \{x \in \mathcal{X} \colon \ : F_j(x, \xi) \leq 0 \ \forall j = 1, 2, \ldots, J \ \forall \xi \in \mathcal{U}\}$$

to denote the (*strictly*) *robust feasible* set.

Example 2.9. In Figure 2.6 an example of an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)) is shown. For three possible solutions $\mathcal{X} := \{x_1, x_2, x_3\}$ (shown in different colors), that are all robust feasible, i.e., $\mathcal{X}_{\mathcal{U}} = \mathcal{X}$, their outcomes are shown for four different scenarios $\mathcal{U} := \{\xi_1, \xi_2, \xi_3, \xi_4\}$ (depicted using different shapes). Let us consider the deterministic problems (P(ξ)), $\xi \in \mathcal{U}$, individually: If the first scenario ξ_1 materializes, solutions x_2 and x_3 are efficient, whereas x_1 is not efficient, since $f(x_1, \xi_1)$ is dominated by $f(x_2, \xi_1)$. In the second scenario ξ_2 solutions x_1 and x_2 are efficient, and in the third scenario ξ_3 solutions x_1 and x_3 are efficient. Finally, in the fourth scenario ξ_4 only x_1 is efficient.

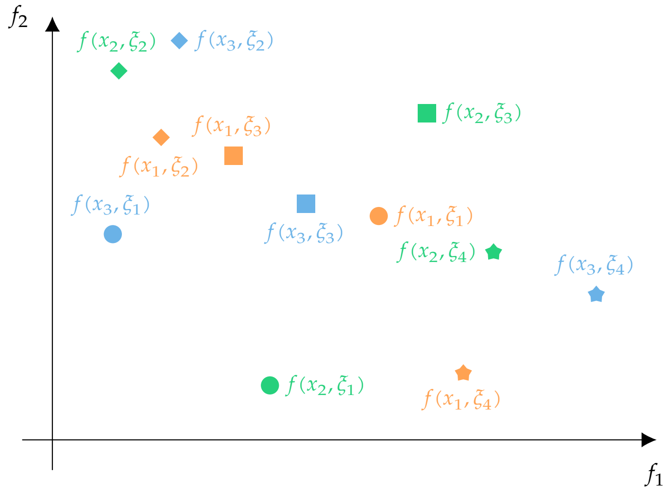


Figure 2.6: An uncertain multi-objective optimization problem

We are interested in finding efficient solutions to the uncertain multi-objective optimization problem which are robust. Since $\mathcal{U}$ is varied within some of the proposed algorithms, we refer to the specific set $\mathcal{U}$ in the notation MOP($\mathcal{U}$).

Several ways to generalize minmax robustness to multi-objective uncertain problems have been proposed (see [IS16; WD16] for surveys, [Eic21, Section 8] and the references therein for a more recent introduction). The concepts relevant to this thesis are introduced in the following.

POINT-BASED MINMAX ROBUST EFFICIENCY. One way to generalize the concept of minmax robustness to multi-objective problems is to consider the worst-case scenario for each objective individually. The resulting problem we call the *multi-objective robust counterpart*

$$\min_{x \in \mathcal{X}_{\mathcal{U}}} \begin{pmatrix} \sup_{\xi \in \mathcal{U}} f_1(x, \xi) \\ \sup_{\xi \in \mathcal{U}} f_2(x, \xi) \\ \vdots \\ \sup_{\xi \in \mathcal{U}} f_p(x, \xi) \end{pmatrix}. \tag{MORC($\mathcal{U}$)}$$

To improve readability, we define

$$f_i^{\mathcal{U}}(x) := \sup_{\xi \in \mathcal{U}} f_i(x, \xi) \tag{2.4}$$

for all $i = 1, 2, \dots, p$ and set

$$f^{\mathcal{U}}(x) := \left(f_1^{\mathcal{U}}(x), f_2^{\mathcal{U}}(x), \dots, f_p^{\mathcal{U}}(x)\right)^\top \tag{2.5}$$

as the vector containing all p objective functions. Note that

$$f^{\mathcal{U}'}(x) \leqq f^{\mathcal{U}}(x) \text{ for } \mathcal{U}' \subseteq \mathcal{U}. \tag{2.6}$$

The multi-objective robust counterpart (MORC($\mathcal{U}$)) can hence be interpreted as a (deterministic) multi-objective problem (MOP) with $\mathcal{X}_{\mathcal{U}} := \bigcap_{\xi \in \mathcal{U}} \mathcal{X}_\xi$ as its feasible set and $g := f^{\mathcal{U}}$ as objective function. This point of view is later taken in Chapters 4 and 5, specifically, in Algorithms 5.2 and 5.3.

The multi-objective robust counterpart leads to the definition of *point-based minmax robust efficiency*.

Definition 2.10 (Kuroiwa and Lee [KL12]; Kuhn, Raith, Schmidt and Schöbel [Kuh+16])**.**

Given an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)), a solution $x \in \mathcal{X}_{\mathcal{U}}$ is called

- *point-based minmax robust efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f^{\mathcal{U}}(x') \lneq f^{\mathcal{U}}(x)$,
- *point-based minmax robust strictly efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f^{\mathcal{U}}(x') \leqq f^{\mathcal{U}}(x)$, and
- *point-based minmax robust weakly efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f^{\mathcal{U}}(x') < f^{\mathcal{U}}(x)$.

Example 2.9 (Continued). Figure 2.7 shows the objective-wise worst-case outcomes $f^{\mathcal{U}}(x)$ for all solutions $x \in \mathcal{X}_{\mathcal{U}}$. One can see that $f^{\mathcal{U}}(x_1)$ dominates both $f^{\mathcal{U}}(x_2)$ and $f^{\mathcal{U}}(x_3)$. Thus, x_1 is the only solution that is point-based minmax robust efficient.

We observe that finding point-based minmax robust (weakly/strictly) efficient solutions to an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)) is tantamount to determining (weakly/strictly) efficient solutions to a deterministic multi-objective optimization problem with a supremum in each objective: the multi-objective robust counterpart (MORC($\mathcal{U}$)).

Point-based minmax robust efficiency will be central in Chapters 3 to 5.

SET-BASED MINMAX ROBUST EFFICIENCY. A different approach to evaluate a robust feasible solution $x \in \mathcal{X}_{\mathcal{U}}$ is to consider the set of possible outcomes

$$f_{\mathcal{U}}(x) := \{f(x, \xi) : \xi \in \mathcal{U}\} \subseteq \mathbb{R}^p. \tag{2.7}$$

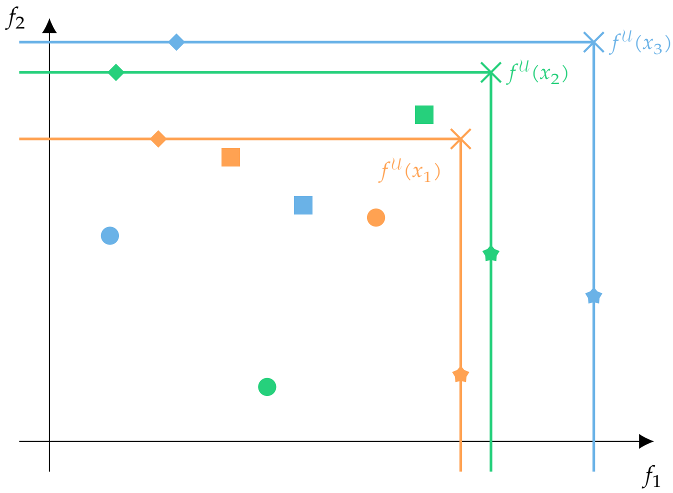


Figure 2.7: Point-based minmax robust efficiency

Analogous to (2.6) we observe

$$f_{\mathcal{U}'}(x) \subseteq f_{\mathcal{U}}(x) \text{ for } \mathcal{U}' \subseteq \mathcal{U}. \tag{2.8}$$

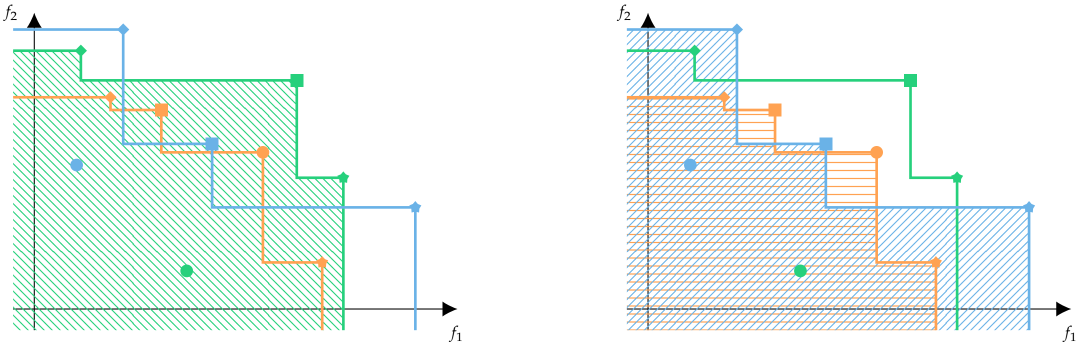
Considering the set $f_{\mathcal{U}}(x)$ instead of the point $f^{\mathcal{U}}(x)$ leads to the definition of set-based minmax robust efficiency.

Definition 2.11 (Ehrgott, Ide and Schöbel [EIS14, Definition 3.1])**.** Given an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)), a solution $x \in \mathcal{X}_{\mathcal{U}}$ is called

- *set-based minmax robust efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(x) - \mathbb{R}^p_{\geq}$,
- *set-based minmax robust strictly efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(x) - \mathbb{R}^p_{\geqq}$, and
- *set-based minmax robust weakly efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(x) - \mathbb{R}^p_{>}$.

Note that set-based minmax robust efficiency has originally been introduced for problems with uncertainty limited to the objectives. However, the generalization made above is straightforward and consistent with both the definition of minmax robust optimality in the single-objective case (Definition 2.7) and the definition of point-based minmax robust efficiency (Definition 2.10).

Example 2.9 (Continued). Figure 2.8 shows the upper boundary of the area $f_{\mathcal{U}}(x) - \mathbb{R}_{\geqq}$ for all solutions x in $\mathcal{X} = \{x_1, x_2, x_3\}$. In Figure 2.8a one can see that all elements of $f_{\mathcal{U}}(x_1)$ lie in the green area, that is, $f_{\mathcal{U}}(x_2) - \mathbb{R}^p_{\geq}$. Therefore, x_2 is not set-based minmax robust efficient, whereas x_1 and x_3 are set-based minmax robust efficient since neither the orange, that is, $f_{\mathcal{U}}(x_1) - \mathbb{R}^p_{\geq}$, nor the blue area, that is, $f_{\mathcal{U}}(x_3) - \mathbb{R}^p_{\geq}$, in Figure 2.8b contain all possible outcomes of another solution.



(a) Solution x_2 is not set-based minmax robust efficient

(b) Solutions x_1 and x_3 are set-based minmax robust efficient

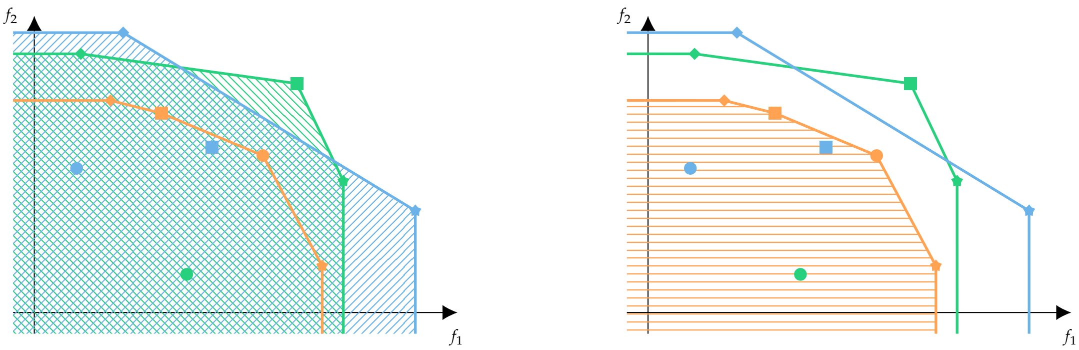
Figure 2.8: Set-based minmax robust efficiency

HULL-BASED MINMAX ROBUST EFFICIENCY. A third approach is to consider the convex hull of the set of possible outcomes $f_{\mathcal{U}}(x)$. This leads to the definition of hull-based minmax robust efficiency.

Definition 2.12 (Bokrantz and Fredriksson [BF17]). Given an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)), a solution $x \in \mathcal{X}_{\mathcal{U}}$ is called

- *hull-based minmax robust efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f_{\mathcal{U}}(x') \subseteq \operatorname{conv}(f_{\mathcal{U}}(x)) - \mathbb{R}^p_{\gneq}$,
- *hull-based minmax robust strictly efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f_{\mathcal{U}}(x') \subseteq \operatorname{conv}(f_{\mathcal{U}}(x)) - \mathbb{R}^p_{\geqq}$, and
- *hull-based minmax robust weakly efficient* if there is no solution $x' \in \mathcal{X}_{\mathcal{U}} \setminus \{x\}$ such that $f_{\mathcal{U}}(x') \subseteq \operatorname{conv}(f_{\mathcal{U}}(x)) - \mathbb{R}^p_{>}$.

Example 2.9 (Continued). For all solutions in $\mathcal{X} = \{x_1, x_2, x_3\}$ Figure 2.9 shows the upper boundary of the set $\operatorname{conv}(f_{\mathcal{U}}(x)) - \mathbb{R}^p_{\geqq}$. Observe that $f_{\mathcal{U}}(x_1) \subseteq \operatorname{conv}(f_{\mathcal{U}}(x_2)) - \mathbb{R}^p_{\gneq}$ and $f_{\mathcal{U}}(x_1) \subseteq \operatorname{conv}(f_{\mathcal{U}}(x_3)) - \mathbb{R}^p_{\gneq}$ (see Figure 2.9a). Thus, x_2 and x_3 are not hull-based minmax robust efficient and x_1 is the only solution that is (see Figure 2.9b).



(a) Solutions x_2 and x_3 are not hull-based minmax robust efficient

(b) Solution x_1 is hull-based minmax robust efficient

Figure 2.9: Hull-based minmax robust efficiency

Set-based and hull-based minmax robust efficiency are investigated in Chapter 6.

Since in the special case $p = 1$ point-based/set-based/hull-based minmax robust efficiency all reduce to minmax robust optimality, these concept all constitute possible generalizations of minmax robustness to multi-objective problems.

It is known that for any solution $x \in \mathcal{X}$, point-based minmax robust *strict* efficiency implies hull-based minmax robust *strict* efficiency and hull-based minmax robust efficiency implies set-based minmax robust efficiency (c.f. [BF17, Proposition 5.2]). In general, the reverse is not true. However, in the case of objective-wise uncertainty, i.e., if $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_p$ and $f_i \colon \mathcal{U}_i \times \mathcal{X} \to \mathbb{R}$, $i = 1, 2, \ldots, p$, all three concepts coincide (see [IS16, Lemma 39]).

REGRET ROBUST EFFICIENCY We recall the regret function in the single-objective case:

$$r(x,\xi) := h(x,\xi) - \min_{x^* \in \mathcal{X}} h(x^*,\xi). \tag{2.3 revisited}$$

In a multi-objective setting this can be generalized to

$$r(x,\xi) = \begin{pmatrix} r_1(x,\xi) \\ r_2(x,\xi) \\ \vdots \\ r_p(x,\xi) \end{pmatrix} := \begin{pmatrix} f_1(x,\xi) - \min_{x^* \in \mathcal{X}} f_1(x^*,\xi) \\ f_2(x,\xi) - \min_{x^* \in \mathcal{X}} f_2(x^*,\xi) \\ \vdots \\ f_p(x,\xi) - \min_{x^* \in \mathcal{X}} f_p(x^*,\xi) \end{pmatrix} = f(x,\xi) - y^I(\xi), \tag{2.9}$$

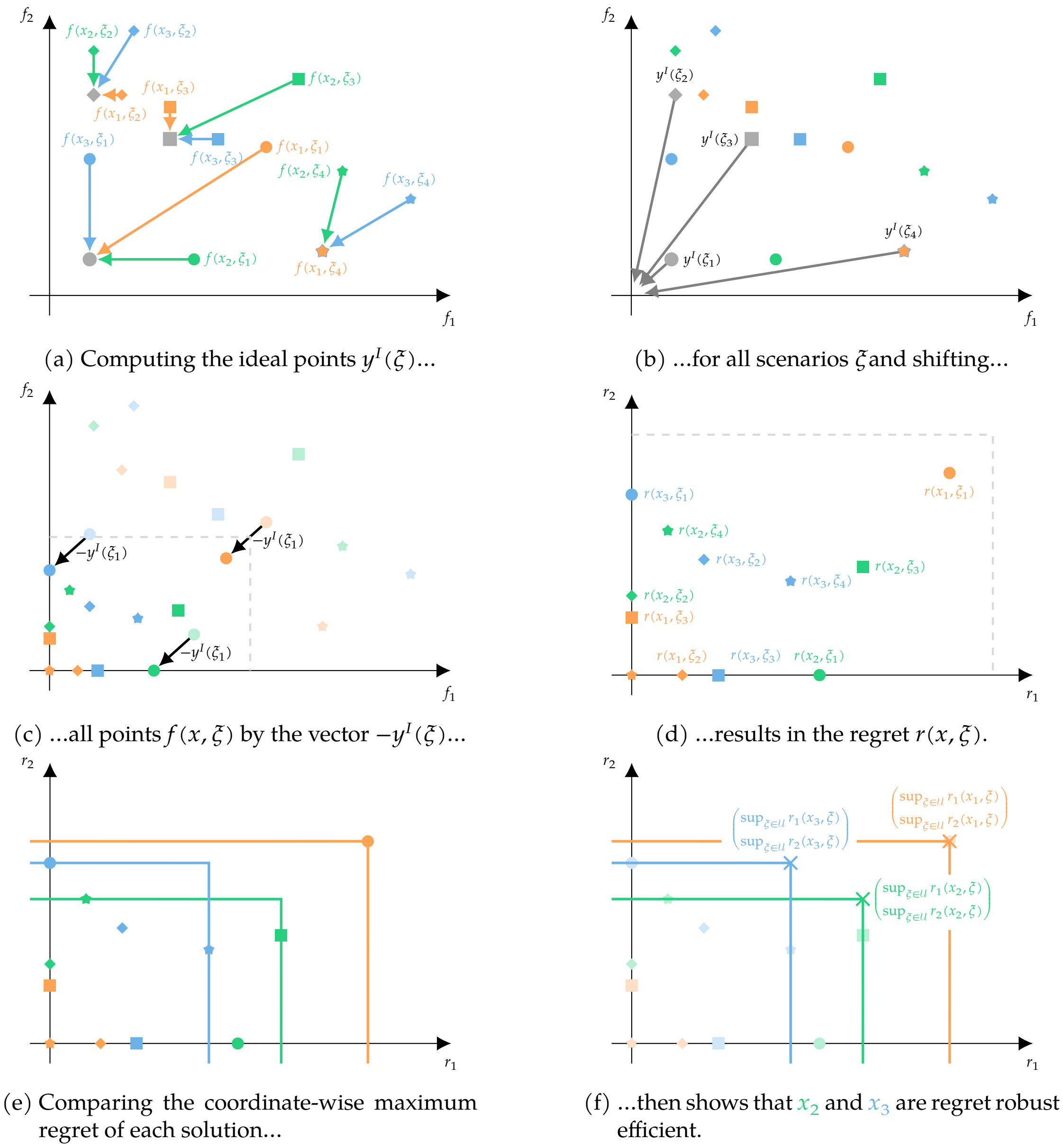
where $y^I(\xi)$ is the ideal point of MOP(ξ). This leads to the *multi-objective regret robust counterpart*

$$\min_{x \in \mathcal{X}_\mathcal{U}} \begin{pmatrix} \sup_{\xi \in \mathcal{U}} \left(f_1(x,\xi) - \min_{x^* \in \mathcal{X}} f_1(x^*,\xi)\right) \\ \sup_{\xi \in \mathcal{U}} \left(f_2(x,\xi) - \min_{x^* \in \mathcal{X}} f_2(x^*,\xi)\right) \\ \vdots \\ \sup_{\xi \in \mathcal{U}} \left(f_p(x,\xi) - \min_{x^* \in \mathcal{X}} f_p(x^*,\xi)\right) \end{pmatrix}. \tag{MORRC($\mathcal{U}$)}$$

Definition 2.13 (Groetzner and Werner [GW22])**.** Given an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)), a solution $x \in \mathcal{X}_\mathcal{U}$ is called (*point-based*) *regret robust* (*strictly*/*weakly*) *efficient* if it is an (strictly/weakly) efficient solution to the multi-objective regret robust counterpart (MORRC($\mathcal{U}$)).

Note that the definition by [GW22] only covers problems where the uncertainty only occurs in the objectives, i.e., $\mathcal{X}_\xi = \mathcal{X}$ for all $\xi \in \mathcal{U}$. Regret robust efficiency is discussed in Chapter 7.

Example 2.9 (Continued)*.* Figure 2.10 visualizes regret robust efficiency. In Figure 2.10a the ideal point $y^I(\xi)$ for each scenario ξ is shown. In Figures 2.10b and 2.10c the points $f(x,\xi)$ are shifted in direction $-y^I(\xi)$. This leads to the multi-objective regret $r(x,\xi) = f(x,\xi) - y^I(\xi)$ which is visualized in Figure 2.10d (enlarged for better visibility). Finally, Figures 2.10e and 2.10f show the objective value of each solution of the regret robust counterpart (MORRC($\mathcal{U}$)). The outcome of solution x_1 is dominated by both x_2 and x_3. The latter two, however, are not dominated. Hence, x_2 and x_3 are regret robust efficient.



(a) Computing the ideal points $y^I(\xi)$...

(b) ...for all scenarios ξ and shifting...

(c) ...all points $f(x, \xi)$ by the vector $-y^I(\xi)$...

(d) ...results in the regret $r(x, \xi)$.

(e) Comparing the coordinate-wise maximum regret of each solution...

(f) ...then shows that x_2 and x_3 are regret robust efficient.

Figure 2.10: Regret robust efficiency

The observations about the problem in Example 2.9 made in Figures 2.6 to 2.10 are summarized in Table 2.2.

A solution x is...	...if $\nexists x' \in \mathcal{X}$ such that	x_1	x_2	x_3
efficient w.r.t. scenario ξ_1	$f(x', \xi_1) \lneq f(x, \xi_1)$		×	×
efficient w.r.t. scenario ξ_2	$f(x', \xi_2) \lneq f(x, \xi_2)$	×	×	
efficient w.r.t. scenario ξ_3	$f(x', \xi_3) \lneq f(x, \xi_3)$	×		×
efficient w.r.t. scenario ξ_4	$f(x', \xi_4) \lneq f(x, \xi_4)$	×		
point-based minmax robust efficient	$f^{\mathcal{U}}(x') \lneq f^{\mathcal{U}}(x)$	×		
set-based minmax robust efficient	$f_{\mathcal{U}}(x') \subseteq f_{\mathcal{U}}(x) - \mathbb{R}^p_{\gneq}$	×		×
hull-based minmax robust efficient	$f_{\mathcal{U}}(x') \subseteq \operatorname{conv}(f_{\mathcal{U}}(x)) - \mathbb{R}^p_{\gneq}$	×		
regret robust efficient	$\sup_{\xi \in \mathcal{U}} r(x', \xi) \lneq \sup_{\xi \in \mathcal{U}} r(x, \xi)$		×	×

Table 2.2: Observations from Example 2.9

2.4 LITERATURE REVIEW

CONCEPTS FOR MULTI-OBJECTIVE ROBUSTNESS. In order to find a good solution for an uncertain multi-objective problem, a notion of what constitutes a *robust efficient* solution has to be formulated first. As we have seen in Section 2.3, this is not trivial, since there is no straightforward way to generalize the concept of Pareto optimality used in multi-objective optimization to uncertain multi-objective problems or to generalize a given notion of robustness to multi-objective problems. In the context of this thesis three different generalizations of minmax robustness to multi-objective problems called *set-based* (see [EIS14]), *hull-based* (see [BF17]), and *point-based minmax efficiency* (see [KL12]), as well as *regret-robust efficiency* (see [RY13; Xid+17; GW22]), are the most important concepts. However, over the years, several concepts for robust multi-objective efficiency have been proposed (see [IS16; WD16] for surveys). Some further approaches based on sets exist, such as *lower set less ordered efficiency*, (*alternative*) *set less ordered efficiency* and *certainly less ordered efficiency* (see [IK14]). The relationship to set-valued optimization has been investigated (see [Ide+14]). A concept related to point-based minmax robust efficiency is called *properly robust efficiency* (see [KL12]). *Lightly robust efficiency* (see [Kuh+16] for the case of only one uncertain objective; see [IS16] for a generalization) has been shown to be a good compromise between more conservative approaches such as minmax robustness and an approach solely based on the nominal scenario (see [SZK21]). The oldest concepts are the notion of *flimsily efficient* (sometimes: *possibly efficient*) and *highly efficient* (sometimes: *necessarily efficient*) solutions (see, e.g., [Bit80; IS96; Kuh+16; ES20]) describing solutions that are efficient for at least one or for all considered scenarios, respectively. Closely related concepts are *local highly* (*robust*) *efficiency* (see [GLP13]) and *relative deviation robustness* (see [NKM13] under the name *robustness with respect to a set of scenarios*). The latter can be seen as a generalization of highly robust weak efficiency.

Other notable concepts include *multi-scenario efficiency*, which builds upon the idea of treating each scenario as an objective (see [BS19], with ideas from [Fad+05; Wie+09]), *multi-criteria adjustable robustness* (see [Hal+24]), *local efficiency w.r.t. the robust counterpart* (see [Chu20] under the name *local robust Pareto solution*), *parameter-robust efficiency* (see [WD16], for the original concepts see [DW09; WOBD13]), *mean efficient regularization robust solutions* (see [DG06]), and *insensitive solutions* (see [GA05]). More concepts, such as *nominal-efficiency under strictly robust feasibility* (see [KDD16]), have been introduced but are inspired by specific applications. An overview of some of these concepts and their relationship is given in a survey by Ide and Schöbel (see [IS16]).

PROPERTIES OF MINMAX ROBUST EFFICIENCY. For point-based minmax robust efficiency many theoretical results exist: Goberna, Jeyakumar, Li and Vicente-Pérez consider specific forms of data uncertainty (box data uncertainty, norm data uncertainty, ellipsoidal uncertainty) and provide deterministic reformulations (see [Gob+15]). Functionwise box uncertainty with a limited sum of deviations has been considered by Hassanzadeh, Nemati and Sun; Fliege and Werner (see [HNS13; FW14]). Antczak, Pandey, Singh and Mishra establish necessary and sufficient conditions for robust ϵ-efficient solutions for uncertain nonsmooth multi-objective optimization problems, but no algorithmic method is provided (see [Ant+20]). Separation results and some characterizations of optimality are developed by Wei, Chen and Li (see [WCL20b; WCL20a]), and the robustness gap for point-based minmax robust efficiency has been introduced by Krüger, Schöbel, Fritzen and Wiecek (see [Krü+23]). The high degree of conservativeness of multi-objective minmax robustness and point-based minmax robust efficient solutions in particular, has lead to research on the *price of multi-objective robustness* (see [SZK21], inspired by [BS04]). Furthermore, for a given solution, its *degree of robustness* has been investigated (see [BA06]). Point-based minmax robust efficiency has been generalized to *efficiency w.r.t. to a general cone* (see [WLC15; Ide+14]) and it has been applied to decision robustness by Eichfelder, Krüger and Schöbel (see [EKS17]).

As a general algorithmic idea, many authors suggest scalarization approaches transferring a robust multi-objective problem to a single-objective robust problem, e.g., [EIS14; Ide+14; Gob+15], but the approaches proposed in those papers are still on an abstract level and only capable of finding some robust efficient solutions while this thesis provides concrete algorithms for determining a representative set of all supported robust efficient solutions. Other algorithmic approaches consider special cases, e.g., cardinality-constrained uncertainty for combinatorial problems (see [Rai+18b]), uncertain multi-objective shortest-path problems [Rai+18a] or cardinality-constrained box uncertainty in the context of portfolio selection problems [HNS14].

For set-based minmax robust efficiency, fewer results exist. Eichfelder and Quintana show that under certain conditions the underlying set optimization problem can be reformulated into a multi-objective optimization problem, which then can be solved by methods from multi-objective optimization (see [EQ24]). Which conditions have to be met for robustification and scalarization to commute is investigated by Caprari, Cerboni Baiardi and Molho (see [CCBM22], building on similar work for point-based minmax robust efficiency in [FW14]).

Some of the reviewed literature will be discussed in more detail at a later point when the context of this thesis warrants it.

3 OPTIMIZATION-PESSIMIZATION FOR MULTI-OBJECTIVE OPTIMIZATION

In this chapter, we take the perspective of a robust optimizer and apply a method known from robust optimization. More precisely, we use a cutting plane approach, called *optimization-pessimization*, which is designed to find minmax robust solutions of uncertain (but single-objective) optimization problems. The approach is reviewed in Section 3.1 and then extended to multi-objective optimization problems. In Section 3.2 multi-objective optimization problems with uncertain objectives are considered. We present a multi-objective version of the optimization-pessimization method enabling us to determine point-based minmax robust (extreme supported) efficient solutions. Subsequently, this is extended to problems with uncertainty in the constraints in Section 3.3.

3.1 DETERMINING ROBUST OPTIMAL SOLUTIONS FOR SINGLE-OBJECTIVE PROBLEMS

This section deals with (single-objective) optimization problems that have an uncertain objective but deterministic constraints, i.e., $\mathcal{X}_\xi = \mathcal{X}$ for all $\xi \in \mathcal{U}$ and, hence, $\mathcal{X}_\mathcal{U} := \cap_{\xi\in\mathcal{U}}\mathcal{X}_\xi = \mathcal{X}$. The problem can then be written as

$$\left\{\min_{x\in\mathcal{X}} h(x,\xi)\right\}_{\xi\in\mathcal{U}}. \qquad (\text{P}(\mathcal{U}) \text{ revisited})$$

Our aim is to determine minmax robust solutions for such problems and, to that end, solve the robust counterpart,

$$\min_{x\in\mathcal{X}} \sup_{\xi\in\mathcal{U}} h(x,\xi). \qquad (\text{RC}(\mathcal{U}) \text{ revisited})$$

We assume that for every fixed $x \in \mathcal{X}$ the function $h(x,\cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$ is continuous and quasi-convex and that $\mathcal{U}$ is compact. Hence, $\sup_{\xi\in\mathcal{U}} h(x,\xi)$ is attained for all $x \in \mathcal{X}$ and from now on we can write $\max_{\xi\in\mathcal{U}} h(x,\xi)$ instead. For a given uncertainty set $\mathcal{U}$ let us denote $z(\mathcal{U}) := \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}} h(x,\xi)$ as optimal objective function value of the robust counterpart ($\text{RC}(\mathcal{U})$).

There exist many approaches for solving $\text{RC}(\mathcal{U})$, which are grouped in [GYD15] into two classes: The first class of algorithms is based on reformulations to avoid the maximum over an (often infinite) set. We follow this approach in Section 5.3. The algorithms of the second class proceed iteratively. They start with a small set of scenarios and add scenarios step by step. These approaches are known under various names such as *cutting set method* (Mutapcic and Boyd [MB09]), *cutting plane method* (Bertsimas, Dunning and Lubin [BDL16]), *scenario relaxation procedure* (Assavapokee, Realff, Ammons and

Hong; Aissi, Bazgan and Vanderpooten [Ass+08; ABV09]), *outer approximation method* (Reemtsen; Bürger, Notarstefano and Allgöwer; Goerigk and Schöbel [Ree94; BNA13; GS16]), (*modified*) *Benders decomposition approach* (Montemanni; Siddiqui, Azarm and Gabriel [Mon06; SAG11]), or *implementor-adversarial framework* (Bienstock [Bie07]).

The earliest mention of this approach known to the author is by Blankenship and Falk and dates back to 1976. Their article on 'Infinitely constrained optimization problems' (see [BF76]) deals with max-min problems outside of the context of robust optimization and is mostly not referenced in the work of robust optimizers mentioned above.

We refer to the approach as *optimization-pessimization*. The idea is to utilize that robust optimization problems are often easier to solve for (very) small uncertainty sets: The routine starts with a reduced set of scenarios $\mathcal{U}' \subseteq \mathcal{U}$ for which a robust solution is determined. For this solution, the routine determines a worst-case scenario out of the full uncertainty set $\mathcal{U}$, which is added to $\mathcal{U}'$. For the new scenario set, a new robust solution is found. This procedure is repeated until the quality of the solution found is good enough, see Figure 3.1 for an illustration.

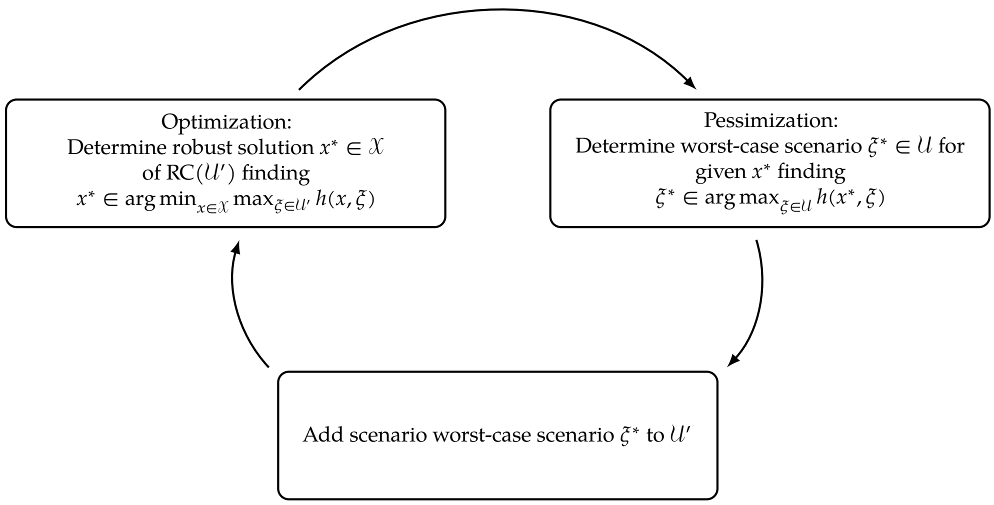


Figure 3.1: Optimization-pessimization for robust single-objective optimization

Formally, the optimization and pessimization problems are defined as follows: For any $\mathcal{U}' \subseteq \mathcal{U}$ the *optimization problem* is defined as

$$z(\mathcal{U}') := \min_{x\in\mathcal{X}} \max_{\xi\in\mathcal{U}'} h(x,\xi). \tag{RC($\mathcal{U}'$)}$$

It is a relaxation of $(\mathrm{RC}(\mathcal{U}))$ and, thus, yields a lower bound for $(\mathrm{RC}(\mathcal{U}))$, i.e.,

$$z(\mathcal{U}') \leq z(\mathcal{U}). \tag{3.1}$$

For a given $x \in \mathcal{X}$, the *pessimization problem*

$$h^{\mathcal{U}}(x) := \max_{\xi\in\mathcal{U}} h(x,\xi) \tag{Pess$^{\mathrm{SO}}(x)$}$$

evaluates x over the complete set of scenarios $\mathcal{U}$ and, thus, provides an upper bound for $z(\mathcal{U})$, i.e.,

$$h^{\mathcal{U}}(x) \geq z(\mathcal{U}). \tag{3.2}$$

Algorithm 3.1 describes how this method can be put to use algorithmically if $\mathcal{U}$ is a polytope or finite.

Algorithm 3.1 Optimization-pessimization for single-objective robust optimization

Require: Robust optimization problem $(\mathrm{P}(\mathcal{U}))$
Require: Finite initial set $\mathcal{U}^{(0)} \subseteq \mathcal{U}$.
Ensure: Either $\mathcal{U}$ finite or $\mathcal{U}$ a polytope and $h(x, \cdot)$ continuous and quasi-convex.

Set $k := 0$.
repeat
  Set $\mathcal{U}^{(k+1)} := \mathcal{U}^{(k)}$.

  Optimization
  Determine $x^k \in \arg\min_{x \in \mathcal{X}} \{\max_{\xi \in \mathcal{U}^k} h(x, \xi)\}$. Set $z(\mathcal{U}^{(k)}) := \max_{\xi \in \mathcal{U}^{(k)}} h(x^k, \xi)$.

  Pessimization
  For given x^k determine solution $\xi^k \in \arg\max_{\mathcal{U}} h(x^k, \xi^k)$. Set $h^{\mathcal{U}}(x^k) := h(x^k, \xi^k)$.
  Add ξ^k to $\mathcal{U}^{(k+1)}$.

  Set $k := k + 1$.
until $h^{\mathcal{U}}(x^{k-1}) = z(\mathcal{U}^{(k-1)})$.
return robust solution x^*.
return set of worst-case scenarios $\mathcal{U}^{\mathrm{FINAL}} := \mathcal{U}^k$.

The routine produces a sequence of sets

$$\mathcal{U}^{(0)} \subseteq \mathcal{U}^{(1)} \subseteq \mathcal{U}^{(2)} \subseteq \cdots \subseteq \mathcal{U}. \tag{3.3}$$

According to (3.1) we receive a sequence of lower bounds

$$z(\mathcal{U}^{(0)}) \leq z(\mathcal{U}^{(1)}) \leq z(\mathcal{U}^{(2)}) \leq \cdots \leq z(\mathcal{U}) \tag{3.4}$$

and a feasible solution x^k in each iteration from which we can derive an upper bound according to (3.2), i.e.,

$$z(\mathcal{U}^{(k)}) \leq z(\mathcal{U}) \leq h^{\mathcal{U}}(x^k). \tag{3.5}$$

We stop when lower and upper bound coincide. Then an optimal solution to the robust counterpart $(\mathrm{RC}(\mathcal{U}))$ and thus a (minmax) robust optimal solution to the underlying uncertain problem $(\mathrm{P}(\mathcal{U}))$ has been found. For more detailed discussions of the method we refer to Bertsimas, Dunning and Lubin; Aissi, Bazgan and Vanderpooten; Pätzold and Schöbel [BDL16; ABV09; PS20]. The finiteness of Algorithm 3.1 for uncertainty sets $\mathcal{U}$ that are polytopes is shown in the following lemma in part (ii).

Lemma 3.1. *Assume that* $(\mathrm{RC}(\mathcal{U}))$ *has an optimal solution and* $(\mathrm{RC}(\mathcal{U}'))$ *has an optimal solution for all finite* $\mathcal{U}' \subseteq \mathcal{U}$.

(*i*) *Let* $\mathcal{U}$ *be finite. Then Algorithm 3.1 returns a minmax robust optimal solution to* $(\mathrm{P}(\mathcal{U}))$ *in at most* $|\mathcal{U}|$ *iterations.*

(*ii*) *Let* $\mathcal{U}$ *be a polytope or finite and let* $\mathrm{ext}(\mathcal{U})$ *be its set of extreme points. Furthermore, let*

$$h(x,\cdot)\colon \mathrm{conv}(\mathcal{U}) \to \mathbb{R},$$

$x \in \mathcal{X}$, *be continuous and quasi-convex. Then Algorithm 3.1 returns a minmax robust optimal solution to* $(\mathrm{P}(\mathcal{U}))$ *in at most* $|ext(\mathcal{U})|$ *iterations if we choose an algorithm for the pessimization problem* $(\mathrm{Pess}^{\mathrm{SO}}(x))$ *which always finds an extreme point of* $\mathcal{U}$.

Proof. Algorithm 3.1 stops if the lower and upper bound for $z(\mathcal{U})$ (see (3.5)) coincide (see line 8 of Algorithm 3.1), i.e., if $h^{\mathcal{U}}(x^k) = z(\mathcal{U}^{(k)})$. We hence have that x^k is an optimal solution to $(\mathrm{MORC}(\mathcal{U}))$. Note that

$$h^{\mathcal{U}}(x^k) = \max_{\xi\in\mathcal{U}} h(x^k,\xi) = \max_{\xi\in\mathcal{U}^{(k)}} h(x^k,\xi) = z(\mathcal{U}^{(k)}), \tag{3.6}$$

if at least one worst-case scenario of $\mathcal{U}$ for x^k is already contained in $\mathcal{U}^{(k)}$. For a finite uncertainty set, in every iteration either a new worst-case scenario is added or (3.6) holds and the procedure stops. The latter happens after at most $|\mathcal{U}|$ iterations which shows (i).

For (ii), consider the pessimization problem $\mathrm{Pess}(x^k)$: here we maximize a continuous function over a compact set $\mathcal{U}$, i.e., a maximum always exists. Since $h(x,\cdot)$ is quasi-convex, a maximum is always attained at an extreme point of $\mathcal{U}$. If we choose an algorithm that returns an extreme point for such optimization problems, we add a new extreme point in each iteration. Since the number of extreme points of $\mathcal{U}$ is finite, the procedure stops when (3.6) holds. As in part (i) this happens after at most $|\,\mathrm{ext}(\mathcal{U})|$ iterations. □

We remark that Algorithm 3.1 also converges for bounded non-polyhedral sets $\mathcal{U}$ under uniform Lipschitz-continuity in x for all fixed values of ξ (see [MB09]).

3.2 DETERMINING POINT-BASED MINMAX ROBUST EFFICIENT SOLUTIONS FOR PROBLEMS WITH UNCERTAIN OBJECTIVES

In this section we aim to find point-based minmax robust efficient solutions to an uncertain multi-objective problem $(\mathrm{MOP}(\mathcal{U}))$ with uncertainty only in the objectives. To this end, we aim to solve a minmax problem with p objective functions, which is the multi-objective robust counterpart $(\mathrm{MORC}(\mathcal{U}))$. Once again, since $\mathcal{X}_\xi = \mathcal{X}$ for all $\xi \in \mathcal{U}$, we have $\mathcal{X}_{\mathcal{U}} := \cap_{\xi\in\mathcal{U}}\mathcal{X}_\xi = \mathcal{X}$ and can, thus, write the multi-objective robust counterpart as follows:

$$\min_{x\in\mathcal{X}} \begin{pmatrix} \sup_{\xi\in\mathcal{U}} f_1(x,\xi) \\ \sup_{\xi\in\mathcal{U}} f_2(x,\xi) \\ \vdots \\ \sup_{\xi\in\mathcal{U}} f_p(x,\xi) \end{pmatrix}. \tag{MORC($\mathcal{U}$) revisited}$$

With the purpose of determining point-based minmax robust efficient solutions to an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)), i.e., efficient solutions to multi-objective robust counterpart (MORC($\mathcal{U}$)), in mind, we develop a generalized version of the optimization-pessimization method presented in Section 3.1.

We will first show in Section 3.2.1 that, as in the single-objective case, solving the (now multi-objective) optimization problem

$$z(\mathcal{U}') := \min_{x\in\mathcal{X}} \begin{pmatrix} \sup_{\xi\in\mathcal{U}'} f_1(x,\xi) \\ \sup_{\xi\in\mathcal{U}'} f_2(x,\xi) \\ \vdots \\ \sup_{\xi\in\mathcal{U}'} f_p(x,\xi) \end{pmatrix}. \tag{MORC($\mathcal{U}'$)}$$

for a smaller uncertainty set $\mathcal{U}' \subseteq \mathcal{U}$ produces a lower bound for the Pareto frontier of MORC($\mathcal{U}$); whereas solving the *pessimization problem*

$$f^{\mathcal{U}}(x) := \begin{pmatrix} \sup_{\xi\in\mathcal{U}} f_1(x,\xi) \\ \sup_{\xi\in\mathcal{U}} f_2(x,\xi) \\ \vdots \\ \sup_{\xi\in\mathcal{U}} f_p(x,\xi) \end{pmatrix}, \tag{Pess$^{\text{pb}}(x)$}$$

which consists of p independent problems (that are all deterministic and single-objective for given $x \in \mathcal{X}$) yields an upper bound for the Pareto frontier of MORC($\mathcal{U}$). Subsequently, we will present an optimization-pessimization method designed to determine *efficient* solutions to the multi-objective robust counterpart (MORC($\mathcal{U}$)) in Section 3.2.2 and an optimization-pessimization method designed to determine *supported efficient* solutions to MORC($\mathcal{U}$) in Section 3.2.3.

3.2.1 *Lower and upper bounds provided by the optimization and the pessimization problem*

We first discuss the optimization and pessimization problems in relation to (MORC($\mathcal{U}$)) which we are interested in solving.

Recall that for single-objective problems (RC($\mathcal{U}$)), solutions to RC($\mathcal{U}'$) and Pess$^{\text{SO}}(x)$ provide lower and upper bounds to RC($\mathcal{U}$) such that for all $x \in \mathcal{X}$ we have

$$z(\mathcal{U}^{(k)}) \leq z(\mathcal{U}) \leq h^{\mathcal{U}}(x^k). \tag{3.5 revisited}$$

In the multi-objective setting, we do not evaluate single solutions, rather we need to evaluate (Pareto) sets. Sets can be compared by set order relations, one of the most commonly used ones is called the *upper set less order* relation.

Definition 3.2 (Kuroiwa [Kur99]). A set $A \subseteq \mathbb{R}^p$ *dominates* a set $B \subseteq \mathbb{R}^p$ with respect to the *upper set less order relation* (we write $A \preceq^{\text{upp}} B$) if $A \subseteq B - \mathbb{R}^p_{\geqq}$.

A related set order relation is the *lower set less order* relation.

Definition 3.3 (Ide and Köbis [IK14]). A set $A \subseteq \mathbb{R}^p$ *dominates* a set $B \subseteq \mathbb{R}^p$ with respect to the *lower set less order relation* (we write $A \preceq^{\text{low}} B$) if $B \subseteq A + \mathbb{R}^p_{\geqq}$.

Put differently,

$$A \preceq^{\text{upp}} B \text{ if for all } a \in A \text{ there exists } b \in B \text{ such that } a \leqq b, \text{ and} \tag{3.7}$$
$$A \preceq^{\text{low}} B \text{ if for all } b \in B \text{ there exists } a \in A \text{ such that } a \leqq b \tag{3.8}$$

In this sense, we can say that the set A is an upper or lower set less order lower bound on the set B.

We now use the *lower set less order* relation to generalize (3.5) showing that for the multi-objective optimization problem MORC$(\mathcal{U})$ we also get a lower and upper bound by solving the optimization problem for a reduced uncertainty set $\mathcal{U}' \subsetneq \mathcal{U}$ and solving the pessimization problem, respectively.

More precisely, let $X^*(\mathcal{U})$ be the set of efficient solutions to the multi-objective robust counterpart (MORC$(\mathcal{U})$). Then $\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U})\}$ describes the Pareto frontier of MORC$(\mathcal{U})$ for which a lower and upper bound exist.

Lemma 3.4. *Let $\mathcal{U}' \subseteq \mathcal{U}$ and let $X^*(\mathcal{U}')$ and $X^*(\mathcal{U})$ denote the sets of efficient solutions to the multi-objective robust counterparts (MORC$(\mathcal{U})$ and MORC$(\mathcal{U}')$, respectively). Assume that MORC$(\mathcal{U}')$, and MORC$(\mathcal{U})$ both satisfy the domination property (2.2). Then the following holds for the lower set less order $\preceq^{\text{low}}$:*

$$\left\{f^{\mathcal{U}'}(x) : x \in X^*(\mathcal{U}')\right\} \preceq^{\text{low}} \left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U})\right\} \preceq^{\text{low}} \left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U}')\right\}. \tag{3.9}$$

Proof. We first show the left hand side of (3.9). To this end, take $x \in X^*(\mathcal{U})$. We want to show that there exists $\tilde{x} \in X^*(\mathcal{U}')$ such that

$$f^{\mathcal{U}'}(\tilde{x}) \leqq f^{\mathcal{U}}(x). \tag{3.10}$$

From $\mathcal{U}' \subseteq \mathcal{U}$ we get that $f^{\mathcal{U}'}(x) \leqq f^{\mathcal{U}}(x)$, see (2.6). Hence, if $x \in X^*(\mathcal{U}')$ we set $\tilde{x} := x$ and are done. Otherwise, $x \notin X^*(\mathcal{U}')$, i.e., x is not an efficient solution to (MORC$(\mathcal{U}')$). Then, due to the domination property, there exists $\tilde{x} \in X^*(\mathcal{U}')$ with $f^{\mathcal{U}'}(\tilde{x}) \leqq f^{\mathcal{U}'}(x) \leqq f^{\mathcal{U}}(x)$ and (3.10) holds.

For the right hand side of (3.9), we take $x \in X^*(\mathcal{U}')$. The goal is to find $\tilde{x} \in X^*(\mathcal{U})$ such that

$$f^{\mathcal{U}}(\tilde{x}) \leqq f^{\mathcal{U}}(x).$$

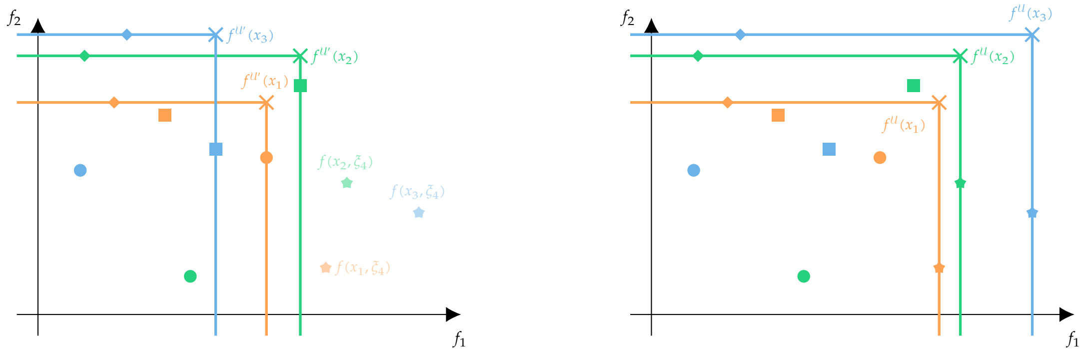
Similar as above, if $x \in X^*(\mathcal{U})$ we set $\tilde{x} := x$ and are done. Otherwise, x is not efficient for MORC$(\mathcal{U})$ and due to the domination property we find $\tilde{x} \in X^*(\mathcal{U})$ with $f^{\mathcal{U}}(\tilde{x}) \leqq f^{\mathcal{U}}(x)$, which finishes the proof. □

We will now show that in general, however, the bounds in (3.9) are *not* bounds for the Pareto front of MORC$(\mathcal{U})$ with respect to the *upper set less order*. To this end, we turn to the following examples.

Example 3.5. Consider the setting of Example 2.9 (illustrated in Figure 2.6) on Page 11 with the reduced uncertainty set $\mathcal{U}' = \{\xi_1, \xi_2, \xi_3\} \subsetneq \mathcal{U}$. Both solutions x_1 and x_3 are efficient for MORC$(\mathcal{U}')$ as can be seen in Figure 3.2a. Thus, $X^*(\mathcal{U}') = \{x_1, x_3\}$. For the full uncertainty set $\mathcal{U}$, however, Figure 3.2b shows that only x_1 is efficient to MORC$(\mathcal{U})$, hence, $X^*(\mathcal{U}) = \{x_1\}$. Since there is no $x \in X^*(\mathcal{U})$ such that $f^{\mathcal{U}}(x) \geqq f^{\mathcal{U}'}(x_3)$ we get

$$\left\{f^{\mathcal{U}'}(x) : x \in X^*(\mathcal{U}')\right\} = \left\{f^{\mathcal{U}'}(x_1), f^{\mathcal{U}'}(x_3)\right\} \not\leq^{\mathrm{upp}} \left\{f^{\mathcal{U}}(x_1)\right\} = \left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U})\right\},$$

which contradicts the left hand side of (3.9) for the upper set less order relation.



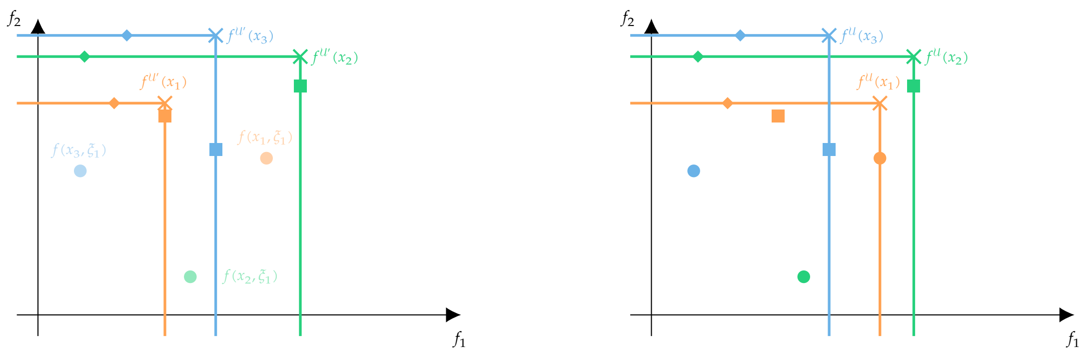
(a) Both x_1 and x_3 are efficient to MORC$(\mathcal{U}')$

(b) Only x_1 is efficient to MORC$(\mathcal{U})$

Figure 3.2: Illustration of Example 3.5

Example 3.6. Now consider Example 2.9 without ξ_4, such that the full uncertainty set is now $\mathcal{U} = \{\xi_1, \xi_2, \xi_3\}$, and the reduced uncertainty set $\mathcal{U}' = \{\xi_2, \xi_3\} \subsetneq \mathcal{U}$. Only x_1 is efficient for MORC$(\mathcal{U}')$ as can be seen in Figure 3.3a. Thus, $X^*(\mathcal{U}') = \{x_1\}$. However, Figure 3.3b shows that both x_1 and x_3 are efficient to MORC$(\mathcal{U})$, hence, $X^*(\mathcal{U}) = \{x_1, x_3\}$. Thus,

$$\left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U})\right\} = \left\{f^{\mathcal{U}}(x_1), f^{\mathcal{U}}(x_3)\right\} \not\leq^{\mathrm{upp}} \left\{f^{\mathcal{U}}(x_1)\right\} = \left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U}')\right\}.$$



(a) Only x_1 is efficient to MORC$(\mathcal{U}')$

(b) Both x_1 and x_3 are efficient to MORC$(\mathcal{U})$

Figure 3.3: Illustration of Example 3.6

Examples 3.5 and 3.6 show that in general the bounds in (3.9) are not *upper set less order* lower and upper bounds. Nor are they bounds with respect to the *set less order* relation (see [You31; EJ11]). Thus, we consider the statement in (3.9) to be the multi-objective analog of (3.5).

3.2.2 *Point-based minmax robust efficient solutions*

In this section we work under the following assumption.

Assumption 1. Let $\mathcal{U}$ is compact. Furthermore, let the functions $f_i(x, \cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$, $i = 1, 2, \dots, p$, be continuous for every fixed $x \in \mathcal{X}$.

This is the same assumption that we made previously in Section 3.1–just for all p objectives. As before, we can now assume that $\sup_{\xi \in \mathcal{U}} f_i(x, \xi)$ is attained for all $i = 1, 2, \dots, p$ and all $x \in \mathcal{X}$. Hence, we can write $f_i^{\mathcal{U}}(x) = \max_{\xi \in \mathcal{U}} f_i(x, \xi)$ instead.

REDUCTION OF THE SCENARIO SET. We examine the conditions under which a reduced uncertainty set $\mathcal{U}' \subseteq \mathcal{U}$ already contains all relevant scenarios, such that the efficient solutions MORC($\mathcal{U}'$) are the same as those of MORC($\mathcal{U}$).

In the single-objective setting we have seen that all relevant scenarios are included in $\mathcal{U}' \subseteq \mathcal{U}$ if for an optimal solution $x^* \in \mathcal{X}$ to the robust counterpart (RC($\mathcal{U}$)) a worst-case scenario is already included in $\mathcal{U}'$ such that (3.6) holds. As a multi-objective analog we formulate the following condition for some $x \in \mathcal{X}$:

The set $\mathcal{U}'$ includes optimal solutions to all p problems in $\mathrm{Pess}^{\mathrm{pb}}(x)$.
(Condition A)

As an immediate consequence of Condition A we observe

$$f^{\mathcal{U}'}(x) = \begin{pmatrix} \max_{\xi \in \mathcal{U}'} f_1(x, \xi) \\ \max_{\xi \in \mathcal{U}'} f_2(x, \xi) \\ \vdots \\ \max_{\xi \in \mathcal{U}'} f_p(x, \xi) \end{pmatrix} \overset{\text{(Condition A)}}{=} \begin{pmatrix} \max_{\xi \in \mathcal{U}} f_1(x, \xi) \\ \max_{\xi \in \mathcal{U}} f_2(x, \xi) \\ \vdots \\ \max_{\xi \in \mathcal{U}} f_p(x, \xi) \end{pmatrix} = f^{\mathcal{U}}(x). \tag{3.11}$$

The following theorem formalizes the above considerations and shows under which conditions the efficient solutions of MORC($\mathcal{U}$) and MORC($\mathcal{U}'$) coincide.

Theorem 3.7. *Let $\mathcal{U}' \subseteq \mathcal{U}$. Consider $x \in \mathcal{X}$. If x satisfies Condition A then under Assumption 1 the following holds:*

x is efficient for MORC($\mathcal{U}'$) $\Rightarrow$ x is efficient for MORC($\mathcal{U}$).

Additionally, if the domination property (2.2) holds for MORC($\mathcal{U}'$) and all *solutions $x \in \mathcal{X}$ that are efficient for MORC($\mathcal{U}'$) satisfy Condition A, then the following holds:*

x is efficient for MORC($\mathcal{U}'$) $\Leftrightarrow$ x is efficient for MORC($\mathcal{U}$) .

Proof. $\underline{\Rightarrow:}$ Let x be efficient for MORC($\mathcal{U}'$) and satisfy Condition A, i.e., $f^{\mathcal{U}'}(x) = f^{\mathcal{U}}(x)$. Assume to the contrary that x is not efficient for MORC($\mathcal{U}$), i.e., there exists $x' \in \mathcal{X}$, such that

$$f^{\mathcal{U}}(x') \lneq f^{\mathcal{U}}(x). \tag{3.12}$$

$\mathcal{U}' \subseteq \mathcal{U}$, hence $f^{\mathcal{U}'}(x') \leqq f^{\mathcal{U}}(x')$, see (2.6). This leads to

$$f^{\mathcal{U}'}(x') \overset{(2.6)}{\leqq} f^{\mathcal{U}}(x') \overset{(3.12)}{\lneq} f^{\mathcal{U}}(x) \overset{(3.11)}{=} f^{\mathcal{U}'}(x),$$

which contradicts efficiency of x for MORC($\mathcal{U}'$).

$\underline{\Leftarrow}$: Let Condition A hold for all solutions which are efficient for (MORC($\mathcal{U}'$)) and let x be efficient for MORC($\mathcal{U}$). Assume to the contrary that $x \in \mathcal{X}$ is not efficient for MORC($\mathcal{U}'$). Then, since the domination property holds, there is a solution $x' \in \mathcal{X}$ that is efficient for MORC($\mathcal{U}'$) such that

$$f^{\mathcal{U}'}(x') \lneq f^{\mathcal{U}'}(x). \tag{3.13}$$

Note that since x' is efficient for MORC($\mathcal{U}'$), it satisfies Condition A. Together with $\mathcal{U}' \subseteq \mathcal{U}$ we receive

$$f^{\mathcal{U}}(x') \overset{(3.11)}{=} f^{\mathcal{U}'}(x') \overset{(3.13)}{\lneq} f^{\mathcal{U}'}(x) \overset{(2.6)}{\leqq} f^{\mathcal{U}}(x).$$

This contradicts the assumption of x being efficient for MORC($\mathcal{U}$). □

Consequently, under the stated assumptions the Pareto frontiers of MORC($\mathcal{U}$) and MORC($\mathcal{U}'$) coincide, if all solutions $x \in \mathcal{X}$ that are efficient for MORC($\mathcal{U}'$) satisfy Condition A.

Remark 3.8. The same results as in the second part of Theorem 3.7 (but only for the image space) could be derived from Lemma 3.4 as follows: Suppose the conditions of Theorem 3.7 are met, i.e., all solutions $x \in \mathcal{X}$ that are efficient for MORC($\mathcal{U}'$) satisfy Condition A. Using the notation from Lemma 3.4 we can write this as

$$f^{\mathcal{U}'}(x) \overset{(3.11)}{=} f^{\mathcal{U}}(x) \text{ for all } x \in X^*(\mathcal{U}'). \tag{3.14}$$

This means that the lower and the upper bound constructed in Lemma 3.4 coincide and we get

$$\begin{aligned} \left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U})\right\} &\overset{(3.9)}{\leq^{\text{low}}} \left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U}')\right\} \\ &\overset{(3.14)}{=} \left\{f^{\mathcal{U}'}(x) : x \in X^*(\mathcal{U}')\right\} \overset{(3.9)}{\leq^{\text{low}}} \left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U})\right\}, \end{aligned} \tag{3.15}$$

which shows equality of the Pareto frontiers

$$\left\{f^{\mathcal{U}'}(x) : x \in X^*(\mathcal{U}')\right\} = \left\{f^{\mathcal{U}}(x) : x \in X^*(\mathcal{U})\right\}. \tag{3.16}$$

Checking *all* efficient solutions of a multi-objective problem is computationally hard (or even impossible). Thus, in the next result we strengthen the above theorem in a fashion that Condition A must only be satisfied for all solutions from a representative set.

Theorem 3.9. *Let $\mathcal{U}' \subseteq \mathcal{U}$ and let the domination property (2.2) be satisfied for MORC($\mathcal{U}$) and MORC($\mathcal{U}'$). If there is a representative set $\mathcal{R}'$ of efficient solutions for MORC($\mathcal{U}'$) whose elements satisfy Condition A, then under Assumption 1 we have:*

(*i*) *$x \in \mathcal{R}' \Rightarrow x$ is efficient for MORC($\mathcal{U}$),*

(*ii*) *x is efficient for MORC($\mathcal{U}$) $\Rightarrow x$ is efficient for MORC($\mathcal{U}'$), and*

(*iii*) *$\mathcal{R}'$ is a representative set of efficient solutions to MORC($\mathcal{U}$).*

Proof.

(i) Let $x \in \mathcal{R}'$. In particular, x is efficient for MORC($\mathcal{U}'$) and by assumption it satisfies Condition A. We can hence apply Theorem 3.7 and conclude that x is efficient for MORC($\mathcal{U}$).

(ii) Let x be efficient for MORC($\mathcal{U}$) and assume x is not efficient for MORC($\mathcal{U}'$). Due to the domination property, there is a solution x' that satisfies $f^{\mathcal{U}'}(x') \lneq f^{\mathcal{U}'}(x)$. Moreover, since $\mathcal{R}'$ is a representative set for MORC($\mathcal{U}'$) we can choose $x' \in \mathcal{R}'$. Hence, Condition A holds for x' and we receive

$$f^{\mathcal{U}}(x') \overset{(3.11)}{=} f^{\mathcal{U}'}(x') \lneq f^{\mathcal{U}'}(x) \overset{(2.6)}{\leqq} f^{\mathcal{U}}(x).$$

This contradicts efficiency of x for MORC($\mathcal{U}$).

(iii) Let $\mathcal{R} \subseteq \mathcal{X}$ be a representative set of efficient solutions for MORC($\mathcal{U}$). We show that $f^{\mathcal{U}}(\mathcal{R}') = f^{\mathcal{U}}(\mathcal{R})$.
$\subseteq$: Let $y' \in f^{\mathcal{U}}(\mathcal{R}')$. Then $y' = f^{\mathcal{U}}(x')$ for some $x' \in \mathcal{R}'$. According to (i), x' is efficient for MORC($\mathcal{U}$), hence $y' \in f^{\mathcal{U}}(\mathcal{R})$.
$\supseteq$: Let $y \in f^{\mathcal{U}}(\mathcal{R})$. Then $y = f^{\mathcal{U}}(x)$ for some x that is efficient for MORC($\mathcal{U}$). According to (ii), x is also efficient for MORC($\mathcal{U}'$). Hence, $x' \in \mathcal{R}'$ exists such that $f^{\mathcal{U}'}(x) = f^{\mathcal{U}'}(x')$. This leads to

$$y = f^{\mathcal{U}}(x) \overset{(2.6)}{\geq} f^{\mathcal{U}'}(x) = f^{\mathcal{U}'}(x') \overset{(3.11)}{=} f^{\mathcal{U}}(x'). \tag{3.17}$$

Since by assumption y is nondominated for MORC($\mathcal{U}$), equality must hold true. Thus, $y = f^{\mathcal{U}}(x')$ for $x' \in \mathcal{R}'$ and, consequently, $y \in f^{\mathcal{U}}(\mathcal{R}')$.

□

Theorem 3.9 shows that it is not necessary to check Condition A for *all* efficient solutions, rather it is sufficient to check it only for a representative set. However, a representative set may still be infinite. Later, in Section 3.2.3, we will therefore strengthen Theorem 3.9 once more.

We can now formulate the multi-objective generalization of optimization-pessimization.

ADAPTION OF OPTIMIZATION-PESSIMIZATION. In order to deal with the multi-objective setting algorithmically, we modify optimization-pessimization for multi-objective problems as it is described in the following (see also Figure 3.4):

When solving the *optimization problem* (MORC($\mathcal{U}'$)) we do not only determine one optimal solution, but a representative set X'^* of efficient solutions. In the subsequent *pessimization step* we consider *all* solutions $x \in X'^*$. For each of them we determine not just

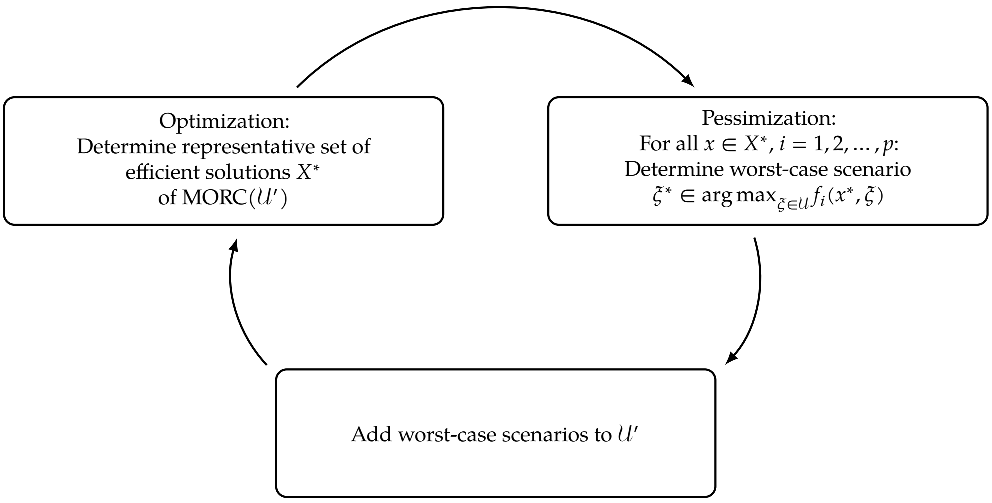


Figure 3.4: Optimization-pessimization: Determining point-based minmax robust efficient solutions for problems with uncertain objectives

one worst-case scenario, but a worst-case scenario for each of the p objective functions independently. All of these $p \cdot |X'^*|$ worst-case scenarios are then added to the uncertainty set.

Algorithm 3.2 describes the exact procedure and the following theorem shows its correctness.

Theorem 3.10. *Let Assumption 1 hold. Let the ideal point property* (2.1) *and the domination property* (2.2) *hold for* (MORC($\mathcal{U}$)) *and for* (MORC($\mathcal{U}'$)) *for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$.*

(*i*) *Let $\mathcal{U}$ be finite. Then Algorithm 3.2 returns a representative set of point-based minmax robust efficient solutions to MOP($\mathcal{U}$) in at most $|\mathcal{U}|$ iterations.*

(*ii*) *Let $\mathcal{U}$ be a polytope or finite and $f_i(x, \cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$, $i = 1, 2, \dots, p$, be continuous and quasi-convex. Then Algorithm 3.2 returns a representative set of point-based minmax robust efficient solutions to MOP($\mathcal{U}$) in at most k iterations where k is the number of extreme points of $\mathcal{U}$, if we choose an algorithm for the pessimization problem which always finds an extreme point of $\mathcal{U}$.*

Proof. Algorithm 3.2 determines a representative set of efficient solutions to $\mathcal{U}^{(k-1)}$ in step k. It stops if

$$f^{\mathcal{U}}(x^*) = f^{\mathcal{U}^{(k-1)}}(x^*) \tag{3.18}$$

for all $x^* \in X^{(k-1)*}$.

Hence, $\mathcal{R}' = X^{(k-1)*}$ is a representative set of efficient solutions to MORC($\mathcal{U}^{(k-1)}$) whose elements satisfy Condition A. We can thus apply Theorem 3.9 (iii) for $\mathcal{U}' = \mathcal{U}^{(k-1)} \subseteq \mathcal{U}$ and, after termination, $\mathcal{R}' = X^{(k-1)*}$ is a representative set of efficient solutions to (MORC($\mathcal{U}$)).

We now show the bounds on the number of iterations.

Algorithm 3.2 Optimization-pessimization: Determining point-based minmax robust efficient solutions for problems with uncertain objectives

Require: Multi-objective robust optimization problem (MORC($\mathcal{U}$)).
Require: Finite initial set $\mathcal{U}^{(0)} \subseteq \mathcal{U}$.
Ensure: Either $\mathcal{U}$ finite or $\mathcal{U}$ a polytope and $f_i(x, \cdot)$, $i = 1, 2, \dots, p$ continuous and quasi-convex.
Ensure: Ideal point property (2.1) and domination property (2.2) hold for (MORC($\mathcal{U}$)) and for (MORC($\mathcal{U}'$)) for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$.
Ensure: There is a finite representative set of extreme supported efficient solution to (MORC($\mathcal{U}'$)) for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$

Set $k := 0$.
repeat
  Set $\mathcal{U}^{(k+1)} := \mathcal{U}^{(k)}$.

  Optimization
  Determine representative set of efficient solutions $X^{(k)*}$ and nondominated points $Y^{(k)*}$ of $P(\mathcal{U}^{(k)})$.

  Pessimization
  for all $x^* \in X^{(k)*}$ **do**
    for all $i = 1, 2, \dots, p$ **do**
      Determine $\xi^* \in \arg\max_{\mathcal{U}} f_i(x^*, \xi)$.
      Add ξ^* to $\mathcal{U}^{(k+1)}$.
    end for
  end for

  $k := k + 1$
until $f^{\mathcal{U}}(x^*) = f^{\mathcal{U}^{(k-1)}}(x^*)$ for all $x^* \in X^{(k-1)*}$.
return $X^{(k-1)*}$: representative set of efficient solutions to (MORC($\mathcal{U}$)).
return $Y^{(k-1)*}$: set of nondominated points to (MORC($\mathcal{U}$)).
return $\mathcal{U}^{\text{FINAL}} := \mathcal{U}^k$: set of worst-case scenarios.

(i) In every iteration, either at least one new worst-case scenario is added or (3.18) holds and the procedure stops. Since $\mathcal{U}$ is finite, the latter happens after at most $|\mathcal{U}|$ iterations.

(ii) Consider the pessimization problem $\text{Pess}^{\text{pb}}(x)$: here we maximize a continuous function over a compact set $\mathcal{U}$, i.e., a maximum always exists. Since $f(x, \cdot)$ is quasi-convex, the maximum is always attained at an extreme point of $\mathcal{U}$. If we choose an algorithm that returns an extreme point for such optimization problems, we add a new extreme point in each iteration until (3.18) holds as in (i).

□

3.2.3 *Point-based minmax robust extreme supported efficient solutions*

In Section 3.2.2 we have shown that a representative set of efficient solutions to MORC($\mathcal{U}'$) is also a represenative set of efficient solutions to MORC($\mathcal{U}$), if all elements of the first set satisfy Condition A. However, the resulting Algorithm 3.2 still depends on our ability to

- determine such a set, and

- check Condition A for all elements of this set.

Since for many problems a representative set will be very large or (in the case of a non-discrete set $\mathcal{X}$) even contain infinitely many elements, it is helpful to further strengthen the statement from Theorems 3.7 and 3.9:

In this section we show that the statement of Theorem 3.7 remains valid if we replace the set of all efficient solutions to MORC($\mathcal{U}'$) not only by a representative set, but even by a representative set of only their *extreme supported efficient* solutions.

For this purpose, we first define *point-based minmax robust (extreme) supported efficiency*.

Definition 3.11. Given an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)), a solution $x \in \mathcal{X}$ is called *point-based minmax robust (extreme) supported efficient* if x is (extreme) supported efficient (in the sense of Definition 2.2) for the multi-objective robust counterpart (MORC($\mathcal{U}$)).

We then adapt the optimization-pessimization method such that we can determine point-based minmax robust extreme supported efficient solutions to the uncertain multi-objective optimization problem (MOP($\mathcal{U}$)), i.e., extreme supported efficient solutions to its robust counterpart (MORC($\mathcal{U}$)), by determining the same set for the multi-objective robust counterpart with a reduced uncertainty set (MORC($\mathcal{U}'$)). Later on, in Section 4.1, with dichotomic search we provide an algorithm for computing extreme supported solutions to MORC($\mathcal{U}'$).

REDUCTION OF THE SCENARIO SET. We start with a preparatory statement investigating the relation between extreme supported nondominated points $\mathcal{Y}_{\text{ESN}}$ and the set of all images $\mathcal{Y}$. Specifically, we want to show that–under some assumptions–for a multi-objective optimization problem (MOP) every outcome $y \in \mathcal{Y}$ can be written as or is dominated by a convex combination of extreme supported nondominated points, i.e.,

$$\mathcal{Y} \subseteq \operatorname{conv}\left(\mathcal{Y}_{\text{ESN}}\right) + \mathbb{R}^p_{\geqq}. \tag{3.19}$$

In Chapter 8 we will show (c.f. Lemma 8.19 on Page 96) that (3.19) holds whenever $\mathcal{Y}$ is compact. Since the proof is technical and requires a closer look on the topic of (extreme) supported nondominance, at this point we provide a shorter proof with the additional, but in the context of this chapter unproblematic, condition that $\mathcal{Y}_{\text{ESN}}$ is finite.

Lemma 3.12. *Let a multi-objective optimization problem* (MOP) *with* $\mathcal{Y} \subsetneq \mathbb{R}^p$ *compact be given and let* $\mathcal{Y}_{\text{ESN}} \neq \emptyset$ *be its set of extreme supported nondominated points. We assume that* $\mathcal{Y}_{\text{ESN}}$ *is finite. Then* $\mathcal{Y} \subseteq \operatorname{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^p_{\geqq}$ *holds.*

Proof. Assume there is a $\bar{y} \in \mathcal{Y}$ that does not lie in $\operatorname{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^p_{\geqq}$. We then can show that there is also $\hat{y}$ outside of $\operatorname{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^p_{\geqq}$ which is extreme supported nondominated, a contradiction.

So, assume to the contrary that $\bar{y} \in \mathcal{Y} \setminus (\operatorname{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^p_{\geqq})$ exists. Then the sets $\{\bar{y}\}$ and $\operatorname{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^p_{\geqq}$ are disjoint, nonempty, closed and convex sets. Hence, a separating hyperplane exists (see [BV04]), i.e., $\nu \in \mathbb{R}^p \setminus \{0\}$ and $s \in \mathbb{R}$ exist such that

$$\nu^\top \bar{y} < s < \nu^\top y, \ \forall y \in \operatorname{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^p_{\geqq}. \tag{3.20}$$

The elements of $\operatorname{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^p_{\geqq}$ can get arbitrarily big in each component, hence $\nu_i \geq 0$ for all $i = 1, 2, \dots, p$. Let now $z_\nu := \min\{\nu^\top y \colon y \in \mathcal{Y}\}$ and $\mathcal{Y}_\nu := \arg\min\{\nu^\top y \colon y \in \mathcal{Y}\}$. Since $\bar{y} \in \mathcal{Y}$, we get

$$\nu^\top y^* \leq \nu^\top \bar{y} < s, \ \forall y^* \in \mathcal{Y}_\nu. \tag{3.21}$$

Together with (3.20) this shows that $\mathcal{Y}_\nu$ can be separated from $\operatorname{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^p_{\geqq}$ and, hence, cannot be extreme supported nondominated themselves. Specifically, the lexicographic minimum, $\hat{y} := \operatorname{lex\,min}_{y \in \mathcal{Y}_\nu}$, i.e., $\hat{y}_j = \min\{y_j \colon y \in \mathcal{Y}_\nu, y_1 = \hat{y}_1, \dots, y_{j-1} = \hat{y}_{j-1}\}$, $j = 1, 2, \dots, p$, is not extreme supported nondominated (the existence of this point follows from compactness of $\mathcal{Y}$).

Hence, a nontrivial convex combination of nondominated points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}$, $\lambda \in \Delta^p$ exists such that

$$\hat{y} \geq \sum_{i=1}^{n} \lambda_i y^{(i)}. \tag{3.22}$$

Now we assume that $y^{(i)} \notin \mathcal{Y}_\nu$ for at least one $i = 1, 2, \dots, n$. Without loss of generality assume $y^{(1)} \notin \mathcal{Y}_\nu$. Then

$$\nu^\top \sum_{i=1}^{n} \lambda_i y^{(i)} = \lambda_1 \underbrace{\nu^\top y^{(1)}}_{> z_\nu} + \sum_{i=2}^{n} \underbrace{\nu^\top \lambda_i y^{(i)}}_{\geq z_\nu} > \sum_{i=1}^{n} \lambda_i z_\nu = z_\nu = \nu^\top \hat{y}. \tag{3.23}$$

Since $\nu_i \geq 0, i = 1, 2, \dots, n$, (3.23) contradicts (3.22). Thus, our assumption that $y^{(i)} \notin \mathcal{Y}_\nu$ for at least one $i = 1, 2, \dots, n$ is contradicted and we have that $y^{(i)} \in \mathcal{Y}_\nu$ for all $i = 1, 2, \dots, n$.

Consequently, a nontrivial convex combination only consisting of nondominated points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_\nu \subseteq \mathcal{Y}$ exists such that (3.22) holds. This, however, is not possible since, by definition, $\hat{y}$ is the lexicographic minimum of $\mathcal{Y}_\nu$ and thus all other elements of $\mathcal{Y}_\nu$ lie in the lexicographic cone $\hat{y} + \{y \in \mathbb{R}^p \colon y_1 = y_2 = \dots y_i = 0, y_{i+1} > 0 \text{ for some } i = 0, 1, \dots, p\}$. $\square$

The following corollary will be of use in the proof of the subsequent theorem.

Corollary 3.13. *Under the assumptions of Lemma 3.12, for any $y \in \mathcal{Y} \setminus \mathcal{Y}_{\text{ESN}}$ a nontrivial convex combination*

$$\sum_{i=1}^{n} \lambda_i y^{(i)} \leqq y$$

with $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_{\text{ESN}}$, $\lambda \in \Delta^p$, exists.

We can now utilize the above corollary and show that the statement of Theorem 3.9 remains valid even if only representative sets of extreme supported efficient solutions are considered.

Theorem 3.14. *Let $\mathcal{U}' \subseteq \mathcal{U}$ and let the domination property (2.2) be satisfied for MORC($\mathcal{U}$) and MORC($\mathcal{U}'$). If $f^{\mathcal{U}'}(\mathcal{X})$ is compact and there is a finite representative set $\mathcal{R}'_{ESE}$ of extreme supported efficient solutions for MORC($\mathcal{U}'$) whose elements satisfy Condition A, then under Assumption 1* (*i*) $x \in \mathcal{R}'_{ESE} \Rightarrow x$ *is extreme supported efficient for MORC*$(\mathcal{U})$,

(*ii*) *x is extreme supported efficient for MORC*$(\mathcal{U}) \Rightarrow x$ *is extreme supported efficient for MORC*$(\mathcal{U}')$, *and*

(*iii*) $\mathcal{R}'_{ESE}$ *is a representative set of extreme supported efficient solutions to MORC*$(\mathcal{U})$.

Proof. (i) Let $x \in \mathcal{R}'_{\text{ESE}}$. Assume to the contrary that x is not extreme supported efficient for MORC$(\mathcal{U})$, i.e., there exists a nontrivial convex combination of solutions efficient for MORC$(\mathcal{U})$ $x'_1, \dots, x'_n \in \mathcal{X}$, $\lambda \in \Delta^p$, such that

$$\sum_{i=1}^{n} \lambda_i f^{\mathcal{U}}(x'_i) \leqq f^{\mathcal{U}}(x), \tag{3.24}$$

and $f^{\mathcal{U}}(x'_i) \neq f^{\mathcal{U}}(x)$ for all $i = 1, 2, \dots, n$.

$\mathcal{U}' \subseteq \mathcal{U}$, hence $f^{\mathcal{U}'}(x'_i) \leqq f^{\mathcal{U}}(x'_i)$, $i = 1, 2 \dots, n$, see (2.6). This leads to

$$\sum_{i=1}^{n} \lambda_i f^{\mathcal{U}'}(x'_i) \overset{(2.6)}{\leqq} \sum_{i=1}^{n} \lambda_i f^{\mathcal{U}}(x'_i) \overset{(3.24)}{\leqq} f^{\mathcal{U}}(x) \overset{(3.11)}{=} f^{\mathcal{U}'}(x). \tag{3.25}$$

Hence, extreme supported efficiency of x for MORC$(\mathcal{U}')$ is contradicted or

$$f^{\mathcal{U}'}(x'_i) = f^{\mathcal{U}'}(x) \text{ for at least one } i = 1, 2, \dots, n \tag{3.26}$$

must hold. Assume that 3.26 holds. Then

$$f^{\mathcal{U}}(x) \overset{(3.11)}{=} f^{\mathcal{U}'}(x) \overset{(3.26)}{=} f^{\mathcal{U}'}(x'_i) \overset{(2.6)}{\leqq} f^{\mathcal{U}}(x'_i)$$

follows. Since x'_i is efficient for MORC$(\mathcal{U})$, equality holds. Hence, $f^{\mathcal{U}}(x'_i) \neq f^{\mathcal{U}}(x)$ for all $i = 1, 2, \dots, n$ is contradicted.

(ii) Let x be extreme supported efficient for MORC$(\mathcal{U})$. Assume to the contrary that $x \in \mathcal{X}$ is not extreme supported efficient for MORC$(\mathcal{U}')$. Since $\mathcal{R}'_{\text{ESE}}$ is finite and $\mathcal{Y} = f^{\mathcal{U}'}(\mathcal{X})$ is compact Corollary 3.13 can be applied to the problem MORC$(\mathcal{U}')$ and there exists a nontrivial convex combination $x'_1, \dots, x'_n \in \mathcal{R}'_{\text{ESE}}$, $\lambda \in \Delta^p$, such that

$$\sum_{i=1}^{n} \lambda_i f^{\mathcal{U}'}(x'_i) \leqq f^{\mathcal{U}'}(x) \tag{3.27}$$

and $f^{\mathcal{U}'}(x'_i) \neq f^{\mathcal{U}'}(x)$ for all $i = 1, 2, \dots, n$.

Note that since $x'_i \in \mathcal{R}'_{\text{ESE}}$, $i = 1, 2, \dots, n$, they satisfy Condition A. Together with $\mathcal{U}' \subseteq \mathcal{U}$ we receive

$$\sum_{i=1}^{n} \lambda_i f^{\mathcal{U}}(x'_i) \overset{(3.11)}{=} \sum_{i=1}^{n} \lambda_i f^{\mathcal{U}'}(x'_i) \overset{(3.27)}{\leqq} f^{\mathcal{U}'}(x) \overset{(2.6)}{\leqq} f^{\mathcal{U}}(x). \tag{3.28}$$

This contradicts the assumption of x being extreme supported efficient for MORC($\mathcal{U}$) or

$$f^{\mathcal{U}}(x'_i) = f^{\mathcal{U}}(x) \text{ for at least one } i = 1, 2, \ldots, n \tag{3.29}$$

must hold.

Assume 3.29 holds. Then

$$f^{\mathcal{U}'}(x) \overset{(2.6)}{\leqq} f^{\mathcal{U}}(x) \overset{(3.29)}{=} f^{\mathcal{U}}(x'_i) \overset{(3.11)}{=} f^{\mathcal{U}'}(x'_i)$$

follows. Since x'_i is efficient for MORC($\mathcal{U}'$), equality holds. Hence, $f^{\mathcal{U}'}(x'_i) \neq f^{\mathcal{U}'}(x)$ for all $i = 1, 2, \ldots, n$ is contradicted.

(iii) Let $\mathcal{R}_{\text{ESE}} \subseteq \mathcal{X}$ be a representative set of extreme supported efficient solutions for MORC($\mathcal{U}$). Analogously to the proof of Theorem 3.9 (iii) we show that $f^{\mathcal{U}}(\mathcal{R}'_{\text{ESE}}) = f^{\mathcal{U}}(\mathcal{R}_{\text{ESE}})$.
$\underline{\subseteq}$: Let $y' \in f^{\mathcal{U}}(\mathcal{R}'_{\text{ESE}})$. Then $y' = f^{\mathcal{U}}(x')$ for some $x' \in \mathcal{R}'_{\text{ESE}}$. According to (i), x' is extreme supported efficient for MORC($\mathcal{U}$), hence $y' \in f^{\mathcal{U}}(\mathcal{R}_{\text{ESE}})$.
$\underline{\supseteq}$: Let $y \in f^{\mathcal{U}}(\mathcal{R}_{\text{ESE}})$. Then $y = f^{\mathcal{U}}(x)$ for some x that is extreme supported efficient for MORC($\mathcal{U}$). According to (ii), x is also extreme supported efficient for MORC($\mathcal{U}'$). Hence, $x' \in \mathcal{R}'_{\text{ESE}}$ exists such that $f^{\mathcal{U}'}(x) = f^{\mathcal{U}'}(x')$. This leads to

$$y = f^{\mathcal{U}}(x) \overset{(2.6)}{\geq} f^{\mathcal{U}'}(x) = f^{\mathcal{U}'}(x') \overset{(3.11)}{=} f^{\mathcal{U}}(x').$$

Since by assumption y is extreme supported nondominated for MORC($\mathcal{U}$), equality must hold true. Thus, $y = f^{\mathcal{U}}(x')$ for $x' \in \mathcal{R}'_{\text{ESE}}$ and, consequently, $y \in f^{\mathcal{U}}(\mathcal{R}'_{\text{ESE}})$. □

ADAPTION OF OPTIMIZATION-PESSIMIZATION. Figure 3.5 illustrates the method, Algorithm 3.3 describes the exact procedure.

Now the *optimization problem* consists of determining a representative set X'^* of *extreme supported efficient* solutions to MORC($\mathcal{U}'$). As before, in the *pessimization step* we consider *all* solutions $x \in X'^*$ and for each of them we determine a worst-case scenario for each of the p objective functions independently. The worst-case scenarios are then added to the uncertainty set.

The following lemma shows correctness of Algorithm 3.3 and is the analog of Theorem 3.10.

Theorem 3.15. *Let Assumption 1 hold. Furthermore, let the ideal point property* (2.1) *and the domination property* (2.2) *hold for MORC*($\mathcal{U}$) *and for MORC*($\mathcal{U}'$) *for any finite subset* $\mathcal{U}' \subseteq \mathcal{U}$. *Furthermore, let there be a finite representative set of extreme supported efficient solutions to MORC*($\mathcal{U}'$).

(*i*) *Let* $\mathcal{U}$ *be finite. Then Algorithm 3.3 returns a representative set of extreme supported efficient solutions to MORC*($\mathcal{U}$) *in at most* $|\mathcal{U}|$ *iterations.*

(*ii*) *Let* $\mathcal{U}$ *be a polytope or finite and* $f_i(x, \cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$, $i = 1, 2, \ldots, p$, *be quasi-convex. Then Algorithm 3.3 returns a representative set of extreme supported efficient solutions*

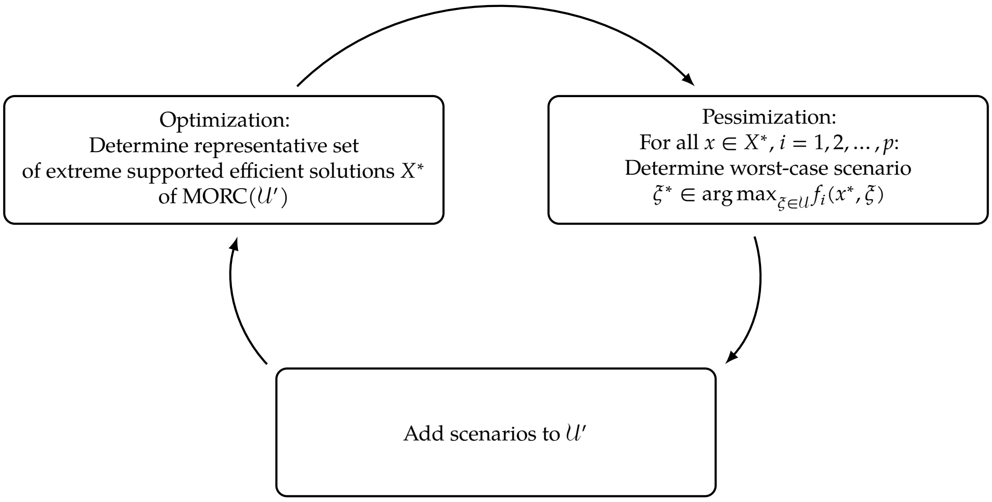


Figure 3.5: Optimization-pessimization: Determining point-based minmax robust extreme supported efficient solutions for problems with uncertain objectives

Algorithm 3.3 Optimization-pessimization: Determining point-based minmax robust extreme supported efficient solutions for problems with uncertain objectives

Require: Multi-objective robust optimization problem (MORC($\mathcal{U}$)).
Require: Finite initial set $\mathcal{U}^{(0)} \subseteq \mathcal{U}$.
Ensure: Either $\mathcal{U}$ finite or $\mathcal{U}$ a polytope and $f_i(x, \cdot)$, $i = 1, 2, \dots, p$ continuous and quasi-convex.
Ensure: ideal point property (2.1) and domination property (2.2) hold for (MORC($\mathcal{U}$)) and for (MORC($\mathcal{U}'$)) for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$.

Set $k := 0$.
repeat
  Set $\mathcal{U}^{(k+1)} := \mathcal{U}^{(k)}$.
  Optimization
  Determine representative set for extreme supported efficient solutions $X^{(k)*}$ and extreme supported nondominated points $Y^{(k)*}$ of $P(\mathcal{U}^{(k)})$.
  Pessimization
  for all $x^* \in X^{(k)*}$ **do**
    for all $i = 1, 2, \dots, p$ **do**
      Determine $\xi^* \in \arg\max_{\mathcal{U}} f_i(x^*, \xi)$.
      Add ξ^* to $\mathcal{U}^{(k+1)}$.
    end for
  end for
  $k := k + 1$
until $f^{\mathcal{U}}(x^*) = f^{\mathcal{U}^{(k-1)}}(x^*)$ for all $x^* \in X^{(k-1)*}$.
return $X^{(k-1)*}$: representative set of extreme supported efficient solutions of (MORC($\mathcal{U}$)).
return $Y^{(k-1)*}$: set of extreme supported nondominated points of (MORC($\mathcal{U}$)).
return $\mathcal{U}^{\text{FINAL}} := \mathcal{U}^k$: set of worst-case scenarios.

to MORC($\mathcal{U}$) in at most k iterations where k is the number of extreme points of $\mathcal{U}$, if we choose an algorithm for the pessimization problem which always finds an extreme point of $\mathcal{U}$.

Proof. Throughout the algorithm $\mathcal{U}^(k)$, $k = 0, 1, 2, \ldots$, is finite. Thus, the functions $f_i^{\mathcal{U}'}$, $i = 1, 2, \ldots, p$ are continuous. It follows that $\mathcal{Y} = f^{\mathcal{U}'}(\mathcal{X})$ is the image of a compact set under a continuous function and therefore compact, too. Since additionally $X^{(k-1)*}$ (the representative set of extreme supported efficient solutions to MORC$(\mathcal{U}')$) is finite, Theorem 3.14 can be applied instead of Theorem 3.9. The rest of the proof is analogous to the proof of Theorem 3.10. □

Algorithm 3.3 provides a method to solve the bi-objective robust mixed-integer linear optimization problem BRO$(\mathcal{U})$, which will be defined in Section 4.2. However, this is still challenging since in each iteration a representative set for all extreme supported efficient solutions to MORC$(\mathcal{U}')$ for some $\mathcal{U}' \subseteq \mathcal{U}$ needs to be found. In Section 5.2 we show how for that purpose dichotomic search can be employed.

3.3 DETERMINING POINT-BASED MINMAX ROBUST EFFICIENT SOLUTIONS FOR PROBLEMS WITH UNCERTAIN OBJECTIVES AND UNCERTAIN CONSTRAINTS

In this section, the results of the previous section will be extended to that end, that multi-objective optimization-pessimization can also be used for problems with uncertainty not only in the objectives, but also in the constraints. Unlike in Section 3.2, we no longer assume that $\mathcal{X}_\xi = \mathcal{X}$ for all $\xi \in \mathcal{U}$. Instead we consider the uncertain multi-objective optimization problem

$$\left\{ \min \left\{ \begin{pmatrix} f_1(x,\xi) \\ f_2(x,\xi) \\ \vdots \\ f_p(x,\xi) \end{pmatrix} : \begin{array}{ll} F_j(x,\xi) \leq 0 & \forall j = 1, 2, \ldots, J \\ x \in \mathcal{X} & \end{array} \right\} \right\}_{\xi \in \mathcal{U}} \tag{MOP($\mathcal{U}$) revisited}$$

and have to deal with feasibility sets that depend on the uncertainty set.

In Section 3.3.1 we will show how in the context of point-based minmax robust efficiency, uncertainty in the objective can be transferred to the constraints. Subsequently, in Section 3.3.2 we will formulate optimization-pessimization for multi-objective problems with uncertainty in both the constraints and the objectives.

3.3.1 *Transferring all uncertainty into the constraints*

In Section 2.2 we noted that in a single-objective setting–when exploring minmax robustness–it can be assumed without loss of generality that uncertainty occurs only in the constraints. This is due to the fact that by the means of introducing bottleneck variables a general uncertain problem (P$(\mathcal{U})$) can equivalently be written as problem with uncertainty only in the constraints (P$^{\text{BN}}(\mathcal{U})$) and their robust counterparts (RC$(\mathcal{U})$ and RC$^{\text{BN}}(\mathcal{U})$, respectively) are equivalent.

We will formally show that in the context of point-based minmax efficiency, this can be generalized to multi-objective problems; and the multi-objective robust counterpart

$$\min \left\{ \begin{pmatrix} \sup_{\xi \in \mathcal{U}} f_1(x,\xi) \\ \sup_{\xi \in \mathcal{U}} f_2(x,\xi) \\ \vdots \\ \sup_{\xi \in \mathcal{U}} f_p(x,\xi) \end{pmatrix} : \begin{array}{c} F_j(x,\xi) \leq 0 \quad \forall \xi \in \mathcal{U},\ \forall j = 1,2,\dots,J \\ x \in \mathcal{X} \end{array} \right\} \tag{MORC($\mathcal{U}$) reformulated}$$

is, in a sense, "equivalent" to

$$\min \left\{ \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix} : \begin{array}{ll} f_i(x,\xi) \leq y_i & \forall \xi \in \mathcal{U},\ \forall i = 1,2,\dots,p \\ F_j(x,\xi) \leq 0 & \forall \xi \in \mathcal{U},\ \forall j = 1,2,\dots,J \\ x \in \mathcal{X} & \\ y \in \mathbb{R}^p & \end{array} \right\}. \tag{MORC$^{\text{BN}}(\mathcal{U})$}$$

Specifically, we show that (extreme supported) efficient solutions to MORC($\mathcal{U}$) can be constructed from (extreme supported) efficient solutions to MORC$^{\text{BN}}(\mathcal{U})$ and vice versa.

We use the following notation for the robust feasible set $\mathcal{X}_{\mathcal{U}'}$ of an uncertain multi-objective optimization problem with uncertainty set $\mathcal{U}' \subseteq \mathcal{U}$:

$$\begin{aligned} \mathcal{X}_{\mathcal{U}} &:= \bigcap_{\xi \in \mathcal{U}} \underbrace{\left\{ x \in \mathcal{X} \colon F_j(x,\xi) \leq 0\ \forall j = 1,2,\dots,J \right\}}_{=\mathcal{X}_\xi} \\ &\subseteq \bigcap_{\xi \in \mathcal{U}'} \left\{ x \in \mathcal{X} \colon F_j(x,\xi) \leq 0\ \forall j = 1,2,\dots,J \right\} =: \mathcal{X}_{\mathcal{U}'} \subseteq \mathcal{X}. \end{aligned} \tag{3.30}$$

Lemma 3.16. *Consider a problem of type MORC($\mathcal{U}$). Let $\mathcal{U}$ be compact and let the functions $f_i(x,\cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$ be continuous for all $x \in \mathcal{X}_{\mathcal{U}} \subseteq \mathcal{X}$, $i = 1,2,\dots,p$. Then the following holds:*

(*i*) *The set of feasible solutions to MORC$^{BN}(\mathcal{U})$ is $\{(x,y) : x \in \mathcal{X}_{\mathcal{U}}, y \geq f^{\mathcal{U}}(x)\} \neq \emptyset$.*

(*ii*) *The set $X \subseteq \mathcal{X}_{\mathcal{U}}$ is the set of efficient solutions to MORC($\mathcal{U}$) if and only if $\{(x,y) : x \in X, y = f^{\mathcal{U}}(x)\}$ is the set of efficient solutions to MORC$^{BN}(\mathcal{U})$. In particular, the set of nondominated points for MORC($\mathcal{U}$) and MORC$^{BN}(\mathcal{U})$ coincide.*

(*iii*) *The set of extreme supported nondominated points for MORC($\mathcal{U}$) and MORC$^{BN}(\mathcal{U})$ coincide.*

(*iv*) *The set $X \subseteq \mathcal{X}_{\mathcal{U}}$ is a representative set of extreme supported efficient solutions to MORC($\mathcal{U}$) if and only if $\{(x,y)\colon x \in X, y = f^{\mathcal{U}}(x)\}$ is a representative set of extreme supported efficient solutions to MORC$^{BN}(\mathcal{U})$.*

Proof.

(i) This follows directly from the definition of MORC$^{\text{BN}}(\mathcal{U})$. Compactness of $\mathcal{U}$ implies that the feasible set of MORC$^{\text{BN}}(\mathcal{U})$ is not empty.

(ii) Let (x, y) be efficient for $\text{MORC}^{\text{BN}}(\mathcal{U})$. We show that this yields $y = f^{\mathcal{U}}(x)$: Clearly, $y \geq f^{\mathcal{U}}(x)$ otherwise (x, y) is not feasible for $\text{MORC}^{\text{BN}}(\mathcal{U})$, (see (i)). Now assume that $y_i > \max_{\xi \in \mathcal{U}} f_i(x, \xi)$ for $i = 1, 2, \dots, p$. Then (x, y) is dominated by the feasible solution $(x, f^{\mathcal{U}}(x))$ and hence not efficient. The set of efficient solutions to $\text{MORC}^{\text{BN}}(\mathcal{U})$ hence is contained in $\{(x, f^{\mathcal{U}}(x)) : x \in \mathcal{X}_{\mathcal{U}}\}$.

Note that $f^{\mathcal{U}}(x)$ is the objective function value of x in $\text{MORC}(\mathcal{U})$ and also of $(x, f^{\mathcal{U}}(x))$ in $\text{MORC}^{\text{BN}}(\mathcal{U})$. This yields that x is efficient to $\text{MORC}(\mathcal{U})$ if and only if $(x, f^{\mathcal{U}}(x))$ is efficient to $\text{MORC}^{\text{BN}}(\mathcal{U})$. Hence, X is the set of efficient solutions to $\text{MORC}(\mathcal{U})$ if and only if $\{(x, f^{\mathcal{U}}(x)) : x \in X\}$ is the set of efficient solutions to $\text{MORC}^{\text{BN}}(\mathcal{U})$ and the sets of nondominated points of both problems coincide.

(iii) The definition of extreme supported nondominated solutions only uses the set of nondominated points in the objective space. Due to (ii) the set of nondominated points for $\text{MORC}(\mathcal{U})$ and $\text{MORC}^{\text{BN}}(\mathcal{U})$ coincide, hence also their extreme supported nondominated points.

(iv) Let $X \subseteq \mathcal{X}_{\mathcal{U}}$ be a representative set of extreme supported efficient solutions to $\text{MORC}(\mathcal{U})$. Then $f^{\mathcal{U}}(X)$ is the set of extreme supported nondominated points for $\text{MORC}(\mathcal{U})$. According to (iii), $f^{\mathcal{U}}(X)$ is also the set of extreme supported nondominated points to $\text{MORC}^{\text{BN}}(\mathcal{U})$. Since $f^{\mathcal{U}}(X)$ is the image of $\{(x, f^{\mathcal{U}}(x)) : x \in X\}$ for $\text{MORC}^{\text{BN}}(\mathcal{U})$, the latter set is a representative set of extreme supported efficient solutions to $\text{MORC}^{\text{BN}}(\mathcal{U})$.

Let a representative set of extreme supported efficient solutions to $\text{MORC}^{\text{BN}}(\mathcal{U})$ be given. By (ii), it takes the form $\{(x, y) : x \in X, y = f^{\mathcal{U}}(x)\}$ for some $X \subseteq \mathcal{X}_{\mathcal{U}}$.

Its image $f^{\mathcal{U}}(X)$ then is the set of extreme supported nondominated solutions to $\text{MORC}^{\text{BN}}(\mathcal{U})$, and according to (iii), also to $\text{MORC}(\mathcal{U})$. Consequently, X is a representative set of extreme supported efficient solutions to $\text{MORC}(\mathcal{U})$. □

We now present an algorithm designed to determine (extreme supported) efficient solutions to $\text{MORC}^{\text{BN}}(\mathcal{U})$. By Lemma 3.16 this algorithm also solves $\text{MORC}(\mathcal{U})$.

3.3.2 *Adaption of optimization-pessimization*

In this section, we once again work under Assumption 1.

We present an optimization-pessimization algorithm that solves $\text{MORC}^{\text{BN}}(\mathcal{U})$ by iteratively increasing the uncertainty set. Again, the algorithmic idea is illustrated in Figure 3.6 and the algorithm is explicitly stated in Algorithm 3.4. It can be used to determine either point-based minmax efficient or point-based minmax extreme supported efficient solutions.

Theorem 3.17. *Let $\mathcal{U}' \subseteq \mathcal{U}$ and let the ideal point property (2.1) and the domination property (2.2) hold for $\text{MORC}(\mathcal{U})$ and for $\text{MORC}(\mathcal{U}')$ for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$. Furthermore, let the objective functions $f_i(x, \cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$, $i = 1, 2, \dots, p$, and the constraint functions $F_j(x, \cdot)$, $j = 1, 2, \dots, J$, be continuous.*

(i) *Let $\mathcal{U}$ be finite. Then Algorithm 3.4 returns a representative set of point-based minmax robust (extreme supported) efficient solutions to $\text{MOP}(\mathcal{U})$ in at most $|\mathcal{U}|$ iterations.*

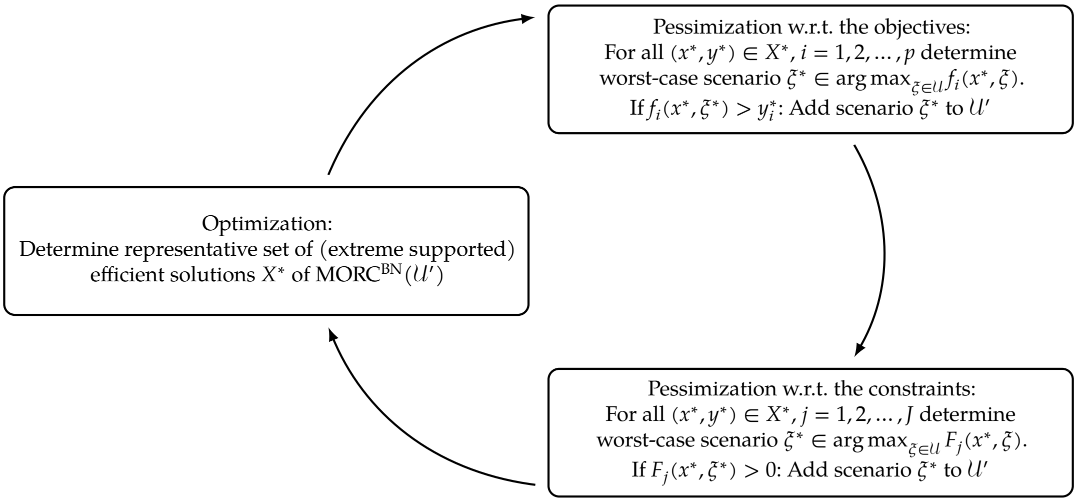


Figure 3.6: Optimization-pessimization for problem with uncertainty in the objectives and the constraints

(*ii*) *Let $\mathcal{U}$ be a polytope or finite and let the objectives $f_i(x, \cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$ for all $i = 1, 2, \dots, p$ and the constraints $F_j(x, \cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$ for all $j = 1, 2, \dots, J$ be continuous and quasi-convex. Then Algorithm 3.4 returns a representative set of point-based minmax robust (extreme supported) efficient solutions to* $(\text{MOP}(\mathcal{U}))$ *in at most k iterations, where k is the number of extreme points of $\mathcal{U}$, if we choose an algorithm for the pessimization problem which always finds an extreme point of $\mathcal{U}$.*

Proof. We first show that the solutions returned after termination of the algorithm are in fact point-based minmax robust (extreme supported) efficient solutions to $\text{MOP}(\mathcal{U})$. We then show termination after the stated number of iterations.

For the first part, note that when the algorithm terminates we have

$$\max_{\xi \in \mathcal{U}} f_i(x^*, \xi) \le y_i^* \qquad \forall i = 1, 2, \dots, p, \tag{3.31}$$

$$\max_{\xi \in \mathcal{U}} F_j(x^*, \xi) \le 0 \qquad \forall j = 1, 2, \dots, J \tag{3.32}$$

for all $(x^*, y^*) \in X^{(k-1)*}$. Thus, all solutions in $X^{(k-1)*}$ are feasible for $\text{MORC}^{\text{BN}}(\mathcal{U})$. Since they are (extreme supported) efficient to $\text{MORC}^{\text{BN}}(\mathcal{U}^{(k)})$–a problem with a smaller feasible set but the same objective as $\text{MORC}^{\text{BN}}(\mathcal{U})$–, it follows that they are (extreme supported) efficient to $\text{MORC}^{\text{BN}}(\mathcal{U})$. By Lemma 3.16 (ii) and (iv), it follows that $\operatorname{proj}_{\mathcal{X}}(\mathcal{X}^{(k-1)*})$, i.e., the first p components of all elements of $\mathcal{X}^{(k-1)*}$, is a representative set of (extreme supported) efficient solutions to $\text{MORC}(\mathcal{U})$, i.e., a representative set of point-based minmax robust (extreme supported) efficient solutions to $\text{MOP}(\mathcal{U})$. By the same argument $\operatorname{proj}_{\mathcal{Y}}(\mathcal{X}^{(k-1)*})$ is the set of point-based minmax robust efficient solutions to $\text{MOP}(\mathcal{U})$.

For the second part note that analogous to the proof of Theorem 3.10 in each iteration at least one scenario is added to $\mathcal{U}^{(k)}$. The statement in (i) follows directly. By continuity and quasi-convexity of the functions $f_i(x, \cdot)$, $i = 1, 2, \dots, p$, and $F_j(x, \cdot)$, $j = 1, 2, \dots, J$, and compactness of $\mathcal{U}$ it follows that we can choose an algorithm that returns an extreme point of $\mathcal{U}$, the statement in (ii) follows. $\square$

Algorithm 3.4 Optimization-pessimization: Determining point-based minmax robust (extreme supported) efficient solutions for problems with uncertain objectives

Require: Multi-objective robust optimization problem $\text{MORC}^{\text{BN}}(\mathcal{U})$
Require: Finite initial set $\mathcal{U}^{(0)} \subseteq \mathcal{U}$.
Ensure: Either $\mathcal{U}$ finite or $\mathcal{U}$ a polytope and $f_i(x,\cdot)$, $i = 1,2,\dots,p$, and $F_j(x,\cdot)$, $j = 1,2,\dots,J$, continuous and quasi-convex.
Ensure: ideal point property (2.1) and domination property (2.2) hold for $(\text{MORC}(\mathcal{U}))$ and for $(\text{MORC}(\mathcal{U}'))$ for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$.

```
Set k := 0.
repeat
   Set U^(k+1) := U^(k).
   ┌ Optimization
   │ Determine representative set of (extreme supported) efficient solutions X^(k)*
   │ of MORC^BN(U).
   └
   ┌ Pessimization
   │ for all (x*, y*) ∈ X^(k)* do
   │    for all i = 1, 2, ..., p do
   │       Determine ξ* ∈ arg max_U f_i(x*, ξ).
   │       if f_i(x*, ξ*) > y*_i then
   │          Add ξ* to U^(k+1).
   │       end if
   │    end for
   │    for all j = 1, 2, ..., J do
   │       Determine ξ* ∈ arg max_U F_j(x*, ξ).
   │       if F_j(x*, ξ*) > 0 then
   │          Add ξ* to U^(k+1).
   │       end if
   │    end for
   │ end for
   └
   k := k + 1
until U^(k) = U^(k+1).
return proj_X(X^(k−1)*): representative set of point-based minmax robust (extreme supported) efficient solutions to MOP(U).
return proj_Y(X^(k−1)*): set of point-based minmax robust (extreme supported) nondominated points of MOP(U).
return U^FINAL := U^k: set of worst-case scenarios.
```

CONCLUSION

In this chapter we have shown that the optimization-pessimization method can be employed to solve the multi-objective robust counterpart $(\text{MORC}(\mathcal{U}))$ of an uncertain multi-objective optimization problem $(\text{MOP}(\mathcal{U}))$ by solving the multi-objective robust counterpart $(\text{MORC}(\mathcal{U}'))$ for a sequence of smaller–and, most notably, finite–uncertainty sets $\mathcal{U}' \subseteq \mathcal{U}$ (optimization step) and solving deterministic single-objective maximization problems $(\text{Pess}^{\text{pb}}(x))$. We showed that the method can be utilized to determine *point-based minmax robust extreme supported efficient* as well as *point-based minmax robust efficient* solutions (see Section 3.2) for problems with uncertainty in the objective and/or in the constraints (see Section 3.3). Under some additional assumptions, most notably constraints and objectives being quasi-convex in ξ for fixed x and $\mathcal{U}$ being compact, this method is finite (see Theorems 3.10, 3.15 and 3.17). Even if optimization-pessimization does not terminate, it provides us with an upper and a lower bound for the Pareto frontier of MORC($\mathcal{U}$) (see Lemma 3.4).

However, the optimization step so far has been treated as a blackbox. We have shown that the optimization problem in each iteration (MORC($\mathcal{U}'$)) can be reformulated as a deterministic multi-objective minimization problem with a finite number, specifically $(p + J) \cdot |\mathcal{U}'|$, of additional constraints (MORC$^{\mathrm{BN}}$($\mathcal{U}'$)). Such problems can be solved using a wide range of multi-objective optimization methods. In particular, scalarization methods (see [PS84; Eic09; BQT19]) such as weighted sum or epsilon constraints, can be considered. Enumeration methods like Benson's (see [Ben98]) or Bökler and Mutzel's method (see [BM15]) are other possible choices.

In the following two chapters, we restrict ourselves to bi-objective robust optimization and employ the dichotomic search method to solve the multi-objective (then bi-objective) robust counterpart.

4 DICHOTOMIC SEARCH FOR BI-OBJECTIVE MINMAX OPTIMIZATION

In order to solve the multi-objective robust counterpart–either for the full uncertainty set $\mathcal{U}$ or for a reduced uncertainty set $\mathcal{U}' \subseteq \mathcal{U}$–of an uncertain bi-objective optimization problem, we turn to dichotomic search. We start this chapter with a review of the dichotomic search method from the literature in Section 4.1 before then turning to bi-objective mixed-integer linear programs with minmax objective, which is a special case of the multi-objective robust counterpart (MORC($\mathcal{U}$)). In Section 4.2 we first formally define the problem to be solved (BRO($\mathcal{U}$)) and study its properties (Section 4.2.1). We then generalize dichotomic search from bi-objective mixed-integer linear optimization to bi-objective mixed-integer linear *minmax* optimization, i.e., to problems of type BRO($\mathcal{U}$) (Section 4.2.2).

4.1 DICHOTOMIC SEARCH FOR BI-OBJECTIVE MIXED-INTEGER LINEAR OPTIMIZATION

In this section we consider a special case of a multi-objective optimization problem (MOP), namely bi-objective linear mixed-integer optimization problems

$$\min_{x \in \mathcal{X}} \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}. \tag{BOP}$$

The feasible set $\mathcal{X} \subseteq \mathbb{R}^n$ is a polyhedron intersected with $\mathbb{Z}^k \times \mathbb{R}^{n-k}$. The objective functions $g_1, g_2 \colon \mathcal{X} \to \mathbb{R}$ are (affinely) linear functions.

A well-known approach to solve such problems is *dichotomic search*, formulated in Algorithm 4.1. The method has first been published by Cohon in 1978 (see [Coh78]) and Aneja and Nair in 1979 (see [AN79]) for more specific problem classes and is now part of multi-objective folklore and sometimes also known as *Aneja and Nair's bicriteria method* (see, e.g., [UT94]) or–named after all three authors–*CAN method* (see, e.g., [ÖK10]). Most frequently, it is used to solve bi-objective *linear* problems. However, it can also be applied to bi-objective *mixed-integer linear* problems where it determines all extreme supported efficient nondominated points Y^* and a representative set of extreme supported nondominated solutions X^*. Dichotomic search takes advantage of the fact that in $\mathbb{R}^2$ sorting nondominated solutions with respect to their first coordinates is the same as reverse sorting by the second coordinate, i.e., for two nondominated solutions $y^l, y^r \in \mathcal{Y} \subseteq \mathbb{R}^2$, $y_1^l < y_1^r$ implies $y_2^l > y_2^r$. The idea is to start with the lexicographically optimal solutions and then in each step find a supported non-dominated point "between" two given supported non-dominated points. The method proceeds iteratively until all extreme supported nondominated points are identified. Algorithmically, first, the lexicographic optimal solutions x^L, x^R for BOP are computed. After that, in each iteration, a tuple (y^l, y^r) of two points known to be supported nondominated is taken and $\lambda = (y^l_2 - y^r_2, y^r_1 - y^l_1)$, corresponding to the slope $\frac{y^l_2 - y^r_2}{y^r_1 - y^l_1}$ of the line segment from y^l to y^r, is chosen. Solving the corresponding weighted-sum (scalarized) problem

$$\min_{x \in \mathcal{X}} \lambda^\top g(x)$$

either finds a new supported nondominated point between y^l and y^r or certifies that there is no such point. The algorithm terminates when all extreme supported nondominated points–each with a corresponding extreme supported efficient solution–have been discovered. It might also find supported nondominated points which are not extreme supported nondominated, but these can be easily identified and removed.

Algorithm 4.1 Dichotomic search

Require: Bi-objective mixed-integer linear optimization problem (BOP).
Ensure: Feasible set $\mathcal{X}$ is a polyhedron intersected with $\mathbb{R}^{n-k} \times \mathbb{Z}^k$ for some $k \in \{0, \dots, n\}$.

```
Initialize 𝓛 := ∅. {𝓛 will contain list of tuple images (y^l, y^r) satisfying y^l_1 < y^r_1, y^l_2 > y^r_2}
                                                        Determine lexicographic solutions
Compute ϵ_1 := min_{x∈𝒳} g_1(x).
Determine x^L ∈ arg min_{x∈𝒳} {g_2(x) : g_1(x) ≤ ϵ_1}.
Set y^L := g(x^L).
Compute ϵ_2 := min_{x∈𝒳} g_2(x).
Determine x^R ∈ arg min_{x∈𝒳} {g_1(x) : g_2(x) ≤ ϵ_2}.
Set y^R := g(x^R).
if y^L = y^R then
    STOP. Only one nondominated image found.
    return Y* = {y^L}, X* = {x^L}.
else
    Y* = {y^L, y^R}, X* = {x^L, x^R}, 𝓛 = {(y^L, y^R)}.
end if
while 𝓛 ≠ ∅ do
    Remove element (y^l, y^r) from 𝓛.
    Compute λ := (y^l_2 − y^r_2, y^r_1 − y^l_1).
                                          Solve weighted-sum problem for weights λ
    Determine x* ∈ arg min_{x∈𝒳} λ^⊤ g(x).
    Set y* := g(x*).
    if λ^⊤ y* < λ^⊤ y^l. then
        Add y* to Y*, add x* to X*.
        Add (y^l, y*), (y*, y^r) to 𝓛.
    end if
end while
return Y*: contains all extreme supported nondominated points.
return X*: contains a representative set of extreme supported efficient solutions.
```

Finiteness and correctness of Algorithm 4.1 follow from the considerations above, which are derived from the literature (see [PKL19; ÖK10]) and are stated in the following lemma. The lemma is valid if BOP satisfies the ideal point property (2.1). This is a slight generalization to [ÖK10] who assumed that BOP is bounded by the origin, i.e., $g_i(x)$, $i = 1, 2$, are non-negative for all $x \in \mathcal{X}$.

Lemma 4.1 (e.g., [ÖK10])**.** *Let a bi-objective mixed-integer linear problem* (BOP) *be given, i.e., a problem with*

- *affinely linear objectives g_1, g_2 and*
- *a feasible set $\mathcal{X}$ that is a polyhedron intersected with $\mathbb{R}^{n-k} \times \mathbb{Z}^k$.*
- *Furthermore, let the ideal point property* (2.1) *hold for BOP.*

Then Algorithm 4.1 returns a set Y^* containing all extreme supported nondominated points and a set X^* containing a representative set of extreme supported efficient solutions after $2|Y^*| - 3$ *iterations* (*lines 15–22*) *if* $|Y^*| > 2$ *and zero iterations if* $|Y^*| = 1$.

It is known that in the case of bi-objective linear optimization problems, the set of all extreme supported nondominated points and a representative set of extreme supported efficient solutions can be used to construct all nondominated points and a representative set of efficient solutions, respectively. We will show a related result in Lemma 5.1 in Chapter 5.

4.2 DICHOTOMIC SEARCH FOR BI-OBJECTIVE MIXED-INTEGER LINEAR MINMAX OPTIMIZATION

In order to solve the multi-objective robust counterpart (MORC($\mathcal{U}$)) of an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)) with $p = 2$ objectives we will now extend dichotomic search to bi-objective mixed-integer linear *minmax* optimization problems. To this end, we will first introduce the problem at hand and study its properties in Section 4.2.1. Subsequently, in Section 4.2.2 we show how the problem can be solved with dichotomic search.

4.2.1 *The problem to be solved: A bi-objective mixed-integer linear robust optimization problem (BRO)*

We consider uncertain bi-objective optimization problems (MOP($\mathcal{U}$) with $p = 2$) with uncertainty only in the objectives. Their robust counterpart (MORC($\mathcal{U}$)) takes the following form:

$$\min_{x \in \mathcal{X}} \begin{pmatrix} \sup_{\xi \in \mathcal{U}} f_1(x, \xi) \\ \sup_{\xi \in \mathcal{U}} f_2(x, \xi) \end{pmatrix}. \tag{BRO($\mathcal{U}$)}$$

Our goal is to determine the Pareto frontier and the associated efficient solutions of BRO($\mathcal{U}$).

For BRO($\mathcal{U}$) we always assume the following:

- (BRO-1) The feasible set takes the form $\mathcal{X} = P \cap (\mathbb{Z}^k \times \mathbb{R}^{n-k})$ where $P \subseteq \mathbb{R}^n$ is a polytope and $0 \leq k \leq n$.

- (BRO-2) The uncertainty set $\mathcal{U} \subseteq \mathbb{R}^m$ is a polytope or finite set.
- (BRO-3) The objective functions $f_1, f_2 \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ are affinely linear in x for every fixed $\xi \in \mathcal{U}$ and quasi-convex and continuous in ξ for every fixed $x \in \mathcal{X}$.

Under the latter two assumptions, (BRO-2) and (BRO-3), the supremum in the definition of BRO($\mathcal{U}$) is always attained and we can write maximum instead, i.e., $f_i^{\mathcal{U}}(x) = \max_{\xi \in \mathcal{U}} f_i(x, \xi)$ for $x \in \mathcal{X}$, $i = 1, 2$. The third condition (BRO-3) guarantees that $f_i \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is jointly continuous in (x, ξ) (see, e.g., [KD69]). Finally, the feasible set $\mathcal{X}$ determines the type of the problem at hand: For $k = 0$ the problem is a bi-objective (pure) *linear* minmax problem, for $k = n$ the problem is a bi-objective *integer linear* minmax problem and for $1 \leq k < n$ we have a bi-objective *mixed-integer linear* minmax problem.

DOMINATION AND IDEAL POINT PROPERTY FOR MULTI-OBJECTIVE ROBUST OPTIMIZATION PROBLEMS. We conclude this section by discussing under which assumptions the ideal point property (2.1) and the domination property (2.2) are satisfied for *robust* multi-objective problems (MORC($\mathcal{U}$)), i.e., for the case that the objective functions of a multi-objective optimization problem (MOP) are given as $g = f^{\mathcal{U}}$. For a discussion of the domination property (2.2) in the context of multi-objective robust optimization, we refer to Schöbel and Zhou-Kangas (see [SZK21]).

Lemma 4.2. *Let either*

(*i*) *$\mathcal{X}$ and $\mathcal{U}$ both be finite,*

(*ii*) *$\mathcal{X}$ be finite, $\mathcal{U}$ compact and $f(x, \cdot)$ continuous in $\mathcal{U}$ for every fixed $x \in \mathcal{X}$,*

(*iii*) *$\mathcal{U}$ be finite, $\mathcal{X}$ compact and $f(\cdot, \xi)$ continuous in $\mathcal{X}$ for every fixed $\xi \in \mathcal{U}$, or*

(*iv*) *$\mathcal{X}$ and $\mathcal{U}$ be compact and $f(\cdot, \cdot)$ jointly continuous in $\mathcal{X}$ and $\mathcal{U}$.*

Then both, the ideal point property (2.1) *and the domination property* (2.2) *are satisfied for a robust multi-objective optimization problem MORC*($\mathcal{U}$).

Proof. We set $g_i(x) := \sup_{\xi \in \mathcal{U}} f_i(x, \xi)$, $i = 1, 2, \ldots, p$, and distinguish two cases:
First, consider the case where $\mathcal{X}$ is finite. Then due to Lemma 2.5, ideal point property (2.1) and domination property (2.2) hold if $g_i(x) = \sup_{\xi \in \mathcal{U}} f_i(x, \xi)$ exists for all $x \in \mathcal{X}$. This is the case since either $\mathcal{U}$ is finite or $\mathcal{U}$ is compact and $f_i(x, \cdot)$ continuous for every fixed $x \in \mathcal{X}$.

Second, assume $\mathcal{X}$ is compact (but not finite). In this case, Lemma 2.5 requires that $g_i(x)$ is continuous. This holds since either

ad (iii) $\mathcal{U}$ is finite, hence $g_i(x)$, $i = 1, 2$, is continuous as the maximum of a finite set of continuous functions $f_i(\cdot, \xi)$, $\xi \in \mathcal{U}$, or

ad (iv) $\mathcal{U}$ is compact and $f_i(\cdot, \cdot)$, $i = 1, 2$, is jointly continuous in (x, ξ) and hence again, $g_i(x)$ is continuous.

□

We conclude that ideal point property (2.1) and domination property (2.2) hold for BRO($\mathcal{U}$).

Corollary 4.3. *BRO($\mathcal{U}$) satisfies both ideal point property* (2.1) *and domination property* (2.2).

Proof. By the assumptions of BRO($\mathcal{U}$), $\mathcal{X}$ and $\mathcal{U}$ are both compact (see (BRO-1) and (BRO-2)) and $f_i : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is jointly continuous in (x, ξ) for $i = 1, 2$ (BRO-3). Lemma 4.2 hence gives the result. □

4.2.2 *Solving BRO with dichotomic search*

Our goal is to apply dichotomic search to a bi-objective mixed-integer linear robust optimization problem (BRO($\mathcal{U}$)). Recall that the functions f_1 and f_2 are affinely linear in x for every fixed $\xi \in \mathcal{U}$ and $\mathcal{X} = P \cap (\mathbb{Z}^k \times \mathbb{R}^{n-k})$ for a polyhedron P and $0 \leq k \leq n$, i.e., without the supremum BRO($\mathcal{U}$) would satisfy the requirements of Lemma 4.1. However, since the functions $f_i^{\mathcal{U}} \colon \mathcal{X} \to \mathbb{R}$, $x \mapsto \sup_{\xi \in \mathcal{U}} f_i(x, \xi)$, $i = 1, 2$, are not (affinely) linear, we aim to transform BRO($\mathcal{U}$) to a bi-objective mixed-integer linear optimization problem (BOP) for which we can apply dichotomic search.

We proceed in two steps. The first step is to transform BRO($\mathcal{U}$) to its bottleneck version, i.e., to

$$\min \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : \begin{array}{ll} x \in \mathcal{X}, & f_1(x, \xi) \leq y_1 \quad \forall \xi \in \mathcal{U}, \\ y \in \mathbb{R}^2, & f_2(x, \xi) \leq y_2 \quad \forall \xi \in \mathcal{U} \end{array} \right\}. \tag{BRO$^{\mathrm{BN}}(\mathcal{U})$}$$

Recall Lemma 3.16 which regarded the relationship of a multi-objective robust counterpart (MORC($\mathcal{U}$)) and its reformulation (MORC$^{\mathrm{BN}}$($\mathcal{U}$)). Since the problems BRO($\mathcal{U}$) and BRO$^{\mathrm{BN}}$($\mathcal{U}$) are special cases of MORC($\mathcal{U}$) and MORC$^{\mathrm{BN}}$($\mathcal{U}$), respectively, Lemma 3.16 is applicable.

We can now turn our focus to the reformulation BRO$^{\mathrm{BN}}$($\mathcal{U}$) which, unlike BRO($\mathcal{U}$), has affinely linear objective functions. However, to ensure its feasible set meets the requirements of Lemma 4.1, we additionally need that the feasible set of BRO$^{\mathrm{BN}}$($\mathcal{U}$) is a polyhedron intersected with $\mathbb{Z}^k \times \mathbb{R}^{n-k}$ for $0 \leq k \leq n$. Then Algorithm 4.1 can be applied to BRO$^{\mathrm{BN}}$($\mathcal{U}$) and determines all its extreme supported nondominated points and a representative set of extreme supported efficient solutions. In the following lemma we show more, namely that we do not need the bottleneck version but can apply Algorithm 4.1 directly to BRO($\mathcal{U}$) to receive the extreme supported nondominated points and a representative set of extreme supported efficient solutions of BRO($\mathcal{U}$) if the set $\mathcal{U}$ of scenarios is finite.

Lemma 4.4. *Let a problem of type BRO($\mathcal{U}$) be given and let (BRO-1) and (BRO-3) hold. We assume that $\mathcal{U}$ is non-empty and finite.*

Then Algorithm 4.1 applied to BRO($\mathcal{U}$) returns a set Y^* containing all extreme supported nondominated points and a set X^* containing a representative set of extreme supported efficient solutions after $2|Y^*| - 3$ iterations (lines 14-23) if $|Y^*| > 2$ and zero iterations if $|Y^*| = 1$.

Proof. The proof is in two parts: First, we show that dichotomic search *applied to the bottleneck version* BRO$^{\mathrm{BN}}$($\mathcal{U}$) of BRO($\mathcal{U}$) returns a representative set of extreme supported efficient solutions and the set of all extreme supported nondominated points for the (non-bottleneck) problem BRO($\mathcal{U}$). Second, we show that applying dichotomic search directly to BRO($\mathcal{U}$) yields the exact same solutions as applying it to the bottleneck version BRO$^{\mathrm{BN}}$($\mathcal{U}$).

For the first part we use that the bottleneck version of the problem, i.e., $\mathrm{BRO}^{\mathrm{BN}}(\mathcal{U})$, meets the requirements of Lemma 4.1: We use the assumptions made for $\mathrm{BRO}(\mathcal{U})$ and see that $\mathrm{BRO}^{\mathrm{BN}}(\mathcal{U})$ is a bi-objective problem with two affinely linear objectives y_1 and y_2. For the feasible set note that the original feasible set $\mathcal{X}$ of $\mathrm{BRO}(\mathcal{U})$ is given as $\mathcal{X} = P \cap (\mathbb{R}^{n-k} \times \mathbb{Z}^k)$. Since we add two variables and two affinely linear constraints for each scenario from the finite set $\mathcal{U}$ (see part (i) of Lemma 3.16), also the resulting feasible set for $\mathrm{BRO}^{\mathrm{BN}}(\mathcal{U})$ can be written as $P' \cap (\mathbb{R}^{n'-k} \times \mathbb{Z}^k)$ with a new polyhedron P' and dimension $n' = n + 2$. Furthermore, (2.1) holds due to Corollary 4.3.

Thus, due to Lemma 4.1, dichotomic search (Algorithm 4.1) can be applied and a set Y^*_{BN} containing all extreme supported nondominated points and a representative set of extreme supported efficient solutions X^*_{BN} for $\mathrm{BRO}^{\mathrm{BN}}(\mathcal{U})$ are determined after $2|Y^*| - 3$ iterations (lines 14-23) if $|Y^*| > 2$ and zero iterations if $|Y^*| = 1$.

Lemma 3.16 (iv) shows that $X^*_{BN} = \{(x, f^{\mathcal{U}}(x)) : x \in X\}$ for some set $X \subseteq \mathcal{X}$ which is a representative set of extreme supported efficient solutions of $\mathrm{BRO}(\mathcal{U})$.

For the second part, note that the difference between using $\mathrm{BRO}(\mathcal{U})$ or $\mathrm{BRO}^{\mathrm{BN}}(\mathcal{U})$ concerns lines 2, 3, 5, 6, and each iteration of line 17 of Algorithm 4.1. However, there is no difference between applying these steps to $\mathrm{BRO}(\mathcal{U})$ and $\mathrm{BRO}^{\mathrm{BN}}(\mathcal{U})$: the feasible set of the latter problem is of higher dimension than the feasible set of the former but their outcomes in the objective space $\mathbb{R}^2$ coincide (see Lemma 3.16) and only those are needed for subsequent computations. □

The lemma above justifies the application of dichotomic search to our problem of interest $\mathrm{BRO}(\mathcal{U})$ if $\mathcal{U}$ is finite. However, in $\mathrm{BRO}(\mathcal{U})$ the set $\mathcal{U}$ may be a polytope. On the other hand, in (BRO-3) we made the additional – and thus far unnecessary – assumption that $f_i(x, \cdot) : \mathcal{U} \to \mathbb{R}$, $i = 1, 2$ are quasi-convex. Utilizing this additional requirement, we now show that Lemma 4.4 is still valid if $\mathcal{U}$ is a polytope instead of a finite set.

Lemma 4.5. *Let a problem of type $\mathrm{BRO}(\mathcal{U})$ be given and let (BRO-1) and (BRO-3) hold. We assume that $\mathcal{U}$ is a polytope.*

Then Algorithm 4.1 applied to $\mathrm{BRO}(\mathcal{U})$ returns a set Y^* containing all extreme supported nondominated points and a set X^* containing a representative set of extreme supported efficient solutions after $2|Y^*| - 3$ iterations (lines 14-23) if $|Y^*| > 2$ and zero iterations if $|Y^*| = 1$.

Proof. If $\mathcal{U}$ is a polytope it has a finite number of (not necessarily known) extreme points $\xi_1, \dots, \xi_l$. Since the functions $f_1(x, \cdot), f_2(x, \cdot) : \mathrm{conv}(\mathcal{U}) \to \mathbb{R}$, $x \in \mathcal{X}$ are quasi-convex, according to [EIS14, Theorem 5.9], $\mathrm{BRO}(\mathcal{U})$ and $\mathrm{BRO}(\{\xi_1, \dots, \xi_l\})$ are equivalent since their objective functions $f^{\mathcal{U}}$ and $f^{\{\xi_1, \dots, \xi_l\}}$ are the same.

Lemma 4.4 justifies that we can apply Algorithm 4.1 to $\mathrm{BRO}(\{\xi_1, \dots, \xi_l\})$ and get all extreme supported nondominated points and a representative set of extreme supported efficient solutions of $\mathrm{BRO}(\{\xi_1, \dots, \xi_l\})$ and hence also of $\mathrm{BRO}(\mathcal{U})$ in $2|Y^*| - 3$ iterations if $|Y^*| > 2$ and zero iterations if $|Y^*| = 1$. This, however, requires that $\xi_1, \dots, \xi_l$ are known. Since finding the vertices of a given polytope, known as *vertex enumeration*, is a hard problem (see [Kha+09]), we apply Algorithm 4.1 directly to $\mathrm{BRO}(\mathcal{U})$ without using the extreme points of $\mathcal{U}$. Luckily, this can be done by using the equivalence of $\mathrm{BRO}(\{\xi_1, \dots, \xi_l\})$ and $\mathrm{BRO}(\mathcal{U})$ once more:

Namely, we replace $\mathrm{BRO}(\{\xi_1, \dots, \xi_l\})$ by $\mathrm{BRO}(\mathcal{U})$ whenever it occurs in Algorithm 4.1, i.e, in Steps 2,3,6,7 and in Step 17 and note that it does not change any result. Summarizing, we can also apply Algorithm 4.1 directly to $\mathrm{BRO}(\mathcal{U})$. □

CONCLUSION

This chapter dealt with the bi-objective minmax problem BRO($\mathcal{U}$) for which we have shown that it can be solved with dichotomic search. However, in each iteration of dichotomic search the minmax problem

$$\min_{x\in\mathcal{X}} \left\{ \lambda_1 \underbrace{\max_{\xi\in\mathcal{U}} f_1(x,\xi)}_{=f_1^{\mathcal{U}}(x)=g_1(x)} + \lambda_2 \underbrace{\max_{\xi\in\mathcal{U}} f_2(x,\xi)}_{=f_2^{\mathcal{U}}(x)=g_2(x)} \right\} \tag{BRO($\mathcal{U},\lambda$)}$$

has to be solved. How to proceed with this task is addressed in the next chapter.

5 ALGORITHMS FOR ROBUST BI-OBJECTIVE OPTIMIZATION

In Sections 3.2 and 4.2 algorithms known from (deterministic) bi-objective and (single-objective) robust optimization, respectively, have been generalized. However, in each iteration of the proposed dichotomic search method (Algorithm 4.1, Lemma 4.5) a robust problem has to be solved and, similarly, in each iteration of the proposed optimization-pessimization method (Algorithm 3.3, Theorem 3.15) a multi-objective problem has to be solved. So far, we treated these steps as if they were performed by an oracle.

In this section, we put these steps into concrete terms and, in doing so, present algorithms designed to solve uncertain bi-objective problems, more specifically, the problem BRO($\mathcal{U}$) as defined in Chapter 4. Throughout this section, we always assume that the assumptions of BRO($\mathcal{U}$), i.e., (BRO-1), (BRO-2), and (BRO-3) (see Page 45), hold.

Specifically, three different approaches to find efficient solutions to BRO($\mathcal{U}$), i.e., point-based minmax robust efficient solutions to an uncertain bi-objective optimization problem satisfying (BRO-1)-(BRO-3), are presented:

- A robust optimizer's approach (ROA): We view the problem BRO($\mathcal{U}$) primarily as a *robust* optimization problem – just with the added difficulty that it has two objective functions – and, consequently, apply a method from robust optimization, namely the generalized optimization-pessimization method (Algorithm 3.3), to the problem BRO($\mathcal{U}$). The subproblem to be solved in each iteration is a *bi-objective* problem BRO($\mathcal{U}'$) with a small uncertainty set $\mathcal{U}' \subseteq \mathcal{U}$, which we tackle by the generalized version of dichotomic search (Algorithm 4.1). This algorithm is presented in Section 5.1.

- A multi-objective optimizer's approach (MOA): We view the problem BRO($\mathcal{U}$) primarily as a *bi-objective* optimization problem – with the added difficulty that we aim to find a *robust* solution and the objective functions, thus, contain a maximum – and, consequently, apply a method from bi-objective optimization, namely the generalized version of dichotomic search (Algorithm 4.1) to the problem BRO($\mathcal{U}$). The subproblem to be solved in each iteration is a single-objective but *uncertain* problem $P(\mathcal{U},\lambda)$ BRO($\mathcal{U},\lambda$), which we tackle by the optimization-pessimization method (Algorithm 3.1). This algorithm is presented in Section 5.2.

- A multi-objective optimizer's approach for bilinear problems using dualization (DA): As in the aforementioned approach, we take the multi-objective optimizer's perspective and apply the generalized version of dichotomic search (Algorithm 4.1) to the problem BRO($\mathcal{U}$). The subproblem BRO($\mathcal{U},\lambda$) is directly solved through a reformulation in each iteration. This algorithm is presented in Section 5.3.

Finally, in Section 5.4, results of computational experiments on Algorithms 5.1 to 5.3 are presented.

Before we present the algorithms, we investigate the relationship between efficient and extreme supported efficient solutions for the specific problem BRO($\mathcal{U}$).

RELATIONSHIP BETWEEN EFFICIENT AND EXTREME SUPPORTED EFFICIENT SOLUTIONS OF BRO. Algorithms 5.1 to 5.3 each determine all extreme supported nondominated points and a corresponding representative set of extreme supported efficient solutions for BRO($\mathcal{U}$). The following lemma shows that these sets can be used to determine *all* nondominated points and a representative set for all efficient solutions of BRO($\mathcal{U}$).

Lemma 5.1. *Let BRO($\mathcal{U}$) be given and let $\mathcal{X}$ be a polytope. Further, let $\mathcal{Y}_{\text{ESN}}$, $|\mathcal{Y}_{\text{ESN}}| < \infty$, be its set of nondominated extreme supported points and $\mathcal{X}_{\text{ESE}}$ a representative set of extreme supported efficient solutions. Let $\mathcal{X}_{\text{ESE}} = \{x^{(1)}, x^{(2)}, \dots, x^{(n)}\}$, $\mathcal{Y}_{\text{ESN}} = \{y^{(1)}, y^{(2)}, \dots, y^{(n)}\}$, $y_1^{(1)} < y_1^{(2)} < \dots < y_1^{(n)}$ and $f^{\mathcal{U}}(x^{(i)}) = y^{(i)}$ for $i = 1, 2, \dots, n$. Then*

$$\mathcal{X}^* := \bigcup_{i=1,2,\dots,n-1} \left\{\lambda x^{(i)} + (1-\lambda)x^{(i+1)} \colon \lambda \in [0,1]\right\}$$

is a representative set (of efficient solutions) and

$$\mathcal{Y}^* := \bigcup_{i=1,2,\dots,n-1} \left\{\lambda y^{(i)} + (1-\lambda)y^{(i+1)} \colon \lambda \in [0,1]\right\}$$

is the set of nondominated points of BRO($\mathcal{U}$).

Proof. Let $\bar{x} \in \mathcal{X}^*$. Then $\bar{x} = \lambda x^{(i)} + (1-\lambda)x^{(i+1)}$ for some $i = 1, 2, \dots, n-1$, $\lambda \in [0,1]$, and

$$\begin{aligned}
\bar{y} := f^{\mathcal{U}}(\bar{x}) &= \max_{\xi \in \mathcal{U}} f(\lambda x^{(i)} + (1-\lambda)x^{(i+1)}, \xi) \\
&\overset{\text{(BRO-3)}}{=} \max_{\xi \in \mathcal{U}} \left\{\lambda f(x^{(i)}, \xi) + (1-\lambda) f(x^{(i+1)}, \xi)\right\} \\
&\leq \max_{\xi \in \mathcal{U}} \lambda f(x^{(i)}, \xi) + \max_{\xi \in \mathcal{U}} (1-\lambda) f(x^{(i+1)}, \xi) \\
&= \lambda f^{\mathcal{U}}(x^{(i)}) + (1-\lambda) f^{\mathcal{U}}(x^{(i+1)}) \\
&= \lambda y^{(i)} + (1-\lambda) y^{(i+1)}.
\end{aligned}$$

However, since by Lemma 3.12 we have $\mathcal{Y} \subseteq \text{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^2_{\geq}$ and since $\{\lambda y^{(i)} + (1-\lambda)y^{(i+1)}\}$ is a facet of $\text{conv}(\mathcal{Y}_{\text{ESN}})$, there is no $y \in \mathcal{Y}$ with $y \lneq \lambda y^{(i)} + (1-\lambda)y^{(i+1)}$. Thus, we have $\bar{y} = \lambda y^{(i)} + (1-\lambda)y^{(i+1)}$ and $\bar{y}$ is nondominated. This shows that the solutions in $\mathcal{X}^*$ are efficient and the points in $\mathcal{Y}^*$ are nondominated.

It remains to be shown that all nondominated points are included in $\mathcal{Y}^*$. This, however, follows directly from the fact that, by Lemma 3.12, $\mathcal{Y} \subseteq \text{conv}(\mathcal{Y}_{\text{ESN}}) + \mathbb{R}^2_{\geq}$. $\square$

5.1 A ROBUST OPTIMIZER'S APPROACH

The robust optimizer's approach is based on the idea of applying the generalization of optimization-pessimization (Algorithm 3.3). In the k-th iteration a representative set of extreme supported efficient solutions to BRO($\mathcal{U}^{(k)}$) has to be determined. For this purpose in Algorithm 5.1 we employ dichotomic search for robust bi-objective linear mixed-integer optimization problems as shown possible in Section 4.2.

Algorithm 5.1 Robust optimizer's approach (ROA)

Require: Bi-objective mixed-integer linear robust optimization problem (BRO($\mathcal{U}$)).
Require: Finite initial set $\mathcal{U}^{(0)} \subseteq \mathcal{U}$.
Ensure: Feasible set $\mathcal{X}$ is a polyhedron intersected with $\mathbb{R}^{n-k} \times \mathbb{Z}^k$ for some $k \in \{0, \dots, n\}$.
Ensure: $\mathcal{U}$ finite or $\mathcal{U}$ a polytope and $f_i(x, \cdot)$, $i = 1, 2, \dots, p$ continuous and quasi-convex.
Ensure: ideal point property (2.1) and domination property (2.2) hold for $P(\mathcal{U})$ and for $P(\mathcal{U}')$ for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$.

repeat
  Set $\mathcal{U}^{(k+1)} := \mathcal{U}^{(k)}$.

  Optimization
  Call dichotomic search (Algorithm 4.1) for BRO($\mathcal{U}^{(k)}$) to determine representative set for extreme supported efficient solutions $X^{(k)*}$ and representative set for extreme supported nondominated points $Y^{(k)*}$.

  Pessimization
  for all $x^* \in X^{(k)*}$ **do**
    for all $i = 1, 2$ **do**
      Determine one $\xi^* \in \arg\max_{\mathcal{U}} f_i(x^*, \xi)$.
      Add ξ^* to $\mathcal{U}^{(k+1)}$.
    end for
  end for

  $k := k + 1$
until $f^{\mathcal{U}}(x^*) = f^{\mathcal{U}^{(k-1)}}(x^*)$ for all $x^* \in X^{(k-1)*}$.
return $X^{(k-1)*}$: representative set of extreme supported efficient solutions of BRO($\mathcal{U}$).
return $Y^{(k-1)*}$: set of extreme supported nondominated points of BRO($\mathcal{U}$).
return $\mathcal{U}^{\mathrm{FINAL}} := \mathcal{U}^{(k)}$: set of worst-case scenarios.

Note that Algorithm 5.1 is just Algorithm 3.3 with the optimization step performed by dichtomic search (Algorithm 4.1). Consequently, the requirements correspond to those of Algorithm 3.3 and Algorithm 4.1 as formulated in Theorem 3.15 and Lemma 4.5, respectively. This is stated in the following lemma.

Lemma 5.2. *Let BRO($\mathcal{U}$) be given.*

(*i*) *Let $\mathcal{U}$ be finite. Then Algorithm 5.1 returns a representative set of extreme supported efficient solutions to BRO($\mathcal{U}$) in at most $|\mathcal{U}|$ iterations.*

(*ii*) *Let $\mathcal{U}$ be a polytope or finite and $f_i(x, \cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$, $i = 1, 2, \dots, p$, be continuous and quasi-convex. Then Algorithm 5.1 returns a representative set of extreme supported efficient solutions to BRO($\mathcal{U}$) in at most k iterations (where k is the number of extreme points of $\mathcal{U}$) if we choose an algorithm for the pessimization problem which always finds an extreme point of $\mathcal{U}$.*

Proof. By Corollary 4.3, BRO($\mathcal{U}$) satisfies ideal point property (2.1) and domination property (2.2). Algorithm 5.1 is the same as Algorithm 3.3, but for $p = 2$ and with dichotomic search (Algorithm 4.1) specified in the optimization step. Lemma 4.5 justifies that dichotomic search works correctly for BRO($\mathcal{U}$). Consequently, we may use dichotomic search in line 4 of Algorithm 3.3. Under ideal point property (2.1) and domination property (2.2) for BRO($\mathcal{U}$) and BRO($\mathcal{U}'$) for all finite sets $\mathcal{U}' \subseteq \mathcal{U}$ Theorem 3.15 gives us correctness of Algorithm 3.3 and hence also of Algorithm 5.1. □

Note that if Algorithm 5.1 is stopped before the stopping criterion in line 11 is met, the sets $\{f^{\mathcal{U}^{(k-1)}}(x) \colon x \in \mathcal{X}^{(k-1)*}\}$ and $\{f^{\mathcal{U}}(x) \colon x \in X^{(k-1)*}\}$ provide lower and upper bounds with respect to the lower set less order, as we have shown in Lemma 3.4. Using convex combinations of subsequent points in these sets like we did in Lemma 5.1 for Y^*, we obtain bounds on the region in which the Pareto frontier Y^* will lie. In this sense, Algorithm 5.1 can be used as an approximation algorithm for BRO($\mathcal{U}$).

5.2 A MULTI-OBJECTIVE OPTIMIZER'S APPROACH

The multi-objective optimizer's approach is based on the idea of applying dichotomic search (Algorithm 4.1) as introduced in Section 4.1 directly to BRO($\mathcal{U}$). In each iteration of dichotomic search, we have to solve the scalarized weighted-sum problem

$$\min_{x \in \mathcal{X}} \lambda_1 f_1^{\mathcal{U}}(x) + \lambda_2 f_2^{\mathcal{U}}(x) + \cdots + \lambda_p f_p^{\mathcal{U}}(x) \tag{P($\mathcal{U}$,λ)}$$

for $p = 2$ and given weights $\lambda \in \mathbb{R}^p_{\geq}$. In order to do this, we utilize optimization-pessimization for single-objective robust optimization as reviewed in Section 3.1: We solve a sequence of problems P($\mathcal{U}^{(0)}$,λ), P($\mathcal{U}^{(1)}$,λ), P($\mathcal{U}^{(2)}$,λ), ..., P($\mathcal{U}^{(k)}$,λ) until it is guaranteed that P($\mathcal{U}^{(k)}$,λ) and P($\mathcal{U}$,λ) share a representative set of extreme supported minmax robust efficient solutions. As in Chapter 3 we exploit the fact that for finite sets $\mathcal{U}'$ a problem P($\mathcal{U}'$,λ) is easier to solve than P($\mathcal{U}$,λ) as it can be written as a problem with finitely many constraints. For solving the scalarization we assumed an oracle in Algorithm 4.1. Now we want to be more specific. We first reformulate problem P($\mathcal{U}$,λ) such that we can apply optimization-pessimization (see Section 3.1) for its solution. This is done in the next lemma.

Lemma 5.3. *Let $\lambda \in \mathbb{R}^2_{\geq}$ be fixed. Then P($\mathcal{U}$,λ) can be transformed to*

$$\min_{x \in \mathcal{X}} \sup_{\bar{\xi} \in \bar{\mathcal{U}}} \bar{f}_\lambda(x, \bar{\xi}), \tag{$\bar{\text{P}}$($\mathcal{U}$,λ)}$$

i.e., a problem of type RC($\mathcal{U}$) as introduced on Page 8, for $\bar{\mathcal{U}} := \times_{i=1,2,\ldots,p} \mathcal{U}$, $\bar{\xi} := (\xi_1, \xi_2, \ldots, \xi_p)$ and $\bar{f}_\lambda(x, \bar{\xi}) := \sum_{i=1}^p \lambda_i f_i(x, \xi_i)$.

Proof. We reformulate P($\mathcal{U}$,λ) as follows:

$$\begin{aligned}
\min_{x \in \mathcal{X}} \left\{\lambda_1 f_1^{\mathcal{U}}(x) + \lambda_2 f_2^{\mathcal{U}}(x) + \cdots + \lambda_p f_p^{\mathcal{U}}(x)\right\} &= \min_{x \in \mathcal{X}} \left\{\sum_{i=1}^p \lambda_i \sup_{\xi \in \mathcal{U}} f_i(x, \xi)\right\} \\
&= \min_{x \in \mathcal{X}} \sup_{(\xi_1, \xi_2, \ldots, \xi_p) \in \mathcal{U}^p} \left\{\sum_{i=1}^p \lambda_i f_i(x, \xi_i)\right\} \\
&= \min_{x \in \mathcal{X}} \sup_{\bar{\xi} \in \bar{\mathcal{U}}} \bar{f}_\lambda(x, \bar{\xi}).
\end{aligned}$$

□

Lemma 5.3 shows that $\mathrm{P}(\mathcal{U},\lambda)$ can be solved by solving a single-objective robust optimization problem $\overline{\mathrm{P}}(\mathcal{U},\lambda)$, i.e., a problem of type $\mathrm{RC}(\mathcal{U})$. For the special case $p = 2$ this implies that instead of $\mathrm{BRO}(\mathcal{U},\lambda)$ the problem

$$\min_{x\in\mathcal{X}} \sup \left\{\lambda_1 f_1(x,\xi_1) + \lambda_2 f_2(x,\xi_2) \colon (\xi_1,\xi_2)^\top \in \mathcal{U}^2\right\} \qquad (\overline{\mathrm{BRO}}(\mathcal{U},\lambda))$$

can be solved.

Algorithm 5.2 describes a *basic version* of the multi-objective optimizer's approach. Its correctness is shown in the following lemma.

Algorithm 5.2 Multi-objective optimizer's approach (MOA)

Require: Bi-objective mixed-integer linear robust optimization problem ($\mathrm{BRO}(\mathcal{U})$).
Require: Finite initial set $\mathcal{U}^{(0)} \subseteq \mathcal{U}$.
Ensure: Feasible set $\mathcal{X}$ is a polyhedron intersected with $\mathbb{R}^{n-k} \times \mathbb{Z}^k$ for some $k \in \{0,\dots,n\}$.
Ensure: $\mathcal{U}$ finite or $\mathcal{U}$ a polytope and $f_i(x,\cdot)$, $i = 1,2,\dots,p$, continuous and quasi-convex.
Ensure: ideal point property (2.1) and domination property (2.2) hold for $P(\mathcal{U})$ and for $P(\mathcal{U}')$ for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$.

Initialize $\mathcal{L} := \emptyset$ $\{\mathcal{L}$ will contain list of tuple images (y^l, y^r) satisfying $y_1^l < y_1^r, y_2^l > y_2^r\}$

Determine lexicographic solutions

Call optimization-pessimization (Algorithm 3.1) on $\min_{x\in\mathcal{X}} f_1^{\mathcal{U}}(x)$ with initial set $\mathcal{U}^{(0)}$ to determine ϵ_1, $\mathcal{U}^{\mathrm{FINAL}}$, and ξ^{WC}.
Call optimization-pessimization (Algorithm 3.1) on $\min_{x\in\mathcal{X}}\{f_2^{\mathcal{U}}(x) \colon \max_{\xi\in\mathcal{U}} f_1(x,\xi) \le \epsilon_1\}$ with initial set $\mathcal{U}^{\mathrm{FINAL}}$ to determine optimal solution x^L.
Set $y^L := f^{\mathcal{U}}(x^L)$.
Call optimization-pessimization (Algorithm 3.1) on $\min_{x\in\mathcal{X}} f_2^{\mathcal{U}}(x)$ with initial set $\mathcal{U}^{(0)}$ to determine ϵ_2, $\mathcal{U}^{\mathrm{FINAL}}$, and ξ^{WC}.
Call optimization-pessimization (Algorithm 3.1) on $\min_{x\in\mathcal{X}}\{f_1^{\mathcal{U}}(x) \colon \max_{\xi\in\mathcal{U}} f_2(x,\xi) \le \epsilon_2\}$ with initial set $\mathcal{U}^{\mathrm{FINAL}}$ to determine optimal solution x^R..
Set $y^R := f^{\mathcal{U}}(x^R)$.

if $y^L = y^R$ **then**
    STOP. Only one nondominated image found.
    return $Y^* = \{y^L\}, X^* = \{x^L\}$.
else
    $Y^* = \{y^L, y^R\}, X^* = \{x^L, x^R\}, \mathcal{L} = \{(y^L, y^R)\}$.
end if
while $\mathcal{L} \neq \emptyset$ **do**
    Remove element (y^l, y^r) from $\mathcal{L}$.
    Compute $\lambda := (y_2^l - y_2^r, y_1^r - y_1^l)$.

Solve weighted-sum problem $\bar{P}(\mathcal{U},\lambda)$

    Call optimization-pessimization (Algorithm 3.1) on $\min_{x\in\mathcal{X}} \bar{f}_\lambda(x)$ with initial set $\mathcal{U}^{(0)}$ to determine optimal solution x^*.
    Set $y^* := \bar{f}_\lambda(x^*)$.

    if $\lambda^\top y^* \neq \lambda^\top y^l$ **then**
        Add y^* to Y^*, add x^* to X^*.
        Add $(y^l, y^*), (y^*, y^r)$ to $\mathcal{L}$
    end if
end while
return X^*: representative set of extreme supported efficient solutions of $\mathrm{BRO}(\mathcal{U})$.
return Y^*: set of extreme supported nondominated points of $\mathrm{BRO}(\mathcal{U})$.

Lemma 5.4. *Let BRO($\mathcal{U}$) be given. Then Algorithm 5.2 returns a representative set of extreme supported efficient solutions to BRO($\mathcal{U}$) after a finite number of iterations.*

Proof. Algorithm 5.2 is dichotomic search (Algorithm 4.1), where we specified the algorithm for steps 2-3, 5-6, 17-18, namely by solving the scalarized (single-objective) problem BRO($\mathcal{U}, \lambda$) by optimization-pessimization (Algorithm 3.1) in each iteration. Since BRO($\mathcal{U}$) meets the requirements of Lemma 4.4 (in case $\mathcal{U}$ is finite) or Lemma 4.5 (in case $\mathcal{U}$ is a polytope), Algorithm 4.1 returns a representative set of extreme supported efficient solutions and a set of extreme supported nondominated solutions after finitely many iterations.

It remains to show that lines 2-3, 5-6 and 17-18 in Algorithm 5.2 are correct specifications of the same lines of Algorithm 4.1.

For lines 2 and 5 this is straightforward as the problems

$$\min_{x \in \mathcal{X}} f_i^{\mathcal{U}}(x), \tag{5.1}$$

$i = 1, 2$, are single-objective robust optimization problems. Since $\mathcal{U}$ is a polytope or finite and $f_i(x, \cdot) : \mathcal{U} \to \mathbb{R}, i = 1, 2$, are continuous and quasi-convex, Lemma 3.1 can be applied and optimization-pessimization (Algorithm 3.1) solves (5.1).

The problems in lines 3 and 6 are also of type (5.1) only with one additional constraint, i.e., with feasible sets

$$\mathcal{X}_j' := \{x \in \mathcal{X} : \max_{\xi \in \mathcal{U}} f_j(x, \xi) \leq \epsilon_j\}, j = 2, 1.$$

In lines 17-18 of Algorithm 4.1 the problem BRO($\mathcal{U}, \lambda$) is to be solved for some $\lambda \in \mathbb{R}^p_{\gneq}$. By Lemma 5.3 this can be done by solving $\overline{\text{BRO}}(\mathcal{U}, \lambda)$ instead which is done in lines 17-18 of Algorithm 5.2. Since continuity and quasi-convexity of $\bar{f}$ are inherited from continuity and quasi-convexity of f_1 and f_2, Theorem 3.15 can be applied and optimization-pessimization returns a robust solution to BRO($\mathcal{U}, \lambda$). □

WARM START MODIFICATIONS. In the basic version of Algorithm 5.2 the cutting plane method is initialized with $\mathcal{U}^{(0)}$ in lines 2,5 and 17. A possible modification of Algorithm 5.2 is to start the cutting plane method with a larger set $\mathcal{U}'$ that includes some additional scenarios that have been generated in previous iterations but that is still guaranteed to be finite. This way, previously generated cutting planes are not forgotten. Specifically, we propose two modifications:

- Variant 1 (MOA-ws1): We initialize optimization-pessimization with all previously generated scenarios. To this end, we modify lines 5 and 17 such that the cutting plane method is initialized with $\mathcal{U}^{\text{FINAL}}$. This way, $\mathcal{U}^{\text{FINAL}}$ grows monotonically.

- Variant 2 (MOA-ws2): We initialize the cutting plane method with those scenarios that turned out to be worst-case scenarios for a previously found optimal solution x. After lines 2-3, 5-6, and 17-18 the worst-case scenarios ξ^{WC} for x^L, x^R, and x^*, respectively, are added to $\mathcal{U}^{(0)}$ and the set grows monotonically, but is much smaller than the set in Variant 1.

As Algorithm 5.2 and Lemma 5.4 above only assume finiteness of the initial uncertainty set, their validity is not affected by these modifications.

5.3 A MULTI-OBJECTIVE OPTIMIZER'S APPROACH FOR BILINEAR PROBLEMS

In this section, we confine ourselves to a special class of problems: bi-objective mixed-integer linear robust optimization problems (BRO($\mathcal{U}$)) which satisfy not only (BRO-1), (BRO-2), and (BRO-3) as before, but also the following additional properties:

- the uncertainty set $\mathcal{U}$ is as a polytope $\mathcal{U} = \{\xi \in \mathbb{R}^m : C\xi \leq d\}$ for a matrix $C \in \mathbb{R}^{m' \times m}$ and a vector $d \in \mathbb{R}^{m'}$, and
- the functions $f_1, f_2 : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ are not only linear in x for every fixed $\xi \in \mathcal{U}$ as required in (BRO-3), but also linear in ξ for each x, i.e., they are *bilinear* functions.

The following lemma shows that under these assumptions a bi-objective mixed-integer linear *minmax* optimization problem can be reformulated as a bi-objective mixed-integer linear *minimization* problem.

Lemma 5.5. *We consider the uncertain multi-objective problem*

$$\min_{x \in \mathcal{X}} f^{\mathcal{U}}(x). \tag{MORC($\mathcal{U}$) revisited}$$

Let the uncertainty set be a non-empty polytope $\mathcal{U} = \{\xi \in \mathbb{R}^m : C\xi \leq d\}$, with $C \in \mathbb{R}^{m' \times m}$, $d \in \mathbb{R}^{m'}$, and let the functions $f_i(x, \xi)$, $i = 1, 2, \ldots, p$, be linear in ξ for each x, i.e.,

$$f_i(x, \xi) := \left[\hat{c}_i(x)\right]^\top \xi \tag{5.2}$$

for functions $\hat{c}_i : \mathcal{X} \to \mathbb{R}^m$, $i = 1, 2, \ldots, p$.

Let $\lambda \in \mathbb{R}^p_{\geq}$. Then a solution $x^* \in \mathcal{X}$ is optimal for the scalarized problem

$$\min_{x \in \mathcal{X}} \lambda^\top f^{\mathcal{U}}(x) \tag{P($\mathcal{U}$,λ) revisited}$$

if and only if there exist $\pi^{(1)*}, \ldots, \pi^{(p)*} \in \mathbb{R}^{m'}$ such that $(x^*, \pi^{(1)*}, \ldots, \pi^{(p)*})$ is optimal for

$$\min_{x \in \mathcal{X}, \pi^{(1)}, \ldots, \pi^{(p)} \in \mathbb{R}^{m'}} \left\{ d^\top \sum_{i=1}^{p} \lambda_i \pi^{(i)} : C^\top \pi^{(i)} = \hat{c}_i(x), \pi^{(i)} \geq 0, i = 1, 2, \ldots, p \right\}. \tag{D($\mathcal{U}$,λ)}$$

More precisely, let x be fixed and let $(\pi^{(1)*}, \ldots, \pi^{(p)*})$ be an optimal solution to

$$\min_{\pi^{(1)}, \ldots, \pi^{(p)} \in \mathbb{R}^{m'}} \left\{ d^\top \sum_{i=1}^{p} \lambda_i \pi^{(i)} : C^\top \pi^{(i)} = \hat{c}_i(x), \pi^{(i)} \geq 0, i = 1, 2, \ldots, p \right\} \tag{5.3}$$

with optimal objective function value z. Then $z = \lambda^\top f^{\mathcal{U}}(x)$ and for all $i = 1, 2, \ldots, p$ with $\lambda_i > 0$

$$d^\top \pi^{(i)*} = \max_{\xi \in \mathcal{U}} \left[\hat{c}_i(x)\right]^\top \xi. \tag{5.4}$$

Proof. First note that $\mathrm{D}(\mathcal{U}, \lambda)$ is equivalent to

$$\min_{x \in \mathcal{X}} \min_{\pi^{(1)}, \dots, \pi^{(p)} \in \mathbb{R}^{m'}} \left\{ d^\top \sum_{i=1}^{p} \lambda_i \pi^{(i)} \colon C^\top \pi^{(i)} = \hat{c}_i(x), \pi^{(i)} \geq 0, i = 1, 2, \dots, p \right\},$$

which can be interpreted as optimization problem

$$\min_{x \in \mathcal{X}} g_\lambda(x) \tag{$\overline{\mathrm{D}}(\mathcal{U}, \lambda)$}$$

with

$$g_\lambda(x) := \min_{\pi^{(1)}, \dots, \pi^{(p)} \in \mathbb{R}^{m'}} \left\{ d^\top \sum_{i=1}^{p} \lambda_i \pi^{(i)} \colon C^\top \pi^{(i)} = \hat{c}_i(x), \pi^{(i)} \geq 0, i = 1, 2, \dots, p \right\}. \tag{5.5}$$

We now need to show that the objective functions and the feasible sets of $\mathrm{P}(\mathcal{U}, \lambda)$ and $\overline{\mathrm{D}}(\mathcal{U}, \lambda)$ coincide. Specifically, we show

$$\lambda^\top f^{\mathcal{U}}(x) = g_\lambda(x) \tag{5.6}$$

for all $x \in \mathcal{X}$.

We first note that $\mathcal{U}$ is a compact set, hence for any fixed $x \in \mathcal{X}$ and any $i = 1, 2, \dots, p$ the linear program

$$\max \left\{ \left[\hat{c}_i(x) \right]^\top \xi \colon \xi \in \mathcal{U} \right\} \tag{5.7}$$

has an optimal solution. Using that $\mathcal{U} = \{\xi \in \mathbb{R}^m \colon C\xi \leq d\}$ we hence get from linear programming duality for $i = 1, 2, \dots, p$ and fixed $x \in \mathcal{X}$ that

$$\max_{\xi \in \mathbb{R}^m} \left\{ \left[\hat{c}_i(x) \right]^\top \xi \colon C\xi \leq d, \right\} = \min_{\pi^{(i)} \geq 0, \pi^{(i)} \in \mathbb{R}^{m'}} \left\{ d^\top \pi^{(i)} \colon C^\top \pi^{(i)} = \hat{c}_i(x) \right\}, \tag{5.8}$$

i.e., for any fixed $x \in \mathcal{X}$ and $i = 1, \dots, p$, an optimal solution $\pi^{(i)*}$ to the right hand side satisfies

$$d^\top \pi^{(i)*} = \max_{\xi \in \mathcal{U}} \left[\hat{c}_i(x) \right]^\top \xi,$$

which shows (5.4). We can now derive

$$\begin{aligned}
\lambda^\top f^{\mathcal{U}}(x) &= \sum_{i=1}^{p} \lambda_i \underbrace{\max_{\xi \in \mathbb{R}^m} \left\{ \left[\hat{c}_i(x) \right]^\top \xi \colon C\xi \leq d \right\}}_{= f_i^{\mathcal{U}}(x)} \\
&\overset{(5.8)}{=} \sum_{i=1}^{p} \lambda_i \min_{\pi^{(i)} \in \mathbb{R}^{m'}} \left\{ d^\top \pi^{(i)} \colon C^\top \pi = \hat{c}_i(x), \pi^{(i)} \geq 0 \right\} \\
&= \min_{\pi^{(1)}, \dots, \pi^{(p)} \in \mathbb{R}^{m'}} \left\{ d^\top \sum_{i=1}^{p} \lambda_i \pi^{(i)} \colon C^\top \pi^{(i)} = \hat{c}_i(x), \pi^{(i)} \geq 0 \right\},
\end{aligned}$$

where the last step puts the single optimization problems together into a bigger (still separable) problem. Thus, for any fixed x the objective values of $\mathrm{P}(\mathcal{U},\lambda)$ and $\overline{\mathrm{D}}(\mathcal{U},\lambda)$ coincide and hence x is optimal to $\mathrm{P}(\mathcal{U},\lambda)$ if and only if it is optimal to $\overline{\mathrm{D}}(\mathcal{U},\lambda)$. □

As in Section 5.2, we apply dichotomic search to MORC($\mathcal{U}$) and solve $\mathrm{P}(\mathcal{U},\lambda)$ for different weights $\lambda \in \mathbb{R}^p_{\geq}$. However, unlike in Section 5.2 we do not solve MORC($\mathcal{U}$) with an iterative approach, but adopt the other approach described by [GYD15]: reformulation of $\mathrm{P}(\mathcal{U},\lambda)$. More specifically, we weaponize Lemma 5.5 and choose to solve

$$z^*(\mathcal{U},\lambda) := \min_{x\in\mathcal{X},\pi^{(1)},\dots,\pi^{(p)}\in\mathbb{R}^{m'}} \left\{ d^\top \sum_{i=1}^{p} \lambda_i \pi^{(i)} \colon C^\top \pi^{(i)} = \hat{c}_i(x), \pi^{(i)} \geq 0, i = 1,2,\dots,p \right\} \tag{$\mathrm{D}(\mathcal{U},\lambda)$ revisited}$$

instead of $\mathrm{P}(\mathcal{U},\lambda)$.

This leads to Algorithm 5.3. The following lemma shows correctness.

Lemma 5.6. *Let BRO($\mathcal{U}$) with an nonempty polytope explicitly stated as $\mathcal{U} = \{\xi \in \mathbb{R}^m \colon C\xi \leq d\}$ for a matrix $C \in \mathbb{R}^{m'\times m}$ and a vector $d \in \mathbb{R}^{m'}$ as uncertainty set and bilinear functions $f_1, f_2 \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ be given. Then Algorithm 5.3 solves BRO($\mathcal{U}$).*

Proof. The assumptions of Lemma 4.5 are satisfied since (BRO-1) and (BRO-3) hold and $\mathcal{U}$ is a polytope. Hence dichotomic search can be applied to $\min_{x\in\mathcal{X}} f^{\mathcal{U}}(x)$. It remains to be shown that $\mathrm{P}(\mathcal{U},\lambda)$ is solved correctly throughout the algorithm. Lemma 5.5 shows that robust solutions of $\mathrm{P}(\mathcal{U},\lambda)$ can be determined by solving $\mathrm{D}(\mathcal{U},\lambda)$ (lines 2, 5, 17) and the corresponding point on the Pareto front can be computed by $y_i^* = d^\top \pi^{(i)*}$ (see line 16). □

5.4 NUMERICAL RESULTS

We implemented Algorithms 5.1 to 5.3 and conducted computational experiments, results of which are presented in this section.

5.4.1 *Problem structure*

We restricted ourselves to a certain class of bi-objective optimization problems: The objective functions $f_i \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}^2$, $i = 1,2$, were assumed to be bilinear, and the feasible set and uncertainty set were polytopes or discrete sets. More specifically, we considered problems

$$\left\{ \min_{x\in\mathcal{X}} \begin{pmatrix} \max_{\xi\in\mathcal{U}} \xi^\top M_1 x \\ \max_{\xi\in\mathcal{U}} \xi^\top M_2 x \end{pmatrix} \right\}_{\xi\in\mathcal{U}}$$

with

$$\begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^n \colon L^x \leq x_i \leq U^x, Ax \leq b\} \text{ or } & \mathcal{X} &= \{x \in \mathbb{Z}^n \colon L^x \leq x_i \leq U^x, Ax \leq b\}, \\ \mathcal{U} &= \{\xi \in \mathbb{R}^m \colon L^\xi \leq \xi_i \leq U^\xi, C\xi \leq d\} \text{ or } & \mathcal{U} &= \{\xi \in \mathbb{Z}^m \colon L^\xi \leq \xi_i \leq U^\xi, C\xi \leq d\}. \end{aligned}$$

Algorithm 5.3 Multi-objective optimizer's approach with dualization (DA)

Require: Bi-objective mixed-integer linear robust optimization problem (BRO($\mathcal{U}$)).
Require: Finite initial set $\mathcal{U}^{(0)} \subseteq \mathcal{U}$.
Ensure: Feasible set $\mathcal{X}$ is a polyhedron intersected with $\mathbb{R}^{n-k} \times \mathbb{Z}^k$ for some $k \in \{0, \dots, n\}$.
Ensure: $\mathcal{U}$ a polytope and $f_i(x, \cdot)$, $i = 1, 2, \dots, p$ continuous and quasi-convex.
Ensure: ideal point property (2.1) and domination property (2.2) hold for $P(\mathcal{U})$.
Ensure: $f(x, \cdot) : \mathcal{U} \to \mathbb{R}^p$ linear

Initialize $\mathcal{L} := \emptyset$ {$\mathcal{L}$ will contain list of tuple images (y^l, y^r) satisfying $y_1^l < y_1^r, y_2^l > y_2^r$}

Determine lexicographic solutions

Determine optimal objective value ϵ_1 of D($\mathcal{U}, \lambda$) for $\lambda = (1, 0)^\top$
Determine $x^L \in \arg\min_{\mathcal{X}} \{g_{(0,1)}(x) : g_{(1,0)}(x) \le \epsilon_1\}$
Set $y^L := (\epsilon_1, g_{(0,1)}(x^L))^\top$
Determine optimal objective value ϵ_2 of D($\mathcal{U}, \lambda$) for $\lambda = (0,)^\top$
Determine $x^R \in \arg\min_{\mathcal{X}} \{g_{(1,0)}(x) : g_{(0,1)}(x) \le \epsilon_2\}$
Set $y^R := (g_{(1,0)}(x^L), \epsilon_2)^\top$

if $y^L = y^R$ **then**
  STOP. Only one nondominated image found
  return $Y^* = \{y^L\}, X^* = \{x^L\}$
else
  $Y^* = \{y^L, y^R\}, X^* = \{x^L, x^R\}, \mathcal{L} = \{(y^L, y^R)\}$
end if
while $\mathcal{L} \neq \emptyset$ **do**
  Remove element (y^l, y^r) from $\mathcal{L}$
  Compute $\lambda := (y_2^l - y_2^r, y_1^r - y_1^l)$.

  Solve D($\mathcal{U}, \lambda$)

  Find one optimal solution $(x^*, \pi^{(1)}, \dots, \pi^{(k)})$ for D($\mathcal{U}, \lambda$).
  Set $y_i^* = d^\top \pi^{(i)*}$ for $i = 1, 2$.

  if $\lambda^\top y^* \neq \lambda^\top y^l$ **then**
    Add y^* to Y^*, add x^* to X^*.
    Add $(y^l, y^*), (y^*, y^r)$ to $\mathcal{L}$
  end if
end while
return X^*: representative set of extreme supported efficient solutions of BRO($\mathcal{U}$).
return Y^*: set of extreme supported nondominated points of BRO($\mathcal{U}$).

The lower and upper bounds L^x, U^x, L^ξ, U^ξ are added to ensure that $\mathcal{X}$ and $\mathcal{U}$ are subsets of the boxes $[L^x, U^x]^n$ and $[L^\xi, U^\xi]^m$, respectively, and, thus, are bounded as it is required. We chose $L^x = 1, U^x = 200, L^\xi = -100$ and $U^\xi = 100$. By doing so we avoid problems where both $0_n \in \mathcal{X}$ and $0_m \in \mathrm{int}(\mathcal{U})$, since this would imply that $x = 0$ is a trivial minimizer of $f_i^{\mathcal{U}}(x) = \max_{\xi \in \mathcal{U}} \xi M_i x$, $i = 1, 2$.

GENERATING INSTANCES We created 100 instances of BRO($\mathcal{U}$) with $A \in \mathbb{Z}^{30 \times 5}$ and $C \in \mathbb{Z}^{30 \times 5}$. To obtain instances with smaller number of constraints, as used in our experiments, we removed constraints from these initial instances. This makes it easier to draw conclusions when comparing algorithm performance for different values of n' and m'. The entries of the matrices $A \in \mathbb{Z}^{n' \times n}$ and $C \in \mathbb{Z}^{m' \times m}$ as well as the entries of $M_1, M_2 \in \mathbb{Z}^{m \times n}$ determining the objective function are randomly and independently generated uniformly distributed integers in $\{-100, -99, \dots, 99, 100\}$.

Equally, $\tilde{b}_i$, $i = 1, 2, \dots, n'$, and $\tilde{d}_j$, $j = 1, 2, \dots, m'$, are randomly generated uniformly distributed integers in $\{50, 51, \dots, 99, 100\}$. We then set $\bar{x} := (100, 100, \dots, 100)^\top \in \mathbb{Z}^n$ and $\bar{\xi} := (0, 0, \dots, 0)^\top \in \mathbb{Z}^m$. Let A_i, $i = 1, 2, \dots, n'$, and C_j, $j = 1, 2, \dots, m'$, denote the rows of A and C. By setting the right hand-side coefficients $b_i := A_i^\top \bar{x} + \tilde{b}_i \|A_i\|_2$ for $i = 1, 2, \dots, n'$ and $d_j := C_j^\top \xi_0 + \tilde{d}_j \left\|C_j\right\|_2$ for $j = 1, 2, \dots, m'$, we guarantee that the spheres $\{x \in \mathbb{R}^n \colon \|x - \bar{x}\|_2 \leq 50\}$ and $\{\xi \in \mathbb{R}^m \colon \left\|\xi - \bar{\xi}\right\|_2 \leq 50\}$ are included in $\mathcal{X}$ and $\mathcal{U}$, respectively. See [CV14] for more on this.

IMPLEMENTATION We used C++ to implement our algorithms. Whenever a linear or integer optimization problem has to be solved, Gurobi 2.3 (see [Gur23]) is called (with default settings). We use Gurobi's capacity to provide solutions that are known to be basic solutions. The implementations were tested on a computer with 16 GB RAM, AMD Ryzen 5 PRO 2500U, 2.00 GHz.

5.4.2 *Evaluation of the algorithms*

In this section we evaluate the performance of the algorithms for instances of different types (polytopal and discrete sets $\mathcal{X}$ and $\mathcal{U}$) and different sizes by varying the number of considered constraints n' and m', respectively.

DISCRETE FEASIBLE SET AND DISCRETE UNCERTAINTY SET First, let us consider problems with a discrete feasible set and a discrete uncertainty set. For such instances, the robust optimizer's approach (ROA, Algorithm 5.1) and the multi-objective optimizer's approach (MOA, Algorithm 5.2) in its baseline version and with its two warm-start modifications are available. The dualization approach (DA, Algorithm 5.3) cannot solve such instances as it requires a polytope as uncertainty set.

Figure 5.1 shows the average running time of our algorithms. Each data point is the average over 100 instances with n' constraints on the feasible set. The number of variables for the feasible set n, the number of variables for the uncertainty set m and the number of constraints for the uncertainty set m' are all fixed and set at 5.

Independently of n', the robust optimizer's approach – where the uncertainty set $\mathcal{U}^{(k)}$ increases monotonously – is faster than the baseline version of the multi-objective optimizer's approach. However, the warm start modifications to the latter method turn out to

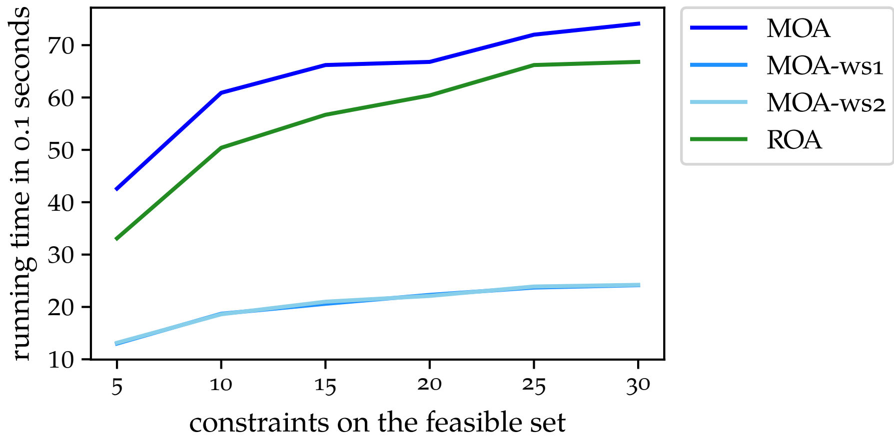


Figure 5.1: Average running time of our four algorithms for 100 instances as a function of n' with $n = m = m' = 5$ fixed, $\mathcal{X}$ and $\mathcal{U}$ discrete (the lines for MOA-ws1 and MOA-ws2 overlap and are hard to see)

be significant improvements over the baseline version: with those the multi-objective optimizer's approach performs faster. We see a clear increase in running time when going from 5 to 10 constraints for all tested methods, but above that point an increasing number of constraints does not seem to make the problem much harder to solve.

Figure 5.2 shows how the number of constraints in the definition of the uncertainty set $\mathcal{U}$ influences the running time.

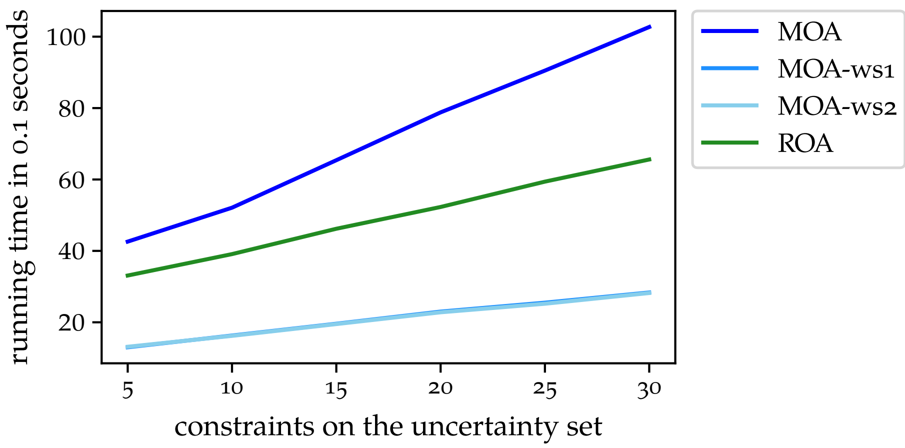


Figure 5.2: Average running time of our four algorithms for 100 instances as a function of m' with $n = m = n' = 5$ fixed, $\mathcal{X}$ and $\mathcal{U}$ discrete

We observe the same pattern: The modified warm-start versions of MOA are by far the fastest algorithms; ROA is still faster than the baseline version of MOA. Clearly, the problem gets harder the more constraints are necessary to describe $\mathcal{U}$. This leads us to conclude that the difficulty of the problem is rooted much more in the complexity of $\mathcal{U}$ than in the one of $\mathcal{X}$.

DISCRETE FEASIBLE SET AND POLYTOPAL UNCERTAINTY SET Now let us turn to problems with a polytope as uncertainty set. On those instances all of the algorithms we introduced can be used. This includes the dualization approach (DA), which is the only algorithm that does not use optimization-pessimization but instead solves the scalarized problem $P(\mathcal{U}, \lambda)$ for each weight λ directly (via the means of dualization of the inner problem).

Figures 5.3 and 5.4 show the average running time of our algorithms on the same instances as in Figures 5.1 and 5.2 – just with the integrality constraint for $\mathcal{U}$ dropped.

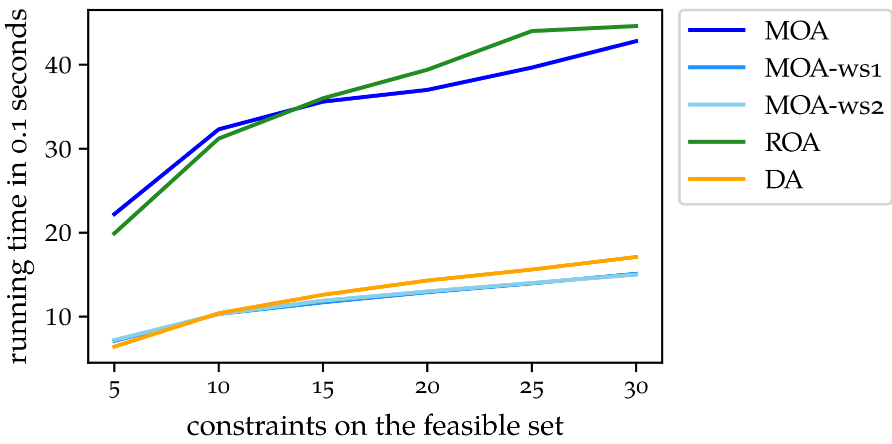


Figure 5.3: Average running time of our five algorithms for 100 instances as a function of n' with $n = m = m' = 5$ fixed, $\mathcal{X}$ discrete, $\mathcal{U}$ polytope

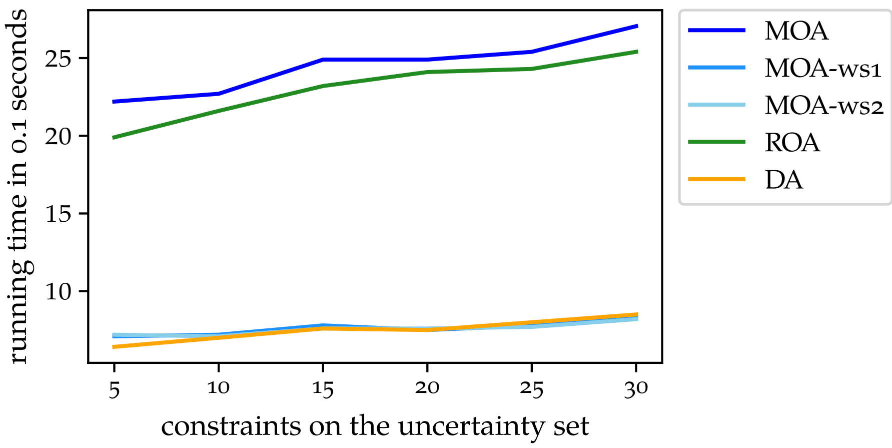


Figure 5.4: Average running time of our five algorithms for 100 instances as a function of m' with $n = m = n' = 5$ fixed, $\mathcal{X}$ discrete, $\mathcal{U}$ polytope

Our experiments show that for such instances DA is effective, but not noticeably better than the modified versions of MOA. The ranking of the other algorithms is essentially the same as before: The modified warm-start versions of MOA outperform ROA which is still faster than MOA's baseline version. Dropping the integrality constraint reduced the overall running time of all algorithms by about factor two. This is while the number of extreme supported nondominated points stayed roughly the same.

The apparent ranking of the proposed algorithms raises the question of whether this applies only on average over a larger number of instances, or if it also applies to each individual instance. For this we turn to Figure 5.5. In this figure we display the running time for the 5 different algorithms on the first 10 of the tested 100 instances. Including all tested instances here does not change the discussed findings, but decreases visibility, which is why we included only the results of ten instances.

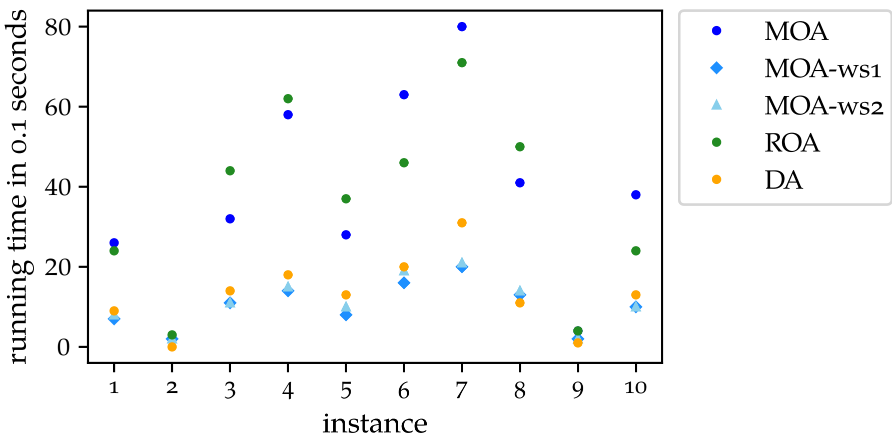


Figure 5.5: Running time of our five algorithms for 10 instances with $n = n' = m = 5, m' = 30$, $\mathcal{X}$ discrete, $\mathcal{U}$ polytope

Each of the ten columns in Figure 5.5 represents one instance (with $n = m = n' = 5, m' = 30$) on which we tested the algorithms. We can see that for all instances either DA or the warm-start modifications of MOA perform best and either ROA or the baseline version of MOA perform worst. The ranking of the algorithms is not the same for all instances.

To get a deeper understanding of this we turn to Figure 5.6.

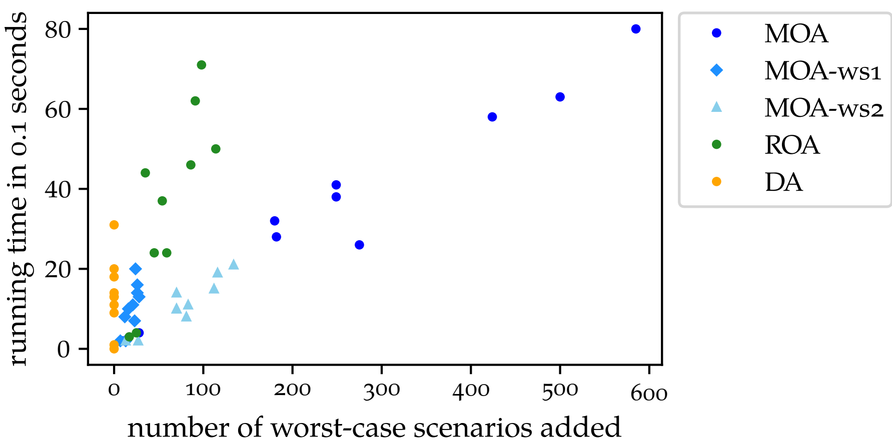


Figure 5.6: Running time vs. number of worst-case scenarios added for 10 instances with $n = n' = m = 5, m' = 30$, $\mathcal{X}$ discrete, $\mathcal{U}$ polytope

For the ROA and all three versions of MOA it shows the running time plotted against the number of times we add a worst-case scenario during the execution of the algorithms.

The strong correlation indicates that the number of pessimization steps decisively determines the overall time required. The two algorithms where the uncertainty set $\mathcal{U}^{(k)}$ grows monotonously, namely MOA-ws1 and ROA, have similarly high costs per added scenario. This can be explained by the fact that the resulting robust optimization problems are harder to solve due to the number of scenarios in $\mathcal{U}^{(k)}$. Vice versa, MOA and MOA-ws2 both "forget" scenarios. Consequently, they need to (re)add more scenarios, but the optimization problems are simpler. For them the ratio between runtime and added scenario is lower. This also explains why the warm-start modifications pay off: Apparently, the additional cost of starting with a larger scenario set $\mathcal{U}^{(k)}$ is more than offset by the less frequent need to execute the pessimization step.

EVALUATION FOR POLYTOPAL FEASIBLE SETS Additionally, we tested the algorithms on instances with feasible sets $\mathcal{X}$ that are polytopes. In this case DA is faster. Apart from that, the observations do not deviate significantly from the ones discussed in the previous paragraphs except that if $\mathcal{U}$ is a polytope too, DA is faster than MOA-ws1 and MOA-ws2 as can be seen in Figure 5.7.

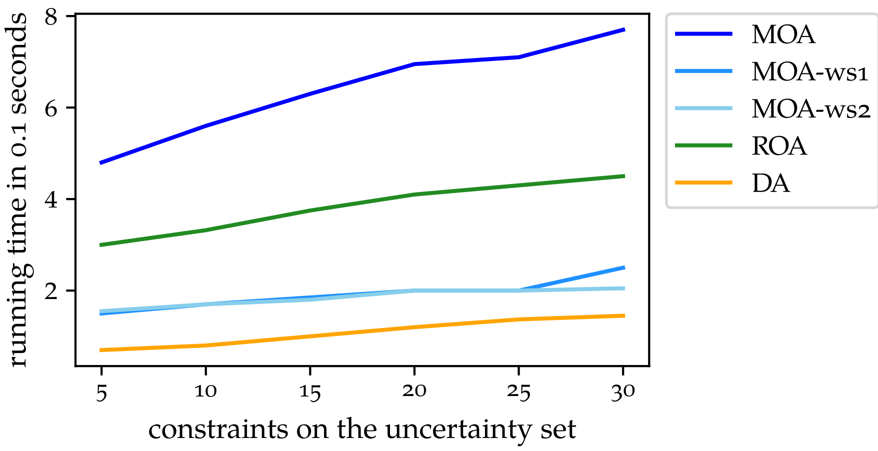


Figure 5.7: Average running time of our five algorithms for 100 instances as a function of m' with $n = m = n' = 5$ fixed, $\mathcal{X}$ and $\mathcal{U}$ polytopes

THE ALGORITHMS AS APPROXIMATION ALGORITHMS Lastly, we want to investigate how soon the algorithms provide a reasonable approximation of the Pareto front. For this we turn to Algorithm 5.1, which in the k-th iteration determines (via dichotomic search) all extreme supported nondominated points of BRO($\mathcal{U}^{(k)}$) and then determines the worst-case outcomes of those points under $\mathcal{U}$. Figure 5.8 shows for an instance with $n = m = n' = 5, m' = 30$ and $\mathcal{X}, \mathcal{U}$ both continuous the lower and upper bound determined in the second and fourth iteration and the robust solutions determined in the final 7th iteration. We can see that our method provides a good approximation to the Pareto front early on.

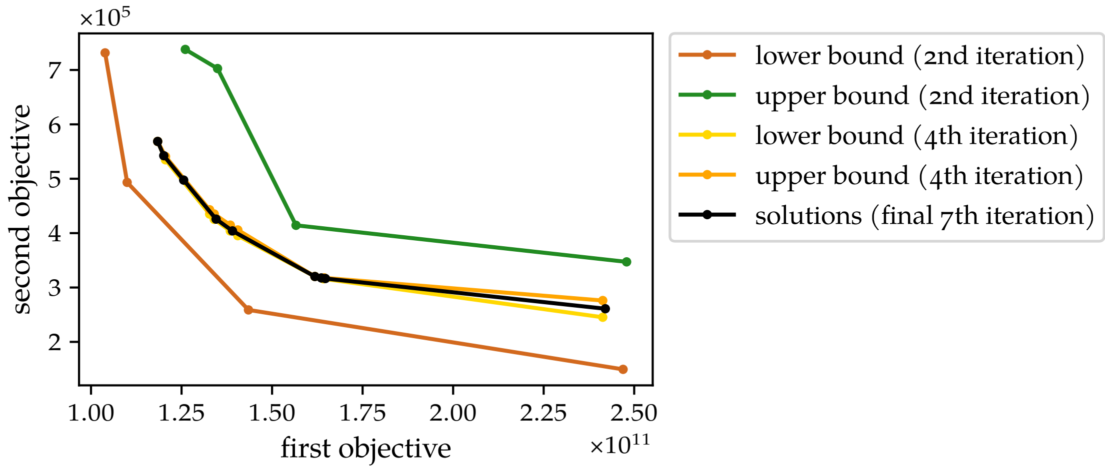


Figure 5.8: Lower bound and upper bound determined in the 2nd, 4th, and in the final (7th) iteration in an instance with $n = m = 10, n' = m' = 20$ and $\mathcal{X}, \mathcal{U}$ continuous

6 SET-BASED AND HULL-BASED MINMAX ROBUST EFFICIENCY

After having considered point-based minmax robust efficiecy so far, this chapter deals with hull-based and set-based minmax robust efficient solutions to uncertain multi-objective optimization problems ($\text{MOP}(\mathcal{U})$). We once again turn to the optimization-pessimization method (c.f. Chapter 3). In Section 6.1 we develop an algorithm that is capable of determining set-based and hull-based minmax robust efficient solutions. Subsequently, in Section 6.2 we show that the algorithms provides upper and lower bounds for the set of set-based and hull-based minmax robust solutions.

As in Sections 3.1, 3.2.2 and 3.3.2, we work under Assumption 1 (see Page 26).

6.1 OPTIMIZATION-PESSIMIZATION

REDUCTION OF THE UNCERTAINTY SET. We show how hull-based and set-based minmax robust efficient solutions can be found by solving $\text{MOP}(\mathcal{U})$ for a reduced uncertainty set $\mathcal{U}' \subseteq \mathcal{U}$. We state conditions under which a subset $\mathcal{U}' \subseteq \mathcal{U}$ already contains all *relevant* scenarios such that set-based minmax robust efficient solutions to $\text{MOP}(\mathcal{U})$ can be found by solving the problem $\text{MOP}(\mathcal{U}')$ instead.

In order to check whether a point-based minmax robust efficient solution to a problem $\text{MOP}(\mathcal{U}')$ is point-based minmax robust efficient to $\text{MOP}(\mathcal{U})$ as well, we had to solve a pessimization problem ($\text{Pess}^{\text{pb}}(x)$ on Page 23).

We will see that for hull-based and set-based minmax robust efficiency the pessimization problem for a fixed solution $x \in \mathcal{X}$ takes a different form, namely

$$\max_{\xi \in \mathcal{U}} \begin{pmatrix} f_1(x,\xi) \\ f_2(x,\xi) \\ \vdots \\ f_p(x,\xi) \end{pmatrix}. \qquad (\text{Pess}^{\text{sb}}(x))$$

Note that unlike the pessimization problem in Section 3.2 ($\text{Pess}^{\text{pb}}(x)$), the problem defined above is truly a multi-objective optimization problem. Yet, for fixed $x \in \mathcal{X}$ it is still deterministic.

We formulate the following conditions:

The set $\mathcal{U}'$ includes a rep. set of extreme supported efficient solutions to $\text{Pess}^{\text{sb}}(x)$. (Condition B)

The set $\mathcal{U}'$ includes a representative set of efficient solutions to $\text{Pess}^{\text{sb}}(x)$.
(Condition C)

We note that the formulated conditions are ordered with Condition C being the strictest and the previously introduced Condition A (c.f. Page 26) being the weakest: Since any representative set of efficient solutions includes a representative set of extreme supported efficient solutions, Condition C is stronger than Condition B. Likewise, a representative set of extreme supported efficient solutions to $\text{Pess}^{\text{sb}}(x)$ necessarily includes solutions that maximize the individual objectives $f_i(x, \cdot)$, $i = 1, 2, \ldots, p$. Thus, Condition B is stronger than Condition A.

The following lemma deals with the implications of Conditions B and C.

Lemma 6.1. *Let $x, \hat{x} \in \mathcal{X}$ be fixed and let $\mathcal{U}$ be compact. Furthermore let $\mathcal{U}' \subseteq \mathcal{U}$ and let Assumption 1 hold. Then if x satisfies Condition B,*

$$f_{\mathcal{U}}(x) \subseteq \operatorname{conv}\left(f_{\mathcal{U}'}(x)\right) - \mathbb{R}^p_{\geqq} \tag{6.1a}$$

follows and if $\hat{x}$ satisfies Condition C,

$$f_{\mathcal{U}}(\hat{x}) \subseteq f_{\mathcal{U}'}(\hat{x}) - \mathbb{R}^p_{\geqq} \tag{6.1b}$$

holds.

Proof. As usual, let $\mathcal{Y}$ denote the outcome set of $\text{Pess}^{\text{sb}}(x)$ and let $\mathcal{Y}_{\text{N}} \subseteq \mathcal{Y}$ and $\mathcal{Y}_{\text{ESN}} \subseteq \mathcal{Y}$ be the set of nondominated and extreme supported nondominated points, respectively. By the previously mentioned Lemma 8.19 (see Page 96) it follows that

$$\mathcal{Y} \subseteq \operatorname{conv}\left(\mathcal{Y}_{\text{ESN}}\right) - \mathbb{R}^p_{\geqq} \tag{6.2a}$$

and since compactness of $\mathcal{U}$ implies the domination property (see Lemma 2.5) we get

$$\mathcal{Y} \subseteq \mathcal{Y}_{\text{N}} - \mathbb{R}^p_{\geqq}. \tag{6.2b}$$

Note that the sign in front of $\mathbb{R}^p_{\geqq}$ is due to the fact that $\text{Pess}^{\text{sb}}(x)$ is a *maximization* problem. Condition B implies

$$\mathcal{U}' \supseteq \mathcal{R}_{\text{ESE}}, \tag{6.3a}$$

for some representative set of extreme supported efficient solutions $\mathcal{R}_{\text{ESE}}$, while Condition C implies

$$\mathcal{U}' \supseteq \mathcal{R}_{\text{E}} \tag{6.3b}$$

for some representative set of efficient solutions $\mathcal{R}_{\text{E}}$. Clearly, $f(x, \mathcal{R}_{\text{ESE}}) = \mathcal{Y}_{\text{ESN}}$ and $f(x, \mathcal{R}_{\text{E}}) = \mathcal{Y}_{\text{N}}$.

Under Condition B the claim follows from

$$\begin{aligned} f_{\mathcal{U}}(x) = f(x,\mathcal{U}) = \mathcal{Y} \overset{(6.2a)}{\subseteq} \operatorname{conv}\left(\mathcal{Y}_{\mathrm{ESN}}\right) - \mathbb{R}^p_{\geqq} &= \operatorname{conv}\left(f(x,\mathcal{R}_{\mathrm{ESE}})\right) - \mathbb{R}^p_{\geqq} \\ &\overset{(6.3a)}{\subseteq} \operatorname{conv}\left(f(x,\mathcal{U}')\right) - \mathbb{R}^p_{\geqq} \\ &= \operatorname{conv}\left(f_{\mathcal{U}'}(x)\right) - \mathbb{R}^p_{\geqq} \end{aligned} \tag{6.4a}$$

and under Condition C the claim follows from

$$\begin{aligned} f_{\mathcal{U}}(\hat{x}) = f(\hat{x},\mathcal{U}) = \mathcal{Y} \overset{(6.2b)}{\subseteq} \mathcal{Y}_{\mathrm{N}} - \mathbb{R}^p_{\geqq} &= f(\hat{x},\mathcal{R}_{\mathrm{E}}) - \mathbb{R}^p_{\geqq} \\ &\overset{(6.3b)}{\subseteq} f(\hat{x},\mathcal{U}') - \mathbb{R}^p_{\geqq} = f_{\mathcal{U}'}(\hat{x}) - \mathbb{R}^p_{\geqq}. \end{aligned} \tag{6.4b}$$

□

The intuition behind Condition B and Condition C is illustrated in the following example.

Example 6.2. Once again, consider the setting of Example 2.9. In Figures 6.1a, 6.1c and 6.1e the (deterministic) multi-objective maximation problems $\mathrm{Pess}^{\mathrm{sb}}(x)$ for $x = x_1, x_2, x_3$ are shown individually. Those solutions in $\mathcal{U} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ that are efficient are encircled. In Figures 6.1b, 6.1d and 6.1f the implication of Lemma 6.1 can be seen: If Condition C is satisfied, i.e., if all encircled scenarios are included in $\mathcal{U}'$, then (6.1b) holds, i.e., the outcomes of all scenarios, that is $f_{\mathcal{U}}(x_i) = \{f(x_i,\xi) : \xi \in \mathcal{U}\}$, are included in $f_{\mathcal{U}'}(x_i) - \mathbb{R}^p_{\geqq}$, $i = 1, 2, 3$.

Figure 6.2 shows the same for Condition B. Note the difference between Figure 6.2e and Figure 6.1e: Solution ξ_3 is *efficient* for $\mathrm{Pess}^{\mathrm{sb}}(x_3)$ (encircled in Figure 6.1e), but not *extreme supported efficient* for $\mathrm{Pess}^{\mathrm{sb}}(x_3)$ (not encircled in Figure 6.2e) since $f(x_3,\xi_3)$ is dominated by a convex combination of $f(x_3,\xi_2)$ and $f(x_3,\xi_4)$. Hence, all outcomes $f_{\mathcal{U}}(x_3)$ are included in $\operatorname{conv}(f_{\mathcal{U}'}(x_3)) - \mathbb{R}^p_{\geqq}$ for a reduced scenario set $\mathcal{U}'$ that contains just ξ_2 and ξ_4 but not necessarily ξ_3 as can be seen in Figure 6.2f.

Before we continue, we introduce the domination property with respect to set-based and hull-based minmax robust efficiency.

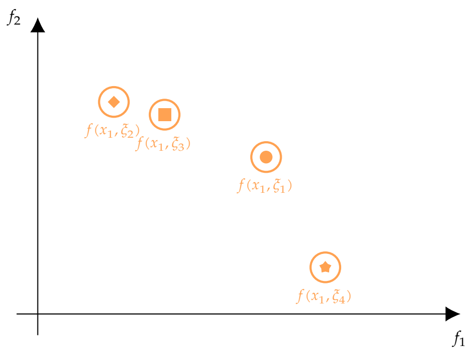
Definition 6.3. An uncertain multi-objective optimization problem $(\mathrm{MOP}(\mathcal{U}))$ has the

- *domination property with respect to hull-based minmax robust efficiency*, if for all $x \in \mathcal{X}$ it holds that either x is hull-based minmax robust efficient to $\mathrm{MOP}(\mathcal{U})$ or there is a hull-based minmax robust efficient solution $x' \in \mathcal{X}$ to $\mathrm{MOP}(\mathcal{U})$ such that

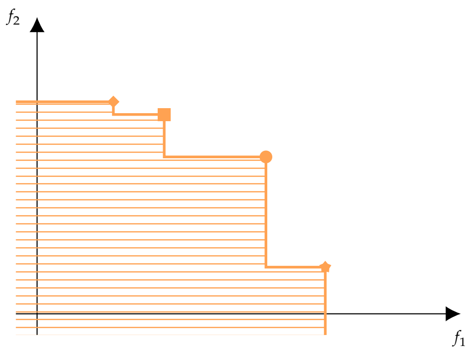
$$f_{\mathcal{U}}(x') \in \operatorname{conv}\left(f_{\mathcal{U}}(x)\right) - \mathbb{R}^p_{\geq}, \text{ and}$$

- *domination property with respect to set-based minmax robust efficiency*, if for all $x \in \mathcal{X}$ it holds that either x is set-based minmax robust efficient to $\mathrm{MOP}(\mathcal{U})$ or there is a set-based minmax robust efficient solution $x' \in \mathcal{X}$ to $\mathrm{MOP}(\mathcal{U})$ such that

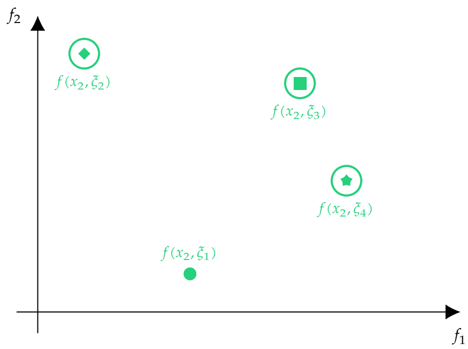
$$f_{\mathcal{U}}(x') \in f_{\mathcal{U}}(x) - \mathbb{R}^p_{\geq}.$$



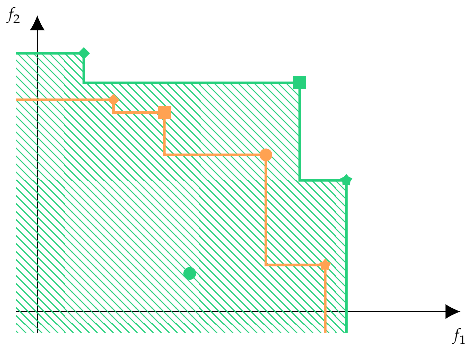
(a) Solutions $\xi_1, \xi_2, \xi_3, \xi_4$ are the ones that are efficient to $\text{Pess}^{\text{sb}}(x_1)$...



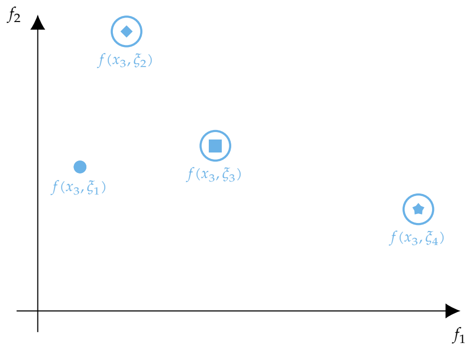
(b) ...thus, if $\mathcal{U}'$ contains $\xi_1, \xi_2, \xi_3, \xi_4$, we have $f^{\mathcal{U}}(x_1) \subseteq f^{\mathcal{U}'}(x_1) - \mathbb{R}^p_{\geq}$.



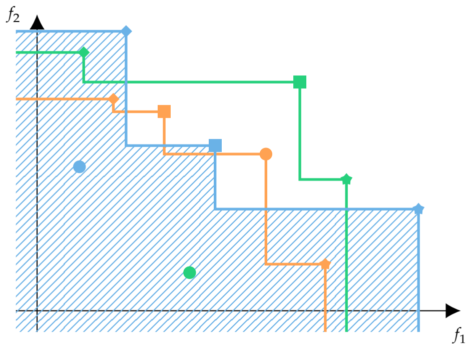
(c) Solutions ξ_2, ξ_3, ξ_4 are the ones that are efficient to $\text{Pess}^{\text{sb}}(x_2)$...



(d) ...thus, if $\mathcal{U}'$ contains ξ_2, ξ_3, ξ_4, we have $f^{\mathcal{U}}(x_2) \subseteq f^{\mathcal{U}'}(x_2) - \mathbb{R}^p_{\geq}$.

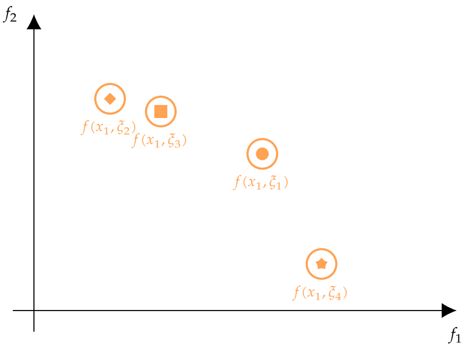


(e) Solutions ξ_2, ξ_3, ξ_4 are the ones that are efficient to $\text{Pess}^{\text{sb}}(x_3)$...

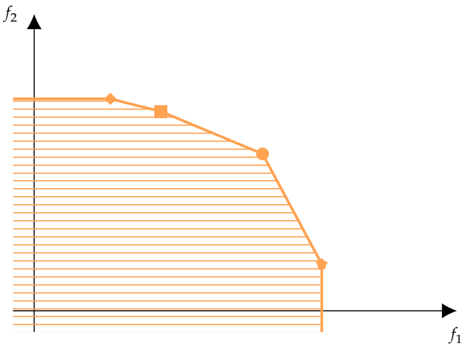


(f) ...thus, if $\mathcal{U}'$ contains ξ_2, ξ_3, ξ_4, we have $f^{\mathcal{U}}(x_3) \subseteq f^{\mathcal{U}'}(x_3) - \mathbb{R}^p_{\geq}$.

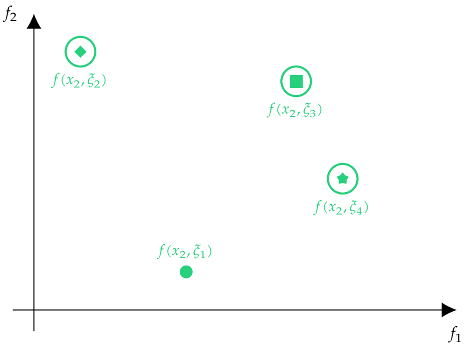
Figure 6.1: Implications of Condition C



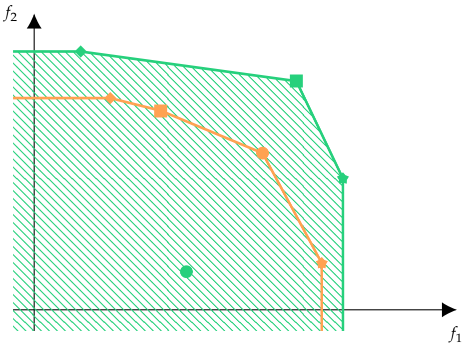
(a) Solutions $\xi_1, \xi_2, \xi_3, \xi_4$ are the ones that are extreme supported efficient to $\text{Pess}^{\text{sb}}(x_1)$...



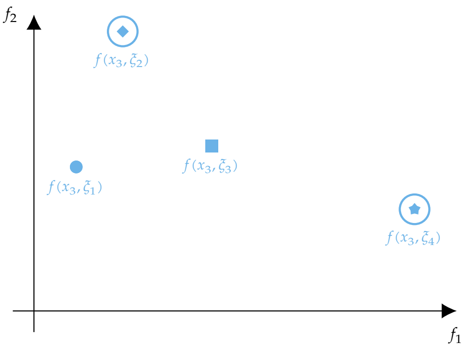
(b) ...thus, if $\mathcal{U}'$ contains $\xi_1, \xi_2, \xi_3, \xi_4$, we have $f^{\mathcal{U}}(x_1) \subseteq f^{\mathcal{U}'}(x_1) - \mathbb{R}^p_{\geqq}$.



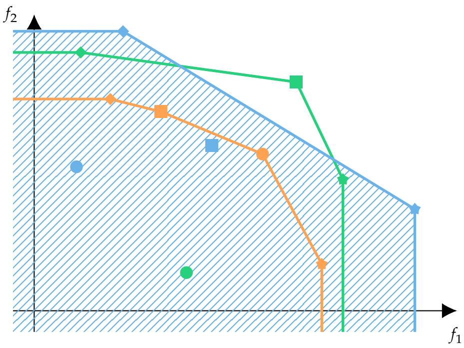
(c) Solutions ξ_2, ξ_3, ξ_4 are the ones that are extreme supported efficient to $\text{Pess}^{\text{sb}}(x_2)$...



(d) ...thus, if $\mathcal{U}'$ contains ξ_2, ξ_3, ξ_4, we have $f^{\mathcal{U}}(x_2) \subseteq f^{\mathcal{U}'}(x_2) - \mathbb{R}^p_{\geqq}$.



(e) Solutions ξ_2 and ξ_4 are the ones that are extreme supported efficient to $\text{Pess}^{\text{sb}}(x_3)$...



(f) ...thus, if $\mathcal{U}'$ contains ξ_2, ξ_4, we have $f^{\mathcal{U}}(x_3) \subseteq f^{\mathcal{U}'}(x_3) - \mathbb{R}^p_{\geqq}$.

Figure 6.2: Implications of Condition B

In Chapter 3 we have seen that under the domination property (2.2), point-based minmax robust efficiency for MOP($\mathcal{U}$) and MOP($\mathcal{U}'$) are equivalent, if all point-based minmax robust efficient solutions to MOP($\mathcal{U}'$) satisfy Condition A. We can now show the analog statement for hull-based and set-based minmax robust efficiency. This is accomplished in the following theorems.

Theorem 6.4 (analog of Theorem 3.7)**.** *Let $\mathcal{U}' \subseteq \mathcal{U}$. Consider $x \in \mathcal{X}$. If x satisfies Condition B, then under Assumption 1 the following holds:*

$$\begin{aligned} &x \text{ is hull-based minmax robust efficient for MOP}(\mathcal{U}') \\ \Longrightarrow &x \text{ is hull-based minmax robust efficient for MOP}(\mathcal{U}). \end{aligned}$$

Additionally, if the domination property with respect to hull-based minmax robust efficiency (see Definition 6.3) holds for MOP($\mathcal{U}'$) and all *solutions $x \in \mathcal{X}$ that are hull-based minmax robust efficient for MOP($\mathcal{U}'$) satisfy Condition B, then the following holds:*

$$\begin{aligned} &x \text{ is hull-based minmax robust efficient for MOP}(\mathcal{U}') \\ \Longleftrightarrow &x \text{ is hull-based minmax robust efficient for MOP}(\mathcal{U}). \end{aligned}$$

Theorem 6.5 (analog of Theorems 3.7 and 6.4)**.** *Let $\mathcal{U}' \subseteq \mathcal{U}$. Consider $x \in \mathcal{X}$. If x satisfies Condition C, then under Assumption 1 the following holds:*

$$\begin{aligned} &x \text{ is set-based minmax robust efficient for MOP}(\mathcal{U}') \\ \Longrightarrow &x \text{ is set-based minmax robust efficient for MOP}(\mathcal{U}). \end{aligned}$$

Additionally, if the domination property with respect to set-based minmax robust efficiency (see Definition 6.3) holds for MOP($\mathcal{U}'$) and all *solutions $x \in \mathcal{X}$ that are set-based minmax robust efficient for MOP($\mathcal{U}'$) satisfy Condition C, then the following holds:*

$$\begin{aligned} &x \text{ is set-based minmax robust efficient for MOP}(\mathcal{U}') \\ \Longleftrightarrow &x \text{ is set-based minmax robust efficient for MOP}(\mathcal{U}). \end{aligned}$$

Since Theorems 6.4 and 6.5 are very similar, we present one proof proving both theorems. Read (6.5a)-(6.10a) for a proof of Theorem 6.4 and (6.5b)-(6.10b) for a proof of Theorem 6.5

Proof of Theorems 6.4 and 6.5. $\underline{\Rightarrow:}$ Let x be hull-based minmax robust efficient for MOP($\mathcal{U}'$) and satisfy Condition B or let $\hat{x}$ be set-based minmax robust efficient for MOP($\mathcal{U}'$) and satisfy Condition C. Then, by Lemma 6.1,

$$f_{\mathcal{U}}(x) \subseteq \text{conv}\,(f_{\mathcal{U}'}(x)) - \mathbb{R}^p_{\geqq} \text{ and} \tag{6.5a}$$

$$f_{\mathcal{U}}(\hat{x}) \subseteq f_{\mathcal{U}'}(\hat{x}) - \mathbb{R}^p_{\geqq} \tag{6.5b}$$

must hold, respectively. Now, assume to the contrary that x is not hull-based or $\hat{x}$ is not set-based minmax robust efficient for MOP($\mathcal{U}$), i.e., there exist $x', \hat{x}' \in \mathcal{X}$, such that

$$f_{\mathcal{U}}(x') \subseteq \operatorname{conv}(f_{\mathcal{U}}(x)) - \mathbb{R}^p_{\geq} \text{ and} \tag{6.6a}$$
$$f_{\mathcal{U}}(\hat{x}') \subseteq f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^p_{\geq}, \tag{6.6b}$$

respectively. From $\mathcal{U}' \subseteq \mathcal{U}$ we get $f_{\mathcal{U}'}(x') \subseteq f_{\mathcal{U}}(x')$ and $f_{\mathcal{U}'}(\hat{x}') \subseteq f_{\mathcal{U}}(\hat{x}')$, see (2.8). If x is hull-based minmax robust efficient for MOP($\mathcal{U}'$) and satisfies Condition B this leads to

$$\begin{aligned} f_{\mathcal{U}'}(x') \overset{(2.8)}{\subseteq} f_{\mathcal{U}}(x') &\overset{(6.6a)}{\subseteq} \operatorname{conv}(f_{\mathcal{U}}(x)) - \mathbb{R}^p_{\geq} \\ &\overset{(6.5a)}{\subseteq} \operatorname{conv}\left(f_{\mathcal{U}'}(x) - \mathbb{R}^p_{\geqq}\right) - \mathbb{R}^p_{\geq} = \operatorname{conv}(f_{\mathcal{U}'}(x)) - \mathbb{R}^p_{\geq}. \end{aligned} \tag{6.7a}$$

Similarly, if $\hat{x}$ is set-based minmax robust efficient for MOP($\mathcal{U}'$) and satisfies Condition C

$$\begin{aligned} f_{\mathcal{U}'}(\hat{x}') \overset{(2.8)}{\subseteq} f_{\mathcal{U}}(\hat{x}') &\overset{(6.6b)}{\subseteq} f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^p_{\geq} \\ &\overset{(6.5b)}{\subseteq} \left(f_{\mathcal{U}'}(\hat{x}) - \mathbb{R}^p_{\geqq}\right) - \mathbb{R}^p_{\geq} = f_{\mathcal{U}'}(\hat{x}) - \mathbb{R}^p_{\geq} \end{aligned} \tag{6.7b}$$

follows. Hull-based minmax robust efficiency of x for MOP($\mathcal{U}'$) is contradicted by (6.7a) and set-based minmax robust efficiency of $\hat{x}$ for MOP($\mathcal{U}'$) is contradicted by (6.7b). This proves "$\Rightarrow$".

<u>$\Leftarrow$:</u> Let Condition B hold for all solutions which are hull-based minmax robust efficient for MOP($\mathcal{U}'$) and let Condition C hold for all solutions which are set-based minmax robust efficient for MOP($\mathcal{U}'$). Furthermore, let x be hull-based minmax robust efficient for MOP($\mathcal{U}$) and let $\hat{x}$ be set-based minmax robust efficient for MOP($\mathcal{U}$). Assume to the contrary that $x, \hat{x} \in \mathcal{X}$ are not hull-based and set-based minmax robust efficient efficient for MOP($\mathcal{U}'$), respectively.

Then, since the domination property w.r.t. to hull-based and set-based minmax robust efficiency (see Definition 6.3) holds, there is a solution $x' \in \mathcal{X}$ that is hull-based minmax robust efficient for MOP($\mathcal{U}'$) such that

$$f_{\mathcal{U}'}(x') \subseteq \operatorname{conv}(f_{\mathcal{U}'}(x)) - \mathbb{R}^p_{\geq} \tag{6.8a}$$

and there is a solution $\hat{x}' \in \mathcal{X}$ that is set-based minmax robust efficient for MOP($\mathcal{U}'$) such that

$$f_{\mathcal{U}'}(\hat{x}') \subseteq f_{\mathcal{U}'}(\hat{x}) - \mathbb{R}^p_{\geq}. \tag{6.8b}$$

Note that since x' and $\hat{x}'$ are hull based and set-based minmax robust efficient for MOP($\mathcal{U}'$) they satisfy Condition B and Condition C, respectively, i.e.,

$$\operatorname{conv}(f_{\mathcal{U}'}(x')) - \mathbb{R}^p_{\geqq} \supseteq f_{\mathcal{U}}(x') \tag{6.9a}$$
$$f_{\mathcal{U}'}(\hat{x}') - \mathbb{R}^p_{\geqq} \supseteq f_{\mathcal{U}}(\hat{x}') \tag{6.9b}$$

Together with $\mathcal{U}' \subseteq \mathcal{U}$ we receive

$$\begin{aligned} f_{\mathcal{U}}(x') &\overset{(6.9a)}{\subseteq} \operatorname{conv}\left(f_{\mathcal{U}'}(x')\right) - \mathbb{R}^p_{\geqq} \overset{(6.8a)}{\subseteq} \operatorname{conv}\left(f_{\mathcal{U}'}(x) - \mathbb{R}^p_{\gneq}\right) - \mathbb{R}^p_{\geqq} \\ &= \operatorname{conv}\left(f_{\mathcal{U}'}(x)\right) - \mathbb{R}^p_{\gneq} \overset{(2.8)}{\subseteq} \operatorname{conv}\left(f_{\mathcal{U}}(x)\right) - \mathbb{R}^p_{\gneq}. \end{aligned} \tag{6.10a}$$

and

$$\begin{aligned} f_{\mathcal{U}}(\hat{x}') &\overset{(6.9b)}{\subseteq} f_{\mathcal{U}'}(\hat{x}') - \mathbb{R}^p_{\geqq} \overset{(6.8b)}{\subseteq} \left(f_{\mathcal{U}'}(\hat{x}) - \mathbb{R}^p_{\gneq}\right) - \mathbb{R}^p_{\geqq} \\ &= f_{\mathcal{U}'}(\hat{x}) - \mathbb{R}^p_{\gneq} \overset{(2.8)}{\subseteq} f_{\mathcal{U}}(\hat{x}) - \mathbb{R}^p_{\gneq}. \end{aligned} \tag{6.10b}$$

This contradicts the assumption of x and $\hat{x}$ being hull-based and set-based minmax robust efficient for MOP($\mathcal{U}$), respectively. □

ADAPTION OF OPTIMIZATION-PESSIMIZATION. Having shown that Condition B and Condition C can be used as termination conditions, we can now formulate an optimization-pessimization algorithm designed to determine hull-based minmax robust efficient solutions. The method is presented in Algorithm 6.1. Its correctness is proven in the subsequent theorem.

Theorem 6.6. *Let $\mathcal{U}' \subseteq \mathcal{U}$ and let the domination property with respect to hull-based [set-based] minmax robust efficiency hold for MOP($\mathcal{U}$) and for MOP($\mathcal{U}'$) for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$. Furthermore, let the objective functions $f_i(x,\cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$, $i = 1,2,\dots,p$, and the constraint functions $F_j(x,\cdot)$, $j = 1,2,\dots,J$, be continuous and let Assumption 1 hold.*

(*i*) *Let $\mathcal{U}$ be finite. Then Algorithm 6.1 returns the set of hull-based [set-based] minmax robust efficient solutions to MOP($\mathcal{U}$) after at most $|\mathcal{U}|$ iterations.*

(*ii*) *Let $\mathcal{U}$ be a polytope or finite and let the objectives $f_i(x,\cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$ for all $i = 1,2,\dots,p$ and the constraints $F_j(x,\cdot)\colon \operatorname{conv}(\mathcal{U}) \to \mathbb{R}$ for all $j = 1,2,\dots,J$ be continuous and quasi-convex. Then Algorithm 6.1 returns the set of hull-based [set-based] minmax robust efficient solutions to MOP($\mathcal{U}$) after at most k iterations, where k is the number of extreme points of $\mathcal{U}$, if we choose an algorithm for the pessimization problem ($\mathrm{Pess}^{\mathrm{sb}}(x)$) which always finds an extreme point of $\mathcal{U}$.*

Proof. As in the proof of Theorem 3.17, we first show that the solutions returned after termination of the algorithm are in fact hull-based [set-based] minmax robust efficient solutions to MOP($\mathcal{U}$). We then show termination after the stated number of iterations.

For the first part, note that when the algorithm terminates we have

$$\max_{\xi \in \mathcal{U}} F_j(x^*, \xi) \leq 0 \ \forall j = 1,2,\dots,J \tag{6.11}$$

for all $(x^*, y^*) \in X^{(k)*}$. Thus, all solutions in $X^{(k-1)*}$ are strictly robust feasible. Furthermore, after termination $\mathcal{U}^{(k)}$ contains a representative set of extreme supported efficient solutions [a representative set of efficient solutions] to $\mathrm{Pess}^{\mathrm{sb}}(x)$ for all $x \in X^{(k)}$. Thus,

Algorithm 6.1 Optimization-pessimization: Determining hull-based [set-based] minmax robust efficient solutions

Require: Uncertain multi-objective optimization problem (MOP($\mathcal{U}$))
Require: Finite initial set $\mathcal{U}^{(0)} \subseteq \mathcal{U}$.
Ensure: Either $\mathcal{U}$ finite or $\mathcal{U}$ a polytope and $f_i(x,\cdot)$, $i = 1,2,\dots,p$, and $F_j(x,\cdot)$, $j = 1,2,\dots,J$, are continuous and quasi-convex.
Ensure: Domination property with respect to hull-based [set-based] minmax robust efficiency (see Definition 6.3) holds for MOP($\mathcal{U}$) and for MOP($\mathcal{U}'$) for any finite subset $\mathcal{U}' \subseteq \mathcal{U}$.

Set $k := 0$.
repeat
  Set $\mathcal{U}^{(k+1)} := \mathcal{U}^{(k)}$.

Optimization

  Determine set $X^{(k)*}$ of hull-based [set-based] minmax robust efficient solutions to MOP($\mathcal{U}'$).

Pessimization

  for all $(x^*, y^*) \in X^{(k)*}$ **do**
    Determine a representative set of extreme supported efficient solutions [a representative set of efficient solutions] $\mathcal{U}^*$ for Pess$^{\text{sb}}(x)$.
    for all $\xi^* \in \mathcal{U}^*$ **do**
      if $\xi^* \notin \mathcal{U}^{(k+1)}$ **then**
        Add ξ^* to $\mathcal{U}^{(k+1)}$.
      end if
    end for
    for all $j = 1,2,\dots,J$ **do**
      Determine $\xi^* \in \arg\max_{\mathcal{U}} F_j(x^*, \xi)$.
      if $F_j(x^*, \xi^*) > 0$ **then**
        Add ξ^* to $\mathcal{U}^{(k+1)}$.
      end if
    end for
  end for

  $k := k + 1$
until $\mathcal{U}^{(k)} = \mathcal{U}^{(k+1)}$.
return $X^{(k-1)*}$: set of hull-based [set-based] minmax robust efficient solutions to MOP($\mathcal{U}$).
return $\mathcal{U}^{\text{FINAL}} := \mathcal{U}^k$: set of worst-case scenarios.

Condition B [Condition C] is satisfied for all solutions which are set-based minmax robust efficient for MOP($\mathcal{U}'$). By Theorem 6.4 [Theorem 6.5], it follows that $x \in X^{(k)}$ is a set of all hull-based [set-based] minmax robust efficient solutions to MOP($\mathcal{U}$).

The second part is analogous to the proof of Theorem 3.17. □

6.2 LOWER AND UPPER BOUNDS PROVIDED BY THE OPTIMIZATION AND THE PESSIMIZATION PROBLEM

In order to state an analog of Lemma 3.4 (c.f. Page 24) we introduce the following order relations:

Definition 6.7. For sets of sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{R}^p)$, we write

$$\mathcal{A} \preceq^{\text{low}}_{\text{hb}} \mathcal{B} \text{ if for all } B \in \mathcal{B} \text{ there exists } A \in \mathcal{A} \text{ such that } A \subseteq \operatorname{conv}(B) - \mathbb{R}^p_{\geqq} \tag{6.12a}$$

and

$$\mathcal{A} \preceq^{\text{low}}_{\text{sb}} \mathcal{B} \text{ if for all } B \in \mathcal{B} \text{ there exists } A \in \mathcal{A} \text{ such that } A \subseteq B - \mathbb{R}^p_{\geqq}. \tag{6.12b}$$

Note that this can be seen as the hull-based and set-based analog to the lower set less order

$$A \preceq^{\text{low}} B \text{ if for all } b \in B \text{ there exists } a \in A \text{ with } a \leqq b, \tag{3.8 revisited}$$

which we used when comparing *points* on the Pareto frontier of the multi-objective robust counterpart (MORC($\mathcal{U}$)).

We can now formulate the following lemma.

Lemma 6.8. *Let $X^*_{hb}(\mathcal{U})$, $X^*_{hb}(\mathcal{U}')$, $X^*_{sb}(\mathcal{U})$, and $X^*_{sb}(\mathcal{U}')$ denote the sets of hull-based and set-based minmax robust efficient solutions to MOP($\mathcal{U}$) and MOP($\mathcal{U}'$), respectively. Then*

$$\left\{f_{\mathcal{U}'}(x) \colon x \in X^*_{hb}(\mathcal{U}')\right\} \preceq^{\text{low}}_{\text{hb}} \left\{f_{\mathcal{U}}(x) \colon x \in X^*_{hb}(\mathcal{U})\right\} \preceq^{\text{low}}_{\text{hb}} \left\{f_{\mathcal{U}}(x) \colon x \in X^*_{hb}(\mathcal{U}')\right\} \textit{ and} \tag{6.13a}$$

$$\left\{f_{\mathcal{U}'}(x) \colon x \in X^*_{sb}(\mathcal{U}')\right\} \preceq^{\text{low}}_{\text{sb}} \left\{f_{\mathcal{U}}(x) \colon x \in X^*_{sb}(\mathcal{U})\right\} \preceq^{\text{low}}_{\text{sb}} \left\{f_{\mathcal{U}}(x) \colon x \in X^*_{sb}(\mathcal{U}')\right\} \tag{6.13b}$$

holds.

Proof. We show (6.13a) and leave it to the reader to transfer the statement (6.13b).

For the left hand side of (6.13a), take $x \in X^*_{\text{hb}}(\mathcal{U})$. We need to show that there is $\tilde{x} \in X^*_{\text{hb}}(\mathcal{U}')$ such that

$$f_{\mathcal{U}'}(\tilde{x}) \subseteq \operatorname{conv}\left(f_{\mathcal{U}}(x)\right) - \mathbb{R}^p_{\geqq}. \tag{6.14}$$

Under the domination property w.r.t to hull-based minmax robust efficiency (see Definition 6.3), either x itself is contained in $X^*_{\text{hb}}(\mathcal{U}')$ or $\tilde{x} \in \mathcal{X}^*_{\text{hb}}(\mathcal{U}')$ exists such that

$$f_{\mathcal{U}'}(\tilde{x}) \subseteq \operatorname{conv}\left(f_{\mathcal{U}'}(x)\right) - \mathbb{R}^p_{\geq} \tag{6.15}$$

holds. In the first case, set $\tilde{x} = x$ and the statement follows immediately. In the second case, the statement in (6.14) follows from

$$f_{\mathcal{U}'}(\tilde{x}) \overset{(6.15)}{\subseteq} \operatorname{conv}\left(f_{\mathcal{U}'}(x)\right) - \mathbb{R}^p_{\geqq} \overset{(2.8)}{\subseteq} \operatorname{conv}\left(f_{\mathcal{U}}(x)\right) - \mathbb{R}^p_{\geqq}.$$

For the right hand side of (6.13b), take $x \in X^*_{\text{hb}}(\mathcal{U}')$. If $x \in X^*_{\text{hb}}(\mathcal{U})$, i.e., if x is also hull-based minmax robust efficient for MOP($\mathcal{U}$), set $\tilde{x} = x$ and we are done. Otherwise, by the domination property with respect to hull-based minmax robust efficiency, existence of an $\tilde{x} \in X^*_{\text{hb}}(\mathcal{U})$ such that

$$f_{\mathcal{U}}(\tilde{x}) \subseteq \operatorname{conv}\left(f_{\mathcal{U}}(x)\right) - \mathbb{R}^p_{\geq}$$

is ensured. This finishes the proof. □

CONCLUSION AND OUTLOOK

In this chapter we showed that optimization-pessimization can be formulated for hull-based and set-based minmax robust efficiency and that solutions detected in the process can be used as lower and upper bounds.

However, in comparison to the algorithms developed in Chapter 3, Algorithm 6.1 is more challenging in two ways (see Table 6.1): First, the pessimization problem in Algorithm 6.1, that is $\mathrm{Pess}^{\mathrm{sb}}(x)$, is multi-objective. Since for any fixed x, it is still deterministic, it can be solved using multi-objective methods (see the Conclusion of Chapter 3). Second, the optimization problem in each iteration consists of finding hull-based or set-based minmax robust solutions for an uncertain multi-objective optimization problem with a reduced uncertainty set $\mathcal{U}' \subseteq \mathcal{U}$.

	Optimization problem	Pessimization problem
Algorithms 3.2 to 3.4	solve $\mathrm{MORC}(\mathcal{U}')$: deterministic, multi-objective	solve $\mathrm{Pess}^{\mathrm{pb}}(x)$: p deterministic and single-objective problems
Algorithm 6.1	find hull-based [set-based] minmax robust efficient solutions to $\mathrm{MOP}(\mathcal{U}')$	solve $\mathrm{Pess}^{\mathrm{sb}}(x)$: deterministic, multi-objective

Table 6.1: Comparison of optimization-pessimization algorithms

In 2014 when set-based minmax robust efficiency was conceptualized, Ehrgott, Ide and Schöbel proposed two scalarization approaches to determine set-based minmax robust efficient solutions. Their first approach utilizes weighted-sum scalarization. The basic idea is to solve

$$\min_{x \in \mathcal{X}} \sup_{\xi \in \mathcal{U}} \sum_{i=1}^{p} \lambda_i f_i(x, \xi) \tag{6.16}$$

for all weights λ from a predefined set of weights $\Lambda \subseteq \mathbb{R}^p_>$. Under Assumption 1, the solutions found in this way are guaranteed to be set-based minmax robustly efficient for $\mathrm{MOP}(\mathcal{U})$ (c.f. [EIS14, Theorem 4.3 (b)]). Yet, in general not all set-based minmax robust efficient solutions can be found by solving (6.16). Those that can be found might well be considered *set-based minmax robust supported efficient*.

The process of solving (6.16) (as suggested by Ehrgott, Ide and Schöbel) might be hard if the uncertainty set is large or infinite. However, if their approach is combined with Algorithm 6.1 (in a similar fashion to Algorithm 5.1), in each iteration of Algorithm 6.1 the problems (6.16) are solved for a finite set $\mathcal{U}' \subseteq \mathcal{U}$. For what is more, while in general not all set-based minmax robust efficient solutions can be found, the bounds established in Lemma 6.8 can give a sense of "how good" the approximation at any state of the algorithm is.

More recently, in 2024, Eichfelder and Quintana showed that under a set of several (alternative) conditions, set-based minmax robust efficient solutions can be found by solving a reformulation that is a multi-objective optimization problem (see [EQ24]). Finiteness of $\mathcal{U}$ being one of these conditions, their approach can be used to reformulate MOP($\mathcal{U}'$), since $\mathcal{U}' \subseteq \mathcal{U}$ is finite throughout Algorithm 6.1.

7

REGRET ROBUST EFFICIENCY

In this chapter we investigate regret robust efficient solutions. Recall Definition 2.13 from Page 15: A solution $x \in \mathcal{X}$ to an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)) is *regret robust efficient*, if it is an efficient solution to the multi-objective regret robust counterpart

$$\min_{x \in \mathcal{X}_{\mathcal{U}}} \begin{pmatrix} \sup_{\xi \in \mathcal{U}} \left(f_1(x, \xi) - \min_{x^* \in \mathcal{X}} f_1(x^*, \xi) \right) \\ \sup_{\xi \in \mathcal{U}} \left(f_2(x, \xi) - \min_{x^* \in \mathcal{X}} f_2(x^*, \xi) \right) \\ \vdots \\ \sup_{\xi \in \mathcal{U}} \left(f_p(x, \xi) - \min_{x^* \in \mathcal{X}} f_p(x^*, \xi) \right) \end{pmatrix}. \qquad \text{(MORRC($\mathcal{U}$) revisited)}$$

For shorter notation we use

$$r_i(x, \xi) = f_i(x, \xi) - \min_{x^* \in \mathcal{X}} f_i(x^*, \xi) \qquad (7.1)$$

for all $i = 1, 2, \ldots, p$.

We first show that the algorithms developed in Chapter 5, specifically Algorithms 5.1 to 5.3, can be used in order to find regret robust efficient solutions in Section 7.1. Subsequently, in Section 7.2, we discuss set-based concepts of regret robust efficiency.

7.1 RELATION TO POINT-BASED MINMAX ROBUST EFFICIENCY

As for point-based minmax robust efficient solutions, regret robust efficient solution can be found by solving a robust counterpart: the multi-objective regret robust counterpart (MORRC($\mathcal{U}$)). The latter has the same structure as the multi-objective robust counterpart (MORC($\mathcal{U}$)), just with the functions $r_i(x, \xi)$, $i = 1, 2, \ldots, p$, in the place of $f_i(x, \xi)$, $i = 1, 2, \ldots, p$.

Consequently, if r meets the conditions set for f in Chapters 3 to 5, the computational methods presented therein should also apply to regret-robust efficiency. We therefore investigate properties of the regret function, see (7.1).

PROPERTIES OF THE REGRET FUNCTION

Lemma 7.1. *Let the function $h \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ be linear in x for every fixed $\xi \in \mathcal{U}$ and quasi-convex and continuous in ξ for every fixed $x \in \mathcal{X}$. Furthermore, let $\mathcal{X}$ and $\mathcal{U}$ be compact. Then*

$$r(x, \xi) := h(x, \xi) - \min_{x^* \in \mathcal{X}} h(x^*, \xi)$$

is affinely linear in x for every fixed $\xi \in \mathcal{U}$ and quasi-convex and continuous in ξ for every fixed $x \in \mathcal{X}$.

Proof. First, let $\bar{\bar{\xi}} \in \mathcal{U}$ be fixed and consider $r(\cdot, \bar{\bar{\xi}}) \colon \mathcal{X} \to \mathbb{R}$. From

$$r(x, \bar{\bar{\xi}}) = \underbrace{h(x, \bar{\bar{\xi}})}_{\text{linear in } x} - \underbrace{\min_{x^* \in \mathcal{X}} h(x^*, \xi)}_{\text{constant in } x}$$

affine linearity of $r(\cdot, \bar{\bar{\xi}})$ follows immediately.

Second, let $\bar{x} \in \mathcal{X}$ be fixed and consider $r(\bar{x}, \cdot) \colon \mathcal{U} \to \mathbb{R}$. Then continuity of

$$r(\bar{x}, \xi) = \underbrace{h(\bar{x}, \xi)}_{\text{cont. in } \xi} - \underbrace{\min_{x^* \in \mathcal{X}} h(x^*, \xi)}_{\text{cont. in } \xi}$$

follows from the fact that both addends are continuous in ξ. The second addend is continuous since joint continuity of $h\colon \mathcal{X} \times \mathcal{U}$ on the compact set $\mathcal{X} \times \mathcal{U}$ implies uniform continuity. Lastly, for any $\xi = \lambda \xi^1 + (1 - \lambda)\xi^2, \lambda \in [0, 1]$, we get

$$\begin{aligned}
r(\bar{x}, \xi) &= h(\bar{x}, \xi) - \min_{x^* \in \mathcal{X}} h(x^*, \xi) = \max_{x^* \in \mathcal{X}} (h(\bar{x}, \xi) - h(x^*, \xi)) \\
&= \max_{x^* \in \mathcal{X}} h(\bar{x} - x^*, \xi) \\
&\leq \max_{x^* \in \mathcal{X}} \max \left\{ h(\bar{x} - x^*, \xi^1), h(\bar{x} - x^*, \xi^2) \right\} \\
&= \max \left\{ \max_{x^* \in \mathcal{X}} h(\bar{x} - x^*, \xi^1), \max_{x^* \in \mathcal{X}} h(\bar{x} - x^*, \xi^2) \right\} \\
&= \max \left\{ \max_{x^* \in \mathcal{X}} (h(\bar{x}, \xi^1) - h(x^*, \xi^1)), \max_{x^* \in \mathcal{X}} (h(\bar{x}, \xi^2) - h(x^*, \xi^2)) \right\} \\
&= \max \left\{ r(\bar{x}, \xi^1), r(\bar{x}, \xi^2) \right\},
\end{aligned}$$

where the equalities in lines two and five follow from linearity of $h(\cdot, \xi)$ for fixed ξ and the inequality in the third line holds due to quasi-convexity of $h(x, \cdot)$ for fixed x. Quasi-convexity of $r(\bar{x}, \cdot)$ follows. □

Recall the bi-objective robust optimization problem (BRO($\mathcal{U}$)) (see Page 45), a special case of the multi-objective robust counterpart (MORC($\mathcal{U}$)):

$$\min_{x \in \mathcal{X}} \begin{pmatrix} \sup_{\xi \in \mathcal{U}} f_1(x, \xi) \\ \sup_{\xi \in \mathcal{U}} f_2(x, \xi) \end{pmatrix}. \qquad \text{(BRO}(\mathcal{U}) \text{ revisited)}$$

For BRO($\mathcal{U}$) we assumed the following:

- (BRO-1) The feasible set takes the form $\mathcal{X} = P \cap (\mathbb{Z}^k \times \mathbb{R}^{n-k})$ where $P \subseteq \mathbb{R}^n$ is a polytope and $0 \leq k \leq n$.
- (BRO-2) The uncertainty set $\mathcal{U} \subseteq \mathbb{R}^m$ is a polytope or finite set.
- (BRO-3) The objective functions $f_1, f_2 \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ are affinely linear in x for every fixed $\xi \in \mathcal{U}$ and quasi-convex and continuous in ξ for every fixed $x \in \mathcal{X}$.

By Lemma 7.1, the properties stated for f_1, f_2 in (BRO-3) are *almost* inherited by the functions r_1, r_2.

Corollary 7.2. *Let an uncertain bi-objective optimization problem* (MOP($\mathcal{U}$)) *be given. If its multi-objective robust counterpart takes the form BRO($\mathcal{U}$) and satisfies conditions (BRO-1), (BRO-2), and (BRO-3), and additionally, $f_i(\cdot,\xi)\colon \mathcal{X} \to \mathbb{R}$, $i = 1,2,\dots,p$, is linear for fixed $\xi \in \mathcal{U}$, then its multi-objective regret robust counterpart*

$$\min_{x\in\mathcal{X}} \begin{pmatrix} \sup_{\xi\in\mathcal{U}} \left(f_1(x,\xi) - \min_{x^*\in\mathcal{X}} f_1(x^*,\xi)\right) \\ \sup_{\xi\in\mathcal{U}} \left(f_2(x,\xi) - \min_{x^*\in\mathcal{X}} f_2(x^*,\xi)\right) \end{pmatrix} \tag{BRRO($\mathcal{U}$)}$$

satisfies conditions (BRO-1), (BRO-2), and (BRO-3), too. Specifically, the regret functions $r_i(x,\xi) = (f_i(x,\xi) - \min_{x^*\in\mathcal{X}} f_i(x^*,\xi)) : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, $i = 1,2$, are affinely linear in x for every fixed $\xi \in \mathcal{U}$ and quasi-convex and continuous in ξ for every fixed $x \in \mathcal{X}$.

Consequently, under just slightly stronger conditions, Algorithms 5.1 to 5.3 that were designed to solve BRO($\mathcal{U}$) can solve BRRO($\mathcal{U}$), too.

7.2 SET-BASED AND HULL-BASED REGRET ROBUST EFFICIENCY

Having observed that regret robust efficiency (as defined in [GW22]) is essentially the same as point-based minmax robust efficiency–just with $r(x,\xi)$ instead of $f(x,\xi)$ as the objective function, we can state the following (alternative) definition:

Definition 7.3. Given an uncertain multi-objective optimization problem

$$\left\{ \min \begin{pmatrix} f_1(x,\xi) \\ f_2(x,\xi) \\ \vdots \\ f_p(x,\xi) \end{pmatrix} \right\}_{\xi\in\mathcal{U}} \tag{MOP($\mathcal{U}$) revisited}$$

a solution $x \in \mathcal{X}_{\mathcal{U}}$ is called (*point-based*) *regret robust* (*strictly/weakly*) *efficient* if it is point-based minmax robust (strictly/weakly) efficient to

$$\left\{ \min \begin{pmatrix} r_1(x,\xi) \\ r_2(x,\xi) \\ \vdots \\ r_p(x,\xi) \end{pmatrix} \right\}_{\xi\in\mathcal{U}} \tag{7.2}$$

for $r(x,\xi) := f(x,\xi) - \min_{x^*\in\mathcal{X}} f(x^*,\xi)$.

Then it is only natural to define set-based and hull-based regret robust efficiency as follows:

Definition 7.4. Given an uncertain multi-objective optimization problem (MOP($\mathcal{U}$)) a solution $x \in \mathcal{X}_{\mathcal{U}}$ is called *set-based* [*hull-based*] *regret robust* (*strictly/weakly*) *efficient* if it is set-based [hull-based] minmax robust (strictly/weakly) efficient to 7.2.

Example 2.9 (Continued). Consider the setting of Example 2.9. In Figure 2.10d (see Page 16) the regret for all solutions $x \in \mathcal{X}$ and all scenarios $\xi \in \mathcal{U}$ was depicted. In Figures 7.1 and 7.2 the lines illustrating the upper border of the sets $\{r(x, \xi \colon \xi \in \mathcal{U})\} - \mathbb{R}^p_{\geqq}$ and $\operatorname{conv}(\{r(x, \xi \colon \xi \in \mathcal{U})\}) - \mathbb{R}^p_{\geqq}$ are shown. One can see that x_2 and x_3 are set-based and hull-based regret robust efficient.

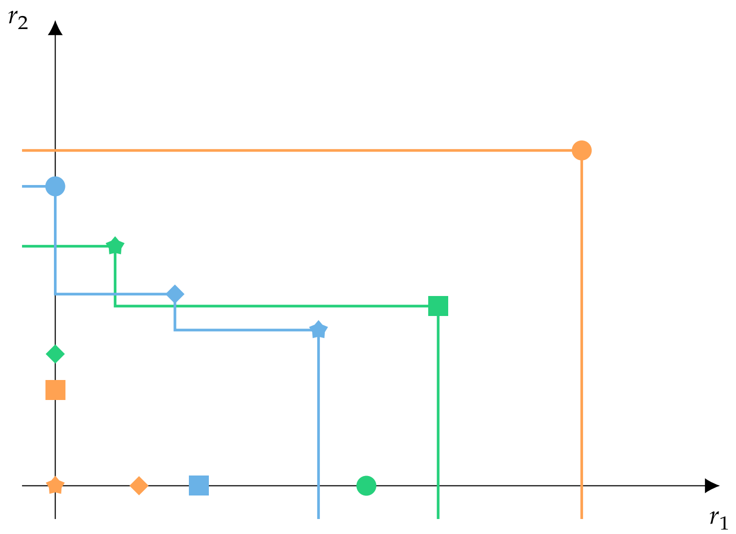


Figure 7.1: Set-based regret robust efficiency

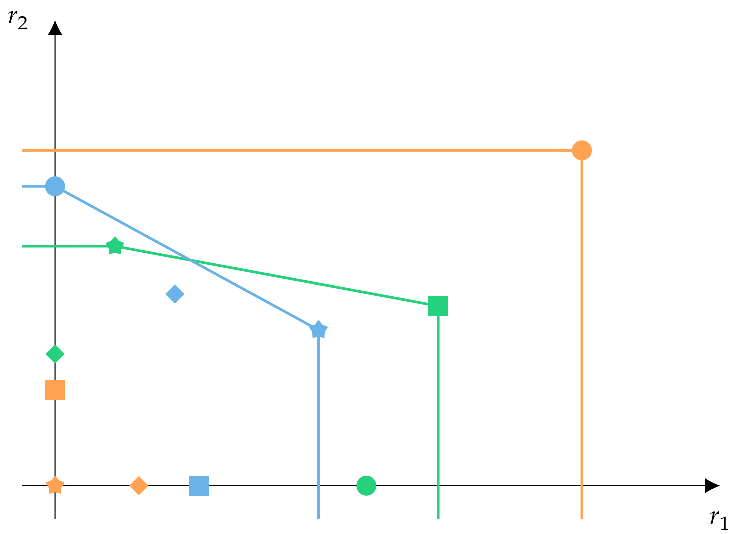


Figure 7.2: Hull-based regret robust efficiency

8 SUPPORTED EFFICIENCY

In previous chapters, we referred to a subset of efficient solutions called (*extreme*) *supported efficient* solutions. Specifically, at two points (in Chapter 3 on Page 31 and in Chapter 6 on Page 68) we used the fact that the image of all feasible solutions of a multi-objective optimization problem (MOP) lies in the convex hull of or is dominated by a convex combination of extreme supported efficient solutions (see (3.19),(6.2a)). In this chapter we will investigate supported efficient and extreme supported efficient solutions in more detail.

8.1 INTRODUCTION AND PRELIMINARIES

We use the following notation: We consider a general version of a (deterministic) multi-objective optimization problem

$$\min_{x \in \mathcal{X}} \begin{pmatrix} g_1(x) \\ \vdots \\ g_p(x) \end{pmatrix}. \tag{MOP revisited}$$

We again assume that the *domination property* (see [Hen86])

For all $y \in \mathcal{Y} \setminus \mathcal{Y}_N$, there exists a point $y' \in \mathcal{Y}_N$ such that $y' \lneq y$. (2.2 revisited)

is satisfied.

As a consequence of (2.2), we have

$$\mathcal{Y} \subseteq \mathcal{Y}_N \cup \mathcal{Y}_N + \mathbb{R}^p_{\gneq} = \mathcal{Y}_N + \mathbb{R}^p_{\geqq}. \tag{8.1}$$

Solutions that are often easier to determine than the set of all efficient solutions $\mathcal{X}_E \subseteq \mathcal{X}$ are *supported efficient solutions*. As stated in the preliminaries (c.f. Page 5), intuitively, the set of supported efficient solutions can be described as the set that contains all solutions to the weighted-sum scalarization

$$\min_{x \in \mathcal{X}} \lambda_1 g_1(x) + \lambda_2 g_2(x) + \cdots + \lambda_p g_p(x). \tag{MOP(λ) revisited}$$

for "reasonable" choices of weights λ_i, $i = 1, 2, \ldots, p$ (what is meant by "reasonable" will be precised in the following). As Ehrgott put it (c.f. [Ehr08]): "Geometrically, supported efficient solutions are efficient solutions with $g(x)$ on the 'lower left' boundary of the convex hull of $\mathcal{Y}$."

Correspondingly, the image $y = g(x) \in \mathcal{Y}$ of a supported efficient solution $x \in \mathcal{X}$, i.e., a solution to

$$\min_{y \in \mathcal{Y}} \sum_{i=1}^{p} \lambda_i y, \tag{8.2}$$

is then called *supported nondominated*.

However, the exact definition of supported efficiency–and supported nondominance–varies from author to author (and sometimes within an author's publications).

Before we proceed to present a collection of nine characterizations of *supported nondominatedness*, we need to introduce one more type of efficiency: *proper efficiency*.

Definition 8.1 (Geoffrion [Geo68]). A feasible solution $\bar{x} \in \mathcal{X}$ is called *properly efficient* and its outcome $\bar{y} = g(x) \in \mathcal{Y}$ is called *properly nondominated*, if it is efficient and there is a real number $M > 0$ such that for all $i = 1, 2, \dots, p$ and $x \in \mathcal{X}$ satisfying $f_i(x) < f_i(\bar{x})$ there exists an index $j = 1, 2, \dots, p$, $j \neq i$, such that $f_j(\bar{x}) < f_j(x)$ and

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leq M$$

We use $\mathcal{X}_{\text{pE}} \subseteq \mathcal{X}$ to denote the set of *properly efficient solutions* and $\mathcal{Y}_{\text{pN}} := g(\mathcal{X}_{\text{pE}}) \subseteq \mathcal{Y}$ for the set of *properly nondominated points*. Naturally, $\mathcal{X}_{\text{pE}} \subseteq \mathcal{X}_{\text{E}} \subseteq \mathcal{X}_{\text{wE}}$ and $\mathcal{Y}_{\text{pN}} \subseteq \mathcal{Y}_{\text{N}} \subseteq \mathcal{Y}_{\text{wN}}$ hold.

8.2 SUPPORTED EFFICIENCY NONDOMINANCE

The following are nine characterizations of supported (weak) efficiency and supported (weak) nondominance (nondominatedness). Five of them can be found in the literature; the other four are natural extensions added by the author.

Definition 8.2. For a multi-objective optimization problem (MOP) a solution $\bar{x} \in \mathcal{X}$ is called *supported efficient* and its outcome $\bar{y} = g(\bar{x}) \in \mathcal{Y}$ is called *supported nondominated* if

(SN-1) $\bar{y} \in \arg\min\{\lambda^\top y \colon y \in \mathcal{Y}\}$ for some $\lambda \in \mathbb{R}^p_>$ (see, e.g., [UT94; Ehr06; RE09a; CPF21; SNR15; PGE10; RSN17]),

(SN-2) $\bar{y} \in \{y' \in \operatorname{conv}(\mathcal{Y}_{\text{N}}) \colon \operatorname{conv}(\mathcal{Y}_{\text{N}}) \cap (y' - \mathbb{R}^p_{\geqq}) = y'\}$ and $\bar{y}$ is nondominated (see, e.g., [HPR07]),

(SN-3) there is no convex combination $\sum \mu_i y^{(i)}$ of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_{\text{N}} \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that $\sum_{i=1}^n \mu_i y^{(i)} \lneq \bar{y}$,

(SN-4) $\bar{y} \in \arg\min\{\lambda^\top y \colon y \in \mathcal{Y}\}$ for some $\lambda \in \mathbb{R}^p_{\gneq}$ and $\bar{y}$ is nondominated,

(SN-5) $\bar{y}$ lies on the boundary of $\operatorname{conv}(\mathcal{Y}_{\text{N}} + \mathbb{R}^p_{\geqq})$ and $\bar{y}$ is nondominated (see, e.g., [EF09a; EF09b; AC16; SKS19]),

(SN-6) there is no convex combination $\sum \mu_i y^{(i)}$ of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_{\text{N}} \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that $\sum_{i=1}^n \mu_i y^{(i)} < \bar{y}$ and $\bar{y}$ is nondominated (see, e.g., [ÖK10]).

A solution $\bar{x} \in \mathcal{X}$ is called *supported weakly efficient* and its outcome $\bar{y} = g(\bar{x}) \in \mathcal{Y}$ is called *supported weakly nondominated* if

(SN-7) $\bar{y} \in \arg\min\{\lambda^\top y \colon y \in \mathcal{Y}\}$ for some $\lambda \in \mathbb{R}^p_{\gneq}$ (see, e.g., [RSS89; Fig+17]).

(SN-8) $\bar{y}$ lies on the boundary of $\operatorname{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_{\geq})$,

(SN-9) there is no convex combination $\sum \mu_i y^{(i)}$ of points $y^{(1)}, \ldots, y^{(n)} \in \mathcal{Y}_\mathrm{N} \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that $\sum_{i=1}^n \mu_i y^{(i)} < \bar{y}$.

Some of the above characterizations have originally been formulated for the decision space $\mathcal{X}$, i.e., they are definitions of *supported efficiency*, while others were originally formulated for the image space $\mathcal{Y}$, i.e., as definitions of *supported nondominatedness*. Since in any case, a solution $x \in \mathcal{X}$ is supported (weakly) efficient *if and only iff* its outcome $y = g(x) \in \mathcal{Y}$ is supported (weakly) nondominated, in the following we will focus on the image space leaving the reader with the task to transfer all definitions and statements to their respective analogon in the decision space.

There are several other possible characterizations that are easy to recognize as being equivalent to the above ones:

Remark 8.3. The following characterizations, namely of a point $\bar{y} \in \mathcal{Y}$ as *supported nondominated* if

(SN-2′) $\bar{y} \in \{y' \in \operatorname{conv}(\mathcal{Y}_\mathrm{N}) \colon \operatorname{conv}(\mathcal{Y}_\mathrm{N}) \cap (y' - \mathbb{R}^p_{\geqq}) = y'\}$,

(SN-5′) $\bar{y}$ lies on the boundary of $\operatorname{conv}(\mathcal{Y})$ and $\bar{y}$ is nondominated (see, e.g., [RE09b; SKS19]),

and as *supported weakly nondominated* if

(SN-8′) $\bar{y}$ lies on the boundary of $\operatorname{conv}(\mathcal{Y})$,

are equivalent to (SN-2), (SN-5), and (SN-8), respectively.

The characterizations of supported (weak) nondominatedness in Definition 8.2 can be sorted into three categories: The first category includes those definitions that characterize supported (weakly) efficient solutions as solutions to the weighted-sum *scalarization* (MOP(λ)) (and/or supported nondominated points as their images under g). These are (SN-1), (SN-4) and (SN-7). In the literature, this seems to be the most common approach. (SN-1) is mainly used in the context of *discrete* multi-objective optimization (see, e.g., [UT94; Ehr06; RE09a; CPF21; SNR15; PGE10]). In Section 8.2.1.4 (c.f. Example 8.12 and Lemma 8.13) it will become clear why that is the case. Solutions that satisfy (SN-7) are referred to by Rosenblatt and Sinuany-Stern (see [RSS89]) but are not given a name. They are introduced as *supported* (weakly) efficient solutions by Figueira, Fonseca, Halffmann et al. (see [Fig+17]). The author did not find (SN-4) in the literature but in several lecture notes.

The second category of characterizations consists of (SN-2), (SN-5) and (SN-8). These definitions concern the *geometric* structure of the convex hull of the nondominated points. Hamacher, Pedersen and Ruzika introduced (SN-2) in the context of multi-objective flow problems (see [HPR07]). The characterization (SN-5) was used by Eusébio and Figueira in an article concerning a bi-objective problem (see [EF09a]) and by Alves and

Costa for a three-objective problem (see [AC16]). In Section 8.2.1.4 we will see that this difference is meaningful, since (SN-5) coincides with definition (SN-2) if $p \leq 2$ but in general not if $p \geq 3$ (c.f. Example 8.14 and Lemma 8.15). Raith and Ehrgott as well as Schulze, Klamroth and Stiglmayr used the equivalent characterization (SN-5′) (see [RE09b; SKS19]). The last geometric definition, (SN-8) (as well as (SN-8′)), was added by the authors of this paper.

Finally, (SN-3), (SN-6) and (SN-9) make up the third category of possible definitions. They identify supported (weakly) nondominated points via the absence of (strictly) dominating *convex combinations of nondominated points*. Özpeynirci and Köksalan introduced *extreme* and *nonextreme supported nondominated* separately in [ÖK10]; their union is described by (SN-6). The other two characterizations, (SN-3) and (SN-9), are natural modifications which we added.

As we will see in Section 8.2.1.1, the definitions added by the authors of this paper–(SN-3), (SN-4), (SN-8), and (SN-9)–are equivalent to ones in the literature.

8.2.1 *Relations between different characterizations of supported efficiency*

To the best of the author's knowledge, literature about the relations between different characterizations of supported efficiency is scarce. The most comprehensive study has been published in 2023 by Könen and Stiglmayr (see [KS23]). In their work they considered four different characterizations, namely (SN-5′), (SN-5), (SN-2), and (SN-1) and focused on the linear case, where the set $\mathcal{Y}$ is a polyhedron or discrete. They stated that in the non-integer case all these characterizations are equivalent. We present a counterexample disproving this statement. Furthermore, Könen and Stiglmayr showed that in the context of bi-objective integer optimization the definitions are equivalent. We will show that this statement holds for bi-objective linear optimization as well.

We now show that the characterizations of supported (weak) efficiency are ordered with (SN-1) being the strictest definition. More specifically, we show that these characterizations can be sorted into four different layers with the characterizations of each layer being equivalent. The order is presented in Figure 8.1.

8.2.1.1 *Equivalence of the characterizations in each layer.*

We start by showing that the characterizations of each layer are equivalent, specifically we show

$$\text{(SN-2)} \iff \text{(SN-3)}, \tag{8.3}$$
$$\text{(SN-4)} \iff \text{(SN-5)} \iff \text{(SN-6)}, \tag{8.4}$$
$$\text{(SN-7)} \iff \text{(SN-8)} \iff \text{(SN-9)}, \tag{8.5}$$

where (SN-i) $\iff$ (SN-j) means: $y \in \mathcal{Y}$ satisfies (SN-i) if and only if it satisfies (SN-j).

We start on the lowest layer and show (8.5) and work up from there showing (8.4) and (8.3).

Lemma 8.4. *Under the domination property* (SN-9) $\iff$ (SN-8) $\iff$ (SN-7)*.*

Proof. We show (SN-9) $\implies$ (SN-8) $\implies$ (SN-7) $\implies$ (SN-9).

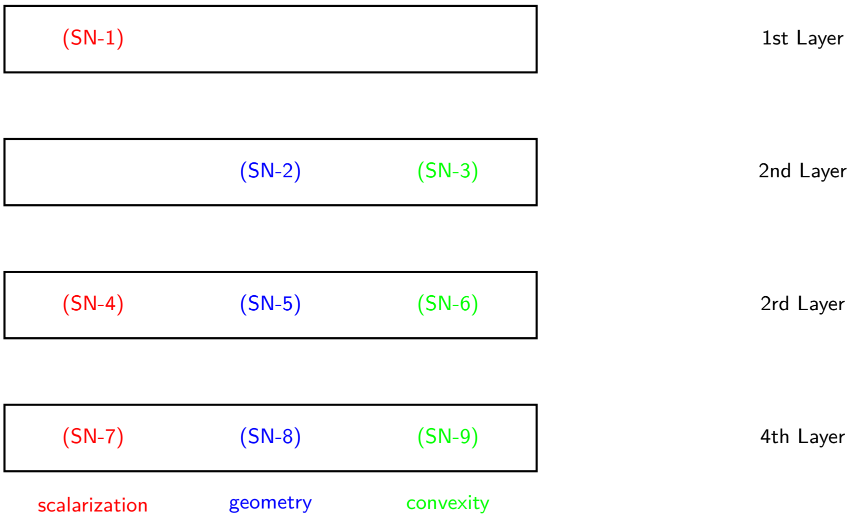


Figure 8.1: Nine different definitions of supported nondominatedness sorted into four different layers

(SN-9) $\implies$ (SN-8): By contradiction. Assume $\bar{y} \in \mathcal{Y}$ does *not* satisfy (SN-8), i.e., $\bar{y} \notin \mathrm{bd}(\mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq))$. Since, by (8.1), $\bar{y} \in \mathcal{Y} \subseteq \mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq \subseteq \mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq)$, it follows that $\bar{y}$ must lie in the interior of $\mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq) = \mathrm{conv}(\mathcal{Y}_\mathrm{N}) + \mathbb{R}^p_\geq$. Consequently, $y' \in \mathrm{conv}(\mathcal{Y}_\mathrm{N})$ exists such that $y' < \bar{y}$. Since $y' \in \mathrm{conv}(\mathcal{Y}) = \mathrm{conv}\left(\mathrm{conv}(\mathcal{Y}_\mathrm{N} \setminus \{\bar{y}\}) \cup \{\bar{y}\}\right)$, there exists $y'' \in \mathrm{conv}(\mathcal{Y}_\mathrm{N} \setminus \{\bar{y}\})$ with $y'' \leq y' < \bar{y}$. This shows that $\bar{y}$ does not satisfy (SN-9).

(SN-8) $\implies$ (SN-7): Let $\bar{y} \in \mathcal{Y}$ satisfy (SN-8), i.e., $\bar{y} \in \mathrm{bd}(\mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq))$.
By the supporting hyperplane theorem (see, e.g., [BV04]) there exists $a \in \mathbb{R}^p \setminus \{0\}$ such that $\{y \in \mathbb{R}^p \colon a^\top y = a^\top \bar{y}\}$ is a supporting hyperplane for the convex set $\mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq)$ at $\bar{y}$, meaning that

$$a^\top y \geq a^\top \bar{y} \qquad \forall y \in \mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq) \tag{8.6}$$

holds.

Now consider two cases: Since $a \neq 0$, either $a \in \mathbb{R}^p \setminus \mathbb{R}^p_\geq$ or $a \in \mathbb{R}^p_\gneq$. In the first case, i.e., if $a \in \mathbb{R}^p \setminus \mathbb{R}^p_\geq$, we have $a_i < 0$ for some $i = 1, 2, \ldots, p$. Then for any $y' \in \mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq)$ and any $M > 0$ we have $y'' := y' + Me_i \in \mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq)$, but by choosing M sufficiently large we get

$$a^\top y'' = a^\top y' + M \underbrace{a_i e_i}_{<0} < a^\top \bar{y},$$

which contradicts (8.6).

In the second case, i.e., if $a \in \mathbb{R}^p_\gneq$, we get

$$\min\{a^\top y \colon y \in \mathcal{Y}\} \overset{(8.1)}{\geq} \min\{a^\top y \colon y \in \mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^p_\geq)\} \overset{(8.6)}{\geq} a^\top \bar{y}$$

and, consequently, $\bar{y}$ is an optimal solution to (8.2) for $\lambda = a \in \mathbb{R}^p_{\gneq}$. (SN-7) follows.

$\underline{\text{(SN-7)} \implies \text{(SN-9)}}$: By contradiction. Let $\bar{y} \in \mathcal{Y}$ satisfy (SN-7) with $\lambda = \bar{\lambda} \in \mathbb{R}_{\gneq}$, i.e., $\bar{y} \in \arg\min\{\bar{\lambda}^\top y \colon y \in \mathcal{Y}\}$, and assume that $\bar{y} \in \mathcal{Y}$ does *not* satisfy (SN-9), i.e., there is a collection of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_\mathrm{N} \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that

$$\sum_{i=1}^{n} \mu_i y^{(i)} < \bar{y}$$

holds. Thus, we have

$$\sum_{i=1}^{n} \mu_i \bar{\lambda}^\top y^{(i)} = \bar{\lambda}^\top \sum_{i=1}^{n} \mu_i y^{(i)} < \bar{\lambda}^\top \bar{y}, \tag{8.7}$$

which implies that there is at least one $y^{(i)} \in \mathcal{Y}_\mathrm{N}$ such we have that $\bar{\lambda}^\top y^{(i)} < \bar{\lambda}^\top \bar{y}$. This is a contradiction to the assumption that $\bar{y}$ minimizes (8.2) for $\lambda = \bar{\lambda}$. □

Lemma 8.5. *Under the domination property* (SN-4) $\iff$ (SN-5) $\iff$ (SN-6).

Proof. Since (SN-4), (SN-5), and (SN-6) are just (SN-7), (SN-8) and (SN-9) with the additional requirement of $\bar{y}$ being nondominated, the equivalence of the latter three (shown in Lemma 8.4) implies the equivalence of the former three. □

Lemma 8.6. *Under the domination property* (SN-2) $\iff$ (SN-3).

Proof. $\underline{\text{(SN-2)} \implies \text{(SN-3)}}$: Let $\bar{y} \in \mathcal{Y}$ satisfy (SN-2). Then $\operatorname{conv}(\mathcal{Y}_\mathrm{N}) \cap (\bar{y} - \mathbb{R}^p_{\gneq})$ is empty and (SN-3) follows directly.

$\underline{\text{(SN-3)} \implies \text{(SN-2)}}$: Let $\bar{y} \in \mathcal{Y}$ satisfy (SN-3), i.e., there is no convex combination of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_\mathrm{N} \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that $\sum_{i=1}^{n} \mu_i y^{(i)} \lneq \bar{y}$. Then there is also no convex combination of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_\mathrm{N}$, $\mu \in \Delta^n$, such that $\sum_{i=1}^{n} \mu_i y^{(i)} \lneq \bar{y}$. Then $\operatorname{conv}(\mathcal{Y}_\mathrm{N}) \cap (\bar{y} - \mathbb{R}^p_{\gneq})$ is empty and, specifically, $\bar{y}$ is nondominated. (SN-2) follows. □

8.2.1.2 *Hierarchy of layers.*

Having sorted the characterizations into four layers, we need to show how the layers relate to each other. Specifically, we show that the layers are sorted in such a way that the characterizations on the top layer are the strictest and those on the bottom layer are the most permissive:

$$\begin{aligned} \underbrace{\text{(SN-1)}}_{\text{layer 1}} \implies \underbrace{\text{(SN-2)} \iff \text{(SN-3)}}_{\text{layer 2}} &\implies \underbrace{\text{(SN-4)} \iff \text{(SN-5)} \iff \text{(SN-6)}}_{\text{layer 3}} \\ &\implies \underbrace{\text{(SN-7)} \iff \text{(SN-8)} \iff \text{(SN-9)}}_{\text{layer 4}}. \end{aligned}$$

Once again, we start at the bottom and work up from there.

Lemma 8.7. *We have* (SN-4) $\implies$ (SN-7), (SN-5) $\implies$ (SN-8) *and* (SN-6) $\implies$ (SN-9).

Proof. trivial. □

Lemma 8.8. *We have* (SN-1) $\implies$ (SN-3).

Proof (similar to the one in Lemma 8.4). By contradiction. Let $\bar{y} \in \mathcal{Y}$ satisfy (SN-1) with $\lambda = \bar{\lambda} \in \mathbb{R}^p_>$, i.e., $\bar{y} \in \arg\min\{\bar{\lambda}^\top y : y \in \mathcal{Y}\}$, and assume $\bar{y} \in \mathcal{Y}$ does *not* satisfy (SN-3), i.e., there is collection of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_N \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that

$$\sum_{i=1}^{n} \mu_i y^{(i)} \lneq \bar{y}$$

holds. Since all components of $\bar{\lambda}$ are strictly positive, (8.7) follows. This implies that there is at least one $y^{(i)} \in \mathcal{Y}_N$ such we have that $\bar{\lambda}^\top y^{(i)} < \bar{\lambda}^\top \bar{y}$. This is a contradiction to the assumption that $\bar{y}$ minimizes (8.2) for $\lambda = \bar{\lambda}$. □

Lemma 8.9. *We have* (SN-3) $\implies$ (SN-6).

Proof. Let $\bar{y} \in \mathcal{Y}$ satisfy (SN-3), i.e., there is no convex combination $\sum \mu_i y^{(i)}$ of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_N \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that $\sum_{i=1}^n \mu_i y^{(i)} \lneq \bar{y}$. Then specifically, there is no convex combination $\sum \mu_i y^{(i)}$ of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_N \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that $\sum_{i=1}^n \mu_i y^{(i)} < \bar{y}$ and $\bar{y}$ is nondominated. □

8.2.1.3 *Relation to nondominatedness and weak nondominatedness.*

The designation of an outcome $y \in \mathcal{Y}$ as *supported (weakly) nondominated* suggests that it is also (weakly) nondominated. In fact, this is the case. The fact that points that are solutions to (8.2) for weight vectors with zero-components, i.e., for $\lambda \in \mathbb{R}^p_{\geq} \setminus \mathbb{R}^p_{>}$, are only guaranteed to be weakly nondominated is well-known in multi-objective optimization community. For completeness, we will however restate this in the following Lemma.

Lemma 8.10. *Let* $\mathcal{Y} \subseteq \mathbb{R}^p$.

(a) *If $y \in \mathcal{Y}$ satisfies (SN-1), it is properly nondominated.*

(b) *If $y \in \mathcal{Y}$ satisfies (SN-i) for some $i = 1, 2, \dots, 6$, it is nondominated.*

(c) *If $y \in \mathcal{Y}$ satisfies (SN-i) for some $i = 1, 2, \dots, 9$, it is weakly nondominated.*

Proof. The first statement was shown by Geoffrion when he introduced the concept of proper efficiency (see [Geo68, Thm. 1]). The second statement is a trivial consequence of Lemmas 8.8 and 8.9. The fact that (SN-7) implies weak efficiency is part of multi-objective folklore (see, e.g., [Ehr05, Thm. 3.4]). Together with Lemmas 8.4 and 8.7 to 8.9 the third statement follows. □

The characterizations of the first, third and fourth layer are indeed equivalent to proper nondominance, nondominance and weak nondominance, respectively, under some convexity assumptions.

Lemma 8.11. *Let* $\mathcal{Y} + \mathbb{R}^p_{\geq}$ *be convex.*

(a) *A point $y \in \mathcal{Y}$ satisfies (SN-1) if and only if y is properly nondominated.*

(b) *A point $y \in \mathcal{Y}$ satisfies (SN-i) for some $i = 4, 5, 6$ if and only if y is nondominated.*

(c) *A point $y \in \mathcal{Y}$ satisfies (SN-i) for some $i = 7, 8, 9$ if and only if y is weakly nondominated.*

Proof. The forwards direction is shown in Lemma 8.10. We only have to show the backwards direction. We show 1., 3., and at last 2.

(a) Let $\mathcal{Y} + \mathbb{R}^p_{\geqq}$ be convex and $\bar{y} \in \mathcal{Y}$ be properly nondominated. Consider the problem (MOP') with $\mathcal{X}' := \mathcal{Y} + \mathbb{R}^p_{\geqq}$ and $f' :=$ id. Since $\bar{y}$ is properly nondominated in $\mathcal{Y}$, it is also properly nondominated in $\mathcal{Y}' := f'(\mathcal{X}') = \mathrm{id}(\mathcal{Y} + \mathbb{R}^p_{\geqq}) = \mathcal{Y} + \mathbb{R}^p_{\geqq}$. The problem (MOP') has a convex feasible set and convex objectives. Hence, a Theorem of Geoffrion (see [Geo68, Thm. 2]) can be applied which states that properly nondominatedness of $\bar{y}$ in $\mathcal{Y}' := f'(\mathcal{X}') = \mathrm{id}(\mathcal{Y} + \mathbb{R}^p_{\geqq}) = \mathcal{Y} + \mathbb{R}^p_{\geqq}$ implies that $\bar{y}$ is optimal to $\min\{\bar{\lambda}^\top y : y \in \mathcal{Y}'\}$ for some $\bar{\lambda} \in \mathbb{R}^p_{>}$. Since $\bar{y} \in \mathcal{Y} \subseteq \mathcal{Y}'$, it is also an optimal solution to $\min\{\bar{\lambda}^\top y : y \in \mathcal{Y}\}$. (SN-8) follows.

(c) This has been shown in [Ehr05, Thm. 3.5].

(b) Let $y \in \mathcal{Y}$ be nondominated. Then it is also weakly nondominated and by (c). it follows that y satisfies (SN-i), $i = 7, 8, 9$. Together with nondominatedness, (SN-i), $i = 4, 5, 6$ follow.

□

8.2.1.4 *Relations between the layers*

So far, we have been able to group nine definitions of supported nondominatedness (and supported efficiency) into four different layers and establish a hierarchy of these layers. However, it still needs to be shown that the characterizations from the various layers are actually different. To that end, in this section we investigate the relation between the layers. Specifically, we show that for a *general* multi-objective optimization problem (MOP) the characterizations of any two layers are non-equivalent and show for which special cases layers coincide.

RELATION BETWEEN THE FIRST AND SECOND LAYER. To show that the first and second layer are *not* equivalent, we present an example of a multi-objective optimization problem (MOP) with points that satisfy (SN-2) and (SN-3), but not (SN-1).

Example 8.12 (A nonlinear problem). Consider a problem where $\mathcal{Y} = \{(y_1, y_2) \in \mathbb{R}^2 : (y_1 - 2)^2 + (y_2 - 2)^2 \leq 1\} \subsetneq \mathbb{R}^2$ (see Figure 8.2). We show that the point $\bar{y} = (1, 2)^\top \in \mathcal{Y}$ does not satisfy (SN-1), but satisfies (SN-3).

- We show that $\bar{y}$ does not satisfy (SN-1) by showing that for every $\lambda \in \mathbb{R}^2_{\geq}$ we can choose $\epsilon > 0$ small enough such that

  $$\hat{y} := \bar{y} + \begin{pmatrix} \epsilon^2 \\ -\epsilon \end{pmatrix}$$

  is both feasible and better solution than $\bar{y}$ for (8.2).

  Clearly, $\hat{y} \in \mathcal{Y}$ for any $0 < \epsilon < 1$ since

  $$(\hat{y}_1 - 2)^2 + (\hat{y}_2 - 2)^2 = ((1 + \epsilon^2) - 2)^2 + ((2 - \epsilon) - 2)^2 = 1 + \epsilon^2 \underbrace{(\epsilon^2 - 1)}_{<0} < 1.$$

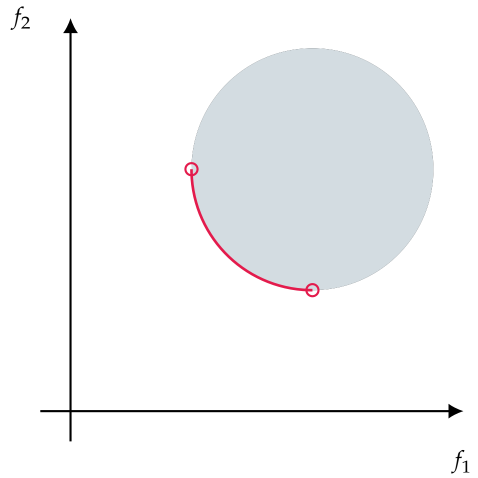


Figure 8.2: A nonlinear problem: Illustration of Example 8.12

Furthermore, for sufficiently small $\epsilon > 0$ we have

$$\min\{\lambda^\top y \colon y \in \mathcal{Y}\} \leq \lambda^\top \hat{y} = \lambda^\top \bar{y} + \lambda_1 \epsilon^2 - \lambda_2 \epsilon = \lambda^\top \bar{y} + \epsilon \underbrace{(\epsilon\lambda_1 - \lambda_2)}_{<0} < \lambda^\top \bar{y}.$$

- However, $\bar{y}$ is clearly nondominated and

  $$\bar{y} \in \left\{y' \in \operatorname{conv}(\mathcal{Y}_N) \colon \operatorname{conv}(\mathcal{Y}_N) \cap (y' - \mathbb{R}^p_{\geqq}) = y'\right\}$$

  . Thus, $\bar{y}$ satisfies (SN-2) (and, by Lemma 8.6 also (SN-3)).

This shows that (SN-1) is indeed stricter than (SN-2) and (SN-3). However, our example utilizes the "smoothness" of $\mathcal{Y}$. We show that any counterexample depends on that, in other words, that (SN-1) is equivalent to (SN-2) (and (SN-3)) for linear or discrete problems.

Lemma 8.13. *Consider a general multi-objective optimization problem* (MOP)*, where* $\mathcal{Y}$ *is finite or a polytope. Then if* $y \in \mathcal{Y}$ *satisfies (SN-3), it satisfies (SN-1).*

Proof. Let $\mathcal{Y}$ be finite or a polytope and let $\bar{y} \in \mathcal{Y}$ satisfy (SN-3). The convex hull of $\mathcal{Y}$ can be described by a finite number of inequalities. At the point $\bar{y} \in \mathcal{Y} \subseteq \operatorname{conv}(\mathcal{Y})$ some of these inequalities must be active (or else $\bar{y}$ lies in the interior of $\operatorname{conv}(\mathcal{Y})$ and $y' \lneq y$, $y' \in \operatorname{conv}(\mathcal{Y})$, exists which contradicts (SN-3)). So without loss of generality we get

$$\operatorname{conv}(\mathcal{Y}) = \left\{y \in \mathbb{R}^p \colon Ay \leq b, \hat{A}y \leq \hat{b}\right\} \tag{8.8}$$

$$A\bar{y} = b \tag{8.9}$$

$$\hat{A}\bar{y} < \hat{b} \tag{8.10}$$

and for $v \in \mathbb{R}^p_{\gneq}$ sufficiently close to zero

$$\hat{A}(\bar{y} - v) = \underbrace{\hat{A}\bar{y}}_{\overset{(8.10)}{<}\hat{b}} - \hat{A}v \leqq \hat{b} \tag{8.11}$$

still holds. However, since $\bar{y} \in \mathcal{Y}$ satisfies (SN-3), there is no $v \in \mathbb{R}^p_{\gneqq}$ such that

$$y := \bar{y} - v \in \operatorname{conv}\left(\mathcal{Y}_\mathrm{N} \setminus \{\bar{y}\}\right) \subseteq \operatorname{conv}\left(\mathcal{Y}\right).$$

This implies that there is no $v \in \mathbb{R}^p_{\gneqq}$ satisfying

$$Av = A(\bar{y} - y) \overset{(8.9)}{=} b - Ay \overset{(8.8)}{\geqq} 0. \tag{8.12}$$

Consequently, we get

$$\max\left\{\sum_{i=1}^{p} v_i \colon Av \geqq 0, v \geqq 0, v \in \mathbb{R}^p\right\} = 0$$

and by linear programming duality

$$\min\left\{0 \colon A^\top x \geqq 1, x \leqq 0, x \in \mathbb{R}^k\right\} = 0$$

follows, showing that there is an $\bar{x} \in \mathbb{R}^k$ such that $A^\top \bar{x} \geqq 1$, $\bar{x} \leqq 0$. Choose $\lambda := A^\top \bar{x} \in \mathbb{R}^p_{>}$. Then for all $y \in \operatorname{conv}\left(\mathcal{Y}\right)$

$$\lambda^\top y = \underbrace{\bar{x}^\top}_{\leqq 0} \underbrace{Ay}_{\leqq b} \overset{(8.8)}{\geqq} \bar{x}^\top b \overset{(8.9)}{=} x^\top A\bar{y} = \lambda^\top \bar{y}$$

follows. Hence, $\bar{y}$ minimizes MOP(λ) for $\lambda \in \mathbb{R}^p_{>}$. □

RELATION BETWEEN THE SECOND AND THIRD LAYER. As in the previous paragraph, to show that the second and third layer are not equivalent we present a counterexample, where there are points that satisfy (SN-5) and (SN-4), but not (SN-2) and (SN-3).

Example 8.14 (A problem with three objectives). Consider the discrete or the continuous problem where

$$\mathcal{Y} = \left\{\begin{pmatrix}1\\4\\1\end{pmatrix}, \begin{pmatrix}4\\1\\1\end{pmatrix}, \begin{pmatrix}3\\3\\1\end{pmatrix}\right\}, \qquad \mathcal{Y} = \operatorname{conv}\left(\left\{\begin{pmatrix}1\\4\\1\end{pmatrix}, \begin{pmatrix}4\\1\\1\end{pmatrix}, \begin{pmatrix}3\\3\\1\end{pmatrix}\right\}\right), \tag{8.13}$$

respectively (see Figure 8.1). We show that $\bar{y} := (3,3,1)^\top \in \mathcal{Y}$ satisfies (SN-4), but not (SN-3):

- Clearly, the point $\bar{y} = (3,3,1)$ is nondominated. Furthermore, for $\lambda = (0,0,1)^\top \in \mathbb{R}^3_{\gneqq}$, the point $\bar{y}$ is an optimal solution to (8.2). Thus, $\bar{y}$ satisfies (SN-4) (and by Lemma 8.5 also (SN-5) and (SN-6)).

Figure 8.3: A problem with three objectives: Illustration of Example 8.14

- However,

$$y' := \frac{1}{2}\begin{pmatrix}1\\4\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}4\\1\\1\end{pmatrix} = \begin{pmatrix}2.5\\2.5\\1\end{pmatrix}$$

  is a convex combination of points in $\mathcal{Y}_\mathrm{N} \setminus \{\bar{y}\}$ and $y' \lneq \bar{y}$. Thus, $\bar{y}$ does not satisfy (SN-3) (and by Lemma 8.6 also not (SN-2)).

The presented counterexample works for discrete, linear and nonlinear problems. However, it requires at least 3 objectives. The following Lemma shows that in the case of bi-objective optimization, the characterizations of the second and third layer are equivalent.

Lemma 8.15. *Consider a problem* (MOP) *with* $p = 2$. *Then if* $y \in \mathcal{Y}$ *satisfies* (*SN*-5), *it satisfies* (*SN*-3).

Proof. Let $\bar{y} \in \mathcal{Y}$ satisfy (SN-5), i.e., $\bar{y} \in \mathrm{bd}(\mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^2_{\geq}))$ and $\bar{y}$ is nondominated.

Apply the supporting hyperplane theorem (see again, e.g., [BV04]): There exists $a \in \mathbb{R}^2 \setminus \{0\}$ such that

$$a^\top \bar{y} \leq a^\top y \text{ for all } y \in \mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^2_{\geq}). \tag{8.14}$$

Consider three cases: Since $a \neq 0$, either we have $a \in \mathbb{R}^2 \setminus \mathbb{R}^2_{\geq}$, $a \in \mathbb{R}^2_{>}$, or $a \in \mathbb{R}^2_{\gneq} \setminus \mathbb{R}^2_{>}$.

In the first case, i.e., if $a \in \mathbb{R}^2 \setminus \mathbb{R}^2_{\geq}$, we have $a_i < 0$ for some $i = 1, 2$. Then for any $y' \in \mathcal{Y}$ we can find $M > 0$ sufficiently large such that

$$Ma_i < a^\top(\bar{y} - y'). \tag{8.15}$$

This leads to $y'' := y' + Me_i \in \mathrm{conv}(\mathcal{Y}_\mathrm{N} + \mathbb{R}^2_{\geq})$, yet

$$a^\top \bar{y} \overset{(8.14)}{\leq} a^\top y'' = a^\top(y' + Me_i) = a^\top y' + Ma_i \overset{(8.15)}{<} a^\top \bar{y},$$

which is a contradiction.

Now consider the second case where $a \in \mathbb{R}^2_>$ and assume that $\bar{y}$ does not satisfy (SN-3), i.e., we have a convex combination $\sum \mu_i y^{(i)}$ of points $y^{(1)}, \dots, y^{(n)} \in \mathcal{Y}_N \setminus \{\bar{y}\}$, $\mu \in \Delta^n$, such that $\sum_{i=1}^n \mu_i y^{(i)} \lneq \bar{y}$. This implies

$$\sum_{i=1}^{n} \mu_i \left(a^\top y^{(i)}\right) = a^\top \sum_{i=1}^{n} \mu_i y^{(i)} < a^\top \bar{y}$$

and, thus, $a^\top y^{(i)} < a^\top \bar{y}$ for at least one $i = 1, 2, \dots, n$ which contradicts (8.14).

In the third case we have $a \in \mathbb{R}^2_{\gneq} \setminus \mathbb{R}^2_>$, i.e., wlog. we can assume $a = (1, 0)^\top$. So (8.14) implies that

$$\bar{y}_1 \leq y_1 \qquad \text{for all } y \in \operatorname{conv}(\mathcal{Y}_N + \mathbb{R}^2_\geqq). \tag{8.16}$$

Since $\bar{y}$ is nondominated, we also have

$$\bar{y}_2 \leq y_2 \qquad \text{for all } y \in \mathcal{Y} \text{ with } y_1 = \bar{y}_1. \tag{8.17}$$

This implies that there cannot be a a convex combination of points $y^{(i)} \in \mathcal{Y}_N \setminus \{y\}$ such that $\sum_{i=1}^n \mu_i y^{(i)} \lneq \bar{y}$. □

Note that in the proof above, $p = 2$ is needed for the third case, where $a \in \mathbb{R}^p_{\gneq} \setminus \mathbb{R}^2_>$.

Taken together, Lemmas 8.13 and 8.15 show that if both $\mathcal{Y}$ is a polytope (or finite) and the problem is bi-objective, all characterizations of supported nondominatedness, i.e., (SN-1)-(SN-6), coincide. For (SN-1), (SN-2), and (SN-5) this has previously been shown in [KS23, Lemma 3.3].

Corollary 8.16. *Consider a problem* (MOP)*, where $\mathcal{Y}$ is finite or a polytope and $p = 2$. Then* (*SN-1*)-(*SN-6*) *coincide.*

RELATION BETWEEN THE THIRD AND FOURTH LAYER. The following example shows that an outcome $y \in \mathcal{Y}$ can satisfy (SN-7) without being nondominated. Thus, there are outcomes $y \in \mathcal{Y}$ that satisfy (SN-7) but do not satisfy (SN-4).

Example 8.17. Consider

$$\mathcal{Y} = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 \colon y_1 + 2y_2 \geq 6, 2y_1 + y_2 \geq 6, 1 \leq y_i \leq 5 \text{ for all } i = 1, 2 \right\}.$$

Then the point $y = (5, 1)^\top$ is feasible and an optimal solution to (8.2) for $\lambda = (0, 1)^\top$. However, $\bar{y} = (4, 1)^\top$ is also feasible and dominates y. Thus, y is an example of a point that satisfies (SN-7) (and, by Lemma 8.4, also (SN-9), (SN-8)), but is not nondominated.

SUMMARY OF ALL RELATIONS AND CONCLUSION. Figure 8.4 shows the characterizations and their relations.

The author believes that Examples 8.12 and 8.14 show that (SN-2) and (SN-3) are the definitions that are consistent with the intuitive understanding of supported nondominance expressed earlier. This is why Definition 2.2 (in Chapter 2) was introduced in this form.

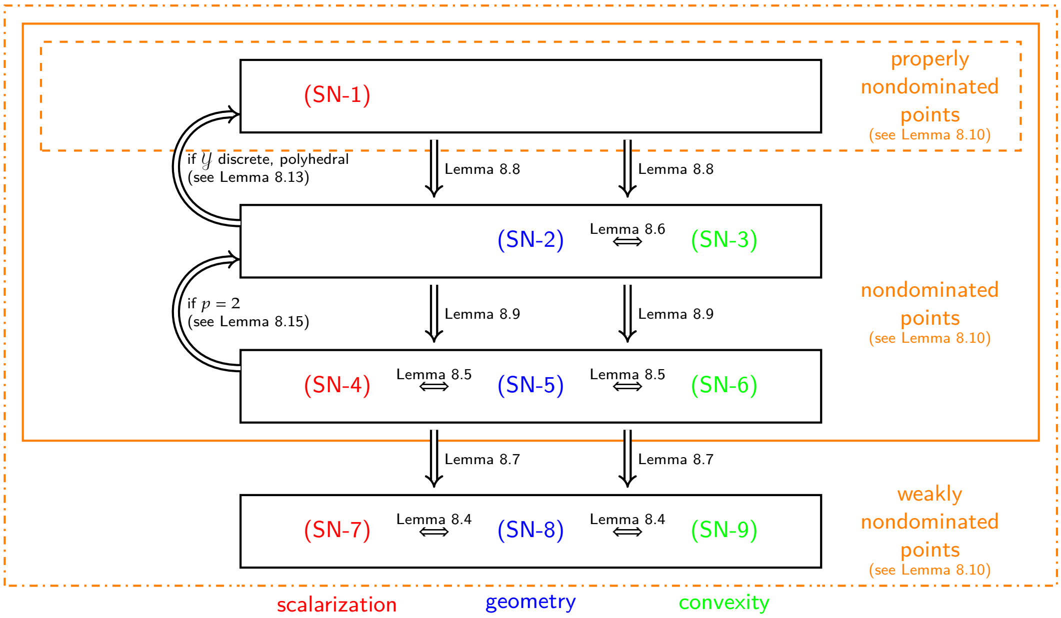


Figure 8.4: Eight different definitions of supported nondominatedness and their relations to each other

We now show that the definition of *extreme* supporteded nondominance that is associated with (SN-3) is also the definition of extreme suppported nondominance that has the desired properties.

8.3 EXTREME SUPPORTED EFFICIENCY

We turn to extreme suppported nondominance. Intuitively, a solution $x \in \mathcal{X}$ to a multi-objective optimization problem (MOP) should be called *extreme supported supported* and its outcome $y = g(x)$ should be called *extreme supported nondominated* if in addition to y being supported nondominated, y is an extreme point of $\mathcal{Y}$ (or of $\mathcal{Y}_{\text{SN}}$).

Examples 8.12 and 8.14 not only shows that the definitions of supported nondominance belong to three different layers, but that the associated definition of extreme supported nondominance are different, too. This follows from the fact that the points in Examples 8.12 and 8.14 that disprove equality of the two layers, are also extreme points. Hence, the definitions of *extreme suppported nondominance* that are associated with ((SN-1))-((SN-9)) differ, too.

We conclude this chapter by showing that the definition of extreme supported nondominance that is associated with (SN-3), has the desired property that

$$\mathcal{Y} \subseteq \operatorname{conv}\left(\mathcal{Y}_{\text{ESN}}\right) + \mathbb{R}^p_{\geqq}. \tag{8.18}$$

We can now show that this definition justifies (8.18). We repeat the definition from Chapter 2.

Definition 2.2 (restated)**.** Given a multi-objective optimization problem (MOP), a solution $x \in \mathcal{X}$ is called

- *supported efficient* and its outcome $y = g(x)$ is called *supported nondominated*, if there is no convex combination of nondominated points $y^{(1)}, y^{(2)}, \dots, y^{(n)} \in \mathcal{Y} \setminus \{y\}$, $\lambda \in \Delta^n$, such that $\sum_{i=1}^n \lambda_i y^{(i)} \lneq y$, and
- *extreme supported efficient* and its outcome $y = g(x)$ is called *extreme supported nondominated*, if there is no convex combination of nondominated points $y^{(1)}, y^{(2)}, \dots,$ $y^{(n)} \in \mathcal{Y} \setminus \{y\}, \lambda \in \Delta^n$, such that $\sum_{i=1}^n \lambda_i y^{(i)} \leqq y$.

Lemma 8.18. *Let $\mathcal{Y}$ be compact. Then $\mathcal{Y}_{ESN} = \operatorname{ext}\left(\operatorname{conv}\left(\mathcal{Y}_N\right) + \mathbb{R}^p_{\geqq}\right)$ holds.*

Proof. Let $\bar{y} \notin \operatorname{ext}\left(\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}\right)$. However, from the domination property (2.2) it follows that $\bar{y} \in \mathcal{Y} \subseteq \operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}$. Thus, $\bar{y}$ can be written as a nontrivial convex combination of points in $\bar{y} \in \operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}$. Then it follows that there is a nontrivial convex combination y of points in $\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right)$, such that $y \leqq \bar{y}$. This is a contradiction to $\hat{y}$ being extreme supported nondominated.

Let $\bar{y} \in \operatorname{ext}\left(\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}\right)$. Then it follows directly that $\bar{y}$ cannot be written as nontrivial convex combination of points in $\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}$. It remains to be shown, that $\bar{y}$ is not dominated by a nontrivial convex combination of points in $\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}$. Assume to the contrary that it is, i.e., there is a point $y \in \operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}$ such that $\bar{y} = y + r$ for some $r \in \mathbb{R}^p_{\gneq}$. Then, by the fact that $y+2r$ is also in $\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}$, it follows that $\bar{y}$ cannot be an extreme point of $\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}$, which is a contradiction. □

Lemma 8.19. *Let $\mathcal{Y}$ be compact. Then $\mathcal{Y} \subseteq \operatorname{conv}(\mathcal{Y}_{ESN}) + \mathbb{R}^p_{\geqq}$ holds.*

Proof. Compactness of $\mathcal{Y} \supseteq \mathcal{Y}_\mathrm{N}$ implies that $\operatorname{conv} \mathcal{Y}_\mathrm{N}$ is bounded. Thus, the characteristic cone of the compact set $\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}$ is $\mathbb{R}^p_{\geqq}$. Since any convex set can be written as sum of the convex hull of its extreme points plus its characteristic cone,

$$\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq} \subseteq \operatorname{conv}\left(\operatorname{ext}\left(\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}\right)\right) + \mathbb{R}^p_{\geqq}$$

follows. Together with the domination property (2.2)–a consequence of compactness of $\mathcal{Y}$–the statement follows:

$$\begin{aligned}
\mathcal{Y} \overset{(2.2)}{\subseteq} \mathcal{Y}_\mathrm{N} + \mathbb{R}^p_{\geqq} \quad &\subseteq \quad \operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq} \\
&\subseteq \quad \operatorname{conv}\left(\operatorname{ext}\left(\operatorname{conv}\left(\mathcal{Y}_\mathrm{N}\right) + \mathbb{R}^p_{\geqq}\right)\right) + \mathbb{R}^p_{\geqq} \\
&\overset{\text{Lemma 8.18}}{=} \operatorname{conv}\left(\mathcal{Y}_\mathrm{ESN}\right) + \mathbb{R}^p_{\geqq}.
\end{aligned}$$

□

9 CONCLUSION

In this thesis, we comprehensively studied solution methods for robust multi-objective optimization. Our focus was on three generalizations of robustness minmax robustness and on regret robustness. While not exhaustive, these concepts hold the potential to advance the field of multi-objective optimization.

We started by investigating point-based minmax robust efficiency. We showed that an iterative optimization-pessimization method designed to find minmax robust optimal solutions to uncertain single-objective optimization problems can be generalized to a multi-objective setting. The proposed procedure solves two problems in each iteration: An optimization problem that is multi-objective and still uncertain but more manageable than the original problem because it has a reduced (and finite) uncertainty set and a pessimization problem that is deterministic and essentially single-objective. In each iteration, the optimization problem delivers a lower bound for the robust Pareto frontier, while the pessimization problem delivers an upper bound. It is shown that the lower bounds are improving over time and, under some assumptions about the uncertainty, that the bounds converge in a finite number of iterations. This method was shown to be applicable for point-based minmax robust solutions and for point-based minmax robust extreme supported efficient solutions. What is more, it applies both to problems with uncertainty in the objectives and uncertainty in the constraints.

We have also used a method from the field of multi-objective optimization, namely dichotomic search, and showed that, under certain conditions, it can be applied to bi-objective problems with a minmax objective, too. While this result is also of interest for other applications, in the case of uncertain bi-objective optimization problems, it implies that dichotomic search can find point-based minmax robust efficient solutions.

We combined optimization-pessimization with dichotomic search and proposed two algorithms: one starts with robustification and the other with scalarization. A third algorithm, specifically for linear problems in the scenario, combines dichotomic search with a reformulation of the minmax problem. All three algorithms are implemented and tested, and their results are compared. Generally speaking, the algorithms taking a multi-objective optimizer's approach, scalarizing first, and then solving single-objective uncertain problems in each iteration performed best.

Future research could build on the content of this thesis by combining optimization-pessimization with multi-objective methods other than dichotomic search in a similar fashion as presented in Chapter 5. It would be interesting to find out if the observation in this thesis that scalarize-first is faster than robustify-first holds if other methods for multi-objective optimization are used. It would also be of interest to examine convergence guarantees of the multi-objective optimization-pessimization approach under different conditions than those in Chapters 3 and 6.

A better understanding of the Pareto frontier for set-based and hull-based minmax robust efficiency would also help us understand the bounds developed in this thesis and a more comprehensive investigation of the relationship between minmax and regret-robust efficiency would help to gain insights into the extent to which the latter is more difficult–or not.

BIBLIOGRAPHY

[ABV09] H. Aissi, C. Bazgan and D. Vanderpooten. 'Min–max and min–max regret versions of combinatorial optimization problems: A survey'. In: *European Journal of Operational Research* 197.2 (2009), pp. 427–438. ISSN: 0377-2217.

[AC16] M. J. Alves and J. P. Costa. 'Graphical exploration of the weight space in three-objective mixed integer linear programs'. In: *European Journal of Operational Research* 248.1 (2016), pp. 72–83.

[AN79] Y. P. Aneja and K. P. K. Nair. 'Bicriteria Transportation Problem'. In: *Management Science* 25.1 (1979), pp. 73–78.

[Ant+20] T. Antczak, Y. Pandey, V. Singh and S. K. Mishra. 'On approximate efficiency for nonsmooth robust vector optimization problems'. In: *Acta Mathematica Scientia* 40.3 (2020), pp. 887–902.

[Ass+08] T. Assavapokee, M. J. Realff, J. C. Ammons and I.-H. Hong. 'Scenario relaxation algorithm for finite scenario-based min–max regret and min–max relative regret robust optimization'. In: *Computers & operations research* 35.6 (2008), pp. 2093–2102.

[BA06] C. Barrico and C. Antunes. 'Robustness Analysis in Multi-Objective Optimization Using a Degree of Robustness Concept'. In: *IEEE Congress on Evolutionary Computation. CEC 2006.* IEEE Computer Society, 2006, pp. 1887 – 1892.

[BDL16] D. Bertsimas, I. Dunning and M. Lubin. 'Reformulation versus cutting-planes for robust optimization'. In: *Computational Management Science* 13.2 (2016), pp. 195–217.

[Ben98] H. P. Benson. 'An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem'. In: *Journal of Global Optimization* 13 (1998), pp. 1–24.

[BF17] R. Bokrantz and A. Fredriksson. 'Necessary and sufficient conditions for Pareto efficiency in robust multiobjective optimization'. In: *European Journal of Operational Research* 262.2 (2017), pp. 682–692.

[BF76] J. W. Blankenship and J. E. Falk. 'Infinitely constrained optimization problems'. In: *Journal of Optimization Theory and Applications* 19 (1976), pp. 261–281.

[BH22] D. Bertsimas and D. den Hertog. *Robust and Adaptive Optimization*. Dynamic Ideas LLC, 2022. ISBN: 9781733788526.

[Bie07] D. Bienstock. 'Histogram models for robust portfolio optimization'. In: *Journal of computational finance* 11.1 (2007), p. 1.

[Bit80] G. R. Bitran. 'Linear Multiple Objective Problems with Interval Coefficients'. In: *Management Science* 26 (1980), pp. 694–706.

[BM15] F. Bökler and P. Mutzel. 'Output-sensitive algorithms for enumerating the extreme nondominated points of multiobjective combinatorial optimization problems'. In: *Algorithms-ESA 2015: 23rd Annual European Symposium, Patras, Greece, September 14-16, 2015, Proceedings*. Springer. 2015, pp. 288–299.

[BNA13] M. Bürger, G. Notarstefano and F. Allgöwer. 'A polyhedral approximation framework for convex and robust distributed optimization'. In: *IEEE Transactions on Automatic Control* 59.2 (2013), pp. 384–395.

[BQT19] G. Bouza, E. Quintana and C. Tammer. 'A unified characterization of nonlinear scalarizing functionals in optimization'. In: *Vietnam Journal of Mathematics* 47 (2019), pp. 683–713.

[BS04] D. Bertsimas and M. Sim. 'The Price of Robustness'. In: *Operations Research* 52.1 (2004), pp. 35–53.

[BS19] M. Botte and A. Schöbel. 'Dominance for multi-objective robust optimization concepts'. In: *European Journal of Operational Research* 273.2 (2019), pp. 430–440. ISSN: 0377-2217.

[BTEN09] A. Ben-Tal, L. El Ghaoui and A. Nemirovski. *Robust Optimization*. Princeton Series in Applied Mathematics. Princeton University Press, 2009.

[BV04] S. P. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[CCBM22] E. Caprari, L. Cerboni Baiardi and E. Molho. 'Scalarization and robustness in uncertain vector optimization problems: a non componentwise approach'. In: *Journal of Global Optimization* 84.2 (2022), pp. 295–320.

[CHSS23] F. Chlumsky-Harttmann, M. Schmidt and A. Schöbel. 'Approaches for Biobjective Integer Linear Robust Optimization'. In: *arXiv preprint arXiv:2311.01883* (2023).

[Chu20] T. D. Chuong. 'Robust Optimality and Duality in Multiobjective Optimization Problems under Data Uncertainty'. In: *SIAM Journal on Optimization* 30.2 (2020), pp. 1501–1526.

[Coh78] J. Cohon. *Multiobjective Programming and Planning*. Academic Press, 1978.

[CPF21] P. Correia, L. Paquete and J. R. Figueira. 'Finding multi-objective supported efficient spanning trees'. In: *Computational Optimization and Applications* 78 (2021), pp. 491–528.

[CV14] K. Chandrasekaran and S. S. Vempala. 'Integer feasibility of random polytopes: random integer programs'. In: *Proceedings of the 5th conference on Innovations in theoretical computer science*. 2014, pp. 449–458.

[DG06] K. Deb and H. Gupta. 'Introducing Robustness in Multi-Objective Optimization'. In: *Evolutionary Computation* 14.4 (2006), pp. 463–494.

[DW09] M. Dellnitz and K. Witting. 'Computation of robust Pareto points'. In: *International Journal of Computing Science and Mathematics* 2.3 (2009), pp. 243–266.

[EF09a] A. Eusébio and J. R. Figueira. 'Finding non-dominated solutions in bi-objective integer network flow problems'. In: *Computers & Operations Research* 36.9 (2009), pp. 2554–2564.

[EF09b] A. Eusébio and J. R. Figueira. 'On the computation of all supported efficient solutions in multi-objective integer network flow problems'. In: *European Journal of Operational Research* 199.1 (2009), pp. 68–76.

[Ehr05] M. Ehrgott. *Multicriteria Optimization*. Springer Nature, 2005.

[Ehr06] M. Ehrgott. 'A discussion of scalarization techniques for multiple objective integer programming'. In: *Annals of Operations Research* 147.1 (2006), pp. 343–360.

[Ehr08] M. Ehrgott. 'Multiobjective optimization'. In: *Ai Magazine* 29.4 (2008), pp. 47–47.

[Eic09] G. Eichfelder. 'Scalarizations for adaptively solving multi-objective optimization problems'. In: *Computational Optimization and Applications* 44 (2009), pp. 249–273.

[Eic21] G. Eichfelder. 'Twenty years of continuous multiobjective optimization in the twenty-first century'. In: *EURO Journal on Computational Optimization* 9 (2021), p. 100014. ISSN: 2192-4406.

[EIS14] M. Ehrgott, J. Ide and A. Schöbel. 'Minmax Robustness for Multi-objective Optimization Problems'. In: *European Journal of Operational Research* 239 (2014), pp. 17–31.

[EJ11] G. Eichfelder and J. Jahn. 'Vector optimization problems and their solution concepts'. In: *Recent Developments in Vector Optimization*. Springer, 2011, pp. 1–27.

[EKS17] G. Eichfelder, C. Krüger and A. Schöbel. 'Decision uncertainty in multiobjective optimization'. In: *Journal of Global Optimization* (2017). online first, pp. 1–26.

[EQ24] G. Eichfelder and E. Quintana. 'Set-based robust optimization of uncertain multiobjective problems via epigraphical reformulations'. In: *European Journal of Operational Research* 313.3 (2024), pp. 871–882.

[ES20] A. Engau and D. Sigler. 'Pareto Solutions in Multicriteria Optimization under Uncertainty'. In: *European Journal of Operational Research* 281.2 (2020), pp. 357–368.

[Fad+05] G. Fadel, I. Haque, V. Blouin and M. Wiecek. 'Multi-criteria multi-scenario approaches in the design of vehicles'. In: *6th World Congress of Structural and Multidisciplinary Optimization, Rio de Janeiro, Brazil*. 2005.

[Fig+17] J. R. Figueira, C. M. Fonseca, P. Halffmann, K. Klamroth, L. Paquete, S. Ruzika, B. Schulze, M. Stiglmayr and D. Willems. 'Easy to say they are hard, but hard to see they are easy—towards a categorization of tractable multiobjective combinatorial optimization problems'. In: *Journal of Multi-Criteria Decision Analysis* 24.1-2 (2017), pp. 82–98.

[FW14] J. Fliege and R. Werner. 'Robust multiobjective optimization & applications in portfolio optimization'. In: *European Journal of Operational Research* 234.2 (2014), pp. 422–433.

[GA05] S. Gunawan and S. Azarm. 'Multi-objective robust optimization using a sensitivity region concept'. In: *Structural and Multidisciplinary Optimization* 29 (2005), pp. 50–60.

[Geo68] A. M. Geoffrion. 'Proper efficiency and the theory of vector maximization'. In: *Journal of mathematical analysis and applications* 22.3 (1968), pp. 618–630.

[GLP13] P. G. Georgiev, D. T. Luc and P. M. Pardalos. 'Robust aspects of solutions in deterministic multiple objective linear programming'. In: *European Journal of Operational Research* 229.1 (2013), pp. 29–36.

[Gob+15] M. Goberna, V. Jeyakumar, G. Li and J. Vicente-Pérez. 'Robust solutions to multi-objective linear programs with uncertain data'. In: *European Journal of Operational Research* 242 (2015), pp. 730–743.

[GS16] M. Goerigk and A. Schöbel. 'Algorithm Engineering in Robust Optimization'. In: *Algorithm Engineering: Selected Results and Surveys*. Ed. by L. Kliemann and P. Sanders. Vol. 9220. LNCS State of the Art. 2016, pp. 245–279.

[Gur23] Gurobi Optimization, LLC. *Gurobi Optimizer Reference Manual*. 2023.

[GW22] P. Groetzner and R. Werner. 'Multiobjective optimization under uncertainty: A multiobjective robust (relative) regret approach'. In: *European Journal of Operational Research* 296.1 (2022), pp. 101–115. ISSN: 0377-2217.

[GYD15] B. L. Gorissen, İ. Yanıkoğlu and D. Den Hertog. 'A practical guide to robust optimization'. In: *Omega* 53 (2015), pp. 124–137.

[Hal+24] E. Halser, E. Finhold, N. Leithäuser, J. Schwientek, K. Teichert and K.-H. Küfer. 'Multicriteria Adjustable Robustness'. In: *arXiv preprint arXiv:2406.07959* (2024).

[Hen86] M. I. Henig. 'The domination property in multicriteria optimization'. In: *Journal of Mathematical Analysis and Applications* 114.1 (1986), pp. 7–16.

[HNS13] F. Hassanzadeh, H. Nemati and M. Sun. 'Robust Optimization for Multiobjective Programming Problems with Imprecise Information'. In: *Procedia Computer Science* 17 (2013), pp. 357 –364. ISSN: 1877-0509.

[HNS14] F. Hassanzadeh, H. Nemati and M. Sun. 'Robust optimization for interactive multiobjective programming with imprecise information applied to R&D project portfolio selection'. In: *European Journal of Operational Research* 238.1 (2014), pp. 41–53.

[HPR07] H. W. Hamacher, C. R. Pedersen and S. Ruzika. 'Multiple objective minimum cost flow problems: A review'. In: *European Journal of Operational Research* 176.3 (2007), pp. 1404–1422.

[Ide+14] J. Ide, E. Köbis, D. Kuroiwa, A. Schöbel and C. Tammer. 'The relationship between multi-objective robustness concepts and set valued optimization'. In: *Fixed Point Theory and Applications* 2014.83 (2014).

[IK14] J. Ide and E. Köbis. 'Concepts of efficiency for uncertain multi-objective optimization problems based on set order relations'. In: *Math Meth Oper Res* 80 (2014), pp. 99–127.

[IS16] J. Ide and A. Schöbel. 'Robustness for uncertain multi-objective optimization: A survey and analysis of different concepts'. In: *OR Spectrum* 38.1 (2016), pp. 235–271.

[IS96] M. Inuiguchi and M. Sakawa. 'Possible and necessary efficiency in possibilistic multiobjective linear programming problems and possible efficiency test'. In: *Fuzzy Sets and Systems* 78.2 (1996). Fuzzy Multiple Criteria Decision Making, pp. 231 –241. ISSN: 0165-0114.

[KD69] R. Kruse and J. Deely. 'Joint continuity of monotonic functions'. In: *The American Mathematical Monthly* 76.1 (1969), pp. 74–76.

[KDD16] M. Kalantari, C. Dong and I. J. Davies. 'Multi-objective robust optimisation of unidirectional carbon/glass fibre reinforced hybrid composites under flexural loading'. In: *Composite Structures* 138 (2016), pp. 264–275.

[Kha+09] L. Khachiyan, E. Boros, K. Borys, V. Gurvich and K. Elbassioni. 'Generating all vertices of a polyhedron is hard'. In: *Twentieth Anniversary Volume: Discrete & Computational Geometry* (2009), pp. 1–17.

[KL12] D. Kuroiwa and G. M. Lee. 'On Robust Multiobjective Optimization'. In: *Vietnam Journal of Mathematics* 40.2&3 (2012), pp. 305–317.

[Krü+23] C. Krüger, A. Schöbel, L. Fritzen and M. M. Wiecek. 'The point-based robustness gap for uncertain multiobjective optimization'. In: *Optimization* (2023), pp. 1–35.

[KS23] D. Könen and M. Stiglmayr. 'An output-polynomial time algorithm to determine all supported efficient solutions for multi-objective integer network flow problems'. In: *arXiv preprint arXiv:2305.12867* (2023).

[Kuh+16] K. Kuhn, A. Raith, M. Schmidt and A. Schöbel. 'Bicriteria robust optimization'. In: *European Journal of Operational Research* 252 (2016), pp. 418–431.

[Kur99] D. Kuroiwa. 'Some duality theorems of set-valued optimization with natural criteria'. In: *Proceedings of the international conference on nonlinear analysis and convex analysis*. World Scientific Singapore. 1999, pp. 221–228.

[KY13] P. Kouvelis and G. Yu. *Robust discrete optimization and its applications*. Vol. 14. Springer Science & Business Media, 2013.

[MB09] A. Mutapcic and S. Boyd. 'Cutting-set methods for robust convex optimization with pessimizing oracles'. In: *Optimization Methods and Software* 24.3 (2009), pp. 381–406.

[Mon06] R. Montemanni. 'A Benders decomposition approach for the robust spanning tree problem with interval data'. In: *European Journal of Operational Research* 174.3 (2006), pp. 1479–1490.

[NKM13] Y. Nikulin, O Karelkina and M. M. Mäkelä. 'On accuracy, robustness and tolerances in vector Boolean optimization'. In: *European Journal of Operational Research* 224.3 (2013), pp. 449–457.

[ÖK10] Ö. Özpeynirci and M. Köksalan. 'An exact algorithm for finding extreme supported nondominated points of multiobjective mixed integer programs'. In: *Management Science* 56.12 (2010), pp. 2302–2315.

[PGE10] A. Przybylski, X. Gandibleux and M. Ehrgott. 'A recursive algorithm for finding all nondominated extreme points in the outcome set of a multiobjective integer programme'. In: *INFORMS Journal on Computing* 22.3 (2010), pp. 371–386.

[PKL19] A. Przybylski, K. Klamroth and R. Lacour. *A simple and efficient dichotomic search algorithm for multi-objective mixed integer linear programs*. 2019. arXiv: `1911.08937 [math.OC]`.

[PS20] J. Pätzold and A. Schöbel. 'Approximate cutting plane approaches for exact solutions to robust optimization problems'. In: *European Journal of Operational Research* 284.1 (2020), pp. 20–30.

[PS84] A. Pascoletti and P. Serafini. 'Scalarizing vector optimization problems'. In: *Journal of Optimization Theory and Applications* 42 (1984), pp. 499–524.

[Rai+18a] A. Raith, M. Schmidt, A. Schöbel and L. Thom. 'Extensions of labeling algorithms for multi-objective uncertain shortest path problems'. In: *Networks* 72.1 (2018), pp. 84–127.

[Rai+18b] A. Raith, M. Schmidt, A. Schöbel and L. Thom. 'Multi-objective minmax robust combinatorial optimization with cardinality-constrained uncertainty'. In: *European Journal of Operational Research* 267.2 (2018), pp. 628–642.

[RE09a] A. Raith and M. Ehrgott. 'A comparison of solution strategies for biobjective shortest path problems'. In: *Computers & Operations Research* 36.4 (2009), pp. 1299–1331.

[RE09b] A. Raith and M. Ehrgott. 'A two-phase algorithm for the biobjective integer minimum cost flow problem'. In: *Computers & Operations Research* 36.6 (2009), pp. 1945–1954.

[Ree94] R. Reemtsen. 'Some outer approximation methods for semi-infinite optimization problems'. In: *Journal of Computational and Applied Mathematics* 53.1 (1994), pp. 87–108.

[RSN17] A. Raith and A. Sedeño-Noda. 'Finding extreme supported solutions of biobjective network flow problems: An enhanced parametric programming approach'. In: *Computers & Operations Research* 82 (2017), pp. 153–166.

[RSS89] M. J. Rosenblatt and Z. Sinuany-Stern. 'Generating the discrete efficient frontier to the capital budgeting problem'. In: *Operations Research* 37.3 (1989), pp. 384–394.

[RY13] S. Rivaz and M. Yaghoobi. 'Minimax regret solution to multiobjective linear programming problems with interval objective functions coefficients'. In: *Central European Journal of Operations Research* 21.3 (2013), pp. 625–649.

[SAG11] S. Siddiqui, S. Azarm and S. Gabriel. 'A modified Benders decomposition method for efficient robust optimization under interval uncertainty'. In: *Structural and Multidisciplinary Optimization* 44 (2011), pp. 259–275.

[Sim+18] G. Simões, M. McDonald, S. Williams, D. Fenn and R. Hauser. 'Relative robust portfolio optimization with benchmark regret'. In: *Quantitative Finance* 18.12 (2018), pp. 1991–2003.

[SKS19] B. Schulze, K. Klamroth and M. Stiglmayr. 'Multi-objective unconstrained combinatorial optimization: a polynomial bound on the number of extreme supported solutions'. In: *Journal of Global Optimization* 74.3 (2019), pp. 495–522.

[SNR15] A. Sedeno-Noda and A. Raith. 'A Dijkstra-like method computing all extreme supported non-dominated solutions of the biobjective shortest path problem'. In: *Computers & Operations Research* 57 (2015), pp. 83–94.

[SZK21] A. Schöbel and Y. Zhou-Kangas. 'The price of multiobjective robustness: Analyzing solution sets to uncertain multiobjective problems'. In: *European Journal of Operational Research* 291.2 (2021), pp. 782–793.

[UT94] E. L. Ulungu and J. Teghem. 'Multi-objective combinatorial optimization problems: A survey'. In: *Journal of Multi-Criteria Decision Analysis* 3.2 (1994), pp. 83–104.

[WCL20a] H.-Z. Wei, C.-R. Chen and S.-J. Li. 'A Unified Approach Through Image Space Analysis to Robustness in Uncertain Optimization Problems'. In: *Journal of Optimization Theory and Applications* 184.2 (2020), pp. 466–493.

[WCL20b] H.-Z. Wei, C.-R. Chen and S.-J. Li. 'Characterizations of multiobjective robustness on vectorization counterparts'. In: *Optimization* 69.3 (2020), pp. 493–518.

[WD16] M. M. Wiecek and G. M. Dranichak. 'Robust Multiobjective Optimization for Decision Making Under Uncertainty and Conflict'. In: *Optimization Challenges in Complex, Networked and Risky Systems*. INFORMS, 2016. Chap. 4, pp. 84–114.

[Wie+09] M. M. Wiecek, V. Y. Blouin, G. M. Fadel, A. Engau, B. J. Hunt and V. Singh. 'Multi-scenario Multi-objective Optimization with Applications in Engineering Design'. In: *Multiobjective Programming and Goal Programming*. Ed. by V. Barichard, M. Ehrgott, X. Gandibleux and V. T'Kindt. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009, pp. 283–298. ISBN: 978-3-540-85646-7.

[WLC15] F. Wang, S. Liu and Y. Chai. 'Robust counterparts and robust efficient solutions in vector optimization under uncertainty'. In: *Operations Research Letters* 43 (2015), pp. 293–298.

[WOBD13] K. Witting, S. Ober-Blöbaum and M. Dellnitz. 'A variational approach to define robustness for parametric multiobjective optimization problems'. In: *Journal of Global Optimization* 57.2 (2013), pp. 331–345.

[Xid+17] P. Xidonas, G. Mavrotas, C. Hassapis and C. Zopounidis. 'Robust multiobjective portfolio optimization: A minimax regret approach'. In: *European Journal of Operational Research* 262.1 (2017), pp. 299 –305.

[You31] R. C. Young. 'The algebra of many-valued quantities'. In: *Mathematische Annalen* 104.1 (1931), pp. 260–290.

Fabian Chlumsky-Harttmann

Curriculum Vitae

since 08/2019	Doctoral Studies in Mathematics, RPTU in Kaiserslautern
since 04/2019	Research Assistant, RPTU in Kaiserslautern
08/2018 – 03/2019	Research Assistant, University of Göttingen
07/2018	Master of Science in Mathematics, University of Göttingen
05/2015 – 09/2017	Teaching Assistant, University of Göttingen
04/2016 – 07/2018	Master Studies in Mathematics, University of Göttingen
03/2016	Bachelor of Science in Mathematics, University of Göttingen
10/2012 – 03/2016	Bachelor Studies in Mathematics, University of Göttingen
06/2012	Abitur, Martin-Niemöller-Schule, Wiesbaden

Fabian Chlumsky-Harttmann

Wissenschaftlicher Werdegang

seit 08/2019	Promotionsstudium in Mathematik, RPTU in Kaiserslautern
seit 04/2019	Wissenschaftlicher Mitarbeiter, RPTU in Kaiserslautern
08/2018 – 03/2019	Wissenschaftlicher Mitarbeiter, Universität Göttingen
07/2018	Master of Science in Mathematik, Universität Göttingen
05/2015 – 09/2017	Studentische Hilfskraft, Universität Göttingen
04/2016 – 07/2018	Masterstudium in Mathematik, Universität Göttingen
03/2016	Bachelor of Science in Mathematik, Universität Göttingen
10/2012 – 03/2016	Bachelorstudium in Mathematik, Universität Göttingen
06/2012	Abitur, Martin-Niemöller-Schule, Wiesbaden

COLOPHON

This document was typeset using the LaTeX package `classicthesis` developed by André Miede and Ivo Pletikosić. It is available at:

`https://bitbucket.org/amiede/classicthesis/`