Significance of zero modes in path–integral quantization of solitonic theories with BRST invariance

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Abstract

The significance of zero modes in the path–integral quantization of some solitonic models is investigated. In particular a Skyrme–like theory with topological vortices in $(1+2)$ dimensions is studied, and with a BRST invariant gauge fixing a well defined transition amplitude is obtained in the one loop approximation. We also present an alternative method which does not necessitate evoking the time–dependence in the functional integral, but is equivalent to the original one in dealing with the quantization in the background of the static classical solution of the non–linear field equations. The considerations given here are particularly useful in – but also limited to –the one–loop approximation.

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1 Introduction

The solutions of static (time–independent) non–linear classical Euler–Lagrange equations are of particular interest in many field theories. The energy of the static classical configuration (named soliton) is higher than the minimum energy of a constant field (the perturbation theory vacuum) by a finite amount. Interpreting the space coordinate x as a Euclidean time in $(1+1)$ dimensional field equations the static classical configuration is the same as the $(1+0)$ dimensional instanton which is responsible for quantum tunneling through barriers [1]–[4]. From translation invariance in $(1+1)$ dimensions it is clear that the solution of the static equation is a function of $x - a$ where a is an arbitrary constant of integration, corresponding to the origin of some reference frame or the position of the soliton. We may set the constant a to zero since no physically interesting quantity can depend on it [5]. However, because of the translation invariance, functional integrals, i. e. transition amplitudes in the path–integral formulation are not well defined as can be seen by recalling that the second variation of the action S about a static solution leads to zero modes which give rise to undefined integrals in the perturbation expansion about the soliton. In order to cure this problem one often resorts to the so–called Faddeev–Popov technique [6] by inserting an identity for 1 with a δ–function integral which transforms the integral over a zero mode into a continous integral over the translation parameter a or the position of the soliton. In this procedure much care has to be taken in calculating the Jacobian of the transformation. In the collective coordinate method with BRST invariance [7] – [10] the shift of the integration variable related to the zero mode is achieved in a natural way by regarding the position of the soliton as a new dynamical variable, namely the collective coordinate, which depends on time. The static solution thus becomes time–dependent through the collective coordinate. Generally speaking, whenever the action of a static field possesses some symmetry the operator of the second variation of the action about the classical static solution has corresponding zero modes [11, 12]. The BRST invariant gauge fixing breaks these symmetries and gives rise to well defined functional integrals. However, applications of the method to specific models are scarce since only very few permit explicit calculations. Below we consider two such models with the intent to expose in particular the vital role played by zero modes.

In the following we therefore investigate the path–integral quantization of some solitonic models and begin with a theory with a Skyrme–like soliton with absolute scale in $(1+2)$ dimensions [12]. The model — motivated by Skyrme–term–modified σ models [13] and useful e. g. as the skeleton of a superconducting cosmic string [14] — involves a complex scalar field and lends itself for some aspects more readily as a less trivial testing ground of the quantization method than the usual soliton model in $(1+1)$ dimensions. In spite of its drawback of being nonrenormalizable in the usual sense it has some intriguing positive aspects which make it a useful laboratory for the study of various phenomena.

Thus the model allows the explicit solution of the classical static equation and the explicit demonstration of the topological stability of the solutions. The nonzero topological charge of these static classical solutions provides a lower bound to the energy integral and renders the solutions stable. But it is also possible to find nontopological static solutions with finite energy which are classically unstable [15]. The topological vortex–type solutions of the model are very similar to those of the Nielsen–Olesen model [16] but are generated by the same complex scalar field and can be calculated explicitly in closed form [12], which makes the model in any case an interesting laboratory. However, the model has another fascinating property. Being a model in $(2+1)$ dimensions it can be supersymmetrized by a well–known method [17]. $(2+1)$ dimensions is the lowest dimensionality for which the topological charge appears as the central charge of an $N=2$ extended supersymmetry of the $N=1$ supersymmetrized theory. The requirement that a theory with a topological charge have a consistent superymmetric extension implies that the theory exhibits a Bogomol'nyi relation [18] which can be shown to be the case. Our Skyrme–like model is a model with these properties. In the case of this Skyrmion theory [12] to be considered below the action has not only the usual translation symmetry but also a global $U(1)$ symmetry which together lead to three zero modes of the second variation operator about the classical configuration and hence to an interesting set of constraints. Though the kinetic energy term is quartic the constraints induced by introducing collective coordinates are seen to be primary and first class. With the help of a BRST invariant gauge fixing a well defined transition amplitude is given up to the one loop approximation.

We then show in the context of a different model that it is not necessary to evoke the time dependence in the functional integrals by regarding transformation parameters as time–dependent dynamical variables. Expanding the action about the classical static–field configuration the classical action can be factored out. The remaining fluctuation part of the action possesses a new and very interesting symmetry under a shift by zero modes (analogous to observations made in [19]). This symmetry can also be broken with BRST invariant gauge fixing to give the desired transition amplitude. The two treatments are seen to be equivalent in the one loop approximation, and the latter is restricted to this.

We add that the significance of zero modes has been discussed in particular by Peskin [20] and Bernard [21]. The BRST quantization of a soliton in $(2+1)$ dimensions has recently been discussed in [22]. General considerations of collective coordinates are discussed in [23].

2 Symmetry of the Skyrme–like soliton theory, collective coordinates and zero modes

The model we consider first is defined by the following Lagrangian density in $(1+2)$ dimensional Minkowski space [12] $(\eta_{00}=1, \eta_{ij}=-\delta_{ij})$,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(i\partial_{[\mu}\varphi\partial_{\nu]}\varphi^*)(i\partial^{[\nu}\varphi\partial^{\mu]}\varphi^*) - U(\varphi\varphi^*) \\ &= -(\partial_0\varphi\partial_i\varphi^* - \partial_i\varphi\partial_0\varphi^*)^2 + \frac{1}{2}(\partial_i\varphi\partial_j\varphi^* - \partial_j\varphi\partial_i\varphi^*)^2 - U(\varphi\varphi^*) \end{aligned} \tag{1}$$

The finite–energy soliton–like solutions we are interested in are the static field configurations φ_c, φ_c^* (with $\partial_0\varphi_c = \partial_0\varphi_c^* = 0$) which minimize the energy, namely the static Hamiltonian

$$H_0 = \int \mathcal{H}_0 d^2x = -\int \mathcal{L}_0 d^2x = S_0 \tag{2}$$

where the Lagrangian density of the static version is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_i\varphi\partial_j\varphi^* - \partial_j\varphi\partial_i\varphi^*)^2 - U(\varphi\varphi^*) \tag{3}$$

and

$$U(\varphi\varphi^*) = \lambda^2(\eta^2 - \varphi\varphi^*)^{2k}, \tag{4}$$

$k > 1$ an integer. The finite energy soliton–like fields φ_c, φ_c^* with vortex shape studied in reference [12] are

$$\varphi_c = \eta R(r)e^{in\theta}, \qquad \varphi_c^* = \eta R(r)e^{-in\theta} \tag{5}$$

and represent maps from S^1 to S^1, thus belonging to equivalence classes of the fundamental homotopy group $\Pi_1(S^1)$ which are characterized by an integer n, the homotopy charge.

The action $S_0 = \int \mathcal{L}_0 d^2x$ is invariant under translations of the origin of the coordinate system, $\vec{x} \to \vec{x} + \vec{a}$, and is also invariant under a global $U(1)$ transformation $\varphi \to e^{i\alpha}\varphi, \varphi^* \to e^{-i\alpha}\varphi^*$. In view of the associated degeneracy of the action, functional integrals representing e. g. the transition amplitude, are not well defined since the symmetries result in zero modes of the second variation operator of the action about φ_c. These zero modes can be analysed in a general context. The classical solutions φ_c, φ_c^*, of course, are obtained by minimizing the action S_0,

$$\left.\frac{\delta S_0}{\delta\varphi}\right|_{\varphi=\varphi_c}\delta\varphi + \left.\frac{\delta S_0}{\delta\varphi^*}\right|_{\varphi^*=\varphi_c^*}\delta\varphi^* = 0 \tag{6}$$

which leads to the equations of motion with solutions φ_c, φ_c^*.

If the action S_0 as well as the equation of motion (6), possess some symmetries, namely, if they are invariant under correspondig transformations $\varphi \to \varphi' = \varphi + \delta\varphi$, the second variation for the zero modes $\delta\Phi_0 = (\delta\varphi_0, \delta\varphi_0^*)$ vanishes, i. e.

$$\int d^2x d^2y \delta\Phi_0 \hat{M}(\varphi_c, \varphi_c^*) \delta^2(x-y) \delta\Phi_0(\eta) = 0 \tag{7}$$

where

$$\hat{M}(\varphi_c, \varphi_c^*) = \begin{pmatrix} \left[\frac{\delta^2 S_0}{\delta\varphi\delta\varphi}\Big|_{\varphi=\varphi_c} \right] & \left[\frac{\delta^2 S_0}{\delta\varphi\delta\varphi^*}\Big|_{\substack{\varphi=\varphi_c \\ \varphi^*=\varphi_c^*}} \right] \\ \left[\frac{\delta^2 S_0}{\delta\varphi^*\delta\varphi}\Big|_{\substack{\varphi=\varphi_c \\ \varphi^*=\varphi_c^*}} \right] & \left[\frac{\delta^2 S_0}{\delta\varphi^*\delta\varphi^*}\Big|_{\varphi^*=\varphi_c^*} \right] \end{pmatrix}. \tag{8}$$

The elements of the operator $\hat{M}$ are given by

$$\left[\frac{\delta^2 S_0}{\delta\varphi\delta\varphi}\Big|_{\varphi=\varphi_c} \right] = -2 \left[\partial_i (\delta_{il}^{jk} \partial_k \varphi_c^* \partial_l \varphi_c^* \partial_j) - \lambda^2 k(2k-1)(\eta^2 - |\varphi_c|^2)^{2(k-1)} \varphi_c^* \varphi_c^* \right] \tag{9}$$

$$\begin{aligned} \left[\frac{\delta^2 S_0}{\delta\varphi\delta\varphi^*}\Big|_{\substack{\varphi=\varphi_c \\ \varphi^*=\varphi_c^*}} \right] &= -2 \Big\{ \partial_i \left[(\delta_{il}^{kj} + \delta_{ij}^{kl}) \partial_k \varphi_c \partial_l \varphi_c^* \partial_j \right] \\ &+ \lambda^2 k (\eta^2 - |\varphi_c|^2)^{2(k-1)} (\eta^2 - 2k|\varphi_c|^2) \Big\} \end{aligned} \tag{10}$$

where $\delta_{il}^{jk} = \delta_i^j \delta_l^k - \delta_i^k \delta_l^j$. The other two elements of (8) are obtained by complex conjugation of (9) and (10). Therefore the operator $\hat{M}$ of the second variation has zero modes

$$\Psi_0^i = \left(\frac{\partial\varphi_c}{\partial a_i}, \frac{\partial\varphi_c^*}{\partial a_i} \right) \tag{11}$$

where $\frac{\partial}{\partial a_i}$ $(i = 1, 2, 3)$ denote the generators of the transformation. In our case the translation invariance gives rise to the two zero modes

$$\Psi_0^j = (\partial_j \varphi_c, \partial_j \varphi_c^*), \qquad j = 1, 2, \tag{12}$$

and the global $U(1)$ symmetry results in the third zero mode

$$\Psi_0^3 = \left(\frac{\partial}{\partial\theta} \varphi_c, \frac{\partial}{\partial\theta} \varphi_c^* \right). \tag{13}$$

Path integral quantization of static non–linear fields about their classical configurations with BRST invariance provides a systematic procedure to remove the degeneracy of the action and therefore leads to a meaningful transition amplitude. In order to see this we first of all elevate the parameters $\vec{a}$ and α to new dynamical variables depending on time. The field about the classical configuration can be written

$$\varphi(\vec{x}, t) = \varphi_c' + \eta(\vec{x}, t) \tag{14}$$

where

$$\varphi_c' = e^{i\alpha(t)}\varphi_c(\vec{x} - \vec{a}(t)) \tag{15}$$

and $\eta(\vec{x}, t)$ is considered as a small fluctuation about φ_c'. The time dependence of $\alpha(t)$ and $\vec{a}(t)$ is determined by the time–dependent Euler–Lagrange equations associated with the Lagrangian (1). In the following we shall consider phase space functional integrals. The conjugate momenta of the fluctuation fields η and η^* are defined as usual,

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \eta)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)}, \quad \pi^* = \frac{\partial \mathcal{L}}{\partial(\partial_0 \eta^*)} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^*)} \tag{16}$$

where

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} = -2\left(\partial_0\varphi\partial_i\varphi^* - \partial_i\varphi\partial_0\varphi^*\right)\partial_i\varphi^* \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi^*)} = 2\left(\partial_0\varphi\partial_i\varphi^* - \partial_i\varphi\partial_0\varphi^*\right)\partial_i\varphi \tag{18}$$

The Hamiltonian density is

$$\mathcal{H} = \pi\partial_0\eta + \pi^*\partial_0\eta^* - \mathcal{L}. \tag{19}$$

Regarding $\vec{a}$ and α as dynamical variables the definition of conjugate momenta for $\vec{a}$ and α leads to constraints:

$$P_i = \frac{\partial L}{\partial \dot{a}_i} = \int d^2x \left[\pi \nabla_i \varphi_c e^{i\alpha} + c.c.\right], \qquad i = 1, 2, 3. \tag{20}$$

The constraints are:

$$\Phi_i = P_i - \int d^2x \left[\pi \nabla_i \varphi_c e^{i\alpha} + c.c.\right], \qquad i = 1, 2, 3 \tag{21}$$

where $L = \int \mathcal{L} d^2x$ is the total Lagrangian and the compact notations for $\dot{a}$ and ∇ are defined by $\dot{a} = (\dot{a}_1, \dot{a}_2, \dot{\alpha})$, $\nabla = (\partial_1, \partial_2, i)$, $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ and $P = (P_1, P_2, P_3)$. The total Hamiltonian is correspondingly

$$H = P \cdot \dot{a} + \int d^2x\, \mathcal{H}. \tag{22}$$

3 Gauge transformations induced by constraints

By direct computation we find the trivial first class constraints algebra

$$\{\Phi_i, \Phi_j\} = 0 \tag{23}$$

where $\{,\}$ denotes a Poisson bracket. The condition that the constraints be maintained in the course of time, i. e.

$$\{\Phi_i, H\} = 0. \tag{24}$$

can be verified by rewriting the Hamiltonian in the form

$$H = \Phi \cdot \dot{a} + H' \tag{25}$$

where

$$H' = \frac{1}{2}\int d^2x\,(\pi\,\sqcap + \pi^*\sqcap^*) - L_0 \tag{26}$$

and

$$\sqcap = \partial_0\varphi = \frac{\vec{A}^2\pi + \vec{A}\cdot\vec{A}^*\pi^*}{2[(\vec{A}\cdot\vec{A}^*)^2 - \vec{A}^{*2}\vec{A}^2]} \tag{27}$$

and

$$A_i = \partial_i\varphi, \qquad \vec{A}^2 = \sum_{i=1}^{2}\partial_i\varphi\partial_i\varphi \tag{28}$$

and $L_0 = \int \mathcal{L}_0 d^2x$ is the original static Lagrangian. The $\dot{a}_i$ play the role of Lagrange multipliers.

The constraints are all primary and first class as they should be. The first class constraints generate the following gauge transformation with time–dependent parameters $\Lambda_i(t)$:

$$\begin{aligned}
\delta\eta &= \{\eta, \Lambda\cdot\Phi\} = -\Lambda\cdot\nabla\varphi_c e^{i\alpha} \\
\delta\eta^* &= \{\eta^*, \Lambda\cdot\Phi\} = -\Lambda\cdot\nabla^*\varphi_c^* e^{-i\alpha} \\
\delta\pi &= \delta\pi^* = \delta P = 0 \\
\delta a &= \{a, \Lambda\cdot\Phi\} = \Lambda
\end{aligned} \tag{29}$$

where we have used the notation $\Lambda\cdot\nabla = \Lambda_1\partial_1 + \Lambda_2\partial_2 - i\Lambda_3$, $\Lambda\cdot\nabla^* = \Lambda_1\partial_1 + \Lambda_2\partial_2 + i\Lambda_3$. The first order Lagrangian

$$L = P\cdot\dot{a} + \int(\pi\dot{\eta} + \pi^*\dot{\eta}^*)d^2x - H \tag{30}$$

can be seen to be invariant under the gauge transformation (29) induced by the constraints Φ. We may add terms into the Lagrangian (30) consisting of Lagrange multipliers λ_i multiplied by the constraints Φ_i such that a new Lagrangian is

$$L^T = L - \lambda\cdot\Phi. \tag{31}$$

Considering the Lagrange multipliers λ_i as new dynamical variables with conjugate momenta P_{λ_i} which imply additional constraints $p_{\lambda_i} = 0$, the action S is invariant, i. e.

$$\delta S = \delta\int L^T dt = 0 \tag{32}$$

under this new gauge transformation provided the additional constraints are added to the generator, i. e. we have

$$\delta\lambda_i = \{\lambda_i, \Lambda\cdot\Phi + \dot{\Lambda}\cdot p_\lambda\} = -\dot{\Lambda}_i. \tag{33}$$

We have thus transformed the symmetries of the action with a static classical field configuration into a gauge symmetry. However, the corresponding total Hamiltonian H^T is not gauge invariant except on the subspace defined by the constraints. The next step is therefore to enforce a BRST invariance after which the next step is to fix the gauge symmetry while maintaining the invariance in the sense of an enlarged BRST invariance. One should also note that the Hamiltonian is not invariant under the transformation (29).

4 BRST invariance and gauge fixing

Since we have enlarged the phase space by elevating the Lagrange multipliers λ_i to dynamical variables we correspondingly obtain three new momenta and constraints from L^T. Thus

$$P_\lambda = \frac{\partial L^T}{\partial\dot{\lambda}} = 0. \tag{34}$$

The gauge transformation for the conjugate momenta of λ is trivial

$$\delta P_\lambda = 0. \tag{35}$$

In order to break the gauge invariance of the action S in (32) and therefore to remove the degeneracy, we introduce the anticommuting ghost variables $c = (c_1, c_2, c_3)$, $\bar{c} = \bar{c}_1, \bar{c}_2, \bar{c}_3)$ and their respective canonical conjugate momenta $\Pi_c = (\Pi_{c_1}, \Pi_{c_2}, \Pi_{c_3})$ and $\Pi_{\bar{c}} = (\Pi_{\bar{c}_1}, \Pi_{\bar{c}_2}, \Pi_{\bar{c}_3})$, such that

$$\{c_i, \Pi_{c_j}\}_+ = \delta_{ij}, \qquad \{\bar{c}_i, \Pi_{\bar{c}_j}\}_+ = \delta_{ij} \tag{36}$$

with

$$\Pi_c \equiv L\frac{\overleftarrow{\partial}}{\partial\dot{c}} = \dot{\bar{c}}, \qquad \Pi_{\bar{c}} \equiv \frac{\vec{\partial}}{\partial\dot{\bar{c}}}L = \dot{c} \tag{37}$$

and vanishing Poisson brackets among c's and Π_c's. The BRST charge Ω is

$$\Omega = c\cdot\Phi + P_\lambda\cdot\Pi_{\bar{c}} \tag{38}$$

constructed such as to generate the following BRST transformation corresponding to the gauge transformation (29):

$$\begin{aligned}
\delta\eta &= \{\eta, \Omega\} = -c\cdot\nabla\varphi_c e^{i\alpha} \\
\delta\eta^* &= -c\cdot\nabla^*\varphi_c^* e^{-i\alpha} \\
\delta a &= c \\
\delta\bar{c} &= P_\lambda \\
\delta\Pi_c &= -\Phi
\end{aligned} \tag{39}$$

and other variations are zero. The gauge symmetry of L^T can be broken by adding to L^T the trivially BRST invariant term

$$\begin{aligned} L_{\mathrm{gf}} &= \delta[\bar{c}\cdot\chi - \lambda\cdot\Pi_c] \\ &= P_\lambda\cdot\chi - \bar{c}\cdot\{\chi,\Phi\}\cdot c + \Pi_{\bar{c}}\cdot\Pi_c + \lambda\cdot\Phi \end{aligned} \tag{40}$$

where $\chi = (\chi_1, \chi_2, \chi_3)$ is the chosen gauge fixing condition and is assumed to be independent of ghost variables and constraints, e. g. $\{\chi, P_\lambda\} = 0$, and

$$\bar{c}\cdot\{\chi,\Phi\}\cdot c \equiv \sum_{i,j=1}^{3} \bar{c}_i\{\chi_i,\Phi_j\}c_j. \tag{41}$$

The effective Hamiltonian is

$$\begin{aligned} H^{\mathrm{eff}} &= P\cdot\dot{a} + P_\lambda\cdot\dot{\lambda} + \Pi_c\cdot\dot{c} + \dot{\bar{c}}\cdot\Pi_{\bar{c}} + \int(\pi\dot{\eta} + \pi^*\dot{\eta}^*)d^2x - L^T - L_{\mathrm{gf}} \\ &= P_\lambda\cdot\dot{\lambda} + \bar{c}\cdot\{\chi,\Phi\}\cdot c - \lambda\cdot\Phi + \Pi_c\cdot\Pi_{\bar{c}} + H. \end{aligned} \tag{42}$$

We end up with the transition amplitude,

$$\begin{aligned} \langle\varphi'(\vec{x})|\varphi(\vec{x})\rangle &= \int_{\varphi(\vec{x})}^{\varphi'(\vec{x})} \mathcal{D}\{\varphi\}\mathcal{D}\{\pi\}\mathcal{D}\{\varphi^*\}\mathcal{D}\{\pi^*\}\mathcal{D}\{P\}\mathcal{D}\{a\}\mathcal{D}\{P_\lambda\}\mathcal{D}\{\lambda\} \\ &\quad\times\mathcal{D}\{\Pi_c\}\mathcal{D}\{c\}\mathcal{D}\{\Pi_{\bar{c}}\}\mathcal{D}\{\bar{c}\}e^{iS_B} \end{aligned} \tag{43}$$

where the action is

$$\begin{aligned} S_B &= \int dt\,(L^T + L_{\mathrm{gf}}) \\ &= \int_t^{t'} dt\;(\Pi_c\cdot\Pi_{\bar{c}} - \bar{c}\cdot\{\chi,\Phi\}\cdot c + \lambda\cdot\Phi + \chi\cdot P_\lambda + L)\,. \end{aligned} \tag{44}$$

Integrating out the ghost variables and P_λ, λ is straightforward and yields

$$\begin{aligned} \langle\varphi'(\vec{x})|\varphi(\vec{x})\rangle &= \int_{\varphi(\vec{x})}^{\varphi'(\vec{x})} \mathcal{D}\{\varphi\}\mathcal{D}\{\pi\}\mathcal{D}\{\varphi^*\}\mathcal{D}\{\pi^*\}\mathcal{D}\{P\}\mathcal{D}\{a\} \\ &\quad\times\left(\prod_k \delta[\Phi(k)]\delta[\chi(k)]\right)\det\{\chi_i,\Phi_j\}e^{i\int_t^{t'}Ldt}. \end{aligned} \tag{45}$$

A natural choice for the gauge fixing functional χ is the following in terms of the zero modes Ψ_0^i

$$\chi_i = \int[\eta\Psi_0^i + \eta^*\Psi_0^{*i}]d^2x \tag{46}$$

since this corresponds to the socalled 'unitary' gauge and requires the fluctuation components along zero modes to vanish.

The matrix elements of the determinant $\det\{\chi_i, \Phi_j\}$ can be evaluated by computing the Poisson brackets, i. e.

$$\det\{\chi_i, \Phi_j\} = \int \left\{[\eta\nabla_j(\Psi_0^i e^{i\alpha}) + c.\ c.\ ] - [\Psi_0^i\Psi_0^j + c.\ c.\ ]\right\} d^2x = -\delta_{ij} + \mathcal{O}(\eta) \tag{47}$$

with appropriate normalization to 1 of the zero modes. The integration of the conjugate momenta P_i of the collective coordinates is readily carried out.

We then express the field (φ, φ^*) as the classical configuration plus quantum fluctuations η as in (14) and then expand the transition amplitude in powers of η and retain terms up to the one–loop approximation. The final result of the transition amplitude is

$$\langle\varphi'(\vec{x})|\varphi(\vec{x})\rangle = \int \mathcal{D}\{a\} e^{iS_c} I \tag{48}$$

where S_c is the classical action evaluated at the Skyrme–likesoliton φ_c. The functional integral of the fluctuation is the factor

$$I = \int \mathcal{D}\{\eta\}\mathcal{D}\{\eta^*\}\mathcal{D}\{\pi\}\mathcal{D}\{\pi^*\} e^{i\int dt \int \frac{1}{2}(\pi\sqcap + \pi^*\sqcap^*)d^2x} \prod_i \delta[\chi(i)] \det(N) e^{i\int y^T \hat{M} y d^2x} \tag{49}$$

where $y = (\eta, \eta^*)$ and the operator $\hat{M}$ is defined by eq. (8) and N_{ij} is an element of the matrix N defined by the normalization integral

$$N_{ij} = \int [\Psi_0^i\Psi_0^j + c.\ c.\ ] d^2x. \tag{50}$$

Here, as previously [9, 10], we observe that the coefficient of $\bar{c}c$ in (44), i. e. the mass of the ghosts, is the Poisson bracket of constraints and gauge fixing conditions which in turn is the Faddeev–Popov determinant in the Faddeev–Popov method. The decoupling of ghosts in the leading loop approximation is achieved only with the unitary gauge, which requires fluctuation components in the directions of the zero modes to vanish.

5 Analysis of the transition amplitude in terms of eigenmodes of the second variation operator

One of the purposes of the BRST invariant gauge fixing in the path–integral quantization of nonlinear fields about classical configurations, as so far employed, is to remove the degeneracy of the action which leads to ill defined functional integrals. In order to demonstrate the latter explicitly we consider now $(1+1)$ dimensional ϕ^4 field theory in order not to blur the main points by complications of the Skyrme–like model.

In ϕ^4 field theory the canonical momentum integral of the fluctuation field η in eq. (49) becomes a Gaussian integral and therefore can be evaluated. The determinant is

$$\det N = \sqrt{M_0} \tag{51}$$

where

$$M_0 = \int \left[\frac{1}{2} \left(\frac{d\phi_c}{dx} \right)^2 + U(\phi_c) \right] dx \tag{52}$$

is the soliton mass, which with the use of the static equation of motion is simply the normalization of the zero mode, i.e. $M_0 = \int (\frac{d\phi_c}{dx})^2 dx$.

We let Ψ_n be an eigenmode of the second variation operator $\hat{M}$ such that

$$\hat{M}\Psi_n = E_n \Psi_n \tag{53}$$

Expanding the fluctuation field η in terms of the set $\{\Psi_n\}$, we have

$$\eta = \sum_n c_n \Psi_n. \tag{54}$$

We then change the variable of integration in the path–integral from η to c_n. The path integral for the fluctuation then becomes

$$\begin{aligned} I &= \left| \frac{\partial \eta}{\partial c_n} \right| \int \mathcal{D}\{c_n\} \delta[c_0] e^{i \int dt \sum c_n^2 E_n} \\ &= \sqrt{M_0} \prod_{n \neq 0} \frac{1}{\sqrt{E_n}} \end{aligned} \tag{55}$$

Since the transformation (54) is linear the associated Jacobian determinant is constant and can be factored out from the integration. In our derivation all constant quantities have been dropped. Here the BRST invariant gauge fixing plays only the role of bringing in the factor $\delta[c_0]$ to remove the undefined integration over the zero mode $\mathcal{D}\{c_0\}$ by replacing it by the integration da.

It is interesting to observe that it suffices to consider only the static Lagrangian without recalling the time dependence. We start from the transition amplitude

$$\langle \phi'(\vec{x}) | \phi(\vec{x}) \rangle = \int \mathcal{D}\{\phi\} e^{iS_0}. \tag{56}$$

After expansion about the classical configuration the classical action can be factored out:

$$\langle \phi'(\vec{x}) | \phi(\vec{x}) \rangle = \int e^{iS_c} \mathcal{D}\{\eta\} e^{i\Delta S} \tag{57}$$

and the remaining part of the action containing the selfadjoint fluctuation operator $\hat{M}$ is

$$\Delta S = \int \eta \hat{M} \eta dx \tag{58}$$

which is invariant under the field shift [13]

$$\eta \rightarrow \eta + \Psi_0 \tag{59}$$

where Ψ_0 denotes the zero mode of operator $\hat{M}$. In order to break this symmetry of the action we again add a BRST invariant term to the Lagrangian. The BRST charge in the present case is defined as

$$\Omega = -cP_a - c\pi\Psi_0(a) + \Pi_{\bar{c}}b \tag{60}$$

where P_a is the momentum conjugate to the translation parameter a, and Ψ_0 is the associated zero mode, and b is the Nakanishi– Lautrup auxiliary field. The nonvanishing Poisson brackets of a,P_a and the ghost variables are taken to be the canonical relations

$$\begin{aligned} \{a, P_a\}_- &= 1 \\ \{\bar{c}, \Pi_{\bar{c}}\}_+ &= 1 \\ \{c, \Pi_c\}_+ &= 1. \end{aligned} \tag{61}$$

The BRST charge generates the following transformations

$$\begin{aligned} \delta\eta &= \{\eta, \Omega\} = -c\Psi_0 \\ \delta a &= -c \\ \delta\bar{c} &= b \\ \delta b &= 0 \\ \delta c &= 0 \\ \delta\Psi_0 &= -c\frac{d\Psi_0}{dx}. \end{aligned} \tag{62}$$

Next we add the following BRST invariant term to the fluctuation Lagrangian

$$\begin{aligned} L_B &= \int \delta[\bar{c}\Psi_0\eta]dx - \frac{1}{2}b^2 \\ &= b\int \Psi_0\eta dx + \bar{c}c\int \Psi_0^2 dx + \bar{c}c\int \eta\frac{d\Psi_0}{dx}dx - \frac{1}{2}b^2 \end{aligned} \tag{63}$$

where the field b can be replaced by the solution of its equation of motion, i. e.

$$b = \int \Psi_0\eta dx. \tag{64}$$

Thus, b is identified with the previous gauge fixing variable χ. The transition amplitude with BRST invariant action is then

$$\langle\phi'|\phi\rangle = \int e^{iS_c}\mathcal{D}\{\eta\}\mathcal{D}\{a\}\mathcal{D}\{c\}\mathcal{D}\{\bar{c}\}e^{i\int \eta\hat{M}\eta dx}e^{i\left\{\frac{1}{2}\left[\int \Psi_0\eta dx\right]^2 + \bar{c}c\int\left(\Psi_0^2 - \Psi_0\frac{d\eta}{dx}\right)dx\right\}}. \tag{65}$$

Integrating out $\mathcal{D}\{c\}$ and $\mathcal{D}\{\bar{c}\}$ we obtain

$$\langle\phi'|\phi\rangle = \int \sqrt{M_0}e^{iS_c}\mathcal{D}\{a\}\int \mathcal{D}\{\eta\}e^{i\int\eta\hat{M}\eta dx}e^{i\frac{1}{2}\left[\int\Psi_0\eta dx\right]^2}. \tag{66}$$

Expanding η in terms of eigenmodes of the operator $\hat{M}$ and a changing integration variables from $\mathcal{D}\{\eta\}$ to $\mathcal{D}\{c_n\}$, we find that integration over the zero mode coefficient, $\mathcal{D}\{c_0\}$, becomes a Gaussian integral and can be carried out. The final result of the transition amplitude is seen to be the same as (55) of the previous method. We therefore conclude that the two treatments are equivalent in the one loop approximation. Of course, since the invariance under the shift (59) occurs only in the one–loop approximation, this is also the limit of its validity. Nonetheless this is an interesting observation which again indicates a novel property of the zero mode.

This second method is particularly useful in the explicit calculation of quantum mechanical tunneling effects with the instanton method. Research along this direction is in progress.

6 Global symmetry and Ward–Takahashi (WT) identity

In the previous section we evaluated in the one–loop approximation the transition amplitude of the complex scalar field in the background of Skyrme–like solitons without external sources. The symmetries of the action were broken to obtain well defined functional integrals. The global $U(1)$ symmetry is a new feature of the model. We therefore add some comments and derive the WT identities related to this symmetry [24]. To this end we start from the generating functional of correlation functions:

$$Z(J) = \int \mathcal{D}\{\varphi\}\mathcal{D}\{\varphi^*\}\exp\left[iS_0(\varphi,\varphi^*) + i\int d^2x(J\varphi + J^*\varphi^*)\right] \tag{67}$$

where S_0 is the action defined by the static Lagrangian (3) and J is the current of external sources. The classical action S_0 is invariant under global $U(1)$ transformations $\varphi' = e^{i\alpha}\varphi$, $\delta\varphi = i\alpha\varphi$, and therefore

$$\int d^2x\left(i\alpha\varphi\frac{\delta S_0}{\delta\varphi} - i\alpha\varphi^*\frac{\delta S_0}{\delta\varphi^*}\right) = 0 \tag{68}$$

The measures of integration $\mathcal{D}\{\varphi\}$ and $\mathcal{D}\{\varphi^*\}$ in the functional integral (67) are invariant under the unimodular transformation $\varphi' = e^{i\alpha}\varphi$. Demanding $\delta Z(J) = 0$ leads to the equation

$$i\alpha\int\mathcal{D}\{\varphi\}\mathcal{D}\{\varphi^*\}\int d^2x(J\varphi - J^*\varphi^*)\exp\left[iS_0 + i\int d^2x(J\varphi + J^*\varphi^*)\right] = 0 \tag{69}$$

We then have

$$\int d^2x \left(J \frac{\delta Z(J)}{\delta J} - J^* \frac{\delta Z(J)}{\delta J^*} \right) = 0 \tag{70}$$

or

$$\int d^2x \left(J \frac{\delta W}{\delta J} - J^* \frac{\delta W}{\delta J^*} \right) = 0 \tag{71}$$

and

$$\int d^2x \left(J \frac{\delta \Gamma}{\delta J} - J^* \frac{\delta \Gamma}{\delta J^*} \right) = 0 \tag{72}$$

where W is the generating functional of the connencted correlation function while Γ denotes proper vertices.

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