KL–TH 96/8

Batalin–Vilkovisky field–antifield quantisation of fluctuations around classical field configurations

F. Zimmerschied

Department of Physics,
University of Kaiserslautern, P. O. Box 3049, D 67653 Kaiserslautern, Germany
E–mail: zimmers@physik.uni–kl.de

Abstract

The Lagrangian field–antifield formalism of Batalin and Vilkovisky (BV) is used to investigate the application of the collective coordinate method to soliton quantisation. In field theories with soliton solutions, the Gaussian fluctuation operator has zero modes due to the breakdown of global symmetries of the Lagrangian in the soliton solutions. It is shown how Noether identities and local symmetries of the Lagrangian arise when collective coordinates are introduced in order to avoid divergences related to these zero modes. This transformation to collective and fluctuation degrees of freedom is interpreted as a canonical transformation in the symplectic field–antifield space which induces a time–local gauge symmetry. Separating the corresponding Lagrangian path integral of the BV scheme in lowest order into harmonic quantum fluctuations and a free motion of the collective coordinate with the classical mass of the soliton, we show how the BV approach clarifies the relation between zero modes, collective coordinates, gauge invariance and the center–of–mass motion of classical solutions in quantum fields. Finally, we apply the procedure to the reduced nonlinear $O(3)$ σ–model.

1 Introduction

One of the most powerful quantisation procedures for gauge theories involving BRST invariance is the Lagrangian field–antifield formalism of Batalin and Vilkovisky (BV) [1, 2]. The main idea is to start from the original Lagrangian, double the number of fields by introducing antifields of opposite parity and then to construct an "extended action" as a bosonic functional in fields and antifields which contains all the information about the dynamics and local symmetries of the original problem.

Together with a special bilinear form, the "antibracket", the fields and antifields generate a symplectic structure which allows the application of well–known symplectic techniques like canonical transformations. Gauge fixing, e. g. , turns out to be a special canonical transformation [2]. Quantization is finally performed in terms of Lagrangian path integrals over the gauge fixed extended action with the unphysical antifields set to zero.

Besides the standard applications of BV quantisation to theories like Yang–Mills field theory or general relativity (which obviously possess gauge symmetry and thus are constrained systems), one can also apply the BV formalism to constrained systems which possess a hidden gauge invariance, a "gauge symmetry without gauge fields" [3], like the example of quantizing a particle around a classical orbit which may be done in a BRST invariant way [4]. More prominent examples (which are nonetheless closely related to the classical orbit model) are field theories with classical, localized solutions, such as e. g. solitons (or sphalerons and bounces, collectively described as solitons here).

"Quantizing solitons", i. e. constructing quantum states of fields in the background of the soliton (instead of the usual field quantum states based on the perturbation theory vacuum) leads to divergences due to zero modes of the Gaussian fluctuation operator which arise because the classical solutions break a global symmetry of the original Lagrangian. In the past 20 years, a lot of work was done to solve and understand this problem [5, 6, 7, 8, 9], resulting in the so–called collective coordinate method which may be understood as a parameter dependent transformation to new (fluctuation) fields. Interpreting the transformation parameters as new dynamical degrees of freedom, the configuration space variables of the transformed theory (i. e. collective coordinates and fluctuations) are overcomplete so that the resulting new Lagrangian is invariant under local transformations and the theory has to be treated as a gauge theory.

The well–known path–integral methods of quantizing a gauge theory, including the choice of gauge fixing, then lead to soliton quantisation in the collective coordinate method. Usually Hamiltonian path integrals are used [10, 11], and the powerful tools of the BRST symmetry [12] in the Hamiltonian Batalin–Fradkin–Vilkovisky (BFV) scheme [15] can be applied to solitons [7, 8]. Particularly in theories with a field dependent mass such as Skyrme–like models [13], it is essential to use the Hamiltonian path integral since here the integration over the field momenta yields an extra measure factor [14], a fact emphasized already 30 years ago [11].

Since field theories start form the (Lorentz invariant) Lagrangian formalism, it is an intriguing task to develop a soliton quantisation procedure which is based on Lagrangian path integrals, avoiding completely the introduction of field momenta. The BV scheme yields precisely such a Lagrangian BRST path integral quantisation which is equivalent to the BFV method [16], so that the measure problem referred to in Lagrangian path integrals should not arise in BV Lagrangian path integrals (although theories with non–trivial measures are also discussed in the context of the BV scheme [17]).

Formally it is straightforward to apply this Lagrangian BV scheme to the quantisation of solitons [9]. We do this in the following by application to a simple field theory model with the intent to study in a transparent way the pecularities of the symplectic field–antifield concept. Thereby, our approach to the method of collective coordinates is different from that in [9], as we introduce collective coordinates and fluctuations together within one and the same transformation from the beginning to the new, overcomplete set of fields. In the field–antifield formalism, this transformation may be understood as a canonical transformation from the old theory (with global symmetries resulting in zero modes of the Gaussian fluctuation operator) to the new theory with local gauge symmetries which are given by the canonical transformation of the extended action in BV.

Then analysing the Lagrangian path integral, we show that the BV scheme emphasises the semiclassical interpretation of solitons as localised particles in Lagrangian field theories: Neglecting quantum corrections to the classical soliton mass in lowest order, we may separate the center–of–mass motion of the soliton (described by its collective coordinate) from the quantum fluctuations around it in the harmonic or one–loop approximation.

Finally, we give some hints concerning the application of the BV scheme to Mottola and Wipf's reduced $O(3)$ σ–model [18] in a spherical field parametrisation which yields a Lagrangian of the general form considered in [9]. Besides collective translation degrees of freedom which we discussed so far, collective rotation degrees in the internal symmetry space arise in this model which in the context of the Hamiltonian BFV technique and for a slightly different σ–model were discussed in [19].

The main advantage of BV quantisation, from the point of view of soliton physics, therefore is not its capability to handle complicated gauge structures like open gauge algebras (which was one of the original aims of Batalin and Vilkovisky), but its simple and straight–forward application to collective coordinates and fluctuations which yields a general procedure and formal structures with clear physical interpretation. The structures of the gauge algebras discussed here are quite simple; no open gauge algebras appear. This allows the investigation of some special topics of the BV formalism such as the relation between infinitesimal canonical transformations, gauge fixing and BRST transformations in the context of a simple example in a more transparent way than in the context of abstract considerations.

2 Collective coordinates, singular properties of the fluctuation Lagrangian and the field–antifield formalism

We consider the scalar field theory of a bosonic ($p(\phi) = 0$) or fermionic ($p(\phi) = 1$, p denoting the Grassmann parity) field $\phi(t, \mathrm{x})$ in $1 + 1$ dimensions with Lagrangian density

$$\tilde{\mathcal{L}}(\phi, \phi'; \dot{\phi}) = \frac{1}{2}(\dot{\phi}^2 - \phi'^2) - V(\phi). \tag{1}$$

Dots denote derivatives with respect to time t, primes with respect to space x. We assume that this model contains a soliton configuration $\varphi(\mathrm{x})$, i. e. a nontrivial, classical, static solution of finite energy (most prominent examples being ϕ^4 and Sine–Gordon theories). The Lagrangian has a global space translation invariance $\mathrm{x} \mapsto \mathrm{x} - a$, $a \in \mathbb{R}$ which is broken by the soliton solution as $\varphi(\mathrm{x} - a)$ is a new solution different from $\varphi(\mathrm{x})$. The parameter a describes the localisation of the soliton.

We now use the classical solution φ to introduce new "fields" $\eta(t, \mathrm{x})$ and $a(t)$ (which in fact is not a field, but a dynamical parameter) by setting

$$\phi(t, \mathrm{x}) = \varphi(\mathrm{x} - a(t)) + \eta(t, \mathrm{x}) \tag{2}$$

which means that we add one more degree of freedom (the "collective coordinate" $a(t)$) to the infinitely many field degrees of freedom. We denote these new "fields" collectively by the symbol $\Phi^i(x) \in \{\eta(t, \mathrm{x}), a(t)\}$; x denoting either spacetime or time.

The transformed Lagrangian density then reads

$$\begin{aligned} \mathcal{L}(\mathrm{x}; \eta, a, \eta'; \dot{a}, \dot{\eta}) &= \frac{1}{2}\left[-\varphi'(\mathrm{x} - a(t))\dot{a}(t) + \dot{\eta}(t, \mathrm{x})\right]^2 \\ &\quad - \frac{1}{2}\left[-\varphi'(\mathrm{x} - a(t)) + \eta'(t, \mathrm{x})\right]^2 \\ &\quad - V(\varphi(\mathrm{x} - a(t)) + \eta(t, \mathrm{x})) \end{aligned} \tag{3}$$

This new Lagrangian density $\mathcal{L}$ depends explicitly on the space coordinate x. This is a first hint to a special feature of collective coordinate techniques: In our model, we have a mixture of fields (the fluctuations $\eta(t, \mathrm{x})$)) and coordinates (the collective coordinate $a(t)$ describing a translation in space). The collective coordinate itself does not depend on space: It is an additionel and redundant degree of freedom to the infinitely many field degrees of freedom labeled by the continuous index "space". In this sense, one may interpret the fluctuation field $\eta(t, \mathrm{x})$ as an infinite set of "fluctuation coordinates" which could be made more explicit by use of the notation $\eta(t, \mathrm{x}) = a_{\mathrm{x}}(t)$. From this point of view, the collective coordinate method is more closely related to Lagrangian mechanics than to Lagrangian field theory. In particular, our model may be regarded as the $N \longrightarrow \infty$ limit of the model of a particle around an orbit in N dimensions [3, 4].

This special role of space position arises again when we evaluate the Euler–Lagrange equations of our model. As usual, the dynamics is determined by extremizing the full action

$$S_0[\eta, a] = \int \mathcal{L}(\mathrm{x}; \eta, a, \eta'; \dot{a}, \dot{\eta}) d\mathrm{x} dt \tag{4}$$

with respect to η and a. Therefore, setting the corresponding functional derivatives to zero,

$$\left(S_0[\eta, a] \frac{\overleftarrow{\delta}}{\delta \eta(t, \mathrm{x})} \right) \stackrel{!}{=} 0, \qquad \left(S_0[\eta, a] \frac{\overleftarrow{\delta}}{\delta a(t)} \right) \stackrel{!}{=} 0 \tag{5}$$

leads to the Euler–Lagrange equations. (In the BV formalism, we have to distinguish between derivatives from the right and from the left even when starting with a bosonic theory: The introduction of antifields later unavoidably leads on to fermionic degrees of freedom. Using derivatives from the right in eq. (5) is more or less a convention which has to be consistent with all later formulae.)

We see that the Euler–Lagrange equation of the collective coordinate does not depend on the position space coordinate x which remains integrated out:

$$\left(S_0[\eta, a] \frac{\overleftarrow{\delta}}{\delta a(t)} \right) = \int \left[\frac{\partial \mathcal{L}}{\partial a} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{a}} \right] d\mathrm{x}. \tag{6}$$

The two equations (5) are not independent: As a trivial consequence of adding one degree of freedom, the relation

$$\int \left(S_0[\eta, a] \frac{\overleftarrow{\delta}}{\delta \eta(t, \mathrm{x})} \right) \cdot \phi'(\mathrm{x} - a(t)) d\mathrm{x} + \left(S_0[\eta, a] \frac{\overleftarrow{\delta}}{\delta a(t)} \right) \cdot 1 = 0 \tag{7}$$

obtained by inspection of the explicit forms of eqs. (5) holds identically. According to the above discussion, we may interpret the integral in eq. (7) as a sum over the continuous index x of the "fluctuation coordinates". Then (7) is an ordinary Noether identity [20] of the form

$$\left(S_0[\Phi] \frac{\overleftarrow{\delta}}{\delta \Phi^i(x)} \right) R^i_\alpha[\Phi](x) \equiv 0, \quad \alpha = 1, \ldots, m \tag{8}$$

($\alpha = m = 1$ in our case) with ordinary Noether generators $R^i_\alpha[\Phi](x)$. These Noether identities ensure that local m–parameter symmetries

$$\delta_{(\epsilon)} \Phi^i(x) = \epsilon^\alpha(x) R^i_\alpha[\Phi](x) \tag{9}$$

hold as can be seen by Taylor expansion of the transformed action $S_0[\Phi + \delta_{(\epsilon)}\Phi]$.

Noether identities of the form (8) are called "ordinary" to distinguish them from generalised Noether identities

$$\int \left(S_0[\Phi] \frac{\overleftarrow{\delta}}{\delta \Phi^i(x')} \right) R^i_\alpha[\Phi](x,x') dx' \equiv 0 \tag{10}$$

with related local symmetries ($R^i_\alpha[\Phi](x,x') \propto \delta(x-x')$)

$$\begin{aligned} \delta_{(\epsilon)}\Phi^i(x) &= \int \epsilon^\alpha(x') R^i_\alpha[\Phi](x,x') dx' \\ &= \epsilon^\alpha(x)^{(0)}R^i_\alpha[\Phi](x) + \partial_\mu \epsilon^\alpha(x)^{(1)}R^{i\mu}_\alpha[\Phi](x) \end{aligned} \tag{11}$$

depending on local parameters $\epsilon^\alpha(x)$ and their derivatives [2, 20]. These are typical symmetries of gauge fields. The fact that only ordinary Noether identities and thus gauge transformations of the form (9) arise in our model justifies the expression "gauge theory without gauge fields" [3]: Although invariant under local transformations, there are no fields which transform like gauge fields in our model.

By comparison of (7) and (8), we can identify the Noether generators of our model,

$$R^{(\eta)}[a](t,\mathrm{x}) = \varphi'(\mathrm{x} - a(t)) \tag{12}$$

$$R^{(a)} = 1 \tag{13}$$

Again, the position space coordinate in (12) should be considered as a continuous index: The fact that it is integrated out in (7) yields that the related gauge symmetry is only local in time:

$$\delta_{(\epsilon)}\eta(t,\mathrm{x}) = \epsilon(t)\varphi'(\mathrm{x} - a(t)) \tag{14}$$

$$\delta_{(\epsilon)}a(t) = \epsilon(t) \tag{15}$$

or in more general notation

$$\delta_{(\epsilon)}\Phi^i(x) = \epsilon^\alpha(t) R^i_\alpha[\Phi](x) \qquad (\alpha = 1). \tag{16}$$

Eq. (14,15) are typical time–local gauge transformations associated with collective translation coordinates: They depend on $\frac{d}{dx}\varphi$, where $\frac{d}{dx}$ is the generator of translations (φ' is also the zero mode of the Gaussian fluctuation operator of the model as we shall discuss later). We observe later that collective rotation coordinates (which arise e. g. in σ–models) also yield typical, time–local gauge transformations related to some generator of rotations.

Certainly, the gauge algebra desribed by (14,15) is trivially abelian. But also in models with more than one collective coordinate (which have more than one Noether identity), the gauge algebras are closed and often abelian (in particular, this is true for σ–models).

In accordance with the usual BRST treatment of gauge theories, we now introduce one ghost "field" (which in fact is a coordinate as it depends only on time) $c(t)$ with parity opposite to that of the symmetry parameter $\epsilon(t)$: $p(c) = p(\epsilon) + 1$. From (15), $p(\epsilon) = p(a)$, and as space has bosonic properties in our model, the ghost $c(t)$ is fermionic. The configuration space now consists of $\{\eta(t, \mathrm{x}), a(t), c(t)\} =: \{Y^A(x)\}$ where we use $Y^a(x)$ as abbreviation for any field in the extended field configuration spaces which we construct in the following (whereas $\Phi^i(x)$ denotes only the original fields, i. e. $\eta(t, \mathrm{x})$ and $a(t)$ in our example).

From the local symmetry (16), we now construct the global BRST symmetry

$$\delta_\lambda Y^A(x) = \lambda \mathcal{S} Y^A(x) \tag{17}$$

where $\mathcal{S}$ is the BRST operator which is defined by the following properties: (1) $\mathcal{S}$ is parity changing: $p(\mathcal{S}Y^A(x)) = p(Y^A(x)) + 1$, (2) $\mathcal{S}$ is nilpotent, $\mathcal{S}^2 = 0$, (3) $\mathcal{S}$ "includes" the gauge symmetry in the sense that $S_0[\Phi]$ is invariant under $\delta_\lambda \Phi^i(x) = \lambda \mathcal{S} \Phi^i(x)$.

From these properties and (9,10) it is easy to conclude that the BRST operator of our model is given by

$$\begin{aligned} \mathcal{S}\eta(t,\mathrm{x}) &= R^{(\eta)}[a](t,\mathrm{x})c(t) = \varphi'(\mathrm{x} - a(t))c(t) && (18) \\ \mathcal{S}a(t) &= R^{(a)}c(t) = c(t) && (19) \\ \mathcal{S}c(t) &= 0 && (20) \end{aligned}$$

Next, we double the number of degrees of freedom of the ghost–enlarged configuration space by introducing "antifields" of opposite parity. Denoting antifields by an asterix, we thus have the "fields" $\{Y^A(x)\} = \{\eta(t, \mathrm{x}), a(t), c(t)\}$ and the "antifields" $\{Y^*_A(x)\} = \{\eta^*(t, \mathrm{x}), a^*(t), c^*(t)\}$ with $p(Y^*_A(x)) = p(Y^A(x)) + 1$. Together with the antibracket of two functionals U, V defined by

$$(U, V) = \int \left\{ \left(U \frac{\overleftarrow{\delta}}{\delta Y^A(x)} \right) \left(\frac{\overrightarrow{\delta}}{\delta Y^*_A(x)} V \right) - \left(U \frac{\overleftarrow{\delta}}{\delta Y^*_A(x)} \right) \left(\frac{\overrightarrow{\delta}}{\delta Y^A(x)} V \right) \right\} dx, \tag{21}$$

the configuration space

$$\mathcal{P} = \{Y^A(x), Y^*_A(x)\} = \{\eta(t, \mathrm{x}), a(t), c(t); \eta^*(t, \mathrm{x}), a^*(t), c^*(t)\} \tag{22}$$

has a symplectic structure similar to that of Hamiltonian phase space with the Poisson bracket. But in contrast to the Poisson bracket, $(U, U) = 0$ holds only for fermionic functionals U with $p(U) = 1$, whereas for bosonic functionals S with $p(S) = 0$

$$(S, S) = 2 \int \left(S \frac{\overleftarrow{\delta}}{\delta Y^A(x)} \right) \left(\frac{\overrightarrow{\delta}}{\delta Y^*_A(x)} S \right). \tag{23}$$

If the nontrivial equation $(S,S)=0$ is fullfilled for some bosonic S, then the identity

$$
\begin{aligned}
0 &= (S,S)\frac{\overleftarrow{\delta}}{\delta Y_B^*(x)} = \left(S, S\frac{\overleftarrow{\delta}}{\delta Y_B^*(x)}\right) \\
&= 2\int \left\{ \left(S\frac{\overleftarrow{\delta}}{\delta Y^A(x')}\right)\left(\frac{\overrightarrow{\delta}}{\delta Y_A^*(x')}S\frac{\overleftarrow{\delta}}{\delta Y^B(x)}\right) \right. \\
&\qquad \left. - \left(S\frac{\overleftarrow{\delta}}{\delta Y_A^*(x')}\right)\left(\frac{\overrightarrow{\delta}}{\delta Y^A(x')}S\frac{\overleftarrow{\delta}}{\delta Y_B^*(x)}\right)\right\} dx' \qquad (24)
\end{aligned}
$$

and a similar one for $(S,S)\frac{\overleftarrow{\delta}}{\delta Y^B(x)}$ holds. Eq. (24) resembles the general form of the Noether identity (8) if $S\frac{\overleftarrow{\delta}}{\delta Y^A(x')}$ is related to the Euler–Lagrange equations (here we see that the directions of the derivatives in (5) and (21) were chosen consistently). This purely structural feature of bosonic functionals on the symplectic field–antifield space (22) is exploited physicswise in the construction of the socalled "extended action" $S_{ext}[Y,Y^*]$. This extended action is a bosonic functional which describes both the dynamics and the gauge symmetries of a given theory. It is defined as solution of the "classical master equation"

$$(S_{ext}[Y,Y^*], S_{ext}[Y,Y^*]) = 0 \qquad (25)$$

with boundary conditions

$$S_{ext}[Y,Y^*]|_{Y^*=0} = S_0[\Phi] \qquad (26)$$

and

$$\left.\frac{\overrightarrow{\delta}}{\delta c^\alpha(x)}S_{ext}[Y,Y^*]\frac{\overleftarrow{\delta}}{\delta \Phi_i^*(x)}\right|_{Y^*=0} = R^i_\alpha[\Phi](x) \qquad (27)$$

where c^α is the general notation for the ghost fields.

The first boundary condition (26) retains the dynamics of the model: In the BV scheme, the antifields have no physical meaning — to obtain physical quantities, they are set to zero (a fact which becomes important in the evaluation of path integrals). The "physical part", in this sense, of the extended action is thus the original action which describes the dynamics of the theory. The second boundary condition (27) uses the relation (24) to enforce the local symmetries of the theory (described by the Noether identities (8)) into the extended action. In a naive way, on could say that the redundant degrees of freedom we gained by introducing antifields enabled us to "add" to the action the important information about the symmetry structure which is contained in the Noether generators, resulting in the "extended action".

The solution of the classical master equation is a crucial step in the BV scheme. Eq. (26) suggests an ansatz in powers of antifields,

$$S_{ext}[Y,Y^*] = \sum_{n=0}^{\infty}\int Y^*_{A_n}(x_n)\cdot\ldots\cdot Y^*_{A_1}(x_1)S^{A_1\ldots A_n}[Y](x_1,\ldots,x_n)dx_1\ldots dx_n \qquad (28)$$

For abelian gauge theories, the structure constants of the gauge algebra vanish, and one can show that no terms higher than those of linear order are necessary in (28) to solve the master equation (this corresponds to the fact that the BRST transformations of the ghost variables are trivial, $\mathcal{S}c^\alpha(x)=0$, in these cases). In particular, we have from (27) that

$$\frac{\overrightarrow{\delta}}{\delta Y_A^*(x)} S_{ext}[Y,Y^*] = \mathcal{S}Y^A(x) \tag{29}$$

($Y^*=0$ is no longer necessary!), and the extended action is simply given by

$$S_{ext}[Y,Y^*] = S_0[\Phi] + \int Y_A^*(x)\left(\mathcal{S}Y^A(x)\right)dx. \tag{30}$$

With (22) and (18,19,20), this implies for our soliton model

$$S_{ext}[\eta,a,c;\eta^*,a^*,c^*] = S_0[\eta,a] + \int a^*(t)c(t)dt + \int \eta^*(t,\mathrm{x})\varphi'(\mathrm{x}-a(t))c(t)dtd\mathrm{x}. \tag{31}$$

From (29) and the definition of the antibracket (21), we see that in the field–antifield formalism, the extended action may be regarded as the analogue of the BRST charge in the Hamiltonian BFV formalism [15] in the sense that it generates the BRST transformations:

$$\mathcal{S}Y^A(x) = \left(Y^A(x), S_{ext}[Y,Y^*]\right) \tag{32}$$

Eq. (32) also holds for nonabelian gauge algebras with more complicated BRST transformations than (19).

It is now obvious how we have to define the BRST transformations of the antifields which have not been defined so far: We simply set

$$\mathcal{S}Y_A^*(x) = \left(Y_A^*(x), S_{ext}[Y,Y^*]\right). \tag{33}$$

With this definition, $\mathcal{S}$ is nilpotent on all functionals over $\mathcal{P}$, and the classical master equation ensures that the extended action is BRST invariant:

$$\mathcal{S}S_{ext}[Y,Y^*] = \left(S_{ext}[Y,Y^*], S_{ext}[Y,Y^*]\right) = 0 \tag{34}$$

3 Canonical field–antifield transformations and collective coordinates

The construction of the extended action demonstrated how purely structural properties (the fact that $(S,S)=0$ is a nontrivial equation for bosonic S) of the symplectic field–antifield space can be used to formulate physical aspects of the theory as dynamics and symmetry by the construction of the extended action which is the main object of the BV formalism.

Another important instrument in the study of symplectic structure is the group of transformations which leave the symplectic form invariant, i. e. the antibracket in the BV scheme. In classical mechanics, these symmetries are called "canonical transformations". As in Hamiltonian mechanics, they are usually defined via a generator which in the BV case has to be a fermionic functional of e. g. old fields $\tilde{Y}^A(x)$ and new antifields $Y^*_A(x)$, $F[\tilde{Y}, Y^*]$. The canonical transformation $\{\tilde{Y}^A(x), \tilde{Y}^*_A(x)\} \mapsto \{Y^A(x), Y^*_A(x)\}$ is then given by [2]

$$\tilde{Y}^*_A(x) = \frac{\overrightarrow{\delta}}{\delta \tilde{Y}^A(x)} F[\tilde{Y}, Y^*], \qquad Y^A(x) = \frac{\overrightarrow{\delta}}{\delta Y^*_A(x)} F[\tilde{Y}, Y^*]. \tag{35}$$

Since F is fermionic, canonical transformations do not change parity.

We now use canonical transformations to introduce collective coordinates and fluctuations, starting from the original Lagrangian (1) and its global symmetry $\mathrm{x} \mapsto \mathrm{x} - a$ which is broken by the soltion solution. For this (gobal!) symmetry, we add one absolutely redundant bosonic symmetry coordinate $A(t)$ — in general, one has to add one such coordinate for each symmetry degree of freedom with corresponding parity.

Since $\tilde{\mathcal{L}}(\phi, \phi'; \dot{\phi})$ does not depend on $A(t)$ at all, we have $\tilde{S}_0[\phi, A] \frac{\overleftarrow{\delta}}{\delta A(t)} = 0$ where $\tilde{S}_0[\phi, A] = \int \tilde{\mathcal{L}}(\phi, \phi'; \dot{\phi}) dt d\mathrm{x}$, and a trivial Noether identity

$$\int \left(\tilde{S}_0[\phi, A] \frac{\overleftarrow{\delta}}{\delta \phi(t, \mathrm{x})} \right) R^{(\phi)} d\mathrm{x} + \left(\tilde{S}_0[\phi, A] \frac{\overleftarrow{\delta}}{\delta A(t)} \right) R^{(A)} = 0 \tag{36}$$

with $R^{(\phi)} = 0$, $R^{(A)} = 1$ holds. In order to obtain the BRST transformations associated with (36), we introduce one fermionic ghost coordinate $C(t)$. Then $\tilde{S}_0[\phi, A]$ is trivially BRST invariant under

$$\begin{aligned} \mathcal{S}\phi(t, \mathrm{x}) &= R^{(\phi)} C(t) = 0 \\ \mathcal{S}A(t) &= R^{(A)} C(t) = C(t) \\ \mathcal{S}C(t) &= 0. \end{aligned} \tag{37}$$

Doubling the (old) fields $\{\tilde{Y}^A(x)\} = \{\phi(t, \mathrm{x}), A(t), C(t)\}$ by introducing antifields $\{\tilde{Y}^*_A(x)\} = \{\phi^*(t, \mathrm{x}), A^*(t), C^*(t)\}$, we can immediately write down the extended action

$$\tilde{S}_{ext}[\phi, A, C; \phi^*, A^*, C^*] = \tilde{S}_0[\phi, A] + \int A^*(t) C(t) dt. \tag{38}$$

We exploit the fact that due to the global symmetry of (1) and the existence of a soliton solution $\varphi(\mathrm{x})$, we have a whole class of solutions $\varphi(\mathrm{x} - a)$, $a \in \mathbb{R}$. Using these we write out the fermionic generator

$$\begin{aligned} & F[\phi, A, C; \eta^*, a^*, c^*] \\ = & \int a^*(t) A(t) dt + \int c^*(t) C(t) dt + \int \eta^*(t, \mathrm{x}) [\phi(t, \mathrm{x}) - \varphi(\mathrm{x} - A(t))] dt d\mathrm{x} \end{aligned} \tag{39}$$

which depends on old fields and new antifields and yields the canonical transformation

$$\begin{aligned}
\phi^*(t,\mathrm{x}) &= \frac{\vec{\delta}}{\delta\phi(t,\mathrm{x})}F[\phi,A,C;\eta^*,a^*,c^*] = \eta^*(t,\mathrm{x}) \\
A^*(t) &= \frac{\vec{\delta}}{\delta A(t)}F[\phi,A,C;\eta^*,a^*,c^*] = a^*(t) + \int \eta^*(t,\mathrm{x})\varphi'(\mathrm{x}-A(t))d\mathrm{x} \\
C^*(t) &= \frac{\vec{\delta}}{\delta C(t)}F[\phi,A,C;\eta^*,a^*,c^*] = c^*(t) \\
\eta(t,\mathrm{x}) &= \frac{\vec{\delta}}{\delta\eta^*(t,\mathrm{x})}F[\phi,A,C;\eta^*,a^*,c^*] = \phi(t,\mathrm{x}) - \varphi(\mathrm{x}-A(t)) \\
a(t) &= \frac{\vec{\delta}}{\delta a^*(t)}F[\phi,A,C;\eta^*,a^*,c^*] = A(t) \\
c(t) &= \frac{\vec{\delta}}{\delta c^*(t)}F[\phi,A,C;\eta^*,a^*,c^*] = C(t).
\end{aligned} \tag{40}$$

We thus have (as in (2)) $\phi(t,\mathrm{x}) = \varphi(\mathrm{x}-a(t)) + \eta(t,\mathrm{x})$. Inserting (40) into the extendend action (40) yields

$$\begin{aligned}
&\tilde{S}_{ext}[\phi,A,C;\phi^*,A^*,C^*] \overset{F}{\mapsto} S_{ext}[\eta,a,c;\eta^*,a^*,c^*] \\
&\quad = S_0[\eta,a] + \int a^*(t)c(t)dt + \int \eta^*(t,\mathrm{x})\varphi'(\mathrm{x}-a(t))c(t)dtd\mathrm{x},
\end{aligned} \tag{41}$$

and from the general form of the extended action in abelian theories, i. e. (30), we can extract from (41) the BRST transformations (18–20) of the classical action in collective and fluctuation degrees of freedom, $S_0[\eta,a]$. We thus obtain the gauge symmetry together with the collective– fluctuation action by a canonical transformation.

The crucial point of this procedure is the construction of the generator (39) which has a simple physical interpretation: The completely redundant coordinate $A(t)$ is correlated to the symmetry parameter a of the original Lagrangian which becomes dynamical, and the difference between the classical soliton solution ϕ at $A(t) = a(t)$ and the original field $\phi(t,\mathrm{x})$ defines the fluctuations $\eta(t,\mathrm{x})$, a definition similar to that of center–of–mass frame coordinates in Lagrangian mechanics and related to the model of the classical orbit [4, 22].

The approach of eq. (2) to define fluctuations which we gained here by a canonical transformation is commonly used in the collective coordinate method [5], but there is also a different classical way to introduce collective coordinates [6] which starts from

$$\phi(t,\mathrm{x}) = \tilde{\phi}(t,\mathrm{x}-a(t)) \tag{42}$$

thus using $\{\tilde{\phi},a\}$ as a new and overcomplete set of configuration space variables. Proceeding along these lines, fluctuations are introduced later by expanding $\tilde{\phi}(t,\rho)$ with $\rho = \mathrm{x}-a(t)$ around $\varphi(\rho)$, where φ again is the soliton solution. This yields

$$\phi(t,\mathrm{x}) = \varphi(\mathrm{x}-a(t)) + \eta(t,\mathrm{x}-a(t)) \tag{43}$$

which is different from (2) since the fluctuations now also depend on the collective coordinate shifted space argument. This may be interpreted as a common frame of reference in which both the soliton and the fluctuations (frequently visualised as analogues of baryon and mesons) are observed. This point of view is useful in the discussion of soliton–fluctuation scattering, but one looses the interpretation of the space coordinate as a continuous index labeling the "fluctuation coordinates" $a_{\mathrm{x}}(t) = \eta(t, \mathrm{x})$ which arises directly from the integral form of the Noether identity (7).

Thus in the BV scheme, the introduction of collective coordinates and fluctuations according to (2), which is related to a canonical transformation in field–antifield space, is more convenient.

Another important application of canonical transformations in the BV scheme is the gauge fixing of the theory which we discuss in the next section. Besides canonical transformations (which are the symmetry transformations of the symplectic structure of the field–antifield space), there is still the BRST symmetry as a symmetry of the extended action. The BRST transformation is not canonical since it changes parity. There is, however, a close relation between the BRST transformation and canonical transformations. To see this, we have to analyse parameter groups of canonical transformations generated by $F_\alpha[\tilde{Y}, Y^*]$ (α being the global parameter of the transformation group). As usual, on can introduce infinitesimal canonical transformations generated by $f[Y, Y^*]$ where f is the linear coefficient in the α–expansion of $F_\alpha[\tilde{Y}, Y^*]$ (replacing $\tilde{Y}$ by Y):

$$\begin{aligned} \delta_\alpha Y^A(x) &= \alpha(Y^A(x), f[Y, Y^*]) \\ \delta_\alpha Y^*_A(x) &= \alpha(Y^*_A(x), f[Y, Y^*]). \end{aligned} \qquad (44)$$

The infinitesimal canonical transformation of a genral functional U is then given by $\delta_\alpha U = \alpha(U, f[Y, Y^*])$. Introducing some condensed notation, we may assign an operator $\mathbf{X}_f$ to the generating functional $f[Y, Y^*]$, defined by $\mathbf{X}_f A := (A, f[Y, Y^*])$. Using this notation naively, we have $\mathcal{S} = \mathbf{X}_{S_{ext}}$ for the BRST operator: But as $S_{ext}[\eta, a, c; \eta^*, a^*, c^*]$ is bosonic, $\mathbf{X}_{S_{ext}}$ changes parity. Parity–changing transformtions in general do not leave the antibracket invariant and thus cannot be canonical.

4 Gauge fixing and path integrals

Before evaluating path integrals to quantize our model, we have to gauge–fix the extended action (31). To do this, it is necessary to add a "trivial system" for each Noether identity (8) of the theory and so to enlarge the symplectic field–antifield space again.

There is only one Noether identity (7) in our model, so one trivial system $\bar{c}(t)$, $b(t)$ with $\mathcal{S}\bar{c}(t) = b(t)$, $\mathcal{S}b(t) = 0$, and the corresponding antifields $\bar{c}^*(t)$, $b^*(t)$ have to be added. $\bar{c}$ and b have opposite parity (which is obvious from their BRST transformations), and $p(\bar{c}) = p(c)$. We "add" the BRST transformation of the trivial system to the extended action, yielding

$$\begin{aligned} & S_{ext}[\eta, a, c, \bar{c}, b; \eta^*, a^*, c^*, \bar{c}^*, b^*] \\ = \; & S_0[\eta, a] + \int a^*(t)c(t)dt + \int \eta^*(t, \mathrm{x})\varphi'(\mathrm{x} - a(t))c(t)dtd\mathrm{x} + \int \bar{c}^*(t)b(t)dt. \end{aligned} \tag{45}$$

The trivial system is needed to construct the "gauge fermion", a fermionic functional $\psi[Y]$ which depends only on fields and not on antifields (Y again denotes all fields, i. e. now including the trivial system). Since ψ is fermionic, it may be regarded as generator of an infinitesimal canonical transformation. Gauge fixing the extended action then simply means performing this transformation:

$$S_{ext}[Y, Y^*] \overset{\psi}{\mapsto} S^{\psi}_{ext}[Y, Y^*] = S_{ext}[Y, Y^*] + \alpha \mathbf{X}_{\psi} S_{ext}[Y, Y^*] \tag{46}$$

where $S^{\psi}_{ext}[Y, Y^*]$ denotes the ψ–gauge fixed extended action. Here, we use infinitesimal canonical transformations to perform the gauge fixing and will set the transformation parameter α to 1 later. It is easy to see that one can also use canonical transformations to perform the gauge fixing [2], but the relation between BRST transformations (which are also given in infinitesimal form) and gauge fixing becomes clearer when infinitesimal canonical transformations are used to gauge fix the extended action.

This relation is due to the fact that the gauge fermion $\psi[Y]$ generates an infinitesimal canonical transformation, whereas the extended action generates the BRST transformation. So

$$\mathrm{X}_{\psi} S_{ext}[Y, Y^*] = (S_{ext}[Y, Y^*], \psi[Y]) = -(\psi[Y], S_{ext}[Y, Y^*]) = -\mathcal{S}\psi[Y]. \tag{47}$$

Gauge fixing, from this point of view, consists of adding the BRST–transformed gauge fermion to the extended action, a term which is trivially BRST invariant due to the nilpotency of $\mathcal{S}$. This is very similar to the gauge fixing procedure in the Hamiltonian BFV scheme [3, 15], and the trivial system fields $\bar{c}(t)$, $b(t)$ are comparable to the "antighost" and the Nakanashi–Lautrup field in BFV ("antighosts" in BFV are not related to the antifields of the ghosts in BV!).

The main task of all gauge fixing procedures is to check that quantized physical quantities (e. g. expectation values defined by path integrals) do not depend on the choice of the specific gauge. (In fact, the proof of this independence is the main subject of the classical BFV papers [15]). In the BV formalism, this means that path integrals have to be invariant under canonical transformations which is not a trivial requirement as the measure $\mathcal{D}\{Y^A(x)\}\mathcal{D}\{Y^*_A(x)\}$ on field–antifield space is not invariant under canonical transformations (different from the Hamiltonian phase space measure for which the Liouville theorem holds). This invariance requirement determines the right measure on the fields $\mu(\mathcal{D}\{Y^A(x)\})$: There is no integration over antifields in BV-Lagrangian path integrals; as we already mentioned, the antifields are unphysical degrees of freedom and have to be set to zero to obtain physical quantities.

The measure μ is usually contained the exponential: One defines the functional $W[Y, Y^*]$ by $\mu(\mathcal{D}\{Y^A(x)\}) \exp\left(\frac{i}{\hbar} S^{\psi}_{ext}[Y, Y^*]\right) = \mathcal{D}\{Y^A(x)\} \exp\left(\frac{i}{\hbar} W[Y, Y^*]\right)$. Then one can check that the expectation value

$$\begin{aligned} \langle\chi\rangle[Y^*] &= \int \mu(\mathcal{D}\{Y^A(x)\})\chi[Y, Y^*] \exp\left(\frac{i}{\hbar} S^{\psi}_{ext}[Y, Y^*]\right) \\ &= \int \mathcal{D}\{Y^A(x)\}\chi[Y, Y^*] \exp\left(\frac{i}{\hbar} W[Y, Y^*]\right) \end{aligned} \tag{48}$$

of an arbitrary functional $\chi[Y, Y^*]$ (the antifields still have to be set to zero) is invariant under infinitesimal canonical transformations if the relations

$$(W[Y, Y^*], W[Y, Y^*]) - 2i\hbar \triangle W[Y, Y^*] = 0 \tag{49}$$

$$\frac{i}{\hbar} (\chi[Y, Y^*], W[Y, Y^*]) + \triangle\chi[Y, Y^*] = 0 \tag{50}$$

with

$$\triangle U[Y, Y^*] = (-1)^{p(Y^A)+p(U)} \int \frac{\overrightarrow{\delta}^2}{\delta Y^*_A(x) \delta Y^A(x)} U[Y, Y^*] dx \tag{51}$$

for any functional $U[Y, Y^*]$ hold. Eq. (49) is the socalled "quantum master equation" and (50) defines the quantum BRST operator

$$\hat{\mathcal{S}} := (\cdot, W[Y, Y^*]) - i\hbar\triangle \tag{52}$$

which is nilpotent if $W[Y, Y^*]$ satisfies the quantum master equation. The cohomology classes of $\hat{\mathcal{S}}$ define the physical observables.

The quantum master equation is usually solved by an ansatz in powers of $\hbar$,

$$W[Y, Y^*] = S^{\psi}_{ext}[Y, Y^*] + \sum_{n=1}^{\infty} \hbar^n M_n[Y, Y^*]. \tag{53}$$

The $\mathcal{O}(\hbar^0)$ order reproduces the classical master equation, and the $\mathcal{O}(\hbar^1)$ order correction is given by

$$\left(S^{\psi}_{ext}[Y, Y^*], M_1[Y, Y^*]\right) = i\triangle S^{\psi}_{ext}[Y, Y^*]. \tag{54}$$

We now apply the general procedure of gauge fixing in the BV formalism to our soliton model. As usual, the physical idea of gauge fixing in this context is to eliminate the fluctuations parallel to the zero mode φ' of the Gaussian fluctuation operator. The corresponding gauge fermion is

$$\psi[\eta, a, \bar{c}] = \int \bar{c}(t) \left(\int \eta(t, \mathrm{x})\varphi'(\mathrm{x} - a(t)) d\mathrm{x}\right) dt \tag{55}$$

where the x–integral is the orthogonality condition $\eta \perp \varphi'$. Another motivation for the choice of gauge fixing will be given later.

From the general formula (46), we obtain the gauge–fixed extended action

$$
\begin{aligned}
& S^{\psi}_{ext}[\eta, a, c, \bar{c}, b; \eta^*, a^*, c^*, \bar{c}^*, b^*] \\
= \; & S_0[\eta, a] + \int \left[a^*(t) + \bar{c}(t)\left(\int \eta(t,\mathrm{x})\varphi''(\mathrm{x}-a(t))d\mathrm{x}\right)\right] c(t)dt \\
& \qquad + \int\int [\eta^*(t,\mathrm{x}) - \bar{c}(t)\varphi'(\mathrm{x}-a(t))]\,\varphi'(\mathrm{x}-a(t))c(t)d\mathrm{x}dt \\
& \qquad + \int \left[\bar{c}^*(t) - \left(\int \eta(t,\mathrm{x})\varphi'(\mathrm{x}-a(t))d\mathrm{x}\right)\right] b(t)dt. \qquad (56)
\end{aligned}
$$

It is easy to check that $\triangle S^{\psi}_{ext}[\eta, a, c, \bar{c}, b; \eta^*, a^*, c^*, \bar{c}^*, b^*] = 0$, so there will be no quantum corrections to the classical master equation in our model. In more general situations, but still with abelian gauge algebras, we have from (30) and (47)

$$
S^{\psi}_{ext}[Y, Y^*] = S_0[\Phi] + \int Y^*_A(x)\left(\mathcal{S}Y^A(x)\right)dx - \mathcal{S}\psi[Y] \qquad (57)
$$

and hence, using the fact that the gauge fermion depends only on fields,

$$
\triangle S^{\psi}_{ext}[Y, Y^*] = (-1)^{p(Y^A)} \int \frac{\vec{\delta}}{\delta Y^A(x)}\left(\mathcal{S}Y^A(x)\right)dx. \qquad (58)
$$

For a large number of collective coordinate models and especially for collective translation and rotation degrees of freedom as we discuss them here, $\mathcal{S}Y^A(x)$ does not depend on $Y^A(x)$ itself. Otherwise, terms proportional to $\frac{\vec{\delta}}{\delta Y^A(x)} Y^A(x) \propto \delta(0)$ arise in (58), and an appropriate regularisation procedure is necessary.

From (58) we see that quantum measure corrections to the BV–Lagrangian path integral arise only from the BRST symmetry structure of the theory and not from the action itself. In particular, there would be no effects due to field dependent mass terms in the kinetic part of the Lagrangian since they arise in Skyrme–like models [13] and lead to measure corrections in the BFV–Hamiltonian path integral [11, 14] when integrating out the field momenta. For these models, the equivalence of the Lagrangian BV and the Hamiltonian BFV methods would have to be analysed again in detail.

Next we evaluate as a simple example of BV Lagrangian path integrals the transition amplitude

$$
\begin{aligned}
\langle \phi^f | \phi^i \rangle \; = \; & \int_{\phi^i}^{\phi^f} \mathcal{D}\{\eta(t,\mathrm{x})\}\mathcal{D}\{a(t)\}\mathcal{D}\{c(t)\}\mathcal{D}\{\bar{c}(t)\}\mathcal{D}\{b(t)\} \\
& \qquad \times \exp\left(\frac{i}{\hbar} S^{\psi}_{ext}[\eta, a, c, \bar{c}, b; \eta^* = 0, a^* = 0, c^* = 0, \bar{c}^* = 0, b^* = 0]\right) \\
= \; & \int_{\phi^i}^{\phi^f} \mathcal{D}\{\eta(t,\mathrm{x})\}\mathcal{D}\{a(t)\}\mathcal{D}\{c(t)\}\mathcal{D}\{\bar{c}(t)\}\mathcal{D}\{b(t)\}
\end{aligned}
$$

$$\times \exp\Big(\frac{i}{\hbar}\Big\{S_0[\eta,a] - \int \bar{c}(t)\Big[\int (\varphi'(\mathrm{x}-a(t)))^2 d\mathrm{x} - \int \eta(t,\mathrm{x})\varphi''(\mathrm{x}-a(t))d\mathrm{x}\Big] c(t)dt - \int \Big[\int \eta(t,\mathrm{x})\varphi'(\mathrm{x}-a(t))d\mathrm{x}\Big] b(t)dt\Big\}\Big). \quad (59)$$

The $b(t)$–integration yields

$$\delta\left(\int \eta(t,\mathrm{x})\varphi'(\mathrm{x}-a(t))d\mathrm{x}\right), \quad (60)$$

and the $\bar{c}(t)c(t)$–integration results in the Faddeev–Popov like determinant $\det_t(M_0 + m_\eta(t)) = M_0 + \mathcal{O}(\eta)$ where

$$M_0 = \int (\varphi'(\rho))^2 d\rho \quad (61)$$

is the classcal soliton mass and

$$m_\eta(t) = \int \eta(t,\mathrm{x})\varphi''(\mathrm{x}-a(t))d\mathrm{x} = \mathcal{O}(\eta) \quad (62)$$

is a quantum correction.

To handle the $\eta(t,\mathrm{x})$ and $a(t)$ integrations, we expand $S_0[\eta,a]$ up to second order in powers of η:

$$\begin{aligned} & S_0[\eta,a] \\ = \;& S_0[0,a] + \int \left[S_0[\eta,a]\frac{\overleftarrow{\delta}}{\delta\eta(t,\mathrm{x})}\right]_{\eta=0} \eta(t,\mathrm{x})dtd\mathrm{x} \\ & + \frac{1}{2}\int \eta(t_1,\mathrm{x}_1)\left[\frac{\overrightarrow{\delta}}{\delta\eta(t_1,\mathrm{x}_1)}S_0[\eta,a]\frac{\overleftarrow{\delta}}{\delta\eta(t_2,\mathrm{x}_2)}\right]_{\eta=0} \eta(t_2,\mathrm{x}_2)dt_1d\mathrm{x}_1dt_2d\mathrm{x}_2 + \mathcal{O}(\eta^3). \end{aligned} \quad (63)$$

The constant contribution is

$$S_0[0,a] = \tilde{S}_0[\varphi] + \int \frac{1}{2}M_0\dot{a}^2(t)dt, \quad (64)$$

the action of the classical soliton solution plus the action of the free motion of a particle with mass M_0.

Due to the fact that φ solves the static classical Euler–Lagrange equations for $\tilde{\mathcal{L}}$, the terms linear in η are

$$\begin{aligned} & \left[S_0[\eta,a]\frac{\overleftarrow{\delta}}{\delta\eta(t,\mathrm{x})}\right]_{\eta=0} \eta(t,\mathrm{x})dtd\mathrm{x} = \int \left[\frac{\partial^2}{\partial t^2}\varphi(\mathrm{x}-a(t))\right]\eta(t,\mathrm{x})dtd\mathrm{x} \\ = \;& \int \ddot{a}(t)\left[\int \eta(t,\mathrm{x})\varphi'(\mathrm{x}-a(t))d\mathrm{x}\right]dt - \int \dot{a}^2(t)\left[\int \eta(t,\mathrm{x})\varphi''(\mathrm{x}-a(t))d\mathrm{x}\right]dt. \end{aligned} \quad (65)$$

The first term is cancelled by the δ–function from the $b(t)$–integration (60), and the second yields a mass correction of $2m_\eta(t) = \mathcal{O}(\eta)$ to the free motion part of (64). This effect is another motivation for the choice of the gauge–fixing by eq. (55): The gauge fermion is chosen such that only "physically meaningful" terms survive in the fluctuation expansion of the classical action. Another possible gauge fixing would be to eliminate the complete linear contribution in the η–expansion: The gauge fermion

$$\psi[\eta, a, \bar{c}] = \int \left[\eta(t, \mathrm{x}) \frac{\partial^2}{\partial t^2} \varphi(\mathrm{x} - a(t)) d\mathrm{x} \right] \bar{c}(t) dt \tag{66}$$

could be a candidate for that idea.

What remains are the quadratic terms yielding harmonic quantum fluctuations in the usual way:

$$\begin{aligned} & \frac{1}{2} \int \eta(t_1, \mathrm{x}_1) \left[\frac{\overrightarrow{\delta}}{\delta \eta(t_1, \mathrm{x}_1)} S_0[\eta, a] \frac{\overleftarrow{\delta}}{\delta \eta(t_2, \mathrm{x}_2)} \right]_{\eta=0} \eta(t_2, \mathrm{x}_2) dt_1 d\mathrm{x}_1 dt_2 d\mathrm{x}_2 \\ = & \ \frac{1}{2} \int \eta(t, \mathrm{x}) \left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} - V''(\varphi(\mathrm{x} - a(t))) \right] \eta(t, \mathrm{x}) dt d\mathrm{x} \end{aligned} \tag{67}$$

The substitution

$$\begin{aligned} \mathrm{x} & \mapsto \rho = \mathrm{x} - a(t) \\ t & \mapsto t \end{aligned} \tag{68}$$

does not change the integral measure since its Jacobi determinant is 1. Under this change of variables, the second derivatives change to

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} - \dot{a}(t) \frac{\partial}{\partial \rho}, \qquad \frac{\partial}{\partial \mathrm{x}} \mapsto \frac{\partial}{\partial \rho} \tag{69}$$

and the harmonic fluctuation integral is

$$\frac{1}{2} \int \eta(t, \rho + a(t)) \left[-\frac{\partial^2}{\partial t^2} - \hat{\Omega} \right] \eta(t, \rho + a(t)) dt d\rho + \int m_{\eta^2}(t) \left(\dot{a}(t) \right)^2 dt + S_0^{(int)}[\eta, a] \tag{70}$$

where $\hat{\Omega}$ is the Gaussian fluctuation operator

$$\hat{\Omega} = -\frac{\partial^2}{\partial \rho^2} + V''(\varphi(\rho)). \tag{71}$$

The change of variables also yields a further quantum correction to the soliton mass (resulting in a modification of the center–of–mass action of the soliton) by

$$m_{\eta^2}(t) = \int \left(\eta'(t, \rho + a(t)) \right)^2 d\rho = \mathcal{O}(\eta^2) \tag{72}$$

and $S_0^{(int)}[\eta, a]$ are coupling terms of second order in the fluctuations:

$$
\begin{aligned}
S_0^{(int)}[\eta, a] &= \frac{1}{2}\int \eta(t,\rho+a(t))\left[\dot{a}(t)\frac{\partial}{\partial t}\frac{\partial}{\partial \rho} + \frac{\partial}{\partial t}\dot{a}(t)\frac{\partial}{\partial \rho}\right]\eta(t,\rho+a(t))dtd\rho \\
&= \mathcal{O}(a, \eta^2).
\end{aligned}
\tag{73}
$$

It is easy to check that $\Psi_0(\rho) = \frac{1}{\sqrt{M_0}}\varphi'(\rho)$ is a normalised zero mode of the Gaussian fluctuation operator, $\hat{\Omega}\Psi_0(\rho) = 0$. Assuming that we know a complete set of (generalised) eigenfunctions of $\hat{\Omega}$, $\hat{\Omega}\Psi_n(\rho) = \omega_n^2\Psi_n(\rho)$, $\int \overline{\Psi_n(\rho)}\Psi_m(\rho)d\rho = \delta_{nm}$, we may expand the fluctuations in terms of Ψ_n:

$$
\eta(t,\rho+a(t)) = \sum_n \alpha_n(t)\Psi_n(\rho). \tag{74}
$$

Eq. (74) may be regarded as a linear substitution in the path integral (59) with Jacobi determinant

$$
\det\left(\frac{\partial\eta(t,\rho+a(t))}{\alpha_n(t)}\right) = J + \mathcal{O}(\eta, a) \tag{75}
$$

with constant J: Physically speaking, we changed the basis in the space of the "fluctuation coordinates" from the "continuous space index" x to a quantum number index n: $a_{\mathrm{x}}(t) \mapsto \alpha_n(t)$, which from the mathematical point of view are the Fourier series coefficients of $\eta(t,\mathrm{x})$ in terms of the basis of eigenfunctions.

Inserting (74) into (70), we obtain

$$
\sum_{n=0}^{\infty}\int\left\{\frac{1}{2}\dot{\alpha}_n^2 - \frac{1}{2}\omega_n^2\alpha_n^2(t)\right\}dt \tag{76}
$$

as quadratic contribution in the η–expansion. The zero mode $\omega_0^2 = 0$ is eliminated by the δ–function in the integrand which in the new $\alpha_n(t)$–variables reads:

$$
\delta\left(\int \eta(t,\mathrm{x})\varphi'(\mathrm{x}-a(t))d\mathrm{x}\right) = \frac{1}{\sqrt{M_0}}\delta(\alpha_0(t)). \tag{77}
$$

Putting everything together, we have in the one–loop approximation

$$
\begin{aligned}
\langle\phi^f|\phi^i\rangle &= \int_{\phi^i}^{\phi^f}\prod_{n=0}^{\infty}\mathcal{D}\{\alpha_n(t)\}\mathcal{D}\{a(t)\}\left(J+\mathcal{O}(\eta,a)\right)\frac{1}{\sqrt{M_0}}\delta(\alpha_0(t))\left(M_0+\mathcal{O}(\eta)\right) \\
&\quad\times\exp\left(\frac{i}{\hbar}\Big\{\tilde{S}_0[\varphi] + \int\frac{1}{2}\left(M_0 + 2m_\eta(t) + 2m_{\eta^2}(t)\right)\dot{a}^2(t)dt\right. \\
&\qquad\left.+\sum_{n=0}^{\infty}\int\left\{\frac{1}{2}\dot{\alpha}_n^2 - \frac{1}{2}\omega_n^2\alpha_n^2(t)\right\}dt + \mathcal{O}(\eta^3) + \mathcal{O}(\eta^2, a)\right)\Big\}
\end{aligned}
\tag{78}
$$

If we neglect quantum corrections to the soliton mass and to the Jacobian in lowest order, the path integral factorises into three parts: The action of the classical solution as a constant factor, $\exp\left(\frac{i}{\hbar}\tilde{S}_0[\varphi]\right)$, the free motion of the center–of–mass of the soliton,

$$\int \mathcal{D}\{a(t)\} \exp\left(\frac{i}{\hbar}\int \frac{1}{2}M_0\dot{a}^2(t)dt\right) \tag{79}$$

and the contribution of harmonic quantum fluctuations

$$\prod_{n\neq 0}\int \mathcal{D}\{\alpha(t)\} \exp\left(\frac{i}{\hbar}\bar{S}_n[\alpha]\right), \tag{80}$$

where

$$\bar{S}_n[\alpha] = \int \left\{\frac{1}{2}\dot{\alpha}_n^2 - \frac{1}{2}\omega_n^2\alpha_n^2(t)\right\} dt \tag{81}$$

is the action of a harmonic oscillator with frequency ω_n. The complete integral is purely Lagrangian and avoids the mixed Lagrangian–Hamiltonian notation which is often used in this context [4, 9].

5 The reduced $O(3)$ σ–model in $1+1$ dimensions, collective rotation coordinates and their zero modes

In the previous sections, we investigated the BV quantisation of a model with one collective coordinate $a(t)$, describing a global translation symmetry which is broken by the soliton solution $\varphi(x)$.

In general the collective coordinate method deals with theories given by an action $S[\Phi]$ which is invariant under the action of a symmetry group $\mathbf{G}$ on the fields $\Phi^i(x)$. Static soliton solutions $\varphi^i(\mathrm{x})$ of the Euler–Lagrange equations $S[\Phi]\frac{\overleftarrow{\delta}}{\delta\Phi^i(x)}$ break this symmetry group to a subgroup $\mathbf{H}\subset\mathbf{G}$, and there are zero modes associated with each element of $\mathbf{G}/\mathbf{H}$ [9, 24]. Collective coordinates, in some sense, "restore" the broken symmetries [25] by introducing a new, *local* symmetry into the theory.

Translation invariance is the symmetry most commonly broken by soliton solutions: This is clear fron the fact that solitons must have finite classical energy and thus are localised static objects in space. But besides translation invariance which is a spacetime symmetry, internal symmetries are important properties of a large class of field theories. The breakdown of internal symmetry degrees of freedom in soliton solutions also leads to important applications of the BV scheme in the context of collective coordinates.

We investigate such an "internal collective coordinate" in the context of th reduced nonlinear $O(3)$ σ–model in $1+1$ dimensions discussed by Mottola and Wipf [18]. It is given by the Lagrange density

$$\mathcal{L}(\vec{\phi}, \partial_\mu\vec{\phi}) = \lambda_1\langle\partial_\mu\vec{\phi}, \partial^\mu\vec{\phi}\rangle - \lambda_0 V(\phi^{(3)}), \quad (\vec{\phi})^2 = 1 \tag{82}$$

with bosonic $\vec{\phi} = (\phi^{(1)}, \phi^{(2)}, \phi^{(3)})$, $\{x^\mu\} = \{t, \mathrm{x}\}$. The bracket $\langle , \rangle$ in the following always denotes the appropriate canonical scalar product in the internal symmetry space. The potential $V(\phi^{(3)}) = 1 - \phi^{(3)}$ breaks the internal $O(3)$ symmetry of the kinetic energy term in the Lagrangian to an $O(2)$ symmetry around the $\phi^{(3)}$ axis. We use this model not only because it has important applications to baryon and lepton number violation calculations, but also because its internal symmetry group is as simple as possible, depending on only one parameter.

In a way different from that of [18], we parametrise the fields $\vec{\phi}(t, \mathrm{x})$ by "spherical parameter fields" $\vec{G} = (G^{(1)}, G^{(2)})$,

$$\vec{\phi}(t,\mathrm{x}) = \begin{pmatrix} \sin G^{(1)}(t,\mathrm{x}) \cos G^{(2)}(t,\mathrm{x}) \\ \sin G^{(1)}(t,\mathrm{x}) \sin G^{(2)}(t,\mathrm{x}) \\ \cos G^{(1)}(t,\mathrm{x}) \end{pmatrix}, \tag{83}$$

to get rid of the constraint $(\vec{\phi})^2 = 1$. This parametrisation is more appropriate to the introduction of collective coordinates in the reduced $O(3)$ σ–model which have been discussed neither in [18] nor in the further investigations of this and related models [26].

In terms of the parameter fields $\vec{G}$, the Lagrangian density can be written

$$\mathcal{L}(\vec{G}, \partial_\mu \vec{G}) = \lambda_1 \langle \partial_\mu \vec{G}, \hat{g}(\vec{G}) \partial^\mu \vec{G} \rangle - \lambda_0 V(\cos G^{(1)}) \tag{84}$$

with a field dependent mass matrix

$$\hat{g}(\vec{G}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 G^{(1)} \end{pmatrix} \tag{85}$$

which defines a Riemannian metric in the internal symmetry space of the parameter fields. Eq. (84) is a Lagrangian of the type discussed in [9]. It is invariant under spacetime transformations $x^\mu \mapsto x^\mu + x_0^\mu$ and constant shifts of the second paramter field, $G^{(2)} \mapsto G^{(2)} + \gamma$, the latter corresponding to an $O(2)$ rotation of $\vec{\phi}$ around $\phi^{(3)}$ with angle γ.

It is well known that the ansatz $G_0^{(1)}(t, \mathrm{x}) = g(\mathrm{x})$, $G_0^{(2)}(t, \mathrm{x}) = \gamma = \mathrm{const}$ yields the Euler–Lagrange equation

$$2g''(\mathrm{x}) - \mu^2 \sin g(\mathrm{x}) = 0, \qquad \mu = \pm \sqrt{\frac{\lambda_0}{\lambda_1}} \tag{86}$$

with solution $g(\mathrm{x}) = 4 \arctan \exp\left(\frac{\mu}{\sqrt{2}}\mathrm{x}\right)$. This corresponds to the sphaleron configuration

$$\vec{\varphi}(\mathrm{x}) = \begin{pmatrix} \sin g(\mathrm{x}) \cos \gamma \\ \sin g(\mathrm{x}) \sin \gamma \\ \cos g(\mathrm{x}) \end{pmatrix} = \begin{pmatrix} 2 \tanh\left(\frac{\mu}{\sqrt{2}}\mathrm{x}\right) \mathrm{sech}\left(\frac{\mu}{\sqrt{2}}\mathrm{x}\right) \cos\gamma \\ 2 \tanh\left(\frac{\mu}{\sqrt{2}}\mathrm{x}\right) \mathrm{sech}\left(\frac{\mu}{\sqrt{2}}\mathrm{x}\right) \sin\gamma \\ 1 - 2\mathrm{sech}^2\left(\frac{\mu}{\sqrt{2}}\mathrm{x}\right) \end{pmatrix}. \tag{87}$$

We immediately see that $\vec{\varphi}(\mathrm{x})$ breaks the space translation invariance and the internal $O(2)$ symmetry of the Lagrangian (82): In terms of the parametrisation fields, $G_0^{(1)}(t,\mathrm{x}) = g(\mathrm{x})$, $G_0^{(2)}(t,\mathrm{x}) = \gamma$ and $G_0^{(1)}(t,\mathrm{x}) = g(\mathrm{x}-a^{(1)})$, $G_0^{(2)}(t,\mathrm{x}) = \gamma + a^{(2)}$ are different solutions. (In the following, we will use the solution with $\gamma = 0$.)

For each of these broken global symmetries, we now introduce one redundant bosonic symmetry coordinate which we write as $\vec{A}(t) = (A^{(1)}(t), A^{(2)}(t))$. Since the Lagrangian density (84) does not depend on $\vec{A}$ at all, we have two trivial Noether identities

$$\int \left(\tilde{S}_0[\vec{G},\vec{A}] \frac{\overleftarrow{\delta}}{\delta G^{(i)}(t,\mathrm{x})} \right) R^{(G^{(i)})} d\mathrm{x} + \left(\tilde{S}_0[\vec{G},\vec{A}] \frac{\overleftarrow{\delta}}{\delta A^{(i)}(t)} \right) R^{(A^{(i)})} \equiv 0, \quad i = 1,2 \quad (88)$$

with $R^{(G^{(i)})} = 0$, $R^{(A^{(i)})} = 1$. The corresponding extended action in the field–antifield space $\{\tilde{Y}^A(x), \tilde{Y}^*_A(x)\} = \{\vec{G}(t,\mathrm{x}), \vec{A}(t), \vec{C}(t); \vec{G}^*(t,\mathrm{x}), \vec{A}^*(t), \vec{C}^*(t)\}$ (we added one fermionic ghost coordinate $C^{(i)}(t)$ for each Noether identity) then reads

$$\tilde{S}_{ext}[\vec{G},\vec{A},\vec{C};\vec{G}^*,\vec{A}^*,\vec{C}^*] = \tilde{S}_0[\vec{G},\vec{A}] + \int \langle \vec{A}^*(t), \vec{C}(t)\rangle dt. \quad (89)$$

To introduce collective coordinates $\vec{a}(t) = (a^{(1)}(t), a^{(2)}(t))$ and fluctuation fields $\vec{\eta}(t,\mathrm{x}) = (\eta^{(1)}(t,\mathrm{x}), \eta^{(2)}(t,\mathrm{x}))$ around the sphaleron solution $\vec{\varphi}(\mathrm{x})$ given by $G_0^{(1)}(t,\mathrm{x}) = g(\mathrm{x})$, $G_0^{(2)}(t,\mathrm{x}) = 0$, we use a canonical transformation

$$\begin{aligned} & \{\tilde{Y}^A(x), \tilde{Y}^*_A(x)\} = \{\vec{G}(t,\mathrm{x}), \vec{A}(t), \vec{C}(t); \vec{G}^*(t,\mathrm{x}), \vec{A}^*(t), \vec{C}^*(t)\} \\ \overset{F}{\mapsto} \; & \{Y^A(x), Y^*_A(x)\} = \{\vec{\eta}(t,\mathrm{x}), \vec{a}(t), \vec{c}(t); \vec{\eta}^*(t,\mathrm{x}), \vec{a}^*(t), \vec{c}^*(t)\} \end{aligned} \quad (90)$$

generated by

$$\begin{aligned} F[\vec{G},\vec{A},\vec{C};\vec{\eta}^*,\vec{a}^*,\vec{c}^*] &= \int \langle \vec{a}^*(t), \vec{A}(t)\rangle dt + \int \langle \vec{c}^*(t), \vec{C}(t)\rangle dt \\ &+ \int \eta^*_{(1)}(t,\mathrm{x}) \left[G^{(1)}(t,\mathrm{x}) - g\left(\mathrm{x} - A^{(1)}(t)\right)\right] dt d\mathrm{x} \\ &+ \int \eta^*_{(2)}(t,\mathrm{x}) \left[G^{(2)}(t,\mathrm{x}) - A^{(2)}(t)\right] dt d\mathrm{x}. \end{aligned} \quad (91)$$

Applying (35), we have $\vec{C}^*(t) = \vec{\eta}^*(t,\mathrm{x})$, $\vec{C}^*(t) = \vec{c}^*(t)$, $\vec{a}(t) = \vec{A}(t)$, $\vec{c}(t) = \vec{C}(t)$ and

$$\eta^{(1)}(t,\mathrm{x}) = G^{(1)}(t,\mathrm{x}) - g\left(\mathrm{x} - A^{(1)}(t)\right) \quad (92)$$

$$\eta^{(2)}(t,\mathrm{x}) = G^{(2)}(t,\mathrm{x}) - A^{(2)}(t) \quad (93)$$

$$A^*_{(1)}(t) = a^*_{(1)}(t) + \int \eta^*_{(1)}(t,\mathrm{x}) g'\left(\mathrm{x} - A^{(1)}(t)\right) d\mathrm{x} \quad (94)$$

$$A^*_{(2)}(t) = a^*_{(2)}(t) - \int \eta^*_{(2)}(t,\mathrm{x}) d\mathrm{x}. \quad (95)$$

Eq. (92) and (93) yield the transformation rules

$$G^{(1)}(t,\mathrm{x}) = g\left(\mathrm{x} - a^{(1)}(t)\right) + \eta^{(1)}(t,\mathrm{x}) \quad (96)$$

$$G^{(2)}(t,\mathrm{x}) = a^{(2)}(t) + \eta^{(2)}(t,\mathrm{x}). \quad (97)$$

Therefore, according to the definition (83) of the parameter fields $\vec{G}(t,\mathrm{x})$, we have to interpret $a^{(1)}(t)$ as *collective translation coordinate* and $\eta^{(1)}(t,\mathrm{x})$ as fluctuations in the "direction of the sphaleron", whereas (since $G^{(2)}(t,\mathrm{x})$ is the angle of the $O(2)$ rotation around the $\phi^{(3)}$ axis in internal symmetry space) $a^{(2)}(t)$ is a *collective rotation coordinate* and $\eta^{(2)}(t,\mathrm{x})$ describes rotational fluctuations of the sphaleron around the fixed axis $\phi^{(3)}$.

Transforming the extended action (89) via (90) yields

$$\begin{aligned} S_{ext}[\vec{\eta},\vec{a},\vec{c};\vec{\eta}^{*},\vec{a}^{*},\vec{c}^{*}] &= S_0[\vec{\eta},\vec{a}] \\ &\quad + \int a^{*}_{(1)}(t)c^{(1)}(t)dt + \int \eta^{*}_{(1)}(t,\mathrm{x})g'\left(\mathrm{x}-A^{(1)}(t)\right)c^{(1)}(t)dtd\mathrm{x} \\ &\quad + \int a^{*}_{(2)}(t)c^{(2)}(t)dt + \int \eta^{*}_{(2)}(t,\mathrm{x})(-1)c^{(2)}(t)dtd\mathrm{x} \end{aligned} \tag{98}$$

where

$$\begin{aligned} & S_0[\vec{\eta},\vec{a}] \\ = & \int \Big\{ \lambda_1 \left[\left(-\dot{a}^{(1)}(t)g'\left(\mathrm{x}-a^{(1)}(t)\right)+\dot{\eta}^{(1)}(t,\mathrm{x})\right)^2 - \left(g'\left(\mathrm{x}-a^{(1)}(t)\right)+\eta^{(1)\prime}(t,\mathrm{x})\right)^2\right] \\ & + \lambda_1 \left(\dot{a}^{(2)}(t)+\dot{\eta}^{(2)}(t,\mathrm{x})\right)^2 - \lambda_1\left(\eta^{(2)\prime}(t,\mathrm{x})\right)^2 \sin^2\left(g\left(\mathrm{x}-a^{(1)}(t)\right)+\eta^{(1)}(t,\mathrm{x})\right) \\ & - \lambda_0 V\left(g\left(\mathrm{x}-a^{(1)}(t)\right)+\eta^{(1)}(t,\mathrm{x})\right) \end{aligned} \tag{99}$$

is the new classical action in terms of collective and fluctuation degrees of freedom.

Comparing (98) with the general ansatz for the extended action (28), we see that only terms linear in the antifields occur which means that the gauge algebra is again abelian. The BRST transformations can be found by comparing (30) and (98), yielding

$$\begin{aligned} \mathcal{S}\eta^{(1)}(t,\mathrm{x}) &= g'\left(\mathrm{x}-a^{(1)}(t)\right)c^{(1)}(t) = R^{(\eta^{(1)})}c^{(1)}(t) \\ \mathcal{S}a^{(1)}(t) &= c^{(1)}(t) = R^{(a^{(1)})}c^{(1)}(t) \\ \mathcal{S}c^{(1)}(t) &= 0 \end{aligned} \tag{100}$$

$$\begin{aligned} \mathcal{S}\eta^{(2)}(t,\mathrm{x}) &= -c^{(2)}(t) = R^{(\eta^{(2)})}c^{(1)}(t) \\ \mathcal{S}a^{(2)}(t) &= c^{(2)}(t) = R^{(a^{(2)})}c^{(2)}(t) \\ \mathcal{S}c^{(2)}(t) &= 0 \end{aligned} \tag{101}$$

with Noether generators $R^{(\eta^{(1)})} = g'\left(\mathrm{x}-a^{(1)}(t)\right)$, $R^{(a^{(1)})} = 1$ which are typically associated with collective translation coordinates (cf. the Noether generators (12), (13) of the simple scalar field theory model in section 2), and $R^{(\eta^{(2)})} = -1$, $R^{(a^{(2)})} = 1$: These are typical Noether generators associated with collective rotation degrees of freedom, they are related to the generator of $O(2)$ rotations, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

It is easy to check that the Noether identities

$$\int \left(S_0[\vec{\eta},\vec{a}] \frac{\overleftarrow{\delta}}{\delta \eta^{(i)}(t,\mathrm{x})} \right) R^{(\eta^{(i)})} dx + \left(S_0[\vec{\eta},\vec{a}] \frac{\overleftarrow{\delta}}{\delta a^{(i)}(t)} \right) R^{(a^{(i)})} \equiv 0 \tag{102}$$

hold for $i = 1, 2$: The x–integration shows that the space position coordinate is again a kind of "continuous index", so that the fluctuation fields can be interpreted as "fluctuation coordinates" $\vec{a}_{\mathrm{x}}(t) = \vec{\eta}(t, \mathrm{x})$.

To quantize the model by path integrals, we have to choose a convenient gauge fermion ψ and fix the gauge by performing the canonical transformation $S_{ext} \overset{\psi}{\mapsto} S^{\psi}_{ext}$. Due to the fact that we now have two Noether identities (102), we have to add two trivial systems $\{\bar{c}^{(i)}(t), b^{(i)}(t)\}$, $i = 1, 2$ to construct the gauge fermion.

Without going into details concerning the evaluation of path integrals in the nonlinear reduced $O(3)$ σ model, we finally consider the zero modes and the choice of gauge fixing in this model. From (100) and (101), we see that the translational and rotational gauge symmetries are independent of each other, and may therefore be gauge fixed independently. For the collective translation degree of freedom, we know a suitable gauge fixing principle which states that fluctuations parallel to the zero mode have to be excluded from the path integration. Denoting the parallel and rotational zero modes by $\Psi_0^{(1)}(\mathrm{x})$ and $\Psi_0^{(2)}(\mathrm{x})$ respectively, the corresponding gauge fermion reads

$$\psi[\vec{\eta},\vec{a},\vec{\bar{c}}] = \sum_{i=1,2} \int \bar{c}^{(1)}(t) \left\{ \int \eta^{(i)}(t,\mathrm{x}) \Psi_0^{(i)}(\mathrm{x} - a^{(i)}(t)) d\mathrm{x} \right\} dt \tag{103}$$

From Section 4, we may conclude that the translational zero mode is $\Psi_0^{(1)} \propto g'$ and thus $\Psi_0^{(1)} \propto R^{(\eta^{(1)})}$. Therefore, we may guess that the rotational zero mode is proportional to the corresponding Noether generator $R^{(\eta^{(2)})} = -1$ which means that this zero mode is constant, $\Psi_0^{(2)} \propto -1$ and thus seems to be not normalizable.

We can see in which sense this is true by investigating the expansion of the classical action $S_0[\vec{\eta},\vec{a}]$ (99) in terms of the fluctuations up to second order (which yields, after the $\vec{b}(t)$ and $\vec{\bar{c}}(t)\vec{c}(t)$ integrations which we skip here, the one loop approximation in the path integral). This expansion may be written as

$$\begin{aligned} S_0[\vec{\eta},\vec{a}] \;&=\; \tilde{S}_0[\vec{\varphi}] + S_0^{(CM1)}[\eta^{(1)}, a^{(1)}] + S_0^{(CM2)}[\eta^{(1)}, a^{(2)}] \\ &\quad + S_0^{(int1)}[\eta^{(1)}, a^{(1)}] + S_0^{(int2)}[\vec{\eta},\vec{a}] \\ &\quad + S_0^{(fluc1)}[\eta^{(1)}, a^{(1)}] + \check{S}_0^{(fluc2)}[\eta^{(2)}, a^{(1)}]. \end{aligned} \tag{104}$$

Here $\tilde{S}_0[\vec{\varphi}] = \int [-\lambda_1 (g'(\rho))^2 - \lambda_0 V(\cos g(\rho))]\, d\rho dt$ is the classical action of the sphaleron, and

$$S_0^{(CM1)}[\eta^{(1)}, a^{(1)}] = \lambda_1 \int \left(\dot{a}^{(1)}\right)^2 [M - 2m_{\eta^{(1)}}(t)] dt \tag{105}$$

with $M = \int (g'(\rho))^2 d\rho$ and

$$m_{\eta^{(1)}}(t) = \int \eta^{(1)}(t,\mathrm{x}) g''\left(\mathrm{x} - a^{(1)}(t)\right) d\mathrm{x} = \mathcal{O}(\eta^{(1)}) \tag{106}$$

is the action of the free center–of–mass translation of the sphaleron, whereas

$$S_0^{(CM2)}[\eta^{(1)}, a^{(2)}] = \lambda_1 \int \left(\dot{a}^{(2)}\right)^2 [I - 2i_{\eta^{(1)}}(t)] dt \tag{107}$$

with $I = \int \sin^2 g(\rho) d\rho$ and

$$\begin{aligned} i_{\eta^{(1)}}(t) &= \int \Big[2\eta^{(1)}(t,\mathrm{x}) \cos g\left(\mathrm{x} - a^{(1)}(t)\right) \sin g\left(\mathrm{x} - a^{(1)}(t)\right) \\ &\qquad + \left(2\eta^{(1)}(t,\mathrm{x})\right)^2 \left(\cos^2 g\left(\mathrm{x} - a^{(1)}(t)\right) - \sin^2 g\left(\mathrm{x} - a^{(1)}(t)\right)\right)\Big] d\mathrm{x} \\ &= \mathcal{O}(\eta^{(1)}) \end{aligned} \tag{108}$$

may be interpreted as the action of the rotation of the sphaleron around the ϕ^3 axis.

Next, we have mixed or interaction terms

$$S_0^{(int1)}[\eta^{(1)}, a(1)] = 2\lambda_1 \int \ddot{a}^{(1)}(t) \left[\int \eta^{(1)}(t,\mathrm{x}) g'\left(\mathrm{x} - a^{(1)}(t)\right) d\mathrm{x}\right] dt \tag{109}$$

$$\begin{aligned} S_0^{(int2)}[\vec{\eta}, \vec{a}] &= 2\lambda_1 \iint \dot{a}^{(2)}(t) \dot{\eta}^{(2)}(t,\mathrm{x}) \sin^2 g\left(\mathrm{x} - a^{(1)}(t)\right) dt d\mathrm{x} \\ &+ 4\lambda_1 \iint \dot{a}^{(2)}(t) \eta^{(1)}(t,\mathrm{x}) \dot{\eta}^{(2)}(t,\mathrm{x}) \\ &\times \cos g\left(\mathrm{x} - a^{(1)}(t)\right) \sin g\left(\mathrm{x} - a^{(1)}(t)\right) dt d\mathrm{x} \end{aligned} \tag{110}$$

The term (109) equals the first integral in (65) which was cancelled by the gauge fixing condition. Since $\Psi_0^{(1)}\left(\mathrm{x} - a^{(1)}(t)\right) \propto g'\left(\mathrm{x} - a^{(1)}(t)\right)$, a gauge fermion of the type (103) removes the interaction integral (109) from the expansion. The second interaction term (110) is due to the collective rotation coordinate and contains quite complicated couplings between all fluctuations to all collective coordinates. One can try to choose the gauge fixing condition such that (110) becomes as simple as possible, but it seems obvious that it is impossible to cancel (110) by gauge fixing.

Finally, we have the quadratic fluctuation parts of the classical action: The translational fluctuations $\eta^{(1)}$ yield, substituting $\rho = \mathrm{x} - a^{(1)}(t)$ (which leads to a further interaction terms as discussed in Section 4),

$$\begin{aligned} S_0^{(fluc1)}[\eta^{(1)}, a^{(1)}] &= \lambda_1 \int \eta^{(1)}\left(t, \rho + a^{(1)}(t)\right) \left[-\frac{\partial^2}{\partial t^2} - \hat{\Omega}^{(1)}\right] \eta^{(1)}\left(t, \rho + a^{(1)}(t)\right) d\rho dt \\ &+ S_0^{(int3)}[\eta^{(1)}, a^{(1)}] \end{aligned} \tag{111}$$

The interaction term is the analogue of (72,73):

$$S_0^{(int3)}[\eta^{(1)}, a^{(1)}]$$

$$
\begin{aligned}
&= \lambda_1 \int \eta^{(1)} \left(t, \rho + a^{(1)}(t)\right) \left[\dot{a}^{(1)}(t) \frac{\partial}{\partial t}\frac{\partial}{\partial \rho} + \frac{\partial}{\partial t}\dot{a}^{(1)}(t)\frac{\partial}{\partial \rho}\right] \eta^{(1)} \left(t, \rho + a^{(1)}(t)\right) d\rho dt \\
&\quad + \lambda_1 \int \left(\int \left(\eta^{(1)\prime}(t, \rho + a(t))\right)^2 d\rho\right) \left(\dot{a}^{(1)}(t)\right)^2 dt \\
&= \mathcal{O}\left((\eta^{(1)})^2, a^{(1)}\right)
\end{aligned} \tag{112}
$$

The Gaussian fluctuation operator is

$$
\hat{\Omega}^{(1)} = -\frac{\partial^2}{\partial \rho^2} + \frac{1}{2}\mu \left[\sin^2 g(\rho) V''(\cos g(\rho)) - \cos g(\rho) V'(\cos g(\rho))\right] \tag{113}
$$

Inserting the potential $V(\phi^{(3)}) = 1 - \phi^{(3)}$ and the appropriate solution g, this yields (with $y = \frac{\rho}{\sqrt{2}}$)

$$
\hat{\Omega}^{(1)} = \frac{\mu^2}{2} \left[\frac{\partial^2}{\partial y^2} + (1 - 2\mathrm{sech}^2 y)\right] \tag{114}
$$

which is the fluctuation operator "parallel to the sphaleron" discussed by Mottola and Wipf [18].

From the rotational fluctuation, we have in the one loop approximation

$$
\check{S}_0^{(fluc2)}[\eta^{(2)}, a^{(1)}] = \lambda_1 \int \eta^{(2)}(t, \mathrm{x}) \left[-\frac{\partial}{\partial t} \sin^2 g\left(\mathrm{x} - a^{(1)}(t)\right) \frac{\partial}{\partial t} - \check{\Omega}^{(2)}\right] \eta^{(2)}(t, \mathrm{x}) d\mathrm{x} dt \tag{115}
$$

with

$$
\check{\Omega}^{(2)} = -\frac{\partial}{\partial \mathrm{x}} \sin^2 g\left(\mathrm{x} - a^{(1)}(t)\right) \frac{\partial}{\partial \mathrm{x}} \tag{116}
$$

This Gaussian fluctuation operator has a constant zero mode, $\check{\Omega}^{(2)}1 = 0$, and the corresponding gauge fixing condition for the $\eta^{(2)}$ fluctuations would be $\int \eta^{(2)}(t, \mathrm{x}) \cdot 1 d\mathrm{x} = 0$.

Although this zero mode is a constant function, one can not conclude that it is not normalizable (a fact from which would follow that the zero mode is part of the continuous spectrum): The eigenvalue problem we have to solve is not simply $\check{\Omega}^{(2)}\Psi = \omega^2\Psi$, but we have to include a measure function on the right hand side [27], because the time derivative in (115) does not have the simple form $-\frac{\partial^2}{\partial t^2}$ which is needed to decompose the fluctuations in harmonic oscillations as shown in Section 4. The measure funtion has to be the (time independent) prefactor of the pure second order time derivative in (115). One can check that in terms of the new integration variable $\rho = \mathrm{x} - a^{(1)}(t)$, the integral (115) becomes

$$
\begin{aligned}
&\check{S}_0^{(fluc2)}[\eta^{(2)}, a^{(1)}] \\
&= \lambda_1 \int \eta^{(2)} \left(t, \rho + a^{(1)}(t)\right) \\
&\qquad \times \left[-\sin^2 g(\rho) \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial \rho} \sin^2 g(\rho) \frac{\partial}{\partial \rho}\right] \eta^{(2)} \left(t, \rho + a^{(1)}(t)\right) dt d\rho \\
&\quad + S_0^{(int4)}[\eta^{(2)}, a^{(1)}]
\end{aligned} \tag{117}
$$

with

$$
\begin{aligned}
& S_0^{(int4)}[\eta^{(2)}, a^{(1)}] \\
= \; & \lambda_1 \int \eta^{(2)} \left(t, \rho + a^{(1)}(t)\right) \left[\dot{a}^{(1)}(t) \frac{\partial}{\partial \rho} \sin^2 g(\rho) \frac{\partial}{\partial t} \right. \\
& \left. \qquad + \sin^2 g(\rho) \frac{\partial}{\partial t} \dot{a}^{(1)}(t) \frac{\partial}{\partial \rho}\right] \eta^{(2)} \left(t, \rho + a^{(1)}(t)\right) dt d\rho \\
& - \lambda_1 \int \left\{ \int \eta^{(2)} \left(t, \rho + a^{(1)}(t)\right) \sin^2 g(\rho) \eta^{(2)\prime\prime} \left(t, \rho + a^{(1)}(t)\right) d\rho \right\} \left(\dot{a}^{(1)}(t)\right)^2 dt \\
= \; & \mathcal{O}\left((\eta^{(2)})^2, a^{(1)}\right), \qquad (118)
\end{aligned}
$$

so the measure function to be inserted into the eigenvalue equation is $r(\rho) = \sin^2 g(\rho) = 4 \tanh^2\left(\frac{\mu}{\sqrt{2}}\mathrm{x}\right) \mathrm{sech}^2\left(\frac{\mu}{\sqrt{2}}\mathrm{x}\right)$. The eigenvalue problem one has to solve is thus

$$
-\frac{\partial}{\partial \rho} r(\rho) \frac{\partial}{\partial \rho} \Psi(\rho) = \omega^2 r(\rho) \Psi(\rho) \qquad (119)
$$

which is a Sturm–Liouville problem. The eigenfunctions are orthogonal with respect to the inner product $(\Psi_a, \Psi_b)_r = \int r(\rho) \overline{\Psi_a(\rho)} \Psi_b(\rho) d\rho$, and in terms of this inner product, the constant zero mode of $\check{\Omega}^{(2)}$ is normalizable: $(1,1)_r < \infty$.

One can simplify the complicated structure of the rotational fluctuation operator by changing the fluctuations to

$$
\xi\left(t, \rho + a^{(1)}(t)\right) = \sin g(\rho) \cdot \eta^{(2)}\left(t, \rho + a^{(1)}(t)\right). \qquad (120)
$$

Inserting this substitution into (117), we obtain

$$
\begin{aligned}
& \lambda_1 \int \eta^{(2)}\left(t, \rho + a^{(1)}(t)\right) \\
& \qquad \times \left[-\sin^2 g(\rho) \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial \rho} \sin^2 g(\rho) \frac{\partial}{\partial \rho}\right] \eta^{(2)}\left(t, \rho + a^{(1)}(t)\right) dt d\rho \\
= \; & \lambda_1 \int \xi\left(t, \rho + a^{(1)}(t)\right) \left[-\frac{\partial^2}{\partial t^2} + \hat{\Omega}^{(2)}\right] \xi\left(t, \rho + a^{(1)}(t)\right) dt d\rho \\
= \; & S_0^{(fluc2)}[\xi, a^{(1)}] \qquad (121)
\end{aligned}
$$

with

$$
\hat{\Omega}^{(2)} = -\frac{\partial^2}{\partial \rho^2} + \left(g''(\rho) \cot g(\rho) - (g(\rho))^2\right) \qquad (122)
$$

which yields, inserting the sphaleron solution $g(\rho)$ and setting $y = \frac{\mu}{\sqrt{2}}\rho$:

$$
\hat{\Omega}^{(2)} = \frac{\mu^2}{2}\left[-\frac{\partial^2}{\partial y^2} + (1 - 6\mathrm{sech}^2 y)\right] \qquad (123)
$$

This is the the second Pöschl-Teller operator discussed in [18, 26]. From $\check{\Omega}^{(2)}1 = 0$ we have the zero mode $\hat{\Omega}^{(2)}\Psi_0^{(2)} = 0$ with

$$\Psi_0^{(2)}(y) = \sin g(y) \cdot 1 = 2 \tanh \left(\frac{\mu}{\sqrt{2}} \mathrm{x} \right) \mathrm{sech}^2 \left(\frac{\mu}{\sqrt{2}} \mathrm{x} \right) \tag{124}$$

which is square integrable in the usual sense: There is no more prefactor to the second time derivative in (121). We also see that the zero mode eigenfunction (124) has one node, so there must be a ground state eigenfunction with no node which has a negative eigenvalue.

6 Conclusions

We have shown how the breakdown of global symmetries of a given Lagrangian in its soliton or sphaleron solutions can be cured by the introduction of collective coordinates. The Lagrangian field–antifield formalism of Batalin and Vilkovisky proved to be an appropriate tool to perform the well-known main steps of the collective coordinate method, such as the introduction of collective and fluctuation degrees of freedom, the investigation of the resulting gauge algebra structure related to the Noether identities of the theory, the BRST treatment of the gauge symmetry, finally the choice of gauge fixing and the evaluation of path integrals. We also used the simple gauge structure of soliton models to explore some pecularities of the BV scheme in concrete detail, in particular canonical transformations were used to introduce collective coordinates and to fix the gauge.

Although the BV scheme clarifies the collective coordinate method from a structural point of view, the explicit evaluation of path integrals remains a complicated task: The special mapping from the original fields to collective coordinates and fluctuation fields (2) was used throughout this article since it may be derived from a canonical transformation, and yields complicated interaction terms when we expand the classical action up to harmonic fluctuations. Some of these vanish due to an appropriate choice of the gauge fixing condition, others may be interpreted as quantum correction to the soliton mass.

We did not go beyond the one loop approximation in the path integral. In fact, it is possible to write down the terms contributing to higher loop expansions, but it will be more difficult to evaluate the corresponding path integrals.

Acknowledgements

This research has been carried out in the framework of the European Human Capital and Mobility Network on "Constrained Dynamical Systems". The author is indebted to F. de Jonghe, J.–Q. Liang, H.J.W. Müller–Kirsten and J.–G. Zhou for helpfull discussions.

References

[1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **102B**(1981)27; **120B**(1983)66; I.A. Batalin and G.A. Vilkovisky, Phys. Rev. D28(1983)2567, E: **D30**(1984)508; I.A. Batalin and G.A. Vilkovisky, Nucl. Phys. **B234**(1984)106; I.A. Batalin and G.A. Vilkovisky, J. Math. Phys. **26**(1985)172.

[2] J. Gomis, J. Paris and S. Samuel, Phys. Reports **259**(1995)213.

[3] H.J.W. Müller–Kirsten and J.–z. Zhang, "Collective Coordinates and BRST transformations or gauge theorie without gauge fields", **in:** J.M. Charap (ed.), "Geometry of constrained dynamical systems", Cambridge University press 1995.

[4] H.J.W. Müller–Kirsten and J.–z. Zhang, Phys. Lett. **339B**(1994)65.

[5] J.L. Gervais and B. Sakita, Phys. Rev. **D10**(1975)2943; A. Hosoya and K. Kikkawa, Nucl. Phys. **B101**(1975)271.

[6] C.G. Callan and D.J. Gross, Nucl. Phys. **B93**(1975)29; E. Tomboulis, Phys. Rev. D12(1975)1678, R. Jackiw, Rev. Mod. Phys. **49**(1977)681.

[7] L. Baulieu and M. Bellon, Phys. Lett. **B202**(1988)67.

[8] J. Alfaro and P.H. Damgaard, Ann. Phys. **202**(1990)398.

[9] J.P. Garrahan, M. Kurczenski and D.R. Bes, hep–th9603041.

[10] L.D. Faddeev, Theor. and Math. Phys. (USSR) **1**(1970)1.

[11] C. Garrod, Rev. Mod. Phys. **38**(1966)483; E.S. Abers and B.W. Lee, Phys. Rep. **9**(1973)1.

[12] C. Becchi, A.Rouet and B. Stora, Phys. Lett. **52B**(1974)344; C. Becchi, A.Rouet and B. Stora, Ann. Phys. (NY) **98**(1976)278; I.V. Tyutin, Lebedev Report FIAN39, 1975 (unpublished).

[13] D.H. Tchrakian and H.J.W. Müller–Kirsten, Phys. Rev. **D44**(1991)1204.

[14] J.G. Zhou, F. Zimmerschied, J.–Q. Liang, H.J.W. Müller–Kirsten and D.H. Tchrakian, hep–th/9603018.

[15] E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. **55B**(1975)224; I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **69B**(1977)309; E.S. Fradkin and T.E. Fradkina, Phys. Lett. **72B**(1978)343; I.A. Batalin and E.S. Fradkin, Phys. Lett. **122B**(1983)157.

[16] F. de Jonghe, Phys. Lett. **316B**(1993)503.

[17] J. Alfaro and P.H. Damgaard, hep–th/9505156.

[18] E. Mottola and A. Wipf, Phys. Rev. **D39**(1988)588.

[19] J.P. Garrahan, M. Kruczenski, C.L. Schat, D.R Bes and N.N. Scoccola, Phys. Rev. **D51**(1995)2950.

[20] K. Sundermeyer, "Constrained Dynamics", Springer, Berlin 1982.

[21] I.A. Batalin, P.M. Lavrov, I.V. Tyutin, J. Math. Phys. **31**(1990)1487.

[22] H.J.W. Müller–Kirsten and A. Wiedemann, Nuov. Cim. **78A**(1983)61.

[23] N. Dragon, hep–th/9602163.

[24] H.J.W. Müller–Kirsten and A. Wiedemann, Fortschr. Phys. **32**(1984)281.

[25] D.R Bes and J. Kurchan, "The treatment of collective coordinates in many–body systems", World Scientific, Singapore 1990.

[26] K. Funakubo, S. Otsuki and F. Toyoda, Prog. Theor. Phys. **83**(1990)118, **84**(1990)1196; K. Funakubo, Prog. Theor. Phys. **83**(1990)286.

[27] D.H. Tchrakian, H.J.W. Müller–Kirsten and F. Zimmerschied, Phys. Lett. **A197**(1995)263.