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Robust adaptive tube tracking model predictive control for piece-wise constant reference signals

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Abstract

Robust tracking of piece-wise constant reference signals for constrained systems with parametric plant uncertainty and additive disturbances is addressed in this paper. The parametric uncertainty is decreased online by set-membership estimation and a nominal model is updated for improving set-point tracking. The online estimated parametric uncertainty is used for an online-determined terminal set which enlarges the set of reachable references close to the system constraints when compared to an offline worst-case consideration. An artificial target state is introduced which can deviate from the nominal target state. This new target state is used to ensure recursive feasibility for unreachable references and changes in the reference signal. Moreover, a novel "recovery mode" is specified which is deployed in case the new nominal model yields an infeasible control problem. Control algorithms are developed for time-invariant systems and systems with arbitrarily fast changing plants but known relative bounds. Constraint satisfaction and l_2-stability are guaranteed for the proposed algorithms. Controlling the engine load of a self-propelled work machine is used as a practical example.

KEYWORDS

adaptive control, l_2-stability, predictive control, uncertain linear systems

1 | INTRODUCTION

Model predictive control (MPC) is an optimization-based control technique which is able to handle systems with multiple inputs and outputs under input, state and output constraints. At each sample step, an open-loop finite-horizon control problem is solved using a model of the system and a sequence of control actions is calculated. The first element from this sequence is applied to the system and the procedure is repeated at the next sample step. Disturbances and model mismatches decrease the control performance[1] and may lead to constraint violations. Robust MPC formulations deal explicitly with disturbances and model uncertainty to ensure recursive feasibility of the control problem. As a downside, the amount of uncertainty considered at the offline design phase may decrease the control performance and leads to small domains of attraction.[2] The introduced amount of conservatism is high for constant or slowly varying systems due to the large and fixed uncertainty description. This is especially an issue for tracking applications when targets close to the constraints cannot be reached.

Adaptive MPC: Adaptive control methods can be used to identify the true system parameters online and to decrease the level of conservatism. An MPC algorithm based on a condition for persistent excitation is used by Marafioti et al.[3] which ensures convergence of the parameter estimation algorithm. An extension is dual MPC[4] which actively performs system identification and exploits the improved models. As a downside, the before mentioned adaptive MPC schemes do not ensure recursive feasibility or stability. This shortcoming is addressed in this article by using an approach based on robust adaptive MPC.

Robust adaptive MPC: The combination of adaptive control and guaranteed recursive feasibility and stability are addressed in robust adaptive MPC schemes.[5,6] Nominal system models are updated online to improve closed-loop performance and constraint satisfaction is ensured by using robust MPC methods for a fixed uncertainty description. However, the two cited algorithms do not fully utilize the benefits of adaptive control as the uncertainty description remains fixed during the online execution phase.

There are several variants of robust adaptive MPC schemes with updates of the uncertainty description.[7-13] The article by Zhu et al.[13] is limited to single-input systems with a special system matrix structure compared to the multi-input case in this work. In the work by Lorenzen et al.,[7] all possible state trajectories are bounded by the homothetic tube approach.[14] The homothetic tube approach scales tube cross sections by checking the evolution of the states at every vertex of the respective cross section. The uncertainty description is updated by set-membership estimation without any fixed shape of the parameter set. This leads to a possibly unbounded number of half spaces for the set of the uncertainty description. The algorithm by Köhler et al.[8] addresses this issue by bounding the state evolution with a computationally efficient but also conservative approach for non-linear control problems.[15] In addition, the shape of the uncertainty description is restricted to hypercubes. The state evolution can be also bounded by flexible tubes[9,10] using a set-inclusion method.[16] The algorithm by Zhang et al.[9] requires a re-calculation of tube parameters and the terminal set at every sample step which is computationally demanding. In contrast, the approach by Lu et al.[10] solves a min-max problem for the tube parameters offline and online, new tube parameters are calculated by a simple convex combination. The set-membership estimation is similar to the work of Köhler et al.[8] but uses the more general shape of a zonotope.

This article uses the method described by Lu et al.[10] to compute the tube parameters due to its balance of computational complexity and conservativeness. However, the least mean square filter given by Lorenzen et al.[7] is used to update the nominal model as tracking performance benefits more than using the projection-based update rule given by Lu et al.[10]

Terminal constraints for tracking MPC: The before mentioned robust adaptive MPC algorithms focus on the regulation to a given set-point. The control problem becomes more difficult if tracking of a given reference is considered as the target steady-state changes every time when a system model update occurs. This poses effectively a new control problem at every time step which needs to be considered for proving stability and recursive feasibility.

Invariant sets are usually deployed as terminal constraints for MPC algorithms to ensure recursive feasibility. The definition of such invariant sets for tracking of varying set-points is challenging as their shape and size depend on the desired set-point. One approach uses a fixed shape of a terminal set for a known linear time-invariant system[17] and scales its size online depending on the set-point. Another approach for known linear time-invariant systems uses a terminal set augmented with a steady-state parameter and a cost function with an artificial target state.[18] This leads to an increased domain of attraction and if a reference is unreachable, the artificial reference will converge to the closest feasible one. This approach has been also used for linear control problems with plant uncertainty,[19] additive disturbances[20] and for non-linear control problems.[21] Another objective for tracking problems may be to ensure that a given error bound will not be violated for varying references. This objective is discussed using a reference governor for linear systems[22] and non-linear systems.[23]

Terminal sets for robust tracking MPC: The definition of a suitable terminal set becomes even more involved for control problems with parametric uncertainty and varying references. A steady-state based on a wrong system model will lead to an equilibrium drift[24](ch. 2.1.3) which may be interpreted as a disturbance depending on the target state and uncertainty. Hanema et al.[25] consider a linear parameter-varying (LPV) system, a terminal set for a given error bound and a reference which is defined based on the reference governor approach.[22] However, no offset-free tracking of the reference can be guaranteed even if perfect model knowledge is available. The algorithm by Limon et al.[18] can be extended to linear systems with parametric uncertainty.[19] The approach allows for varying, piece-wise constant references but offset-free tracking can only be achieved if the true system matches a nominal model defined offline. Moreover, the terminal set is defined offline and cannot be updated if the uncertainty decreases online.

A robust adaptive MPC algorithm for tracking different set-points is given by Köhler et al.[8] However, the target steady-state needs to be defined offline and the considered uncertainty is fixed. This may lead to a performance decrease if set-points close to the system constraints should be reached. Those may be infeasible under the offline uncertainty but become feasible during operation when the uncertainty is decreased.

This article provides a solution to the limitations of the previously mentioned methods by defining a suitable terminal set. The terminal set is based on a steady-state which utilizes the latest update of a nominal model. This ensures offset-free tracking if the nominal model converges to the true one. In addition, the terminal set is based on the latest update of the online uncertainty estimate which decreases the conservativeness over time. The specific set-point does not have to be considered during the offline design phase and can be freely chosen online which is important in many applications.

Contribution: This paper addresses the problem of tracking piece-wise constant references for systems with additive disturbances and parametric plant uncertainty. The main goals are

- ensuring constraint satisfaction when the nominal model and set-point switch, and
- the definition of a terminal set for varying uncertainties, nominal models and set-points, and
- asymptotic stability and offset-free tracking if no additive disturbances and modeling errors exist.

Two types of systems are considered: linear time-invariant systems with unknown parameters and linear time-varying systems with known relative bounds but unknown center which has not been addressed yet in robust adaptive MPC literature. In order to achieve good control performance, for example, offset-free tracking with a low level of conservatism, adaptive control methods are deployed. A set-membership algorithm is used to decrease the uncertainty online and point estimates are used to estimate the true parameter vector. As the plant uncertainty and nominal model get updated at every sample step, a terminal set is defined online which depends on the plant uncertainty and target state. A "recovery mode" is introduced when the target state changes due to an update of the nominal parameter vector and the original optimization problem becomes infeasible.

Uncertainty in the output matrix is considered in this work which leads to constraints which depend on the online estimated uncertainty. The proposed algorithm utilizes the online estimated uncertainty for bounding the state evolution, terminal set definition and system constraint definition. This is especially valuable if targets close to the system constraints should be tracked which is also shown in a practical example.

This paper is organized as follows. The notation used in this work is presented in Section 2 and the considered system in Section 3. The basic definition of tubes and their adaptation for the tracking problem is described in Section 5. The adaptive algorithms and the tube construction are given in Section 4. The terminal set definition, the MPC algorithm and its main properties are presented in Section 6 and its variants in Section 7. The results are illustrated in Section 8 with a practical example.

2 | NOTATION

The notations $||x||_R^2 = x'Rx$ and $|x| = ||x||_1$ are used where x' denotes the transpose of a vector x or a matrix. In addition, the ith row of a matrix is written as $[\cdot]_i$ and I_n is the identity matrix of dimension $n \times n$. A vector only containing ones is denoted as $\underline{1}$. A set which is convex, compact and contains the origin in its interior is called a C-Set. A polytopic C-set $S = \{x : Vx \leq \underline{1}\}$ can be scaled by a scaling factor $\lambda \in [0, \infty)$ with $\lambda S = \{x : Vx \leq \lambda\underline{1}\}$. A C-set S is λ-contractive for system $x_{k+1} = Ax_k$ and $\lambda \in [0, 1)$ if $x_{k+1} \in \lambda S$ holds for all $x \in S$. For $\lambda = 1$ the set S is positively invariant. The convex hull of a set S is written as $\mathrm{Co}\{S\}$ and $\mathrm{Pre}\{S\}$ denotes the set of states which evolve into S in one time step. The Minkowski sum for two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ is written as $\mathcal{S}_1 \oplus \mathcal{S}_2$. The set of integers from l to m is denoted as $\mathbb{N}_l^m$ and $\mathbb{N}_0$ is the set of non-negative integers.

3 | SYSTEM DESCRIPTION

A discrete-time, uncertain linear time-invariant system

$$\begin{aligned} x_{k+1} &= A(\theta^*)x_k + B(\theta^*)u_k + w_k, \\ y_k &= C(\theta_C^*)x_k + e_k \end{aligned} \tag{1}$$

is considered with $\theta^* \in \mathbb{R}^{n_\mathrm{P}}$ and $\theta^*_\mathrm{C} \in \mathbb{R}^{n_\mathrm{C}}$ as unknown but constant parameters.

Assumption 1. The true parameters θ^* and θ^*_C are contained in the respective known bounded convex sets $\Theta \subset \mathbb{R}^{n_\mathrm{P}}$ and $\Theta_\mathrm{C} \subset \mathbb{R}^{n_\mathrm{C}}$.

The variable $x \in \mathbb{R}^{n_\mathrm{x}}$ denotes the state vector, $u \in \mathbb{R}^{n_\mathrm{u}}$ the input vector, $y \in \mathbb{R}^{n_\mathrm{y}}$ the output vector and $k \in \mathbb{N}_0$ denotes the time step.

Assumption 2. The state vector x_k can be measured at time k.

Assumption 3. System matrix $A \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ and input matrix $B \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{u}}$ are affine functions of $\theta \in \Theta$ with

$$(A(\theta), B(\theta)) = \left(A^{(0)}, B^{(0)}\right) + \sum_{j=1}^{n_\mathrm{P}} [\theta]_j \left(\Delta A^{(j)}, \Delta B^{(j)}\right), \tag{2}$$

and known matrices $\Delta A^{(j)}, \Delta B^{(j)}, j \in \mathbb{N}_1^{n_\mathrm{P}}$. Output matrix $C(\theta_\mathrm{C}) \in \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{x}}$ is analogously defined as

$$C(\theta_\mathrm{C}) = C^{(0)} + \sum_{j=1}^{n_\mathrm{C}} [\theta_\mathrm{C}]_j \Delta C^{(j)} \tag{3}$$

with known matrices $\Delta C^{(j)}, j \in \mathbb{N}_1^{n_\mathrm{C}}$. The additive disturbance w_k and measurement noise e_k lie in the respective C-sets

$$\mathbb{W} = \{w : M_\mathrm{w} w \leq b_\mathrm{w}\} \tag{4}$$

and

$$\mathbb{E} = \{e : M_\mathrm{e} e \leq b_\mathrm{e}\} \tag{5}$$

with $M_\mathrm{w} \in \mathbb{R}^{q_\mathrm{w} \times n_\mathrm{x}}, b_\mathrm{w} \in \mathbb{R}^{q_\mathrm{w}}$ and $M_\mathrm{e} \in \mathbb{R}^{q_\mathrm{e} \times n_\mathrm{y}}, b_\mathrm{e} \in \mathbb{R}^{q_\mathrm{e}}$. State constraints, input constraints and output constraints are defined by the respective compact polytopic sets $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$. Possible couplings between the sets are incorporated by defining

$$\begin{bmatrix} F \\ F_\mathrm{C} C(\theta^*_\mathrm{C}) \end{bmatrix} x_k + G u_k \leq \underline{1}. \tag{6}$$

Assumption 4. Output constraints can be always expressed as state constraints: $\mathcal{Y} = \{x : F_\mathrm{C} C(\theta^*_\mathrm{C}) x \leq \underline{1}\}$.

Assumption 5. For the number of outputs n_y and number of inputs n_u, it holds that $n_\mathrm{y} \leq n_\mathrm{u}$ and

$$\operatorname{rank} \begin{bmatrix} A(\theta) - I_{n_\mathrm{x}} & B(\theta) \\ C(\theta_\mathrm{C}) & 0 \end{bmatrix} = n_\mathrm{x} + n_\mathrm{y} \ \forall\ \theta \in \Theta\ ,\ \theta_\mathrm{C} \in \Theta_\mathrm{C}.$$

Assumption 5 ensures the existence of a steady-state x_T and input u_T such that $C(\theta_\mathrm{C}) x_\mathrm{T} = r, x_\mathrm{T} = A(\theta) x_\mathrm{T} + B(\theta) u_\mathrm{T}$ holds for given parameters $\theta, \theta_\mathrm{C}$ and reference r.

The control objective is to control the system output y of the uncertain plant (1) to the reference r while considering state constraints, input constraints and output constraints (6). The following constrained optimization problem can be used to express the control objective:

$$\begin{aligned} \underline{u}_k^* &= \arg\min J(x_k, x_\mathrm{T}, u_\mathrm{T}), \\ \text{s.t.} \quad x_{l+1|k} &= A(\theta^*)\, x_{l|k} + B(\theta^*)\, u_{l|k}, \forall\, l \in \mathbb{N}_0^{N-1}, \\ x_{l|k} &\in \mathcal{X},\ u_{l|k} \in \mathcal{U},\ y_{l|k} \in \mathcal{Y}, \forall\, l \in \mathbb{N}_0^{N-1}, \\ x_{N|k} &\in \mathbb{X}_\mathrm{T}\ , \end{aligned} \tag{7}$$

where $J(x_k, x_\mathrm{T}, u_\mathrm{T})$ is a finite horizon cost with prediction horizon length $N \in \mathbb{N}_0$ which is linked to the predicted states $x_{l|k}$ and predicted inputs $\underline{u}_k = \{u_{0|k}, \dots, u_{N-1|k}\}$. As the true parameter vector θ^* is unknown, a parameter estimate $\hat{\theta}_k$ needs to be used in problem (7) which is defined in Section 4. Moreover, the constraints on states and outputs cannot be directly implemented due to the uncertain plant parameters which leads to a set of possible states at every prediction step. A method for finding tighter estimates of the uncertainty is presented in Section 4 and the resulting uncertainty estimates are then used in Section 5 to bound the evolution of states and outputs. A terminal set $\mathbb{X}_\mathrm{T}$ is used to ensure recursive feasibility after the end of the prediction horizon and is defined in Section 6.

4 | ADAPTIVE ALGORITHM

Set-membership algorithms are commonly used in adaptive control to find tighter bounds on the system uncertainty. For robust adaptive control, set-membership estimation is used to update the time-varying set of all possible plant realizations.[7,10] In case this set does not have a fixed structure as in the work by Lorenzen et al.,[7] the number the number of constraints may increase tremendously over time. In contrast, the algorithm from Lu et al.[10] requires the set to be a zonotope, for example, a hypercube, which adds conservatism but yields a fixed computational complexity. In addition, the conversion from a half-space description of polytopes to vertex representation can be easily done for zonotopes. In this section, the algorithm by Lu et al.[10] is shortly presented and adapted for systems with uncertainties in the output matrix. Persistent excitation is a sufficient condition for convergence of the adaptive algorithm[7] and can be enforced by choosing an appropriate cost function.[11] It should be noted that persistent excitation is not necessary to ensure stability and recursive feasibility for the presented control algorithm but improves tracking performance.

4.1 | Set-membership estimation

Two zonotopes

$$\Theta_k = \{\theta : M_\Theta \theta \leq b_{\Theta,k}\} = \mathrm{Co}\{\theta_k^{(1)}, \dots \theta_k^{(m)}\} \tag{8}$$

and

$$\Theta_{\mathrm{C},k} = \{\theta_\mathrm{C} : M_{\Theta_\mathrm{C}} \theta_\mathrm{C} \leq b_{\Theta_\mathrm{C},k}\} = \mathrm{Co}\{\theta_{\mathrm{C},k}^{(1)}, \dots \theta_{\mathrm{C},k}^{(m_\mathrm{C})}\} \tag{9}$$

are defined with initial conditions $\Theta \subseteq \Theta_0$, $\Theta_\mathrm{C} \subseteq \Theta_{\mathrm{C},0}$ and number of vertices m and m_C. In the following, Θ_k and $\Theta_{\mathrm{C},k}$ are assumed to be hyperrectangles with $M_\theta = \begin{bmatrix} I_{n_\mathrm{P}} & -I_{n_\mathrm{P}} \end{bmatrix}'$ and $M_{\Theta_\mathrm{C}} = [I_{n_\mathrm{C}} \quad -I_{n_\mathrm{C}}]'$ to keep the computational complexity low. In principle, any polytope of fixed shape can be used. The level of conservativeness introduced by the over-approximation heavily depends on the original shape of Θ and Θ_C and needs to be balanced with the computational complexity of more involved approximations.

Matrices with all possible extreme state and output deviations compared to the nominal model for the adaptive algorithm are defined with

$$D(x,u) = \left[\Delta A^{(1)} x + \Delta B^{(1)} u, \ \dots, \Delta A^{(n_\mathrm{P})} x + \Delta B^{(n_\mathrm{P})} u\right] \tag{10}$$

and

$$D_\mathrm{C}(x) = \left[\Delta C^{(1)} x, \ \dots, \Delta C^{(n_\mathrm{C})} x\right], \tag{11}$$

where $D(x,u) \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{P}}$ and $D_\mathrm{C}(x) \in \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{C}}$. Moreover, the nominal state prediction error is defined as

$$d_k = A^{(0)} x_{k-1} + B^{(0)} u_{k-1} - x_k \tag{12}$$

and the nominal output prediction error as

$$d_{\mathrm{C},\,k} = C^{(0)} x_k - y_k. \tag{13}$$

The sets of unfalsified parameters

$$\begin{aligned}\phi_k &= \{\theta : x_k - (A(\theta)x_{k-1} + B(\theta)u_{k-1}) \in \mathbb{W}\} \\ &= \{\theta : -M_{\mathrm{w}} D(x_{k-1}, u_{k-1})\theta \le b_{\mathrm{w}} + M_{\mathrm{w}} d_k\} \\ &= \{\theta : M_{\phi,k}\theta \le b_{\phi,k}\}\end{aligned} \tag{14}$$

and

$$\begin{aligned}\phi_{\mathrm{C,k}} &= \{\theta_{\mathrm{C}} : y_k - C(\theta_{\mathrm{C}})x_k \in \mathbb{E}\} \\ &= \{\theta : -M_{\mathrm{e}} D_{\mathrm{C}}(x_k)\theta \le b_{\mathrm{e}} + M_{\mathrm{e}} d_{\mathrm{C},k}\} \\ &= \{\theta : M_{\phi_{\mathrm{C}},k}\theta \le b_{\phi_{\mathrm{C}},k}\}\end{aligned} \tag{15}$$

must lead to prediction errors which lie in the respective disturbance sets $\mathbb{W}$ and $\mathbb{E}$. Tighter estimation results are achievable if several measurements are considered in a block-window.[26]

The requirements $\Theta_{k+1} \subseteq \Theta_k$ and $\Theta_{\mathrm{C},k+1} \subseteq \Theta_{\mathrm{C},k}$ are needed to ensure recursive feasibility. This can be achieved by computing $\tilde{\Theta}_{k+1} = \Theta_k \cap \phi_k$ first and then using the following proposition[10]:

Proposition 1 *(27)*. Given two polytopes $S_1 = \{x : F_1 x \le g_1\}$ and $S_2 = \{x : F_2 x \le g_2\}$, then $S_1 \subseteq S_2$ if and only if there exists a non-negative matrix H such that $HF_1 = F_2$ and $Hg_1 \le g_2$.

As pointed out by Kouvaritakis et al.[2(ch. 5.5)], a row-wise minimization

$$\begin{aligned}[b_{\Theta,k+1}]_i = &\min_{b, H_i \in \mathbb{R}^{1\times(2n_{\mathrm{P}}+q_{\mathrm{w}})}} b \\ \text{s.t.} \quad & H_i \begin{bmatrix} M_\Theta \\ M_{\phi,k}\end{bmatrix} = [M_\Theta]_i,\ H_i \begin{bmatrix} b_{\Theta,k} \\ b_{\phi,k}\end{bmatrix} \le b,\ H_i \ge 0\end{aligned} \tag{16}$$

can be performed to calculate a non-negative matrix H for Proposition 1 and a corresponding vector $b_{\theta,k+1}$. This leads to a fixed complexity of Θ_k as only $b_{\theta,k}$ is updated. The number of linear programs to be solved is $2n_{\mathrm{P}}$ for box constrained sets.

The same steps needs to be analogously performed for the parameter set $\Theta_{\mathrm{C},k}$ which yields

$$\begin{aligned}[b_{\Theta_{\mathrm{C}},k+1}]_i = &\min_{b, H_i \in \mathbb{R}^{1\times(2n_{\mathrm{C}}+q_e)}} b \\ \text{s.t.} \quad & H_i \begin{bmatrix} M_{\Theta_{\mathrm{C}}} \\ M_{\phi_{\mathrm{C}},k}\end{bmatrix} = [M_{\Theta_{\mathrm{C}}}]_i,\ H_i \begin{bmatrix} b_{\Theta_{\mathrm{C}},k} \\ b_{\phi_{\mathrm{C}},k}\end{bmatrix} \le b,\ H_i \ge 0\ .\end{aligned} \tag{17}$$

4.2 | Update for nominal parameter vector

Offset-set free tracking of reference signals is a common goal for control applications. In contrast to regulation to the origin, a correct nominal model is needed for tracking problems to reach the reference in absence of additive disturbances. Here, a least mean square (LMS) filter is used to find a nominal model for the dynamics (26). In existing literature, least mean square filters are used for FIR models[28] and for robust adaptive control.[7] A new parameter estimate $\hat{\theta}_k$ for nominal system (26) is recursively defined

$$\tilde{\theta}_k = \hat{\theta}_{k-1} + \mu D(x_{k-1}, u_{k-1})'(x_k - \hat{x}_{1|k-1}) \tag{18a}$$

$$\hat{\theta}_k = \arg\min_{\theta \in \Theta_k} \|\theta - \tilde{\theta}_k\| \tag{18b}$$

with update gain $\mu \in \mathbb{R}_{>0}$ and nominal state $\hat{x}_{1|k-1} = A(\hat{\theta}_{k-1})x_{k-1} + B(\hat{\theta}_{k-1})u_{k-1}$.

Proposition 2 (7). For a symmetric, positive definite matrix $Q \in \mathbb{R}^{n\times n}$ and vectors $a, b \in \mathbb{R}^n$, there exist $\epsilon > 0$ such that

$$\|a+b\|_Q^2 \le \left(1+\frac{1}{\epsilon}\right)\|a\|_Q^2 + (1+\epsilon)\|b\|_Q^2.$$

The proof for Proposition 2 can be found in Appendix A.

Proposition 3. If $(x_k, u_k) \in \mathcal{X}\times\mathcal{U}$ and $\theta^* \in \Theta_k$, then parameter estimation (18) for $\hat{\theta}_k$ is bounded and there exists a constant ϵ such that

$$\|\hat{\theta}_{k+1} - \hat{\theta}_k\|^2 < (1+\epsilon)\mu^2\mu_D^2\|\hat{\theta}_k - \theta^*\|^2 + \left(1+\frac{1}{\epsilon}\right)\mu^2\mu_D\|w_k\|^2 \tag{19}$$

holds with $\mu_D > \sup_{x\in\mathcal{X}, u\in\mathcal{U}} \|D(x,u)\|^2$.

Proof. It is

$$\begin{aligned}
\|\hat{\theta}_{k+1} - \hat{\theta}_k\|^2 &\le \|\tilde{\theta}_{k+1} - \hat{\theta}_k\|^2 \\
&= \|\mu D(x_k,u_k)'(x_{k+1} - \hat{x}_{1|k})\|^2 \\
&\le \mu^2\|D(x_k,u_k)\|^2\|\tilde{x}_{1|k} + w_k\|^2 \\
&< \mu^2\mu_D\|\tilde{x}_{1|k} + w_k\|^2 \\
&= \mu^2\mu_D\|D(x_k,u_k)'(\theta^* - \hat{\theta}_k) + w_k\|^2 \\
&\le (1+\epsilon)\mu^2\mu_D^2\|\theta^* - \hat{\theta}_k\|^2 + \left(1+\frac{1}{\epsilon}\right)\mu^2\mu_D\|w_k\|^2.
\end{aligned} \tag{20}$$

The first inequality follows from the projection on Θ_k in (18b) and the first equality from (18a). The second inequality is due to definitions $\hat{x}_{1|k} = A(\theta_k)x_k + B(\theta_k)u_k$, $\tilde{x}_{1|k} = A(\theta^*)x_k + B(\theta^*)u_k - \hat{x}_{1|k}$ and the Cauchy–Schwarz inequality. The third inequality follows by the definition of μ_D and the second equality follows from rewriting $\tilde{x}_{1|k}$. The fourth inequality follows from Proposition 2 and considering $\|\cdot\|^2 = \|\cdot\|_I^2$.

Boundedness follows from the set update $\Theta_k \subseteq \Theta_{k-1}$ and the projection of $\tilde{\theta}_k$ on Θ_k. ■

This bound on the change of the parameter estimate is necessary for proving l_2-stability in Section 6. Algorithm (18) can be analogously defined for estimating the nominal output matrix

$$\tilde{\theta}_{C,k} = \hat{\theta}_{C,k-1} + \mu_C D_C(x_k)'(y_k - C(\hat{\theta}_{C,k-1})x_k), \tag{21a}$$

$$\hat{\theta}_{C,k} = \arg\min_{\theta\in\Theta_{C,k}} \|\theta - \tilde{\theta}_{C,k}\| . \tag{21b}$$

with update gain $\mu_C \in \mathbb{R}_{>0}$.

Proposition 4. If $x_k \in \mathcal{X}$, $y \in \mathcal{Y}\oplus\mathbb{E}$ and $\theta_C^* \in \Theta_{C,k}$, then the parameter estimate $\hat{\theta}_{C,k}$ using (21) is bounded and the change in parameters is bounded by some constants μ_{θ_C}, μ_e with

$$\|\hat{\theta}_{C,k+1} - \hat{\theta}_{C,k}\|^2 < \mu_{\theta_C}\|\hat{\theta}_{C,k} - \theta_C^*\|^2 + \mu_e\|e_k\|^2. \tag{22}$$

Proof. Proof is similar to Proposition 3. ■

Remark 1. It is also possible to use an update law based on projection on the parameter set as proposed by Lu et al.[11] This does not affect recursive feasibility and only marginally changes the stability result. However, the LMS update may yield the correct parameter vector even if the parameter set does not converge to a point which reduces the steady-state error.

5 | POLYTOPIC TUBES

In this work, a tube-based approach for bounding the state evolution is used with the following tube definition:

Definition 1 (29). Consider a sequence of control inputs $\{u_{0|k}, u_{1|k}, \dots, u_{N|k}\}$ and a sequence of sets (cross-sections) $\{\mathbb{X}_{0|k}, \mathbb{X}_{1|k}, \dots, \mathbb{X}_{N|k}\}$ with $\mathbb{X}_{l|k} \subset \mathbb{R}^{n_x}$. This sequence is called a tube for the uncertain system (1) if $A(\Theta)\mathbb{X}_{l|k} \oplus B(\Theta)u_{l|k} \oplus \mathbb{W} \subseteq \mathbb{X}_{l+1|k}$ for all $l \in \mathbb{N}_0^{N-1}$.

For robust adaptive control with varying uncertainty, the tube can be re-computed for every update of the uncertainty set.

A simple approach of updating the tube structure computes all extreme tube structures offline[30] and online, new matrices are calculated by a convex combination of the vertex realizations. This approach adds conservatism and the computation of the offline-defined terminal set is complex as every combination of nominal plant and uncertainty needs to be considered.

The conservatism induced by the above approach can be reduced by solving a min-max problem for determining the extreme realizations of the tube structures.[10] A proposition based on Farkas' lemma is used for this purpose:

Proposition 5 (10). Given two polytopes $S_1 = \{x : F_1 x \leq f_1\}$ and $S_2(\theta) = \{x : F_2(\theta)x \leq f_2\}$, then $S_1 \subseteq S_2(\theta) \ \ \forall \theta \in \Theta \subset \mathbb{R}^{n_p}$ if and only if there exists $\bar{H}_i$ such that

$$\theta' \bar{H}_i \geq 0, \quad \bar{H}_i F_1 = \bar{F}_{2,i}, \quad \theta' \bar{H}_i f_1 \leq [f_2]_i \tag{23}$$

holds for all $i \in \mathbb{N}_1^{n_2}$ and $\theta \in \Theta$ with $F_1 \in \mathbb{R}^{n_1 \times n_x}$, $F_2(\theta) \in \mathbb{R}^{n_2 \times n_x}$, $\theta' \bar{F}_{2,i} = [F_2(\theta)]_i$ and $\bar{H}_i \in \mathbb{R}^{n_p \times n_1}$.

In contrast to the robust adaptive MPC algorithm using homothetic tubes in Lorenzen et al.,[7] the presented approach does not check all vertices of the tube at each prediction step but directly operates on the half space representation of the polytope. This results in less inequality constraints for the MPC algorithm but may add conservatism.[8]

Tubes can be constructed to be invariant regarding disturbances or hyperparameters.[16,19] These tubes are centered around a nominal system trajectory which makes it difficult to prove recursive feasibility when the nominal model changes. This issue can be circumvented for adaptive control when no nominal model for the tube construction is considered.[10] This work extends the algorithm by Lu et al.[10] for tracking applications when target states and target inputs are changing.

5.1 | Tube construction

The planned system input

$$u_{l|k} = K(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k}) + c_{l|k} + \hat{u}_{\mathrm{T},k} \, , l \in \mathbb{N}_0^{N-1} \tag{24}$$

consists of a linear feedback term $K(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k})$ for pre-stabilization around an artificial target state $\hat{x}_{\mathrm{T},k} \in \mathbb{R}^{n_x}$ with artificial target input $\hat{u}_{\mathrm{T},k} \in \mathbb{R}^{n_u}$. The artificial target state and target input are decision variables of the optimization problem and are used to ensure recursive feasibility for changing references or nominal parameters.

In addition, a free control move $c_{l|k}$ is used for optimizing the trajectory of the system over the prediction horizon N. The gain K is chosen such that

$$\Phi(\theta_0^{(j)}) P \Phi(\theta_0^{(j)}) - P \preceq -(Q + K'RK) \ \forall j \in \mathbb{N}_1^m \tag{25}$$

is fulfilled with positive definite state weight Q, input weight R, terminal weight P, $\Phi(\theta) = A(\theta) + B(\theta)K$ as the closed-loop system matrix and m is the number of vertices as defined in (8). The variable $\hat{x}_{l|k} \in \mathbb{R}^{n_x}$ denotes the nominal system state defined by the nominal system

$$\hat{x}_{l+1|k} - \hat{x}_{\mathrm{T},k} = \Phi(\hat{\theta}_k)(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k}) + B(\hat{\theta}_k)c_{l|k} \tag{26}$$

with $\hat{\theta}_k$ as an estimate of the true system parameter derived by (18).

Definition 2. A set $\mathcal{S} = \{x : Vx \leq \underline{1}\}$ is called robust λ-contractive for system (1) with $u = Kx \in \mathcal{U}$ if $\forall x \in \mathcal{S} : \Phi(\theta)x \in \lambda\mathcal{S},\ \forall\ \theta \in \Theta_0$.

Remark 2. Note that satisfaction of state constraints like $\mathcal{S} \subseteq \mathcal{X}$ is not required in Definition 2. State constraint satisfaction will be ensured online by the MPC algorithm and the set S is only used to obtain a suitable structure for the tube cross-sections and the terminal set.

In order to ensure bounded growth of the tube under additive disturbances, the tube cross-section is required to be a robust λ-contractive set for the system under parametric uncertainty but without additive disturbances. The tube cross-section $\mathbb{X}_{l|k}$, which contains all possible state realizations at time predicted time $l|k$, is defined as

$$\mathbb{X}_{l|k} = \{x \in \mathbb{R}^{n_x} : Vx \leq \alpha_{l|k}\}, \tag{27}$$

where the vector $\alpha_{l|k}$ is chosen online and scales each half space and $V \in \mathbb{R}^{n_\alpha \times n_x}$. This allows for a flexible shape of the tube which consists of all consecutive $\mathbb{X}_{l|k}$, $l = 0, \dots, N$. All states evolving from $\mathbb{X}_{l|k}$ must be contained in $\mathbb{X}_{l+1|k}$ which requires

$$\mathbb{X}_{l+1|k} \supseteq A(\theta)\mathbb{X}_{l|k} \oplus B(\theta)u_{l|k} \oplus \mathbb{W} \quad \forall\ \theta \in \Theta_k. \tag{28}$$

Condition (28) holds true if there exists an input $u_{l|k}$ such that

$$\begin{aligned} & V\left(A(\theta)x + B(\theta)u_{l|k} + w\right) \leq \alpha_{l+1|k}, \quad \forall \theta \in \Theta_k, x \in \mathbb{X}_{l|k}, w \in \mathbb{W}, \\ \Leftrightarrow\ & V\left(\Phi(\theta)x + (A(\theta) - \Phi(\theta))\hat{x}_{\mathrm{T},k} + B(\theta)c_{l|k} + B(\theta)\hat{u}_{\mathrm{T},k} + \bar{w}\right) \leq \alpha_{l+1|k},\ \forall \theta \in \Theta_k, x \in \mathbb{X}_{l|k}\ . \end{aligned} \tag{29}$$

This follows from the input parametrization (24) and w_k is bounded by $\bar{w}$ using

$$[\bar{w}]_i = \max_{w \in \mathbb{W}} [V]_i w. \tag{30}$$

Similar to the work by Lu et al.,[10] the tube construction is given by the following Proposition:

Proposition 6. The sequence of predicted state sets $\{\mathbb{X}_{0|k}, \mathbb{X}_{1|k}, \dots, \mathbb{X}_{N|k}\}$ satisfies tube condition (28) if for all $j \in \mathbb{N}_1^m, l \in \mathbb{N}_0^{N-1}$ it holds that

$$H(\bar{\theta}_k^{(j)})V = V\Phi(\theta_k^{(j)})\ , \tag{31a}$$

$$\alpha_{l+1|k} \geq H(\bar{\theta}_k^{(j)})\alpha_{l|k} + \bar{w} + V(I_{n_x} - \Phi(\theta_k^{(j)}))\hat{x}_{\mathrm{T},k} + VB(\theta_k^{(j)})c_{l|k}\ , \tag{31b}$$

where $\bar{\theta}_k^{(j)\prime} = [1\ \ \theta_k^{(j)\prime}]$ and $[H(\bar{\theta}_k^{(j)})]_i = \bar{\theta}_k^{(j)\prime}\bar{H}_i$. The min-max problem for determining each $\bar{H}_i$ can be stated as the linear program

$$\begin{aligned} \bar{H}_i = & \underset{H \in \mathbb{R}^{(n_p+1)\times n_\alpha}}{arg\ min}\ \max_{j \in \mathbb{N}_1^m} \bar{\theta}_0^{(j)\prime} H \underline{1} \\ s.t. & \quad \bar{\theta}_0^{(j)\prime} H \geq 0 \quad \forall\ j \in \mathbb{N}_1^m, \end{aligned} \tag{32a}$$

$$HV = \begin{bmatrix} [V]_i\Phi^{(0)} \\ [V]_i\Delta\Phi^{(1)} \\ \vdots \\ [V]_i\Delta\Phi^{(n_p)} \end{bmatrix} \tag{32b}$$

with $\Delta\Phi^{(j)} = \Delta A^{(j)} + \Delta B^{(j)}K$.

Proof. For each matrix row, conditions (31) fulfill the inclusion conditions stated in Proposition 5 for $S_1 = \mathbb{X}_{l|k}$ and $S_2 = \{x : V(\Phi(\theta)x + (I_{n_x} - \Phi(\theta))\hat{x}_{T,k} + B(\theta)c_{l|k}) + \bar{w} \le \alpha_{l+1|k}\}$. Due to $\Theta_k \subseteq \Theta_0$ and convexity of Θ_k, inequalities (32a) ensure that $\bar{\theta}'\bar{H}_i \ge 0$ from Proposition 5 holds for all $\theta \in \Theta_k \subseteq \Theta_0$. Moreover, equality (32b) ensures that the first equality in (23) holds with $F_1 = V$ and $F_2(\theta) = V\Phi(\theta)$.

The last condition $\bar{\theta}'\bar{H}_i f_1 \le [f_2]_i$ from Proposition 5 is fulfilled by (31b) for $f_1 = \alpha_{l|k}$ and $f_2 = \alpha_{l+1|k}$. ∎

5.2 | Constraints

System constraints (6) need to hold for all possible realization of $\theta_C \in \Theta_{C,k}$. Hence, it needs to hold that

$$\bar{F}(\theta_C)x_{l|k} + Gu_{l|k} \le \underline{1} \quad \forall x_{l|k} \in \mathbb{X}_{l|k},\ \theta_C \in \Theta_{C,k} \tag{33}$$

with

$$\bar{F}(\theta_C) = \begin{bmatrix} F \\ F_C C(\theta_C) \end{bmatrix} \in \mathbb{R}^{n_F \times n_x}.$$

Inequality (33) needs to hold for all possible $x_{l|k} \in \mathbb{X}_{l|k}$. Proposition 5 can be again applied by exploiting convexity of $\Theta_{C,k}$ and requiring that $\mathbb{X}_{l|k}$ is a subset of (33) which yields

$$H_C(\bar{\theta}^{(j)}_{C,k})V = \bar{F}(\theta^{(j)}_{C,k}) + GK \tag{34}$$

and

$$H_C(\bar{\theta}^{(j)}_{C,k})\alpha_{l|k} - GK\hat{x}_{T,k} + Gc_{l|k} + G\hat{u}_{T,k} \le \underline{1} \tag{35}$$

for all $j \in \mathbb{N}_1^{n_C}$, $l \in \mathbb{N}_0^N$ with $[H_C(\bar{\theta}^{(j)}_C)]_i = \bar{\theta}^{(j)\prime}_C \bar{H}_{C,i}$, $\bar{\theta}^{(j)\prime}_C = [1 \ \ \theta^{(j)\prime}_C]$.

Matrices $\bar{H}_{C,i}$ can be determined similar to (32) by solving the linear program

$$\begin{aligned} \bar{H}_{C,i} = & \mathop{\arg\min}_{H \in \mathbb{R}^{(n_C+1)\times n_\alpha}} \ \max_{j \in \mathbb{N}_1^{m_C}} \bar{\theta}^{(j)\prime}_{C,0} H\underline{1} \\ \text{s.t.} \quad & \bar{\theta}^{(j)\prime}_{C,0} H \ge 0 \quad \forall j \in \mathbb{N}_1^{m_C}, \\ HV = & \begin{bmatrix} [\bar{F}(0) + GK]_i \\ [\Delta\bar{F}^{(1)}]_i \\ \vdots \\ [\Delta\bar{F}^{(n_C)}]_i \end{bmatrix} \end{aligned} \tag{36}$$

with and $\Delta\bar{F}^{(j)} = \bar{F}(\theta^{(j)}_{C,0}) - \bar{F}(0)$.

6 | CONTROL ALGORITHM

In order to increase the domain of attraction[18] and to improve controller performance for changing targets, a cost function is defined with an artificial target state $\hat{x}_{T,k}$ as

$$J_N = \sum_{l=0}^{N-1} \|\hat{x}_{l|k} - \hat{x}_{T,k}\|_Q^2 + \|u_{l|k} - \hat{u}_{T,k}\|_R^2 + \|\hat{x}_{N|k} - \hat{x}_{T,k}\|_P^2 + \|\hat{x}_{T,k} - x_{T,k}\|_{T_x}^2, \tag{37}$$

where P solves the Lyapunov equation (25). The positive definite weight T_x penalizes deviations from the artificial target state to the true target state $x_{T,k}$. The level of sub-optimality introduced by the artificial target state can be reduced by choosing $T_x = \gamma_T P$ with γ_T as a large scalar value.

The term $\|\hat{x}_{l|k} - \hat{x}_{T,k}\|_Q^2$ can be interpreted as the deviation between the nominal state to a time-varying, feasible steady-state. The common steady-state equation

$$\begin{pmatrix} x_T(\theta, \theta_C) \\ u_T(\theta, \theta_C) \end{pmatrix} = \begin{pmatrix} A(\hat{\theta}) - I & B(\hat{\theta}) \\ C(\hat{\theta}_C) & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ r \end{pmatrix}, \tag{38}$$

is used to compute the targets $x_{T,k} = x_T(\hat{\theta}_k, \hat{\theta}_{C,k})$ and $u_{T,k} = u_T(\hat{\theta}_k, \hat{\theta}_{C,k})$. Note that the solution to (38) is always defined due to Assumption 5.

Assumption 6. All references r lie in a compact set $\mathcal{R}$.

Assumption 6 ensures that the set of all possible target states $\mathcal{Z} = \{x_T : (38),\ r \in \mathcal{R}, \hat{\theta}_k \in \Theta_0, \hat{\theta}_{C,k} \in \Theta_{C,0}\}$ is bounded.

6.1 | Modes of operation

When the nominal model is updated, a new target state is computed by (38) which may result in an infeasible optimization problem. One solution could be to find an intermediate model update with a feasible target state. However, this is a non-linear problem with a significant computation time. Instead, the approach presented in this work uses an artificial target state and exploits that all steady-state manifolds intersect at the origin. First, one tries to directly use the new nominal model while the algorithm is in “normal operation” as shown in Figure 1. In case no feasible solution can be found, the algorithm enters a “recovery mode” and a new feasible input is constructed based on the solution from the previous time step which is always feasible. At the next time step, the control problem is formulated using the last feasible nominal model and the origin as a target state which always has a feasible solution due to the use of an artificial target state. While the system is steered to the origin, the optimization problem is re-formulated using the latest nominal model. If a feasible solution exists, the algorithms “jumps” to a new steady-state manifold as depicted in Figure 2. Otherwise, a control input can be constructed from the last feasible solution which steers the system further to the origin. After N steps, a new solution needs to be calculated using the last feasible nominal model and the origin as a target state. In the worst case, the steady-state manifold can be always switched when the nominal system state reaches the origin. However, it is expected that a new feasible solution can be found earlier as the distance between the steady-state manifolds decrease while approaching the origin. When the new nominal model yields a feasible solution, the artificial target state converges to the true target state or to a neighborhood of it. If the true target state is infeasible, the final artificial target state will be the closest steady-state with respect to T_x.

In this “recovery” scheme, one MPC problem has to be to solved at every sample step. However, several other variants can be constructed depending on computation time constraints.

6.2 | Terminal set

In this section, a terminal set is defined online based on the current parameter sets Θ_k and $\Theta_{C,k}$. As a downside, the proposed terminal set is not the maximum robust positive invariant set but a subset of the former. However, the online determined terminal set might be larger during run-time than the offline-defined maximum terminal set as the plant uncertainty decreases over time.

A necessary property of the terminal set is positive set invariance.[31] Set invariance can be checked by backward propagation of all states inside a candidate set. This sufficient condition can be stated as[27]

$$\mathbb{X}_T \subseteq \text{Pre}\{\mathbb{X}_T\}, \tag{39}$$

where

$$\mathbb{X}_T = \{x : Vx \leq \alpha_N\} \tag{40}$$

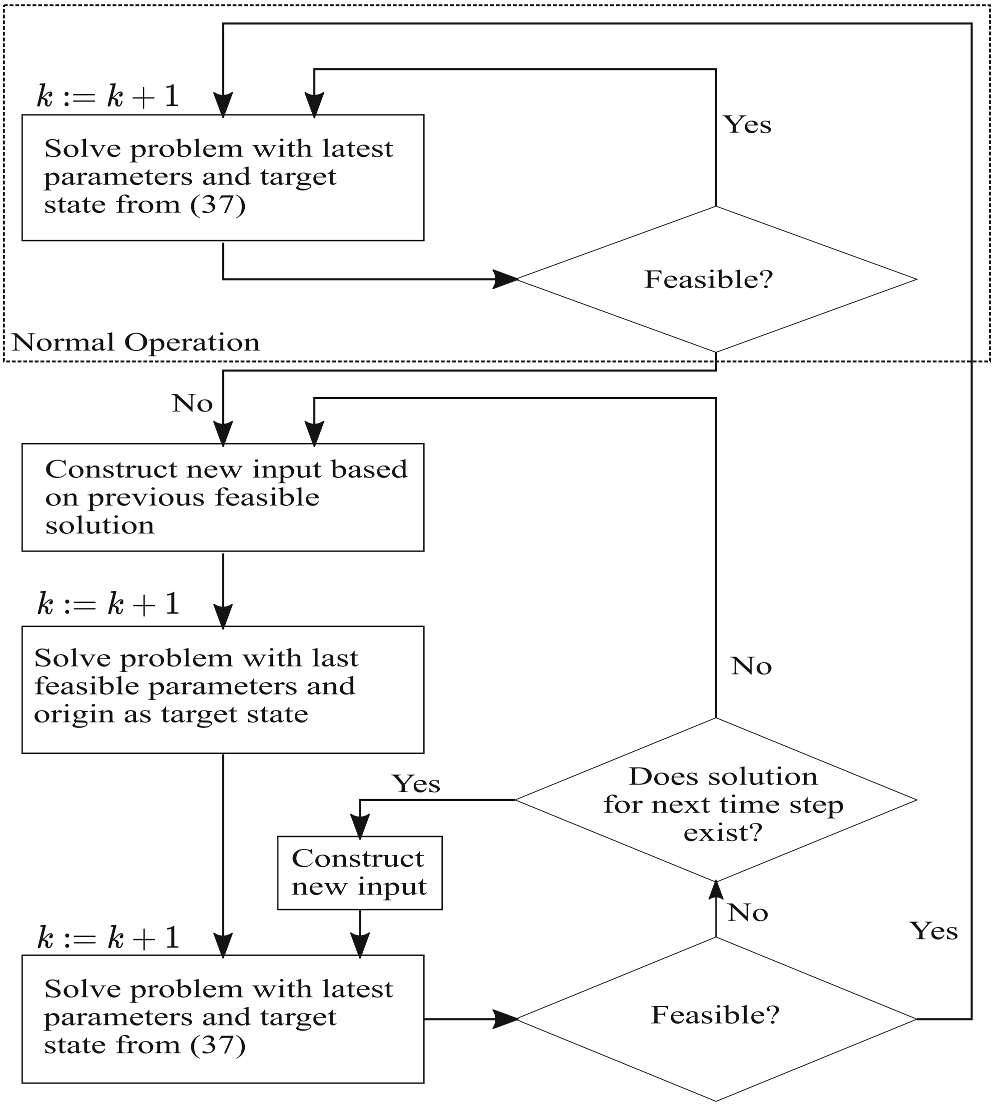

FIGURE 1 Flow chart of the control algorithm. In case a new target state yields an infeasible optimization problem, the parts outside the box describe a "Recovery Mode". This mode transitions the control algorithm to a new stead-state manifold.

denotes the structure of the terminal set. Using condition (31b) for the state evolution then gives

$$H(\bar{\theta}_k^{(j)})\alpha_{N|k} + V\left(I_{n_x} - \Phi(\theta_k^{(j)})\right)\hat{x}_{T,k} + \bar{w} \leq \alpha_{N|k} \tag{41}$$

with $c_{N|k} = 0$ and $j \in \mathbb{N}_1^m$. The conservatism added by having a non-maximum invariant set is further reduced by allowing $\hat{x}_{T,k}$ to deviate from the current target state $x_{T,k}$.

System constraints inside the terminal set are considered by

$$H_C(\bar{\theta}_{C,k}^{(j)})\alpha_{N|k} - GK\hat{x}_{T,k} + G\hat{u}_{T,k} \leq \underline{1}, \quad \forall j \in \mathbb{N}_1^{m_C} \tag{42}$$

which can be derived from (35) and considering $c_{N|k} = 0$. Hence, the terminal set directly depends on $H(\bar{\theta}_k^{(j)})$ and $H_C(\bar{\theta}_{C,k}^{(j)})$ and changes its size and shape depending on the estimated uncertainty.

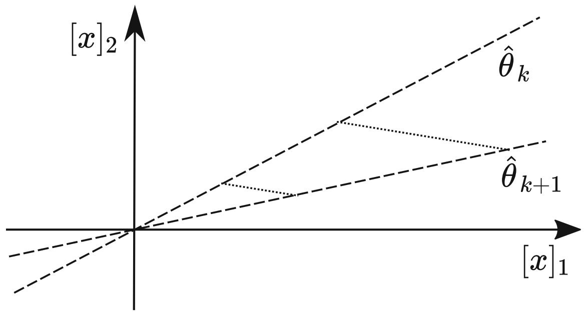

FIGURE 2 Exemplary steady-state manifolds for two different parameter vectors. The dotted lines indicate possible transitions between the manifolds.

6.3 Algorithm

The control algorithm can be now stated as:

Offline:

1. Choose a robustly stabilizing gain K and terminal weight P for system (1) such that equation (25) is fulfilled.
2. Compute the robust λ-contractive set S and matrices $\bar{H}_i$ and $\bar{H}_{\mathrm{C,i}}$ according to (32) and (36).
3. Define zonotopes Θ_0 and $\Theta_{\mathrm{C},0}$ with $\Theta_0 \supseteq \Theta$ and $\Theta_{\mathrm{C},0} \supseteq \Theta_{\mathrm{C}}$, and initial parameter vectors $\hat{\theta}_0 \in \Theta_0$ and $\hat{\theta}_{\mathrm{C},0} \in \Theta_{\mathrm{C},0}$.

*Online Algorithm **A***:

1. Measure state vector x_k. Update Θ_k and $\Theta_{\mathrm{C},k}$ according to the respective set-membership estimation (16) and (17). Use LMS algorithms (18) and (21) to update the respective point estimates $\hat{\theta}_k$ and $\hat{\theta}_{\mathrm{C},k}$.
2. Depending on the flow chart in Figure 1, either
   (a) compute matrices $H(\bar{\theta}_k^{(j)})$, $H_{\mathrm{C}}(\bar{\theta}_k^{(j_{\mathrm{C}})})$ based on the vertices of Θ_k and $\Theta_{\mathrm{C,k}}$. Use the nominal parameter estimates $\hat{\theta}_k$ and $\hat{\theta}_{\mathrm{C},k}$ for computing the nominal plant models $A(\hat{\theta}_k), B(\hat{\theta}_k), C(\hat{\theta}_{\mathrm{C},k})$ and current target tuple $(x_{\mathrm{T},k}, u_{\mathrm{T},k})$ based on (38). Solve

$$\min_{\mathbf{d}_k} J_N$$

$$\text{s.t. } \forall l \in \mathbb{N}_0^{N-1}, j \in \mathbb{N}_1^{m}, j_{\mathrm{C}} \in \mathbb{N}_1^{m_{\mathrm{C}}} \quad :$$

$$\hat{x}_{l+1|k} = \Phi(\hat{\theta}_k)(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k}) + B(\hat{\theta}_k)c_{l|k} + A(\hat{\theta}_k)\hat{x}_{\mathrm{T},k} + B(\hat{\theta}_k)\hat{u}_{\mathrm{T},k} \ , \tag{43a}$$

$$(A(\hat{\theta}_k) - I_{n_x})\hat{x}_{\mathrm{T},k} + B(\hat{\theta}_k)\hat{u}_{\mathrm{T},k} = 0 \tag{43b}$$

$$H(\bar{\theta}_k^{(j)})\alpha_{l|k} + V\left(I_{n_x} - \Phi(\theta_k^{(j)})\right)\hat{x}_{\mathrm{T},k} + VB(\theta_k^{(j)})c_{l|k} + \bar{w} \leq \alpha_{l+1|k} \ , \tag{43c}$$

$$H_{\mathrm{C}}(\bar{\theta}_{\mathrm{C},k}^{(j_{\mathrm{C}})})\alpha_{l|k} - GK\hat{x}_{\mathrm{T},k} + Gc_{l|k} + G\hat{u}_{\mathrm{T},k} \leq \underline{1} \ , \tag{43d}$$

$$H(\bar{\theta}_k^{(j)})\alpha_{N|k} + V\left(I_{n_x} - \Phi(\theta_k^{(j)})\right)\hat{x}_{\mathrm{T},k} + \bar{w} \leq \alpha_{N|k} \ , \tag{43e}$$

$$H_{\mathrm{C}}(\bar{\theta}_{\mathrm{C},k}^{(j_{\mathrm{C}})})\alpha_{N|k} - GK\hat{x}_{\mathrm{T},k} + G\hat{u}_{\mathrm{T},k} \leq \underline{1}, \tag{43f}$$

$$Vx_k \leq \alpha_{0|k}, \tag{43g}$$

$$x_k = \hat{x}_{0|k}, \tag{43h}$$

   or

   (b) use the last feasible nominal models and current parameter sets. Set the origin as the target tuple $(x_{\mathrm{T},k}, u_{\mathrm{T},k})$. Replace (43h) with $V\hat{x}_{0|k} \leq \alpha_{0|k}$ and solve the modified quadratic program (43).

3. If the problem is feasible, then

$$u_k = K(x_k - \hat{x}_{\mathrm{T},k}) + c_{0|k} + \hat{u}_{\mathrm{T},k}, \tag{44}$$

else

$$u_k = K(x_k - \hat{x}_{\mathrm{T},k-i}) + c_{i|k-i} + \hat{u}_{\mathrm{T},k-i}, \tag{45}$$

where i denotes the number of samples to the last feasible solution.

The decision variable

$$\mathbf{d}_k = \{\underline{\hat{x}}_k, \underline{c}_k, \underline{\alpha}_k, \hat{x}_{\mathrm{T},k}, \hat{u}_{\mathrm{T},k}\} \in \mathbb{D}(x_k, \Theta_k, \Theta_{\mathrm{C},k}, \hat{\theta}_k) \tag{46}$$

is from the set of all admissible decision variables

$$\mathbb{D}(x_k, \Theta_k, \Theta_{\mathrm{C},k}, \hat{\theta}_k) = \{\mathbf{d}_k : (43a) \text{ to } (43h)\}. \tag{47}$$

The underlined variables denote sequences over the prediction horizon of the respective variable.

Assumption 7. Consider two non-empty feasible sets $\mathcal{S}^1 = \mathbb{D}(x^1, \Theta^1, \Theta^1_{\mathrm{C}}, \hat{\theta}^1)$ and $\mathcal{S}^2 = \mathbb{D}(x^1, \Theta^1, \Theta^1_{\mathrm{C}}, \hat{\theta}^2)$. Then for every input sequence $c^1 \in \mathcal{S}^1$ and artificial steady-state $\hat{x}^1_{\mathrm{T}} \in \mathcal{S}^1$, there exist $c^2 \in \mathcal{S}^2$, $\hat{x}^2_{\mathrm{T}} \in \mathcal{S}^2$ and a constant $L \in \mathbb{R}_{>0}$ such that $\| \begin{bmatrix} c^1 & \hat{x}^1_{\mathrm{T}} \end{bmatrix}' - \begin{bmatrix} c^2 & \hat{x}^2_{\mathrm{T}} \end{bmatrix}' \|_2 \leq L \| \begin{bmatrix} \hat{\theta}^1 & x^1_{\mathrm{T}} \end{bmatrix}' - \begin{bmatrix} \hat{\theta}^2 & x^2_{\mathrm{T}} \end{bmatrix}' \|_2$ holds for all $x^1 \in \mathcal{X}, \Theta^1 \subseteq \Theta_0, \Theta^1_{\mathrm{C}} \subseteq \Theta_{\mathrm{C},0}$ and $\theta^1, \theta^2 \in \Theta_0$, and $x^1_{\mathrm{T}} = x_{\mathrm{T}}(\theta^1, \theta_{\mathrm{C}}), x^2_{\mathrm{T}} = x_{\mathrm{T}}(\theta^2, \theta_{\mathrm{C}}), \theta_{\mathrm{C}} \in \Theta_{\mathrm{C},0}$.

Remark 3. The feasible set of a convex multi-parametric quadratic program with affine inequality constraints is always Lipschitz continuous.[32] It is very hard to verify similar properties for multi-parametric quadratic programs with non-linear constraints regarding the parameter and optimizer as in (43).

Proposition 7. If problem (43) is feasible at time $k+1$, then for $\|\delta x_{l|k+1}\|_Q = \|\hat{x}_{l|k+1} - \hat{x}_{l+1|k}\|_Q$ and $l \in \mathbb{N}_0^N$ there exist positive constants $\sigma_1, \sigma_{\delta_x}, \sigma_{\theta,1}, \ \sigma_{\theta,2}, \sigma_{\mathrm{w},1}$ such that

$$\|\delta x_{l|k+1}\|_Q^2 \leq \sigma_1^l \sigma_{\delta_x} \|\theta^* - \hat{\theta}_k\|_Q^2 + \sigma_1^l \sigma_{\mathrm{w},1} \|w_k\|_Q^2 + \sum_{i=0}^{l-1} \sigma_1^i (\sigma_{\theta,1} \|\hat{\theta}_{k+1} - \hat{\theta}_k\|_Q^2 + \sigma_{\theta,2} \|\hat{\theta}_{\mathrm{C},k+1} - \hat{\theta}_{\mathrm{C},k}\|_Q^2) \tag{48}$$

holds.

Proposition 7 builds on the results from Reference 11 and the proof for Proposition 7 can be found in the Appendix A.

Lemma 1. Let Assumptions 1 to 5 hold and consider the case for constant parameters $\hat{\theta}_k = \hat{\theta}_{k+1}, \hat{\theta}_{\mathrm{C},k} = \hat{\theta}_{\mathrm{C},k+1}$ and parameter sets $\Theta_{k+1} \subseteq \Theta_k$ and $\Theta_{\mathrm{C},k+1} \subseteq \Theta_{\mathrm{C},k}$. If $\theta^* \in \Theta_0$, $\hat{\theta}_k \in \Theta_{k+1}$, $\theta^*_{\mathrm{C}} \in \Theta_{\mathrm{C},0}$ and problem (43) is feasible at time $k = 0$ with $\mathbb{D}(x_{\mathrm{m},k}, \Theta_k, \Theta_{\mathrm{C},k}, \hat{\theta}_k) \neq \emptyset$, then $\mathbb{D}(x_{\mathrm{m},k}, \Theta_k, \Theta_{\mathrm{C},k}, \hat{\theta}_k) \neq \emptyset$ for all $k \geq 1$.

Proof. A shifted, possibly sub-optimal solution $\tilde{\mathbf{d}}_{k+1}$ is considered. It consists of the shifted sequences $\underline{\alpha}_{k+1}, \underline{c}_{k+1}$ and $\underline{\hat{x}}_{k+1}$. Repeated targets $\hat{x}_{\mathrm{T},k+1} = \hat{x}_{\mathrm{T},k}, \hat{u}_{\mathrm{T},k+1} = \hat{u}_{\mathrm{T},k}$ are used. As required in the derivation of the terminal set condition (43e), new values $\alpha_{N|k+1}$ can be derived by repeating the previous values $\alpha_{N|k}$. In addition, $c_{N|k+1} = 0$ holds as defined in the derivation of (41). As shown in Appendix B the shifted sequences form a feasible solution for (43c) and (43e) due to convexity of Θ_k, $\Theta_{\mathrm{C},k}$ and $\Theta_{k+1} \subseteq \Theta_k$, $\Theta_{\mathrm{C},k+1} \subseteq \Theta_{\mathrm{C},k}$. Then, the imposed system constraints (43d) and terminal constraints (43f) also hold with $c_{N|k+1} = 0$. ∎

Theorem 1. *Let Assumptions 1 to 5 hold and consider algorithm* ***A****. If $\theta^* \in \Theta_0$, $\theta^*_{\mathrm{C}} \in \Theta_{\mathrm{C},0}$ and problem (43) is feasible at time $k = 0$ with $\mathbb{D}(x_{\mathrm{m},k}, \Theta_k, \Theta_{\mathrm{C},k}, x_{\mathrm{T},k}, \hat{\theta}_k) \neq \emptyset$, then algorithm* ***A*** *always yields a feasible solution.*

Proof. Starting from "normal operation" in Figure 1, the solution of problem (43) can be either feasible or infeasible. If infeasible, a feasible input can be constructed from the previous solution by choosing $c_{0|k} = c_{1|k-1}$, $\hat{u}_{T,k} = \hat{u}_{T,k-1}$ as $\Theta_k \subseteq \Theta_{k-1}$, $\Theta_{\mathrm{C},k} \subseteq \Theta_{\mathrm{C},k-1}$ and Lemma 1 holds. At the next sample step, the origin is chosen as the target state using the last feasible parameters for problem (43). This always yields a feasible solution as the artificial target state may be the same as from the last feasible solution.

At the next sample step, problem (43) is formulated using the latest parameters $\hat{\theta}_k$, $\hat{\theta}_{\mathrm{C},k}$, Θ_k and $\Theta_{\mathrm{C},k}$. If no solution exists, a control input can be constructed from the solution of the regulation problem which is again feasible as the state always lies inside the predicted tube. If the solution vector of the regulation problem has no predicted control input for the next sample step, then the regulation problem has to be solved again using the last feasible parameters which always has a feasible solution. ∎

Theorem 2. *Let Assumptions 1 to 7 hold. If problem (43) is feasible and algorithm* $\boldsymbol{A}$ *is in "normal operation" and the reference r is constant, then system (1) is finite gain* l_2*-stable with respect to the nominal trajectory* $\hat{x}_{\mathrm{T},k}$ *for all* $K \in \mathbb{N}$ *with*

$$\sum_{k=0}^{K} \|x_k - \hat{x}_{\mathrm{T},k}\|^2 \leq \sigma_0 + \sum_{k=0}^{K} \sigma_\theta \|\hat{\theta}_k - \theta^*\|^2 + \sum_{k=0}^{K} \sigma_{\mathrm{w}} \|w_k\|^2 + \sum_{k=0}^{K} \sigma_{\theta_{\mathrm{C}}} \|\hat{\theta}_{\mathrm{C},k} - \theta_{\mathrm{C}}^*\|^2 + \sum_{k=0}^{K} \sigma_{\mathrm{e}} \|e_k\|^2, \tag{49}$$

where $\sigma_0, \sigma_\theta, \sigma_{\mathrm{w}}, \sigma_{\theta_{\mathrm{C}}}, \sigma_{\mathrm{e}} \in \mathbb{R}_{>0}$.
Asymptotic stability without additive disturbances and model mismatch:

If the parameter vectors converge to their true values $\hat{\theta}_k = \theta^*$, $\hat{\theta}_{\mathrm{C},k} = \theta_{\mathrm{C}}^*$ *and* $w_k = 0$ *holds, and (43e) and (43f) are non-empty using* x_{T}^**, then system state* x_k *converges to* x_{T}^**. If* x_{T}^* *is infeasible for algorithm* $\boldsymbol{A}$*, then the system is steered towards the closest feasible steady-state* $\hat{x}_{\mathrm{T},k}$ *depending on the choice of* T_{x}*.*
Recovery mode: When algorithm $\boldsymbol{A}$ *is in "recovery mode", then it will return to "normal operation" using the latest model parameters after a finite time.*

Proof. Finite l_2*-gain for "normal operation"*: The difference in the value function of the quadratic program (43) for consecutive time steps can be bounded by $V_N(x_{0|k+1}, \hat{\theta}_{k+1}, \Theta_{k+1}, \Theta_{\mathrm{C},k+1}, x_{\mathrm{T},k+1}) - V_N(x_{0|k}, \hat{\theta}_k, \Theta_k, \Theta_{\mathrm{C},k}, x_{\mathrm{T},k}) \leq -\sigma_0 \|\hat{x}_{0|k} - \hat{x}_{\mathrm{T},k}\|^2 + \sigma_{\theta,0} \|\hat{\theta}_k - \theta^*\|^2 + \sigma_{\mathrm{w},0} \|w_k\|^2 + \sigma_{\theta_{\mathrm{C}},0} \|\hat{\theta}_{\mathrm{C},k} - \theta_{\mathrm{C}}^*\|^2 + \sigma_{\mathrm{e},0} \|e_k\|^2 + \|\hat{x}_{\mathrm{T},k+1} - x_{\mathrm{T},k+1}\|_{T_{\mathrm{x}}}^2 - \|\hat{x}_{\mathrm{T},k} - x_{\mathrm{T},k}\|_{T_{\mathrm{x}}}^2$. The complete derivation for the difference of the value function is given in the Appendix and the main steps are summarized here. Using Proposition 2, the predicted state trajectory at time $k+1$ is separated into to a trajectory based on the previous solution and a prediction error trajectory. Then Proposition 7 and Assumption 7 are used bound the prediction error, changes in the free control move and artificial target states by changes in the parameter vectors $\hat{\theta}_k$ and $\hat{\theta}_{\mathrm{C},k}$. Then, inequality (49) can be derived summing up the cost function decrease from 0 to K and bounding the final deviation $\hat{x}_{\mathrm{T},K} - x_{\mathrm{T},K}$ by a constant due to compactness of $\mathcal{Z}$ and $\mathcal{X}$.

Asymptotic stability without additive disturbances and model-mismatch: In case no model-mismatch and additive disturbances exist, it follows from standard stability proofs for MPC that x_k converges to $\hat{x}_{\mathrm{T},k}$ which lies on the same steady-state manifold as $x_{\mathrm{T},k} = x_{\mathrm{T}}^*$. From the main theorem in Limon et al.[18] it follows that the artificial steady state $\hat{x}_{\mathrm{T},k}$ converges to the true target state x_{T}^* if it is feasible with respect to the terminal inequalities (43e) and (43f). If no feasible terminal set exists for x_{T}^*, then the system is steered towards the closest feasible steady-state depending on the choice of T_{x} as shown in Reference 33.

Finite time "recovery mode": During "recovery mode", the system is steered to the origin by solving problem (43) with constant parameter vectors, the origin as the target state and the first nominal state as an additional decision variable. Using standard stability proofs for constant parameter vectors, it can be shown that the nominal system state $\hat{x}_{0|k}$ converges to $\hat{x}_{\mathrm{T},k}$. In addition, using the main theorem in Limon et al.[18] and that $\hat{x}_{0|k} = \hat{x}_{\mathrm{T},k}$ can be chosen if $\hat{x}_{0|k}$ enters the terminal set for $\hat{x}_{\mathrm{T},k}$, it can be shown that the artificial steady state $\hat{x}_{\mathrm{T},k}$ converges to the origin. At some point in time, the state x_k lies in the terminal set for the origin and then, problem (43) is also feasible for the updated parameter vectors as all constraints are fulfilled if the origin is chosen as the artificial target state. ∎

Remark 4. During "normal operation", no convergence of $\hat{x}_{\mathrm{T},k}$ to $x_{\mathrm{T},k}$ can be guaranteed if $\hat{\theta}_k \neq \theta^*$, $\hat{\theta}_{\mathrm{C},k} \neq \theta_{\mathrm{C}}^*$ and $w_k \neq 0$. In this case, only convergence to a neighborhood of $x_{\mathrm{T},k}$ can be guaranteed which depends on the magnitude of T_{x}.

Remark 5. Changes in the reference r can be incorporated similar to changes in $\hat{\theta}_{\mathrm{C},k}$ for the stability proof. A new term regarding the future change $r_{k+1} - r_k$ would appear in in (49).

The online quadratic program (43) has $n_\mathrm{F}(N+1)m_\mathrm{C} + n_\alpha((N+1)m+1)$ inequality constraints, $n_\mathrm{x}(N+2)$ equality constraints and $n_\mathrm{x}(N+2) + n_\mathrm{u}(N+1) + (N+1)n_\alpha$ decision variables.

7 | VARIANT: BOUNDED TIME-VARYING PARAMETERS

A robust adaptive MPC algorithm for time-varying parameters with a bounded rate of variation has been considered in Lorenzen et al.[7] In this section, the main algorithm is extended for time-varying parameters which lie in a known convex set with unknown center. No bounds on the rate of variation are assumed and in addition, no worst-case terminal set is employed but an online-adapted terminal set based on the current parameter set estimate.

A discrete-time, linear system with bounded time-varying parameters

$$\begin{aligned} x_{k+1} &= A(\theta_k^*)x_k + B(\theta_k^*)u_k + w_k, \\ y_k &= C(\theta_\mathrm{C}^*)x_k + e_k \end{aligned} \tag{50}$$

is considered. In the following, only time-variance of the parameter vector θ_k^* is considered for conciseness but the results can be easily extended for a time-varying parameter vector θ_C^*. The parameter vector θ_k^* consists of an unknown time-invariant vector θ_B and known relative bounds for possible deviations.

Assumption 8. For the time-varying parameter θ_k^* holds that

$$\theta_k^* \in \theta_\mathrm{B} \oplus \mathbb{B} \subseteq \Theta_\mathrm{B}, \tag{51}$$

where θ_B is an unknown but constant parameter, $\mathbb{B}$ a known convex set containing possible time-varying variations and Θ_B a convex set containing all possible realizations.

Assumption 9. The set

$$\mathbb{B} = \{\theta : M_\mathrm{B}\theta \leq b_\mathrm{B}\} \tag{52}$$

is centrally symmetric with $M_\mathrm{B} \in \mathbb{R}^{q_\mathrm{B} \times n_\mathrm{P}}$.

The set-membership algorithm from Section 4 needs to be adapted due to the time variance of the parameter. At every sample step, the set of unfalsified parameters ϕ_k depending on the current measurement is computed as before. However, it may hold that $\phi_k \cap \phi_{k+1} = \emptyset$ due to the time-variance of the scheduling parameter θ. Hence, the set ϕ_k needs to be enlarged by the known bounds $\mathbb{B}$. As the relative position of ϕ_k in $\mathbb{B}$ is not known, the worst-case is considered by computing $\phi_k \oplus 2\mathbb{B}$ which always includes $\theta_\mathrm{B} \oplus \mathbb{B}$ due to Assumption 9. In order to ensure recursive feasibility and finding tighter set estimates, the updated parameter set is defined by $\Theta_{k+1} = (\phi_k \oplus 2\mathbb{B}) \cap \Theta_k$. The Minkowski sum is computationally expensive for arbitrary polytopes but can be easily computed for polytopes with the same matrix for the half-space representation:

Proposition 8. Given two polytopes $S_1 = \{a : M_1 a \leq b_1\}$ and $S_2 = \{b : M_1 b \leq b_2\}$, the Minkowski sum $S_3 = S_1 \oplus S_2 = \{x = a + b : a \in A, b \in B\}$ is defined by

$$S_3 = \{x : M_1 x \leq b_1 + b_2\}. \tag{53}$$

Proof. The result can be derived by adding S_1 and S_2 in half-space representation. ∎

In order to exploit Proposition 8, a set $\bar{\phi}_k = \{\theta : M_\mathrm{B}\theta \leq b_{\bar{\phi},k}\} \supseteq \phi_k$ is computed first by solving

$$\begin{aligned} [b_{\bar{\phi},k}]_i = &\min_{b, H_i \in \mathbb{R}^{1 \times q_w}} b \\ &\text{s.t.} \quad H_i M_{\phi,k} = [M_\mathrm{B}]_i,\ H_i b_{\phi,k} \leq b,\ H_i \geq 0\ . \end{aligned} \tag{54}$$

Remark 6. In case $\mathrm{rank}(M_{\phi,k}) \neq \mathrm{rank}(M_\mathrm{B})$ due to the particular definition of $\left(\Delta A^{(j)}, \Delta B^{(j)}\right)$, the set $\bar{\phi}_k \supseteq (\phi_k \cap \Theta_k)$ can be computed instead.

Then, the updated parameter set $\Theta_{k+1} \supseteq (\bar{\phi}_k \oplus 2\mathbb{B}) \cap \Theta_k$ is given by

$$[b_{\Theta,k+1}]_i = \min_{b,H_i \in \mathbb{R}^{1\times(2n_p+q_B)}} b$$
$$\text{s.t.} \quad H_i \begin{bmatrix} M_\Theta \\ M_B \end{bmatrix} = [M_\Theta]_i, \; H_i \begin{bmatrix} b_{\Theta,k} \\ b_{\bar{\phi},k} + 2b_B \end{bmatrix} \leq b, \; H_i \geq 0 \tag{55}$$

and can be used for the MPC algorithm (43).

The result of Proposition 3 using the LMS algorithm does not hold anymore due to the time-variance of θ_k^*. For this reason, a simple projection on the current parameter set

$$\hat{\theta}_k = \arg\min_{\theta \in \Theta_k} \|\theta - \hat{\theta}_{k-1}\| \tag{56}$$

is used to derive a new estimated parameter $\hat{\theta}_k$ which has converged to its final value if $\Theta_k = \theta_B \oplus \mathbb{B}$. It holds that $\|\hat{\theta}_{k+1} - \hat{\theta}_k\| \leq \|\hat{\theta}_k - \hat{\theta}_k^*\|$. The same parameter update can be performed for $\hat{\theta}_{C,k}$ with

$$\hat{\theta}_k = \arg\min_{\theta_C \in \Theta_{C,k}} \|\theta_C - \hat{\theta}_{C,k-1}\|. \tag{57}$$

Theorem 3. *Let Assumptions 1 to 9 hold. If $\theta_k^* \in \theta_B \oplus \mathbb{B} \subseteq \Theta_0$ and problem (43) is feasible at time $k = 0$ with $\mathbb{D}(x_{m,k}, \Theta_k, \Theta_{C,k}, \hat{\theta}_k) \neq \emptyset$, then algorithm **A** always yields a feasible solution for system (50) with Θ_k estimated by (54), (55) and parameter vector $\hat{\theta}_k$ and $\hat{\theta}_{C,k}$ updated by (56) and (57). If algorithm **A** is in "normal operation" and the reference r is constant, then system (50) is finite gain l_2-stable with respect to the nominal trajectory $\hat{x}_{T,k}$ for all $K \in \mathbb{N}$ with*

$$\sum_{k=0}^{K} \|x_k - \hat{x}_{T,k}\|^2 \leq \sigma_0 + \sum_{k=0}^{K} \sigma_\theta \|\hat{\theta}_k - \theta_k^*\|^2 + \sum_{k=0}^{K} \sigma_w \|w_k\|^2 + \sum_{k=0}^{K} c_{\theta_C} \|\hat{\theta}_{C,k} - \theta_C^*\|^2 \tag{58}$$

where $\sigma_0, \sigma_w, \sigma_\theta, \sigma_{\theta_C} \in \mathbb{R}_{>0}$.

*Recovery mode: When algorithm **A** is in "recovery mode", then it will return to "normal operation" using the latest model parameters after a finite time.*

Proof. Feasibility: The same proof as for Theorem 1 holds as $\Theta_{k+1} \subseteq \Theta_k$ is always ensured by (55).
Stability: The proof is analogous to Theorem 2 using $\|\hat{\theta}_{k+1} - \hat{\theta}_k\|^2 \leq \|\hat{\theta}_k - \hat{\theta}_k^*\|^2$ and $\|\hat{\theta}_{C,k+1} - \hat{\theta}_{C,k}\|^2 \leq \|\hat{\theta}_{C,k} - \hat{\theta}_C^*\|^2$. ■

8 PRACTICAL EXAMPLE

The proposed algorithm is evaluated for controlling the engine load of a self-propelled work machine with velocity-dependent processing power. Examples of such machines are milling machines for removing asphalt layers on roads and combine harvesters for cutting and threshing of crops. Self-propelled work machines share the properties of hard constraints on the maximum engine load and an uncertain throughput which depends on the velocity and changing environmental conditions, for example, changing crop yields in a field. In addition, the relationship between throughput and engine load depends on machine settings, which are selected differently for each working condition. For example, a combine harvester is configured differently for each crop type which results in a new relationship between throughput and engine load. Moreover, it also common that the desired engine load changes for process quality or machine feeding reasons during operation. In the following, a simple model of a self-propelled work machine is given.[34] It is not possible to measure all states in many practical applications. In this case, a state observer for uncertain systems needs to be added which is addressed in robust output MPC, for example in Kögel et al.[35]

The two main components of a self-propelled work machine are the throughput unit and the propulsion system

$$\begin{bmatrix} x_{k+1}^{\mathrm{th}} \\ x_{k+1}^{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} A^{\mathrm{th}}(\theta) & B^{\mathrm{th}}(\theta)C^{\mathrm{p}} \\ 0 & A^{\mathrm{p}} \end{bmatrix} \begin{bmatrix} x_k^{\mathrm{th}} \\ x_k^{\mathrm{p}} \end{bmatrix} + \begin{bmatrix} 0 \\ B^{\mathrm{p}} \end{bmatrix} u_k + w_k, \tag{59}$$

where the superscripts "th" and "p" denote states and matrices belonging to the throughput unit and the propulsion system, respectively. The propulsion system usually consists of a hydrostatic transmission and is modeled here as a second-order system. Measurable states of the hydrostatic transmission are the rotational velocity of the hydraulic motor and the pressure difference inside the hydraulic circuit. The input u_k denotes the speed command sent to a transmission controller. The throughput system is modeled as an uncertain first-order system and its input depends on the machine's velocity and uncertain environment conditions.

The propulsion system is described by

$$A^{\mathrm{p}} = \begin{bmatrix} 0.2034 & -0.2972 \\ 0.1902 & 0.1294 \end{bmatrix}, \qquad B^{\mathrm{p}} = \begin{bmatrix} 0.0951 \\ 0.2786 \end{bmatrix}, \qquad C^{\mathrm{p}} = \begin{bmatrix} 0 & 3.125 \end{bmatrix} \tag{60}$$

and the uncertain throughput system by

$$\begin{aligned} &\begin{bmatrix} A^{\mathrm{th}}(\theta_0^{(1)}) & B^{\mathrm{th}}(\theta_0^{(1)})C^{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} 0.6065 & 0 & 0.0781 \end{bmatrix}, \qquad \begin{bmatrix} A^{\mathrm{th}}(\theta_0^{(2)}) & B^{\mathrm{th}}(\theta_0^{(2)})C^{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} 0.8465 & 0 & 0.0781 \end{bmatrix}, \\ &\begin{bmatrix} A^{\mathrm{th}}(\theta_0^{(3)}) & B^{\mathrm{th}}(\theta_0^{(3)})C^{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} 0.8465 & 0 & 0.3125 \end{bmatrix}, \qquad \begin{bmatrix} A^{\mathrm{th}}(\theta_0^{(4)}) & B^{\mathrm{th}}(\theta_0^{(4)})C^{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} 0.6065 & 0 & 0.3125 \end{bmatrix}, \end{aligned} \tag{61}$$

which describes the uncertainty in the filling dynamics of the throughput system and uncertainty in the relationship between throughput and velocity.

In order to ensure a comfortable ride for the machine operator, the control problem is stated using the Δu formulation. This allows to limit and weight changes in the ground speed command. This leads to the augmented system

$$\begin{bmatrix} x_{k+1}^{\mathrm{th}} \\ x_{k+1}^{\mathrm{p}} \\ u_k \end{bmatrix} = \begin{bmatrix} A^{\mathrm{th}}(\theta) & B^{\mathrm{th}}(\theta)C^{\mathrm{p}} & 0 \\ 0 & A^{\mathrm{p}} & B^{\mathrm{p}} \\ 0 & 0 & I_{n_{\mathrm{u}}} \end{bmatrix} \begin{bmatrix} x_k^{\mathrm{th}} \\ x_k^{\mathrm{p}} \\ u_{k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ B^{\mathrm{p}} \\ I_{n_{\mathrm{u}}} \end{bmatrix} \Delta u_k \tag{62}$$

with the augmented state $\bar{x}_k = \begin{bmatrix} x_k^{\mathrm{th}} & x_k^{\mathrm{p}} & u_{k-1} \end{bmatrix}'$. The system output is the engine load, which consists of power related to throughput and propulsion

$$C(\theta_{\mathrm{C},0}^{(1)}) = \begin{bmatrix} 1.0 & 0 & 0.3125 & 0 \end{bmatrix}, C(\theta_{\mathrm{C},0}^{(1)}) = \begin{bmatrix} 0.9 & 0 & 0.3125 & 0 \end{bmatrix} \tag{63}$$

where the uncertainty lies in the relationship between throughput and engine load.

The system output should not exceed 100% engine load with $\|y\|_\infty \leq 1$ and the velocity should not exceed 10 kph and be above 0 kph with $0 \geq C^{\mathrm{p}} x_k^{\mathrm{p}} \geq 10$. The system states are bounded by $\|\bar{x}_k\|_\infty \leq 10$ and the change in input by $\|\Delta u_k\|_\infty \leq 0.5$. The additive disturbance acts on all states with $\|w_k\|_\infty \leq 0.01$ and the measurement noise belongs to the set $\|e_k\|_\infty \leq 0.01$. In simulation, the disturbances are realized as pseudorandom sequences of the respective vertices.

The true parameter vectors are $\theta^* = \begin{bmatrix} 0 & 1 \end{bmatrix}'$ and $\theta_{\mathrm{C}}^* = 0$. The initial parameter set is a hypercube with $M_\Theta = \begin{bmatrix} I_2 & -I_2 \end{bmatrix}'$ and $b_{\Theta,0} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}'$. The initial parameter vectors are chosen as $\hat{\theta}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}'$ and $\hat{\theta}_{\mathrm{C},0} = 1$. Weighting matrices are defined as $Q = 10 I_{n_{\mathrm{x}}}$, $R = I_{n_{\mathrm{u}}}$, $T_{\mathrm{x}} = 1e2P$ and for the pre-stabilizing gain $K = \begin{bmatrix} -0.058 & -0.031 & -0.0144 & -0.9234 \end{bmatrix}$ holds. A prediction horizon of $N = 6$ is used.

First, the effect of the adaptive algorithms is evaluated. A reference signal r is used which starts at 0.5, then increases to 0.9 and 1 as depicted in Figure 3. It should be noted that the reference 1 is also a output constraint. Hence, it is desired that the system output will be steered as close as possible to the reference without reaching it. The proposed control algorithm from Section 6 without any adaptive elements converges to a state much higher than the desired reference of 0.5. This is due to mismatches in the nominal model which leads to a wrong target state calculation. When the reference increases to 0.9, the output of the non-adaptive algorithm only increases slightly. When the reference increases to 1, no increase

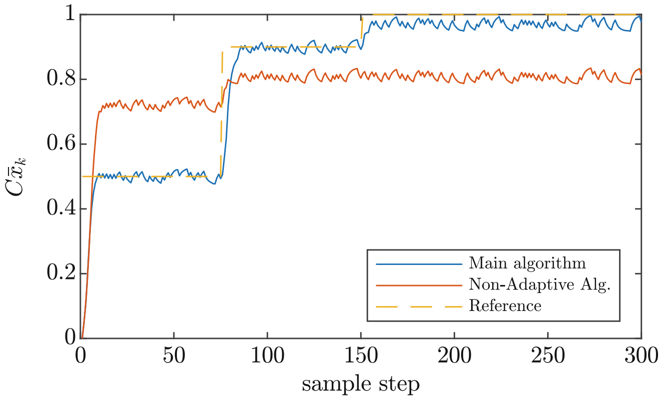

FIGURE 3 Reference trajectory and system output for Algorithm **A** with and without adaptive elements.

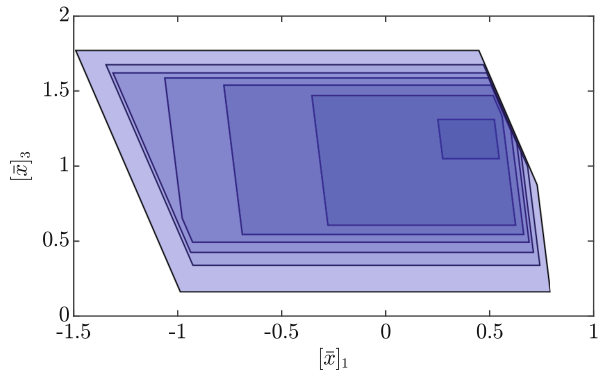

FIGURE 4 Tube evolution over the prediction horizon at sample step 100.

in the output is visible. This is due to the large uncertainty which keeps the artificial target state $\hat{x}_{\mathrm{T},k}$ far away from the desired set-point $x_{\mathrm{T},k}$ in order to ensure satisfaction of the output constraints. In contrast, the proposed algorithm yields a system output which is close to the reference value considering the magnitude of the additive disturbance. The parameter estimate converges to the true values which yields a correct steady-state target and an artificial target state close to the constraints.

The state tube at sample step 100 is given in Figure 4, where areas with lighter colors are tube cross-sections at the end of the prediction horizon. In addition, the four-dimensional tube-cross sections are projected on the first and third state. The first tube cross-section is the smallest one and almost rectangular. The following tube cross-sections increase in area due to the uncertain state evolution and share a noticeable boundary at the upper right side due to the output constraints. In addition, the shape of the tube cross-sections changes over time as each half space of the polytope is scaled individually by $\alpha_{l|k}$.

In Figure 5, the system operates close to the output constraints at sample step 200 and the resulting state tube is given. Compared to Figure 4, the area of the tube cross-sections is much smaller which leads to a tight state evolution close to the system constraints.

The same setup is repeated to evaluate the algorithm for systems with relative parameter bounds which are given by $M_{\mathrm{B}} = \begin{bmatrix} I_2 & -I_2 \end{bmatrix}'$, $b_{\mathrm{B}} = \begin{bmatrix} 0.05 & 0.05 & 0.05 & 0.05 \end{bmatrix}'$ and $\theta_{\mathrm{B}} = \begin{bmatrix} 0.05 & 0.95 \end{bmatrix}'$. The set-estimation of the variant with relative bounds is shown in Figure 6. The shaded areas indicate the parameter sets Θ_k and the red crosses are the parameter vector estimate $\hat{\theta}_k$. The estimated parameter set does converge to the minimal one but remains slightly larger. In addition, the parameter vector estimate $\hat{\theta}_k$ does not converge to the central parameter θ_{B} but stays at the boundary of $\mathbb{B}$.

This result also affects the tracking performance as shown in Figure 7. The reference is not tracked without offset on average in contrast to Figure 3. In addition, the distance from the output constraint to the output signal is larger when the reference equals to the output constraint.

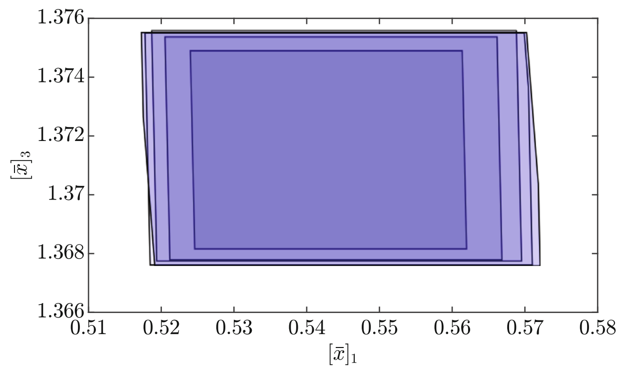

FIGURE 5 Tube evolution over the prediction horizon at sample step 200.

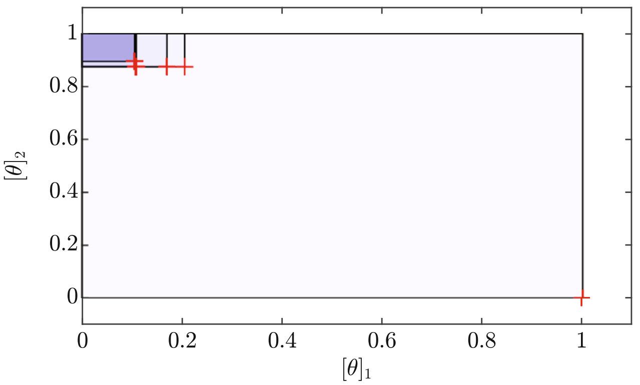

FIGURE 6 Estimated parameter sets with relative bounds for fist 26 iterations.

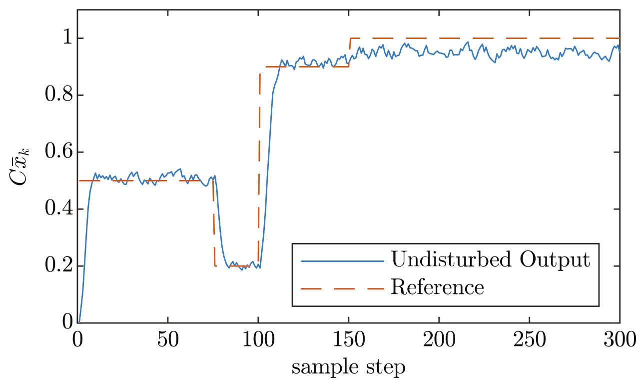

FIGURE 7 Reference trajectory and system output for algorithm from Section 7.

The simulations were run in MATLAB using “quadprog” with the “interior-point-convex”- algorithm on a Intel Core i7-6820. The maximum computation time for the quadratic program of the main algorithm is 0.09s and the average computation time is 0.04s.

9 | CONCLUDING REMARKS

Robust adaptive model predictive control algorithms for tracking of piece-wise constant reference signals have been presented which provide l_2-stability and recursive constraint satisfaction. The adaption of the parameter set decreases the uncertainty online and a least-mean-squares filter is used to update a nominal model for time-invariant systems. This increases tracking performance and operation close to the system constraints becomes less conservative. An artificial target state and input are utilized to enhance recursive feasibility if the nominal system model or reference signal change. This online-defined target state is especially useful in combination with the proposed terminal set which depends on the estimated parameter set. When the uncertainty decreases, target states become feasible which had been initially infeasible.

The assumption on time-invariance of the plant is removed in a proposed variant of the algorithm. Instead, known relative bounds but an unknown center of these bounds are assumed. Convergence of the parameter estimation to the true bounds has been shown in a numerical example.

Future work may be devoted to combine robust adaptive tracking MPC with output feedback MPC when no state measurements are available. Another interesting area is the incorporation of estimates about future variations of the plant or uncertainty.

CONFLICT OF INTEREST STATEMENT

There is no conflict of interest to declare.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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APPENDIX A. PROOFS FOR PROPOSITIONS

Proof of Proposition 2. It holds that

$$
\begin{aligned}
\|a+b\|_Q^2 &= |\|a\|_Q^2 + \|b\|_Q^2 + a'Qb + b'Qa| \\
&\leq \|a\|_Q^2 + \|b\|_Q^2 + \frac{1}{2\epsilon}\|(a'Q^{0.5})'\|^2 + \frac{\epsilon}{2}\|(Q^{0.5}b)\|^2 + \frac{\epsilon}{2}\|(b'Q^{0.5})'\|^2 + \frac{1}{2\epsilon}\|(Q^{0.5}a)\|^2 \\
&= \|a\|_Q^2 + \|b\|_Q^2 + \frac{1}{2\epsilon}\|a\|_Q^2 + \frac{\epsilon}{2}\|b\|_Q^2 + \frac{\epsilon}{2}\|b\|_Q^2 + \frac{1}{2\epsilon}\|a\|_Q^2 \\
&= \left(1+\frac{1}{\epsilon}\right)\|a\|_Q^2 + (1+\epsilon)\|b\|_Q^2 .
\end{aligned}
$$

The first equality follows from positive definiteness of Q. The first inequality follows from the triangle inequality, Cauchy-Schwarz inequality and from Young's inequality. The second equality follows from the symmetry of Q and $Q^{0.5}$ and from the definition of $\|\cdot\|^2$.

Proof of Proposition 7. First, the case $l = 0$ is considered with

$$\|\delta x_{0|k+1}\|^2 = \|\hat{x}_{0|k+1} - \hat{x}_{1|k}\|_Q^2$$

$$
\begin{aligned}
&= \|A(\hat{\theta}^*)x_{0|k} + B(\hat{\theta}^*)(c_{0|k} + K(x_{0|k} - \hat{x}_{\mathrm{T},k}) + \hat{u}_{\mathrm{T},k}) + w_k - \left(A(\hat{\theta}_k)x_{0|k} + B(\hat{\theta}_k)(c_{0|k} + K(x_{0|k} - \hat{x}_{\mathrm{T},k}) + \hat{u}_{\mathrm{T},k})\right)\|_Q^2 \\
&= \|\sum_i [(\hat{\theta}^* - \hat{\theta}_k)]_i \Delta A^{(i)} x_{0|k} + \sum_i [(\hat{\theta}^* - \hat{\theta}_k)]_i \Delta B^{(i)}(c_{0|k} + K(x_{0|k} - \hat{x}_{\mathrm{T},k}) + \hat{u}_{\mathrm{T},k}) + w_k\|_Q^2 \\
&\leq 2\|\sum_i [(\hat{\theta}^* - \hat{\theta}_k)]_i \Delta A^{(i)} x_{0|k}\|_Q^2 + 4\|\sum_i [(\hat{\theta}^* - \hat{\theta}_k)]_i \Delta B^{(i)}(c_{0|k} + K(x_{0|k} - \hat{x}_{\mathrm{T},k}) + \hat{u}_{\mathrm{T},k})\|_Q^2 + 4\|w_k\|_Q^2 \\
&\leq \sigma_{\delta_{\mathrm{x}}} \|\hat{\theta}^* - \hat{\theta}_k\|_Q^2 + \sigma_{\mathrm{w},1}\|w_k\|_Q^2 .
\end{aligned}
$$

The first inequality results from using Proposition 2. The second inequality follows from the existence of constants $\sigma_{\delta_{\mathrm{x}}}, \sigma_{\mathrm{w},1} \in \mathbb{R}_{>0}$ due to compactness of $\mathcal{X}, \mathcal{U}$ and $\mathcal{Z}$.

Now, the case $1 \leq l \leq N$ is considered for a feasible new target state and nominal states:

$$
\begin{aligned}
\|\delta x_{l|k+1}\|_Q^2 &= \|\hat{x}_{l|k+1} - \hat{x}_{l+1|k}\|_Q^2 \\
&= \|A(\hat{\theta}_{k+1})\hat{x}_{l-1|k+1} + B(\hat{\theta}_{k+1})(c_{l-1|k+1} + K(\hat{x}_{l-1|k+1} - \hat{x}_{\mathrm{T},k+1}) + \hat{u}_{\mathrm{T},k+1}) \\
&\quad - \left(A(\hat{\theta}_k)\hat{x}_{l|k} + B(\hat{\theta}_k)(c_{l|k} + K(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k}) + \hat{u}_{\mathrm{T},k})\right)\|_Q^2 \\
&= \|\left(A(\hat{\theta}_{k+1}) + B(\hat{\theta}_{k+1})K\right)\left(\hat{x}_{l-1|k+1} - \hat{x}_{l|k}\right) + \sum_i [\hat{\theta}_{k+1} - \hat{\theta}_k]_i (\Delta A^{(i)}\hat{x}_{l|k} + \Delta B^{(i)} c_{l|k} + \Delta B^{(i)} K\hat{x}_{l|k}) \\
&\quad + B(\hat{\theta}_{k+1})(c_{l-1|k+1} - c_{l|k} - K(\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}) + \hat{u}_{\mathrm{T},k+1} - \hat{u}_{\mathrm{T},k}) + \sum_i [\hat{\theta}_{k+1} - \hat{\theta}_k]_i \Delta B^{(i)}(\hat{u}_{\mathrm{T},k} - K\hat{x}_{\mathrm{T},k})\|_Q^2 \\
&\leq \sigma_1\|\delta x_{l-1|k+1}\|_Q^2 + \sigma_2\|\hat{\theta}_{k+1} - \hat{\theta}_k\|_Q^2 + \sigma_3\|\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}\|_Q^2 + \sigma_4\|\hat{u}_{\mathrm{T},k+1} - \hat{u}_{\mathrm{T},k}\|_Q^2 + \sigma_5\|\delta c_{l-1|k+1}\|_Q^2 \\
&\leq \sigma_1\|\delta x_{l-1|k+1}\|_Q^2 + \sigma_{\theta,1}\|\hat{\theta}_{k+1} - \hat{\theta}_k\|_Q^2 + \sigma_{\theta,2}\|\hat{\theta}_{\mathrm{C},k+1} - \hat{\theta}_{\mathrm{C},k}\|_Q^2 \\
&\leq \sigma_1(\sigma_1\|\delta x_{l-2|k+1}\|_Q^2 + \sigma_{\theta,1}\|\hat{\theta}_{k+1} - \hat{\theta}_k\|_Q^2 + \sigma_{\theta,2}\|\hat{\theta}_{\mathrm{C},k+1} - \hat{\theta}_{\mathrm{C},k}\|_Q^2) + \sigma_{\theta,1}\|\hat{\theta}_{k+1} - \hat{\theta}_k\|_Q^2 + \sigma_{\theta,2}\|\hat{\theta}_{\mathrm{C},k+1} - \hat{\theta}_{\mathrm{C},k}\|_Q^2 \\
&\leq \sigma_1^l \sigma_{\delta_{\mathrm{x}}}\|\theta^* - \hat{\theta}_k\|_Q^2 + \sigma_1^l \sigma_{\mathrm{w},1}\|w_k\|_Q^2 + \sum_{i=0}^{l-1} \sigma_1^i (\sigma_{\theta,1}\|\hat{\theta}_{k+1} - \hat{\theta}_k\|_Q^2 + \sigma_{\theta,2}\|\hat{\theta}_{\mathrm{C},k+1} - \hat{\theta}_{\mathrm{C},k}\|_Q^2) .
\end{aligned}
$$

The first inequality follows from Proposition 2 and there exist suitable positive constants σ_1 to $\sigma_5 \in \mathbb{R}_{>0}$ due to compactness of $\mathcal{X}, \mathcal{U}, \mathcal{Z}$ and Θ.

For the second inequality, differences $\delta c_{l-1|k+1} = c_{l-1|k+1} - c_{l|k}$, $\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}$, $\hat{u}_{\mathrm{T},k+1} - \hat{u}_{\mathrm{T},k}$ can be set to zero for constant parameters as the shifted sequence of the solution at time k can be used as a feasible candidate solution at time $k+1$. For varying parameters, the shifted sequence might not be feasible due to a new target state. Consider two feasible sub-optimal solutions $\tilde{\mathbf{d}}_{k+1}(x_{k+1}, \Theta_{k+1} = \Theta_k, \Theta_{\mathrm{C},k+1} = \Theta_{\mathrm{C},k}, \hat{\theta}_{k+1})$ and $\tilde{\mathbf{d}}_{k+1}(x_{k+1}, \Theta_{k+1} = \Theta_k, \Theta_{\mathrm{C},k+1} = \Theta_{\mathrm{C},k}, \hat{\theta}_k)$ where the latter consists of the shifted sequence of the previous solution. Denote $c(\hat{\theta}_{k+1})_{l|k+1}$ and $c(\hat{\theta}_k)_{l|k+1}$ as the elements from the respective solution vectors. Then it holds that $\|\delta c_{l-1|k+1}\|^2 = \|c(\hat{\theta}_{k+1})_{l-1|k+1} - c(\hat{\theta}_k)_{l-1|k+1} + c(\hat{\theta}_k)_{l-1|k+1} - c(\hat{\theta}_k)_{l|k}\|^2 = \|c(\hat{\theta}_{k+1})_{l-1|k+1} - c(\hat{\theta}_k)_{l-1|k+1}\|^2 \leq L(\|\hat{\theta}_{k+1} - \hat{\theta}_k\|^2 + \|\hat{\theta}_{\mathrm{C},k+1} - \hat{\theta}_{\mathrm{C},k}\|^2)$ due to Assumption 7, $c(\hat{\theta}_k)_{l-1|k+1} = c(\hat{\theta}_k)_{l|k}$ and as $x_{\mathrm{T},k}$ is a Lipschitz continuous function of $\hat{\theta}_k, \hat{\theta}_{\mathrm{C},k}$. The same reasoning can be applied for the differences $\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}$ and $\hat{u}_{\mathrm{T},k+1} - \hat{u}_{\mathrm{T},k}$. The third and fourth inequalities follow from recursion.

APPENDIX B. PROOF OF THEOREMS

B.1 Feasibility proof

The same shifted sequence as in Section 6 is considered. The proof is given here for the terminal set dynamics and can be easily adapted for the tube dynamics (43c).

$$
\begin{aligned}
&[H(\bar{\theta}_{k+1}^{(j)})\alpha_{N|k} + \bar{w} + V(I_{n_{\mathrm{x}}} - \Phi(\theta_{k+1}^{(j)}))\hat{x}_{\mathrm{T},k}]_i \\
&\quad = [H(\bar{\theta}_{k+1}^{(j)})]_i \alpha_{N|k} + [\bar{w}]_i + [V(I_{n_{\mathrm{x}}} - \Phi(\theta_{k+1}^{(j)}))]_i \hat{x}_{\mathrm{T},k} \\
&\quad = \left[\sum_{l=1}^m \lambda^{(l)} H(\bar{\theta}_k^{(l)})\right]_i \alpha_{N|k} + [\bar{w}]_i + \left[V\left(\sum_{l=1}^m \lambda^{(l)}(I_{n_{\mathrm{x}}} - \Phi(\theta_k^{(l)}))\right)\right]_i \hat{x}_{\mathrm{T},k}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max_l [H(\bar{\theta}_k^{(l)})]_i \alpha_{N|k} + [\bar{w}]_i + [V]_i \left(I_{n_x} - \Phi(\theta_k^{(l)}) \right) \hat{x}_{\mathrm{T},k} \\
&= [\alpha_{N|k}]_i
\end{aligned}
$$

The second equality follows from convexity. The first inequality follows from the linear programming principle that a maximum value lies on vertex of a convex set. The last equality follows from the consideration of all vertices in (43f).

B.2 Stability proof

Lemma 2. It holds that

$$
\begin{aligned}
&V_N(x_{0|k+1}, \hat{\theta}_{k+1}, \Theta_{k+1}, \Theta_{\mathrm{C},k+1}, x_{\mathrm{T},k+1}) - V_N(x_{0|k}, \hat{\theta}_k, \Theta_k, \Theta_{\mathrm{C},k}, x_{\mathrm{T},k}) \\
&\leq -\sigma_0 \|\hat{x}_{0|k} - \hat{x}_{\mathrm{T},k}\|_2 + \sigma_{\theta,0} \|\hat{\theta}_k - \theta^*\|_2 + \sigma_{\mathrm{w},0} \|w_k\|_2 + \sigma_{\theta_{\mathrm{C}},0} \|\hat{\theta}_{\mathrm{C},k} - \theta_{\mathrm{C}}^*\|_2 + \sigma_{\mathrm{e},0} \|e_k\|_2 + \|\hat{x}_{\mathrm{T},k+1} - x_{\mathrm{T},k+1}\|_{T_x}^2 - \|\hat{x}_{\mathrm{T},k} - x_{\mathrm{T},k}\|_{T_x}^2 .
\end{aligned}
$$

with constants $\sigma_0, \sigma_{\theta,0}, \sigma_{\theta_{\mathrm{C}},0}, \sigma_{\mathrm{w},0}, \sigma_{\mathrm{e},0} \in \mathbb{R}_{>0}$.

Proof.

$$
\begin{aligned}
&V_N(x_{0|k+1}, \hat{\theta}_{k+1}, \Theta_{k+1}, \Theta_{\mathrm{C},k+1}, x_{\mathrm{T},k+1}) - V_N(x_{0|k}, \hat{\theta}_k, \Theta_k, \Theta_{\mathrm{C},k}, x_{\mathrm{T},k}) \\
&\leq \tilde{V}_N(x_{0|k+1}, \hat{\theta}_k, \Theta_{k+1}, \Theta_{\mathrm{C},k}, x_{\mathrm{T},k+1}) - V_N(x_{0|k}, \hat{\theta}_k, \Theta_k, \Theta_{\mathrm{C},k}, x_{\mathrm{T},k}) \\
&= \sum_{l=0}^{N-1} \|\hat{x}_{l|k+1} - \hat{x}_{\mathrm{T},k+1}\|_Q^2 + \|c_{l|k+1} + K(\hat{x}_{l|k+1} - \hat{x}_{\mathrm{T},k+1})\|_R^2 + \|\hat{x}_{N|k+1} - \hat{x}_{\mathrm{T},k+1}\|_P^2 + \|\hat{x}_{\mathrm{T},k+1} - x_{\mathrm{T},k+1}\|_{T_x}^2 - \|\hat{x}_{0|k} - \hat{x}_{\mathrm{T},k}\|_Q^2 \\
&\quad - \|c_{0|k} + K(\hat{x}_{0|k} - \hat{x}_{\mathrm{T},k})\|_R^2 - \sum_{l=1}^{N-1} \|\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k}\|_Q^2 + \|c_{l|k} + K(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k})\|_R^2 - \|\hat{x}_{N|k} - \hat{x}_{\mathrm{T},k}\|_P^2 - \|\hat{x}_{\mathrm{T},k} - x_{\mathrm{T},k}\|_{T_x}^2 \\
&= -\|\hat{x}_{0|k} - \hat{x}_{\mathrm{T},k}\|_Q^2 \\
&\quad + \sum_{l=0}^{N-2} \|\hat{x}_{l+1|k} + \delta x_{l|k+1} - \hat{x}_{\mathrm{T},k} + \hat{x}_{\mathrm{T},k} - \hat{x}_{\mathrm{T},k+1}\|_Q^2 + \|c_{l+1|k} + \delta c_{l|k+1} + K(\hat{x}_{l+1|k} + \delta x_{l|k+1} - \hat{x}_{\mathrm{T},k} + \hat{x}_{\mathrm{T},k} - \hat{x}_{\mathrm{T},k+1})\|_R^2 \\
&\quad - \sum_{l=1}^{N-1} \|\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k}\|_Q^2 + \|c_{l|k} + K(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k})\|_R^2 \\
&\quad + \|\hat{x}_{N|k} + \delta x_{N-1|k+1} - \hat{x}_{\mathrm{T},k} + \hat{x}_{\mathrm{T},k} - \hat{x}_{\mathrm{T},k+1}\|_Q^2 + \|\delta c_{N-1|k+1} + K(\hat{x}_{N|k} + \delta x_{N-1|k+1} - \hat{x}_{\mathrm{T},k} + \hat{x}_{\mathrm{T},k} - \hat{x}_{\mathrm{T},k+1})\|_R^2 \\
&\quad + \|\hat{x}_{N+1|k} + \delta x_{N|k+1} - \hat{x}_{\mathrm{T},k} + \hat{x}_{\mathrm{T},k} - \hat{x}_{\mathrm{T},k+1}\|_P^2 - \|\hat{x}_{N|k} - \hat{x}_{\mathrm{T},k}\|_P^2 \\
&\quad + \|\hat{x}_{\mathrm{T},k+1} - x_{\mathrm{T},k+1}\|_{T_x}^2 - \|\hat{x}_{\mathrm{T},k} - x_{\mathrm{T},k}\|_{T_x}^2 \\
&\leq -\|\hat{x}_{0|k} - \hat{x}_{\mathrm{T},k}\|_Q^2 + \epsilon \sum_{l=1}^{N-1} \|\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k}\|_Q^2 + \|c_{l|k} + K(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k})\|_R^2 + (N-1)\left(1 + \frac{1}{\epsilon}\right)(1+\epsilon)\|\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}\|_Q^2 \\
&\quad + (N-1)\left(1 + \frac{1}{\epsilon}\right)(1+\epsilon)^2 \|\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}\|_{K'RK}^2 \\
&\quad + \left(1 + \frac{1}{\epsilon}\right)(1+\epsilon)\|\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}\|_P^2 + (1+\epsilon)\left(1 + \frac{1}{\epsilon}\right)\|\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}\|_{Q+K'RK}^2 \\
&\quad + (1+\epsilon)\|\hat{x}_{N|k} - \hat{x}_{\mathrm{T},k}\|_{Q+K'RK}^2 + +(1+\epsilon)\|\hat{x}_{N+1|k} - \hat{x}_{\mathrm{T},k}\|_P^2 - \|\hat{x}_{N|k} - \hat{x}_{\mathrm{T},k}\|_P^2 + (1+\epsilon)^2 \|\delta x_{N|k+1}\|_P^2 \\
&\quad + \left(1 + \frac{1}{\epsilon}\right)^2 \sum_{l=0}^{N-1} \|\delta x_{l|k+1}\|_{Q+K'RK}^2 + \left(1 + \frac{1}{\epsilon}\right)^2 (1+\epsilon) \sum_{l=0}^{N-1} \|\delta c_{l|k+1}\|_R^2 + \|\hat{x}_{\mathrm{T},k+1} - x_{\mathrm{T},k+1}\|_{T_x}^2 - \|\hat{x}_{\mathrm{T},k} - x_{\mathrm{T},k}\|_{T_x}^2 \\
&\leq -\|\hat{x}_{0|k} - \hat{x}_{\mathrm{T},k}\|_Q^2 + \epsilon \left(\sum_{l=1}^{N-1} \|\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k}\|_Q^2 + \|c_{l|k} + K(\hat{x}_{l|k} - \hat{x}_{\mathrm{T},k})\|_R^2 + \|x_{N|k} - \hat{x}_{\mathrm{T},k}\|_P^2 \right) \\
&\quad + (1+\epsilon)\left(-\|\hat{x}_{N|k} - \hat{x}_{\mathrm{T},k}\|_P^2 + \|\hat{x}_{N|k} - \hat{x}_{\mathrm{T},k}\|_{\Phi(\hat{\theta}_k)' P \Phi(\hat{\theta}_k)}^2 + \|\hat{x}_{N|k} - \hat{x}_{\mathrm{T},k}\|_{Q+K'RK}^2 \right) \\
&\quad + (1+\epsilon)\left(1 + \frac{1}{\epsilon}\right) N \|\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}\|_{Q+K'RK}^2 + \left(1 + \frac{1}{\epsilon}\right)(1+\epsilon)\|\hat{x}_{\mathrm{T},k+1} - \hat{x}_{\mathrm{T},k}\|_P^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(1+\frac{1}{\epsilon}\right)^2\sigma_{\mathrm{QP}}\sum_{l=0}^{N}\|\delta x_{l|k+1}\|^2+\left(1+\frac{1}{\epsilon}\right)^2(1+\epsilon)\sigma_{\mathrm{R}}\sum_{l=0}^{N-1}\|\delta c_{l|k+1}\|^2\\
&+\|\hat{x}_{\mathrm{T},k+1}-x_{\mathrm{T},k+1}\|_{T_{\mathrm{x}}}^2-\|\hat{x}_{\mathrm{T},k}-x_{\mathrm{T},k}\|_{T_{\mathrm{x}}}^2\\
\leq{}&-\sigma_{\mathrm{Q}}\|\hat{x}_{0|k}-\hat{x}_{\mathrm{T},k}\|^2+\epsilon\bar{V}_N(x_{0|k},\hat{\theta}_k,\Theta_k,\Theta_{\mathrm{C},k},\hat{x}_{\mathrm{T},k})+\left(1+\frac{1}{\epsilon}\right)^2\sigma_{\mathrm{QP}}\sum_{l=0}^{N}\|\delta x_{l|k+1}\|^2+\left(1+\frac{1}{\epsilon}\right)(1+\epsilon)\sigma_{\mathrm{R}}\sum_{l=0}^{N-1}\|\delta c_{l|k+1}\|^2\\
&+(N+1)\left(1+\frac{1}{\epsilon}\right)(1+\epsilon)\sigma_{\mathrm{QP}}\|\hat{x}_{\mathrm{T},k}-\hat{x}_{\mathrm{T},k+1}\|^2+\|\hat{x}_{\mathrm{T},k+1}-x_{\mathrm{T},k+1}\|_{T_{\mathrm{x}}}^2-\|\hat{x}_{\mathrm{T},k}-x_{\mathrm{T},k}\|_{T_{\mathrm{x}}}^2\\
\leq{}&-\sigma_0\|\hat{x}_{0|k}-\hat{x}_{\mathrm{T},k}\|^2+\sigma_{11}\sum_{l=0}^{N}\left[\sigma_1^l\sigma_{\delta_{\mathrm{x}}}\|\hat{\theta}_k-\theta^*\|^2+\sigma_1^l\sigma_{\mathrm{w},1}\|w_k\|^2+\sum_{i=0}^{l-1}\sigma_1^i(\sigma_{\theta,1}\|\hat{\theta}_{k+1}-\hat{\theta}_k\|^2+\sigma_{\theta,2}\|\hat{\theta}_{\mathrm{C},k+1}-\hat{\theta}_{\mathrm{C},k}\|_Q^2)\right]\\
&+\sigma_{\theta,3}(\|\hat{\theta}_{k+1}-\hat{\theta}_k\|^2+\|\hat{\theta}_{\mathrm{C},k+1}-\hat{\theta}_{\mathrm{C},k}\|^2)+\|\hat{x}_{\mathrm{T},k+1}-x_{\mathrm{T},k+1}\|_{T_{\mathrm{x}}}^2-\|\hat{x}_{\mathrm{T},k}-x_{\mathrm{T},k}\|_{T_{\mathrm{x}}}^2\\
\leq{}&-\sigma_0\|\hat{x}_{0|k}-\hat{x}_{\mathrm{T},k}\|^2+\sigma_{\theta,0}\|\hat{\theta}_k-\theta^*\|^2+\sigma_{\mathrm{w},0}\|w_k\|^2+\sigma_{\theta_{\mathrm{C}},0}\|\hat{\theta}_{\mathrm{C},k}-\theta_{\mathrm{C}}^*\|^2+\sigma_{\mathrm{e},0}\|e_k\|^2\\
&+\|\hat{x}_{\mathrm{T},k+1}-x_{\mathrm{T},k+1}\|_{T_{\mathrm{x}}}^2-\|\hat{x}_{\mathrm{T},k}-x_{\mathrm{T},k}\|_{T_{\mathrm{x}}}^2.
\end{aligned}
$$

The first inequality is due to the consideration of a feasible, possibly sub-optimal solution $\tilde{V}_N(x_{0|k+1},\hat{\theta}_{k+1},\Theta_k,\Theta_{\mathrm{C},k},x_{\mathrm{T},k+1})$. The second inequality follows from repetitive application of Proposition 2 and the third inequality from upper bounding $\|\cdot\|_P^2$ and $\|\cdot\|_{Q+KR'K}^2$ by $\sigma_{\mathrm{QP}}\|\cdot\|^2$ and $\|\cdot\|_R^2$ by $\sigma_{\mathrm{R}}\|\cdot\|^2$. The fourth inequality follows Lyapunov inequality (25). In addition, $\sum_{l=1}^{N-1}\|\hat{x}_{l|k}-\hat{x}_{\mathrm{T},k}\|_Q^2+\|c_{l|k}+K(\hat{x}_{l|k}-\hat{x}_{\mathrm{T},k})\|_R^2+\|x_{N|k}-\hat{x}_{\mathrm{T},k}\|_P^2$ is the cost function of standard regulation problem to $\hat{x}_{\mathrm{T},k}$ and written as $\bar{V}_N(x_{0|k},\hat{\theta}_k,\Theta_k,\Theta_{\mathrm{C},k},\hat{x}_{\mathrm{T},k})$.

Following the proof of Theorem 14 in Lorenzen et al.,[7] the cost $\bar{V}_N(x_{0|k},\hat{\theta}_k,\Theta_k,\Theta_{\mathrm{C},k},\hat{x}_{\mathrm{T},k})$ is a piece-wise quadratic function of $\hat{x}_{0|k}-\hat{x}_{\mathrm{T},k}$[36] for every $\theta\in\Theta_0$ and there exists a quadratic function which gives an upper bound on the feasible set. Hence, on the feasible set there exists a constant σ_{V} such that $\bar{V}_N(x_{0|k},\hat{\theta}_k,\Theta_k,\Theta_{\mathrm{C},k},\hat{x}_{\mathrm{T},k})\leq\sigma_{\mathrm{V}}\|\hat{x}_{0|k}-\hat{x}_{\mathrm{T},k}\|^2$ holds due to compactness of Θ_k, $\Theta_{\mathrm{C},k}$ and $\mathcal{Z}$.

The fifth inequality is due to the existence of constants $\epsilon,\sigma_0\in\mathbb{R}_{>0}$ such that $-\sigma_{\mathrm{Q}}\|\hat{x}_{0|k}-\hat{x}_{\mathrm{T},k}\|^2+\epsilon\sigma_{\mathrm{V}}\|\hat{x}_{0|k}-\hat{x}_{\mathrm{T},k}\|^2\leq-\sigma_0\|\hat{x}_{0|k}-\hat{x}_{\mathrm{T},k}\|^2$ holds. In addition, the terms containing $\delta c_{l|k}$ and $\hat{x}_{\mathrm{T},k+1}-\hat{x}_{\mathrm{T},k}$ are bounded as in the proof for Proposition 7 and the terms containing $\delta x_{l|k}$ by Proposition 7.

The last inequality follows from Propositions 3 and 4. ■