Optimizing Bus Line Infrastructure and Tariffs in Public Transport

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Abstract

Designing attractive public transport services is an important task towards the Paris Climate Accords and the 17 Sustainable Development Goals. Both the interests of passengers and the financial concerns of transport operators must be taken into account. In this thesis, models for infrastructure optimization of a bus rapid transit line and for tariff optimization are developed.

Motivated by the development of a new bus rapid transit (BRT) line around Copenhagen, Part I of this thesis deals with the infrastructure optimization for such a BRT line. The municipalities that are involved in the BRT line have to decide which segments of the route of the BRT line should be upgraded with, for example, dedicated bus lanes and priority at intersections. There is a trade-off between attracting as many new passengers as possible by upgrading the bus infrastructure and at the same time keeping the required budget small. This problem is formulated as a bi-objective model. An ϵ-constraint-based solution method is developed that can compute the complete Pareto front composed of the number of passengers and the budget. A theoretical analysis is performed for the bi-objective model as well as for the single-objective optimization problem that is solved within the algorithm. The model is applied and evaluated in computational experiments on artificial instances as well as on realistic instances based on the case study of the BRT line around Copenhagen.

Part II deals with two models for the optimization of flat, distance and zone tariffs in public transport. The first model minimizes the absolute deviation from given reference prices. The resulting problems for flat and distance tariffs can be identified as median problems, which allows for linear solvability. For zone tariffs, different variants are compared with each other, and it is shown that the problem is in general NP-hard for zone tariffs. The second model examines the trade-off between the revenue and the number of passengers willing to use public transport depending on the fare. While the Pareto front of this bi-objective problem can be computed using the ϵ-constraint method, additional algorithms are developed for the optimization of flat, distance and zone tariffs that exploit the structure of the tariffs. Computational experiments for flat and distance tariffs show a good performance of the corresponding algorithms, especially for instances with more than one non-dominated point.

Zusammenfassung

Die Gestaltung eines attraktiven öffentlichen Verkehrsangebots ist eine wichtige Aufgabe, um das Pariser Klimaabkommen und die 17 Nachhaltigkeitsziele umzusetzen. Dabei müssen sowohl die Interessen der Passagiere als auch die finanziellen Belange der Verkehrsbetreiber berücksichtigt werden. In dieser Arbeit werden Modelle für die Infrastrukturplanung einer Schnellbuslinie und die Gestaltung von Tarifen im öffentlichen Verkehr entwickelt.

Motiviert durch das Projekt einer neuen Schnellbuslinie bei Kopenhagen beschäftigt sich Teil I der Arbeit mit der Infrastrukturplanung für eine solche Buslinie. Die Gemeinden, durch die die Schnellbuslinie führt, müssen dabei entscheiden, welche Straßenabschnitte beispielsweise mit Busspuren und Vorrangschaltungen an Lichtsignalanlagen ausgestattet werden sollen. Es ergibt sich ein Zielkonflikt, möglichst viele neue Passagiere durch einen guten Ausbau der Businfrastruktur zu gewinnen und gleichzeitig das benötigte Budget gering zu halten. Diese Fragestellung wird als bikriterielles Modell formuliert. Mithilfe eines Algorithmus basierend auf der ϵ-beschränkten Methode kann die vollständige Pareto-Front, die sich aus der Anzahl an Passagieren und dem Budget zusammensetzt, berechnet werden. Es wird eine theoretische Analyse für das bikriterielle Modell sowie für das einkriterielle Optimierungsproblem, das innerhalb des Algorithmus gelöst werden muss, durchgeführt. Das Modell wird in Rechenexperimenten auf künstliche sowie realistische Instanzen basierend auf der Fallstudie bei Kopenhagen angewendet und evaluiert.

Teil II befasst sich mit zwei Modellen für die Gestaltung von Einheits-, Distanz- und Zonentarifen im öffentlichen Verkehr. Das erste Modell minimiert die absolute Abweichung von gegebenen Referenzpreisen. Die resultierenden Probleme für Einheits- und Distanztarife können als Median-Probleme interpretiert werden, was eine lineare Laufzeit zum Lösen ermöglicht. Für Zonentarife werden verschiedene Varianten miteinander verglichen und es wird gezeigt, dass das Tarifproblem für Zonentarife im Allgemeinen NP-schwer ist. Das zweite Modell untersucht den Trade-off zwischen den Einnahmen und der Anzahl an Passagieren, die je nach Preis bereit sind, den öffentlichen Verkehr zu nutzen. Während die Pareto-Front dieses bikriteriellen Problems ebenfalls mit der ϵ-beschränkten Methode berechnet werden kann, werden zusätzliche Algorithmen für die Optimierung von Einheits-, Distanz- und Zonentarifen entwickelt, die die Struktur der Tarife ausnutzen. Rechenexperimente für Einheits- und Distanztarife zeigen eine gute Performance der entsprechenden Algorithmen, besonders für Instanzen mit mehr als einer Pareto-Lösung.

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Chapter 1

Introduction

The transition towards sustainable transport modes is a major task towards the Paris Climate Accords and the 17 Sustainable Development Goals (SDGs). A report of the UITP (Union Internale des Transports Publics, English: International Association of Public Transport), published in 2023, explains that public transport can directly contribute to 14 of the 17 SDGs [UIT23]. In addition to environmental effects like a reduction of greenhouse gas emissions, public transport can, for example, be a means of enabling participation in society by increasing mobility of vulnerable groups such as children and elderly people. It is, therefore, essential to provide an attractive public transport and to improve the passengers' experiences.

To achieve these goals, optimization models can cater as a decision support tool and can be applied to point out different options and their impact. Within this thesis, two main perspectives are considered. On the one hand, the goal is to attract (new) passengers by a pleasant experience and a good and affordable public transport service. On the other hand, the financial concerns and requirements of the public transport operator have to be considered. Reducing costs or increasing revenue is of great interest. In this thesis, both objectives are applied in bi-objective and single-objective models in the settings of infrastructure optimization of a new bus rapid transit (BRT) line and of tariff optimization for public transport.

In Part I of this thesis, the optimization of investments in dedicated infrastructure for a BRT line is considered. Based on a real-world case of planning a new BRT line around Copenhagen, the question arises which segments of the route of the new BRT line should be implemented as dedicated BRT infrastructure, such as dedicated bus lanes or priority at intersections. Multiple municipalities are involved that decide about their own infrastructure investments. While the aim is to attract new passengers, there is also an interest to reduce the amount of money spent. For this bi-objective setting in Part I, a thorough introduction is given in Chapter 3.

In Part II, we consider tariff optimization for public transport. We deal with the task to determine fares, i.e., the ticket prices (based on paths), for flat tariffs, affine distance tariffs and zone tariffs, which are well-known and popular fare structures in Germany as well as throughout the world. Here, we assume that the infrastructure is fixed and means of travel exist along given passengers' paths. The problem of optimizing tariffs is approached by two different models. The first one relies on given reference prices for each origin-destination (OD) pair and the aim is to design new fares such that the absolute deviation between the newly determined fares and the given reference prices is minimized yielding a single-objective problem. The second approach starts from the premise that passengers have a limited willingness to pay and do not use public transport if the fare exceeds the willingness to pay. For this, a bi-objective model is considered that maximizes the revenue as well as the number of passengers. For Part II, an introduction to tariff optimization is given in Chapter 8.

Related Literature

We refer to Chapter 3 for an overview of the literature on the infrastructure optimization of a BRT line and to Chapter 8 for literature on tariff optimization. The design of the infrastructure and the fares are two of several steps in public transport planning. Further planning steps, which are not discussed in this thesis, are stop location, line planning, timetabling, vehicle and crew scheduling, delay management as well as intermediate steps of demand estimation and passenger routing. The general public transport planning process and the occurring optimization problems are described and reviewed, for example, in [BWZ97; Sch06; CCM15; Sch20; Gki22].

Publications

Part I is based on the publications [Hoo+22; Hoo+23; Hoo+24], which are joint work together with Rowan Hoogervorst, Evelien van der Hurk, Philine Schiewe and Anita Schöbel. For this thesis, the results have been revised and extended. The author of this thesis has been responsible for the modeling and the theoretical results in Chapters 4 and 5 and has contributed to the interpretation of the computational experiments in Chapter 6. The implementation of the computational experiments has been done by Rowan Hoogervorst.

In Part II, a paper based on Chapter 10, which is joint work together with Anita Schöbel, is currently under review [SU25]. The tariff optimization models of Chapter 10 are implemented in LinTim [Sch+; Sch+24] and are publicly available. Sections 11.2 to 11.4 are based on and extend the publication [SSU24], which is joint work together with Philine Schiewe and Anita Schöbel. For Theorem 11.10, two new proofs that offer additional insights are provided in this thesis. The computational experiments in Section 11.4 have been extended. The author of this thesis is the main author of both papers [SSU24; SU25]. Section 11.5 is newly developed for this thesis.

Chapter 2

Preliminaries

In this chapter, we introduce the notation and recall the main concepts used throughout this thesis. We assume basic knowledge on linear and integer programming and multi-objective optimization as well as graph theory.

For simplicity, we introduce the following shorthand notation for sets:

$$[n] := \{1, \dots, n\} \text{ for all } n \in \mathbb{N}_{\geq 1},$$
$$[0] := \emptyset.$$

2.1 Linear and Integer Programming

In this thesis, we apply linear and integer programming to formalize optimization problems and to obtain solution methods. In this section, we recall the main concepts of linear and integer programming that are used based on [GLS93; NW99], which provide a fundamental introduction to this topic.

First, we distinguish the following types of mathematical programs: *linear programs* (LP), which have continuous variables, *integer programs* (IP), which have discrete variables, and *mixed-integer linear programs* (MILP), which have continuous as well as discrete variables. The constraints and objective functions are linear in all cases.

It is well-known that Karmarkar's interior point method combined with an optimal rounding method is able to solve LPs in polynomial time finding an optimal solution that is an extreme point. This can also be exploited for IPs with the following property:

Definition 2.1 (Totally unimodular)**.** A matrix $A \in \mathbb{Z}^{k \times l}$ with $k, l \in \mathbb{N}_{\geq 1}$ is *totally unimodular* if the determinant of each square submatrix of A is in $\{0, 1, -1\}$.

If the coefficient matrix of an IP is totally unimodular and its right-hand side is integral, the integrality theorem of Hoffman and Kruskal states that the corresponding polyhedron is integral, i.e., all of its extreme points are integral. Hence, solving the LP-relaxation of the IP with Karmarkar's algorithm yields an integral solution in polynomial time.

A sufficient criterion for a matrix to be totally unimodular is the consecutive ones property, which is well known in the literature with various applications, see, e.g., [RS04; Sch05; Dom+08].

Definition 2.2 (Consecutive ones property)**.** A matrix $A \in \{0,1\}^{k \times l}$ with $k, l \in \mathbb{N}_{\geq 1}$ satisfies the consecutive ones property on the rows if for all rows $i \in [k]$ it holds: If $A_{i,j} = 1$ and $A_{i,j'} = 1$ for some $j, j' \in [l]$ with $j < j'$, then $A_{i,\bar{j}} = 1$ for all $j \leq \bar{j} \leq j'$.

Note that a matrix with the consecutive ones property is also called an *interval matrix*, e.g., in [NW99, Def. 2.2 in Sec. III.1.2].

Lemma 2.3. *If a matrix $A \in \{0,1\}^{k \times l}$ with $k, l \in \mathbb{N}_{\geq 1}$ satisfies the consecutive ones property, then A is totally unimodular.*

Proof. A proof is given in [NW99, Cor. 2.10 in Sec. III.1.2]. □

2.2 Bi-objective Optimization

In this section, we recall some definitions and results of bi-objective optimization. For an in-depth introduction to multi-objective optimization, we refer to [Ehr05].

A general bi-objective optimization problem is given as

$$\begin{aligned} \max \quad & \phi_1(x) \\ \max \quad & \phi_2(x) \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{2.1}$$

where $\phi_1 \colon X \to \mathbb{R}$ and $\phi_2 \colon X \to \mathbb{R}$ are the objective functions and X denotes the solution space. In particular, we aim to find a feasible solution $x \in X$ that maximizes two objectives at the same time. One or both of the maximizations could also be replaced with minimizations. For the sake of simplicity, we stick to maximization within this section.

Each feasible solution $x \in X$ induces a two-dimensional vector of objective function values. Hence, in this bi-objective setting, we need a different notion of "optimality". As usual in multi-objective optimization, we are interested in finding the Pareto front and corresponding efficient solutions. Generally speaking, we aim to find those feasible solutions that do not allow to improve one objective function without deteriorating the other.

Definition 2.4 (Domination)**.** Let a bi-objective optimization problem (2.1) be given, and let $x, x' \in X$. We say that x' *dominates* x and $(\phi_1(x'), \phi_2(x'))$ *dominates* $(\phi_1(x), \phi_2(x))$ if $\phi_1(x) \leq \phi_1(x')$ and $\phi_2(x) \leq \phi_2(x')$ and at least one inequality holds strictly.

Definition 2.5 (Efficient solution, non-dominated point and Pareto front)**.** Let a bi-objective optimization problem (2.1) be given. A feasible solution $x \in X$ is called *efficient* and its objective function value $(\phi_1(x), \phi_2(x))$ is called *non-dominated* if there does not exist another feasible solution $x' \in X$ that dominates x. The set of all non-dominated points is also called the *Pareto front*.

Note that there are several options what it means to "solve" a bi-objective optimization problem [Ser87, Sec. 2; Ehr05, p. 24]. Within this thesis, the aim of solving a bi-objective optimization problem is to determine the complete set of non-dominated points, i.e., the Pareto front. Additionally, an (arbitrary) efficient solution is sometimes stored for each non-dominated point.

In addition to efficient solutions, also *weakly efficient* solutions (Definition 2.6) are considered in multi-objective optimization. By definition, every efficient solution is also weakly efficient. When determining the Pareto front of a problem, we need to make sure that we actually obtain efficient solutions and not just weakly efficient ones.

Definition 2.6 (Weakly efficient solution, weakly non-dominated point)**.** Let a bi-objective optimization problem (2.1) be given. A feasible solution $x \in X$ is called *weakly efficient* and its objective function value $(\phi_1(x), \phi_2(x))$ is called *weakly non-dominated* if there does not exist another feasible solution $x' \in X$ such that $\phi_1(x) < \phi_1(x')$ and $\phi_2(x) < \phi_2(x')$.

2.2.1 ϵ-Constraint Method

As a well-known solution method for bi-objective optimization problems, we apply the ϵ-constraint method (see, e.g., [CH83, Sec. 6.3; Ehr05, Sec. 4.1; LTZ06; BGP09]). It is a scalarization method that considers the following single-objective ϵ-constraint problems $(P_i(\epsilon_j))$ in order to determine non-dominated points of a bi-objective optimization problem (2.1):

$$
\begin{aligned}
(P_i(\epsilon_j)) \qquad \max_{x} \quad & \phi_i(x) \\
\text{s.t.} \quad & \phi_j(x) \geq \epsilon_j \\
& x \in X
\end{aligned}
\tag{2.2}
$$

with $i, j \in \{1, 2\}$, $i \neq j$ and $\epsilon_j \in \mathbb{R}$. The single-objective optimization problem is obtained by moving one objective function to the constraints and requiring its value to be at least ϵ_j.

While an optimal solution to one of the problems $P_1(\epsilon_2)$ or $P_2(\epsilon_1)$ for some $\epsilon_1, \epsilon_2 \in \mathbb{R}$ is a weakly efficient solution for problem (2.1) by Lemma 2.7, the result can be strengthened to obtain efficient solutions by Theorem 2.8. In particular, every efficient solution can be found by solving ϵ-constraint problems with an appropriate choice of ϵ_1, ϵ_2 by Theorem 2.8.

Algorithm 2.1: ϵ-Constraint method

Input : Bi-objective optimization problem
Output: Set Γ of all non-dominated points

1 Initialize $\Gamma \leftarrow \emptyset$.
2 Compute $\bar{z}_2 \leftarrow \max\{\phi_2(x) : x \in X\}$.
3 Set $\epsilon_2 \leftarrow \phi_2(x_1)$ for some $x_1 \in \arg\max_{x \in X} \phi_1(x)$.
4 **while** $\epsilon_2 \leq \bar{z}_2$ **do**
5   Compute an optimal solution x_1 to $P_1(\epsilon_2)$ with $z_1 \leftarrow \phi_1(x_1)$.
6   Compute an optimal solution x_2 to $P_2(z_1)$ with $z_2 \leftarrow \phi_2(x_2)$.
7   Update $\Gamma \leftarrow \Gamma \cup \{(z_1, z_2)\}$.
8   Choose $\epsilon_2 > z_2$ such that no non-dominated point that is not yet in Γ is cut off.
9 **return** Γ

Lemma 2.7 ([Ehr05, Thm. 4.3])**.** *Let $i, j \in \{1, 2\}$ with $i \neq j$ and $\epsilon_j \in \mathbb{R}$ be given. If $x \in X$ is an optimal solution to problem $P_i(\epsilon_j)$, then x is weakly efficient for problem* (2.1)*.*

Theorem 2.8 ([Ehr05, Thm. 4.5])**.** *A feasible solution $x \in X$ is efficient for problem* (2.1) *if and only if there exist $\epsilon_1, \epsilon_2 \in \mathbb{R}$ such that x is an optimal solution to $P_1(\epsilon_2)$ and $P_2(\epsilon_1)$.*

Choosing the correct values for ϵ_1, ϵ_2 is a crucial task. For bi-objective optimization problems with a finite set of non-dominated points, one way of applying the ϵ-constraint method is given in Algorithm 2.1. When a point (z_1, z_2) is added to Γ in line 7, it is a non-dominated point because the corresponding feasible solution (x_1, x_2) is efficient by Theorem 2.8. The algorithm finds all non-dominated points because we choose ϵ_2 in line 8 such that the non-dominated points that are already added to Γ are infeasible but no other non-dominated point is cut of. Because the number of non-dominated points is finite by assumption and a different non-dominated point is added to Γ in every iteration, of which one non-dominated point is $(z_1, \bar{z}_2)$ with $z_1 \in \max\{\phi_1(x) : x \in X \text{ and } \phi_2(x) = \bar{z}_2\}$, we eventually have $\epsilon_2 > \bar{z}_2$ and the algorithm terminates. The way ϵ_2 is chosen in line 8 is problem dependent and is discussed when Algorithm 2.1 is applied.

Remark 2.9. While Algorithm 2.1 computes one non-dominated point in every iteration, there are also other options for implementing the ϵ-constraint method. If we drop the computation of a second optimization problem in line 6 and set $z_2 = \phi_2(x_1)$, we still obtain weakly efficient solutions in every iteration by Lemma 2.7. Solutions that are weakly efficient but not efficient can be filtered during the iterations as implemented in Algorithm 5.1 or afterwards with Algorithm 2.2. Note that in order to ensure that the algorithm terminates, we are only allowed to look at finitely many weakly efficient solutions.

Algorithm 2.2: Filtering for non-dominated points

Input : Finite superset $\Gamma' \subseteq \mathbb{R}^2$ of the set of non-dominated points of a bi-objective optimization problem (2.1), $|\Gamma'| = n$

Output: Set $\Gamma \subseteq \Gamma'$ of all non-dominated points

1 Let $(y_1, z_1), \dots, (y_n, z_n)$ be a sorting of Γ' such that $y_1 \geq \dots \geq y_n$ and such that for all $i \in [n-1]$ with $y_i = y_{i+1}$, it holds that $z_i > z_{i+1}$.
2 Set $\Gamma \leftarrow \{(y_1, z_1)\}$.
3 Set $\bar{z} \leftarrow z_1$.
4 **for** $i = 2, \dots, n$ **do**
5   **if** $z_i > \bar{z}$ **then**
6   Update $\Gamma \leftarrow \Gamma \cup \{(y_i, z_i)\}$.
7   Update $\bar{z} \leftarrow z_i$.
8 **return** Γ

2.2.2 Filtering for Non-dominated Points

Let a bi-objective optimization problem (2.1) with a *finite* set of non-dominated points be given. If $\Gamma' \subseteq \mathbb{R}^2$ is a finite superset of the set of non-dominated points, it can be filtered for the non-dominated points as described in Algorithm 2.2. The points in Γ' are first sorted in decreasing order by the first component and as a second criterion by a decreasing second component. We then iterate over this sorted list and add a point to Γ whenever the second component is higher than the highest value so far. This ensures that, while the first component decreases, the second component needs to increase to add a point to Γ. Hence, when the algorithm terminates, Γ is the set of non-dominated points of the bi-objective optimization problem (2.1).

The running time of Algorithm 2.2 is dominated by the sorting in line 1, which can be done in $\mathcal{O}(n \cdot \log(n))$ for a set Γ' with n elements. Lines 2 to 7 are then performed in $\mathcal{O}(n)$, which leads to an overall running time of $\mathcal{O}(n \cdot \log(n))$.

2.3 Complexity and Tractability

In this thesis, we investigate whether there are polynomial time algorithms for the analyzed problems (i.e., the problems are in P) or whether they are NP-hard. For an introduction to complexity theory and to the classes P and NP, we refer to [GJ79].

In general terms, a decision problem P is in NP if a solution to P can be verified in polynomial time. It is NP-*hard* if it is at least as hard as every problem P' in NP, meaning that there is a polynomial time reduction from P' to P. In this case, we cannot expect to find a polynomial time algorithm for the problem P (unless P $=$ NP). We say that a decision problem P is NP-*complete* if P is in NP and NP-hard. Additionally, we call a single-objective optimization problem

$$\begin{aligned} \max_x \quad & \phi(x) \qquad (\text{with } \phi\colon X \to \mathbb{R}) \\ \text{s.t.} \quad & x \in X \end{aligned}$$

NP-*hard* if its decision version "Given $J \in \mathbb{R}$, is there a feasible solution $x \in X$ with $\phi(x) \geq J$?" is NP-complete. We use the same name for the optimization problem and its decision version as it is clear from the context which is meant.

For a bi-objective optimization problem (2.1), we consider its canonical decision problem "Given $J_1, J_2 \in \mathbb{R}$, is there a feasible solution $x \in X$ with $\phi_1(x) \geq J_1$ and $\phi_2(x) \geq J_2$?" (see [Bök17, Def. 3]). Note that the canonical decision problem of problem (2.1) and the decision version of its single-objective ϵ-constraint problem (2.2) with $\epsilon_1 = J_1$ and $\epsilon_2 = J_2$ coincide.

Tractability While the optimal objective function value is unique for single-objective optimization problems, we are searching for a set of non-dominated points for bi-objective optimization problems. The bi-objective problems that we consider in this thesis have a finite Pareto front. Therefore, we make use of the notion of (in-)tractability for multi-objective combinatorial optimization problems (see, e.g., [Ehr05, Def. 8.13]). We call a bi-objective optimization problem with a finite Pareto front *intractable* if the size of the Pareto front can be exponential in the size of the problem instance. In this case, because of the exponential number of non-dominated points, there is no polynomial time algorithm for finding the complete Pareto front.

List of Problems We now state some problems that are known to be NP-complete, which we use for polynomial time reductions:

Problem 2.10 (0-1 Knapsack)**.** Given k elements with rewards $g_i \in \mathbb{N}_{\geq 1}$ and weights $w_i \in \mathbb{N}_{\geq 1}$ for all $i \in [k]$, a weight budget W and a bound H, is there a set $F \subseteq [k]$ such that $\sum_{i \in F} w_i \leq W$ and $\sum_{i \in F} h_i \geq H$? This problem is NP-complete by [GJ79, Problem MP9].

Problem 2.11 (Bipartite Subgraph)**.** Given a graph $G = (V, E)$, a positive integer $Q' \in \mathbb{N}_{\geq 1}$ with $Q' \leq |E|$, is there a subset $E' \subseteq E$ with $|E'| \geq Q'$ such that (V, E') is bipartite? This problem is NP-complete by [GJ79, Problem GT25].

Remark 2.12. The problem Bipartite Subgraph asks whether G has a bipartite subgraph with at least Q' edges. In other words: Can we obtain a bipartite subgraph of G by deleting at most $|E| - Q'$ edges? Setting $Q := |E| - Q'$ and

$$\operatorname{int}(A, B) := \{\{v_1, v_2\} \in E : v_1, v_2 \in A\} \cup \{\{v_1, v_2\} \in E : v_1, v_2 \in B\},$$

it is hence equivalent to ask whether there is a bipartition (A, B) of V, i.e., $A \cup B = V$ and $A \cap B = \emptyset$, such that $|\operatorname{int}(A, B)| \leq Q$.

Problem 2.13 (Multicut (special case))**.** Given a star graph $G = (V, E)$, a set of source-terminal pairs $\mathcal{C} \subseteq V \times V$ and a non-negative integer $Q \in \mathbb{N}_{\geq 1}$, is there a subset $\bar{E} \subseteq E$ with $|E| \leq Q$ such that all source-terminal pairs in $\mathcal{C}$ are separated by $\bar{E}$? This problem is NP-complete by [GVY97, Thm. 3.1].

Problem 2.14 (Partition)**.** Given a set $A = \{a_1, \ldots, a_K\} \subseteq \mathbb{N}_{\geq 1}$ with $K \in \mathbb{N}_{\geq 1}$ elements, is there a subset $A' \subseteq A$ such that $\sum_{a \in A'} a = \frac{1}{2} \sum_{a \in A} a = \sum_{a \in A \setminus A'} a$? This problem is NP-complete by [GJ79, Problem SP12].

2.4 Public Transport Networks and Origin-Destination Data

This section defines a public transport network (PTN) and origin-destination (OD) data, which provide the central information for public transport planning. A PTN represents the infrastructure and OD data gives information about the (potential) passengers.

Definition 2.15 (PTN)**.** A *public transport network (PTN)* $G = (V, E)$ is an undirected graph with a node set V given by a set of stops or stations and an edge set E of direct connections between them. We assume that a PTN is connected and has neither loops nor parallel edges. A subset of nodes $Z \subseteq V$ is called *connected* if its induced subgraph $G[Z]$ is connected.

The PTN can be used to model railway, tram, or bus networks. In the following, we call the nodes of the PTN stations, even if bus networks with stops are under consideration.

Definition 2.16 (Path)**.** Let an undirected graph $G = (V, E)$ that has neither loops nor parallel edges be given.

- A *path* W in G is a sequence $(v_1, e_1, v_2, \ldots, v_{n-1}, e_{n-1}, v_n)$ of nodes and edges with $n \in \mathbb{N}_{\geq 1}$ so that for all $i \in [n-1]$, we have that e_i is an edge from v_i to v_{i+1}. The *node set* of W is the multiset $V(W) = \{v_1, v_2, \ldots, v_n\}$ and its *edge set* is the multiset $E(W) = \{e_1, e_2, \ldots, e_{n-1}\}$. We denote the *set of all paths* in G by $\mathcal{W}$. Since we assumed that G does not have parallel edges or loops, a path is uniquely determined by its nodes, and we write $W = (v_1, v_2, \ldots, v_n)$.
- A path is called *simple* if no edge appears more than once.
- A path is called *elementary* if no node occurs more than once.

In public transport, paths are often simple or elementary. Nevertheless, if not stated explicitly, the results of this thesis hold for arbitrary paths.

Definition 2.17 (OD data)**.** For a given graph $G = (V, E)$, we call the following information the *origin-destination (OD) data*:

- a set $D \subseteq (V \times V) \setminus \{(v, v) : v \in V\}$ with $D \neq \emptyset$,
- for all $d = (v_1, v_2) \in D$, a path $W_d \in \mathcal{W}$ from v_1 to v_2,
- for all $d \in D$, a number $t_d \in \mathbb{N}_{\geq 1}$.

We call the elements of D the *OD pairs*. Passengers of OD pair $d = (v_1, v_2) \in D$ travel from their origin v_1 along W_d to their destination v_2. The number t_d is the *demand*. This can for example be the current number of passengers or an estimated passenger potential. We write (D, W_d, t_d) as a shorthand notation for $(D, (W_d)_{d\in D}, (t_d)_{d\in D})$.

Part I

Infrastructure Optimization of a Bus Rapid Transit Line

Bus rapid transit (BRT) systems can provide a fast and reliable service to passengers at low investment costs compared to tram, metro and train systems. Therefore, they can be of great value to attract more passengers to use public transport. In this part, we thus focus on the BRT investment model: Which segments of a single bus line should be upgraded when maximizing the number of newly attracted passengers and minimizing the costs?

Chapter 3

Introduction

Increasing the modal share of public transport is widely recognized as an important path towards reducing greenhouse gas emissions [Mes+19]. Bus rapid transit (BRT) lines can contribute to this goal because they can offer an attractive service to passengers at relatively low investment costs compared to rail-based alternatives [DN11]. A BRT line generally uses dedicated lanes for a large share of its route and is therefore not sensitive to delays as a result of traffic jams caused by private vehicles. Moreover, BRT lines often get priority at crossings. Therefore, BRT lines are characterized by higher speed, higher frequency and higher reliability of service in comparison to traditional buses.

This part of the thesis addresses the planning of a single BRT line. Specifically, the *BRT investment model* poses the question of which segments of the BRT line should be upgraded to a full BRT standard and which could remain as a traditional mixed-traffic bus segment with the objectives to maximize the ridership and to minimize the costs. Upgrading a segment includes investments to establish separate bus lanes as well as to adapt intersections and traffic lights to allow for priority of the BRT line. Thus, the number and the location of upgraded segments have a direct impact on the quality of a passenger's journey, and thereby on the expected ridership of the BRT line. While there is a base amount of ridership independent of upgrades, we focus on the number of passengers that can be attracted additionally because of the improvements.

The BRT investment model is particularly motivated by the development of a new BRT line in the urban area of Copenhagen (Greater Copenhagen), which will connect multiple municipalities surrounding the city of Copenhagen [Mov20]. Each of these municipalities is responsible for the investments required for upgrading segments on its territory. Because these investments come out of the general budgets of the municipalities, which also cover other municipal expenses, municipalities must weigh the costs of upgrades against the societal benefits they generate. The willingness of the municipalities to work together towards a social optimum is shown through the collaboration within the transport agency Movia, which is funded by the collective of municipalities in the Capital Region. Moreover, due to the expertise available within the agency, Movia overall takes a leading role in the design of the new BRT line and thereby provides suggestions that then need to be approved by the municipalities. This process can be iterative: Municipalities discuss solutions and revise their budget, followed by new suggestions from Movia. Due to the conflict of goals between increasing ridership and reducing costs, we aim to quantify the impact of investments through constructing the Pareto front between the number of attracted passengers and the investment budget that is split over all municipalities.

Additionally, a separate investment budget per municipality could lead to a bus line that often blends in and out of mixed traffic, which may not make passengers experience the line as very different from a traditional bus line. Therefore, the BRT investment model also includes a BRT component constraint, which limits the number of separate sequences of upgraded segments.

In the following, we formulate the BRT investment model as a bi-objective MILP for two potential passenger responses to upgrades on the line: a linear and a threshold relation. While an upgraded segment leads to a proportional number of newly attracted passengers under the linear passenger response (Linear), passengers are only attracted to the BRT line in the threshold passenger response (MinImprov) if a certain minimum level of improvements is realized along their journey. The latter can be interpreted as a mode choice being made by a group of homogeneous passengers, where the passengers only switch to using the BRT line when it becomes their fastest option. Considering these two different passenger responses leads to two different variants of the BRT investment model, allowing us to analyze the impact of the passenger response on the trade-off between attracted passengers and investment budget.

The proposed model is intended to be used as a decision support tool within the planning process of a new BRT line. While it can be applied in a setting with a global decision maker without municipalities to find a social optimum, its main application is the case of municipalities collaborating through a transport agency, such as in Greater Copenhagen with the transport agency Movia.

Contribution and Outline

In Part I of this thesis, we propose the bi-objective *BRT investment model* with a BRT component constraint and multiple investing municipalities for two alternative passenger responses to upgrades. The objectives reflect on the respective passenger response and the overall investment budget. In Chapter 4, we define the BRT investment model formally, introduce the two different passenger responses Linear and MinImprov and provide corresponding bi-objective MILP formulations. In Chapter 5, we propose an ϵ-constraint-based algorithm to compute the complete Pareto front of the BRT investment model. This is followed by a theoretical analysis: we determine tractable and intractable cases of the BRT investment model and identify both NP-hard and polynomially solvable cases of the single-objective problem solved within the ϵ-constraint-based algorithm. In Chapter 6, we perform an extensive computational study on artificial instances and realistic instances based on the Greater Copenhagen BRT line, where we analyze among others the impact of the passenger response, the BRT component constraint, the demand pattern, and the budget split among the municipalities. Ideas for future research related to the BRT investment model are discussed in Chapter 7.

Related Literature

The BRT investment model is most closely related to the network design step of public transport planning, in which the public transport network (PTN) is determined. An overview of the problem of PTN design and the models and solution methods used to solve it is given in [LMO00] and [LM19].

While the focus in network design has traditionally been on designing a PTN from scratch, recent work has increasingly focused on the improvement of existing PTNs. A number of papers focuses on adding dedicated bus lines within an existing multi-modal network that is used by buses as well as private modes. In [Yao+12], a bilevel programming model that determines the allocation of dedicated bus lanes and bus frequencies in a multi-modal network minimizing the sum of travel costs and transit operating costs is presented. A bi-objective problem minimizing the travel time of bus and non-bus traffic is proposed in [KTM14]. Additionally to the allocation of dedicated bus lanes, time periods are determined during which lanes are exclusive available for buses. In [BG21], a bilevel problem is under consideration that aims to determine the allocation of dedicated bus lanes in order to reduce the total travel time within the network while accounting for traffic dynamics. The paper [TKG21] deals with the trade-off between prioritizing buses and resulting traffic congestion when allocating dedicated bus lanes. While the BRT investment model shares the main topic of upgrading segments to dedicated bus lanes, it focuses on a different objective: the trade-off between the number of attracted passengers and the investment budget. Moreover, it focuses on the context of a single line and considers the effect of a BRT component constraint.

Another relevant addition to the network design problem is the consideration of multiple investing parties. While it is typically assumed that all investment decisions are made by one central authority, [WZ17] considers local authorities that can only make upgrade decisions for their own parts of the network. In a game-theoretic setting, the interaction of the local authorities is formulated in a cooperative, competitive and chronological way (among others). Here, the aim of the local authorities is to minimize the travel time by increasing the capacity of edges under a budget constraint. In the BRT investment model, we take into account the effect of multiple municipalities through separate municipality budgets and through investigating different budget splits. Our setting differs in considering a bi-objective problem on a single line and through the addition of a BRT component constraint.

The underlying mathematical structure of the BRT investment model also shows similarities to the more general network improvement problem. This problem consists of choosing edges (and nodes) in a network to be upgraded while minimizing costs or satisfying budget constraints [Kru+98; ZYC04; Bal+22]. The problem has seen applications, e.g., in the area of road network optimization, where restricted resources can be used to upgrade edges in order to minimize the travel time between certain source-destination pairs [LM15] or where roads can be upgraded to all-weather roads to improve the accessibility of health services [MC09]. The BRT investment model differs from the network improvement problem through being bi-objective and through the consideration of the BRT component constraint. Moreover, one of our passenger responses depends in a non-linear way on the realized improvements.

Chapter 4

BRT Investment Model

This chapter establishes the foundations of the BRT investment model. In Section 4.1, we give a formal definition of the BRT investment model, in which one objective reflects the passenger response and the other the investment budget. We introduce two different passenger responses, namely LINEAR and MINIMPROV, in Section 4.2 and notation for different variants of the BRT investment model in Section 4.3. Further, we discuss the difference between the investment budget and the investment costs in Section 4.4. Finally, in Section 4.5, we provide a bi-objective MILP and prove its correctness.

4.1 Problem Definition

The BRT investment model models the allocation of upgrades along a bus line. We denote the bus line by a linear graph $G = (V, E)$. Upgrading a segment results in a BRT segment, where the vehicles of the BRT line can operate independently of other modes of transportation. We denote the costs of upgrading a segment $e \in E$ by $c_e \in \mathbb{N}_{\geq 1}$, which encompasses all costs related to creating BRT infrastructure for the segment. Note that the integrality of the costs is later relevant for the ϵ-constraint based solution method in Section 5.1.

We consider a BRT line that crosses *multiple municipalities*, each of them being responsible for investments in their respective parts of the line. We denote the set of municipalities by M and let $E_m \subseteq E$ denote the set of segments within municipality $m \in M$. We assume that the sets E_m contain consecutive segments and form a partition of the set E, which means that the sets E_m with $m \in M$ are non-empty, pairwise disjoint and $E = \bigcup_{m \in M} E_m$. This can often be achieved by splitting the segments at the borders of the municipalities. Furthermore, we suppose that each municipality is allocated a fixed budget share $s_m \in \mathbb{R}_{>0}$ of a (total) investment budget $b \in \mathbb{R}_{\geq 0}$ such that $\sum_{m \in M} s_m = 1$. For further insights into the evaluation of investments, we refer to Section 4.4.

We additionally include a *BRT component constraint* that limits the number of separate sequences of upgraded segments. We denote the maximum number of separate sequences by $Z \in \mathbb{N}_{\geq 1}$. As a result of the different municipalities, each with its own budget, the upgraded segments might become spread out over the BRT line without such a constraint. Passengers may experience such a line that constantly mixes in and out of blended traffic as not much different from a general bus line. Moreover, such mixing into blended traffic might create delays, reducing the reliability of the BRT line and thus making connected upgrades more desirable. It might also be easier from an organizational perspective to realize upgrades along several consecutive segments than on many (short) scattered segments.

The number of new passengers that are attracted to the BRT line depends on the set of segments chosen to be upgraded. We refer to this as the *passenger response* to upgrades and let $\mathsf{pass}(F)$ denote the number of passengers that are newly attracted when the segments in $F \subseteq E$ are upgraded. We evaluate two possible passenger responses: a passenger response LINEAR, in which the number of attracted passengers scales relatively to the improvement achieved on the passengers' paths, and a passenger response MINIMPROV, where passengers are attracted when a certain minimum improvement is realized along their path. These passenger responses are defined in Section 4.2.

We are now able to define the BRT investment model formally:

Definition 4.1 (BRT investment model)**.** Let the following be given:

Infrastructure:

- a linear graph $G = (V, E)$, where $V = [n]$ for $n \in \mathbb{N}_{\geq 1}$ denotes the set of stations and $E = \{e_i = \{i, i+1\} : i \in [n-1]\}$ the set of segments between the stations,
- upgrade costs $c_e \in \mathbb{N}_{\geq 1}$ for all $e \in E$,
- an upper bound $Z \in \mathbb{N}_{\geq 1}$ on the number of BRT components,

Municipalities:

- a set of municipalities M,
- a non-empty set of consecutive segments $E_m \subseteq E$ for all $m \in M$ with $\bigcup_{m \in M} E_m = E$ and such that the sets E_m are pairwise disjoint,
- a budget share $s_m \in \mathbb{R}_{>0}$ for all $m \in M$ such that $\sum_{m \in M} s_m = 1$,

Passenger Response:

- a function $\mathsf{pass}\colon 2^E \to \mathbb{R}_{\geq 0}$ that determines the number of newly attracted passengers, i.e., there are $\mathsf{pass}(F)$ newly attracted passengers when upgrading the segments in $F \subseteq E$.

The aim is to determine combinations (F, b) of upgraded segments $F \subseteq E$ and an investment budget $b \in \mathbb{R}_{\geq 0}$ that

max $\mathsf{pass}(F)$ (maximize the number of newly attracted passengers)
min b (minimize the investment budget)

and satisfy the following constraints:

- The *budget constraints*

$$\sum_{e \in F \cap E_m} c_e \leq s_m b \text{ for all } m \in M$$

  restrict the investment of each municipality, where $s_m b$ is the budget of municipality $m \in M$.

- The *BRT component constraint* restricts the subgraph $G[F]$ induced by the set of segments F, i.e., the subgraph of G containing all edges in F and their incident nodes, to have at most Z connected components.

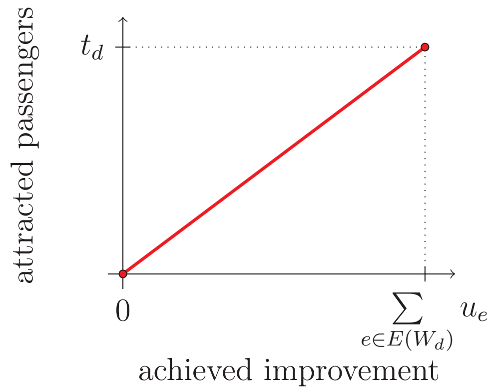
In order to simplify notation, we call the connected components of $G[F]$ the *BRT components* of F. Hence, the BRT component constraint limits the number of BRT components of F to at most Z.

In the following, we are interested in finding the non-dominated points and corresponding efficient solutions that constitute the Pareto front with respect to the number of newly attracted passengers and the investment budget.

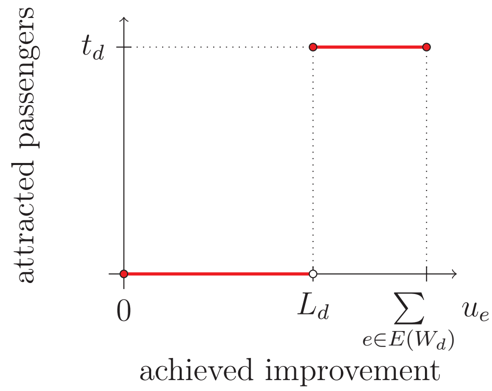
4.2 Objective Functions Reflecting the Passenger Response

It remains to define the passenger response functions. Information about potential passengers is given as OD data (D, W_d, t_d) (Definition 2.17). As we consider a single line, for each OD pair $d \in D$, we assume the path $W_d \in \mathcal{W}$ to be the unique simple path along the line. The number $t_d \in \mathbb{N}_{\geq 1}$ reflects the number of potential passengers who would like to travel along OD pair $d \in D$ in case the full set of segments is upgraded, which we assume to be known. Such an estimate could, for example, be derived from a traffic study which assumes that all sections are upgraded.

Additionally, for all segments $e \in E$, we assume a value $u_e \in \mathbb{R}_{>0}$ to be given, which we call the infrastructure improvement of segment e. If a segment e is upgraded, passengers benefit from the infrastructure improvement u_e that encompasses the reduction in travel time due to upgrading the segment e to a full BRT standard but it could, for example, also represent the improved reliability as a result of upgrading the segment.



(a) Linear



(b) MinImprov

Figure 4.1: Illustration of the passenger responses Linear and MinImprov for a fixed OD pair $d \in D$.

The passenger responses Linear and MinImprov determine the number of newly attracted passengers for each OD pair $d \in D$ based on the passenger potential t_d and the sum of infrastructure improvements u_e realized along the path W_d. These two passenger responses are illustrated in Figure 4.1. The passenger response Linear leads to a number of newly attracted passengers that is proportional to the realized infrastructure improvements, i.e., realizing $x\%$ of the potential improvements leads to $x\%$ of the potential passengers being attracted. The passenger response MinImprov instead relies on a threshold $L_d \in \mathbb{R}_{>0}$, which represents the point at which potential passengers switch over to actually using the BRT line. An infrastructure improvement below this threshold leads to no passengers being attracted, while all potential passengers are attracted if the realized infrastructure improvement exceeds the threshold.

We now formally define the passenger responses Linear and MinImprov:

Definition 4.2 (Passenger response)**.** Let the following be given:

- OD data (D, W_d, t_d) (Definition 2.17), where the paths $W_d \in \mathcal{W}$ are the unique simple paths from v_1 to v_2 along the line for all $d = (v_1, v_2) \in D$,
- infrastructure improvements $u_e \in \mathbb{R}_{>0}$ for all $e \in E$,

and additionally for the passenger response MinImprov:

- an improvement threshold level $L_d \in \mathbb{R}_{>0}$ for each $d \in D$ that satisfies $L_d \leq \sum_{e\in E(W_d)} u_e$.

Let $F \subseteq E$ be the set of upgraded segments, and let an OD pair $d \in D$ be given. In Linear, the number of newly attracted passengers of OD pair $d \in D$ is

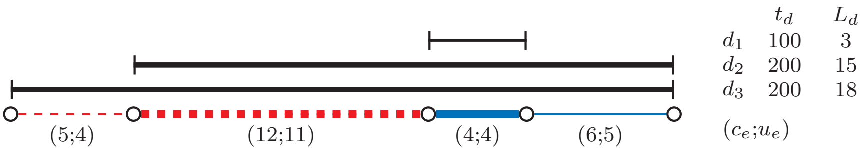


Figure 4.2: Example instance for the BRT investment model.

OD pair d	$\sum_{e\in E(W_d)} u_e$	$\sum_{e\in F\cap E(W_d)} u_e$	L_d	t_d	$\mathsf{pass}_d(F)$	
					Linear	MinImprov
d_1	4	4	3	100	100	100
d_2	20	15	15	200	150	200
d_3	24	15	18	200	125	0

Table 4.1: Infrastructure improvements and number of attracted passengers per OD pair for the example instance in Example 4.3.

determined by

$$\mathsf{pass}_d(F) := \frac{\sum_{e\in F\cap E(W_d)} u_e}{\sum_{e'\in E(W_d)} u_{e'}} \cdot t_d.$$

In MinImprov, the number of newly attracted passengers of OD pair $d \in D$ is determined by

$$\mathsf{pass}_d(F) := \begin{cases} t_d & \text{if } L_d \leq \sum_{e\in F\cap E(W_d)} u_e, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the total number of newly attracted passengers dependent on the set of upgraded segments is given by $\mathsf{pass}\colon 2^E \to \mathbb{R}_{\geq 0},\ F \mapsto \sum_{d\in D} \mathsf{pass}_d(F)$.

An example of both passenger responses as well as the notation introduced in Definitions 4.1 and 4.2 is given in Example 4.3.

Example 4.3. Consider the example instance given in Figure 4.2. The graph (V, E) with five nodes is given at the bottom with costs c_e and infrastructure improvements u_e below the edges. The red, dashed segments belong to municipality m_1 and the blue, solid segments belong to municipality m_2. The bold edges form the set F of segments to be upgraded. Three OD pairs are given above, where the line width corresponds to the number of potential passengers t_d.

In this example, municipality m_1 invests 12 and municipality m_2 invests 4. Because both upgraded segments in F are next to each other, F has only one BRT component, i.e., F satisfies the BRT component constraint for any $Z \in \mathbb{N}_{\geq 1}$. Table 4.1 shows the infrastructure improvements for each OD pair as well as the number of newly attracted passengers $\mathsf{pass}_d(F)$ for Linear and MinImprov.

Parameter	Value	Explanation
λ_1	LINEAR	passenger response LINEAR
	MINIMPROV	passenger response MINIMPROV
	$\star$	any of the passenger responses
λ_2	$Z \geq 1$	any limit on the number of BRT components
	$Z = k$	fixed limit k on the number of BRT components
	$Z = \infty$	no limit on the number of BRT components
λ_3	$\lvert M\rvert \geq 1$	any number of municipalities
	$\lvert M\rvert = k$	fixed number k of municipalities

Table 4.2: Overview of the allowed values in the classification of the problem variants.

4.3 Problem Variants

We consider several problem variants for which we use a scheduling-like notation. Each variant of the BRT investment model is classified as BRT($\lambda_1/\lambda_2/\lambda_3$) as follows:

λ_1: The function chosen to represent the passenger response.

λ_2: The upper bound on the number of BRT components of the BRT line.

λ_3: The number of municipalities that are present.

An overview of the possible values that $\lambda_1, \lambda_2, \lambda_3$ can take is given in Table 4.2. We remark that we use the symbolic notation "$Z = \infty$" to denote the setting in which the BRT component constraint is not applied, i.e., it indicates a model without a constraint limiting the number of BRT components. Lemma 4.4 shows that the BRT component constraint is always fulfilled for values of $Z \geq \left\lceil \frac{|E|}{2} \right\rceil$. Therefore, we regard BRT($\star/Z = \infty/|M| \geq 1$) as a special case of BRT($\star/Z \geq 1/|M| \geq 1$).

Lemma 4.4. *Let an instance of* BRT($\star/Z = k/|M| \geq 1$) *be given. The BRT component constraint is always fulfilled if* $Z \geq \left\lceil \frac{|E|}{2} \right\rceil$.

Proof. Let an arbitrary subset $F \subseteq E$ be given. Then $G[F]$ has the maximum number of connected components if F is a maximum matching in G, which would be to take every second segment. This yields at most $\left\lceil \frac{|E|}{2} \right\rceil$ connected components. Hence, the number of connected components of $G[F]$ is always less or equal Z if $Z \geq \left\lceil \frac{|E|}{2} \right\rceil$. In this case, the BRT component constraint is satisfied. □

In our solution method (Algorithm 5.1), we also encounter a *single-objective* BRT investment model that maximizes the number of passengers pass given a fixed budget b, which is part of the input. These single-objective variants are classified with an asterisk, i.e., as BRT*($\lambda_1/\lambda_2/\lambda_3$).

4.4 Evaluating the Investment

An efficient solution (F, b) to the BRT investment model and its objective function value $(\mathsf{pass}(F), b)$ represent the set of upgraded segments, the number of newly attracted passengers and the investment budget. For a given set of upgraded segments F, the corresponding *minimum investment budget* b is the minimum budget such that all budget constraints are satisfied for F, i.e.,

$$\begin{aligned} b &= \min \left\{ b' \in \mathbb{R} : \sum_{e \in F \cap E_m} c_e \leq s_m b' \text{ for all } m \in M \right\} \\ &= \max \left\{ \frac{1}{s_m} \sum_{e \in F \cap E_m} c_e : m \in M \right\}. \end{aligned}$$

Note that in any efficient solution (F, b), we have that b is the minimum investment budget corresponding to F and is uniquely determined by F. For practical applications, however, the *investment costs* $\mathsf{cost}(F)$ given as

$$\mathsf{cost}(F) := \sum_{e \in F} c_e,$$

which state the actual costs incurred by upgrading the segments in F, are another important figure. Because of the budget split among the municipalities based on the budget shares, for a fixed set of upgraded segments F, the investment costs $\mathsf{cost}(F)$ can be less than the available investment budget b.

By solving the BRT investment model, we obtain the Pareto front with respect to the investment budget. It is not immediately clear if this Pareto front overlaps with the one where the investment costs $\mathsf{cost}(F)$ constitute the second objective function. We show that both Pareto fronts coincide when there is a global decision maker (i.e., $|M| = 1$) in Lemma 4.5. However, this is generally not the case when there are multiple municipalities, which we illustrate with a counterexample in Example 4.6.

Lemma 4.5. *The problems* BRT($\star/Z \geq 1/|M| = 1$) *and*

$$\begin{aligned} \max \quad & \mathsf{pass}(F) \\ \min \quad & \mathsf{cost}(F) \qquad\qquad (4.1) \\ \text{s.t.} \quad & \text{there are at most } Z \text{ BRT components,} \end{aligned}$$

where we minimize the investment costs instead of the investment budget, are equivalent in the sense that for every efficient solution of one problem there is an efficient solution of the other problem with the same objective function value and with the same set of upgraded segments $F \subseteq E$. In particular, in this case, the sets of non-dominated points coincide.

Proof. Let (F, b) be an efficient solution to the BRT investment model with its corresponding non-dominated point $(\mathsf{pass}(F), b)$. Because $|M| = 1$, the budget constraint reduces to $\mathsf{cost}(F) \leq b$. Because (F, b) is efficient, the constraint needs to hold with equality, i.e., $\mathsf{cost}(F) = b$. We show that F is efficient and $(\mathsf{pass}(F), b) = (\mathsf{pass}(F), \mathsf{cost}(F))$ is a non-dominated point of problem (4.1). Assume that F is not efficient. Then there is a feasible solution F' to problem (4.1) such that $\mathsf{pass}(F') \geq \mathsf{pass}(F)$ and $\mathsf{cost}(F') \leq \mathsf{cost}(F)$ and at least one inequality holds strictly. In both cases, we have a contradiction to (F, b) being efficient because the solution $(F', \mathsf{cost}(F'))$ would dominate (F, b).

Now let F be an efficient solution to problem (4.1) with its corresponding non-dominated point $(\mathsf{pass}(F), \mathsf{cost}(F))$. We set $b := \mathsf{cost}(F)$. Assume that (F, b) is not an efficient solution to the BRT investment model. Then there is a feasible solution (F', b') to $\text{BRT}(\star/Z \geq 1/|M| = 1)$ such that $\mathsf{pass}(F') \geq \mathsf{pass}(F)$ and $b' \leq b$ and at least one inequality holds strictly. Again, we have a contradiction to F being efficient because the solution F' would dominate F because $\mathsf{cost}(F') \leq b' \leq b = \mathsf{cost}(F)$. □

Example 4.6. We consider the instance given in Figure 4.3 with two municipalities $M = \{m_1, m_2\}$ and their corresponding segments $E_{m_1} = \{e_1\}$ (red, dashed) and $E_{m_2} = \{e_2\}$ (blue, solid). Moreover, the investment budget is split such that municipality m_1 receives two thirds and municipality m_2 one third, i.e., $s_{m_1} = \frac{2}{3}$ and $s_{m_2} = \frac{1}{3}$.

The different sets of upgraded segments with their corresponding investment costs, minimum investment budgets and numbers of passengers as well as an evaluation which solutions are efficient for the respective problems are given in Table 4.3. This yields for the BRT investment model that the set of non-dominated points of the form $(\mathsf{pass}(F), b)$ is $\{(3, 3), (0, 0)\}$ for the passenger response Linear as well as for MinImprov. When considering problem (4.1), the set of non-dominated points of the form $(\mathsf{pass}(F), \mathsf{cost}(F))$ are given by $\{(3, 3), (2, 2), (1, 1), (0, 0)\}$ for the passenger response Linear and by $\{(3, 2), (2, 1), (0, 0)\}$ for the passenger response MinImprov. Comparing the sets of non-dominated points, where the second objective is once the investment budget and once the investment costs, we see that they do not coincide, neither for the passenger response Linear nor for the passenger response MinImprov. One does not even need to be contained in the other.

In contrast to Lemma 4.5, also a set of upgraded segments $F \subseteq E$ may lead to an efficient solution in one problem but not in the other problem, which Table 4.3 shows for MinImprov in both directions and for Linear in one direction

			pass(F)		efficient solutions			
					LINEAR		MINIMPROV	
F	cost(F)	b	LINEAR	MINIMPROV	BRT	cost	BRT	cost
$\emptyset$	0	0	0	0	x	x	x	x
$\{e_1\}$	2	3	2	3		x	x	x
$\{e_2\}$	1	3	1	2		x		x
$\{e_1, e_2\}$	3	3	3	3	x	x	x	

Table 4.3: Overview of the solutions of Example 4.6. For all sets $F \subseteq E$ of upgraded segments, the investment costs cost(E), the corresponding minimum investment budget b and the number of attracted passengers pass(F) for LINEAR and MINIMPROV are given. The solutions are evaluated whether they are efficient solutions in BRT($\star/Z \geq 1/|M| = 1$) and in the investment cost related problem (4.1) for LINEAR and MINIMPROV.

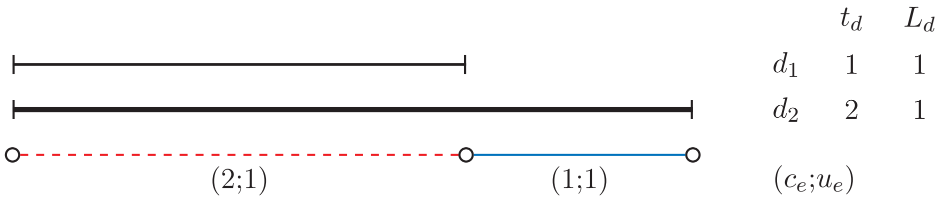


Figure 4.3: Instance for Example 4.6 with municipality m_1 containing segment e_1 (red, dashed) and municipality m_2 containing segment e_2 (blue, solid).

in this example. We remark that it is possible to also construct an instance with an efficient solution to the BRT investment model that is not efficient for problem (4.1).

The idea of the BRT investment model using the investment budget b as an objective function is that the municipality budgets are relative to each other, for example, depending on sociocultural, economical or political factors. In the computational experiments in Sections 6.1 and 6.2, we compute the efficient solutions and the Pareto fronts with respect to the investment budget b. Because of the practical relevance, we evaluate the results, however, also with respect to the investment costs cost(F).

4.5 MILP Formulation

We now provide a bi-objective MILP for the BRT investment model. This formulation uses the following variables:

- a binary variable $x_e \in \{0,1\}$ for all $e \in E$ that denotes whether segment e is upgraded,
- an auxiliary binary variable $z_i \in \{0,1\}$ for all $i \in [n-2]$, which has value 1 if exactly one of the segments e_i and e_{i+1} is upgraded,
- [only if $\lambda_1 = \textsc{MinImprov}$:] an auxiliary binary variable $y_d \in \{0,1\}$ for all $d \in D$, which satisfies in every efficient solution that $y_d = 1$ if and only if $L_d \leq \sum_{e \in F \cap E(W_d)} u_e$ for the set $F \subseteq E$ of upgraded segments,
- a continuous variable $b \in \mathbb{R}_{\geq 0}$ denoting the investment budget.

We obtain the following MILP formulation, which differs with respect to the passenger response. An explanation of the constraints is given below.

objective function

$\lambda_1 = \textsc{Linear}$:

$$\max \sum_{e \in E} \tilde{u}_e x_e$$
$$\min \ b$$
$$\text{with } \tilde{u}_e := u_e \cdot \sum_{\substack{d \in D: \\ e \in E(W_d)}} \frac{t_d}{\sum_{e' \in E(W_d)} u_{e'}}$$

$\lambda_1 = \textsc{MinImprov}$:

$$\max \sum_{d \in D} t_d y_d$$
$$\min \ b$$
$$\text{s.t.} \quad L_d y_d \leq \sum_{e \in E(W_d)} u_e x_e \quad \text{for all } d \in D$$

budget constraints

$$\sum_{e \in E_m} c_e x_e \leq s_m b \quad \text{for all } m \in M$$

BRT component constraints

$$x_{e_i} - x_{e_{i+1}} \leq z_i \quad \text{for all } i \in [n-2] \tag{4.2a}$$
$$x_{e_{i+1}} - x_{e_i} \leq z_i \quad \text{for all } i \in [n-2] \tag{4.2b}$$
$$x_{e_1} + \sum_{i=1}^{n-2} z_i + x_{e_{n-1}} \leq 2Z \tag{4.2c}$$

variable domains

$$x_e, z_i, y_d \in \{0,1\} \quad \text{for all } e \in E, i \in [n-2], d \in D$$
$$b \in \mathbb{R}_{\geq 0}.$$

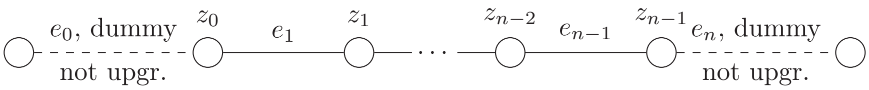


Figure 4.4: Visualization of the BRT component constraints (4.2) for Lemma 4.7.

In these bi-objective formulations of the BRT investment model, the objectives are to maximize the number of attracted passengers and to minimize the investment budget. The number of attracted passengers is determined either according to the passenger response LINEAR or MINIMPROV. Note that the objective regarding the number of attracted passengers for $\lambda_1 = $ LINEAR is reformulated as

$$\sum_{d \in D} \left(t_d \cdot \frac{\sum_{e \in E(W_d)} u_e x_e}{\sum_{e' \in E(W_d)} u_{e'}} \right) = \sum_{e \in E} \left(\sum_{\substack{d \in D: \\ e \in E(W_d)}} t_d \cdot \frac{u_e}{\sum_{e' \in E(W_d)} u_{e'}} \right) x_e = \sum_{e \in E} \tilde{u}_e x_e .$$

For $\lambda_1 = $ MINIMPROV, a constraint is added to ensure that the variable y_d takes value 1 only if the minimum improvement L_d is realized for an OD pair $d \in D$. The remaining constraints are the same for both passenger responses. The budget constraints determine the available budget for each municipality based on the budget shares s_m. Moreover, the BRT component constraints (4.2) ensure that the number of BRT components is not larger than Z. They are based on the observation that it suffices to count the number of times where an upgraded segment is succeeded by a segment that is not upgraded and vice versa. We present the idea and its correctness formally in the following lemma:

Lemma 4.7. *Let an instance of the BRT investment model and $F \subseteq E$ be given. We reflect F by setting* $x_e := \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{if } e \in E \setminus F. \end{cases}$
Then it holds that F has at most Z BRT components if and only if there is a vector $z \in \{0,1\}^{n-2}$ such that the BRT component constraints (4.2) are satisfied.

Proof. Let $F \subseteq E$ be given with K BRT components, i.e., $G[F]$ has K connected components, denoted by $F_1, \ldots, F_K$. We modify the linear graph by adding dummy edges e_0 and e_n at the front and end as depicted in Figure 4.4 that are not upgradable, i.e., we fix $x_{e_0} = x_{e_n} = 0$, and we add the binary variables $z_0 \in \{0,1\}$ and $z_{n-1} \in \{0,1\}$. Based on that, we define the vector $\bar{z} \in \{0,1\}^n$ by $\bar{z}_i := |x_{e_i} - x_{e_{i+1}}|$ for all $i \in \{0, \ldots, n-1\}$. By definition, $\bar{z}$ is feasible for the constraints (4.2a) and (4.2b), and it has exactly $2K$ entries with value 1, namely one for each start and end of a BRT component F_i with $i \in [K]$. Furthermore, because $x_{e_0} = x_{e_n} = 0$, we have $\bar{z}_0 = x_{e_1}$ and $\bar{z}_{n-1} = x_{e_{n-1}}$.

For the first direction, let F have at most Z BRT components, i.e., $K \leq Z$. Then

$$x_{e_1} + \sum_{i=1}^{n-2} \bar{z}_i + x_{e_{n-1}} = \sum_{i=0}^{n-1} \bar{z}_i = 2K \leq 2Z.$$

Hence, the constraints (4.2) are satisfied for the vector $(\bar{z}_1, \ldots, \bar{z}_{n-2}) \in \{0,1\}^{n-2}$.

For the second direction, we suppose that there is some $z^* \in \{0,1\}^{n-2}$ such that the constraints (4.2) hold. Due to the constraints (4.2a) and (4.2b), for all $i \in [n-2]$, we have that $|x_{e_i} - x_{e_{i+1}}| = 1$ implies $z_i^* = 1$. Hence, $\bar{z}_i \leq z_i^*$ for all $i \in [n-2]$. Then $K \leq Z$ because

$$2K = \sum_{i=0}^{n-1} \bar{z}_i \leq x_{e_1} + \sum_{i=1}^{n-2} z_i^* + x_{e_{n-1}} \leq 2Z. \qquad \square$$

Chapter 5

Solution Methods and Theoretical Analysis

In this chapter, we conduct a theoretical analysis of the BRT investment model and provide solution methods. In Section 5.1, we present an algorithm based on the ϵ-constraint method to solve the BRT investment model, and we analyze the size of its Pareto front. The impact of the BRT component constraint is investigated in Section 5.2, and the complexity of the single-objective problems solved within the ϵ-constraint method is studied in Section 5.3. In Section 5.4, we consider relaxations of the single-objective problem. A summary of the results of Chapter 5 is given in Section 5.5.

5.1 Solution Method and Tractability

Solving the BRT investment model requires computing the set of non-dominated points of an instance of $\text{BRT}(\star/Z \geq 1/|M| \geq 1)$. To do so, we employ the ϵ-constraint method (see Algorithm 2.1 with $\phi_1 = \mathsf{pass}$ and Remark 2.9). Note that the Pareto front of $\text{BRT}(\star/Z \geq 1/|M| \geq 1)$ is finite because there are only finitely many options to choose the set of upgraded segments $F \subseteq E$, and thus $\mathsf{pass}(F)$ attains only finitely many values.

The method for solving the BRT investment model is given in Algorithm 5.1. In this algorithm, we place an upper bound on the investment budget objective, this means that we solve single-objective problems $\text{BRT*}(\star/Z \geq 1/|M| \geq 1)$ (see Section 4.3), which take b as input. We start by finding the investment budget b at which all segments can be upgraded, meaning that all passengers will be attracted. In every iteration of the algorithm, we then reduce b in such a way that no non-dominated points are missed. This is repeated as long as the investment budget b is non-negative. We store weakly non-dominated points in (p^*, b^*) until we know that we have a non-dominated point, which we then add to Γ.

In this algorithm, we use the integrality of the upgrade costs to identify a step size δ in each iteration that does not cut off any non-dominated point. To do so, we first identify the minimum budget at which the current solution remains feasible and the municipalities for which this minimum budget is tight. Due to the integrality of the upgrade costs, we know that the individual budget for each such tight municipality can be reduced by 1 without cutting off any non-dominated point that we have not found yet. Similarly, we can reduce the individual budget for each non-tight municipality to the next integer level without cutting off any non-dominated point. We then choose the step size as the minimum value that leads to a budget satisfying these conditions for each municipality.

We formally prove that the algorithm is able to find the complete set of non-dominated points in Theorem 5.1.

Theorem 5.1. *Algorithm 5.1 computes the set of all non-dominated points for* $\mathrm{BRT}(\star/Z \geq 1/|M| \geq 1)$.

Proof. The algorithm starts with $p^* = \sum_{d \in D} t_d$, which is the upper limit on the number of attracted passengers that can be realized by all municipalities $m \in M$ upgrading all segments, this means investing $\sum_{e \in E_m} c_e$. This investment is possible for each municipality if the investment budget is set to $b^* = b = \max\left\{\frac{1}{s_m} \cdot \sum\limits_{e \in E_m} c_e : m \in M\right\}$. The idea of the algorithm is to iteratively compute all non-dominated points by solving $\mathrm{BRT}^*(\star/Z \geq 1/|M| \geq 1)$ for a budget b and then reducing b by δ.

For this ϵ-constraint based method, we have to make sure that $\delta > 0$ does not cut off any non-dominated points (Step 1) and that the algorithm only considers a finite number of weakly non-dominated points and thus terminates (Step 2).

Step 1: No non-dominated point is cut off. The step width δ is computed in line 10. Because $\frac{1}{s_m} > 0$ for all $m \in M$ and $x - \lceil x - 1 \rceil > 0$ for all $x \in \mathbb{R}_{\geq 0}$, it holds that $\delta > 0$. We need to ensure that the step width δ does not cut off solutions with a budget $b' < \bar{b}$. We do so by showing that if a set of upgraded segments $F' \subseteq E$ is feasible for a budget $b' < \bar{b}$, it is also feasible for the investment budget $\bar{b} - \delta$. So let $F' \subseteq E$ be feasible with a corresponding investment budget $b' < \bar{b}$ and let $m \in M$ be arbitrary. It holds that

$$\sum_{e \in F' \cap E_m} c_e \leq s_m \cdot b' < s_m \cdot \bar{b}.$$

Because $c_e \in \mathbb{N}_{\geq 1}$ for all $e \in E$, we obtain

$$\sum_{e \in F' \cap E_m} c_e \leq \lfloor s_m \cdot b' \rfloor \leq \lceil s_m \cdot \bar{b} - 1 \rceil.$$

Now let δ be chosen as in line 10 and update $b := \bar{b} - \delta$. This means that the right-hand side of the budget constraint of municipality m in the next iteration is

$$s_m \cdot (\bar{b} - \delta) \geq \begin{cases} s_m \cdot \left(\bar{b} - \frac{1}{s_m}\right) = s_m \cdot \bar{b} - 1 \overset{(*)}{=} \lceil s_m \cdot \bar{b} - 1 \rceil & \text{if } m \in \bar{M}, \\ s_m \cdot \left(\bar{b} - \frac{s_m \cdot \bar{b} - \lceil s_m \cdot \bar{b} - 1 \rceil}{s_m}\right) = \lceil s_m \cdot \bar{b} - 1 \rceil & \text{if } m \in M \setminus \bar{M}, \end{cases}$$

Algorithm 5.1: Computing the set of non-dominated points for BRT($\star/Z \geq 1/|M| \geq 1$)

Input : Instance I of BRT($\star/Z \geq 1/|M| \geq 1$)

Output: Set Γ of all non-dominated points

Initialize

$\Gamma \leftarrow \emptyset$,

$p^* \leftarrow \sum_{d \in D} t_d$,

$b^* \leftarrow \max \left\{ \frac{1}{s_m} \cdot \sum\limits_{e \in E_m} c_e : m \in M \right\}$,

$b \leftarrow b^*$.

while $b \geq 0$ **do**

  Compute BRT*($\star/Z \geq 1/|M| \geq 1$) for instance I with investment budget b. Let $F \subseteq E$ be an optimal solution and $\bar{p}$ be the optimal objective function value.

  Compute the minimum investment budget $\bar{b}$ such that F remains feasible as

$$\bar{b} \leftarrow \max \left\{ \frac{1}{s_m} \cdot \sum_{e \in F \cap E_m} c_e : m \in M \right\}.$$

  Determine the set of municipalities $\bar{M}$ for which the investment budget $\bar{b}$ is tight as

$$\bar{M} \leftarrow \left\{ m \in M : \sum_{e \in F \cap E_m} c_e = s_m \cdot \bar{b} \right\}.$$

  Compute the step width δ as

$$\delta \leftarrow \min \left\{ \min_{m \in \bar{M}} \left(\frac{1}{s_m} \right), \min_{m \in M \setminus \bar{M}} \left(\frac{s_m \cdot \bar{b} - \lceil s_m \cdot \bar{b} - 1 \rceil}{s_m} \right) \right\}.$$

  if $\bar{p} < p^*$ **then**

    Update $\Gamma \leftarrow \Gamma \cup \{(p^*, b^*)\}$.

    Update $p^* \leftarrow \bar{p}$.

  Update $b^* \leftarrow \bar{b}$.

  Update $b \leftarrow \bar{b} - \delta$.

Update $\Gamma \leftarrow \Gamma \cup \{(p^*, b^*)\}$.

return Γ

where $(*)$ holds because $m \in \bar{M}$ and thus $s_m \cdot \bar{b} = \sum_{e \in F \cap E_m} c_e \in \mathbb{N}_{\geq 0}$. Hence, the solution F' with investment budget $b' < \bar{b}$ is not cut off.

Step 2: The algorithm terminates. In every iteration, we consider a different (weakly) efficient solution to $\mathrm{BRT}(\star/Z \geq 1/|M| \geq 1)$: In lines 7 and 8, a set of upgraded segments F and its minimum investment budget $\bar{b}$ such that F remains feasible are computed. In line 15, the bound b on the investment budget is then reduced by $\delta > 0$ so that F is not a feasible solution to $\mathrm{BRT}^*(\star/Z \geq 1/|M| \geq 1)$ with the investment budget $b = \bar{b} - \delta$ in the next iteration. Because there are only finitely many sets $F \subseteq E$ of which one is $F = \emptyset$ with $\bar{b} = 0$, the stopping criterion $b < 0$ is met after a finite number of iterations and the algorithm terminates. □

Note that the algorithm simplifies for the special case in which there is a global decision maker, i.e., for $\mathrm{BRT}(\star/Z \geq 1/|M| = 1)$. In this special case, lines 8 and 10 in Algorithm 5.1 simplify to

$$\bar{b} \leftarrow \sum_{e \in F} c_e,$$
$$\delta \leftarrow 1.$$

This means that the minimum investment budget for a given solution corresponds to the investment costs to realize it and we can always choose the step size to be equal to 1 because of the integral costs $c_e \in \mathbb{N}_{\geq 1}$. This finding relates to Lemma 4.5, in which we found that the Pareto front with respect to the investment costs coincides with the one for the investment budget.

To analyze the running time of Algorithm 5.1, we consider the complexity of the single-objective problem solved in line 7, the number of iterations and the number of non-dominated points of $\mathrm{BRT}(\star/Z \geq 1/|M| \geq 1)$. We give upper bounds on the last two values in Lemma 5.2. Theorem 5.3 shows that 2^{n-1} is a tight bound and that the BRT investment model is generally intractable, meaning that its Pareto front may contain an exponential number of non-dominated points. In Lemma 5.4, we deduce from the upper bound (5.1) that the number of non-dominated points is polynomial for the special case where all segment upgrade costs are equal

Lemma 5.2. *The number of non-dominated points of* $\mathrm{BRT}(\star/Z \geq 1/|M| \geq 1)$ *and the number of iterations of Algorithm 5.1 are limited by* 2^{n-1} *and*

$$\left| \left\{ \frac{1}{s_m} \sum_{e \in F \cap E_m} c_e : m \in M, F \subseteq E_m \right\} \right|. \tag{5.1}$$

Proof. For every non-dominated point $(\mathsf{pass}(F), \bar{b})$ with a set of upgraded segments F, we have that $\bar{b}$ is the minimum investment budget such that all segments in F can still be upgraded. Moreover, in every iteration of Algorithm 5.1, we consider a different minimum investment budget $\bar{b}$, which is strictly decreasing with every iteration. Therefore, the number of non-dominated points and the number of iterations of Algorithm 5.1 are limited by the number μ of different minimum investment budgets that can occur:

$$\mu = \left| \left\{ \max \left\{ \frac{1}{s_m} \sum_{e \in F \cap E_m} c_e : m \in M \right\} : F \subseteq E \right\} \right|.$$

We obtain upper bounds on μ by taking one value for every set $F \subseteq E$:

$$\mu \leq |\{F \subseteq E\}| = 2^{n-1},$$

and by dropping the maximization and taking one value $\bar{b}$ for every municipality $m \in M$ and every set $F \subseteq E_m$:

$$\mu \leq \left| \left\{ \frac{1}{s_m} \sum_{e \in F \cap E_m} c_e : m \in M, F \subseteq E_m \right\} \right|. \qquad \square$$

Theorem 5.3. BRT($\star/Z \geq 1/|M| \geq 1$) *is intractable, even if* $Z = \infty$ *and* $|M| = 1$.

Proof. We consider an instance of BRT($\star/Z = \infty/|M| = 1$) with

- a linear graph (V, E) with $|V| = n$,
- $u_e = 1$ for all $e \in E$,
- $D := \{d_i = (i, i+1) : i \in [n-1]\}$,
- $t_{d_i} := 2^{i-1}$ and $c_{e_i} := 2^{i-1}$ for all $i \in [n-1]$,
- $L_d = 1$ for all $d \in D$ (if applicable).

Because the paths of all OD pairs only contain one segment, upgrading a segment $e_i \in E$ results in attracting $t_{d_i} = 2^{i-1}$ passengers both for Linear and MinImprov. Upgrading any set of segments $F \subseteq E$ hence results in attracting $\sum_{i \in I} 2^{i-1}$ passengers with investment costs and hence also an investment budget of $\sum_{i \in I} 2^{i-1}$ with $I = \{i \in [n-1] : e_i \in F\}$. Because

$$\left\{ \sum_{i \in I} 2^{i-1} : I \subseteq [n-1] \right\} = \{0, \dots, 2^{n-1} - 1\},$$

there is a solution of BRT($\star/Z = \infty/|M| = 1$) with objective function value (k, k) for each number $k \in \{0, \dots, 2^{n-1} - 1\}$. We can easily see that all these points are non-dominated. Thus the set of non-dominated points has size 2^{n-1}. $\square$

Lemma 5.4. *Let $c \in \mathbb{N}_{\geq 1}$. For* BRT$(\star/Z \geq 1/|M| \geq 1)$ *with $c_e = c$ for all $e \in E$ and $s_m = \frac{1}{|M|}$ for all $m \in M$, then there are at most n non-dominated points and Algorithm 5.1 terminates after at most n iterations, where n is the number of nodes in the linear graph.*

Proof. We plug the input into the bound (5.1):

$$\begin{aligned}(5.1) &= \left|\left\{|M| \cdot |F \cap E_m| \cdot c : m \in M, F \subseteq E_m\right\}\right| \\ &= \max\{|E_m| : m \in M\} + 1 \leq n.\end{aligned}$$

Thus, by Lemma 5.2, the number of non-dominated points and the number of iterations of Algorithm 5.1 are limited by n. □

Note, in particular, that Lemma 5.4 is applicable for BRT$(\star/z \geq 1/|M| = 1)$ with $c_e = c$ for all $e \in E$.

Next, we identify two special cases for Linear with only one municipality in which the Pareto front can be computed in polynomial time, see Lemmas 5.5 and 5.6. The setting in item 2 of Lemma 5.5 occurs for example if we consider unit infrastructure improvements $u_e = 1$ for all $e \in E$ and a maximal set of OD pairs $D = V \times V \setminus \{(v,v) : v \in V\}$ with a passenger potential that is distributed evenly over all OD pairs, i.e., $t_d = t_{d'}$ for all $d, d' \in D$. A cost pattern as in item 2 of Lemma 5.6 occurs for example if the costs are less expensive in the middle of the line but are increasingly expensive towards its ends. Recall that $\mathsf{cost}(F) = \sum_{e \in F} c_e$ and $\mathsf{pass}(F) = \sum_{e \in F} \tilde{u}_e$ for all $F \subseteq E$ with $\tilde{u}_e := u_e \cdot \sum_{\substack{d \in D: \\ e \in E(W_d)}} \frac{t_d}{\sum_{e' \in E(W_d)} u_{e'}}$ for all $e \in E$.

Lemma 5.5. *Let an instance of* BRT(Linear$/Z = \infty/|M| = 1$) *with unit costs $c_e := 1$ for all $e \in E$ be given. Let $e_{(i)}$ for $i \in [n-1]$ denote a sorting of the segments such that $\tilde{u}_{e_{(1)}} \geq \ldots \geq \tilde{u}_{e_{(n-1)}}$.*

1. *If $b \in \{0, \ldots, n-1\}$ and $F = \{e_{(i)} : i \in [b]\}$, then (F, b) is an efficient solution.*
2. *If there is some $\bar{i} \in [n-1]$ such that $\tilde{u}_{e_j} \leq \tilde{u}_{e_{j'}}$ for all $j \leq j' \leq \bar{i}$ and $\tilde{u}_{e_j} \geq \tilde{u}_{e_{j'}}$ for all $\bar{i} \leq j \leq j'$, then for every non-dominated point (p, b), there is an efficient solution (F, b) with $\mathsf{pass}(F) = p$ such that the segments in F are connected.*
3. *The instance can be solved in polynomial time.*

Proof. Note that for all $F \subseteq E$, we have $\mathsf{cost}(F) = |F|$.

1. First, (F, b) is feasible because $\mathsf{cost}(F) = |F| = b$, hence, the budget constraint is satisfied. Second, suppose it is not efficient. Then it is dominated by some solution (F', b'). First, assume $b' < b$, then

$$|F'| = \mathsf{cost}(F') \leq b' < b = \mathsf{cost}(F) = |F|$$

and hence by definition of the $e_{(i)}$ also

$$\mathsf{pass}(F') = \sum_{e \in F'} \tilde{u}_e < \sum_{i=1}^{b} \tilde{u}_{e_{(i)}} = \sum_{e \in F} \tilde{u}_e = \mathsf{pass}(F).$$

Next, assume $\mathsf{pass}(F') > \mathsf{pass}(F)$. In this case, $|F'| > |F|$ because F contains the b segments with the highest value $\tilde{u}_e$. This implies

$$b' \geq \mathsf{cost}(F') > \mathsf{cost}(F) = b.$$

Because we have a contradiction in both cases, there cannot be a solution that dominates (F, b), and (F, b) is efficient.

2. Let (p, b) be a non-dominated point. Thus, $b \in \{0, \dots, n-1\}$ because the budget constraint must be satisfied with equality. By assumption, we can suppose that $e_{(1)} = e_{\bar{i}}$. Because $\tilde{u}_e$ increases monotonically until $e_{\bar{i}}$ and decreases monotonically afterwards, we can assume that $e_{(2)} \in \{e_{\bar{i}-1}, e_{\bar{i}+1}\}$. Iteratively, we get that if $\{e_{(1)}, \dots, e_{(k)}\} = \{e_j : j \in \{l, \dots, l+k-1\}\}$ for some $l \in [n-1]$, then $e_{(k+1)} \in \{e_{l-1}, e_{l+k}\}$. Therefore, $F := \{e_{(i)} : i \in [b]\}$ is connected. Because of item 1, (F, b) is an efficient solution, and thus $\mathsf{pass}(F) = p$.

3. From Lemma 5.4, we know that there are at most n non-dominated points, one for each investment budget value $b \in \{0, \dots, n-1\}$. The sorting of the segments with respect to the values $\tilde{u}_e$ can be done in $\mathcal{O}(n \cdot \log(n))$. Because we can find the optimal set of upgraded segments F corresponding to a fixed investment budget value b as shown in item 1, we can find all non-dominated points in polynomial time by iterating over the investment budget values $b \in \{0, \dots, n-1\}$, determining F and then computing $\mathsf{pass}(F)$. □

Lemma 5.6. *Let an instance of* BRT(LINEAR/$Z = \infty$/$|M| = 1$) *with* $\tilde{u}_e := 1$ *for all* $e \in E$ *be given. Let* $e_{(i)}$ *for* $i \in [n-1]$ *denote a sorting of the segments such that* $c_{e_{(1)}} \leq \dots \leq c_{e_{(n-1)}}$.

1. *Let* $k \in \{0, \dots, n-1\}$. *If* $b = \sum_{i=1}^{k} c_{e_{(i)}}$ *and* $F = \{e_{(i)} : i \in [k]\}$, *then* (F, b) *is an efficient solution.*

2. *If there is some* $\bar{i} \in [n-1]$ *such that* $c_{e_j} \geq c_{e_{j'}}$ *for all* $j \leq j' \leq \bar{i}$ *and* $c_{e_j} \leq c_{e_{j'}}$ *for all* $\bar{i} \leq j \leq j'$, *then for each non-dominated point* (p, b), *there is some efficient solution* (F, b) *with* $\mathsf{pass}(F) = p$ *such that the segments in* F *are connected.*

3. *The instance can be solved in polynomial time.*

Proof. Note that for all $F \subseteq E$, we have $\mathsf{pass}(F) = |F|$.

1. First, (F, b) is feasible by construction. Second, suppose it is not efficient. Then it is dominated by some solution (F', b'). First, assume $\mathsf{pass}(F') > \mathsf{pass}(F)$. This implies $|F'| > |F| = k$, and hence by definition of the $e_{(i)}$ also
$$b' \geq \mathsf{cost}(F') = \sum_{e \in F'} c_e > \sum_{i=1}^{k} c_{e_{(i)}} = b.$$
Next, assume $b' < b$. In this case $|F'| < |F|$ because F contains the k segments with the lowest costs c_e and $\mathsf{cost}(F) = b$. This implies that $\mathsf{pass}(F') < \mathsf{pass}(F)$. Therefore, there cannot be a solution that dominates (F, b), and (F, b) is efficient.

2. The proof is analogous to the proof of Lemma 5.5, item 2.

3. Because there are n different values for the number of newly attracted passengers, namely $\mathsf{pass}(F) = |F| \in \{0, \ldots, n-1\}$ for all $F \subseteq E$, and $c_e > 0$ for all $e \in E$, there are n non-dominated points, one per number of segments in F. We can sort the segments according to their costs in $\mathcal{O}(n \cdot \log(n))$. Using the formulas for b and F in item 1, we can then find all non-dominated points in polynomial time by iterating over the number of upgraded segments $k \in \{0, \ldots, n-1\}$, determining b and F and then computing $\mathsf{pass}(F)$. □

5.2 Exploiting the Structure of the BRT Component Constraint

The BRT component constraint limits the number of BRT components and, as a result, it also limits the number of feasible sets of upgraded segments $F \subseteq E$. This can have an impact on both the number of non-dominated points and the running time needed to solve the single-objective problems in Algorithm 5.1. For that reason, we further analyze the complexity of the BRT investment model in the context of this component constraint.

We start by considering BRT($\star/Z = k/|M| \geq 1$), where k is fixed and not part of the input, and show that all non-dominated points can be found in polynomial time by an enumeration algorithm, see Theorem 5.7. This means that the problem is "slice-wise polynomial" and, hence, in the complexity class XP [DF13, Ch. 27; Cyg+15, Sec. 1.1]. We remark that Theorem 5.7 does not imply that BRT($\star/Z \geq 1/|M| \geq 1$) is in FPT, the set of fixed-parameter tractable problems, because the degree of the polynomial depends on the parameter k.

Theorem 5.7. BRT($\star/Z = k/|M| \geq 1$) *can be solved in polynomial time for a fixed* $k \in \mathbb{N}_{\geq 1}$.

Proof. Let $k \in \mathbb{N}_{\geq 1}$ be fixed. We consider an instance of BRT($\star/Z = k/|M| \geq 1$), which has $|V| = n$ and $|E| = n - 1$. Each BRT component is uniquely defined by a pair $(q, q') \in V \times V$ with $q < q'$ marking the first and last station of the BRT component. A feasible set of upgraded segments $F \subseteq E$ can have at most k BRT components determined by $(q_1, q_1'), \ldots, (q_k, q_k')$. There are at most n possible values for each q_i and q_i' for all $i \in [k]$. Hence, the number of sets satisfying the BRT component constraint is in $\mathcal{O}(n^{2k})$. For each such set F, we can compute the minimum investment budget in $\mathcal{O}(n-1)$ (see Section 4.4) and the number of attracted passengers in $\mathcal{O}(|D| \cdot (n-1))$. A solution (F, b) can only be efficient if b is the minimum investment budget for which F is feasible. Therefore, the above procedure gives a superset of the set of non-dominated points of BRT($\star/Z = k/|M| \geq 1$), which we can filter with Algorithm 2.2 to obtain the Pareto front. Hence, BRT($\star/Z = k/|M| \geq 1$) can be solved in polynomial time for a fixed $k \in \mathbb{N}_{\geq 1}$. □

Note that the result of Theorem 5.7 is especially useful for finding the set of non-dominated points for instances with a small value of k. We next consider the case with many BRT components. From Lemma 4.4 we know that the BRT component constraint is always fulfilled if $Z \geq \left\lceil \frac{|E|}{2} \right\rceil$. For the passenger response Linear, Lemma 5.8 shows how we can use BRT*(Linear$/Z = \infty/|M| \geq 1$) to obtain bounds on the optimal objective function value of the single-objective problem BRT*(Linear$/Z = k/|M| \geq 1$) for any fixed $k \in \mathbb{N}_{\geq 1}$. Such a bound could, e.g., be used to obtain an approximate Pareto front for instances in which it is hard to solve the single-objective problems in Algorithm 5.1 to optimality.

Lemma 5.8. *Let an instance of* BRT*(Linear$/Z = \infty/|M| \geq 1$) *be given. Let p be the optimal objective function value of* BRT*(Linear$/Z = \infty/|M| \geq 1$) *with an optimal solution F, and let p_k be the optimal objective function value of* BRT*(Linear$/Z = k/|M| \geq 1$) *for a fixed $k \in \mathbb{N}_{\geq 1}$. Let K be the number of BRT components of F. Then $p_k = p$ if $k \geq K$, and $p \geq p_k \geq \frac{k}{K} p$ if $k < K$.*

Proof. Dropping the BRT component constraint is clearly a relaxation, hence, $p \geq p_k$. Also if $k \geq K$, then F is still feasible for the restricted problem BRT*(Linear$/Z = k/|M| \geq 1$). Therefore, $p_k = p$ in that case.

So let $k < K$, and let $F_1, \ldots, F_K$ be the BRT components of F. For every $i \in [K]$, we define $g_i := \sum_{e \in F_i} \tilde{u}_e$ as the gain in passengers when upgrading the i-th BRT component of F. We assume that they are sorted such that $g_1 \geq \ldots \geq g_K \geq 0$. Allowing k BRT components means that $F_1 \cup \ldots \cup F_k$ is a feasible solution as it has exactly k BRT components. This yields that

$$p_k \geq \sum_{i=1}^{k} g_i \geq \frac{k}{K} p.$$

Here, the last inequality holds because of the following argument: Assume that it is not true, i.e., $\sum_{i=1}^{k} g_i < \frac{k}{K} p$. This implies $\sum_{i=k+1}^{K} g_i = p - \sum_{i=1}^{k} g_i > \frac{K-k}{K} p$.

We then have that $g_k < \frac{1}{K}p$ because we would have $\sum_{i=1}^{k} g_i \geq k \cdot \frac{1}{K}p$ otherwise, and $g_{k+1} > \frac{1}{K}p$ analogously. This is a contradiction to $g_k \geq g_{k+1}$. □

5.3 Complexity of the Single-Objective Problem

While we showed in the previous sections that BRT(Linear$/Z = \infty/|M| = 1$) with a special structure (see Lemmas 5.5 and 5.6) and BRT($\star/Z = k/|M| \geq 1$) for small values of k (see Theorem 5.7) can be solved in polynomial time, the time needed to solve the single-objective problem BRT*($\star/Z = \infty/|M| \geq 1$) has a large impact on the running time of Algorithm 5.1 in other cases. In this section, we show that the single-objective BRT investment model is related to the well-known 0-1 Knapsack problem and is NP-hard in general, see Theorems 5.9 and 5.11. However, we also identify polynomially solvable cases in Lemmas 5.10 and 5.12.

Theorem 5.9. *BRT*(Linear$/Z \geq 1/|M| \geq 1$) is NP-hard, even if $Z = \infty$, $|M| = 1$ and $u_e = 1$ for all $e \in E$.*

Proof. We show that the decision version of BRT*(Linear$/Z \geq 1/|M| \geq 1$) with a lower bound p on the objective function value is NP-complete.

Given a solution to BRT*(Linear$/Z \geq 1/|M| \geq 1$), we can check in polynomial time whether the budget constraints and the BRT component constraint are satisfied and a certain value in the objective function is reached. Hence, the problem is in NP.

We reduce 0-1 Knapsack (Problem 2.10) to BRT*(Linear$/Z \geq 1/|M| \geq 1$). Let k elements with rewards $h_i \in \mathbb{N}_{\geq 1}$ and weights $w_i \in \mathbb{N}_{\geq 1}$ for all $i \in [k]$, a weight budget W and a bound H be given. We construct an instance of BRT*(Linear$/Z \geq 1/|M| \geq 1$) as follows: We set

- $n := k + 1$, so $V = [k + 1]$ and $E = \{e_i : i \in [k]\}$,
- $c_{e_i} := w_i$ and $u_{e_i} := 1$ for all $i \in [k]$,
- $Z := k$,
- $M := \{1\}$ and $s_1 := 1$,
- $D := \{(i, i + 1) : i \in [k]\}$,
- $t_d := h_i$ for all $d = (i, i + 1)$ with $i \in [k]$,
- $p := H$,
- $b := W$.

We show that every feasible solution $F' \subseteq [k]$ of 0-1 Knapsack with an objective function value of at least H corresponds to a feasible solution $F \subseteq E$ of BRT*(Linear$/Z \geq 1/|M| \geq 1$) with an objective function value of at least p. The solutions F' and F correspond to each other as follows: $i \in F'$ if and only if $e_i \in F$. Then the claim holds because $\sum_{i \in F'} w_i = \sum_{i \in F'} c_{e_i} = \sum_{e \in F} c_e$ and

$$\sum_{i \in F'} h_i = \sum_{e_i \in F} t_{(i,i+1)} = \sum_{\substack{d=(i,i+1):\\ i \in [k]}} \left(\frac{\sum_{e \in F \cap \{e_i\}} 1}{1} \cdot t_d \right)$$
$$= \sum_{d \in D} \left(\frac{\sum_{e \in F \cap E(W_d)} u_e}{\sum_{e \in E(W_d)} u_e} \cdot t_d \right).$$

Thus, the decision version of BRT*(Linear$/Z \geq 1/|M| \geq 1$) is NP-complete. □

From Section 4.5, we obtain the following IP for the single-objective problem BRT*(Linear$/Z = \infty/|M| \geq 1$) with a fixed $b \in \mathbb{R}_{\geq 0}$:

$$\begin{aligned} \max_{x_e} \quad & \sum_{e \in E} \tilde{u}_e x_e \\ \text{s.t.} \quad & \sum_{e \in E_m} c_e x_e \leq s_m b \qquad \text{for all } m \in M \\ & x_e \in \{0,1\} \qquad \text{for all } e \in E. \end{aligned} \tag{5.2}$$

From this formulation we can see that BRT*(Linear$/Z = \infty/|M| \geq 1$) and BRT*(Linear$/Z = \infty/|M| = 1$) are (multi-dimensional) 0-1 Knapsack problems. Moreover, because the sets E_m for $m \in M$ are pairwise disjoint, BRT*(Linear$/Z = \infty/|M| \geq 1$) can be decomposed into $|M|$ independent 0-1 Knapsack problems and hence can be solved in pseudo-polynomial time by dynamic programming [KPP04, Sec. 9.3.2].

The special case of BRT*(Linear$/Z = \infty/|M| \geq 1$) in which all segments have unit upgrade costs can even be solved in polynomial time:

Lemma 5.10. BRT*(Linear$/Z = \infty/|M| \geq 1$) *can be solved in polynomial time if $c_e = 1$ for all $e \in E$.*

Proof. Consider IP (5.2). Let $A \in \mathbb{R}^{|M| \times |E|}$ be the coefficient matrix of the budget constraints, i.e., for all $m \in M$ and $e \in E$, we have $A_{m,e} = 1$ if $e \in E_m$, and $A_{m,e} = 0$ otherwise. Because of the assumption that the municipalities contain only consecutive segments, the matrix A satisfies the consecutive ones property. Therefore, the problem can be solved in polynomial time (see Section 2.1). □

Theorem 5.11. BRT*(MinImprov$/Z \geq 1/|M| \geq 1$) *is* NP*-hard, even if $Z = \infty$, $M = 1$, $u_e = 1$ for all $e \in E$ and $L_d = 1$ for all $d \in D$.*

Proof. As in the proof of Theorem 5.9, BRT*(MinImprov$/Z \geq 1/|M| \geq 1$) is in NP. Further, we apply the same reduction from 0-1 Knapsack (Problem 2.10) to BRT*(MinImprov$/Z \geq 1/|M| \geq 1$) and additionally choose $L_d := 1$ for all $d \in D$. It remains to show that the objective function value is the same for solutions that correspond to each other. We have that

$$\sum_{\substack{d \in D: \\ L_d \leq \sum\limits_{e \in F \cap E(W_d)} u_e}} t_d = \sum_{i \in F'} h_i,$$

because

$$\left\{ d \in D : L_d \leq \sum_{e \in F \cap E(W_d)} u_e \right\} = \left\{ (i, i+1) : i \in [k] \text{ and } 1 \leq \sum_{e \in F \cap \{e_i\}} 1 \right\}$$
$$= \{(i, i+1) : i \in [k] \text{ and } e_i \in F\} = \{(i, i+1) : i \in F'\}. \qquad \square$$

Also a special case of BRT*(MinImprov$/Z = \infty/|M| \geq 1$) can be solved in polynomial time, namely when all segments have unit upgrade costs and unit improvements and at the same time only a single segment has to be upgraded to attract the passengers for each OD pair:

Lemma 5.12. BRT*(MinImprov$/Z = \infty/|M| \geq 1$) *can be solved in polynomial time if* $c_e = 1$ *and* $u_e = 1$ *for all* $e \in E$ *and* $L_d = 1$ *for all* $d \in D$.

Proof. The special case under consideration yields the following simplified formulation:

$$\begin{aligned}
\max_{x_e, y_d} \quad & \sum_{d \in D} t_d y_d \\
\text{s.t.} \quad & \sum_{e \in E_m} x_e \leq s_m b && \text{for all } m \in M \\
& \sum_{e \in E(W_d)} (-x_e) + y_d \leq 0 && \text{for all } d \in D \\
& x_e \in \{0, 1\} && \text{for all } e \in E \\
& y_d \in \{0, 1\} && \text{for all } d \in D.
\end{aligned}$$

The coefficient matrix of the budget constraints and of the constraints for the objective is of the form $A = \begin{bmatrix} A^1 & \mathbf{0} \\ -A^2 & I \end{bmatrix}$, where $I \in \{0,1\}^{|D| \times |D|}$ is the unit matrix, $A^1 \in \{0,1\}^{|M| \times |E|}$ denotes whether a segment belongs to a municipality, and $A^2 \in \{0,1\}^{|D| \times |E|}$ denotes whether a segment is on the path of an OD pair. Formally, we have for all $m \in M$, $d \in D$ and $e \in E$ that

$$A^1_{m,e} = \begin{cases} 1 & \text{if } e \in E_m, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad A^2_{d,e} = \begin{cases} 1 & \text{if } e \in E(W_d), \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A^1 has the consecutive ones property because of the assumption that municipalities contain only consecutive segments, and A^2 has it because the considered graph is a linear graph. As multiplying a row of a matrix by -1 only influences the sign of the determinant of the matrix and its submatrices, the matrix $\begin{bmatrix} A^1 \\ -A^2 \end{bmatrix}$ is totally unimodular by Lemma 2.3. The coefficient matrix A, which we obtain by appending a part of a unit matrix to this totally unimodular matrix, is then also totally unimodular. Therefore, the problem can be solved in polynomial time (see Section 2.1). □

Lemmas 5.10 and 5.12 can also be applied if $c_e = c$ and $u_e = u$ for all $e \in E$ and $L_d = u$ for all $d \in D$ for some arbitrary but fixed $c \in \mathbb{N}_{\geq 1}$ and $u \in \mathbb{R}_{>0}$. In this case, we can replace the budget constraints by $\sum_{e \in E_m} x_e \leq s_m \cdot {}^{b}\!/_{c}$ for all $m \in M$, and we can replace the objective function constraints $L_d y_d \leq \sum_{e \in E(W_d)} u_e x_e$ by $y_d \leq \sum_{e \in E(W_d)} x_e$. Then the assumptions of Lemmas 5.10 and 5.12 are satisfied.

If we consider a global decision maker (i.e., $|M| = 1$) in addition to the assumptions of Lemmas 5.10 and 5.12, those special cases also satisfy the conditions of Lemma 5.4, and thus the complete Pareto front of BRT($\star/Z = \infty/|M| = 1$) can be constructed in polynomial time.

5.4 Relaxations of the Single-Objective Problem

Because BRT*($\star/Z \geq 1/|M| \geq 1$) is NP-hard for both passenger responses, we study different relaxations, which yield dual bounds on the objective function value, this is, upper bounds on the number of passengers. The trivial lower and upper bounds are 0 and $\sum_{d \in D} t_d$, respectively.

First, it is easy to see that BRT*($\star/Z = \infty/|M| \geq 1$) is a relaxation of BRT*($\star/Z \geq k/|M| \geq 1$) because the BRT component constraint is omitted, which expands the feasible set but the objective function stays the same. Hence, BRT*($\star/Z = \infty/|M| \geq 1$) yields an upper bound on the number of newly attracted passengers in BRT*($\star/Z \geq 1/|M| \geq 1$). However, BRT*($\star/Z = \infty/|M| \geq 1$) is NP-hard itself for both passenger responses. Thus, we consider the special cases of Lemmas 5.10 and 5.12, which are relaxations of BRT*(LINEAR$/Z = \infty/|M| \geq 1$) and BRT*(MINIMPROV$/Z = \infty/|M| \geq 1$), respectively, as the following results show.

Lemma 5.13. *Let $m \in M$. If $F \subseteq E$ satisfies the budget constraint of municipality m, then it also satisfies $|F \cap E_m| \leq \frac{s_m b}{\min\{c_e : e \in E_m\}}$.*

Proof. By assumption, we have

$$s_m b \geq \sum_{e \in F \cap E_m} c_e \geq \sum_{e \in F \cap E_m} \min\{c_{e'} : e' \in E_m\}.$$

Hence, we also have that

$$\frac{s_m b}{\min\{c_e : e \in E_m\}} \geq \sum_{e \in F \cap E_m} 1 = |F \cap E_m|. \qquad \square$$

From Lemma 5.13, we obtain the following relaxations:

Corollary 5.14. *The problem*

$$\begin{aligned} \max_{F} \quad & \mathsf{pass}(F) \\ \text{s.t.} \quad & |F \cap E_m| \leq \frac{s_m b}{\min\{c_e : e \in E_m\}} \quad \text{for all } m \in M \\ & F \subseteq E \end{aligned}$$

is a relaxation of BRT*($\star/Z = \infty/|M| \geq 1$).

For BRT*(Linear$/Z = \infty/|M| \geq 1$), the relaxation in Corollary 5.14 is of the same form as the problem considered in Lemma 5.10 and can, hence, be solved in polynomial time.

Additionally, for MinImprov, we can relax the requirement of gaining at least L_d infrastructure improvement to attract passengers of OD pair $d \in D$ to needing one upgraded edge on the path. This can be modeled by unit infrastructure improvements $u_e = 1$ for all $e \in E$ and $L_d = 1$ for all $d \in D$. This is indeed a relaxation because, if $0 < L_d \leq \sum_{e \in F \cap E(W_d)} u_e$, then we have $F \cap E(W_d) \neq \emptyset$. We apply this in the following corollary:

Corollary 5.15. *The problem*

$$\begin{aligned} \max_{F} \quad & \sum_{\substack{d \in D: \\ F \cap E(W_d) \neq \emptyset}} t_d \\ \text{s.t.} \quad & |F \cap E_m| \leq \frac{s_m b}{\min\{c_e : e \in E_m\}} \quad \text{for all } m \in M \\ & F \subseteq E \end{aligned}$$

is a relaxation of BRT*(MinImprov$/Z = \infty/|M| \geq 1$).

The relaxation in Corollary 5.15 is of the same form as the problem considered in Lemma 5.12 and can, hence, be solved in polynomial time.

5.5 Summary

To determine the Pareto front of BRT*($\star/Z \geq 1/|M| \geq 1$), an ϵ-constraint-based solution method is presented in Algorithm 5.1, and an enumeration method is considered in Theorem 5.7. Bounds on the objective function value of the single-objective problem are given in Lemma 5.8 and in Section 5.4. We summarize the tractability and complexity results in Table 5.1.

	passenger response	Z	$\|M\|$	add. assump.	tractability/ complexity	reference
BRT	$\star$	≥ 1	≥ 1		intractable	Thm. 5.3
BRT	$\star$	$= \infty$	$= 1$		intractable	Thm. 5.3
BRT	$\star$	≥ 1	≥ 1	$c_e = c$, $s_m = \frac{1}{\|M\|}$	tractable	Thm. 5.4
BRT	$\star$	≥ 1	$= 1$	$c_e = c$	tractable	Thm. 5.4
BRT	$\star$	$= k$	≥ 1		in XP	Lemma 5.7
BRT*	Linear	≥ 1	≥ 1		NP-hard	Thm. 5.9
BRT*	Linear	$= \infty$	$= 1$	$u_e = 1$	NP-hard	Thm. 5.9
BRT*	Linear	$= \infty$	≥ 1	$c_e = c$	polynomial	Lemma 5.10
BRT*	MinImprov	≥ 1	≥ 1		NP-hard	Thm. 5.11
BRT*	MinImprov	$= \infty$	$= 1$	$u_e = 1$, $L_d = 1$	NP-hard	Thm. 5.11
BRT*	MinImprov	$= \infty$	≥ 1	$c_e = c$, $u_e = u$, $L_d = u$	polynomial	Lemma 5.12

Table 5.1: Overview of the tractability and complexity results for the BRT investment model. In the first column, BRT stands for the bi-objective BRT investment model, and BRT* stands for the single-objective BRT investment model encountered in Algorithm 5.1, which maximizes the number of passengers pass given a fixed budget b.

Chapter 6

Computational Experiments

In this chapter, we conduct computational experiments on artificial instances in Section 6.1 as well as for a case study on the planned BRT line around Copenhagen in Section 6.2. The insights of the computational experiments are summarized in Section 6.3.

6.1 Experiments on Artificial Instances

The Pareto front and the impact of the passenger response, the upper bound on the number of BRT components and the existence of municipalities are at the center of the computational experiments. These are analyzed in the context of a large collection of artificial instances with different interplays between the passenger potential and the upgrade costs. Moreover, to investigate the impact of municipalities, we consider different options to split the investment budget among them.

6.1.1 Description of Instances

All artificial instances consider a line (V, E) consisting of 25 stations with a complete set of OD pairs $D = V \times V \setminus \{(v, v) : v \in V\}$. The infrastructure improvement u_e is randomly chosen between 20 and 50 for each segment $e \in E$ and is the same in all instances. The artificial instances however differ in terms of the graph scenario $\alpha = (\alpha_1, \alpha_2)$, which determines a cost pattern for the upgrade costs per segment and a demand pattern, as well as in terms of the budget scenario β, which determined the budget split among five municipalities. The values considered for these parameters are given in Table 6.1.

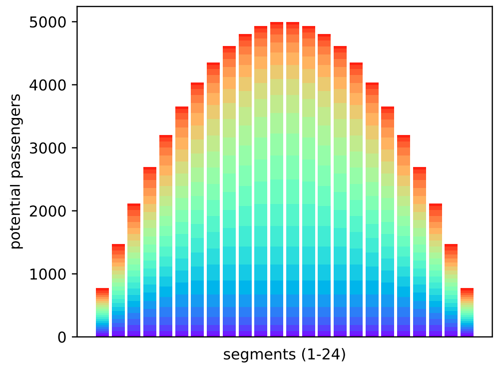
The cost pattern α_1 varies between uniform upgrade costs per segment (UNIT), a pattern with higher upgrade costs in the center of the line (MIDDLE), and a pattern with the highest upgrade costs at the ends of the line (ENDS). The cost pattern MIDDLE could, for example, model the situation that segments in the inner city are more complicated and hence more expensive to upgrade, while ENDS could represent that long highway segments outside a city are expensive to upgrade. The cost patterns together with the infrastructure improvements are depicted in Figure A.1 in Appendix A.

Parameter		Value	Explanation
α_1	cost pattern	UNIT	unit costs $c_e = 1$ for all $e \in E$
		MIDDLE	more expensive towards the middle of the line
		ENDS	more expensive towards the end stations of the line
α_2	demand pattern	EVEN	same passenger potential for all OD pairs
		HUBS	centered around large stations, passengers distributed according to the gravity model [Rod20]
		TERMINI	high passenger potential between end stations of the line
β	budget split	EQUAL	budget distributed equally among municipalities, i.e., equal budget shares s_m
		COST	budget shares s_m proportional to the costs of the segments in municipality m
		PASS	budget shares s_m proportional to the number of potential passengers entering or exiting in municipality m

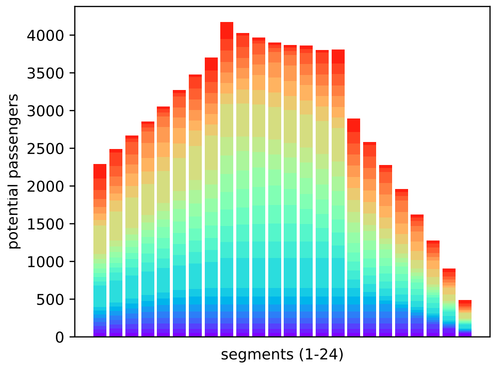
Table 6.1: Parameters for generating artificial instances.

We consider three different demand patterns α_2 that determine the number of potential passengers t_d for each OD pair $d \in D$, namely EVEN, HUBS and TERMINI, where the potential demand is evenly distributed over all OD pairs (EVEN), centered around three large stations (HUBS) or especially high between the end stations of the line (TERMINI). Figure A.2 in Appendix A shows the location of the large stations on the line. Additionally, Figure 6.1 shows the load profiles, this means the number of passengers traveling along each edge, resulting from the three demand patterns for α_2, where the height of a bar indicates the load of a segment, and the colored shading indicates the length of the boarded passengers' paths. We can see that HUBS generally leads to shorter path lengths than TERMINI and EVEN, whereas TERMINI has especially many passengers traveling from one end station to the other one by design. Moreover, EVEN has fewer passengers traveling around the terminals of the line than the other two, which have large stations at one or both ends. For additional information regarding the travel distances of passengers for these demand patterns, see Figure A.3 in Appendix A.

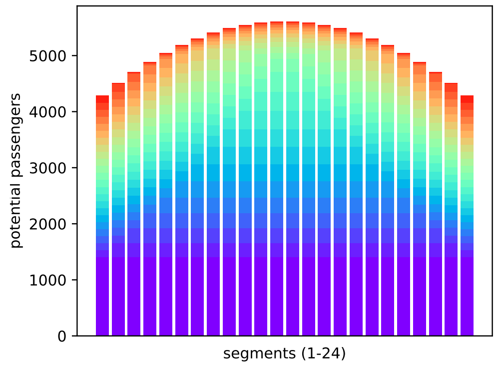
The budget split β describes the distribution of the total available investment budget b among the municipalities. We consider a distribution according to



(a) EVEN



(b) HUBS



(c) TERMINI

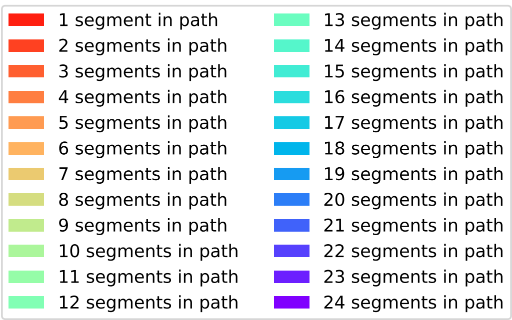


Figure 6.1: Load profiles of all demand patterns for the artificial instances. The horizontal axis contains each of the 24 segments of the BRT line. Each bar represents the number of potential passengers using the respective segment, and the coloring of a bar depicts the total travel distance of passengers using that segment.

equal budget shares (EQUAL), proportional to the costs required for upgrading all segments in a municipality (COST), and according to the total potential passenger volume that flows in and out of the stations that are assigned to a municipality (PASS).

6.1.2 Computational Study Design

In our computational experiments, we compute the Pareto fronts for all instances, focusing on the part where the investment budget does not exceed the investment costs for upgrading all segments, i.e., $b \leq \mathsf{cost}(E)$. As a consequence, dependent on the budget split, not all municipalities may have enough budget to upgrade all of their segments. In addition to varying characteristics of the instances (Section 6.1.1), we consider the different problem variants BRT($\lambda_1/\lambda_2/\lambda_3$) from Section 4.3:

Passenger Response (λ_1) All instances are evaluated for the objective functions Linear and MinImprov. For MinImprov, we require that a minimum of roughly 75% of the potential infrastructure improvements is achieved through upgrades before the passengers corresponding to that OD pair are attracted, in concrete terms:

$$L_d := \left\lfloor 0.75 \cdot \sum_{e \in E(W_d)} u_e \right\rfloor$$

for all OD pairs $d \in D$.

Number of BRT components (λ_2) We consider upper bounds on the number of BRT components $Z \in \{1, 2, 3, \infty\}$. Our experiments show that the difference between $Z = 3$ and $Z = \infty$ is generally small (see Section 6.1.4 on the influence of the number of BRT components), and therefore including more options for Z than $\{1, 2, 3, \infty\}$ would not lead to further insights in our setting.

Municipalities (λ_3) In order to determine the impact of the separate municipality budgets, each instance is evaluated both in the context of a global decision maker with a single budget ($|M| = 1$) as well as in the original context, where each municipality has its own budget constraint ($|M| = 5$). In the former setting, the global decision maker can spend the whole investment budget b, i.e., there is a single municipality with $s_1 = 1$, while in the latter setting, the investment budget is distributed among the municipalities according to the budget split β (see Table 6.1).

The combination of three cost patterns, three demand patterns, three options for the budget split plus the the scenario of a global decision maker ($|M| = 1$) and four upper bounds on the number of BRT components yield a total of $3^2 \cdot 4^2 = 144$ artificial instances that are evaluated regarding both passenger responses. The data is available at `https://doi.org/10.11583/DTU.23653893`.

6.1.3 Running Time

All instances are solved with Algorithm 5.1 using the solver CPLEX 22.1 on a computer with an Intel Xeon Gold 6126 processor, using 12 CPU cores and a total of 24 GB of RAM. The running time for computing the non-dominated points for the artificial instances is shown in Table 6.2. Here, we give the average time to

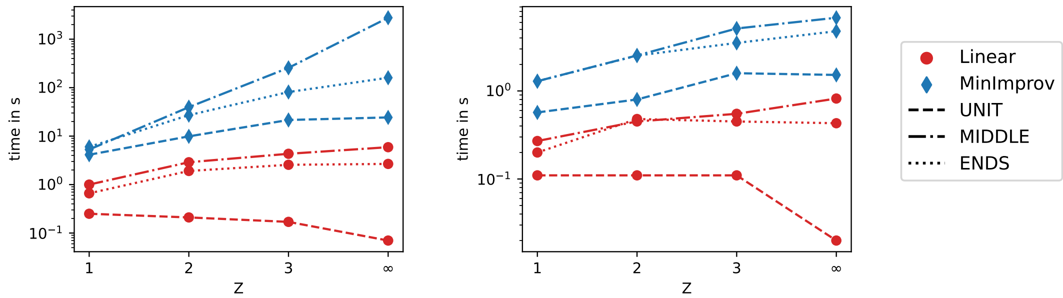
passenger response λ_1	cost pattern α_1	BRT($\star/Z \geq 1/\lvert M\rvert = 1$)			BRT($\star/Z \geq 1/\lvert M\rvert = 5$)		
		all points	# points	per point	all points	# points	per point
LINEAR	UNIT	0.17	25.00	0.007	0.09	7.47	0.012
LINEAR	MIDDLE	3.53	179.83	0.020	0.52	37.53	0.014
LINEAR	ENDS	1.95	107.00	0.016	0.39	28.25	0.014
MINIMPROV	UNIT	14.96	25.00	0.599	1.12	7.42	0.152
MINIMPROV	MIDDLE	768.00	105.42	5.098	3.93	26.08	0.139
MINIMPROV	ENDS	68.65	77.17	0.639	3.02	25.00	0.117

Table 6.2: Running time in seconds, number of obtained Pareto points and running time per Pareto point for problem variants BRT($\star/Z \geq 1/|M| = 1$) and BRT($\star/Z \geq 1/|M| = 5$). The results have been averaged over artificial instances sharing the same cost pattern α_1.

find the Pareto front, the average number of points on the Pareto front and the average time for obtaining a single non-dominated point for each passenger response, each cost pattern α_1 and both municipality scenarios $|M| \in \{1, 5\}$. Note that the reported values are averaged over all three demand patterns α_2 and over the considered upper bound on the number of BRT components $Z \in \{1, 2, 3, \infty\}$. Additionally, for the setting with municipalities ($|M| = 5$), the results are also averaged over the different budget splits β. The data is also presented in Figures 6.2 to 6.4 dependent on the upper limit Z on the number of BRT components.

The results in Table 6.2 show that the Pareto fronts can overall be computed quickly, especially for the passenger response LINEAR. Moreover, it can be seen that the introduction of separate municipality budgets ($|M| = 5$) consistently leads to a faster running time and fewer points on the Pareto front than the consideration of a global decision maker ($|M| = 1$). This is likely a result of the smaller solution space with separate municipality budgets, especially in combination with the upper bounds $Z \in \{1, 2, 3\}$ on the number of BRT components, where fewer combinations of items fit within the individual municipality budgets. The longest running times can be observed for the cost pattern MIDDLE in combination with the passenger response MINIMPROV, where especially the long running time for the setting with a global decision maker ($|M| = 1$) stands out. This might be explained by the middle segments often having the highest passenger load as well as being the most expensive to upgrade when considering the cost pattern MIDDLE. Looking at the number of points on the Pareto front, it can be observed that the number of non-dominated points is often significantly lower for the UNIT cost pattern than for the other cost patterns, which is in line with Lemma 5.4.

In addition, Figures 6.2 to 6.4 give a better insight on how the number of non-dominated points and the running time per point contribute to the total running time for the different choices of the upper limit Z on the number of



(a) $|M| = 1$ (b) $|M| = 5$

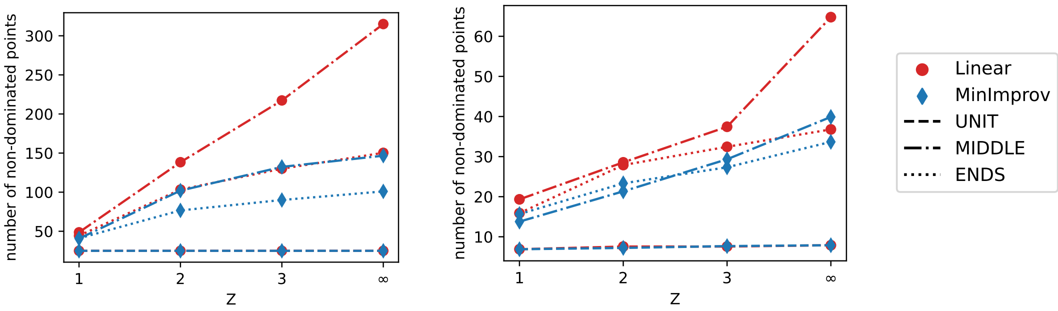
Figure 6.2: Running time in seconds of BRT($\star/|M| = 1/Z \geq 1$) and BRT($\star/|M| = 5/Z \geq 1$) with a logarithmic scale on the vertical axis. The values are averaged over all demand patterns and budget splits (if applicable).

BRT components. The red plots represent the passenger response LINEAR and the blue plots represent the passenger response MINIMPROV, while the line style represents the number of allowed BRT components Z. Figure 6.3 shows that there are at least as many non-dominated points for LINEAR as for MINIMPROV for the artificial instances, which can be explained by the greater number of values attainable by pass for LINEAR. Considering the running time per point in Figure 6.4, we however see that LINEAR is faster, which might be due to the discrete behavior of MINIMPROV and the additional discrete variable needed. For $Z = \infty$ together with the passenger response LINEAR, we see an especially small running time per point. Lemma 5.10 shows that this setting is a polynomial time special case.

Looking at the total running time in Figure 6.2, we can see that the average running time increases with Z, which is mainly because of the growing number of non-dominated points (see Figure 6.3), except for the linear special case of LINEAR together with the cost pattern UNIT, in which it decreases. We further see that the low running times per point for LINEAR lead to lower total running times even though more non-dominated points need to be computed.

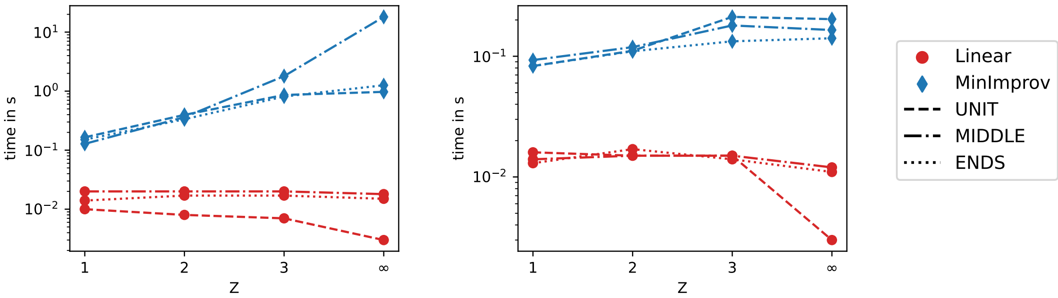
6.1.4 Analysis of Pareto Fronts

In this section, we analyze the influence of the passenger response, the number of BRT components, the demand pattern and the municipalities on the Pareto front. As described in Section 4.4, we compute the efficient solutions and the Pareto fronts with respect to the investment budget, but we evaluate the results with respect to the investment costs. Therefore, the following figures show the investment costs on the horizontal axis and the newly attracted passengers on the vertical axis. Both are given as percentage of the total number of potential



(a) $|M| = 1$ (b) $|M| = 5$

Figure 6.3: Number of non-dominated points of $\mathrm{BRT}(\star/|M| = 1/Z \geq 1)$ and $\mathrm{BRT}(\star/|M| = 5/Z \geq 1)$. The values are averaged over all demand patterns and budget splits (if applicable).



(a) $|M| = 1$ (b) $|M| = 5$

Figure 6.4: Running time per non-dominated point of $\mathrm{BRT}(\star/|M| = 1/Z \geq 1)$ and $\mathrm{BRT}(\star/|M| = 5/Z \geq 1)$ with a logarithmic scale on the vertical axis. The values are averaged over all demand patterns and budget splits (if applicable).

passengers and costs for upgrading all segments, respectively. Figure 6.5 shows the evaluation for a global decision maker ($|M| = 1$). The red plots again represent the passenger response Linear and the blue plots represent the passenger response MinImprov, while the line style represents the number of allowed BRT components Z. All graphs in a row share the same cost pattern α_1, and all graphs in a column share the same demand pattern α_2.

Influence of the Passenger Response In general, the non-linear objective MinImprov leads to solutions with fewer passengers per investment budget than Linear, with the exception of high level investments of at least around 75% of the total budget or more. This cut-off point at 75% correlates with the minimum improvement L_d of 75% required within MinImprov, for $(\alpha_1, \alpha_2) = (\mathrm{ENDS}, \mathrm{TERMINI})$ the cut-off point is a bit lower. Furthermore, the shape of the curve is typically more convex over a large range for MinImprov,

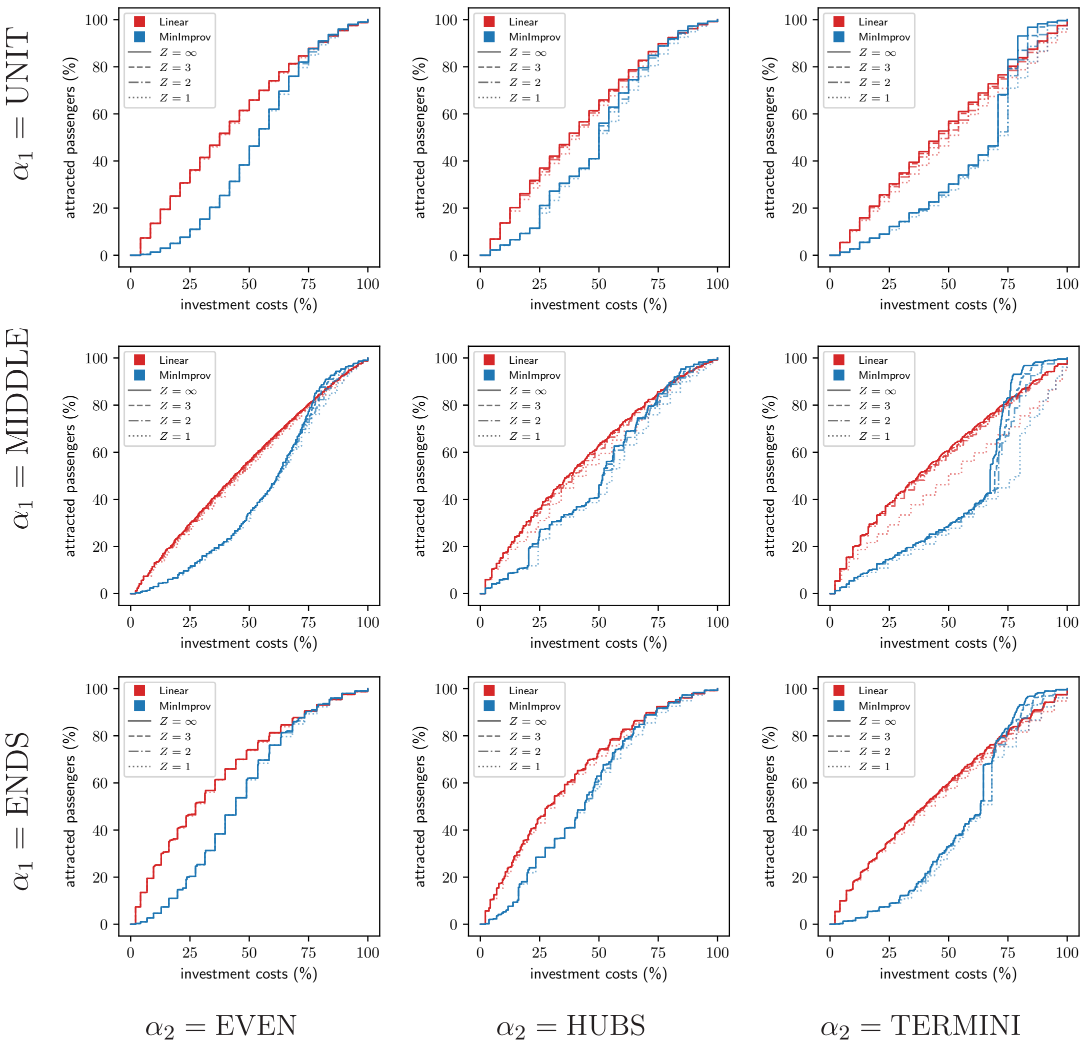


Figure 6.5: Evaluation of the non-dominated points of BRT(Linear$/Z \geq 1/|M| = 1$) (red) and BRT(MinImprov$/Z \geq 1/|M| = 1$) (blue) for artificial instances representing all choices for parameters α_1, α_2 and Z. Both attracted passengers and investment costs are given as percentage of the total number of potential passengers and costs for upgrading all segments, respectively.

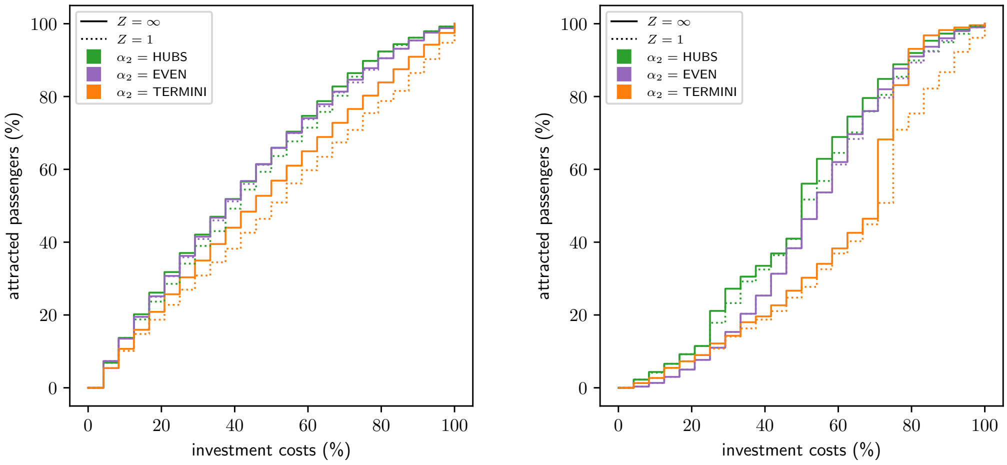
in which the return on investment, meaning the increase in the number of passengers attracted per investment, is generally increasing and only starts to reduce much later than for the passenger response LINEAR. LINEAR rather shows a higher return on investments at the lower investment levels. This can be explained by looking at the passenger responses. For the passenger response LINEAR, passengers of all OD pairs that are affected by upgrades are attracted in proportion to the realized infrastructure improvements. For the passenger response MINIMPROV, however, mainly passengers of OD pairs with a short travel distance that are affected by an upgrade are attracted at a low investment budget level. Only at higher investment budget levels, when sufficiently many segments can be upgraded, long-distance travelers are also attracted. Because the demand pattern TERMINI has around 14% of all passengers traveling along all 24 segments and the aggregated demand over the other OD pairs decreases only slowly in the path length (see Figure A.3 in Appendix A), the convexity effect is most strongly pronounced for this demand pattern. In comparison, the convexity effect is a bit less pronounced for EVEN and only weakly present for HUBS, for which the demand of long-distance journeys is generally decreasing with the path length. The gap between the two passenger responses is generally smaller for high investment budgets.

These results indicate that the passenger response has a strong impact on the trade-off between attracted passengers and investments. Investigating the passenger behavior as part of BRT feasibility studies would thus be important to determine an appropriate investment level.

Influence of the Number of BRT Components The impact of the upper bound on the number of BRT components Z diminishes quickly with size, where the numbers of attracted passengers and the investment costs of non-dominated points for $Z = 3$ and $Z = \infty$ are almost identical. The impact of Z is higher for larger investment budgets and also more prevailing for the passenger response MINIMPROV. Additionally, for the cost pattern MIDDLE and the demand pattern TERMINI we can see a big impact of the BRT component constraint.

In general, we see that restricting the number of BRT components to $Z \in \{2, 3\}$ comes at small costs in the setting of a global decision maker ($|M| = 1$), while it could lead to lines that may be considered of higher quality from a passenger perspective. Finally, from a computational perspective fixing Z can reduce the computational complexity, as shown earlier in Theorem 5.7 and in Section 6.1.3.

Influence of the Demand Pattern Figure 6.6 depicts the effect of the demand pattern on the sets of non-dominated points for BRT($\star/Z = \infty/|M| = 1$) (solid lines) and BRT($\star/Z = 1/|M| = 1$) (dotted lines) for the cost pattern UNIT. The results for the cost patterns ENDS and MIDDLE are similar and can be seen in Figure A.4 in Appendix A. For the passenger response LINEAR, the demand



(a) LINEAR (b) MINIMPROV

Figure 6.6: Evaluation of the non-dominated points of BRT($\star/Z \geq 1/|M| = 1$) for artificial instances with cost pattern α_1 = UNIT and $Z \in \{1, \infty\}$ and all choices for the demand pattern α_2. Both attracted passengers and investment costs are given as percentage of the total number of potential passengers and costs for upgrading all segments, respectively.

patterns behave similarly. The only thing that stands out is that TERMINI leads to slightly fewer attracted passengers compared to HUBS and EVEN. This can also be seen for MINIMPROV. In addition, we see a large jump in attracted passengers for MINIMPROV with demand pattern TERMINI when around 75% of the budget is invested. This is due to the relatively high number of passengers that travel along all 24 segments (about 14% of all passengers) because realizing 75% of the potential improvement suffices to attract all those passengers according to the definition of MINIMPROV. The influence of restricting the number of connected components to $Z = 1$ is especially pronounced for the demand pattern TERMINI. Here again, the high number of passengers using all 24 segments is affected most by restricting the set of upgraded segments.

Influence of Municipalities The impact of the distribution of the investment budget among the municipalities is depicted in Figure 6.7, which is similar in set-up to Figure 6.5 with the difference that the line styles now represent different budget splits among the municipalities, with the solid line representing the case of a global decision maker. Moreover, all results in Figure 6.7 are obtained without the BRT component constraint ($Z = \infty$).

The introduction of municipalities generally leads to lower numbers of attracted passengers. Because of the distribution of the investment budget among the municipalities, compared to the case of a global decision maker, only a smaller share can be invested and not always in the segments that would attract the most pas-

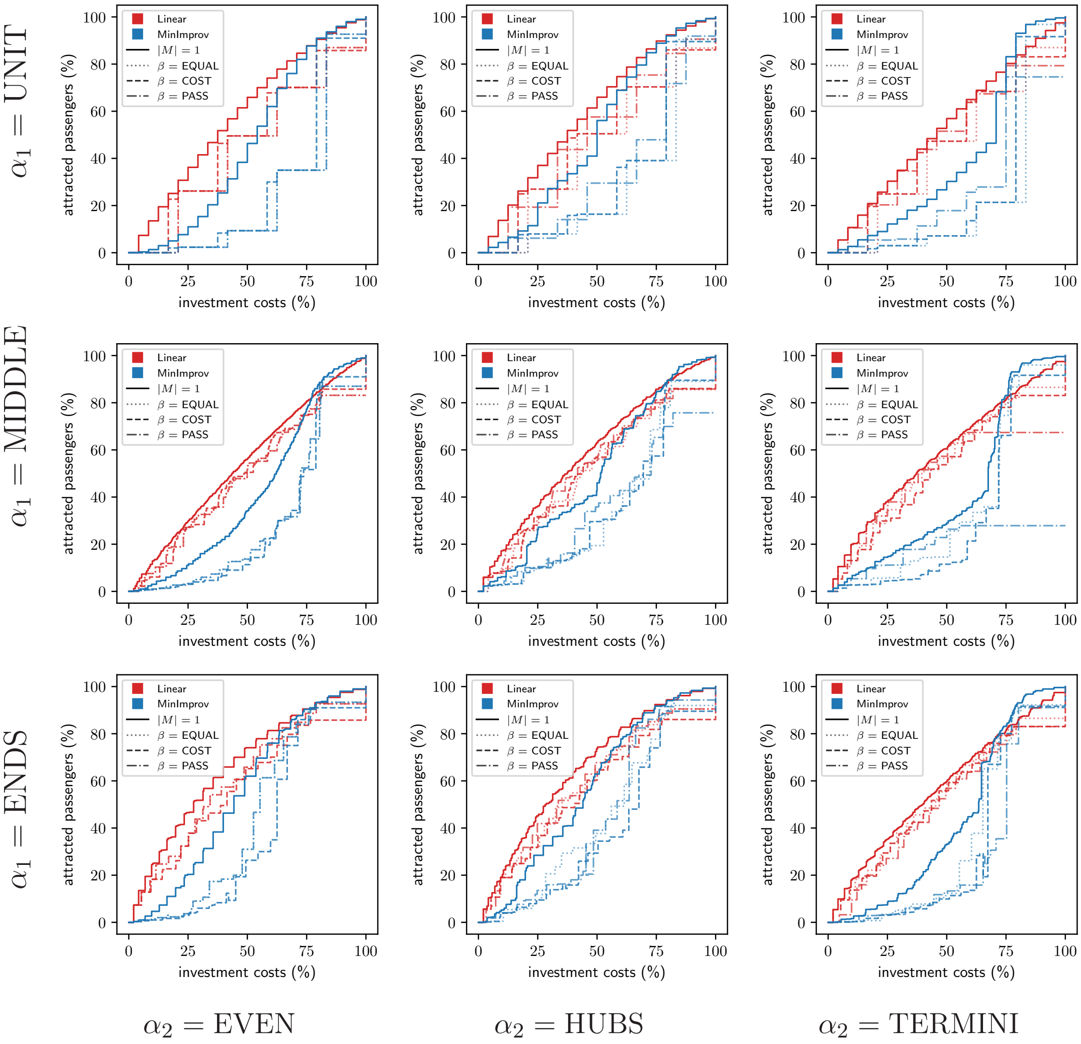


Figure 6.7: Evaluation of the non-dominated points of BRT(LINEAR/$Z = \infty$/$|M| \geq 1$) (red) and BRT(MINIMPROV/$Z = \infty$/$|M| \geq 1$) (blue) for artificial instances with $|M| \in \{1, 5\}$ and all choices for parameters α_1, α_2 and β. Both attracted passengers and investment costs are given as percentage of the total number of potential passengers and costs for upgrading all segments, respectively.

sengers. Moreover, considering several municipalities intensifies the findings of the case with a global decision maker: MinImprov requires higher investments for the same number of passengers until around 75% of investments and is characterized by a return on investment that follows a more convex shape over a large range compared to the passenger response Linear, with the same explanations as for the case of a global decision maker. The impact of the chosen budget split among the municipalities is typically higher for MinImprov as well. For the budget split COST, all segments can be upgraded at an investment budget of 100% of the total upgrade costs by definition of COST. However, if the investment budget is only slightly smaller, there is one segment for each municipality, i.e., five in total, that cannot be upgraded anymore, which leads to the jump in the number of passengers for the budget split COST. For budget splits other than COST, the full upgrade may not be achievable even at an investment budget equal to 100% of the total upgrade costs because individual municipalities may not have enough money available to upgrade all segments belonging to them. This is specifically visible for α_1 = MIDDLE, α_2 = TERMINI, β = PASS: In this case, only about 30% of the passenger potential are attracted at 100% investment budget, showing that the investment costs stagnate at 55% because of the interplay between the budget split and the demand pattern. We also see that the budget split COST

The results indicate that, especially in case of a non-linear relationship between BRT upgrades and attracted passengers, establishing a framework for collaboration and co-commitment has a large influence on the number of attracted passengers and thereby on the line potential.

6.2 Greater Copenhagen Case Study

We now focus on the case study for the planned BRT line in Greater Copenhagen. We first describe the case study and the corresponding instances in Section 6.2.1. Afterwards, we analyze the Pareto plots that are obtained for these instances in Section 6.2.2.

6.2.1 Description of Instances

Currently, the Capital Region in Denmark (a regional government) is planning to build a set of BRT lines within Copenhagen and the urban area surrounding it, i.e., Greater Copenhagen. One of these new BRT lines will run foremost along the route of the bus line 400S, which is currently a traditional mixed traffic line. A pre-assessment study was conducted for the BRT line that calculated the expected costs, travel durations and number of passengers per station for five different route alternatives [VRM22]. These five route alternatives are shown in Figure 6.8.

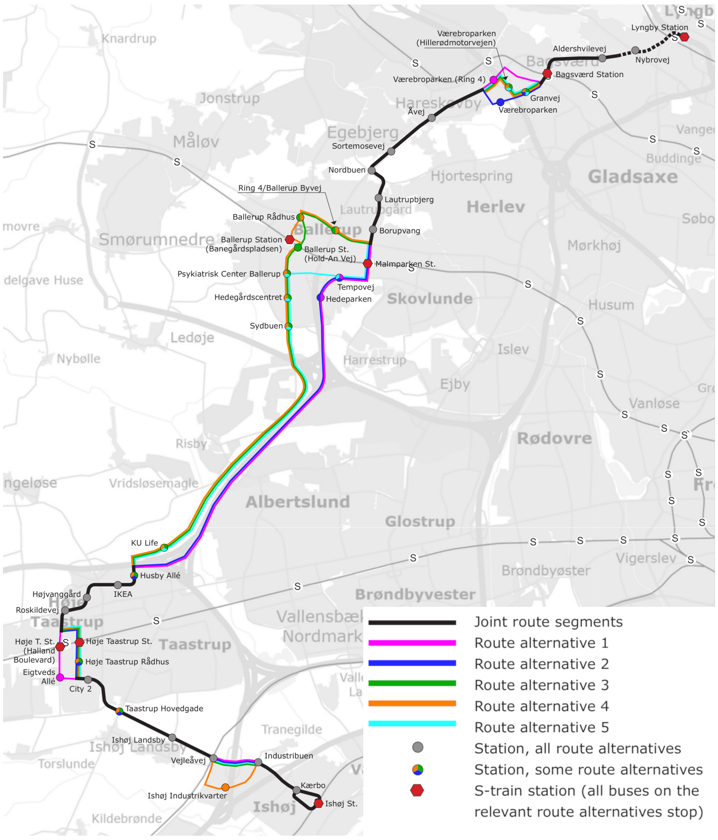


Figure 6.8: Route alternatives for a new BRT line in Greater Copenhagen. Adapted from [VRM22].

All five route alternatives run through a total of eight municipalities. These municipalities have authority over their local investments in the BRT line, and investments in local infrastructure would not be possible without their involvement. In addition to the municipalities, also the Capital Region, the central Danish government and Movia are involved in the planning process. Movia is a public transport agency funded by the collective of municipalities in the Capital Region, which highlights the willingness of the municipalities to work together to find socially optimal solutions for public transport in the region. Moreover, due to the expertise available within the agency, Movia overall takes a leading role in the design of the new BRT line and thereby provides suggestions that then need to be approved by the municipalities. This process can be iterative: Municipalities discuss solutions and revise their budget levels, followed by new suggestions from Movia. Hereby, the proposed model can advise Movia on how sensitively the number of newly attracted passengers reacts to a reduction in the investment budget. Thereby, it can illustrate the importance of achieving a high upgrade level and may aid the transport agency in selecting its strategy.

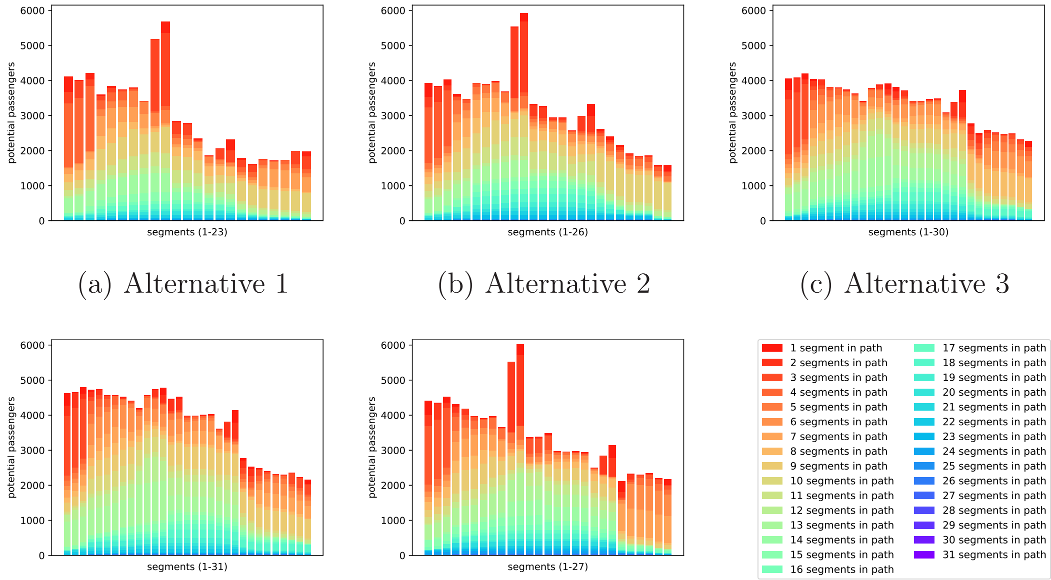
We use the data from the pre-assessment study to derive instances for the BRT investment model for each of the five route alternatives. These instances contain between 24 and 32 stations, depending on the route. The current plan includes connecting the BRT line via Nybrovej to Lyngby station, even though the responsible municipality has indicated it is not willing to invest in upgrading the infrastructure on their segments. Therefore, our case includes two segments that cannot be upgraded. The remaining seven municipalities are willing to partake in the BRT project. The upgrade costs per segment are derived from the required infrastructure investments for the line provided in the pre-assessment. Moreover, the potential benefit of upgrading a segment, this means the infrastructure improvement, is defined by the difference between expected travel time of the current mixed traffic line and the new expected travel times of the BRT line as defined in the pre-assessment. The upgrade costs and infrastructure improvements are depicted in Figure A.5 in Appendix A.

In addition, we constructed an estimate of the future OD data by combining the estimated passenger demand per station from the pre-assessment study with the current OD data on the existing bus line 400S. A customized mapping was built for OD pairs that do not yet exist on the bus line 400S. The resulting load profiles were determined based on conversations with Movia. The obtained load profiles are shown in Figure 6.9, where the height again indicates the load per segment, and the coloring indicates the length of the boarded passengers' paths. Compared to Figure 6.1 for the artificial instances, more passengers travel only short distances and fewer travel for more than 20 segments. We assume that a fixed percentage of each OD pair can be attributed to passengers newly attracted by infrastructure improvements.

We consider two potential budget splits between the seven municipalities that are willing to invest in the line. These are the cost-based and the passenger-based budget splits $\beta \in \{\text{COST}, \text{PASS}\}$, as described in Table 6.1. Because the actual costs and the number of passengers per municipality vary strongly, considering the EQUAL budget split is unrealistic. Additionally, we consider the scenario of a global decision maker ($|M|$). The impact of the number of allowed BRT components is evaluated for $Z \in \{1, 2, 3, \infty\}$. There are thus $3 \cdot 4 = 12$ instances for each of the five route alternatives, which yields a total of 60 instances for the case study. The instances are available at `https://doi.org/10.11583/DTU.23664069`.

6.2.2 Analysis of Pareto Fronts

We now look at the results of our experiments, where our aim is to analyze and compare the investment trade-offs for the five BRT route alternatives, taking into account the effect of the different passenger responses and budget splits over the municipalities. Here, we use a similar computational set-up as described for the artificial instances in Section 6.1.2. Moreover, the investment budget is limited



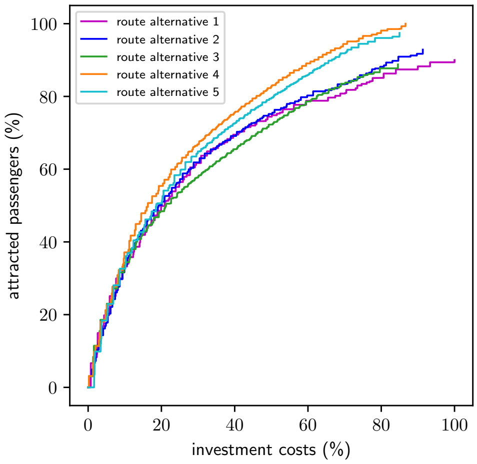
(a) Alternative 1 (b) Alternative 2 (c) Alternative 3

(d) Alternative 4 (e) Alternative 5

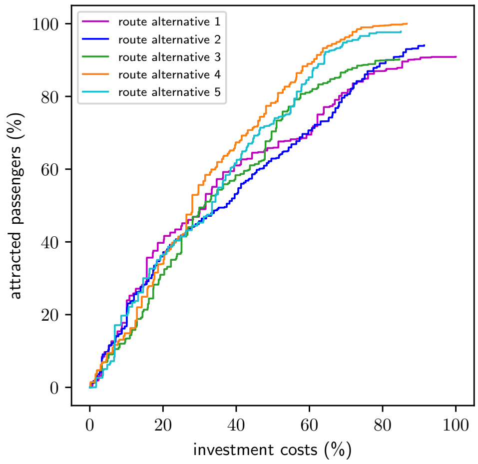
Figure 6.9: Load profiles for the five route alternatives. The horizontal axis contains each segment of the considered route alternative from north (Lyngby St.) to south (Ishøj St.). Each bar represents the number of potential passengers using the respective segment, and the coloring of a bar depicts the total travel distance of passengers using that segment.

to the investment costs for upgrading all segments of the most expensive route alternative for better comparability between the route alternatives. Dependent on the budget split, not all municipalities may have enough budget to upgrade all of their segments. The resulting Pareto fronts for the two passenger responses, with and without municipalities, evaluated regarding the investment costs are given in Figure 6.10 for the setting without a BRT component constraint ($Z = \infty$). Here, the plots in the top row provide the results when there is a global decision maker ($|M| = 1$), and the plots in the bottom row are for the case with municipalities ($|M| = 7$). Each graph indicates the investment costs as a percentage of the costs for upgrading all segments of the most expensive route alternative on the horizontal axis. The vertical axis indicates the attracted passengers relative to the maximum number of potential passengers over all route alternatives. This scaling on both axes allows to directly compare the route alternatives to each other.

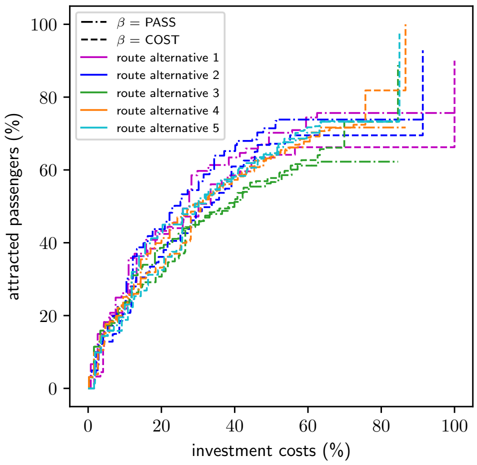
The obtained Pareto plots show that many of the observations from the artificial results carry over to the Greater Copenhagen case study: It can be seen that the number of attracted passengers is again in general higher for the passenger response Linear than for the passenger response MinImprov, except for invest-



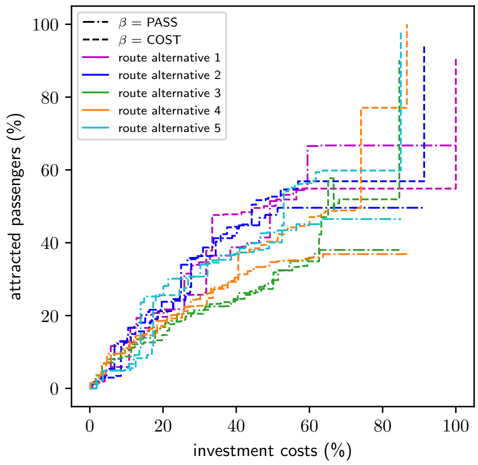
(a) $|M| = 1$, Linear



(b) $|M| = 1$, MinImprov



(c) $|M| = 7$, Linear



(d) $|M| = 7$, MinImprov

Figure 6.10: Comparing investment costs and attracted passengers for the five different route alternatives for $Z = \infty$. The investment costs are given as a percentage of the costs for upgrading all segments of the most expensive route alternative, and the attracted passengers are given as a percentage of the maximum number of potential passengers over all route alternatives.

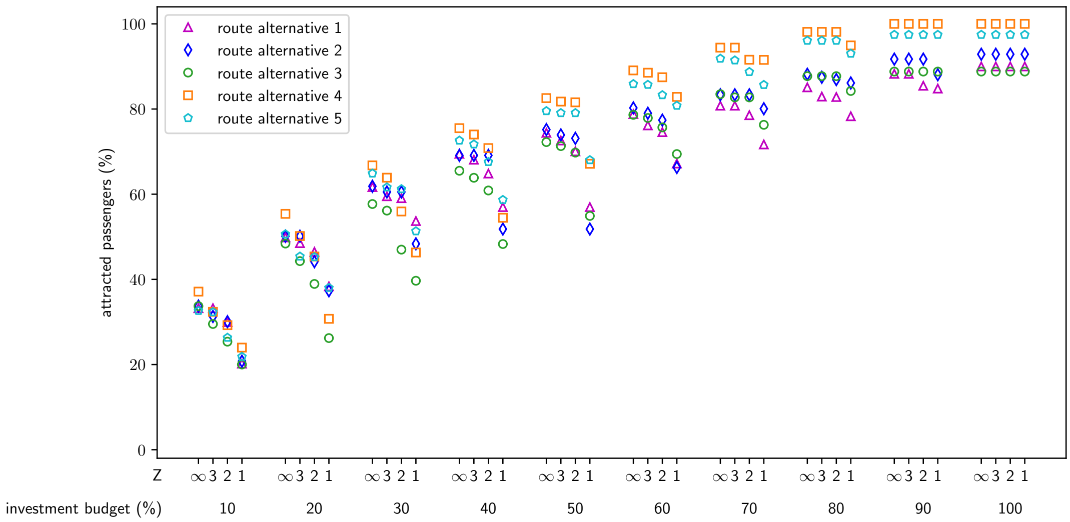
ment levels that are above 75% to 80%, and that this effect is more pronounced when including the different municipalities. Moreover, the introduction of municipality budgets has a significant impact on the number of attracted passengers, especially under the passenger response MinImprov. However, especially apparent in these case study results is the ability of the budget split $\beta = \text{COST}$ to achieve a significantly higher number of passengers at higher investment levels. This effect can be attributed to the presence of segments with very high upgrade costs, which are hard to upgrade for municipalities when they are not awarded a budget share that is in line with these upgrade costs. Also for budget split COST, we can again see a big jump in the number of attracted passengers when the investment budget reaches the total costs of a route alternative, which is the case because slightly decreasing the investment budget simultaneously prevents all seven municipalities from upgrading all of their segments. Furthermore, for MinImprov, we cannot observe an apparent cut-off point as in some cases for the artificial instances. This is most likely due to the higher number of passengers that travel only a few segments and the small number of passengers that travel long distances. In contrast, the demand pattern TERMINI, for which the cut-off at around 75% of the investment costs is the most pronounced (see Figures 6.5 and 6.7), has a high number of passengers that travel for many segments.

When focusing on the comparison of the route alternatives, Figure 6.10 shows that there is not a universal ordering of the route alternatives with respect to the number of attracted passengers. Instead, this ordering depends on both the investment level and the passenger response. For example, it can be seen that the route alternatives 4 and 5 lead to the largest number of attracted passengers for middle to high investment levels under both passenger responses for $|M| = 1$, which can be explained by the higher total passenger potential for these alternatives. However, for $|M| = 7$ and at an investment level between 30% and 70%, the route alternatives 1 and 2 yield the highest numbers of attracted passengers for both passenger responses. For low investment levels, the numbers of attracted passengers deviate less between the route alternatives, but it depends on the precise investment level, which route alternative is best.

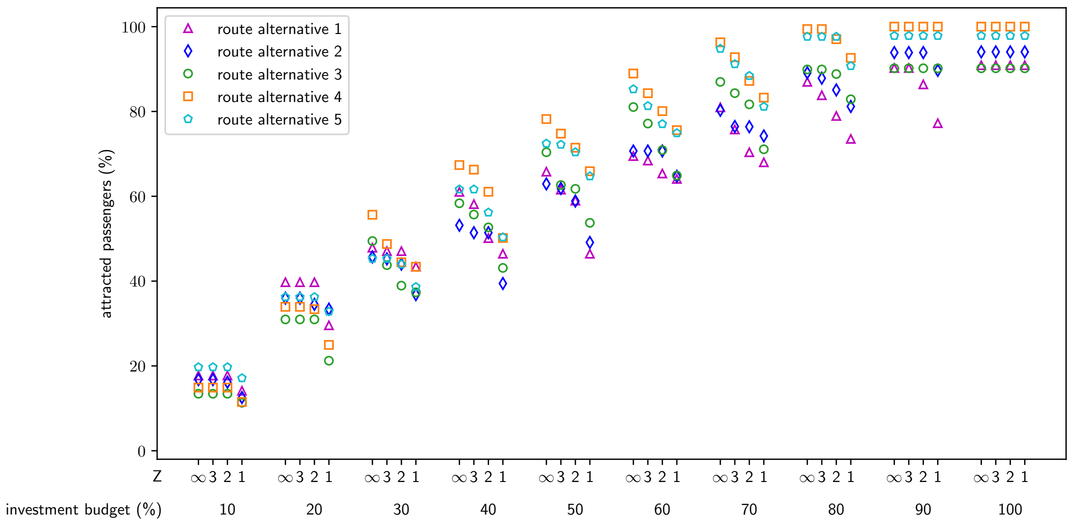
Our results thus show the importance of obtaining knowledge about the passenger response and the willingness of municipalities to invest before a final route alternative is chosen for the BRT line.

Influence of the Number of BRT Components It remains to analyze the impact of the BRT component constraint for the Greater Copenhagen case study. This effect is depicted in Figure 6.11, which analyzes the effect of the number of allowed BRT components Z on the number of attracted passengers for each of the five route alternatives and for both passenger responses. The results are computed for the setting of a global decision maker ($|M| = 1$). Moreover, to make the impact of the BRT component constraint more visible, this figure condenses the Pareto plots to ten investment budget levels and shows the solution with the highest number of attracted passengers with at most this investment budget level.

As with the artificial instances, Figure 6.11 shows that restricting the number of components leads to a reduced number of attracted passengers for all route alternatives. This effect is strongest for investment budget levels that are closer to the middle and lower end. By design, no effect can be seen for the highest investment level because all segments are upgraded. Comparing the passenger responses Linear and MinImprov, an interesting difference is that the impact



(a) LINEAR



(b) MINIMPROV

Figure 6.11: Influence of the upper bound on the number of BRT components on the percentage of attracted passengers for the five route alternatives. This figure condenses Pareto plots to ten investment budget levels and shows the solution with the highest number of attracted passengers with at most this investment budget level. The investment budget is given as a percentage of the costs for upgrading all segments of the most expensive route alternative, and the attracted passengers are given as a percentage of the maximum number of potential passengers over all route alternatives.

of restricting the number of BRT components is stronger for Linear for the very low investment levels. A reason for this is that Linear gains passengers proportional to the infrastructure improvement realized. In contrast to MinImprov, Linear does not have an incentive to upgrade adjacent segments so that the BRT component constraint poses a stronger restriction. In addition, it can be seen that it is again especially the restriction to a single component that leads to a strong reduction in the number of passengers. For instances of the case study, there is a difference between allowing 2, 3 or arbitrarily many BRT components for most available budget levels, although the solution for at most 3 BRT components comes close to that of allowing arbitrarily many BRT components.

6.3 Summary

The computational experiments for artificial instances as well as for the Greater Copenhagen case study analyze the impact of the passenger response, the separate municipality budgets and the BRT component constraint.

The experiments indicate that splitting the investment budget over municipalities significantly reduces the number of attracted passengers, which underlines the importance of collaboration between the municipalities.

Also the requirement to only have one BRT component results in a significant reduction in the number of attracted passengers in some cases. However, as soon as two or three BRT components are allowed, the impact is far smaller. Therefore, it seems reasonable to include a restriction on the number of BRT components in order to obtain a more consistent BRT infrastructure and to improve the passenger experience.

Regarding the artificial instances, the chosen threshold of 75% of the infrastructure improvements being necessary to attract passengers for the passenger response MinImprov is visible in the results: For investment costs below 75% of the total costs for upgrading all segments, the passenger response Linear indicates higher numbers of attracted passengers than the passenger response MinImprov. This changes for higher investment costs, but the values are quite close. The passenger response and the demand pattern contain crucial information about the behavior of (potential) passengers and have a strong impact on the results of the experiments. This emphasizes the need to further investigate passenger behavior.

The Greater Copenhagen case study confirms many of these observations, showing that they translate to real-world instances. Furthermore, the Greater Copenhagen case study shows that the ranking of the route alternatives is highly dependent on both the passenger response and the available investment budget. Hence, obtaining a good estimate on how passengers respond to the upgrades and on the extent to which municipalities are willing to invest is crucial for selecting the best route alternative.

Chapter 7

Outlook

In Part I of this thesis, we study the bi-objective BRT investment model, which focuses on determining the set of segments to be upgraded for a BRT line balancing the number of attracted passengers and the investment budget. Municipalities are considered in this problem through separate municipality budgets. Moreover, this problem allows the restriction of the number of separate sequences of upgraded segments to prevent frequent switching between upgraded and non-upgraded segments.

We analyze the two extreme cases of a linear (Linear) and a threshold (MinImprov) passenger response to upgrades. Considering mixes of these two passenger response functions would be a natural next step. Such a mix could, e.g., be a piecewise linear response function to upgrades, where the impact of an upgrade depends on the overall extent to which upgrades are realized. This would include the special case where the number of passengers grows linearly as soon as a certain threshold of infrastructure improvements is reached. Note that piecewise linear objective functions can be integrated into the algorithmic approach suggested in Section 5.1, meaning that the suggested ϵ-constraint-based algorithm can still be used to find the complete Pareto front.

We assume an interest of the municipalities in a social optimum, which might occur when there is a third party, such as a transport agency, that makes suggestions to the municipalities as to which segments should be upgraded. With the idea of acknowledging the individual interests of the municipalities and the need to justify investment decisions, a variant of the BRT investment model is considered in [Hoo+22], which is coauthored by the author of this thesis. In [Hoo+22], the single-objective setting BRT*$(\star/Z = \infty/|M| \geq 1)$ without the BRT component constraint, a new constraint is considered for each municipality that determines the municipality budget available based on the number of passengers that are attracted. More precisely, for every municipality, a set of OD pairs is defined, which represents the target group of a municipality, for example, all OD pairs that start or end in the respective municipality. For every attracted passenger from the target group, a certain amount of budget is made available for a municipality and can be invested. While a high investment of one municipality might encourage investments of other municipalities, the model does not fully reflect on the individual interests of the municipalities. Note that not only the impact of the municipality's own investments is taken into account for releasing budget, but also investments of other municipalities make budget available. For future work it would thus be interesting to focus on a game-theoretic setting that models the relation between the attracted passengers and the available budget as well as interactions between the municipalities. Moreover, setting incentives for investments could be considered within the context of a central government. The aim could be to find a subsidy scheme to attract the most passengers with minimum subsidies subject to the internal competition between municipalities and their individual interests.

Another interesting direction of future work could be considering the investment problem in a network context instead of for a single line, either by including the determination of the route of the BRT line or by considering that other lines could (be rerouted to) profit from the upgraded BRT infrastructure as well. In a network setting, other models and aspects for passenger behavior could be considered, e.g., including the travel time [SS20] and fares [SU22] as well as route and mode choice. Also, the inclusion of operating constraints considering load profiles, e.g., in the setting of self-driving minibuses with innovative operating modes [GSH22] could be an interesting direction.

While this thesis is focused on the application of public transport, the BRT investment model may as well be applied to the optimization of, for example, infrastructure of bicycle highways. These are bicycle paths that are meant for long-distance traffic. A well-developed cycling infrastructure increases safety of cyclists (see, e.g., [Sch+21]) and provides an alternative for commuting. Here, too, there is a need to determine where an upgrade of the infrastructure is particularly beneficial and financially feasible. Similar to the BRT line case study in Greater Copenhagen, the BRT investment model could be used in feasibility studies for cycling infrastructure. Here, route alternatives can be analyzed and evaluated to provide additional information to decision makers.

Part II

Fare Planning

Fare structures in public transport are an important design element that involves the interests of both (potential) passengers and operators alike. For passengers, fares are one among several criteria for mode and route choice. The affordability and the perceived fairness of fares influence people's decisions to opt for public transport over other modes of transport. In this part, we study two models to optimize fares: the fare deviation model and the revenue-passenger model.

Chapter 8

Introduction

Fares for public transport usage affect the number of people traveling by public transport as well as the passenger satisfaction, and they are important for covering the costs of the public transport operator. A variety of fare structures (which assign prices to paths) are implemented worldwide, each with a different focus and purpose. These can be flat or differential fares, where the latter are further distinguished according to whether they depend on, e.g., a distance, duration, time (e.g., time of day or season) or quality (e.g., standard/express train) component of the journey [Fle+96; Sch+16; CB21]. In this thesis, we focus on flat, distance and zone tariffs, which are very popular in many countries.

In a *flat tariff*, every ticket has the same price. This has the advantage that it is very easy to understand, but on the other hand it is often perceived as unfair because passengers with a short journey pay the same price as passengers with a long journey.

At the other end of the spectrum, there are *distance tariffs*, which may, for example, depend on the actual distance traveled in the network (network distance tariff) or on the Euclidean distance between the start and the end station of the journey (beeline distance tariff). In this thesis, we consider *affine* distance tariffs, which are composed of a base amount and an additional distance price which is multiplied with the network distance or the Euclidean distance, respectively. Hence, the fare is directly linked to the length of a journey. The rising popularity of mobile tickets and smart cards has led to an increased interest in distance tariffs because check-in/check-out systems can be used to determine the distance.

Zone tariffs group stations to zones and set fares based on the traversed zones. While for affine distance tariffs all station pairs have an individual price, there are fewer different price levels in zone tariffs. In this thesis, we consider a *counting-zones pricing*, which means that the area of the public transport operator is divided into zones and the price of a journey is determined according to the number of zones that are traversed. Hence, the number of zones can be seen as an approximate distance measure. We distinguish between the option that a zone is counted each time that it is entered (multiple counting) and that the total number of different zones is counted, i.e., counting a zone at most once (single counting). Furthermore, we consider zone tariffs with and without requiring connected zones. In practice, zones are not always required to be connected which leads to separate parts of the same zone that can only be reached from each other by traveling through other zones.

As two important criteria in practice for a fare structure to be fair and consistent, we regard the no-elongation property and the no-stopover property introduced in [SU20; SU22]. While the no-elongation property ensures that it is not beneficial to buy a ticket for a longer journey that contains the actual path, the no-stopover property makes sure that it is not beneficial to split a ticket into several tickets for subpaths. When designing fare structures, we deem these properties important because, on the one hand, they prevent the undercutting of tariffs and, on the other hand, they make it easier to buy tickets because passengers do not need to check whether other tickets are cheaper than that for their actual journey. Note that in practice, these properties are not always fulfilled. Furthermore, for insights into horizontal and vertical equity in fare planning, we refer to [RSC20], which examines the geographical and distributional fairness of flat, distance and zone tariffs.

In Part II, we address the optimization of tariffs. We consider two models: the fare deviation model and the revenue-passenger model.

Fare Deviation Model The fare deviation model is a reference price model, which means that the objective is to design fares that are close to reference prices given for all OD pairs under the assumption of a fixed demand. The notion of reference prices has been introduced in [HS95] and has further been used in [BK03; HS04; Pal13; PM17; GMS17]. The reference prices can, for example, be chosen as prices that are considered fair, in order to increase the acceptance of the fares and attract more passengers, or as prices of a former fare structure to maintain the prices passengers are used to. Here, in the fare deviation model, an objective function is implemented that minimizes the weighted sum of absolute deviations from reference prices.

Revenue-Passenger Model The affordability and the perceived fairness of fares influence people's decisions to opt for public transport over other modes of transport, for example, their own car. When the fares exceed a certain price limit (willingness to pay), it is reasonable to assume a deterrent effect leading to a reduction in the attractiveness of public transport and, therefore, ridership. Conversely, for operators, fares directly impact the revenue. An increase of prices, for example, increases the income per sold ticket but might decrease the ridership and therefore the total number of sold tickets. With the revenue-passenger model, we investigate the trade-off between revenue and number of passengers. For each OD pair, we consider multiple demand groups that differ in their willingness to pay. If the fare for an OD pair exceeds the willingness to pay of a demand group, this group does not use public transport. These demand groups could, for example, be captive and choice passengers, where the willingness to pay is dependent on whether or not an alternative mode is available, e.g., a car. Another categorization of demand groups could be based on age and income. We introduce a bi-objective model that optimizes fare structures and considers the Pareto front of revenue and number of passengers.

Contribution and Outline

In Part II, we study two tariff optimization models that balance the interests of passengers and operators. To do so, in Chapter 9, we introduce formal definitions of fare structures, in particular flat, affine distance and zone tariffs, and their properties, which lays the foundation of the later modeling and analysis.

In Chapter 10, we study the *fare deviation model*, which minimizes the weighted sum of absolute deviations from given reference prices. Under consideration are flat, affine distance and zone tariffs. We identify the fare deviation model for flat (F-FDM) and affine distance tariffs (D-FDM) as median problems and thus show the solvability in linear time. Further, we analyze and compare different variants of the fare deviation model for zone tariffs (Z-FDM) that appear in the literature. Moreover, we investigate the complexity of the fare deviation model for zone tariffs. In particular, we show NP-hardness in general, and develop a polynomial time algorithm for the price-setting subproblem with fixed zones satisfying the no-elongation property.

In Chapter 11, we introduce the *revenue-passenger model*, which is a bi-objective model maximizing the revenue and the number of passengers. We formulate a general model that can be applied for any fare strategy. Its complete Pareto front can be determined with the ϵ-constraint method. We then study the specific problem for flat tariffs (F-RPM) and affine distance tariffs (D-RPM). In both cases, we identify a finite candidate set, based on which we develop algorithms that compute the Pareto front in quasilinear or cubic time, respectively. We also perform computational experiments on structured datasets and analyze the number of non-dominated points and their respective efficient solutions. The experiments emphasize the advantage in running time of the specialized solution methods for affine distance tariffs compared to the MILP-based ϵ-constraint method. Finally, we consider the revenue-passenger model for zone tariffs (Z-RPM), which is NP-hard but admits a pseudo-polynomial time algorithm.

Possible extensions of the tariff optimization models are discussed in Chapter 12.

Related Literature

In the literature, various optimization models are presented with the aim to determine a fare structure. The maximization of revenue, demand, fairness or social welfare can be objectives pursued in fare planning. Thereby, passenger choices are considered to varying degrees: from fixed paths, maybe with a reference price, a willingness to pay or elasticities to route choice subproblems and equilibrium constraints. We first review publications that perform simulation studies and regression analysis to examine *given* fare structures as well as publications on cheapest paths. Our main focus then is on the fare *planning* literature that applies optimization models and techniques to determine optimal fares with respect to the above mentioned objectives. We separately give an overview of literature on distance- and zone-based fare structures.

Examination of Given Fare Structures A simulation study to evaluate the impact of implementing a distance or a zone tariff on ridership and revenue is conducted in [GM07]. Similarly, [CLC16] performs computational experiments changing from a flat to a distance tariff in a congested network. Besides the impact on ridership and profit, also implications for different segments of the population are investigated. In [MS20], the effects of changes from a zone to a distance tariff on route choice, total travel time, total fares paid and the amount of walking are illustrated in a simulation study. In [CB21], the authors group fare structures of eleven cities in Poland into flat, distance-, quality-, time- and zone-based fare structures and apply regression analysis to evaluate by which function type (linear, power, polynomial, logarithmic or exponential in the respective unit) they are best described.

The computation of cheapest paths is considered for distance tariffs in a railway context in [MS06] and for zone tariffs in [DPW15; DDP19]. In [EB19; ELB24], the so-called ticket graph is presented which models transitions between tickets via transition functions over partially ordered monoids and allows the design of an algorithm for finding cheapest paths in fare structures that do not have the subpath-optimality property. The cheapest ticket problem as well as the no-elongation property and the no-stopover property are studied in [SU20; SU22] for distance- and zone-based fare structures.

Tariff Optimization for Distance-based Fare Structures For distance-based tariffs, [DSH88] presents a quadratic model for determining a base amount, a distance price and a price per transfer maximizing the revenue. As an increasing fare reduces the demand, additional constraints for lower bounds on the demand as well as lower and upper bounds on the fares are applied. In [Pal13; PM17; HB18], a distance-based fare structure with two fare levels or an arbitrary but fixed number of fare levels, respectively, based on the number of traversed edges/stations is determined for a line. While [Pal13; PM17] minimize the weighted sum of squared deviations from reference prices, which are given by a granular fare structure with a fare level for each number of traversed edges, [HB18] maximizes the revenue or the demand while in each case the other one is kept fixed. In [YH14], a bilevel approach for maximizing the demand that is solved by a genetic algorithm is presented. The upper level problem determines distance fares composed of a base amount, a mileage price and surcharges for premier services, while passengers choose routes that minimize their generalized user costs including various time components in the lower level problem. Non-linear beeline distance fares are designed in [Hua+16] while simultaneously optimizing the service frequency. The problem is modeled as a three-player game between the transport authority, the transit enterprise and the passengers. The authors show that the problem is NP-hard and propose an artificial bee colony algorithm to solve it.

Tariff Optimization for Zone-based Fare Structures Research on the design of zone-based fare structures has started with [HS95; Sch96; HS04]. They introduce the objective of minimizing the deviation from given reference prices as applied in this thesis. The sum of absolute and squared deviations is considered, but the main focus lies on minimizing the maximum absolute deviation, where passengers minimize the number of traversed zones. An arbitrary pricing, where the price between each pair of zones is set individually, is applied as well as a counting-zones pricing with multiple counting. While the prices can be determined by an explicit formula once the zones are given, the integrated problem of determining zones and prices simultaneously is shown to be NP-hard for the objective of minimizing the maximum absolute deviation for all fixed numbers of zones greater than or equal to three. A greedy, a clustering and a spanning-tree heuristic for the problem of determining zones are presented. These results are expanded by [BK03] which considers the zone tariff design problem with arbitrary pricing, connected zones and minimizing the maximum absolute deviation. The paper further investigates the complexity regarding NP-completeness and polynomial cases on special graph structures. In [Pra04], the problem of finding zones and prices is formulated as a bilevel program that is solved with a simulated annealing heuristic. The objective is to minimize a weighted sum of revenue increase and decrease. Furthermore, the authors of [Tav+07] point out that districting problems occur in the context of different application areas, in particular, as the problem of finding zones for a zone-based fare structure. The districting problem is modeled as a multi-objective problem, and a local search evolutionary algorithm is proposed to solve it. In [Koh13], an IP is presented with the aim to determine zones for fixed prices given as a base amount and a price per zone that minimize the maximum or average absolute deviation between zone fares and reference prices, which are given by former distance fares. The constructed zones need not be connected. Also for given prices, [CJS14] searches for a zone partition with connected zones along a linear graph that maximizes the revenue, where the fares are determined according to a counting-zones pricing. A tree-based model is used to enumerate the options for the zones. [Yan+20] faces a similar problem and solves it with a local search method moving zone borders. The prices are then chosen as average former ticket costs between pairs of zones. Another local search heuristic with tabu search is developed by [GMS17] for the connected zone design problem with given prices on a general graph. Fares are computed according to a counting-zones pricing and the objective function is to minimize the maximum absolute deviation from reference prices. The local search heuristic improves the found solutions over the heuristics from [HS04]. The authors of [AM18] encounter the problem of finding zones and setting prices (arbitrary pricing) in the context of air cargo. The goal is to maximize the revenue. The problem is solved to optimality by Benders decomposition, and a branch-and-bound method is developed that outperforms the Benders decomposition. The theoretical research of [HS04] and [BK03] is further extended by [OB17]. The authors model different zone-based fare strategies, namely with connected or ring zones and with counting-zones pricing (single counting), cumulative pricing or maximum pricing. The objective function maximizes the revenue considering the willingness to pay of the customers in the constraints. For these problems, MILP formulations are provided and the complexity is analyzed. Recently, [MHR22] tackled the problem of determining connected zones with a counting-zones pricing (multiple counting) by modeling the problem in the dual graph. An IP formulation and a heuristic are developed during which the prices are kept fixed. Iterating over a list of price options, the goal is to find the zone tariff that maximizes the revenue or demand. The authors propose an option to enforce certain spatial patterns such as rings and stripes.

General Tariff Optimization A very general model for optimizing tariffs that can deal with different fare strategies and objectives is proposed by [BKP12]. Also monthly and reduced fares can be incorporated in the non-linear optimization model based on a discrete route choice model.

Chapter 9

Fare Structures and Properties

This chapter introduces the basic knowledge, terminology and notation for tariff optimization. The definitions have originally been given in [SU20; SU22]. Here, we assume a PTN (Definition 2.15) and a set of all paths in the PTN $\mathcal{W}$ (Definition 2.16) to be given. We start by defining a fare structure.

Definition 9.1 (Fare structure)**.** Let a PTN be given, and let $\mathcal{W}$ be the set of all paths in the PTN. A *fare structure* is a function $\pi\colon \mathcal{W} \to \mathbb{R}_{\geq 0}$ that assigns a price to every path in the PTN.

Additionally, a *fare strategy* stipulates requirements (constraints) on a fare structure. A fare structure π is of a certain fare strategy if π satisfies the corresponding strategy constraints. In the following, we study the design of fare structures regarding flat, affine distance, affine beeline and counting zones fare strategies, which we formally define in Sections 9.1 to 9.3.

With the aim to establish fairness and consistency, we want to design fare structures that satisfy the no-elongation property and the no-stopover property.

Definition 9.2 (No-elongation property and no-stopover property)**.** Let a PTN be given, and let $\mathcal{W}$ be the set of all paths in the PTN.

1. A fare structure π satisfies the *no-elongation property* if

$$\pi((v_1, \ldots, v_{n-1})) \leq \pi((v_1, \ldots, v_n))$$

   for all paths $(v_1, \ldots, v_n) \in \mathcal{W}$ with $n \in \mathbb{N}_{\geq 2}$.

2. A fare structure π satisfies the *no-stopover property* if

$$\pi((v_1, \ldots, v_n)) \leq \pi((v_1, \ldots, v_i)) + \pi((v_i, \ldots, v_n))$$

   for all paths $(v_1, \ldots, v_n) \in \mathcal{W}$ with $n \in \mathbb{N}_{\geq 3}$ and all intermediate stations v_i with $i \in \{2, \ldots, n-1\}$.

The no-elongation property ensures that the fare for a path is not allowed to be cheaper than the fare for any subpath. The no-stopover property ensures that it is not beneficial to split a ticket into several ones. This means that buying several tickets to cover a path or making a stopover never decreases the total fare that has to be paid for a path. From the perspective of the passengers, the properties yield transparency and make the fare structure easier to understand. As shown in [SU22, Thm. 1], it is cheapest to buy a ticket for exactly the path intended if both properties are satisfied. From the perspective of a public transport operator this yields consistency in the sense that passengers cannot undercut the fares. This is, for example, relevant to correctly estimate the revenue.

9.1 Flat Tariffs

The simplest fare strategy is the flat tariff, in which all paths are assigned the same fixed price. On the one hand, it is easy to understand and to apply. On the other hand, it can be perceived as unfair because short trips are as expensive as long trips. This can for example be used to attract more passengers with longer journeys while at the same time not incentivizing pedestrians or cyclists to use public transport.

Definition 9.3 (Flat tariff)**.** Let a PTN be given, and let $\mathcal{W}$ be the set of all paths in the PTN. A fare structure π is a *flat tariff* w.r.t. the fixed price $f \in \mathbb{R}_{\geq 0}$ if $\pi(W) = f$ for all $W \in \mathcal{W}$.

From [SU20, Thm. 12], we know that a flat tariff always satisfies the no-elongation property and the no-stopover property. We hence need not consider the no-elongation property or the no-stopover property explicitly when we optimize flat tariffs.

9.2 Distance Tariffs

Distance-based fare structures are widely perceived as fair because the ticket price correlates with the distance between the start and the end station. This might also encourage passengers traveling a short distance to use public transport.

We consider two options to determine the distance associated with a path:

Definition 9.4 (Network distance)**.** Let a PTN (V, E) with weights $l_e \in \mathbb{R}_{>0}$ for all edges $e \in E$ be given. We define the *network distance* of a path $W \in \mathcal{W}$ as $l(W) := \sum_{e \in E(W)} l_e$.

Definition 9.5 (Metric distance)**.** Let a *metric* $\mathsf{dist}\colon V \times V \to \mathbb{R}_{\geq 0}$ on a set of nodes V be given, i.e., for all $v_1, v_2, v_3 \in V$ it holds

$$\begin{aligned}
&\mathsf{dist}(v_1, v_2) = 0 \iff v_1 = v_2 && \text{(identity of indiscernibles)},\\
&\mathsf{dist}(v_1, v_2) = \mathsf{dist}(v_2, v_1) && \text{(symmetry)},\\
&\mathsf{dist}(v_1, v_2) \leq \mathsf{dist}(v_1, v_3) + \mathsf{dist}(v_3, v_2) && \text{(triangle inequality)}.
\end{aligned}$$

Then we define the *metric distance* of a path $W = (v_1, \dots, v_n) \in \mathcal{W}$ as $l(W) := \mathsf{dist}(v_1, v_n)$, which is the distance between the start station v_1 and the end station v_n measured by the metric dist.

Definition 9.6 (Affine distance tariff)**.** Let a PTN be given, and let $\mathcal{W}$ be the set of all paths in the PTN. Let $l\colon \mathcal{W} \to \mathbb{R}_{\geq 0}$ be a function that either measures the network distance or the metric distance of a path. A fare structure π is an *affine (network/metric) distance tariff* w.r.t. a distance price $p \in \mathbb{R}_{\geq 0}$ and a base amount $f \in \mathbb{R}_{\geq 0}$ if $\pi(W) = p \cdot l(W) + f$ for all $W \in \mathcal{W}$.

Note that if the edge lengths are given as the lengths of tracks or streets, then $l(W)$ is the actual travel distance in the network along a path $W \in \mathcal{W}$. We further remark that for a metric distance tariff, the price of a ticket is independent of the actual path but depends just on its start and end station. If the stations are embedded in $\mathbb{R}^2$ and dist is the Euclidean distance, we call the corresponding distance tariff a *beeline distance tariff*.

By [SU20, Thm. 15], an affine network distance tariff always satisfies the no-elongation property and the no-stopover property. In [Urb20, Thm. 6.3; SU20, Thm. 17], it is shown that an affine metric/beeline distance tariff satisfies the no-stopover property. The no-elongation property is not satisfied for an affine metric/beeline distance tariff [Urb20, Thm. 6.4; SU20, Ex. 18] but can be recovered by using a check-in/check-out system. We hence need not consider the no-elongation property or the no-stopover property explicitly when we optimize affine distance tariffs.

9.3 Zone Tariffs

Zone-based fare structures combine the properties of flat and distance-based fare structures. The region of the PTN is divided into tariff zones. Here, we consider a *counting-zones pricing*, in which the price of a path depends on the number of zones traversed by the path. This means that within a zone (as well as for each fixed number of zones) a flat tariff is applied while on a general path the distance is approximated by the number of traversed zones. We distinguish between two types of zone tariffs: with multiple counting and with single counting. In the multiple counting case, we count a zone each time that it is entered, as it is similarly done in Boston (USA) by MBTA for commuter rail. In the single counting case, each zone is counted at most once, which is for example used in Greater Copenhagen (Denmark) by DOT and by many German transport associations, e.g., VRN and saarVV.

Let a PTN (V, E) be given. Formally, we regard the tariff zones as a partition $\mathcal{Z} = \{Z_1, \dots, Z_L\}$ with $L \in \mathbb{N}_{\geq 1}$ of the set of stations V, i.e., $V = \bigcup_{i \in [L]} Z_i$ and the Z_i are non-empty and pairwise disjoint. We call $\mathcal{Z}$ a *zone partition*. Further, we say that a path $W = (v_1, \dots, v_n) \in \mathcal{W}$ traverses a zone $Z \in \mathcal{Z}$ if

$v_i \in Z$ for some $i \in [n]$. If there are $i, j, k \in [n]$ with $i < j < k$ and $v_i, v_k \in Z$ and $v_k \notin Z$ for some $Z \in \mathcal{Z}$, we say that W traverses the zone Z multiple times. Next, we define a *zone function* $\sigma\colon \mathcal{W} \to \mathbb{N}_{\geq 0}$, which counts the number of traversed zones on any path $W \in \mathcal{W}$ with its multiset of nodes $V(W)$ and its multiset of edges $E(W)$. It is different for the two ways of counting.

Multiple Counting Let $e = \{v_1, v_2\} \in E$ be an edge. We define the zone border weight

$$b(e) = b(v_1, v_2) := \begin{cases} 0 & \text{if } v_1 \text{ and } v_2 \text{ are in the same zone,} \\ 1 & \text{otherwise.} \end{cases}$$

From that, we derive for a path $W \in \mathcal{W}$ the *zone function*

$$\sigma(W) := 1 + b(W), \text{ where } b(W) := \sum_{e \in E(W)} b(e).$$

Single Counting For every path $W \in \mathcal{W}$, the *zone function* that counts the number of *different* zones that are traversed is defined as

$$\sigma(W) := |\{Z \in \mathcal{Z} : V(W) \cap Z \neq \emptyset\}| .$$

Definition 9.7 (Zone tariff)**.** Let a PTN together with a zone partition $\mathcal{Z}$ be given, and let $\mathcal{W}$ be the set of all paths in the PTN. A fare structure π is a *zone tariff with multiple/single counting* w.r.t. a price function $P\colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ if $\pi(W) = P(\sigma(W))$ for all $W \in \mathcal{W}$.

Theorem 9.8 states conditions for the no-elongation property and the no-stopover property to hold in case of zone tariffs. These are independent of the PTN and the zone partition $\mathcal{Z}$ and only depend on the price function P.

Theorem 9.8 ([Urb20; SU22])**.** *Let a PTN, a zone partition $\mathcal{Z}$ and price function P be given.*

1. *The zone tariff (with multiple or single counting) w.r.t. $\mathcal{Z}$ and P satisfies the no-elongation property if P is monotonically increasing.*

2. *The zone tariff with multiple counting w.r.t. $\mathcal{Z}$ and P satisfies the no-stopover property if*

$$P(k) \leq P(i) + P(k - i + 1) \text{ for all } k \in \mathbb{N}_{\geq 1},\ i \in [k].$$

3. *The zone tariff with single counting w.r.t. $\mathcal{Z}$ and P satisfies the no-stopover property if*

$$P(k) \leq P(i_1) + P(i_2) \text{ for all } k \in \mathbb{N}_{\geq 1}, i_1, i_2 \in [k] \text{ with } i_1 + i_2 \geq k + 1.$$

Remark 9.9. It has been shown that an equivalence is obtained in Theorem 9.8 if the PTN and the zone partition are not fixed ([Urb20; SU22]). The sufficient conditions of Theorem 9.8 are however not necessary for a specific PTN with a fixed zone tariff because not all numbers of traversed zones are attained by paths in the PTN: For example, a zone tariff with only one zone and a price function with $P(1) = 1$, $P(2) = 3$ and $P(3) = 2$ satisfies the no-elongation property although the prices are not increasing. Thus, a stronger proposition could also be gained by only considering P for input values $\sigma(W)$ for $W \in \mathcal{W}$. The conditions of Theorem 9.8 are used in MILPs and algorithms to ensure the no-elongation property and/or the no-stopover property.

In Lemma 9.10, we show that reducing the values of a price function to a given value $\bar{r}$ preserves the no-elongation property and the no-stopover property.

Lemma 9.10. *Let a PTN (V, E) and $\bar{r} \in \mathbb{R}_{\geq 0}$ be given. Let π be a zone tariff with a zone partition $\mathcal{Z}$ and a price function P. Then the zone tariff π' with the same zone partition $\mathcal{Z}$ and the price function $P' \colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ defined by*

$$P'(k) := \begin{cases} P(k) & \text{if } P(k) \leq \bar{r} \\ \bar{r} & \text{if } P(k) > \bar{r} \end{cases}$$

for all $k \in \mathbb{N}_{\geq 1}$ satisfies the no-elongation property and the no-stopover property if π satisfies them.

Proof. First, let π satisfy the no-elongation property. By Definition 9.2, it holds for all paths $W_1 = (v_1, \dots, v_n), W_2 = (v_1, \dots, v_{n-1}) \in \mathcal{W}$ with $n \geq 2$ and $k_1 := \sigma(W_1)$ and $k_2 := \sigma(W_2)$ that $P(k_2) \leq P(k_1)$. If $P(k_1) > \bar{r}$, then $P'(k_2) \leq \bar{r} = P'(k_2)$. If $P(k_1) \leq \bar{r}$, then $P'(k_2) = P(k_2) \leq P(k_1) = P'(k_1)$. Hence, π' satisfies the no-elongation property.

Second, let π satisfy the no-stopover property. By Definition 9.2, it holds for all paths $W = (v_1, \dots, v_n) \in \mathcal{W}$ and subpaths $W_1 = (v_1, \dots, v_i)$, $W_2 = (v_i, \dots, v_n)$ of W with $n \geq 3$, $i \in \{2, \dots, n-1\}$ and with $k := \sigma(W)$, $k_1 := \sigma(W_1)$ and $k_2 := \sigma(W_2)$ that $P(k) \leq P(k_1) + P(k_2)$. If $P(k_1) > \bar{r}$ or $P(k_2) > \bar{r}$, then we have $P'(k) \leq \bar{r} \leq P'(k_1) + P'(k_2)$. If $P(k_1) \leq \bar{r}$ and $P(k_2) \leq \bar{r}$, then we have $P'(k) \leq P(k) \leq P(k_1) + P(k_2) = P'(k_1) + P'(k_2)$. Hence, π' satisfies the no-stopover property. □

If a PTN with zones and OD data (Definition 2.17) is given, then the zone function can determine the number of zones that each OD pair traverses. We introduce the following notation, which is useful if a zone partition is given and only the price function of a zone tariff is optimized:

Definition 9.11 (Notation for given zones)**.** Let a PTN (V, E), a zone partition $\mathcal{Z}$, a zone function σ (with single or multiple counting) and OD data

(D, W_d, t_d) be given. We define $K := \max_{d \in D} \sigma(W_d)$ as the maximum number of zones that are traversed along a path. For $k \in [K]$, we set

$$D_k := \{d \in D : \sigma(W_d) = k\},$$

which is the set of all OD pairs that traverse k zones.

In general, K is bounded from above by the maximum number of nodes of a path W_d for $d \in D$, which is at most $|V|$ if the paths are elementary. In the single counting case, note that $K \leq L$, where L is the number of zones in the zone partition $\mathcal{Z}$.

Further, we are usually only interested in a price function with a finite number of different values. We hence introduce the notion of a price list.

Definition 9.12. (Price list) Let $K \in \mathbb{N}_{\geq 1}$ and $p_k \in \mathbb{R}_{\geq 0}$ for all $k \in [K]$ be given. Let P be a price function. We identify the list $(p_1, \ldots, p_K)$ with P if

$$P(k) = \begin{cases} p_k & \text{if } k \leq K \\ p_K & \text{if } k > K. \end{cases}$$

In particular, in this case $P(k) = P(K)$ for all $k \geq K$. We call $(p_1, \ldots, p_K)$ a *price list*.

While zone tariffs with a counting-zones pricing presented here are very popular, note that many additional specifications of zone-based fare structures exist, for example: special metropolitan zones, overlapping zones, empty zones, cumulative or maximum pricing with individual prices for the zones.

Chapter 10

Fare Deviation Model

This chapter studies the fare deviation model, which aims to minimize the weighted sum of absolute deviations of fares from given reference prices. The general problem description is given in Section 10.1. We investigate the fare deviation model for flat tariffs in Section 10.2, for affine distance tariffs in Section 10.3 and for zone tariffs in Section 10.4. A summary of the results of Chapter 10 is given in Section 10.5.

10.1 Problem Definition

In the fare deviation model, we ask for a fare structure that minimizes the sum of absolute deviation from given reference prices. These reference prices can for example be the fares in a former fare structure when changing to a new one, or prices that are considered as fair or otherwise desirable. We formally define the fare deviation model as follows:

Definition 10.1 (FDM)**.** The *fare deviation model* (FDM) is defined as follows: Given a PTN $G = (V, E)$ (Definition 2.15), OD data (D, W_d, t_d) (Definition 2.17) and reference prices $r_d \in \mathbb{R}_{\geq 0}$ for all OD pairs $d \in D$ as well as potentially specific input depending on the desired fare strategy, determine a fare structure π regarding the desired fare strategy that minimizes the weighted sum of absolute deviations from the reference prices, this means it minimizes $\sum_{d \in D} t_d |r_d - \pi(W_d)|$.

The fare deviation model minimizing the weighted sum of absolute deviations from reference prices is closely related to median problems as we show in the following sections. We therefore recall the definition of a weighted median.

Definition 10.2 (Weighted median)**.** Given a non-empty index set D, the set of weighted medians of numbers $(r_d)_{d \in D}$ with weights $(t_d)_{d \in D}$ denoted by $\text{w-median}_{d \in D}(r_d, t_d)$ contains all values $\bar{p}$ that satisfy

$$\sum_{d \in D:\, r_d < \bar{p}} t_d \leq \frac{\sum_{d \in D} t_d}{2} \quad \text{and} \quad \sum_{d \in D:\, r_d > \bar{p}} t_d \leq \frac{\sum_{d \in D} t_d}{2}. \tag{10.1}$$

We use the shorthand notation $\mathsf{w\text{-}median}(D)$ if the numbers $(r_d)_{d\in D}$ and weights $(t_d)_{d\in D}$ are clear from the context. Instead of taking a weighted median, one can equivalently consider the median of the values $(\underbrace{r_d,\ldots,r_d}_{t_d \text{ times}} : d \in D)$. Also note that $\mathsf{w\text{-}median}_{d\in D}(r_d, t_d)$ can consist of a single value or, because formula (10.1) is a convex condition, a whole interval $[\bar{p}_1, \bar{p}_2]$, where $\bar{p}_1$ satisfies the first inequality of formula (10.1) with strict inequality and the second with equality, and vice versa for $\bar{p}_2$. Then $\bar{p}_1$ is called the lower weighted median and $\bar{p}_2$ the upper weighted median.

In the following sections, we consider the fare deviation model regarding flat, distance and zone fare strategies.

10.2 Flat Tariffs

In this section, we consider the fare deviation model for flat tariffs (Definition 9.3). Because the price is the same for all passengers independent of their paths in a flat tariff, the PTN is not explicitly required as input to the fare deviation model for flat tariffs but the OD data (even without paths) is sufficient:

Definition 10.3 (F-FDM)**.** The *fare deviation model for flat tariffs* (F-FDM) is defined as follows: Given are a PTN with OD data (D, t_d) and a reference price $r_d \in \mathbb{R}_{\geq 0}$ for all OD pairs $d \in D$. The aim is to determine a price $f \in \mathbb{R}_{\geq 0}$ such that $\sum_{d\in D} t_d |r_d - f|$ is minimized.

If we drop the non-negativity requirement $f \geq 0$ from F-FDM, we obtain a *weighted median problem* as used in statistics (e.g., [Gur90]), or, equivalently, a *one-dimensional 1-median problem*, where especially the two-dimensional version is well known as *Weber problem*, as used in location theory (e.g., [Pla95; Dre+02]).

The function $\varphi\colon \mathbb{R} \to \mathbb{R}_{\geq 0}, f \mapsto \sum_{d\in D} t_d |r_d - f|$ is a continuous, piecewise linear and convex function, for which it is known that the set of optimal solutions to the weighted median (or one-dimensional 1-median) problem is equal to the set of weighted medians $\mathsf{w\text{-}median}_{d\in D}(r_d, t_d)$ (e.g., [Gur90]). Hence, for all optimal solutions f^*, it holds that $f^* \geq \min\{r_d : d \in D\}$, and there is always an optimal solution f^* with $f^* \in \{r_d : d \in D\}$. Because $\min\{r_d : d \in D\} \geq 0$, the non-negativity requirement of F-FDM is not necessary and can be omitted. Therefore, we can regard F-FDM as a weighted median problem. In case that the weighted median is not unique, the lower median f_1 leads to lower prices for the passengers, whereas the operator generates a higher income by implementing the price f_2 of the upper median. These two values as well as all $f \in [f_1, f_2]$ yield the same objective function value because we consider the sum of absolute deviations.

The following well-known LP formulation uses a variable $f \in \mathbb{R}$ for the fixed price and an auxiliary variable $y_d \in \mathbb{R}$ for all $d \in D$ to linearize the objective function:

$$\begin{aligned}
\min_{f,p,y_d} \quad & \sum_{d \in D} t_d y_d \\
\text{s.t.} \quad & r_d - f \leq y_d \quad \text{for all } d \in D \\
& f - r_d \leq y_d \quad \text{for all } d \in D \\
& f \in \mathbb{R}_{\geq 0} \\
& y_d \in \mathbb{R} \quad \text{for all } d \in D.
\end{aligned}$$

Apart from solving the LP formulation in polynomial time, also specialized solution methods for finding a (weighted) median can be used. A linear time selection method, which is in particular able to find a median, is developed by [Blu+73]. Several methods to compute a weighted median in linear time are reviewed by [Gur90]. Consequently, F-FDM can be solved in linear time $\mathcal{O}(|D|)$.

10.3 Distance Tariffs

We now turn our attention to the design of affine distance tariffs (Definition 9.6).

Definition 10.4 (D-FDM)**.** The *fare deviation model for affine distance tariffs* (D-FDM) is defined as follows: Given are a PTN with OD data (D, W_d, t_d), a (network/metric) distance function l (Definitions 9.4 and 9.5) and a reference price $r_d \in \mathbb{R}_{\geq 0}$ for all OD pairs $d \in D$. The aim is to determine a distance price $p \in \mathbb{R}_{\geq 0}$ and a base amount $f \in \mathbb{R}_{\geq 0}$ such that $\sum_{d \in D} t_d |r_d - (p \cdot l(W_d) + f)|$ is minimized.

In the following, we use for all $d \in D$ the shorthand notation $l_d := l(W_d)$.

Note that in contrast to F-FDM, for affine distance tariffs neither $p \geq 0$ nor $f \geq 0$ is ensured by non-negative reference prices $r_d \geq 0$ for all $d \in D$.

When we allow negative values for p and f, i.e., $p, f \in \mathbb{R}$, the fare deviation model for affine distance tariffs is a *least absolute deviations regression* with the linear expression $p \cdot x + f$ as used in statistics (e.g., [Kar58; Wag59]), or equivalently, a *1-median-line location problem with vertical distances* as used in location theory (e.g., [MT83; Sch99a]). This means, one searches for a line

$$L_{p,f} := \{(x, y) \in \mathbb{R}^2 : y = p \cdot x + f\}.$$

Requiring $p, f \geq 0$ leads to a *restricted* line location problem as studied in [Sch99b].

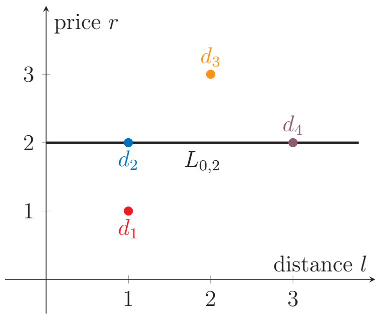
In Theorem 10.5, we derive a finite candidate set for the set of optimal solutions of D-FDM. To do so, we say that a solution (p, f) *meets the reference price* r_d of an OD pair $d \in D$ exactly if $r_d = p \cdot l_d + f$, i.e., the line $L_{p,f}$ passes through the point (l_d, r_d).

Theorem 10.5. *There is always an optimal solution* (p^*, f^*) *to* D-FDM *such that one of the following holds: The reference price of*

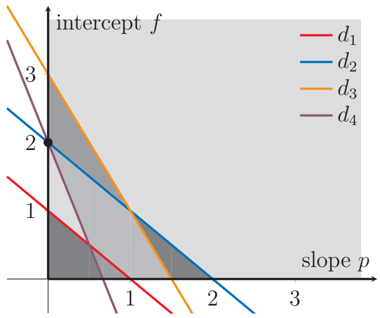
- *at least two OD pairs* d, d' *with* $l_d \neq l_{d'}$ *is met exactly,*
- *at least one OD pair is met exactly and, additionally,* $p^* = 0$ *or* $f^* = 0$*.*

Proof. We regard D-FDM as 1-median-line location problem with the additional requirement that $p, f \geq 0$. For this proof, we adopt the dual interpretation from [Sch99a, Sec. 2.2]. We consider the transformation T that maps a point $(l, r) \in \mathbb{R}^2$ to a line $T((l, r)) := L_{-l,r}$ and a non-vertical line $L_{p,f}$ to a point $T(L_{p,f}) := (p, f)$. The space of the transformed points and lines is called *dual* space. We call the original space the *primal* space. An example is given in Figure 10.1. The vertical deviation between a point (l, r) and a line $L_{p,f}$ in the primal space is the same as the vertical deviation between the transformed line $L_{-l,r}$ and the transformed point (p, f) in the dual space because $r-(p\cdot l+f) = r-p\cdot l-f = (-l\cdot p+r)-f$. Hence, it is equivalent to search for a line $L_{p,f}$ with $p, f \in \mathbb{R}_{\geq 0}$ minimizing the weighted sum of absolute deviations from the points (l_d, r_d) for $d \in D$ in the primal space or to search for a point (p, f) with $p, f \in \mathbb{R}_{\geq 0}$ (i.e., in the first quadrant) that minimizes the weighted sum of absolute deviations from the lines L_{-l_d,r_d} for $d \in D$. The feasible space $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ in the dual space is divided into two-dimensional polyhedra (cells) by the lines L_{-l_d,r_d} for $d \in D$ (see Figure 10.1(b)). We can determine an optimal solution to the overall problem by searching for an optimal solution in each cell and identifying the one with the best optimal objective function value. In each cell, the sign of $r_d - p \cdot l_d - f$ does not change for all $d \in D$ because no line is crossed, which means that the objective function can be written without the absolute value in each cell. Hence, in each cell, the problem is feasible, the objective function is linear and the optimal objective function value is finite. By the fundamental theorem of linear programming, in each cell, there is an optimal solution that is an extreme point of this cell. This is either the intersection of two lines, of a line with an axis, or the origin. Let (p^*, f^*) be the best of all the optimal solutions of all cells. Interpreting the solution for D-FDM, this means that one of the following holds for the solution (p^*, f^*), i.e., with distance price p^* and base amount f^* :

- two reference prices r_{d_1}, r_{d_2} with $d_1, d_2 \in D$ are met exactly (if (p^*, f^*) is the intersection $L_{-l_{d_1},r_{d_1}} \cap L_{-l_{d_2},r_{d_2}}$ of two lines in the dual space, in particular $l_{d_1} \neq l_{d_2}$),
- one reference price r_d with $d \in D$ is met exactly and $p^* = 0$ (if (p^*, f^*) is the intersection of L_{-l_d,r_d} with the f-axis in the dual space),
- one reference price r_d with $d \in D$ is met exactly and $f^* = 0$ (if (p^*, f^*) is the intersection of L_{-l_d,r_d} with the p-axis in the dual space),



(a) Primal space: points (l_d, r_d) for four OD pairs $d \in \{d_1, d_2, d_3, d_4\}$ and the line $L_{0,2}$ of the optimal solution



(b) Dual space: The four points (l_d, r_d) with $d \in \{d_1, d_2, d_3, d_4\}$ are transformed to four lines L_{-l_d,r_d}. The point $(0, 2)$ is the optimal solution and can be transformed back to the line $L_{0,2}$ in the primal space.

Figure 10.1: Example with four OD pairs with one passenger each illustrating the proof of Theorem 10.5. The optimal solution is $(p^*, f^*) = (0, 2)$. This shows that the three options stated in Theorem 10.5 are not mutually exclusive.

- $p^* = 0$ and $f^* = 0$ (if $(p^*, f^*) = (0, 0)$ is the origin in the dual space).

Note that $p^* = 0$ and $f^* = 0$ can only be an optimal solution to D-FDM if there is an OD pair $d \in D$ with $r_d = 0$: Assume that $(p^*, f^*) = (0, 0)$ is an optimal solution and $r_d > 0$ for all $d \in D$. For $f' := \min\{r_d : d \in D\} > 0$, the objective function value of $(0, f')$ is smaller than the objective function value of $(0, 0)$ because $0 < f' \leq r_d$ for all $d \in D$. Hence, there is an OD pair $d \in D$ with $r_d = 0$ and this reference price is met exactly. Therefore the third case is already contained in the second case. □

We remark that, if an optimal solution to the restricted problem $(p, f \in \mathbb{R}_{\geq 0})$ is also an optimal solution to the unrestricted problem $(p, f \in \mathbb{R})$, then

$$\sum_{d \in D:\, r_d < p \cdot l_d + f} t_d \leq \frac{\sum_{d \in D} t_d}{2} \quad \text{and} \quad \sum_{d \in D:\, r_d > p \cdot l_d + f} t_d \leq \frac{\sum_{d \in D} t_d}{2}$$

by [Sch98; Sch99a], similar to the weighted median (Definition 10.2). This means that half of the passengers pay at most as much as their reference price and half of the passengers pay at least as much in this case.

D-FDM can be solved by means of the finite candidate set derived in Theorem 10.5. It can also be formulated as an LP:

$$\begin{aligned}
\min_{p,f,y_d} \quad & \sum_{d\in D} t_d y_d \\
\text{s.t.} \quad & r_d - p\cdot l_d - f \le y_d && \text{for all } d\in D \\
& p\cdot l_d + f - r_d \le y_d && \text{for all } d\in D \\
& p, f \in \mathbb{R}_{\ge 0} \\
& y_d \in \mathbb{R} && \text{for all } d \in D.
\end{aligned} \tag{10.2}$$

Using this LP and a result of [Zem84], we can even show that the problem can be solved in linear time $\mathcal{O}(|D|)$:

Theorem 10.6. D-FDM *can be solved in linear time* $\mathcal{O}(|D|)$.

Proof. In [Zem84], the author presents an algorithm that solves the dual of the s-dimensional Multiple Choice Linear Programming Problem (sMCLPP) in linear time with respect to the number of constraints. We now show that the LP of D-FDM is of the required form for the algorithm and can hence be solved in $\mathcal{O}(|D|)$. To do so, we consider 2MCLPP (i.e., $s = 2$), using the notation of [Zem84]:

$$\begin{aligned}
\max_{x_j} \quad & \sum_{j\in N} c_j x_j \\
\text{s.t.} \quad & \sum_{j\in N} a_j^i x_j = a_0^i && \text{for all } i\in\{1,2\} \\
& \sum_{j\in J_k} b_j x_j = b_0^k && \text{for all } k\in[r] \\
& x_j \ge 0 && \text{for all } j\in N \\
& x_j \in \mathbb{R} && \text{for all } j\in N.
\end{aligned}$$

with $r \in \mathbb{N}_{\ge 1}$ and $N = J_0 \cup J_1 \cup \ldots \cup J_r$ with $0 \notin N$ and the sets J_k with $k \in \{0,\ldots,r\}$ are pairwise disjoint. Dualizing the LP yields

$$\begin{aligned}
\min_{w_i, v_k} \quad & a_0^1 w_1 + a_0^2 w_2 + \sum_{k=1}^{r} b_0^k v_k \\
\text{s.t.} \quad & a_j^1 w_1 + a_j^2 w_2 \ge c_j && \text{for all } j\in J_0 \\
& a_j^1 w_1 + a_j^2 w_2 + b_j v_k \ge c_j && \text{for all } j\in J_k, k\in[r] \\
& w_i, v_k \in \mathbb{R} && \text{for all } i\in\{1,2\}, k\in[r].
\end{aligned}$$

We consider the special case with $J_0 = \{1, 2\}$, $a_0^1 = a_0^2 = 0$, $a_1^1 = a_2^2 = 1$, $a_2^1 = a_1^2 = 0$, $c_1 = c_2 = 0$, $b_j = 1$ for all $j \in J_k$, $k \in [r]$, which yields

$$\begin{aligned} \min_{w_i, v_k} \quad & \sum_{k=1}^{r} b_0^k v_k \\ \text{s.t.} \quad & c_j - a_j^1 w_1 - a_j^2 w_2 \le v_k && \text{for all } j \in J_k \\ & w_1, w_2 \ge 0 \\ & w_i, v_k \in \mathbb{R} && \text{for all } i \in \{1, 2\}, k \in [r]. \end{aligned}$$

Let now $r = |D|$, and for each $d \in D$, let there be a $k \in [r]$ such that $J_k = \{d^+, d^-\}$ and $b_0^k = t_d$. For $k \in [r]$ and $j \in J_k$, we set

$$c_j = \begin{cases} r_d & \text{if } j = d^+, \\ -r_d & \text{if } j = d^-, \end{cases} \qquad a_j^1 = \begin{cases} l_d & \text{if } j = d^+, \\ -l_d & \text{if } j = d^-, \end{cases} \qquad a_j^2 = \begin{cases} 1 & \text{if } j = d^+, \\ -1 & \text{if } j = d^-, \end{cases}$$

and $p = w_1$, $f = w_2$ and $y_d = v_k$ for $d \in D$ and $k \in [r]$ with $J_k = \{d^+, d^-\}$. With this, we obtain LP (10.2). Hence, by [Zem84], LP (10.2) can be solved in linear time in the number of constraints, i.e., in $\mathcal{O}(|D|)$. □

10.4 Zone Tariffs

Finally, we study the fare deviation model for zone tariffs (Definition 9.7).

Definition 10.7 (Z-FDM)**.** The *fare deviation model for zone tariffs* (Z-FDM) is defined as follows: Given are a PTN $G = (V, E)$, OD data (D, W_d, t_d), reference prices $r_d \in \mathbb{R}_{\ge 0}$ for all OD pairs $d \in D$ and an upper bound $N \in \mathbb{N}_{\ge 1}$ on the number of zones. The aim is to determine a zone partition $\mathcal{Z}$ with at most N zones and a price function $P \colon \mathbb{N}_{\ge 1} \to \mathbb{R}_{\ge 0}$ such that $\sum_{d \in D} t_d |r_d - P(\sigma(W_d))|$ is minimized. Note that σ depends on the zone partition $\mathcal{Z}$.

Remark 10.8. It suffices to consider $N \in [|V|]$ because the zones form a zone partition and it is not possible to have more than $|V|$ sets in the partition. Further, it is enough to determine the price function P for input values up to K, where K is the maximum number of nodes of a path W_d for $d \in D$. Because the value $P(k)$ for all $k \ge K$ has no influence on the objective function value of Z-FDM, we can simply set $P(k) = P(K)$ for all $k \ge K$. Therefore, we only need to consider price functions P that attain a finite number of different values and can hence be represented by a price list $(p_1, \ldots, p_K)$ (Definition 9.12). Note that if a zone tariff with a price function P satisfies the no-elongation or the no-stopover property, then so does the zone tariff with the price function represented by the price list $(P(1), \ldots, P(K))$.

		multiple counting (M)	single counting (S)
arbitrary zones	(A)	Z-FDM(MA)	Z-FDM(SA)
connected zones	(C)	Z-FDM(MC)	Z-FDM(SC)

Table 10.1: Variants of Z-FDM.

In addition to distinguishing between multiple and single counting, we also distinguish two versions concerning the zones. Given a PTN $G = (V, E)$, we say that a zone $Z \subseteq V$ is connected if its induced subgraph $G[Z]$ is connected. While there are no requirements in the problem formulation in Definition 10.7, which allows *arbitrary zones*, we also consider that *connected zones* are demanded. This means that each zone $Z \in \mathcal{Z}$ needs to be connected. We obtain the four variants shown in Table 10.1.

Further, we consider Z-FDM with and without the requirement that the resulting fare structure satisfies the no-elongation property or the no-stopover property.

Subproblems of Z-FDM are the zone-partition subproblem and the price-setting subproblem: In the *zone-partition subproblem*, we assume the price function P to be given and only optimize the zone partition $\mathcal{Z}$. Conversely, the zone partition $\mathcal{Z}$ is given in the *price-setting subproblem* and the prices P are optimized.

10.4.1 General Properties and Relations Between the Different Problem Variants

In this section, we observe some general properties of Z-FDM and explore the relations between the different problem variants and the behavior of the objective function values. We start by stating that the definitions of zone functions in the case of single and multiple counting coincide under some circumstances.

Lemma 10.9. *Let a PTN with a zone partition $\mathcal{Z}$ be given. Both definitions of a zone function σ for multiple and single counting coincide for paths $W \in \mathcal{W}$ that do not traverse a zone multiple times. In particular, this is the case when $W = (v_1, v_2)$ for some edge $\{v_1, v_2\} \in E$.*

Proof. Let a path $W \in \mathcal{W}$ that does not traverse a zone multiple times be given. In this case, every time that $b(e) = 1$ for $e \in E(W)$, a new zone is entered that has not been traversed before. Hence, $1 + \sum_{e \in E(W)} b(e)$ is the number of different zones on the path W, which is equal to $|\{Z \in \mathcal{Z} : V(W) \cap Z \neq \emptyset\}|$. □

Next, we prove that an upper bound equal to the maximum reference price $\bar{r} := \max\{r_d : d \in D\}$ is valid for the price function of an optimal solution.

Lemma 10.10. *For all optimal solutions $\mathcal{Z}, P$ to* Z-FDM(XY) *with* $\mathrm{X} \in \{\mathrm{M, S}\}$ *and* $\mathrm{Y} \in \{\mathrm{A,C}\}$ *with/without requiring the no-elongation property and with/without requiring the no-stopover property, it holds that $P(k) \leq \bar{r} := \max\{r_d : d \in D\}$ for all $k \in \mathbb{N}_{\geq 1}$ with $D_k \neq \emptyset$ (see Definition 9.11).*

In particular, there is always an optimal solution $\mathcal{Z}, P$ with $P(k) \leq \bar{r}$ for all $k \in \mathbb{N}_{\geq 1}$.

Proof. Let $\mathcal{Z}, P$ be an optimal solution to Z-FDM(XY) with $\mathrm{X} \in \{\mathrm{M, S}\}$ and $\mathrm{Y} \in \{\mathrm{A,C}\}$. Assume that there is some $k \in \mathbb{N}_{\geq 1}$ with $P(k) > \bar{r}$ and $D_k \neq \emptyset$. We define a new price function $P' \colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ by

$$P'(k) := \begin{cases} P(k) & \text{if } P(k) \leq \bar{r}, \\ \bar{r} & \text{if } P(k) > \bar{r} \end{cases}$$

for all $k \in \mathbb{N}_{\geq 1}$. In order to prove the claim, we show that the zone tariff π' w.r.t. $\mathcal{Z}, P'$ has the same properties and yields a better objective function value than the zone tariff π w.r.t. $\mathcal{Z}, P$, which leads to a contradiction.

First, we consider OD pairs $d \in D$ with $P(\sigma(W_d)) > \bar{r}$. By assumption, there is at least one such OD pair. Because $P(\sigma(W_d)) > \bar{r} \geq r_d$ and $P'(\sigma(W_d)) = \bar{r}$, we have $|P(\sigma(W_d)) - r_d| > |P'(\sigma(W_d)) - r_d|$. Second, for all $d \in D$ with $P(\sigma(W_d)) \leq \bar{r}$, nothing is changed. Hence, replacing P with P' decreases the objective function value.

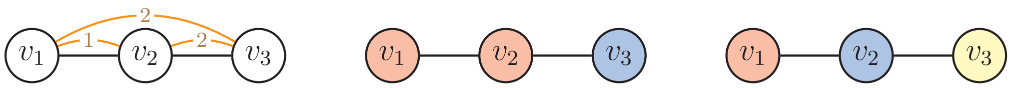
By Lemma 9.10, it holds that π' satisfies the no-elongation property and the no-stopover property if π satisfies them. □

Note that we later show a stronger result for Z-FDM without the requirement of the no-elongation property or the no-stopover property in Theorem 10.24, namely that we can choose the values of the price function from the given reference prices. This is particularly helpful for constructing zone tariffs in examples.

Regarding the upper bound N on the number of zones, the optimal objective function value of each problem Z-FDM(XY) with $\mathrm{X} \in \{\mathrm{M,S}\}$ and $\mathrm{Y} \in \{\mathrm{A,C}\}$ with/without requiring the no-elongation property and with/without requiring the no-stopover property is monotonically decreasing with increasing N because increasing the upper bound on the number of zones extends the feasible domain.

Note that this does not hold if the zone partition needs to consist of *exactly* N non-empty sets as the following example shows: Consider the linear graph depicted in Figure 10.2(a), where the OD pairs with their reference prices and paths are marked in orange. Every OD pair has one passenger, i.e., $t_d = 1$. With two zones $\mathcal{Z} = \{\{1, 2\}, \{3\}\}$ (Figure 10.2(b)) the objective function value is 0, but for three zones the only choice of zones is $\mathcal{Z} = \{\{1\}, \{2\}, \{3\}\}$ (Figure 10.2(c)) with an objective function value of 1. However, here, N is an upper bound on the number of zones, which need not be met with equality.

In the remainder of this section, we compare the optimal objective function values of the four variants of Z-FDM (see Table 10.1). Given an instance of

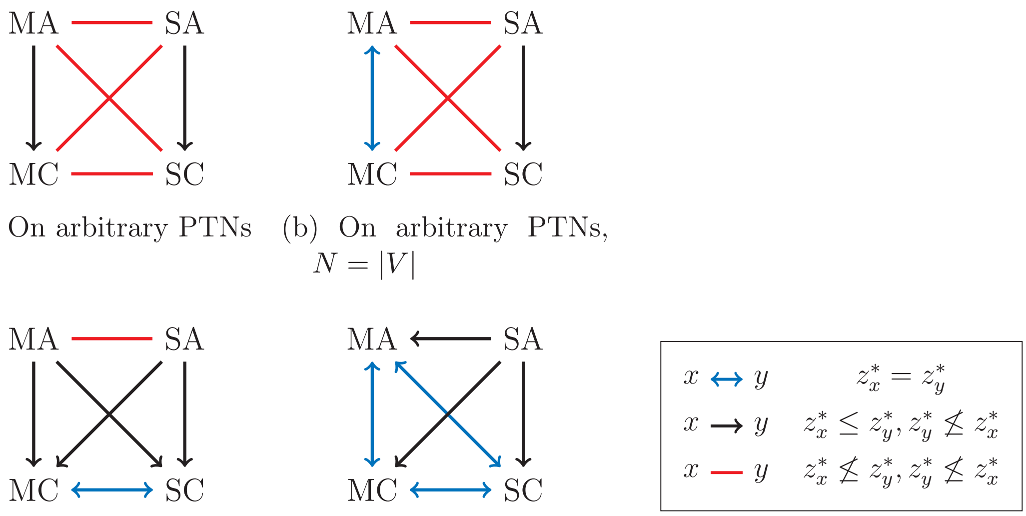


(a) PTN with OD data (orange)

(b) Two zones

(c) Three zones

Figure 10.2: Instance showing that it can be better to implement fewer zones.



(a) On arbitrary PTNs

(b) On arbitrary PTNs, $N = |V|$

(c) On trees

(d) On trees, $N = |V|$

Figure 10.3: Relationships between the optimal objective function values of the four problem variants of Z-FDM in different cases (arbitrary PTN or tree, arbitrary $N \in \mathbb{N}_{\geq 1}$ or $N = |V|$). Note that the results on trees are for the case that the paths W_d are the unique simple paths for all OD pairs d (see Lemma 10.11).

Z-FDM, let z^*_{MA}, z^*_{MC}, z^*_{SA} and z^*_{SC} be the respective optimal objective function values of the four variants. The results of Lemma 10.11, Theorem 10.12 and Examples 10.13 to 10.18 are summarized in Figure 10.3. All these results hold with/without requiring the no-elongation property and with/without requiring the no-stopover property (which is not explicitly mentioned in each result for the sake of shortness).

Lemma 10.11 provides information about the relationship between the objective function values of the different problem variants in general as well as on trees.

Lemma 10.11. *Let an instance of* Z-FDM *be given. Then we have:*

1. $z^*_{\text{MA}} \leq z^*_{\text{MC}}$ *and* $z^*_{\text{SA}} \leq z^*_{\text{SC}}$. *Both may hold strictly.*
2. $z^*_{\text{MC}} = z^*_{\text{SC}}$ *if the graph is a tree and the paths* W_d *for* $d \in D$ *are the unique simple paths.*

(a) PTN with OD data (orange) (b) Optimal zones for Z-FDM(MA)

Figure 10.4: Instance for Example 10.13.

Proof.

1. Requiring connectedness of the zones is a restriction on the solution space. Examples 10.13 and 10.15 show that the inequalities can hold strictly.

2. Follows from Lemma 10.9. □

If $N = |V|$, we also obtain equality for the multiple counting cases, i.e., $z^*_{\text{MA}} = z^*_{\text{MC}}$, as Theorem 10.12 shows. Note that it still may happen that less than N zones are used, as in the example in Figure 10.2.

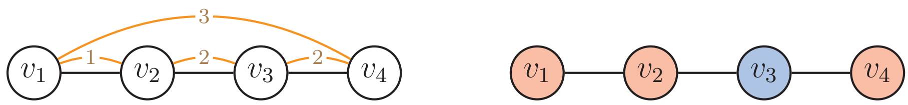
Theorem 10.12. *Let an instance of* Z-FDM *with upper bound* $N \in \mathbb{N}_{\geq 1}$ *on the number of zones be given. For every solution* $\mathcal{Z}, P$ *to* Z-FDM(MA) *with upper bound* N *on the number of zones, there is a zone partition* $\mathcal{Z}'$ *such that* $\mathcal{Z}', P$ *is feasible to* Z-FDM(MC) *with upper bound* $|V|$ *on the number of zones and has the same objective function value. In particular, if* $N = |V|$*, we have* $z^*_{\text{MA}} = z^*_{\text{MC}}$.

Proof. Let $\mathcal{Z}, P$ be a solution to Z-FDM(MA) regarding N with $\mathcal{Z} = \{Z_1, \ldots, Z_L\}$ and $L \leq N$. We enumerate all connected components of all zones: For all $i \in [L]$, let $l_i, k_i \in \mathbb{N}_{\geq 1}$ with $l_i \leq k_i$, $l_1 = 1$ and $l_{i+1} = k_i + 1$ for $i \in [L-1]$ such that $Z'_{l_i}, \ldots, Z'_{k_i}$ denote all connected components of $G[Z_i]$. We set $\mathcal{Z}' := \{Z'_{l_i}, \ldots, Z'_{k_i} : i \in [L]\}$. Then $L \leq |V|$ and all $Z \in \mathcal{Z}'$ are connected. For each OD pair, the number of zone borders that are crossed on its path are the same for $\mathcal{Z}$ and $\mathcal{Z}'$ because no connected parts are split in the new zone partition. Therefore, also the objective function value remains unchanged. □

The following series of Examples 10.13 to 10.18 shows that there are no further inequalities that hold in general, on trees or with $N = |V|$.

Example 10.13 (Example for $z^*_{\text{MA}} < z^*_{\text{MC}}$ on a tree)**.** Consider the PTN depicted in Figure 10.4(a), which is a tree. The OD pairs with their reference prices and paths are marked in orange. Every OD pair has one passenger, i.e., $t_d = 1$. Let $N := 2$.

Then $\mathcal{Z} = \{\{v_1, v_3\}, \{v_2\}\}$ (Figure 10.4(b)) with $(p^*_1, p^*_2, p^*_3) = (1, 1, 2)$ is an optimal solution to Z-FDM(MA) with objective function value $z^*_{\text{MA}} = 0$. The only two structurally different connected zone partitions for Z-FDM(MC) are $\mathcal{Z} = \{\{v_1, v_2, v_3\}\}$ and $\mathcal{Z} = \{\{v_1, v_2\}, \{v_3\}\}$. In both cases it is optimal to set $(p^*_1, p^*_2, p^*_3) = (1, 1, 1)$, yielding an optimal objective function value of 1. Hence, $z^*_{\text{MA}} < z^*_{\text{MC}}$.



(a) PTN with OD data (orange) (b) Optimal zones for Z-FDM(MA)

Figure 10.5: Instance for Example 10.14.

(a) PTN with OD data (orange) (b) Optimal zones for Z-FDM(SA)

Figure 10.6: Instance for Example 10.15

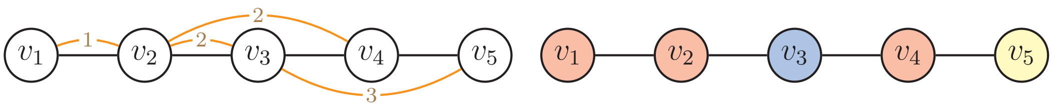
Example 10.14 (Example for $z^*_{\mathrm{MA}} < z^*_{\mathrm{SA}}$ on a tree)**.** Consider the PTN depicted in Figure 10.5(a), which is a tree. The OD pairs with their reference prices and paths are marked in orange. Every OD pair has one passenger, i.e., $t_d = 1$. Let $N := 2$.

Then $\mathcal{Z} = \{\{v_1, v_2, v_4\}, \{v_3\}\}$ (Figure 10.5(b)) with $(p^*_1, p^*_2, p^*_3) = (1, 2, 3)$ is an optimal solution to Z-FDM(MA) with objective function value $z^*_{\mathrm{MA}} = 0$. Because it is not possible to observe three different prices with only two zones for the single counting case, we have $z^*_{\mathrm{MA}} < z^*_{\mathrm{SA}}$.

Example 10.15 (Example for $z^*_{\mathrm{SA}} < z^*_{\mathrm{SC}}$ on a tree with $N = |V|$)**.** Consider the PTN (V, E) depicted in Figure 10.6(a), which is a tree. The OD pairs with their reference prices and paths are marked in orange. Every OD pair has one passenger, i.e., $t_d = 1$. Let $N := 5 = |V|$.

Then $\mathcal{Z} = \{\{v_1, v_4\}, \{v_2, v_3\}, \{v_5\}\}$ (Figure 10.6(b)) with $(p^*_1, p^*_2, p^*_3) = (1, 2, 3)$ is an optimal solution to Z-FDM(SA) with objective function value $z^*_{\mathrm{SA}} = 0$. In order to obtain $z^*_{\mathrm{SC}} = 0$, we need to find a zone partition $\mathcal{Z}$ and a price function P such that $P(\sigma(W_d)) = r_d$ for all $d \in D$. Consider the OD pairs $d' \in \{(v_1, v_2), (v_2, v_4), (v_4, v_5)\}$ with $r_{d'} = 2$, and $\bar{d} = (v_2, v_3)$ with $r_{\bar{d}} = 1$. Then $r_{d'} \neq r_{\bar{d}}$, and we hence need $\sigma(W_{d'}) \neq \sigma(W_{\bar{d}})$. For $\sigma(W_{d'}) = 1$, we must choose $\mathcal{Z} = \{\{v_1, v_2, v_4, v_5\}, \{v_3\}\}$, which does not yield an objective function value of 0 because we cannot observe three different prices with only two zones. Hence, $\sigma(W_{d'}) = 2$, for which we must choose $\mathcal{Z} = \{\{v_1\}, \{v_2, v_3\}, \{v_4\}, \{v_5\}\}$. However, $\sigma(W_{(v_1,v_4)}) = 3 = \sigma(W_{(v_2,v_5)})$ but $r_{(v_1,v_4)} = 2 \neq 3 = r_{(v_2,v_5)}$. Therefore, there is no zone partition $\mathcal{Z}$ and price function P with objective function value 0, and we have $z^*_{\mathrm{SA}} < z^*_{\mathrm{SC}}$.

Example 10.16 (Example for $z^*_{\mathrm{SA}} < z^*_{\mathrm{MA}}$ on a tree with $N = |V|$)**.** Consider the PTN depicted in Figure 10.7(a), which is a tree. The OD pairs with their reference



(a) PTN with OD data (orange) (b) Optimal zones for Z-FDM(SA)

Figure 10.7: Instance for Example 10.16.

(a) PTN with OD data (orange) (b) Optimal zones for Z-FDM(MA) and Z-FDM(MC)

Figure 10.8: Instance for Example 10.17

prices and paths are marked in orange. Every OD pair has one passenger, i.e., $t_d = 1$. Let $N := 5 = |V|$.

Then $\mathcal{Z} = \{\{v_1, v_2, v_4\}, \{v_3\}, \{v_5\}\}$ (Figure 10.7(b)) with $(p_1^*, p_2^*, p_3^*) = (1, 2, 3)$ is an optimal solution to Z-FDM(SA) with objective function value $z_{\text{SA}}^* = 0$. In order to obtain $z_{\text{MA}}^* = 0$, we need to find a zone partition $\mathcal{Z}$ and a price function P such that $P(\sigma(W_d)) = r_d$ for all $d \in D$. If $\sigma(W_{(v_1,v_2)}) = 1$, then $\sigma(W_{(v_2,v_3)}) = 2$ and $\sigma(W_{(v_3,v_5)}) = 3$ because these OD pairs all have different reference prices. But then also $\sigma(W_{(v_2,v_4)}) = 3$, which yields an objective function value of 1. Hence, $\sigma(W_{(v_1,v_2)}) = 2$, $\sigma(W_{(v_2,v_3)}) = 1$, $\sigma(W_{(v_3,v_5)}) = 3$, and thus $\sigma(W_{(v_2,v_4)}) = 2$, which again yields an objective function value of 1. Therefore, there is no zone partition $\mathcal{Z}$ and price function P with objective function value 0, and we have $z_{\text{SA}}^* < z_{\text{MA}}^*$.

Example 10.17 (Example for $z_{\text{MY}_1}^* < z_{\text{SY}_2}^*$ with $N = |V|$ for $\text{Y}_1, \text{Y}_2 \in \{\text{A}, \text{C}\}$)**.** Consider the PTN depicted in Figure 10.8(a). The OD pairs with their reference prices and paths are marked in orange. Every OD pair has one passenger, i.e., $t_d = 1$. Let $N := 5 = |V|$.

Then $\mathcal{Z} = \{\{v_1\}, \{v_2, v_3, v_5\}, \{v_4\}\}$ (Figure 10.8(b)) with $(p_1^*, p_2^*, p_3^*) = (1, 2, 3)$ is an optimal solution to Z-FDM(MA) and Z-FDM(MC) with objective function value $z_{\text{MA}}^* = z_{\text{MC}}^* = 0$. In order to obtain $z_{\text{SY}_2}^* = 0$, we need to find a zone partition $\mathcal{Z}$ and a price function P such that $P(\sigma(W_d)) = r_d$ for all $d \in D$. If $\sigma(W_{(v_1,v_2)}) = 1$, then $\sigma(W_{(v_2,v_3)}) = 2$ because these OD pairs have different reference prices. But then also $\sigma(W_{(v_1,v_3)}) = 2$, which yields an objective function value of at least 1. Hence, $\sigma(W_{(v_1,v_2)}) = 2$, $\sigma(W_{(v_2,v_3)}) = 1$, $\sigma(W_{(v_1,v_3)}) = 2$, and also $\sigma(W_{(v_3,v_5)}) = 1$, which means that v_2, v_3 and v_5 are in the same zone. Thus, $\sigma(W_{(v_2,v_3)}) \in \{1, 2\}$, which both yield an objective function value of at

(a) PTN with OD data (orange) (b) Optimal zones for Z-FDM(SC)

Figure 10.9: Instance for Example 10.18

least 1 because $r_{(d_2,d_5)} \notin \{1,2\}$. Therefore, there is no zone partition $\mathcal{Z}$ and price function P with objective function value 0, which yields $z^*_{\mathrm{MA}} = z^*_{\mathrm{MC}} < z^*_{\mathrm{SA}}$ and $z^*_{\mathrm{MA}} = z^*_{\mathrm{MC}} < z^*_{\mathrm{SC}}$.

Example 10.18 (Example for $z^*_{\mathrm{SY}_1} < z^*_{\mathrm{MY}_2}$ with $N = |V|$ for $\mathrm{Y}_1, \mathrm{Y}_2 \in \{\mathrm{A}, \mathrm{C}\}$)**.** Consider the PTN depicted in Figure 10.9(a). The OD pairs with their reference prices and paths are marked in orange. Every OD pair has one passenger, i.e., $t_d = 1$. Let $N := 5 = |V|$.

Then $\mathcal{Z} = \{\{v_1\}, \{v_2, v_3, v_5\}, \{v_4\}\}$ (Figure 10.9(b)) with $(p_1^*, p_2^*, p_3^*) = (1, 2, 3)$ is an optimal solution to Z-FDM(SA) and Z-FDM(SC) with objective function value $z^*_{\mathrm{SA}} = z^*_{\mathrm{SC}} = 0$. In order to obtain $z^*_{\mathrm{MY}_2} = 0$, we need to find a zone partition $\mathcal{Z}$ and a price function P such that $P(\sigma(W_d)) = r_d$ for all $d \in D$. As in Example 10.17, $\sigma(W_{(v_1,v_2)}) = 1$ does not yield an objective function value of 0. Hence, $\sigma(W_{(v_1,v_2)}) = 2$, $\sigma(W_{(v_2,v_3)}) = 1$, $\sigma(W_{(v_1,v_3)}) = 2$, and also $\sigma(W_{(v_3,v_5)}) = 1$, which means that v_2, v_3 and v_5 are in the same zone. Further, $\sigma(W_{(v_1,v_4)}) = 3$ because $r_{(v_1,v_4)} \notin \{1,2\}$, but then $\sigma(W_{(v_2,v_5)}) = 3$, which yields an objective function value of 1. Therefore, there is no zone partition $\mathcal{Z}$ and price function P with objective function value 0, which yields $z^*_{\mathrm{SA}} = z^*_{\mathrm{SC}} < z^*_{\mathrm{MA}}$ and $z^*_{\mathrm{SA}} = z^*_{\mathrm{SC}} < z^*_{\mathrm{MC}}$.

Because the price lists constructed in Examples 10.13 to 10.18 clearly satisfy the no-elongation property and the no-stopover property, all these results hold with/without requiring the no-elongation property and with/without requiring the no-stopover property.

10.4.2 Complexity and Solution Methods

In [HS04, Thm. 2] it is shown that Z-FDM(MA) and Z-FDM(MC) with the objective to *minimize the maximum absolute deviation* from reference prices are NP-hard in case that $N \geq 3$ is fixed, the zone partition needs to consist of exactly N (non-empty) sets and the passengers' paths minimize the number of traversed zones. Further, for fixed passengers' paths, [OB17, Thm. 1] proves NP-hardness of Z-FDM(SA) (even if $N = 2$) and Z-FDM(SC) with the objective to *maximize the revenue*, where passengers have a limited willingness to pay. Also, [OB17, Thm. 4] shows that setting the prices can be done in $\mathcal{O}(K \cdot |D|)$ in case the zones are given, where K is the maximum number of traversed zones. We contribute to these results with a complexity analysis of Z-FDM(XY) for X $\in$ {M, S} and Y $\in$ {A,C} that *minimize the weighted sum of absolute deviations* from given reference prices including the no-elongation property and the no-stopover property.

In order to analyze the complexity, we consider the decision version of Z-FDM, which we call Z-FDM as well. The problem changes such that we have an additional input parameter $J \in \mathbb{R}_{\geq 0}$ and search for a zone partition and a price function such that $\sum_{d \in D} t_d |r_d - P(\sigma(W_d))| \leq J$.

Lemma 10.19. Z-FDM(XY) *with* X $\in$ {M, S} *and* Y $\in$ {A,C} *with/without requiring the sufficient conditions of the no-elongation property and the no-stopover property (Theorem 9.8) as well as the corresponding zone-partition and price-setting subproblems are in* NP.

Proof. Let a certificate $\mathcal{Z} = \{Z_1, \dots, Z_L\}, P$, where the price function is determined by a price list $(p_1, \dots, p_K)$, where $L \leq N$ and K is the maximum number of stations on a path, be given. We can check in polynomial time:

- $\bigcup_{i \in [L]} Z_i = V$ as well as $Z_i \cap Z_j = \emptyset$ and $Z_i \neq \emptyset$ for all $i, j \in [L]$ with $i \neq j$,
- $\sum_{d \in D} t_d |r_d - P(\sigma(W_d)))| \leq J$,
- connectedness of $G[Z_i]$ for all $i \in [L]$,
- the no-elongation property and the no-stopover property according to Theorem 9.8. □

First, we show that Z-FDM with arbitrary zones is NP-hard.

Theorem 10.20. *The problems* Z-FDM(MA) *and* Z-FDM(SA) *are* NP*-hard*

- *with/without requiring the no-elongation property,*
- *with/without requiring the no-stopover property,*
- *even if* $N = 2$.

Proof. We show that the decision versions of Z-FDM(MA) and Z-FDM(SA) with an upper bound J on the objective function value are NP-complete.

By Lemma 10.19, the problems are in NP.

We use a reduction from Bipartite Subgraph (Problem 2.11) using the equivalent formulation of Remark 2.12. Let an instance $G' = (V', E'), Q$ of Bipartite Subgraph be given. We construct an instance of Z-FDM(XA),

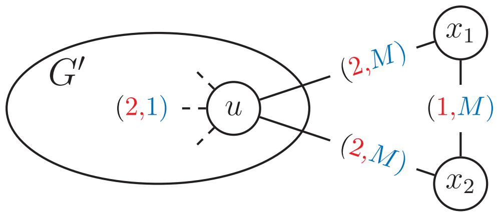


Figure 10.10: Graph construction for the proof of Theorem 10.20. The node u is an arbitrary but fixed node in the graph G' (indicated by the ellipsoid). The additional nodes x_1 and x_2 are connected with each other and with u. The OD pairs correspond to the edges. The reference prices r_d (red) and numbers of passengers t_d (blue) per OD pair $d \in D$ are given on the edges as (r_d, t_d).

X $\in$ {M, S}. Let x_1 and x_2 be additional nodes not contained in V', and let $u \in V'$ be chosen arbitrarily but fixed. We define:

$$\begin{aligned}
V &:= V' \cup \{x_1, x_2\},\\
E &:= E' \cup \{\{u, x_1\}, \{u, x_2\}, \{x_1, x_2\}\},\\
D &:= \{(v_1, v_2) : \{v_1, v_2\} \in E\},\\
r_d &:= \begin{cases} 2 & \text{for all } d \in D \setminus \{(x_1, x_2)\},\\ 1 & \text{for } d = (x_1, x_2),\end{cases}\\
t_d &:= \begin{cases} 1 & \text{for all } d \in D \setminus \{(u, x_1), (u, x_2), (x_1, x_2)\},\\ M := |E'| + 1 & \text{for all } d \in \{(u, x_1), (u, x_2), (x_1, x_2)\},\end{cases}\\
W_d &:= d \quad \text{for all } d \in D,\\
N &:= 2,\\
J &:= Q \le |E'|.
\end{aligned}$$

This means that each OD pair corresponds to an edge and each edge in the new network is a path used by one passenger with reference price 2 for the edges in E', and by $M > |E'|$ passengers with reference price 1 or 2 for the edges in $E \setminus E'$. For the maximum number K of nodes of a path, we thus have $K = 2$. The construction is shown in Figure 10.10. Solving Z-FDM on this instance, we need to determine two zones A and B and two prices (p_1, p_2) (see Remark 10.8). For the zone partition, we have the following three options for x_1, x_2, u:

- First, x_1, x_2, u are in the same zone. Because $\sigma((u, x_1)) = \sigma((x_1, x_2)) = 1$, the contribution to the objective function value is at least

$$M \cdot |2 - p_1| + M \cdot |1 - p_1| \ge M = |E'| + 1 > Q = J.$$

  Hence, this option does not yield a feasible solution.

- Second, $x_1, u \in A, x_2 \in B$ (analogously for swapping A and B and for swapping x_1 and x_2). Because $\sigma((u,x_2)) = \sigma((x_1,x_2)) = 2$, the contribution to the objective function value is at least

$$M \cdot |2 - p_2| + M \cdot |1 - p_2| \geq M = |E'| + 1 > Q = J.$$

  Hence, this option does also not yield a feasible solution.

- Third, $x_1, x_2 \in A, u \in B$ (analogously for swapping A and B). The objective function value then is

$$\begin{aligned}\sum_{d \in D} t_d |r_d - P(\sigma(W_d))| &= M \cdot |1 - p_1| + 2 \cdot M \cdot |2 - p_2| \\ &\quad + \sum_{e \in \operatorname{int}(A,B)} |2 - p_1| + \sum_{e \in E' \setminus \operatorname{int}(A,B)} |2 - p_2|.\end{aligned}$$

  If $p_1 = 2$ or $p_2 = 1$, then the objective function value exceeds J because $M = |E'| + 1 > Q = J$. We therefore choose $p_1^* = 1$ and $p_2^* = 2$. This yields an overall objective function value of $|\operatorname{int}(A,B)|$.

It is hence only possible to obtain a feasible solution if x_1 and x_2 are in the same zone and u is in the other zone. A solution to the remaining problem in this third option is a solution to Bipartite Subgraph and the other way around: In both cases, a bipartition of the nodes V into sets A and B has to be found such that $|\operatorname{int}(A,B)| \leq Q$.

Note that the resulting prices satisfy the no-elongation property (due to the monotonicity of the price list) and the no-stopover property (because the price $P(k)$ is constant for $k \geq 2$). Hence, the construction does the same if these properties are required. Because each path W_d with $d \in D$ consists of only one edge, the number of zones with multiple counting and with single counting are the same by Lemma 10.9. □

Next, we show NP-hardness of Z-FDM with the requirement of connected zones. This does not follow from the proof of Theorem 10.20 because the resulting zones will in most cases not be connected. For the NP-hardness proof, we make use of the NP-complete problem Multicut, which was also used in [OB17].

Theorem 10.21. *The problems* Z-FDM(MC) *and* Z-FDM(SC) *for connected zones are* NP*-hard*

- *with/without requiring the no-elongation property,*
- *with/without requiring the no-stopover property,*
- *even if the graph is a tree.*

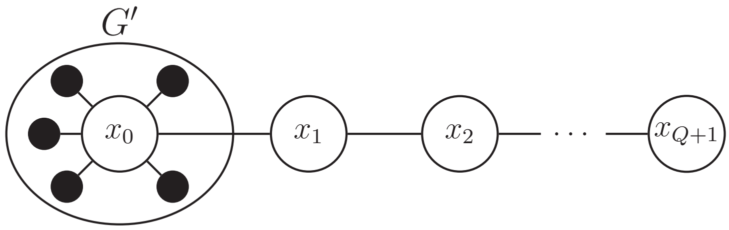


Figure 10.11: Graph construction for the proof of Theorem 10.21. The star graph G' with center x_0 is indicated in the ellipsoid. It is extended by a path consisting of $x_0, \dots, x_{Q+1}$.

Proof. We show that the decision versions of Z-FDM(MC) and Z-FDM(SC) with an upper bound J on the objective function value are NP-complete.

By Lemma 10.19, the problems are in NP.

We use a reduction from MULTICUT (Problem 2.13). Let an instance of MULTICUT consisting of a star graph $G' = (V', E')$, source-terminal pairs $\mathcal{C}$, a non-negative integer Q be given. We construct an instance of Z-FDM(XC), X $\in$ {M, S}. Let $x_1, \dots, x_{Q+1}$ be additional nodes not contained in V, and let $x_0 \in V$ be the center of the star graph G'. We define:

$$
\begin{aligned}
V &:= V' \cup V_x \quad \text{with } V_x := \{x_j : j \in [Q+1]\},\\
E &:= E' \cup E_x \quad \text{with } E_x := \{\{x_j, x_{j+1}\} : j \in \{0, \dots, Q\}\},\\
G &:= (V, E)\\
D &:= \mathcal{C} \cup D_x \quad \text{with } D_x := \{(x_j, x_{j+1}) : j \in \{0, \dots, Q\}\},\\
r_d &:= \begin{cases} 1 & \text{for all } d \in D_x,\\ 2 & \text{for all } d \in \mathcal{C},\end{cases}\\
t_d &:= 1 \quad \text{for all } d \in D,\\
W_d &\text{ is the unique simple path in } (V, E) \text{ for all } d \in D,\\
N &:= Q + 1,\\
J &:= 0.
\end{aligned}
$$

This means that each OD pair has one passenger and corresponds either to a source-terminal pair with a reference price of 2, or to a newly added edge with a reference price of 1. Hence, for the maximum number K of nodes of a path, we have $K \leq 3$. Because $Q < |\mathcal{C}|$, this is a polynomial reduction. The construction is depicted in Figure 10.11. Solving Z-FDM on this instance, we need to determine at most N zones and a price list (p_1, p_2, p_3) (see Remark 10.8).

We show that there is a solution $\bar{E}$ to MULTICUT if and only if there is a solution $\mathcal{Z}, P$ to Z-FDM(XC) with X $\in$ {M, S}.

For the first direction, let $\bar{E} \subseteq E'$ be a solution to MULTICUT. Deleting an edge $e \in \bar{E}$ in the star graph generates a new connected component. Thus

$(V, E \setminus \bar{E})$ has $L := 1 + |\bar{E}| \leq 1 + Q = N$ connected components. We define the connected components to be the zones $Z_1, \dots, Z_L$, in particular $G[Z_i]$ is connected for all $i \in [L]$. Because all pairs in $\mathcal{C}$ are separated by $\bar{E}$ and $\bar{E} \cap E_x = \emptyset$, we have $\sigma(W_d) \in \{2, 3\}$ for all $d \in D$, and $\sigma(W_d) = 1$ for all $d \in D_x$. By setting $p_1^* = 1$ and $p_2^* = p_3^* = 2 =: p_{2,3}^*$, we obtain a feasible solution to Z-FDM(XC) with $\mathrm{X} \in \{\mathrm{M}, \mathrm{S}\}$:

$$\begin{aligned}\sum_{d \in D} t_d |r_d - P(\sigma(W_d))| &= \sum_{d \in \mathcal{C}} |2 - p_{2,3}^*| + \sum_{d \in D_x} |1 - p_1^*| \\ &= \sum_{d \in \mathcal{C}} |2 - 2| + \sum_{d \in D_x} |1 - 1| = 0.\end{aligned}$$

For the other direction, let a zone partition $\mathcal{Z} = \{Z_1, \dots, Z_L\}$ with $L \leq N$ and a price function P given by a price list (p_1^*, p_2^*, p_3^*) be a solution to Z-FDM(XC) with $\mathrm{X} \in \{\mathrm{M}, \mathrm{S}\}$. Because $J = 0$, we have $P(\sigma(W_d)) = r_d$ for all $d \in D$. We set

$$\bar{E} := \{\{v_1, v_2\} \in E' : v_1 \in Z_i, v_2 \in Z_j, i \neq j\}.$$

It holds that $|\bar{E}| \leq Q = N - 1$ because $G' = (V', E')$ is a star graph and $G[Z_i]$ is connected for all $i \in [L]$: Otherwise, if $|\bar{E}| > N - 1$, there would be more than N connected components, which would mean that at least one zone Z_i would not be connected. Next, we show that all source-terminal pairs in $\mathcal{C}$ are separated by $\bar{E}$. Note that it cannot happen that each of the nodes $x_1, \dots, x_{Q+1}$ forms a singleton zone only containing that node because then at least $Q+2$ zones would be needed, which is not feasible because $Q+2 > N$. Hence, there is an $i \in [L]$ and $j \in \{0, \dots, Q\}$ such that $\{x_j, x_{j+1}\} \subseteq Z_i$. Therefore, $\sigma(W_{(x_j, x_{j+1})}) = 1$. Because $J = 0$, we have that $p_1^* = r_{(x_j, x_{j+1})} = 1$. Again because $J = 0$ and $r_d = 2 \neq 1 = p_1^*$ for all $d \in \mathcal{C}$, we obtain $\sigma(W_d) \in \{2, 3\}$ for all $d \in \mathcal{C}$ and $p_2^* = p_3^* = 2$. Hence, no source-terminal pair is in the same zone. This means that for all $d \in \mathcal{C}$, there exists an edge $\{v_1, v_2\} \in E'(W_d)$ with v_1 and v_2 in different zones, and hence $\{v_1, v_2\} \in \bar{E}$. Therefore, all source-terminal pairs in $\mathcal{C}$ are separated by $\bar{E}$.

Note that the resulting prices satisfy the no-elongation property and the no-stopover property. Because the zones are connected and the paths are the unique simple paths in the tree, the number of zones with multiple counting and with single counting are the same by Lemma 10.11. □

Zone-Partition Subproblem Let us now consider the zone-partition subproblem of Z-FDM, i.e., Z-FDM with a given price function and the task to optimize the zone partition. For $N = 1$, this is simple because there is exactly one feasible solution, namely all nodes are in the same zone. However, the zone-partition subproblem with arbitrary zones is already NP-hard if we have $N = 2$, and the zone-partition subproblem with connected zones is already NP-hard if the graph is a tree as the following corollaries show.

Corollary 10.22. *The zone-partition subproblem of* Z-FDM(MA) *and of* Z-FDM(SA) *(with arbitrary zones) is* NP*-hard, even if* $N = 2$.

Proof. The proof of Theorem 10.20 works analogously if we set $p_1^* := 1$ and $p_2^* := 2$ already in the construction of the instance of Z-FDM. □

Corollary 10.23. *The zone-partition subproblem of* Z-FDM(MC) *and of* Z-FDM(SC) *(with connected zones) is* NP*-hard, even if the graph is a tree.*

Proof. The proof of Theorem 10.21 works analogously if we set $p_1^* := 1$ and $p_2^* = p_3^* := 2$ already in the construction of the instance of Z-FDM with connected zones. □

We remark that this is also true for the zone-partition subproblems with single counting and with connected or ring zones with the objective of maximizing the revenue in [OB17] because there the prices are always set to the same values independent of the instance, as well.

Price-Setting Subproblem We have seen that the zone-partition subproblem of Z-FDM remains NP-hard in both cases with and without the requirement of connected zones. We now study the price-setting subproblem of Z-FDM, so let a zone partition $\mathcal{Z}$ be given. In this setting, we use the notation of Definition 9.11. The price-setting subproblem of Z-FDM takes OD data (D, W_d, t_d) and a partition $D_1, \ldots, D_K$ of D as input and searches for a price list $p = (p_1, \ldots, p_K)$, where K is the maximum number of zones traversed by an OD pair and D_k with $k \in [K]$ contains all OD pairs $d \in D$ with $\sigma(W_d) = k$ for the zone partition $\mathcal{Z}$. Note that only for determining the sets D_k with $k \in [K]$ and for the according no-stopover property it is important whether multiple or single counting is considered. Otherwise, the price-setting subproblem does not differ between the two variants.

The price-setting subproblem of Z-FDM without requiring the no-elongation property and without requiring the no-stopover property, which we call the *unrestricted price-setting subproblem*, is

$$\begin{aligned} \min_{p_k} \quad & \sum_{k=1}^{K} \sum_{d \in D_k} t_d |r_d - p_k| \\ \text{s.t.} \quad & p_k \in \mathbb{R}_{\geq 0} \quad \text{for all } k \in [K], \end{aligned}$$

which breaks down into a problem of the form of F-FDM (see Section 10.2) for each number of traversed zones k (see [HS04, Thm. 1]) and can hence be solved in linear time by Theorem 10.24.

Theorem 10.24. *A price list* $(p_1, \ldots, p_K)$ *is an optimal solution to the unrestricted price-setting subproblem of* Z-FDM *if and only if* $p_k \in \mathsf{w\text{-}median}_{d \in D_k}(r_d, t_d)$

if $D_k \neq \emptyset$, and $p_k \in \mathbb{R}_{\geq 0}$ arbitrary otherwise. Hence, the unrestricted price-setting subproblem of Z-FDM *can be solved in $\mathcal{O}(|D|)$.*

In particular, there is an optimal solution with $\{p_1, \ldots, p_K\} \subseteq \{r_d : d \in D\}$.

Proof. The unrestricted price-setting subproblem breaks down into a problem of the form of F-FDM for each number of traversed zones $k \in [K]$. As discussed in Section 10.2, for each $k \in [K]$ with $D_k \neq \emptyset$, we have that p_k is an optimal solution to F-FDM corresponding to D_k if and only if $p_k \in \mathsf{w\text{-}median}(D_k)$. These values can be computed in $\sum_{k\in[K]} \mathcal{O}(|D_k|) = \mathcal{O}(|D|)$. □

If the no-elongation property and the no-stopover property are required, which are implemented with their sufficient conditions (see Theorem 9.8, with different constraints for multiple counting [M] and single counting [S]), the price-setting subproblem can be solved in polynomial time with respect to $|D|$ and K by the following LP formulation:

$$\begin{aligned}
\min_{p_k,\, y_d} \quad & \sum_{d \in D} t_d y_d \\
\text{s.t.} \quad & r_d - p_k \leq y_d && \text{for all } d \in D_k, k \in [K] \\
& p_k - r_d \leq y_d && \text{for all } d \in D_k, k \in [K] \\
& p_k \leq p_{k+1} && \text{for all } k \in [K-1] \\
[\mathrm{M}] \quad & p_k \leq p_i + p_{k-i+1} && \text{for all } k \in [K],\ i \in [k] \\
[\mathrm{S}] \quad & p_k \leq p_{i_1} + p_{i_2} && \text{for all } k \in [K],\ i_1, i_2 \in [k] \\
& && \quad \text{with } i_1 + i_2 \geq k+1 \\
& p_k \in \mathbb{R}_{\geq 0} && \text{for all } k \in [K] \\
& y_d \in \mathbb{R} && \text{for all } d \in D.
\end{aligned} \tag{10.3}$$

The number of constraints for the no-stopover property in case of multiple counting can be reduced slightly by only enforcing the constraint for $k \in \{3, \ldots, K\}$ and $i \in \{2, \ldots, \lfloor \frac{k+1}{2} \rfloor\}$ as shown in [SU20, Thm. 19]. Similarly, the range of the no-stopover property constraints for the single counting case can be reduced to $k \in \{3, \ldots, K\}$ and $i_1 \in \{\lceil \frac{k+1}{2} \rceil, \ldots, k\}$, $i_2 \in \{k+1-i_1, \ldots, i_1\}$.

Because of the relevance of increasing prices in practice, we have a closer look at the price-setting subproblem enforcing the no-elongation property by its sufficient condition of a monotonically increasing price function (see Theorem 9.8), which we call the *monotonic price-setting subproblem*:

$$\begin{aligned}
\min_{p_k} \quad & \sum_{k=1}^{K} \sum_{d \in D_k} t_d |r_d - p_k| \\
\text{s.t.} \quad & p_k \leq p_{k+1} && \text{for all } k \in [K-1] \\
& p_k \in \mathbb{R}_{\geq 0} && \text{for all } k \in [K].
\end{aligned} \tag{10.4}$$

The LP formulation of this problem still has $|D| + K$ variables and $2 \cdot |D| + K$ constraints. Therefore, we aim to find a better solution method, which is motivated by Lemma 10.25 and Remark 10.26.

For the unrestricted price-setting subproblem, we know by Theorem 10.24 that in an optimal solution we have $p_k \in \text{w-median}(D_k)$ for all $k \in [K]$ with $D_k \neq \emptyset$. Note that this does not hold in general for the monotonic price-setting subproblem as the following simple example shows: Let $K := 3$ and we set $D_1 = \{d_1\}$, $t_{d_1} = 1$, $r_{d_1} = 2$ and $D_2 = \{d_2\}$, $t_{d_2} = 2$, $r_{d_2} = 1$ and $D_3 = \{d_3\}$, $t_{d_3} = 3$, $r_{d_3} = 3$. Then $p = (1, 1, 3)$ is an optimal solution to the monotonic price-setting subproblem but $1 \notin \{2\} = \text{w-median}(D_1)$.

Lemma 10.25. *Let $p = (p_1^*, \dots, p_K^*)$ be an optimal solution to the monotonic price-setting subproblem of* Z-FDM. *If $p_1^* < \dots < p_K^*$, then $p_k^* \in \text{w-median}(D_k)$ for all $k \in [K]$.*

Proof. Let $k \in [K]$ with $D_k \neq \emptyset$. Assume $p_k^* \notin \text{w-median}(D_k)$, and let $p_k' \in \text{w-median}(D_k)$. By Section 10.2, p_k' minimizes $\varphi_k \colon p_k \mapsto \sum_{d \in D_k} t_d |r_d - p_k|$, and $\varphi_k(p_k') < \varphi_k(p_k^*)$. Because $p_{k-1}^* < p_k^* < p_{k+1}^*$ (with $p_0^* := -\infty$ or $p_{K+1}^* := +\infty$ if necessary), we can increase or decrease p_k^* towards p_k' so that the order of the prices remains increasing. Because φ_k is convex and hence strictly decreasing on $\mathbb{R}_{\leq p_k'} \setminus \text{w-median}(D_k)$ and strictly increasing on $\mathbb{R}_{\geq p_k'} \setminus \text{w-median}(D_k)$, this leads to a reduction of the objective function value, which is a contradiction to $(p_1^*, \dots, p_K^*)$ being an optimal solution. □

If we consider the example from before, we see that by merging D_1 and D_2 to a common price level we indeed get $1 \in \text{w-median}(D_1 \cup D_2)$. We formalize this idea in Remark 10.26.

Remark 10.26. We can generalize the result of Lemma 10.25 by considering distinct price levels as follows: Let $0 \leq p_1^* \leq \dots \leq p_K^*$ be an optimal solution to the monotonic price-setting subproblem of Z-FDM. Let $q_1, \dots, q_L \in \mathbb{R}_{\geq 0}$ with $L \leq K$ be the distinct price levels that satisfy $\{p_1^*, \dots, p_K^*\} = \{q_1, \dots, q_L\}$ and $q_1 < \dots < q_L$. For all $l \in [L]$, we set $I_l := \{k \in [K] : p_k = q_l\}$. Analogously to Lemma 10.25, it holds that $q_l \in \text{w-median}(\bigcup_{k \in I_l} D_k)$.

Note however that Remark 10.26 only yields a necessary but not a sufficient condition for optimal solutions of the monotonic price-setting subproblem: In the previous example, we could set $I_1 = \{1\}$, $I_2 = \{2, 3\}$ with $q_1 = 1$, $q_2 = 3$. Then $q_1 < q_2$ and $q_1 \in \text{w-median}(D_1)$, $q_2 \in \text{w-median}(D_1 \cup D_2)$, however $p = (1, 3, 3)$ is not an optimal solution. It is crucial that we only form a common price level for consecutive prices that violate monotonicity if it is not enforced, as we see in the rest of this section.

The relation of optimal solutions to the monotonic price-setting subproblem to weighted medians motivates the development of Algorithm 10.1, which computes an optimal solution in $\mathcal{O}(K \cdot |D|)$ as we prove in Theorem 10.29, which is polynomial in the size of the input.

The idea of Algorithm 10.1 is to start with K price levels, one for each number of traversed zones, and to add constraints ensuring monotonicity when it is violated. The list P of price levels (sorted by number of traversed zones) is checked whether it is increasing. Every time a price level p_k and its successive price level p_{k+1} in P are not increasing, they are combined to one common price level, meaning that the price must be the same for all OD pairs assigned to this new combined level although they traverse different numbers of zones. The new price is computed and the list of price levels P is checked again for monotonicity. During the algorithm, D stores a list of corresponding OD pairs for every price level, and I stores the corresponding index sets. The algorithm terminates when the list of price levels is monotonically increasing.

Algorithm 10.1 consists of two main operations, which are performed within the while-loop. We call lines 7 to 13 the *merge* operation and lines 14 to 15 the *move* operation. For simplicity, we assume that the input sets $D_1, \ldots, D_K$ are not empty. This condition can be achieved by already merging the empty levels to neighboring ones, i.e., with the next lower or higher number of traversed zones.

Example 10.27. Before we prove correctness, we illustrate Algorithm 10.1 with an example. The data for the sets of OD pairs traversing a certain number of zones is derived from a PTN with fixed zones. To simplify notation, every OD pair has one passenger ($t_d = 1$), and instead of the OD pairs, we here give a list R of reference prices belonging to the OD pairs, and sort all lists. The initial state is shown in Table 10.2(a). The OD pairs traverse between 1 and 6 zones, the reference price within each level is the same and a weighted median is shown in the last row. Because $1 \leq 3$, a move operation is performed in the first iteration. Hence, the lists do not change. In the second iteration, however, we have $3 > 1$, which leads to a merge operation. The levels for traversing 2 and 3 zones are combined, resulting in a new common price level of 3, shown in Table 10.2(b). The next three iterations are again move operations because $1 \leq 3$, $3 \leq 5$ and $5 \leq 6$, so the state does not change. In the sixth iteration, a merge operation is necessary because $6 > 4$, resulting in Table 10.2(c). Because also $5 > 4$ in the seventh iteration, another merge operation is performed leading to Table 10.2(d). Because then $3 \leq 4$, the while-loop terminates and $(1, 3, 3, 4, 4, 4)$ is returned as the final price list.

We prepare the correctness proof of Algorithm 10.1 by means of Lemma 10.28.

Lemma 10.28. *Let $(p'_1, \ldots, p'_K)$ be an optimal solution to the unrestricted price-setting subproblem of* Z-FDM. *If $p'_k > p'_{k+1}$ for some $k \in [K-1]$, then there is an optimal solution $(p^*_1, \ldots, p^*_K)$ to the monotonic price-setting subproblem of* Z-FDM *that has $p^*_k = p^*_{k+1}$.*

Proof. Let $k \in [K-1]$ with $p'_k > p'_{k+1}$ for an optimal solution $(p'_1, \ldots, p'_K)$ to the unrestricted price-setting subproblem be arbitrary but fixed. Let $(p^*_1, \ldots, p^*_K)$

Algorithm 10.1: Computing an optimal solution for the price-setting subproblem of Z-FDM requiring increasing prices

```
Input  : Set of OD pairs D with a partition D_1, ..., D_K ≠ ∅, numbers of
         passengers t_d and reference prices r_d for all d ∈ D
Output: Monotonically increasing price list (p_1, ..., p_K)
Initialize
I ← [{1}, ..., {K}],                                       // index sets
D ← [D_1, ..., D_K],                                         // OD pairs
P ← [p_1, ..., p_K] with p_k ∈ w-median(D_k) for all k ∈ [K],      // prices
k ← 1.                                      // Start list indexing at 1
while k ≠ length(P) do
    // Check whether the prices are increasing
    if P[k] > P[k + 1] then
        // Merge operation:
        // For lists I, D, P, merge the entries at position k
           and k + 1 and store them at position k
        Update I[k] ← I[k] ∪ I[k + 1].
        Update D[k] ← D[k] ∪ D[k + 1].
        Update P[k] ← p_k with p_k ∈ w-median(D[k]).
        // Delete the entry at position k + 1 and thus shorten
           the lists
        I.delete(k + 1), D.delete(k + 1), P.delete(k + 1)
        if k ≠ 1 then
            Update k ← k − 1.
    else
        // Move operation:
        Update k ← k + 1.
for l = 1, ..., length(P) do
    for k ∈ I[l] do
        Set p_k ← P[l].
return (p_1, ..., p_K)
```

I	[1]	[2]	[3]	[4]	[5]	[6]
R	[1]	[3, 3]	[1]	[5]	[6, 6]	[4, 4, 4, 4]
P	1	3	1	5	6	4

(a) State after the initialization

I	[1]	[2, 3]	[4]	[5]	[6]
R	[1]	[1, 3, 3]	[5]	[6, 6]	[4, 4, 4, 4]
P	1	3	5	6	4

(b) State after the second iteration

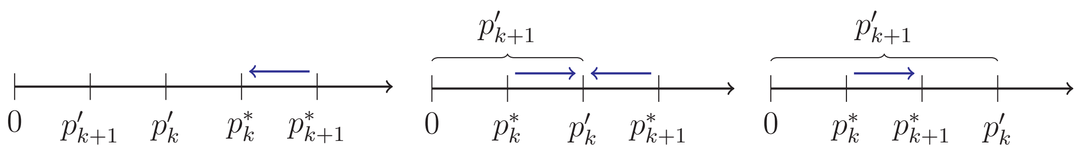
I	[1]	[2, 3]	[4]	[5, 6]
R	[1]	[1, 3, 3]	[5]	[4, 4, 4, 4, 6, 6]
P	1	3	5	4

(c) State after the sixth iteration

I	[1]	[2, 3]	[4, 5, 6]
R	[1]	[1, 3, 3]	[4, 4, 4, 4, 5, 6, 6]
P	1	3	4

(d) State after the seventh iteration and final state

Table 10.2: States during performing Algorithm 10.1 in Example 10.27.



(a) (b) (c)

Figure 10.12: Case distinction of the orders of the values $p'_k, p'_{k+1}, p^*_k, p^*_{k+1}$ in Lemma 10.28.

be an optimal solution to the monotonic price-setting subproblem. If $p^*_k = p^*_{k+1}$, we are done. So now assume that $p^*_k < p^*_{k+1}$. We show that we can modify the solution until $p^*_k = p^*_{k+1}$. For $i \in \{k, k+1\}$, we have that p'_i is a minimum of $\varphi_i : p_i \mapsto \sum_{d \in D_i} t_d |r_d - p_i|$ and we can increase/decrease p^*_i towards p'_i without worsening the objective function value by Section 10.2. The following cases can occur:

- If $p'_k \leq p^*_k$, then we obtain the order depicted in Figure 10.12(a). We decrease p^*_{k+1} to p^*_k.
- If $p^*_k < p'_k < p^*_{k+1}$, we obtain the order depicted in Figure 10.12(b). We increase p^*_k to p'_k and decrease p^*_{k+1} to p'_k.
- If $p^*_k < p^*_{k+1} \leq p'_k$, we obtain the order depicted in Figure 10.12(c). We increase p^*_k to p^*_{k+1}.

Because we only move p^*_k and p^*_{k+1} towards each other, the price list remains increasing. Hence, there is an optimal solution to the monotonic price-setting subproblem with $p^*_k = p^*_{k+1}$. □

Theorem 10.29. *Algorithm 10.1 solves the monotonic price-setting subproblem of* Z-FDM *in* $\mathcal{O}(K \cdot |D|)$.

Proof. Running time: In every iteration of the while-loop, either a merge or a move operation is performed. Merge can be performed at most $K-1$ times because the length of the lists P, D and I is reduced by 1 by every merge operation. Move also is performed at most $K-1$ times in total because the number of levels to still look at, which is length$(P) - k$, is reduced by 1 by every move operation and is also not increased by merge operations. Indeed, in every merge operation, the difference is decreased by 1 if $k = 1$ and remains the same if $k > 1$. Hence, Algorithm 10.1 terminates after at most $2K - 2$ iterations. The theoretical running time is composed of $\mathcal{O}(|D|)$ for the initialization in lines 2 to 4, $(K-1) \cdot \mathcal{O}(|D|)$ for at most $K-1$ merge operations, $(K-1) \cdot \mathcal{O}(1)$ for at most $K-1$ move operations and $\mathcal{O}(K)$ for setting the final price list in lines 16 to 18. This yields a total running time of $\mathcal{O}(K \cdot |D|)$.

Correctness: The aim is to solve the monotonic price-setting subproblem. In line 4 of Algorithm 10.1, the prices are set to weighted medians, which is an optimal solution to the unrestricted price-setting subproblem by Theorem 10.24. If the initialized price list $P = (p'_1, \ldots, p'_K)$ is increasing, this yields an optimal solution also to the monotonic price-setting subproblem, and the algorithm terminates after several move operations without changing the price list. Therefore, we consider now the case that it is not increasing. Let k be minimal such that $p'_k > p'_{k+1}$. By Lemma 10.28, there is an optimal solution $(p^*_1, \ldots, p^*_K)$ to the monotonic price-setting subproblem with $p^*_k = p^*_{k+1}$. Therefore, we can ensure monotonicity by adding the constraint $p_k = p_{k+1}$ to the problem formulation. Equivalently, this means we condense the variables p_k and p_{k+1} to a common variable $p_{k,k+1}$ with $D_{k,k+1} = D_k \cup D_{k+1}$, hence reducing the number of variables (price levels) by one. In Algorithm 10.1, this is implemented in form of the merge operation. Computing the new median, the new list of price levels P is an optimal solution to the unrestricted price-setting subproblem with the condensed input. This process is repeated. If we always started checking monotonicity from the beginning of the list P, we would be done. However, we can do a bit better. Because we always search for the smallest level for which monotonicity is violated, it suffices to decrease the index that is currently looked at by 1 in line 15. This is because when we perform a merge operation for index k, nothing is changed for smaller indices k' with $k' \leq k-2$, and those prices are still increasing after the iteration. Hence, upon termination, Algorithm 10.1 returns an optimal solution to the price-setting subproblem with increasing prices. □

Remark 10.30. For fixed K, Algorithm 10.1 is a linear time algorithm in $\mathcal{O}(|D|)$. Note that the solution method of [Zem84] also can be applied to the LP formulation of problem (10.4) if K is fixed analogously to the proof Theorem 10.6 but with $s = K$, solving the problem in $\mathcal{O}(|D|)$ as well.

10.4.3 MILP Formulation

We now provide MILP formulations for Z-FDM(XY) with X $\in$ {M, S} and Y $\in$ {A,C} with constraints ensuring the no-elongation property and the no-stopover property. It extends the formulation of [OB17] for connected zones and single counting to all four cases of Table 10.1 and can also ensure the no-elongation property and the no-stopover property by implementing their sufficient conditions (see Theorem 9.8).

According to Lemma 10.10, we define $\bar{r} := \max\{r_d : d \in D\}$, and the maximum number of zones traversed on a path is at most

$$K := \begin{cases} \max_{d\in D} |V(W_d)| & \text{if multiple counting,} \\ \min\{N, \max_{d\in D} |V(W_d)|\} & \text{if single counting.} \end{cases}$$

As in [OB17] (except for renaming), the following variables are used, where only the variables for multiple counting are new:

- a binary variable $x_{vz} \in \{0,1\}$ for all stations $v \in V$ and zones $z \in [N]$ that is 1 if and only if station v is assigned to zone Z_z,
- [only for connected zones:] variables to model a single-commodity flow as explained below the MILP, we assume $0 \notin V$:
  - a continuous variable $f_{v_1 v_2} \in \mathbb{R}_{\geq 0}$ for all stations $v_1, v_2 \in V$,
  - a binary variable $s_v \in \{0,1\}$ for all stations $v \in V$,
  - a continuous variable $f_{0v} \in \mathbb{R}_{\geq 0}$ for all $v \in V$,
- [only if multiple counting:] a binary variable $b_e \in \{0,1\}$ for all edges $e = \{v_1, v_2\} \in E$ that is 1 if and only if the stations v_1 and v_2 are in different zones,
- [only if single counting:] a binary variable $b_d^z \in \{0,1\}$ for all OD pairs $d \in D$ and zones $z \in [N]$ that is 1 if and only if the path of OD pair d traverses zone Z_z,
- a binary variable $c_d^k \in \{0,1\}$ for all OD pairs $d \in D$ and numbers of traversed zones $k \in [K]$ that is 1 if and only if the path of OD pair d traverses exactly k zones,
- a continuous variable $p_k \in \mathbb{R}_{\geq 0}$ for all numbers of traversed zones $k \in [K]$ that denotes the price for traversing k zones,
- a continuous variable $\pi_d \in \mathbb{R}_{\geq 0}$ for all OD pairs $d \in D$ that denotes the price for traveling for OD pair d,

- a continuous variable $y_d \in \mathbb{R}$ for all OD pairs $d \in D$ for linearizing the objective function.

We present the MILP formulation in a modular way, where constraints (10.6), (10.7) and (10.9) have previously been used in [OB17]. An explanation of the constraints is given below.

objective function

$$\begin{aligned} \min \quad & \sum_{d \in D} t_d \cdot y_d \\ \text{s.t.} \quad & r_d - \pi_d \leq y_d \quad \text{for all } d \in D \\ & \pi_d - r_d \leq y_d \quad \text{for all } d \in D \end{aligned} \tag{10.5}$$

station assignment

$$\sum_{z=1}^{N} x_{vz} = 1 \quad \text{for all } v \in V \tag{10.6}$$

connected zones (optional)

$$f_{0v} \leq s_v \cdot |V| \quad \text{for all } v \in V \tag{10.7a}$$

$$3 \geq s_{v_1} + s_{v_2} + x_{v_1 z} + x_{v_2 z} \quad \text{for all } v_1, v_2 \in V, \; v_1 \neq v_2, z \in [N] \tag{10.7b}$$

$$f_{v_1 v_2} = 0 \quad \text{for all } \{v_1, v_2\} \notin E \tag{10.7c}$$

$$f_{v_1 v_2} \leq (1 + x_{v_1 z} - x_{v_2 z}) \cdot |V| \quad \text{for all } \{v_1, v_2\} \in E, z \in [N] \tag{10.7d}$$

$$\sum_{v_2 \in V \cup \{0\}} f_{v_2 v_1} = 1 + \sum_{v_2 \in V} f_{v_1 v_2} \quad \text{for all } v_1 \in V \tag{10.7e}$$

multiple counting (Alternative 1)

$$x_{v_1 z} - x_{v_2 z} \leq b_e \quad \text{for all } e = \{v_1, v_2\} \in E, z \in [N] \tag{10.8a}$$

$$b_e \leq 2 - x_{v_1 z} - x_{v_2 z} \quad \text{for all } e = \{v_1, v_2\} \in E, z \in [N] \tag{10.8b}$$

$$\sum_{k=1}^{K} c_d^k = 1 \quad \text{for all } d \in D \tag{10.8c}$$

$$\sum_{e \in E(W_d)} b_e = \sum_{k=1}^{K} (k-1) \cdot c_d^k \quad \text{for all } d \in D \tag{10.8d}$$

single counting (Alternative 2)

$$\sum_{v \in V(W_d)} x_{vz} \leq b_d^z \cdot |V| \qquad \text{for all } d \in D, z \in [N] \tag{10.9a}$$

$$b_d^z \leq \sum_{v \in V(W_d)} x_{vz} \qquad \text{for all } d \in D, z \in [N] \tag{10.9b}$$

$$\sum_{k=1}^{K} c_d^k = 1 \qquad \text{for all } d \in D \tag{10.9c}$$

$$\sum_{z=1}^{N} b_d^z = \sum_{k=1}^{K} k \cdot c_d^k \qquad \text{for all } d \in D \tag{10.9d}$$

price assignment

$$\pi_d \leq p_k + (1 - c_d^k) \cdot \bar{r} \quad \text{for all } d \in D, k \in [K] \tag{10.10a}$$

$$p_k \leq \pi_d + (1 - c_d^k) \cdot \bar{r} \quad \text{for all } d \in D, k \in [K] \tag{10.10b}$$

no-elongation property (optional)

$$p_k \leq p_{k+1} \quad \text{for all } k \in [K-1] \tag{10.11}$$

no-stopover property (optional)

$$\text{[M]} \quad p_k \leq p_i + p_{k-i+1} \quad \text{for all } k \in \{3, \ldots, K\}, i \in \{2, \ldots, \lfloor {}^{k+1}/_2 \rfloor\} \tag{10.12a}$$

$$\text{[S]} \quad p_k \leq p_{i_1} + p_{i_2} \quad \text{for all } k, i_1, i_2 \in [K] \text{ with } i_1, i_2 \leq k \text{ and } i_1 + i_2 \geq k+1 \tag{10.12b}$$

variable domains

$$\begin{aligned} x_{vz}, s_v, b_d^k, b_e, c_d^k &\in \{0,1\} && \text{for all } v \in V, e \in E, z \in [N], d \in D, k \in [K] \\ p_k, \pi_d, f_{v_1 v_2} &\in \mathbb{R}_{\geq 0} && \text{for all } k \in [K], d \in D, v_1 \in V \cup \{0\}, v_2 \in V \\ y_d &\in \mathbb{R} && \end{aligned} \tag{10.13}$$

The objective function to minimize $\sum_{d \in D} t_d |r_d - \pi_d|$ is linearized in the constraints (10.5). The constrains (10.6) specify that each station is assigned to exactly one zone.

Constraints for connected subgraphs based on single- or multi-commodity flows or cuts are applied in many areas besides fare planning, for example, the connected k-cut problem [Hoj+21], forest planning and wildlife conservation [Con+07; DG10; Car+13] and price zones of electricity markets [Gri+17; KS19]. Recently, [BSS23] (for vertex covering with capacitated trees) and [VB22] (for political districting) gave an overview on different formulations for connected subgraphs and their performances, where single-commodity flows performed well. The single-commodity flow constraints (10.7) to ensure connected zones in this thesis have been adopted from [OB17]. The idea is to model a flow from an additional source 0 to each station. It is not allowed to cross zone borders. Flow starting from the source 0 can only be sent to stations that are assigned to the source (10.7a), and at most one station per zone is assigned to the source (10.7b). Flow can only be sent along edges of the PTN (10.7c) and only if the stations of an edge belong to the same zone (10.7d). To see this, let an edge $\{v_1, v_2\} \in E$ be given. If there is some $z \in [N]$ such that $v_1, v_2 \in Z_z$, i.e., $x_{v_1z} = x_{v_2z} = 1$, then $1 + x_{v_1z'} - x_{v_2z'} = 1$ for all $z' \in [N]$ and $f_{v_1v_2}$ is only restricted by the number of stations $|V|$. On the other hand, if v_1 and v_2 are not in the same zone, then there is some $z \in [N]$ such that $1 + x_{v_1z} - x_{v_2z} = 0$, and the flow $f_{v_1v_2}$ is set to 0. Flow conservation with a demand of one flow unit is modeled in the constraints (10.7e). Hence, because each station needs to receive one unit of flow and at most one station per zone gets flow from the source and the flow cannot be sent across zone borders, it is enforced that zones are connected. The constraints (10.7) can be omitted if connected zones are not required.

The constraints (10.8) and (10.9) determine the counting variables for the multiple and the single counting case, respectively. In case of multiple counting, a variable $b_{v_1v_2}$ corresponding to an edge $\{v_1, v_2\} \in E$ is set to 1 if the stations v_1 and v_2 belong to different zones (10.8a), and to 0 if they are in the same zone (10.8b). The constraints (10.8c) and (10.8d) count the number of zone border crossings and set the variable c_d^k for each OD pair $d \in D$ to 1 if the path W_d crosses $k - 1$ zone borders, which means that it traverses k zones. In case of single counting, for all OD pairs $d \in D$ and zones $z \in [N]$, the variable b_d^z is set to 1 if there is a station along the path W_d that is assigned to zone Z_z (10.9a). It is set to 0 otherwise (10.9b). The constraints (10.9c) and (10.9d) determine the total number of different zones that are traversed by each OD pair $d \in D$ by setting the corresponding variable c_d^k to 1 if k different zones are traversed and 0 otherwise.

Based on the number of traversed zones, the price of an OD pair $d \in D$ is assigned to the according price level by the constraints (10.10a) and (10.10b). If $c_d^k = 1$ for some $d \in D$ and $k \in [K]$, then the constraints resolve to $\pi_d = p_k$. For $c_d^k = 0$, the constraints are $\pi_d \leq p_k + \bar{r}$ and $p_k \leq \pi_d + \bar{r}$, which poses no restriction due to Lemma 10.10. The constraints ensuring the no-elongation property (10.11) and the no-stopover property in case of multiple counting (10.12a) or single counting (10.12b) are optional.

	Z-FDM MA	MC	SA	SC	no-elong.	no-stop.	complexity	reference
general	x		x		w/wo	w/wo	NP-hard	Thm. 10.20
zone-partition	x		x		w/wo	w/wo	NP-hard	Cor. 10.22
general		x		x	w/wo	w/wo	NP-hard	Thm. 10.21
zone-partition		x		x	w/wo	w/wo	NP-hard	Cor. 10.23
price-setting	x	x	x	x	wo	wo	$\mathcal{O}(\|D\|)$	Thm. 10.24
price-setting	x	x	x	x	w	wo	$\mathcal{O}(K \cdot \|D\|)$	Thm. 10.29
price-setting	x	x	x	x	w/wo	w	polynomial	LP (10.3)

Table 10.3: Overview of the complexity results for the variants and subproblems of Z-FDM. Abbreviations: w = with, wo = without.

10.5 Summary

We summarize the results of Chapter 10 for the fare deviation model.

F-FDM and D-FDM can be solved in polynomial time $\mathcal{O}(|D|)$ by Section 10.2 and Theorem 10.6, respectively. The complexity results for the variants and subproblems of Z-FDM ranging from NP-hardness to linear solvability are summarized in Table 10.3.

An optimal fixed price of a flat tariff can always be chosen from the set of reference prices (Section 10.2). For an affine distance tariff, there is always an optimal solution that satisfies one of the following cases: either the reference price of at least two OD pairs with different distances is met exactly; or the reference price of at least one OD pair is met exactly and, additionally, the distance price is zero or the base amount is zero (Theorem 10.5). For Z-FDM, the price for each occurring number of traversed zones is at most the maximum of all reference prices and is related to weighted medians (see Theorem 10.24 and Lemma 10.25). A summary of the relationships of the optimal objective function values of the four problem variants of Z-FDM is provided in Figure 10.3 in Section 10.4.1.

Chapter 11

Revenue-Passenger Model

The revenue-passenger model investigates the trade-off between revenue and number of passengers. In Section 11.1, the bi-objective problem definition is introduced. Afterwards, we consider the revenue-passenger model for the case of flat and affine distance tariffs in Sections 11.2 and 11.3, respectively. Based on the solution methods developed to compute the Pareto fronts, we perform computational experiments in Section 11.4 to evaluate the running time and the structure of the Pareto front. In Section 11.5, we study the revenue-passenger model for zone tariffs. A summary of the results in Chapter 11 is given in Section 11.6.

11.1 Problem Definition

Let a PTN be given. To define the revenue-passenger model, we need OD data that is more detailed than in Definition 2.17. Instead of a homogeneous demand for each OD pair, the potential passengers of an OD pair can be distinguished by their willingness to pay. This could for example reflect the degree of dependence on public transport or the income. We define *extended OD data* in Definition 11.1.

Definition 11.1 (Extended OD data)**.** For a given PTN (V, E), we call the following information *extended origin-destination (OD) data*:

- a set $D \subseteq (V \times V) \setminus \{(v, v) : v \in V\}$ with $D \neq \emptyset$,
- for all $d = (v_1, v_2) \in D$, a path $W_d \in \mathcal{W}$ from v_1 to v_2,
- for all $d \in D$, a number $G_d \in \mathbb{N}_{\geq 1}$,
- for all $d \in D$ and $g \in [G_d]$, a number $t_d^g \in \mathbb{N}_{\geq 1}$ and a number $w_d^g \in \mathbb{R}_{\geq 0}$ such that $w_d^g \neq w_d^{g'}$ for all $g, g' \in [G_d]$ with $g \neq g'$.

We call the elements of D the *OD pairs*. Passengers of an OD pair $d = (v_1, v_2) \in D$ travel from their origin v_1 along W_d to their destination v_2. For each OD pair $d \in D$, there are G_d *demand groups*, each with a number of passengers t_d^g and a *willingness to pay* w_d^g. We write $(D, W_d, G_d, t_d^g, w_d^g)$ as shorthand notation for $(D, (W_d)_{d\in D}, (G_d)_{d\in D}, (t_d^g)_{d\in D, g\in[G_d]}, (w_d^g)_{d\in D, g\in[G_d]})$.

Additionally, we introduce the useful notation

$$S_{\text{dem}} := \{(d, g) : d \in D, g \in [G_d]\},$$

which is the set of all demand groups.

The revenue-passenger model is based on the assumption that a demand group uses public transport whenever the ticket price does not exceed its willingness to pay. The objective is to maximize the revenue and the number of passengers simultaneously. While the revenue is the key objective of the operator, the number of passengers serves as an indicator of the success of the transition towards sustainable transport modes. This is particularly significant when public transport is used instead of private motorized transport modes such that the environmental impact of traveling is reduced.

Given a fare structure π, the number of attracted passengers for OD pair $d \in D$ is determined as

$$\mathsf{pass}_d(\pi(W_d)) := \sum_{\substack{g \in [G_d]: \\ \pi(W_d) \le w_d^g}} t_d^g.$$

The total number of passengers with respect to the fare structure π is

$$\mathsf{pass}(\pi) := \sum_{d \in D} \mathsf{pass}_d(\pi(W_d))$$

and the total revenue is

$$\mathsf{rev}(\pi) := \sum_{d \in D} \mathsf{pass}_d(\pi(W_d)) \cdot \pi(W_d).$$

With this, we can now define the revenue-passenger model formally.

Definition 11.2 (Revenue-passenger model (RPM))**.** Let a PTN (V, E) (Definition 2.15), extended OD data $(D, W_d, G_d, t_d^g, w_d^g)$ (Definition 11.1) as well as potentially specific input depending on the fare strategy be given. The aim of the *revenue-passenger model* (RPM) is to determine fare structures π that maximize the revenue $\mathsf{rev}(\pi)$ and the number of passengers $\mathsf{pass}(\pi)$, where a desired fare strategy might be required. The bi-objective model RPM hence is given by:

$$\begin{aligned} \max \quad & \mathsf{rev}(\pi) \\ \max \quad & \mathsf{pass}(\pi) \\ \text{s.t.} \quad & \pi \text{ is of a desired fare strategy} \\ & \pi(W_d) \in \mathbb{R}_{\ge 0} \quad \text{for all } d \in D. \end{aligned}$$

Note that the assumptions $w_d^g \neq w_d^{g'}$ for all $d \in D$ and $g, g' \in [G_d]$ with $g \neq g'$ made in Definition 11.1 hold without loss of generality: If $w_d^g = w_d^{g'}$ for some $d \in D$ and $g, g' \in [G_d]$ with $g \neq g'$, then the demand groups can be merged to one group with demand $t_d^g + t_d^{g'}$.

Further, we remark that $\sum_{d \in D} G_d$ is linear in the input because the numbers t_d^g, w_d^g are explicitly given for all demand groups $(d, g) \in S_{\text{dem}}$ by the extended OD data. This is, for example, relevant for the running time of the algorithms developed in this chapter.

Because there are only finitely many demand groups and pass calculates sums over the number of passengers t_d^g of (some of) these demand groups, pass attains finitely many values. Thus, the Pareto front of RPM is finite. The whole Pareto front of RPM can be computed with the ϵ-constraint method (see Section 2.2.1). This is done by setting $\phi_1 = \mathsf{rev}$ and $\phi_2 = \mathsf{pass}$. Because the number of passengers t_d^g is a natural number for all demand groups $(d, g) \in S_{\text{dem}}$, the objective function pass always attains integral values. In this case, by increasing ϵ with a step width of 1, i.e., choosing $\epsilon_2 = z_2 + 1$ in line 8 of Algorithm 2.1, we do not miss any non-dominated point.

In the following, we consider RPM in more detail for flat, affine distance and zone tariffs.

11.2 Flat Tariffs

In this section, we study the revenue-passenger model with a flat tariff (Definition 9.3) as the desired fare strategy. Because a flat tariff π with fixed price $f \in \mathbb{R}_{\geq 0}$ assigns the same price f to all paths in the PTN, we have $\pi(W_d) = f$ for all $d \in D$. Hence, we can drop the information to which OD pair a demand group belongs.

We define $S_{\text{will}} := \{w_d^g : (d, g) \in S_{\text{dem}}\}$ as the set of all willingness to pay values, and let

$$S := \left\{ (w, t) : w \in S_{\text{will}}, t = \sum_{(d,g) \in S_{\text{dem}} : \, w_d^g = w} t_d^g \right\} \tag{11.1}$$

be the set of all pairs of a willingness to pay and the respective demand with exactly this willingness to pay. In particular, we have $|S| \leq \sum_{d \in D} G_d$, with equality if and only if the willingness to pay is different for every demand group.

For a flat tariff π with fixed price $f \in \mathbb{R}_{\geq 0}$, the objective functions simplify to

$$\mathsf{rev}(\pi) = f \cdot \sum_{\substack{(w,t) \in S: \\ f \leq w}} t \quad \text{and} \quad \mathsf{pass}(\pi) = \sum_{\substack{(w,t) \in S: \\ f \leq w}} t.$$

Because a flat tariff π is uniquely determined by f, we write $\mathsf{rev}(f)$ and $\mathsf{pass}(f)$ instead of $\mathsf{rev}(\pi)$ and $\mathsf{pass}(\pi)$.

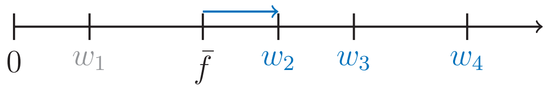


Figure 11.1: Visualization of (the proof of) Lemma 11.4. Increasing $\bar{f} \notin S_{\text{will}}$ to the next higher willingness to pay.

Definition 11.3 (F-RPM)**.** Given a PTN with extended OD data, which yields a set S as defined in formula (11.1), the bi-objective *revenue-passenger model for flat tariffs* (F-RPM) is the following:

$$\begin{aligned} \max \quad & \mathsf{rev}(f) = f \cdot \sum_{\substack{(w,t)\in S:\\ f\leq w}} t \\ \max \quad & \mathsf{pass}(f) = \sum_{\substack{(w,t)\in S:\\ f\leq w}} t \\ \text{s.t.} \quad & f \in \mathbb{R}_{\geq 0}. \end{aligned}$$

We now derive a finite candidate set for F-RPM, meaning a finite superset of the set of efficient solutions. To do this, let $(w_1, t_1), \ldots, (w_{|S|}, t_{|S|})$ be a sorting of S such that $w_1 < \ldots < w_{|S|}$.

Lemma 11.4. *For all efficient solutions $\bar{f}$ to* F-RPM*, it holds that $\bar{f} \in S_{\text{will}}$.*

Proof. Let $\bar{f}$ be an efficient solution, and assume that $\bar{f} \notin S_{\text{will}}$. First, we have that $\bar{f} < \max S_{\text{will}}$: Suppose that $\bar{f} > \max S_{\text{will}}$, which yields the objective function values $(0, 0)$. Then $\bar{f}$ is dominated by $f := \max S_{\text{will}} = w_{|S|}$ with the objective function values $(\mathsf{rev}(f), \mathsf{pass}(f)) = (t_{|S|} \cdot w_{|S|}, t_{|S|})$, which is a contradiction to $\bar{f}$ being efficient. Hence, $\bar{f} < \max S_{\text{will}}$ and $f' := \min\{w \in S_{\text{will}} : w > \bar{f}\}$ is well-defined and is the next higher price compared to $\bar{f}$ that is contained in S_{will} (see Figure 11.1 for an example). By definition of f' and because $\bar{f} \notin S_{\text{will}}$, we then have $\bar{f} < f'$ and $\{(w, t) \in S : \bar{f} \leq w\} = \{(w, t) \in S : f' \leq w\}$. This yields $\mathsf{pass}(\bar{f}) = \mathsf{pass}(f')$ and

$$\mathsf{rev}(\bar{f}) = \bar{f} \cdot \mathsf{pass}(\bar{f}) < f' \cdot \mathsf{pass}(\bar{f}) = \mathsf{rev}(f'),$$

which is a contradiction to $\bar{f}$ being efficient. □

Corollary 11.5. F-RPM *is tractable, i.e., the number of non-dominated points is polynomial in the input, namely in $\mathcal{O}(|S|)$.*

Proof. The claim follows from Lemma 11.4 because there are at most $|S_{\text{will}}| = |S|$ different efficient solutions and hence also non-dominated points. □

From Lemma 11.4, we derive Algorithm 11.1, which computes the Pareto front in $\mathcal{O}(|S| \cdot \log(|S|))$. Note that $|S| \leq \sum_{d\in D} G_d$ and hence $|S|$ is polynomial in the input.

Algorithm 11.1: Computing the set of non-dominated points for F-RPM

Input : Set S (as defined in formula (11.1)) as instance of F-RPM
Output: Set Γ of all non-dominated points

1 Sort $S = \{(w_1, t_1), \ldots, (w_{|S|}, t_{|S|})\}$ such that $w_1 < \ldots < w_{|S|}$.
2 Initialize $\overline{\mathsf{pass}} \leftarrow \sum_{s=1}^{|S|} t_s$; $\overline{\mathsf{rev}} \leftarrow w_1 \cdot \overline{\mathsf{pass}}$; $\Gamma \leftarrow \{(\overline{\mathsf{rev}}, \overline{\mathsf{pass}})\}$; $\mathsf{rev}^* \leftarrow \overline{\mathsf{rev}}$.
3 **for** $s = 2, \ldots, |S|$ **do**
4   Update $\overline{\mathsf{pass}} \leftarrow \overline{\mathsf{pass}} - t_{s-1}$.
5   Update $\overline{\mathsf{rev}} \leftarrow w_s \cdot \overline{\mathsf{pass}}$.
6   **if** $\overline{\mathsf{rev}} > \mathsf{rev}^*$ **then**
7   Update $\Gamma \leftarrow \Gamma \cup \{(\overline{\mathsf{rev}}, \overline{\mathsf{pass}})\}$.
8   Update $\mathsf{rev}^* \leftarrow \overline{\mathsf{rev}}$.
9 **return** Γ

Theorem 11.6. *Algorithm 11.1 computes the set of all non-dominated points for* F-RPM *in* $\mathcal{O}(|S| \cdot \log(|S|))$.

Proof. Correctness: By Lemma 11.4 it suffices to consider the willingness to pay values $w_s \in S_{\text{will}}$ as fixed prices of the flat tariff. Because w_1 is the unique optimum with respect to the objective function pass, we have that $(\mathsf{rev}(w_1), \mathsf{pass}(w_1))$ is a non-dominated point and is added to Γ in line 2. In rev^* we store the maximum revenue that has occurred so far. Increasing the fixed price from w_{s-1} to w_s reduces the number of passengers by those that have a willingness to pay of w_{s-1}, which are t_{s-1} many. Hence, after the updates in lines 4 to 5, $\overline{\mathsf{rev}}$ and $\overline{\mathsf{pass}}$ are the revenue and the number of passengers for a flat tariff with fixed price w_s. Because the number of passengers is strictly decreased in every iteration, the pair $(\overline{\mathsf{rev}}, \overline{\mathsf{pass}})$ is non-dominated whenever the revenue $\overline{\mathsf{rev}}$ is larger than any previous revenue, i.e., if $\overline{\mathsf{rev}} > \mathsf{rev}^*$. Therefore, in this case, the pair is added to Γ and the maximum revenue rev^* is updated.

Running time: Sorting S can be done in $\mathcal{O}(|S| \cdot \log(|S|))$ (see, e.g., [Cor+09]). The initialization of $\overline{\mathsf{pass}}$ in line 2 is executed in $\mathcal{O}(|S|)$, whereas all other initializations and updates are in $\mathcal{O}(1)$. Hence, the for-loop takes $\mathcal{O}(|S|)$ in total. Overall, we obtain a running time of $\mathcal{O}(|S| \cdot \log(|S|))$. □

11.3 Distance Tariffs

We consider the revenue-passenger model with an affine distance tariff (Definition 9.6) as the desired fare strategy.

Definition 11.7 (D-RPM)**.** Given a PTN with extended OD data $(D, W_d, G_d, t_d^g, w_d^g)$ and a (network/metric) distance function l (Definitions 9.4

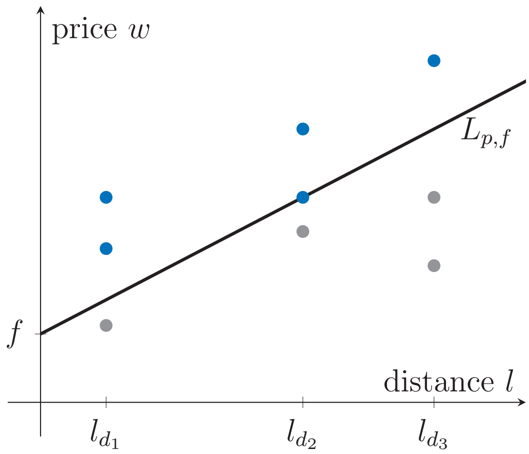


Figure 11.2: Example of three OD pairs with three demand groups each. Every demand group (d, g) is represented by a point (l_d, w_d^g). The line $L_{p,f}$ represents the affine distance tariff with distance price p and base amount f. Demand groups corresponding to points marked in blue are attracted, and demand groups corresponding to points marked in gray are not attracted.

and 9.5), the bi-objective *revenue-passenger model for affine distance tariffs* (D-RPM) is the following:

$$\begin{aligned} \max \quad & \mathsf{rev}(\pi) \\ \max \quad & \mathsf{pass}(\pi) \\ \text{s.t.} \quad & \pi(W_d) = p \cdot l(W_d) + f \quad \text{for all } d \in D \\ & \quad p, f \in \mathbb{R}_{\geq 0}. \end{aligned}$$

In the following, we use for all $d \in D$ the shorthand notation $l_d := l(W_d)$. Here, we also write $\mathsf{rev}(p, f)$ and $\mathsf{pass}(p, f)$ instead of $\mathsf{rev}(\pi)$ and $\mathsf{pass}(\pi)$ because an affine distance tariff is uniquely determined by f and p.

To get an intuition and to better understand D-RPM, we consider Figure 11.2: Three OD pairs with three demand groups each are visualized based on their distance l_d and willingness to pay w_d^g as points (l_d, w_d^g). An affine distance tariff with distance price p and base amount f is drawn as a line

$$L_{p,f} = \{(x, y) \in \mathbb{R}^2 : y = p \cdot x + f\}$$

in the same figure. All demand groups whose willingness to pay is as least as high as the price of the affine distance tariff lie above the line and are marked in blue. These demand groups are attracted as passengers, use public transport and contribute to the revenue. The points that lie below the line (marked in gray) belong to demand groups with a lower willingness to pay. Hence, these demand groups do not use public transport for the given affine distance tariff.

We introduce some additional useful notation for $p, f \in \mathbb{R}_{\geq 0}$:

$$S_{\text{dem}}(p, f) := \{(d, g) \in S_{\text{dem}} : w_d^g \geq p \cdot l_d + f\}, \tag{11.2}$$

which is the set of demand groups that are attracted in case of an affine distance tariff with distance price p and base amount f, i.e., $\mathsf{pass}(p, f) = \sum_{(d,g) \in S_{\text{dem}}(p,f)} t_d^g$.

Lemma 11.8. *For all efficient solutions* $(\bar{p}, \bar{f})$ *to* D-RPM, *it holds that* $S_{\text{dem}}(\bar{p}, \bar{f}) \neq \emptyset$.

Proof. Assume that $S_{\text{dem}}(\bar{p}, \bar{f}) = \emptyset$. The objective function value then is $(0, 0)$ and is dominated by the solution $(p', f') := (0, 0)$ with objective function values $(0, T')$ with $T' := \sum_{(d,g) \in S_{\text{dem}}} t_d^g > 0$. □

11.3.1 MILP Formulation

For the ϵ-constraint method (Algorithm 2.1), the following MILP formulation with a sufficiently large constant $M \in \mathbb{R}$ may be used, which is explained below:

$$\begin{aligned}
\max \quad & \sum_{(d,g) \in S_{\text{dem}}} t_d^g \cdot \pi_d^g && && (11.3a)\\
\max \quad & \sum_{(d,g) \in S_{\text{dem}}} t_d^g \cdot y_d^g && && \\
\text{s.t.} \quad & p \cdot l_d + f \leq w_d^g + M \cdot (1 - y_d^g) && \text{for all } (d, g) \in S_{\text{dem}} && (11.3b)\\
& \pi_d^g \leq p \cdot l_d + f && \text{for all } (d, g) \in S_{\text{dem}} && (11.3c)\\
& \pi_d^g \leq M \cdot y_d^g && \text{for all } (d, g) \in S_{\text{dem}} && (11.3d)\\
& y_d^g \in \{0, 1\} && \text{for all } (d, g) \in S_{\text{dem}} && \\
& p, f, \pi_d^g \in \mathbb{R}_{\geq 0} && \text{for all } (d, g) \in S_{\text{dem}}. &&
\end{aligned}$$

The variables p and f determine the distance price and the base amount of the affine distance tariff. In any efficient solution, the binary variable $y_d^g \in \{0, 1\}$ is 1 if and only if demand group (d, g) uses public transport. Finally, the variable π_d^g stores the price that is actually paid by the demand group (d, g). The constraints (11.3b) ensure that y_d^g is set to 1 only if the price according to the affine distance tariff does not exceed the willingness to pay, for which a sufficiently large constant $M \in \mathbb{R}$ is necessary. We show in Lemma 11.9 that such a constant exists and can be determined based on the input. The constraints (11.3c) limit the price of a demand group to the price of the affine distance tariff and the constraints (11.3d) set the price paid by a demand group to 0 if it does not use public transport. Together the constraints (11.3c) and (11.3d) set the price paid by a demand group to either 0 or the distance tariff price. Because of the interaction with the objective functions, we also have in any efficient solution that $y_d^g = 1$ if the price according to the affine distance tariff does not exceed the willingness to pay.

Lemma 11.9. *For MILP (11.3), the constant $M \in \mathbb{R}$ can be chosen sufficiently large based on the input, namely the extended OD data $(D, W_d, G_d, t_d^g, w_d^g)$ and the distances l_d for all $d \in D$.*

Proof. Let $(\bar{p}, \bar{f})$ be an efficient solution to D-RPM. By Lemma 11.8, it holds that $S_{\text{dem}}(\bar{p}, \bar{f}) \neq \emptyset$. Therefore, $\bar{f} \leq f^{\max} := \max\{w_d^g : (d,g) \in S_{\text{dem}}\}$ and $\bar{p} \leq p^{\max} := \max\left\{\frac{w_d^g}{l_d} : (d,g) \in S_{\text{dem}}\right\}$. Let $l^{\max} := \max\{l_d : d \in D\}$. Setting $M := p^{\max} \cdot l^{\max} + f^{\max}$, we have for all $(d,g) \in S_{\text{dem}}$ that

$$\pi_d^g \overset{(11.3c)}{\leq} \bar{p} \cdot l_d + \bar{f} \leq p^{\max} \cdot l^{\max} + f^{\max} = M.$$

Therefore, M is sufficiently large for the constraints (11.3b) and (11.3d). □

MILP (11.3) can be strengthened by implementing the following set of inequalities

$$y_{d_2}^{g_2} \leq y_{d_1}^{g_1} \quad \text{for all } (d_1, g_1), (d_2, g_2) \in S_{\text{dem}} \text{ with } l_{d_1} \leq l_{d_2}, w_{d_1}^{g_1} \geq w_{d_2}^{g_2}. \tag{11.4}$$

The inequalities (11.4) are valid for MILP (11.3), i.e., they do not cut off any efficient solution. To see this, let an efficient solution $\bar{p}$, $\bar{f}$, $(\pi_d^g)_{(d,g)\in S_{\text{dem}}}$, $(y_d^g)_{(d,g)\in S_{\text{dem}}}$ be given, and let $(d_1, g_1), (d_2, g_2) \in S_{\text{dem}}$ with $l_{d_1} \leq l_{d_2}$ and $w_{d_1}^{g_1} \geq w_{d_2}^{g_2}$. If $y_{d_2}^{g_2} = 1$, then

$$\bar{p} \cdot l_{d_1} + \bar{f} \leq \bar{p} \cdot l_{d_2} + \bar{f} \overset{(11.3b)}{\leq} w_{d_2}^{g_2} \leq w_{d_1}^{g_1},$$

and hence $y_{d_1}^{g_1} = 1$.

11.3.2 Specialized Solution Method for D-RPM

In addition to the MILP-based ϵ-constraint method, we now develop a specialized solution method for D-RPM by exploiting the structure of the problem. As a basis for the algorithm, we identify a finite candidate set that contains at least one efficient solution for every non-dominated point of D-RPM. To do so, we say that a solution (p, f) *meets the willingness to pay* w_d^g of the demand group $(d, g) \in S_{\text{dem}}$ *exactly* if $w_d^g = p \cdot l_d + f$, i.e., if the line $L_{p,f}$ passes through the point (l_d, w_d^g). The finite candidate set is given by Theorem 11.10. In particular, it follows from Theorem 11.10 that D-RPM is tractable (see Corollary 11.11).

Theorem 11.10. *For every non-dominated point of* D-RPM, *there is an efficient solution $(\bar{p}, \bar{f})$ such that one of the following holds: The willingness to pay of*

- *at least two demand groups $(d, g), (d', g') \in S_{\text{dem}}$ with $l_d \neq l_{d'}$ is met exactly,*
- *at least one demand group is met exactly and, additionally, $\bar{p} = 0$ or $\bar{f} = 0$.*

We approach the task of proving Theorem 11.10 in two different ways: first from an illustrative perspective by shifting and rotating lines $L_{p,f}$ corresponding to affine distance tariffs (p, f), and afterwards from a line location perspective using a dual interpretation similar to the proof of Theorem 10.5.

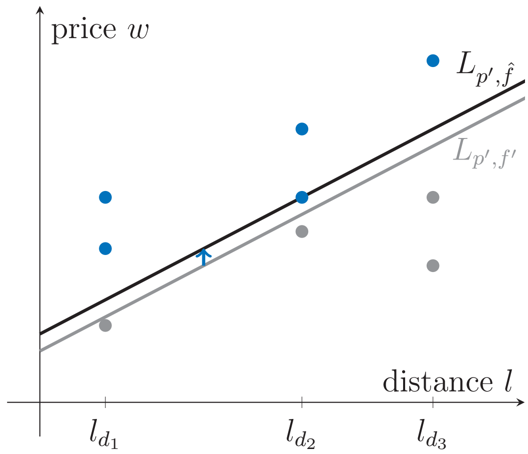


Figure 11.3: Every efficient solution (p', f') has to meet at least one willingness to pay exactly. Otherwise, we can increase the base amount from f' to $\hat{f}$ so that at least one willingness to pay is met exactly. This does not change the demand groups that are attracted but increases the revenue because a higher price is charged.

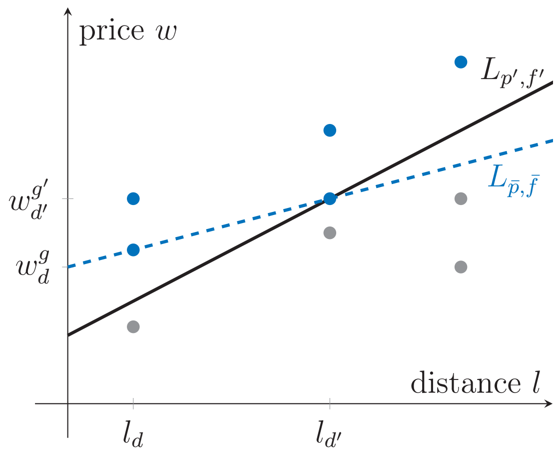


Figure 11.4: The line $L_{p',f'}$ (black, solid) passes through the point $(l_{d'}, w_{d'}^{g'})$ of one demand group (d', g'). We fix the point $(l_{d'}, w_{d'}^{g'})$ as a center of rotation. The line $L_{p',f'}$ can then be rotated clockwise to the line $L_{\bar{p},\bar{f}}$ (blue, dashed), which passes through two points corresponding to two different demand groups (d', g') and (d, g).

First Proof We explain the idea of the first proof using the visualization in Figure 11.2. For an efficient solution (p', f'), we first show that the line $L_{p',f'}$ must pass through a point $(l_{d'}, w_{d'}^{g'})$: If this was not the case, we could shift the line upwards by increasing the base amount to $\hat{f}$ until it passes through a point corresponding to a demand group as depicted in Figure 11.3. Next, we assume that (p', f') only meets the willingness to pay $w_{d'}^{g'}$ of exactly one demand group $(d', g') \in S_{\text{dem}}$ and that $p' \neq 0$ and $f' \neq 0$. Then, we can rotate the line $L_{p',f'}$ clockwise around the point $(l_{d'}, w_{d'}^{g'})$ until either the slope is zero or the line passes through an additional point (l_d, w_d^g) so that all points that lie above $L_{p',f'}$ still lie above the new line. In other words, this means we decrease p' and adapt f' accordingly to obtain a new line $L_{\bar{p},\bar{f}}$ with $S_{\text{dem}}(\bar{p}, \bar{f}) \supseteq S_{\text{dem}}(p', f')$ (see formula (11.2) for the definition of S_{dem}) that either satisfies $\bar{p} = 0$ or passes through an additional point (l_d, w_d^g) corresponding to a demand group $(d, g) \in S_{\text{dem}}$ with $l_d \neq l_{d'}$ as depicted in Figure 11.4. We show that the new solution $(\bar{p}, \bar{f})$ has the same objective function values as (p', f').

Proof of Theorem 11.10. Let (p', f') be an efficient solution.

Step 1: Efficient solutions meet the willingness to pay of at least one demand group exactly. Assume that no willingness to pay is met exactly by (p', f'). By Lemma 11.8, we have $S_{\text{dem}}(p', f') \neq \emptyset$, and $|S_{\text{dem}}(p', f')| < \infty$. Hence, we can set $\delta := \min\{w_d^g - f' - p' \cdot l_d : (d, g) \in S_{\text{dem}}(p', f')\}$. Because (p', f') does not meet a willingness to pay exactly by assumption, we have that $\delta > 0$. Increasing f' to $\hat{f} := f' + \delta$, it holds that $S_{\text{dem}}(p', f') = S_{\text{dem}}\big(p', \hat{f}\,\big)$, and therefore $\mathsf{pass}(p', f') = \mathsf{pass}\big(p', \hat{f}\,\big)$ and

$$\mathsf{rev}(p', f') < \mathsf{rev}(p', f') + \delta \cdot \mathsf{pass}(p', f') = \mathsf{rev}\big(p', \hat{f}\,\big),$$

which is a contradiction to (p', f') being an efficient solution.

Thus, there is at least one willingness to pay $w' := w_{d'}^{g'}$ of a demand group $(d', g') \in S_{\text{dem}}$ with distance $l' := l_{d'}$ that is met exactly. We consider the case that (p', f') does not meet the willingness to pay of any demand group $(d, g) \in S_{\text{dem}}$ with $l_d \neq l'$ exactly and that $p' \neq 0$ and $f' \neq 0$. In the following steps, we show that there is an efficient solution $(\bar{p}, \bar{f})$ with the same objective function values $\mathsf{pass}(\bar{p}, \bar{f}) = \mathsf{pass}(p', f')$ and $\mathsf{rev}(\bar{p}, \bar{f}) = \mathsf{rev}(p', f')$ that meets the willingness to pay of the demand group (d', g') exactly and either also meets the willingness to pay of an additional demand group $(d, g) \in S_{\text{dem}}$ with $l_d \neq l'$ exactly or has $\bar{p} = 0$ or $\bar{f} = 0$.

Step 2: Considering lines that meet the willingness to pay of (d', g'). All lines $L_{p,f}$ with $p, f \in \mathbb{R}_{\geq 0}$ that pass through the point (l', w') are uniquely determined by p because $f = w' - p \cdot l'$. Moreover:

$$0 \leq p \text{ and } 0 \leq f = w' - p \cdot l' \iff 0 \leq p \leq \frac{w'}{l'}. \tag{11.5}$$

Note that $l' \neq 0$ by definition. We set $S_{\text{dem}}(p) := S_{\text{dem}}(p, w' - p \cdot l')$ and consider the function $\mathsf{rev}_{p'}$, which only takes p as input, defined by

$$\mathsf{rev}_{p'}(p) := \sum_{(d,g) \in S_{\text{dem}}(p')} t_d^g \cdot \underbrace{(w' + p \cdot (l_d - l'))}_{= p \cdot l_d + f}.$$

By definition, $\mathsf{rev}_{p'}$ is an affine function, and it holds for all $p \in \left[0, \frac{w'}{l'}\right]$ with $S_{\text{dem}}(p) = S_{\text{dem}}(p')$ that $\mathsf{rev}_{p'}(p) = \mathsf{rev}(p, w' - p \cdot l')$.

Step 3: $S_{\text{dem}}(p)$ and prices that meet a willingness to pay exactly. We define

$$P := \left\{ p \in \left[0, \frac{w'}{l'}\right] : w_d^g = w' + p \cdot (l_d - l') \text{ for some } (d, g) \in S_{\text{dem}} \text{ with } l_d \neq l' \right\},$$

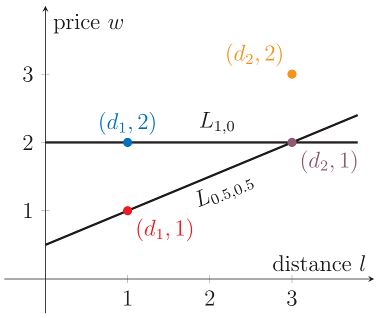
which is the set of all feasible distance prices corresponding to a solution that meets the willingness to pay of (d', g') and of an additional demand group (d, g) with $l_d \neq l'$ exactly. In particular, $p' \notin P$. Because $|S_{\text{dem}}| < \infty$ and for all $(d, g) \in S_{\text{dem}}$ there is at most one line $L_{p,w'-p \cdot l'}$ with a feasible distance price p that passes through $(l_{d'}, w_{d'}^{g'})$ and (l_d, w_d^g), we also have $|P| < \infty$. Let $P = \{p_1, \ldots, p_L\}$ be an enumeration of P with $0 \leq p_1 < \ldots < p_L \leq \frac{w'}{l'}$.

We claim that for all $i \in [L-1]$ and $q_1, q_2 \in \mathbb{R}$ with $p_i < q_1 < q_2 < p_{i+1}$ it holds that $S_{\text{dem}}(q_1) = S_{\text{dem}}(q_2)$. We show the claim by a proof of contradiction. Assume that $S_{\text{dem}}(q_1) \subsetneq S_{\text{dem}}(q_2)$. This means that there is a $(d, g) \in S_{\text{dem}}(q_2) \setminus S_{\text{dem}}(q_1)$, which yields that $w' + q_2 \cdot (l_d - l') \leq w_d^g < w' + q_1 \cdot (l_d - l')$. Because $p \mapsto w' + p \cdot (l_d - l')$ is an affine function, there is a $q_3 \in \mathbb{R}$ with $p_1 < q_3 \leq q_2$ and $w' + q_3 \cdot (l_d - l') = w_d^g$. This is a contradiction because $p_i < q_3 < p_{i+1}$ and thus $q_3 \notin P$. The case for $S_{\text{dem}}(q_1) \supsetneq S_{\text{dem}}(q_2)$ is analogous. Therefore, $S_{\text{dem}}(q_1) = S_{\text{dem}}(q_2)$, which proves the claim.

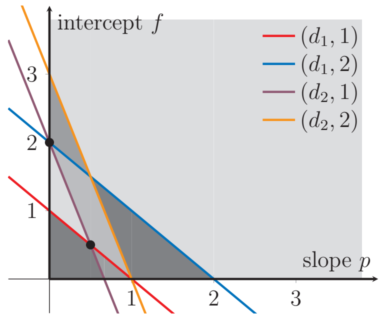
Step 4: The slope of $\mathsf{rev}_{p'}$ is 0. Assume that the slope of $\mathsf{rev}_{p'}$ is not 0. Because of $p', f' > 0$ together with (11.5) and because of $p' \notin P$ and $|P| < \infty$, there are $q_1, q_2 \in \left[0, \frac{w'}{l'}\right]$ with $q_1 < p' < q_2$ and $S_{\text{dem}}(q_1) = S_{\text{dem}}(p') = S_{\text{dem}}(q_2)$ by Step 3. By definition, $\mathsf{rev}_{p'}$ is an affine function, so we have that either $\mathsf{rev}_{p'}(q_1) > \mathsf{rev}_{p'}(p')$ or $\mathsf{rev}_{p'}(q_2) > \mathsf{rev}_{p'}(p')$, which is a contradiction to (p', f') being an efficient solution. Hence, the slope of $\mathsf{rev}_{p'}$ is 0, meaning that $\mathsf{rev}_{p'}$ is constant.

Step 5: There is an efficient solution satisfying one of the characteristics as sought. Let $\bar{p} := \max\{p \in P \cup \{0\} : p < p'\}$, which exists because $p' > 0$, $p' \notin P$ and $|P| < \infty$. Assume that $S_{\text{dem}}(\bar{p}) \subsetneq S_{\text{dem}}(p')$. This means that there is a $(d, g) \in S_{\text{dem}}(p') \setminus S_{\text{dem}}(\bar{p})$ with $l_d \neq l_{d'}$. Thus $w' + p' \cdot (l_d - l') < w_d^g < w' + \bar{p} \cdot (l_d - l')$, where the first inequality holds by assumption that (p', f') does not meet the willingness to pay of any $(\hat{d}, \hat{g})$ with $l_{\hat{d}} \neq l'$, and the second holds by assumption that $(d, g) \notin S_{\text{dem}}(\bar{p}, \bar{f})$. This yields that there is a $\hat{p} \in \mathbb{R}$ with $\bar{p} < \hat{p} < p'$ and $w' + \hat{p} \cdot (l_d - l') = w_d^g$, which is a contradiction because $\hat{p} \notin P$ by choice of $\bar{p}$.

Therefore, $S_{\text{dem}}(p') \subseteq S_{\text{dem}}(\bar{p})$. Assuming $S_{\text{dem}}(p') \subsetneq S_{\text{dem}}(\bar{p})$ yields that $\mathsf{pass}(p', w' - p' \cdot l') < \mathsf{pass}(\bar{p}, w' - \bar{p} \cdot l')$ and $\mathsf{rev}_{p'}(p') = \mathsf{rev}_{p'}(\bar{p}) \leq \mathsf{rev}(\bar{p})$ by Step 4, which is a contradiction to (p', f') being efficient. Therefore, it holds that $S_{\text{dem}}(p') = S_{\text{dem}}(\bar{p})$ and thus $\mathsf{pass}(p', w' - p' \cdot l') = \mathsf{pass}(\bar{p}, w' - \bar{p} \cdot l')$ and, again



(a) Primal space: Points (l_d, w_d^g) for four demand groups (d, g) with $d \in \{d_1, d_2\}$ and $g \in [2]$ and lines $L_{0.5,0.5}$ and $L_{0,2}$ of the efficient solutions



(b) Dual space: The four points (l_d, w_d^g) with $d \in \{d_1, d_2\}$ and $g \in [2]$ are transformed to four lines L_{-l_d, w_d^g}. The points $(0.5, 0.5)$ and $(0, 2)$ are the efficient solutions and can be transformed back to the lines $L_{0.5,0.5}$ and $L_{0,2}$ in the primal space.

Figure 11.5: Example with two OD pairs and two demand groups per OD pair illustrating the proof of Theorem 11.10 using the dual interpretation. For $d \in \{d_1, d_2\}$, we set $t_d^g := 1$ if $g = 1$ and $t_d^g := 2$ if $g = 2$. The efficient solutions are $(\bar{p}_1, \bar{f}_1) = (0.5, 0.5)$ and $(\bar{p}_2, \bar{f}_2) = (0, 2)$.

by Step 4, also $\mathsf{rev}(p', f') = \mathsf{rev}(\bar{p}, w' - \bar{p} \cdot l')$. Hence, $(\bar{p}, w' - \bar{p} \cdot l')$ is an efficient solution as sought. □

Alternative Proof Using the Dual Interpretation Similarly to the proof of Theorem 10.5, we can use a dual interpretation to obtain the result of Theorem 11.10, which we restate here for convenience. For better readability independent of the proof of Theorem 10.5, we also repeat parts of the proof introducing the transformation and the dual space.

Theorem 11.10. *For every non-dominated point of* D-RPM, *there is an efficient solution $(\bar{p}, \bar{f})$ such that one of the following holds: The willingness to pay of*

- *at least two demand groups $(d, g), (d', g') \in S_{\text{dem}}$ with $l_d \neq l_{d'}$ is met exactly,*
- *at least one demand group is met exactly and, additionally, $\bar{p} = 0$ or $\bar{f} = 0$.*

Alternative proof of Theorem 11.10. We first introduce some notation (Steps 1 and 2) that is then used to prove that there is an efficient solution as sought (Step 3).

Step 1: Introducing the transformation and the dual space. For this proof, we adopt the dual interpretation from [Sch99a, Sec. 2.2]. We consider the transformation T that maps a point $(l, w) \in \mathbb{R}^2$ to a line $T((l, w)) := L_{-l,w}$ and a non-vertical line $L_{p,f}$ to a point $T(L_{p,f}) := (p, f)$. The space of the transformed points and lines is called *dual* space. We call the original space the *primal* space. An example is given in Figure 11.5.

The vertical deviation between a point (l, w) and a line $L_{p,f}$ in the primal space is the same as the vertical deviation between the transformed line $L_{-l,w}$ and the transformed point (p, f) in the dual space because $w - (p \cdot l + f) = w - f - p \cdot l = (-l \cdot p + w) - f$. Hence, a demand group $(d, g) \in S_{\text{dem}}$ is attracted in an affine distance tariff with distance price p and base amount f if the point (l_d, w_d^g) is on or above the line $L_{p,f}$ in the primal space, or equivalently if the line L_{-l_d,w_d^g} passes through the point (p, f) or is above it in the dual space.

Step 2: Considering the division of the dual space into cells. The feasible space $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ in the dual space is divided into two-dimensional polyhedra (cells) by the lines L_{-l_d,w_d^g} with $(d, g) \in S_{\text{dem}}$ (see Figure 11.5(b)). Let $\mathcal{C}$ denote the set of all cells. For all $C \in \mathcal{C}$ and $(p, f) \in C$, we define:

$$\begin{aligned}
B(C) &:= \{(d, g) \in S_{\text{dem}} : w_d^g \leq p \cdot l_d + f \text{ for all } (p, f) \in C\},\\
b(C) &:= \left\{(p, f) \in C : f \leq \hat{f} \text{ for all } \left(p, \hat{f}\,\right) \in C\right\} \cap \bigcup_{(d,g) \in B(C)} L_{-l_d,w_d^g},\\
S_{\text{dem}}(C) &:= \{(d, g) \in S_{\text{dem}} : w_d^g \geq p \cdot l_d + f \text{ for all } (p, f) \in C\},\\
\mathsf{pass}(C) &:= \sum_{(d,g) \in S_{\text{dem}}(C)} t_d^g,\\
\mathsf{rev}_C(p, f) &:= \sum_{(d,g) \in S_{\text{dem}}(C)} t_d^g \cdot (p \cdot l_d + f),
\end{aligned}$$

where

- $B(C)$ is the set of all demand groups whose willingness to pay is at most the price of the affine distance tariff with distance price p and base amount f for all $(p, f) \in C$,
- $b(C)$ describes the bottom of the cell C intersected with the union of all supporting hyperplanes L_{-l_d,w_d^g} of C with $(d, g) \in B(C)$ (i.e., the hyperplanes at the bottom of C),
- $S_{\text{dem}}(C)$ is the set of all demand groups whose willingness to pay is at least the price of the affine distance tariff with distance price p and base amount f for all $(p, f) \in C$,
- $\mathsf{pass}(C)$ is the total number of passenger of demand groups in $S_{\text{dem}}(C)$ and
- $\mathsf{rev}_C(p, f)$ is the revenue generated by the demand groups in $S_{\text{dem}}(C)$ for the affine distance tariff with distance price p and base amount f.

Note that $B(C) \cup S_{\text{dem}}(C) = S_{\text{dem}}$ and $B(C) \cap S_{\text{dem}}(C) = \emptyset$ for all $C \in \mathcal{C}$. Further, we have for all $C \in \mathcal{C}$ and $(p', f') \in C \setminus b(C)$ and $(p, f) \in b(C)$ that

$$\begin{aligned} S_{\text{dem}}(p', f') &= S_{\text{dem}}(C), & S_{\text{dem}}(p, f) &\supsetneq S_{\text{dem}}(C), \\ \mathsf{pass}(p', f') &= \mathsf{pass}(C), & \mathsf{pass}(p, f) &> \mathsf{pass}(C), \quad (11.6) \\ \mathsf{rev}(p', f') &= \mathsf{rev}_C(p', f'), & \mathsf{rev}(p, f) &> \mathsf{rev}_C(p, f). \quad (11.7) \end{aligned}$$

Step 3: There is an efficient solution satisfying one of the characteristics of the claim. Let $(p', f') \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ with $(p', f') \neq (0, 0)$. There is a cell $C \in \mathcal{C}$ with $(p, f) \in C \setminus b(C)$ because the sets $C \setminus b(C)$ partition the space. We consider the single-objective optimization problem

$$\begin{aligned} \max \quad & \mathsf{rev}_C(p, f) \\ \text{s.t.} \quad & (p, f) \in C, \end{aligned} \quad (11.8)$$

which is feasible, the objective function is linear and the optimal objective function value is finite (note that the unbounded cell C' on the upper right has $S_{\text{dem}}(C') = \emptyset$, so $\mathsf{rev}_{C'}(p, f) = 0$ for all $(p, f) \in C'$). By the fundamental theorem of linear programming, there is an optimal solution $(\bar{p}, \bar{f})$ to problem (11.8) that is an extreme point of C.

Case 1: If $(\bar{p}, \bar{f}) \in C \setminus b(C)$, then $\mathsf{pass}(\bar{p}, \bar{f}) = \mathsf{pass}(p', f')$ by formula (11.6), and $\mathsf{rev}(\bar{p}, \bar{f}) = \mathsf{rev}_C(\bar{p}, \bar{f}) \geq \mathsf{rev}_C(p', f') = \mathsf{rev}(p', f')$ by formula (11.7) and because $(\bar{p}, \bar{f})$ is an optimal solution to problem (11.8).

Case 2: If $(\bar{p}, \bar{f}) \in b(C)$, then $\mathsf{pass}(\bar{p}, \bar{f}) > \mathsf{pass}(p', f')$ by formula (11.6), and $\mathsf{rev}(\bar{p}, \bar{f}) \geq \mathsf{rev}_C(\bar{p}, \bar{f}) \geq \mathsf{rev}_C(p', f') = \mathsf{rev}(p', f')$ by formula (11.7) and because $(\bar{p}, \bar{f})$ is an optimal solution to problem (11.8).

In both cases, the objective function values of $(\bar{p}, \bar{f})$ are both at least as good as the objective function values of (p', f'). This shows that there is always an efficient solution that is an extreme point of a cell, which is either the intersection of two lines, of a line with an axis or the origin.

Assume that $(0, 0)$ is efficient, then there is a demand group $(d, g) \in S_{\text{dem}}$ with $w_d^g = 0$ because otherwise $(0, 0)$ would be dominated by the solution $(0, \bar{w})$ with $\bar{w} := \min\{w_d^g : (d, g) \in S_{\text{dem}}\} > 0$, which still attracts all passengers but generates a positive revenue. Hence, $(0, 0)$ meets the willingness to pay of one demand group, and the case that the extreme point is the origin is contained in the case of the intersection of a line with an axis.

Interpreting the extreme point solution $(\bar{p}, \bar{f})$ for D-RPM, this means that one of the following holds for the affine distance tariff with distance price $\bar{p}$ and base amount $\bar{f}$:

- the willingness to pay of two demand groups $(d, g), (d', g') \in S_{\text{dem}}$ with $l_d \neq l_{d'}$ is met exactly (if $(\bar{p}, \bar{f})$ is the intersection $L_{-l_d, w_d^g} \cap L_{-l_{d'}, w_{d'}^{g'}}$ of two lines in the dual space, hence $l_d \neq l_{d'}$),

- the willingness to pay of one demand group (d, g) is met exactly and $\bar{p} = 0$ (if $(\bar{p}, \bar{f})$ is the intersection of L_{-l_d, w_d^g} with the f-axis in the dual space),
- the willingness to pay of one demand group (d, g) is met exactly and $\bar{f} = 0$ (if $(\bar{p}, \bar{f})$ is the intersection of L_{-l_d, w_d^g} with the p-axis in the dual space). □

Corollary 11.11. D-RPM *is tractable, i.e., the number of non-dominated points is polynomial in the input, namely in* $\mathcal{O}((\sum_{d\in D} G_d)^2)$.

Proof. The claim follows from Theorem 11.10 because there are at most $\sum_{d\in D} G_d$ non-dominated points for efficient solutions $(\bar{p}, \bar{f})$ with $\bar{p} = 0$ and as many with $\bar{f} = 0$, and at most $(\sum_{d\in D} G_d)^2$ non-dominated points that meet the willingness to pay of two demand groups exactly. □

Based on Theorem 11.10, we can now formulate Algorithm 11.2 to solve D-RPM.

Theorem 11.12. *Algorithm 11.2 computes the set of all non-dominated points of* D-RPM *in* $\mathcal{O}((\sum_{d\in D} G_d)^3)$.

Proof. Correctness: Theorem 11.10 gives a characterization of efficient solutions from which all non-dominated points can be determined. In lines 2 to 5 of Algorithm 11.2, a superset of all non-dominated points with an efficient solution (p, f) with $p = 0$ is determined, in lines 6 to 9 a superset of all those with $f = 0$ and in lines 10 to 15 of all those that meet the willingness to pay of at least two groups exactly are computed. Combinations of demand groups with the same distance are omitted because this would yield an infeasible vertical line. Therefore, Γ' contains all non-dominated points. In line 16, all dominated solutions are removed and, hence, Γ is the set of all non-dominated points.

Running time: The computations in lines 2 to 9 are in $\mathcal{O}((\sum_{d\in D} G_d)^2)$. In lines 10 to 15, we iterate over the combinations of two demand groups and again iterate over the demand groups for determining the revenue and the number of passengers in line 15. This is done in $\mathcal{O}((\sum_{d\in D} G_d)^3)$. Filtering Γ' for non-dominated points in line 16 is done in $\mathcal{O}((\sum_{d\in D} G_d)^2 \cdot \log(\sum_{d\in D} G_d))$. Hence, in total, the algorithm is in $\mathcal{O}((\sum_{d\in D} G_d)^3)$. □

Remark 11.13. Note that the running time is significantly influenced by the number of OD pairs with the same distance because the for-loop in line 10 of Algorithm 11.2 is only performed for OD pairs d' with a larger distance than that of OD pair d, but not for those with the same distance. Hence, the loops over the demand groups and the computation of the objective function value are omitted for OD pair combinations with the same distance.

Algorithm 11.2: Computing the set of non-dominated points for D-RPM

Input : Extended OD data $(D, W_d, G_d, t_d^g, w_d^g)$ and a distance $l_d \in \mathbb{R}_{>0}$ for all $d \in D$ as instance of D-RPM

Output: Set Γ of all non-dominated points

Initialize $\Gamma' \leftarrow \emptyset$.

`// Determine points with a solution with` $p = 0$. `This can also be done with Algorithm 11.1.`

for $d \in D$ **do**
  for $g \in [G_d]$ **do**
    Set $f \leftarrow w_d^g$.
    Update $\Gamma' \leftarrow \Gamma' \cup \{(\mathsf{rev}(0, f), \mathsf{pass}(0, f))\}$.

`// Determine points with a solution with` $f = 0$.

for $d \in D$ **do**
  for $g \in [G_d]$ **do**
    Set $p \leftarrow \frac{w_d^g}{l_d}$.
    Update $\Gamma' \leftarrow \Gamma' \cup \{(\mathsf{rev}(p, 0), \mathsf{pass}(p, 0))\}$.

`// Determine points with a solution that meets the willingness to pay of two groups exactly.`

for $d, d' \in D$ with $l_d < l_{d'}$ **do**
  for $g \in [G_d]$, $g' \in [G_{d'}]$ **do**
    Set $p \leftarrow \frac{w_{d'}^{g'} - w_d^g}{l_{d'} - l_d}$.
    Set $f \leftarrow w_d^g - p \cdot l_d$.
    if $f > 0$ and $p > 0$ **then**
      Update $\Gamma' \leftarrow \Gamma' \cup \{(\mathsf{rev}(p, f), \mathsf{pass}(p, f))\}$.

`// Filter for non-dominated points.`

Apply Algorithm 2.2 to filter Γ' for non-dominated points and let Γ be its result.

return Γ

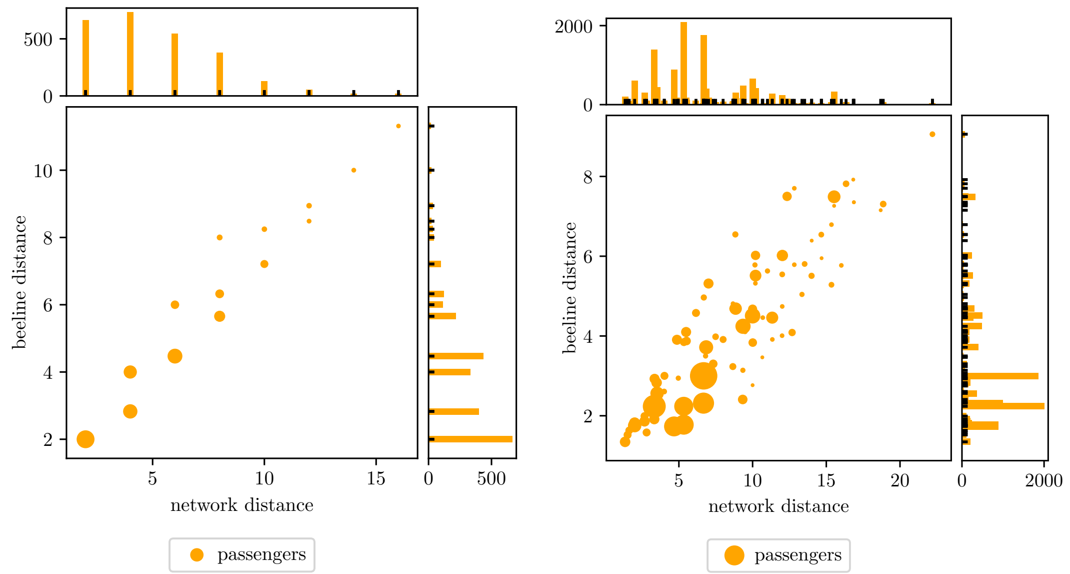
11.4 Computational Experiments

We perform computational experiments for F-RPM and D-RPM on artificial instances based on the datasets `grid` and `mandl` from the open source software library LinTim [Sch+; Sch+24] in order to compare the running times of the different solution methods and to learn more about the Pareto fronts. The PTNs provided for each of the datasets (see Figure B.1 in Appendix B) can be used to compute network and beeline distances between any pair of stations. The distribution of the demand with respect to the network and beeline distances is shown

Parameter	Value	Explanation
demand groups	$G \in \{1,3,5\}$	number of groups $G_d = G$ for all OD pairs $d \in D$
demand split	EQUAL	$\forall d \in D, \forall g \in [G] : t_d^g = \left\lceil \frac{t_d}{G} \right\rceil$
	RANDOM	$\forall d \in D, \forall g \in [G] : t_d^g \in [t_d]$ random with $\sum_{g=1}^{G} t_d^g = t_d$
	INCREASING	$\forall d \in D, \forall g \in \{2,\dots,G\} : t_d^g = \left\lceil \frac{t_d}{2^{G+1-g}} \right\rceil$ and $t_d^1 = \left\lfloor \frac{t_d}{2^{G-1}} \right\rfloor$
	DECREASING	$\forall d \in D, \forall g \in [G-1] : t_d^g = \left\lceil \frac{t_d}{2^g} \right\rceil$ and $t_d^G = \left\lfloor \frac{t_d}{2^{G-1}} \right\rfloor$
willingness to pay	w-FLAT w-NETWORK w-BEELINE	fare strategy used to generate willingness to pay
tariff parameters	A	$\forall g \in [G] : f_g = g, p_g = 0.2$
	B	$\forall g \in [G] : f_g = g, p_g = 0.6 - 0.1g$
	C	$\forall g \in [G] : f_g = 1, p_g = 0.1g$

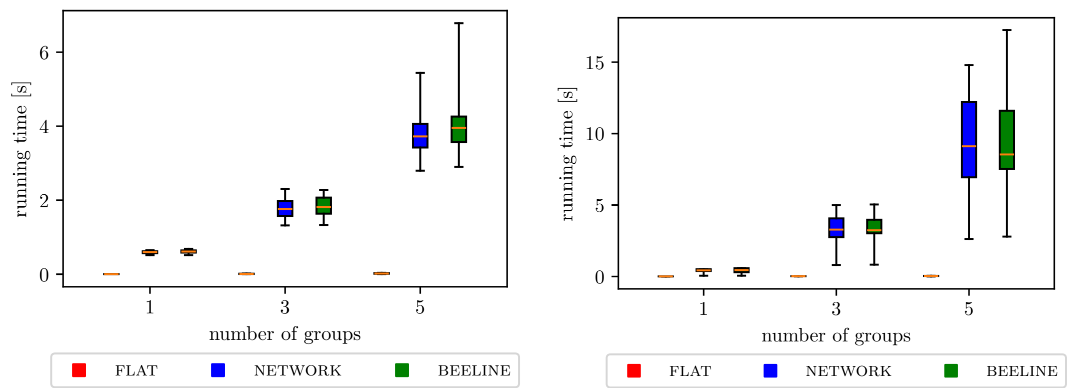
Table 11.1: Parameters for generating artificial instances.

in Figure 11.6. Data set `mandl` consists of 172 OD pairs that have 72 different network distances and 84 beeline distances. While dataset `grid` even has 567 OD pairs, these belong only to 8 network distances and 14 beeline distances. An overview of the parameters for generating the artificial instances is given in Table 11.1: The demand data provided in LinTim is split into $G \in \{1,3,5\}$ demand groups to create the input demand data of the revenue-passenger model in four different ways (EQUAL, RANDOM, INCREASING, DECREASING). The willingness to pay for each group is generated using a flat tariff (w-FLAT) or an affine distance tariff, where the distance is derived from the network distance (w-NETWORK) or the Euclidean distance (w-BEELINE). The parameters f_g and p_g, which determine the tariff for generating the willingness to pay of demand group $g \in [G]$, are chosen from three options (A, B, C) for affine distance tariffs and one option ($\forall g \in [G] : f_g = g$) for flat tariffs. In total, we obtain 84 instances per dataset. The instances are solved for F-RPM and D-RPM, determining FLAT, NETWORK distance and BEELINE distance tariffs. As solution methods, we apply Algorithms 11.1 and 11.2, respectively (which we call ALGO). For D-RPM, we additionally evaluate the running time of the ϵ-constraint method (Algorithm 2.1) with MILP (11.3) (which we call MILP1) and with MILP (11.3) including the additional constraints (11.4) (which we call MILP2). The solution methods are implemented in Python, the MILP-based approaches use Gurobi 9.1.2 [Gur24] for solving the MILPs, and the experiments are run on a machine with an Intel(R) Core(TM) i5-10310U and 16 GB of RAM.



(a) Demand data for dataset `grid` (b) Demand data for dataset `mandl`

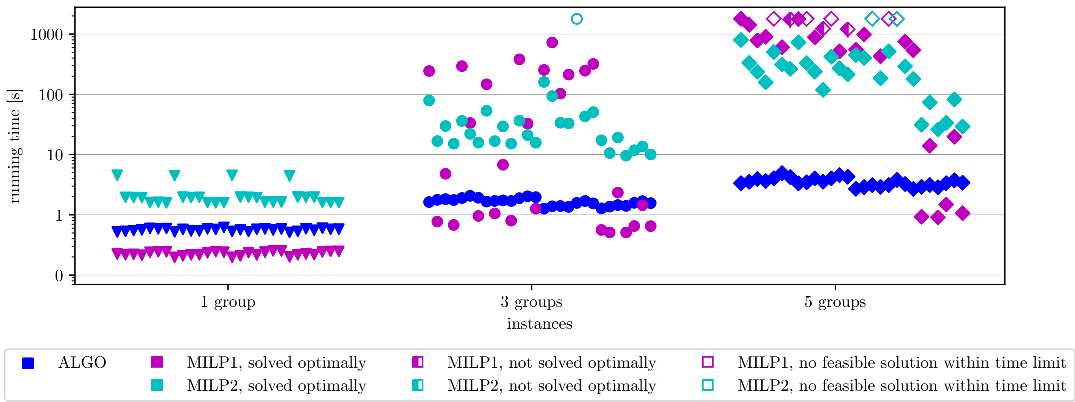
Figure 11.6: Demand data with respect to the different PTNs. The size of a point reflects on the demand. Above and on the right hand side of the plots, the demand with the same network or beeline distance, respectively, is aggregated.



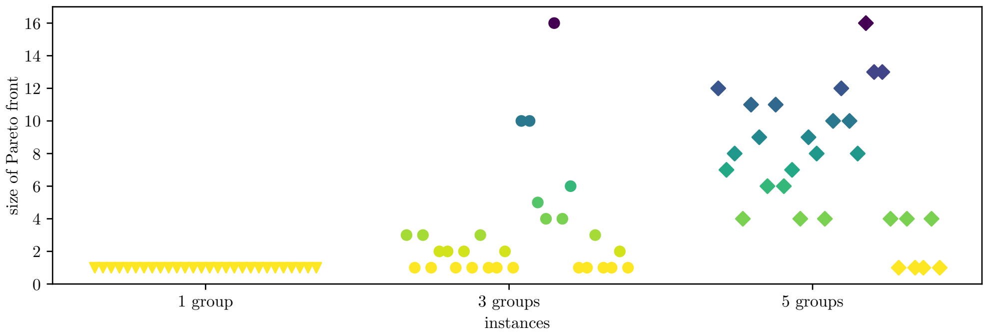
(a) Data set `grid` (b) Data set `mandl`

Figure 11.7: Running times in seconds for computing the complete Pareto fronts with Algorithm 11.1 for F-RPM and with Algorithm 11.2 for NETWORK and BEELINE D-RPM.

Running Time The running times of Algorithms 11.1 and 11.2 are depicted in Figure 11.7. According to Theorems 11.6 and 11.12 the running time of Algorithm 11.1 is quasilinear in the total number of demand groups while the running time of Algorithm 11.2 is cubic. This can be observed in the running times:



(a) Running time



(b) Size of the Pareto front

Figure 11.8: Running time and size of the Pareto front per instance for NETWORK D-RPM with on dataset `grid`. The order of the instances is given by iterating over the demand splits [EQUAL, RANDOM, INCREASING, DECREASING] in an outer loop and over the willingness to pay options [w-FLAT, w-NETWORK (A, B, C), w-BEELINE (A, B, C)] in an inner loop. Note that for $G = 1$, all demand splits yield the same demand $t_d^1 = t_d$.

Figure 11.8(a) shows the running time per instance in seconds of NETWORK D-RPM with Algorithm 11.2 (ALGO) and with the MILP-based method (MILP) on dataset `grid` with a logarithmic scale. Each marker represents the running time for computing the Pareto front of a single instance. The time limit for solving each MILP within the ϵ-constraint method is set to 300 seconds. If a MILP could not be solved to optimality within this time limit but a feasible solution was found, then we continue with this feasible solution and label the instance as “MILP, not solved optimally”. If no feasible solution is found, the procedure terminates and we label the instance as “MILP, no feasible solution within time limit” and depict it in this figure with the maximum running time that occurred for any instance.

Figure 11.8(b) shows the size of the Pareto front per instance. A darker color indicates a larger number of non-dominated points.

	ALGO			MILP1			MILP2		
G	mean	min	max	mean	min	max	mean	min	max
1	0.56	0.51	0.62	0.23	0.20	0.25	2.16	1.58	4.54
3	1.65	1.26	2.07	111.70	0.51	725.24	33.68	9.59	160.47
5	3.53	2.69	4.91	630.27	0.91	1793.72	277.91	26.11	799.46

Table 11.2: Mean, minimum and maximum running times in seconds for solving NETWORK D-RPM on the `grid` instances with Algorithm 11.2 (ALGO) and with the MILP-based method (MILP1 and MILP2). Only the instances that were solved optimally are considered.

F-RPM can be solved in 0.004/0.02/0.04 seconds for each `grid` instance and in 0.01/0.03/0.06 seconds for each `mandl` instance with 1/3/5 demand groups, while the running times of D-RPM are 0.67/2.31/6.78 seconds for `grid` and 0.6/5.04/17.24 seconds for `mandl` with 1/3/5 demand groups, respectively.

Figure 11.6 shows that the input data of `grid` is very structured and that only a few different distances occur, especially for the network distance. As suggested in Remark 11.13, this affects the running time, which is smaller for `grid` than for `mandl`, even though `grid` has roughly three times as many OD pairs (see Figure 11.7).

Figure 11.8(a) and Table 11.2 show the running times of ALGO, MILP1 and MILP2 for NETWORK D-RPM. For many instances, the running times of ALGO are orders of magnitude smaller compared to the running times of the ϵ-constraint based approaches of MILP1 and MILP2. For MILP1 (MILP2), for 6 (2) of the 28 instances of `grid` with 5 demand groups, it was in some iteration not possible to even determine any feasible solution within the time limit. With 3 demand groups this still happened for 1 (1) of the 28 instances. In case of 5 demand groups with MILP1, also 3 instances terminated with a feasible but not necessarily optimal solution. In Figure 11.8(a), we see that the running time of ALGO only shows small deviations for instances with the same number of demand groups. In comparison, the running times of MILP1 and MILP2 depend more on the specific instances and are often orders of magnitude higher than for ALGO. One difference between the instances is the size of their Pareto fronts, which is shown in Figure 11.8(b). MILP1 always performs best and MILP2 always performs worst for instances with only one non-dominated point. For these instances, the buildup of the additional constraints in MILP2 slows down the running time compared to MILP1, whereas MILP2 is faster for all but 6 instances with at least two non-dominated points. While the ϵ-constraint method benefits from a small size of the Pareto front because only a few iterations have to be carried out, ALGO is not affected by the size of the Pareto front but only by the number of demand groups. The specialized solution method ALGO demonstrates a consistent running time for all instances with the same number of demand groups and a better overall scaling than MILP1 and MILP2.

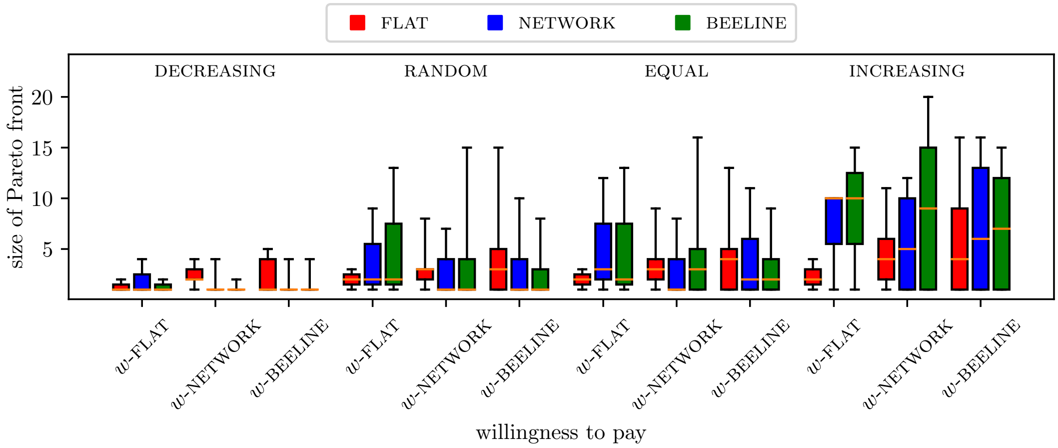
Size of the Pareto Front Figure 11.9 shows the number of points on the Pareto front for the different options for the demand splits and for the generation of the willingness to pay. We can observe two main effects:

First, a DECREASING demand split leads to a small size of the Pareto front. This is because the price increase cannot compensate for losing large demand groups with a low willingness to pay. We see the reverse effect for INCREASING, which leads to the most points on the Pareto front because losing only small demand groups with a low willingness to pay is compensated in the revenue by the increased price.

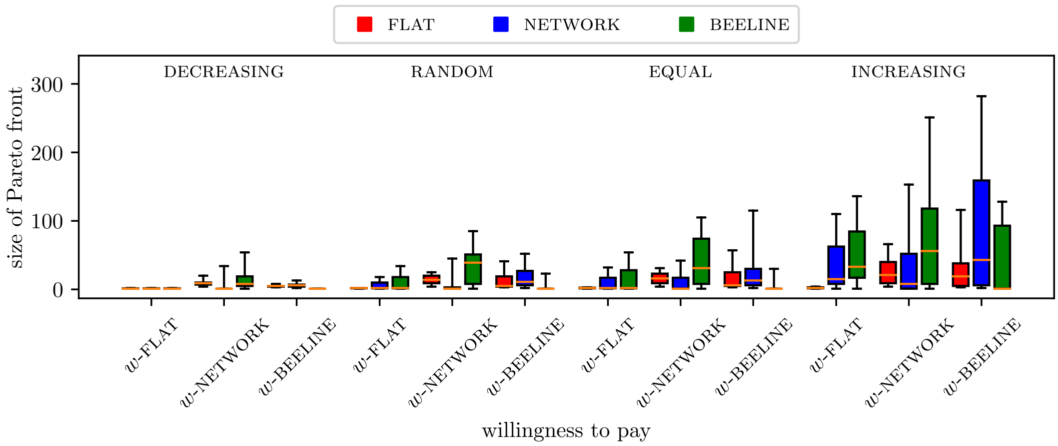
Second, if the fare strategy of the input tariff (for generating the willingness to pay) and the output tariff coincide, the size of the Pareto front is smaller. In this case, an output tariff might be chosen exactly as the willingness to pay of one demand group level, which is in general not possible if they differ.

Structure of the Pareto Front, Efficient Tariffs and Input Data Figures 11.10 to 11.13 show the Pareto fronts in (a) and corresponding efficient solutions in (b) and (c) for selected parameter settings for the `mandl` instances. Additionally, (b) and (c) show the demand as points (l_d, w_d^g) weighted with the number of potential passengers t_d^g. The respective figures for `grid` are given in Figures B.2 to B.5 in Appendix B. In these cases, we can see well that coinciding input and output tariffs lead to a small sized Pareto front that even dominates many of the points of the other tariff types. For w-BEELINE, the Pareto front of BEELINE D-RPM dominates the Pareto front of NETWORK D-RPM, and vice versa for w-NETWORK. Particularly in Figure 11.10, it is visible that the distinct points on the Pareto front belong to solutions that are a FLAT tariff. Only in this setting with the willingness to pay being generated by w-FLAT, we obtain a Pareto front for F-RPM that is not dominated by the Pareto fronts of BEELINE and NETWORK D-RPM. This is however not surprising because a flat tariff is a special case of a distance tariff.

Moreover, in many cases, the efficient tariffs are located on the lower levels of the demand groups, meaning that it is not beneficial to increase the price to the highest willingness to pay. Figure 11.10 constitutes an exception, where it is an efficient solution to choose a flat tariff with a fixed price equal to the second highest willingness to pay. However, we also see here that the highest willingness to pay does not lead to an efficient solution.

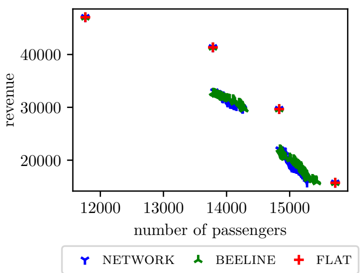


(a) Data set `grid`

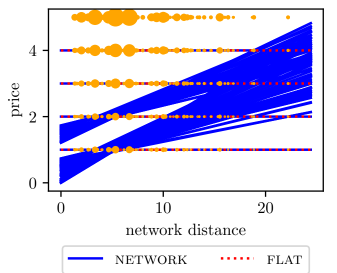


(b) Data set `mandl`

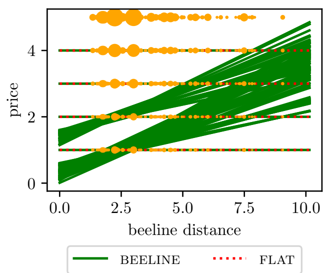
Figure 11.9: Size of the Pareto front dependent on the demand split and the tariff used to generate the willingness to pay.



(a) Pareto fronts

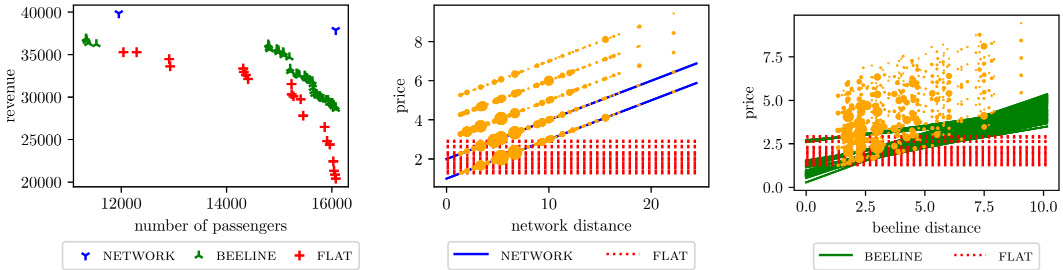


(b) NETWORK and FLAT



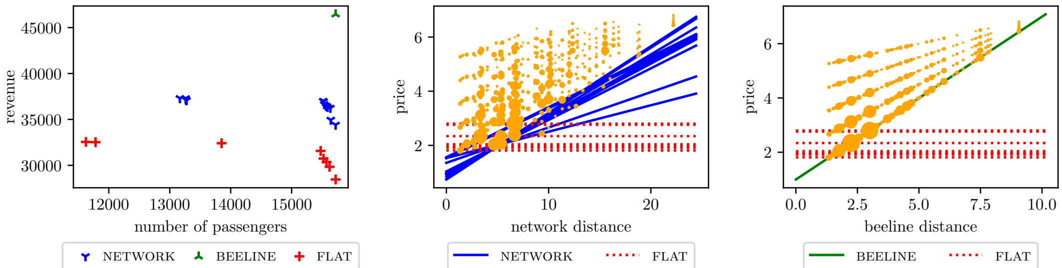
(c) BEELINE and FLAT

Figure 11.10: Instance of `mandl` with parameters 5/INCREASING/w-FLAT/A.



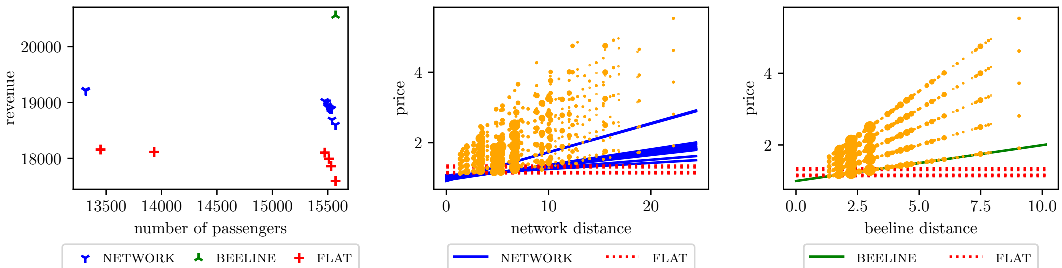
(a) Pareto fronts (b) NETWORK and FLAT (c) BEELINE and FLAT

Figure 11.11: Instance of `mandl` with parameters 5/RANDOM/w-NETWORK/A.



(a) Pareto fronts (b) NETWORK and FLAT (c) BEELINE and FLAT

Figure 11.12: Instance of `mandl` with parameters 5/DECREASING/w-BEELINE/B.



(a) Pareto fronts (b) NETWORK and FLAT (c) BEELINE and FLAT

Figure 11.13: Instance of `mandl` with parameters 5/EQUAL/w-BEELINE/C.

11.5 Zone Tariffs

Finally, we consider the revenue-passenger model with a zone tariff (Definition 9.7) as the desired fare strategy. The aim is to determine zone tariffs, meaning zone partitions $\mathcal{Z}$ and price functions P that maximize the objective functions rev and pass. Note that in [OB17], the single-objective problem of finding a zone tariff that maximizes the revenue with connected zones or ring zones is examined without explicitly considering the total number of passengers. Here, we study a bi-objective setting.

Definition 11.14 (Z-RPM)**.** Given a PTN (V, E) with extended OD data $(D, W_d, G_d, t_d^g, w_d^g)$ and an upper bound $N \in \mathbb{N}_{\geq 1}$ on the number of zones, the bi-objective *revenue-passenger model for zone tariffs* (Z-RPM) is the following:

$$\begin{array}{ll} \max & \mathsf{rev}(\pi) \\ \max & \mathsf{pass}(\pi) \\ \text{s.t.} & \pi \text{ is a zone tariff with a zone partition } \mathcal{Z} \text{ and a price list } P, \\ & \mathcal{Z} \text{ consists of at most } N \text{ zones}, \\ & \text{[optional:] requiring connected zones}, \\ & \text{[optional:] requiring the no-elongation property}, \\ & \text{[optional:] requiring the no-stopover property}. \end{array}$$

Remark 10.8 also applies to Z-RPM, so we may consider price functions P that attain a finite number of different values and can hence be represented by a price list (Definition 9.12).

Lemma 11.15 shows that we can assume that the prices $P(k)$ are bounded from above by the maximum of all willingness to pay values.

Lemma 11.15. *For every non-dominated point of* Z-RPM *with/without requiring the no-elongation property and with/without requiring the no-stopover property, there is a corresponding efficient solution π with a zone partition $\mathcal{Z}$ and a price function P with*

$$P(k) \leq \bar{r} := \max\{w_d^g : d \in D, g \in [G_d]\}$$

for all $k \in \mathbb{N}_{\geq 1}$.

Proof. Let π' w.r.t. a zone partition $\mathcal{Z}$ and a price function P' be an efficient solution to Z-RPM corresponding to a non-dominated point $(\overline{\mathsf{pass}}, \overline{\mathsf{rev}})$. Suppose that there is some $k \in \mathbb{N}_{\geq 1}$ with $P'(k) > \bar{r}$. We define a new price function $P\colon \mathbb{N}_{\geq 1} \to \mathbb{R}_{\geq 0}$ by

$$P(k) := \begin{cases} P'(k) & \text{if } P'(k) \leq \bar{r}, \\ \bar{r} & \text{if } P'(k) > \bar{r} \end{cases}$$

for all $k \in \mathbb{N}_{\geq 1}$. Let π be the zone tariff w.r.t. $\mathcal{Z}, P$. By Lemma 9.10, we have that π satisfies the no-elongation property and the no-stopover property if π' satisfies them. In order to prove the claim, we show that π has the same objective function values as π', namely $(\overline{\mathsf{pass}}, \overline{\mathsf{rev}})$. Because the zone partition $\mathcal{Z}$ is fixed, we can use the notation from Definition 9.11.

Let $k \in \mathbb{N}_{\geq 1}$. Assume that $P'(k) > \bar{r}$ and that there is some $d \in D_k$ and $g \in [G_d]$ with $w_d^g = \bar{r}$. Then we have $\mathsf{pass}_d(P'(k)) = 0 < t_d^g = \mathsf{pass}_d(P(k))$ for the number of passengers and $0 \cdot P'(k) = 0 \leq t_d^g \cdot P(k)$ for the revenue generated by OD pair d. Note that all other demand groups $g' \in [G_d] \setminus \{g\}$ are not attracted because by Definition 11.1, we have $w_d^{g'} \neq w_d^g = \bar{r}$ and hence $w_d^{g'} < w_d^g$. This means that the objective function values improve when changing $P'(k)$ to $P(k)$. Thus, π' dominates π, which is a contradiction to π being efficient. Therefore, we have $P'(k) > \bar{r}$ and $w_d^g < \bar{r}$ for all $d \in D_k$ and $g \in [G_d]$, or we have $P'(k) \leq \bar{r}$. In this case, changing $P'(k)$ to $P(k)$ does not affect the objective function values, and the objective function values of π and π' coincide, which means that π w.r.t. $\mathcal{Z}, P$ is an efficient solution with $P(k) \leq \bar{r}$ for all $k \in \mathbb{N}_{\geq 1}$. ☐

In Lemma 11.16, we show how Z-RPM is related to F-RPM.

Lemma 11.16. *Let a zone tariff π w.r.t. a zone partition $\mathcal{Z}$ and a price function P be an efficient solution to* Z-RPM *without requiring the no-elongation property and without requiring the no-stopover property, and let D_k for all $k \in [K]$ be derived from $\mathcal{Z}$ (see Definition 9.11). Then it holds for all $k \in [K]$ with $D_k \neq \emptyset$ that $P(k)$ is an efficient solution to* F-RPM *with the extended OD data $(D_k, W_d, G_d, t_d^g, w_d^g)$ as input. In particular, $P(k) \in \{w_d^g : d \in D_k, g \in [G_d]\}$.*

Proof. Assume that $P(k')$ for some $k' \in [K]$ with $D_{k'} \neq \emptyset$ is not an efficient solution to F-RPM. Then there is another fixed price $f \in \mathbb{R}_{\geq 0}$ that dominates $P(k')$ regarding Z-RPM, i.e., f yields a higher number of passengers or a higher revenue for the OD pairs in $D_{k'}$. Therefore, for the zone tariff π' w.r.t. $\mathcal{Z}$ and P' with $P'(k') := f$ and $P'(k) := P(k)$ for $k \in \mathbb{N}_{\geq 1} \setminus \{k'\}$, we have $\mathsf{rev}(\pi') \geq \mathsf{rev}(\pi)$ and $\mathsf{pass}(\pi') \geq \mathsf{pass}(\pi)$ with one inequality holding strictly. This means that π' dominates π, which is a contradiction to π being an efficient solution. ☐

For the ϵ-constraint method (Section 2.2.1), the following MILP, which is explained below, can be used with $\bar{r} := \max\{w_d^g : (d, g) \in S_{\text{dem}}\}$:

$$
\begin{aligned}
\max \quad & \sum_{(d,g)\in S_{\text{dem}}} t_d^g \cdot \pi_d^g \\
\max \quad & \sum_{(d,g)\in S_{\text{dem}}} t_d^g \cdot y_d^g \\
\text{s.t.} \quad & \text{constraints (10.6) to (10.13)} \\
& \pi_d \le w_d^g + \bar{r}\cdot(1-y_d^g) \quad \text{for all } (d,g)\in S_{\text{dem}} && \text{(11.9a)}\\
& \pi_d^g \le \pi_d \quad \text{for all } (d,g)\in S_{\text{dem}} && \text{(11.9b)}\\
& \pi_d^g \le y_d^g \cdot \bar{r} \quad \text{for all } (d,g)\in S_{\text{dem}} && \text{(11.9c)}\\
& \pi_d, p_k \le \bar{r} \quad \text{for all } (d,g)\in S_{\text{dem}}, k\in[K] && \text{(11.9d)}\\
& y_d^g \in \{0,1\} \quad \text{for all } (d,g)\in S_{\text{dem}}\\
& \pi_d^g \in \mathbb{R}_{\ge 0} \quad \text{for all } (d,g)\in S_{\text{dem}}
\end{aligned}
$$

The constraints (10.6) to (10.13) from Section 10.4.3 for determining a zone partition and for setting the values p_k of the price list and the prices π_d of the OD pairs can be adopted. For Z-RPM, we add a continuous variable $\pi_d^g \in \mathbb{R}_{\ge 0}$ that specifies the price paid by a demand group $(d,g) \in S_{\text{dem}}$ with $\pi_d^g = \pi_d$ if $\pi_d \le w_d^g$ and $\pi_d^g = 0$ otherwise. This relation is ensured by the binary variable $y_d^g \in \{0,1\}$ with $y_d^g = 1$ if $\pi_d \le w_d^g$, and $y_d^g = 0$ otherwise. To see this, we consider a demand group $(d,g) \in S_{\text{dem}}$:

- If $y_d^g = 1$, constraint (11.9a) requires $\pi_d \le w_d^g$, constraint (11.9c) is $\pi_d^g \le \bar{r}$ and redundant because $\pi_d^g \le \pi_d \le \bar{r}$, and the objective function for the revenue is maximized by setting $\pi_d^g = \pi_d$.
- If $y_d^g = 0$, constraint (11.9a) becomes redundant with $\pi_d \le w_d^g + \bar{r}$ and constraint (11.9c) enforces $\pi_d^g = 0$.

Note that, if $c_d^k = 0$, the constraints (10.10) (price assignment) also reduce to the redundant constraints $\pi_d \le p_k + \bar{r}$ and $p_k \le \pi_d + \bar{r}$. By Lemma 11.15, the restrictions $\pi_d, p_k \le \bar{r}$ in constraints (11.9d) are valid in the sense that the MILP admits at least one feasible efficient solution for each non-dominated point.

The following constraints can be added to strengthen the formulation:

$$y_d^{g_1} \le y_d^{g_2} \text{ for all } d \in D, g_1, g_2 \in [G_d] \text{ with } w_d^{g_1} \le w_d^{g_2}.$$

These are valid because, if $y_d^{g_1} = 1$, we have $\pi_d \le w_d^{g_1} \le w_d^{g_2}$ and hence also $y_d^{g_2} = 1$.

Price-Setting Subproblem In addition to Z-RPM, which determines zones and prices, we also consider the price-setting subproblem of Z-RPM. This means that we let a zone partition $\mathcal{Z}$ be given and only optimize the price function P. Using the notation of Definition 9.11 and Remark 10.8, the price-setting subproblem of Z-RPM takes extended OD data $(D, W_d, G_d, t_d^g, w_d^g)$ and a partition $D_1, \dots, D_K$ of D as input and searches for a price list $p = (p_1, \dots, p_K)$, where K is the maximum number of zones traversed by an OD pair and D_k with $k \in [K]$ contains all OD pairs $d \in D$ with $\sigma(W_d) = k$ for the zone partition $\mathcal{Z}$. Here, we consider the price-setting subproblem without requiring the no-stopover property, so it is only important for determining the sets D_k with $k \in [K]$ whether multiple or single counting is considered. Because the zone partition is fixed in the price-setting subproblem, we write $\mathsf{rev}(p)$ and $\mathsf{pass}(p)$ instead of $\mathsf{rev}(\pi)$ and $\mathsf{pass}(\pi)$.

11.5.1 Complexity and Intractability

In [OB17, Thm. 1], it has been shown that the single-objective problem maximizing the revenue (without explicitly considering the total number of passengers) with single counting, requiring monotonically increasing prices and with connected or ring zones is NP-hard, even on star graphs. In this section, we show that the canonical decision problem of Z-RPM is NP-complete for the specifications stated in Theorem 11.17. Also the canonical decision problem of the price-setting subproblem of Z-RPM is NP-hard, which is surprising because the price-setting subproblem of Z-FDM is polynomially solvable (see Table 10.3), and the price-setting subproblem in [OB17, Thm. 4] is polynomially solvable as well. Moreover, we show that Z-RPM and the price-setting subproblem of Z-RPM are intractable.

Theorem 11.17. *The canonical decision problem of* Z-RPM *is* NP*-complete*

- *with multiple or single counting,*
- *with connected or arbitrary zones,*
- *without requiring the no-elongation property,*
- *without requiring the no-stopover property,*
- *even if the graph is linear and $G_d = 2$ for all $d \in D$.*

Proof. In the canonical decision problem of Z-RPM, we denote the lower bounds on the objective function values by $\overline{\mathsf{rev}}$ and $\overline{\mathsf{pass}}$, respectively.

Let a certificate $\mathcal{Z} = \{Z_1, \dots, Z_L\}, P$ with $L \leq N$ be given. We can check in polynomial time:

- $\bigcup_{i \in [L]} Z_i = V$ and $Z_i \cap Z_j = \emptyset$ and $Z_i \neq \emptyset$ for all $i, j \in [L]$ with $i \neq j$,
- $\sum_{\substack{(d,g) \in S_{\text{dem}}: \\ P(\sigma(W_d)) \leq w_d^g}} t_d^g \geq \overline{\mathsf{pass}}$,

- $\sum_{\substack{(d,g)\in S_{\text{dem}}:\\ P(\sigma(W_d))\leq w_d^g}} t_d^g \cdot P(\sigma(W_d)) \geq \overline{\mathsf{rev}}$,
- connectedness of $G[Z_i]$ for all $i \in [L]$.

Hence, the problem is in NP.

We use a reduction from Partition (Problem 2.14). Given an instance $A = \{a_1, \dots, a_K\}$ of Partition, we construct an instance of Z-RPM. Consider a linear graph (V, E) with $V := \{v_1, \dots, v_{K+1}\}$ and $E := \{\{v_i, v_{i+1}\} : i \in [K]\}$, and set $N := K + 1 = |V|$. Set $D := \{d_1, \dots, d_K\}$ with $d_k := (v_1, v_{k+1})$ for all $k \in [K]$. The paths W_d for $d \in D$ are the unique simple paths in the linear graph (V, E). We set $G_d := 2$ for all $d \in D$ and

$$t_{d_k}^1 := a_k, \quad w_{d_k}^1 := \frac{1}{a_k + 1},$$
$$t_{d_k}^2 := 1, \quad w_{d_k}^2 := a_k + 1.$$

Finally, we define constants $\overline{\mathsf{rev}} := \overline{\mathsf{pass}} := K + \frac{1}{2}\sum_{a\in A} a$. We show that there is a solution A' to Partition if and only if there is a solution $\mathcal{Z}, P$ to the Z-RPM with lower bounds $\overline{\mathsf{rev}}$ and $\overline{\mathsf{pass}}$ on the objective function values.

For the first direction, let $A' \subseteq A$ be a solution to Partition. We define the zone partition $\mathcal{Z} := \{Z_1, \dots, Z_K\}$ by $Z_1 := \{v_1, v_2\}$ and $Z_k := \{v_{k+1}\}$ for all $k \in \{2, \dots, K\}$. This yields $D_k = \{d_k\}$ for all $k \in [K]$, and the prices of the OD pairs can be chosen independently. For all $k \in [K]$, we set

$$p_k := \begin{cases} w_{d_k}^1 & \text{if } a_k \in A', \\ w_{d_k}^2 & \text{if } a_k \in A \setminus A'. \end{cases}$$

and let the price function P be given by the price list $p = (p_1, \dots, p_K)$ (Definition 9.12). Then the zone tariff π w.r.t. $\mathcal{Z}, P$ satisfies

$$\begin{aligned}
\mathsf{pass}(\pi) &= \sum_{k\in[K]} \mathsf{pass}_{d_k}(p_k) = \sum_{\substack{k\in[K]:\\ a_k\in A'}} (t_{d_k}^1 + t_{d_k}^2) + \sum_{\substack{k\in[k]:\\ a_k\in A\setminus A'}} t_{d_k}^2 \\
&= \sum_{a\in A'} (a+1) + \sum_{a\in A\setminus A'} 1 = K + \sum_{a\in A'} a = K + \frac{1}{2}\sum_{a\in A} a = \overline{\mathsf{pass}}, \\
\mathsf{rev}(\pi) &= \sum_{k\in[K]} \mathsf{pass}_{d_k}(p_k)\cdot p_k = \sum_{\substack{k\in[K]:\\ a_k\in A'}} (t_{d_k}^1 + t_{d_k}^2) w_{d_k}^1 + \sum_{\substack{k\in[K]:\\ a_k\in A\setminus A'}} t_{d_k}^2 w_{d_k}^2 \\
&= \sum_{a\in A'} (a+1)\cdot\frac{1}{a+1} + \sum_{a\in A\setminus A'} 1\cdot(a+1) = K + \sum_{a\in A\setminus A'} a \\
&= K + \frac{1}{2}\sum_{a\in A} a = \overline{\mathsf{rev}}.
\end{aligned}$$

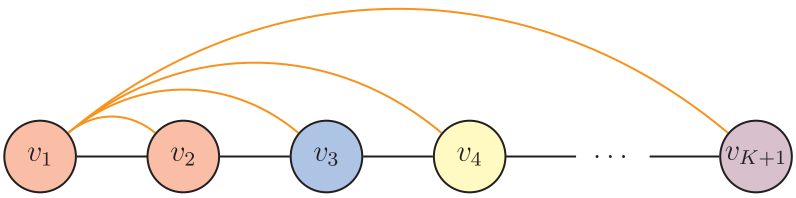


Figure 11.14: Construction of the PTN and the OD pairs in the proof of Theorems 11.17 and 11.19.

For the other direction, let a zone tariff π w.r.t. a zone partition $\mathcal{Z}$ and a price function P be a solution to Z-RPM. We set

$$A' := \{a_k : k \in [K] \text{ with } P(\sigma(W_{d_k})) \leq w^1_{d_k}\}.$$

Then it holds that

$$\begin{aligned}\sum_{a\in A'} a &= -K + \sum_{a\in A'}(a+1) + \sum_{a\in A\setminus A'} 1 \overset{(\star)}{=} -K + \sum_{\substack{k\in[K]:\\ a_k\in A'}} (t^1_{d_k} + t^2_{d_k}) + \sum_{\substack{k\in[K]:\\ a_k\in A\setminus A'}} t^2_{d_k} \\ &\geq -K + \mathsf{pass}(\pi) \geq -K + \overline{\mathsf{pass}} = -K + K + \frac{1}{2}\sum_{a\in A} a = \frac{1}{2}\sum_{a\in A} a,\end{aligned}$$

and

$$\begin{aligned}\sum_{a\in A\setminus A'} a &= -K + \sum_{a\in A}(a+1)\cdot\frac{1}{a+1} + \sum_{a\in A\setminus A'} 1\cdot(a+1) \\ &\overset{(\star)}{=} -K + \sum_{\substack{k\in[K]:\\ a_k\in A'}} (t^1_{d_k} + t^2_{d_k})w^1_{d_k} + \sum_{\substack{k\in[K]:\\ a_k\in A\setminus A'}} t^2_{d_k} w^2_{d_k} \\ &\geq -K + \mathsf{rev}(\pi) \geq -K + \overline{\mathsf{rev}} = -K + K + \frac{1}{2}\sum_{a\in A} a = \frac{1}{2}\sum_{a\in A} a,\end{aligned}$$

where we first add a zero, namely $0 = -K + \sum_{a\in A'} 1 + \sum_{a\in A\setminus A'} 1$, and then both equalities $(\star)$ hold by choice of t^g_d for $(d,g) \in S_{\text{dem}}$. Finally, both formulas hold with equality because $\sum_{a\in A'} a + \sum_{a\in A\setminus A'} a = \sum_{a\in A} a$. Hence, A' is a solution to Partition.

Note that the proof does not differentiate between multiple or single counting. The constructed zone partition in the first direction is connected, thus the proof works for arbitrary zones as well as with the requirement of connected zones. By choice of the willingness to pay values of the demand groups, the no-elongation property and the no-stopover property will in most cases not be satisfied. □

Corollary 11.18. *The price-setting subproblem of* Z-RPM *with multiple or single counting and without requiring the no-elongation property and without requiring the no-stopover property is* NP*-hard, even if the graph is linear, $G_d = 2$ for all $d \in D$ and $|D_k| = 1$ for all $k \in [K]$.*

Proof. The claim follows from the proof of Theorem 11.17 by fixing the zone partition $\mathcal{Z} := \{Z_1, \ldots, Z_K\}$ with $Z_1 := \{v_1, v_2\}$ and $Z_k := \{v_{k+1}\}$ for all $k \in \{2, \ldots, K-1\}$ beforehand. □

In contrast to the revenue-passenger model for flat and affine distance tariffs, Z-RPM and its price-setting subproblem are intractable (see Section 2.3).

Theorem 11.19. Z-RPM *is intractable*

- *with multiple or single counting,*
- *with connected or arbitrary zones,*
- *without requiring the no-elongation property,*
- *without requiring the no-stopover property,*
- *even if the graph is linear and $G_d = 2$ for all $d \in D$.*

Proof. We present an instance of Z-RPM such that there are exponentially many non-dominated points. Consider a linear graph (V, E) with $V := \{v_1, \ldots, v_{K+1}\}$ and $E := \{\{v_i, v_{i+1}\} : i \in [K]\}$. Set $N := K + 1 = |V|$. Let $D := \{d_0, \ldots, d_{K-1}\}$ with $d_k := (v_1, v_{k+2})$ for all $k \in \{0, \ldots, K-1\}$. The paths W_d for $d \in D$ are the unique simple paths in the linear graph (V, E). Let $G_d = 2$ for all $d \in D$. We define the numbers of passengers and the willingness to pay for all $k \in \{0, \ldots, K-1\}$ by $t^1_{d_k} := t^2_{d_k} := 2^{2k}$ and $w^1_{d_k} := 1$ and $w^2_{d_k} := 3$. For this input, we choose the zone partition $\mathcal{Z} := \{Z_1, \ldots, Z_K\}$ with $Z_1 := \{v_1, v_2\}$ and $Z_k := \{v_{k+1}\}$ for all $k \in \{2, \ldots, K-1\}$. This yields $D_k = \{d_{k-1}\}$ for all $k \in [K]$, and the prices of the OD pairs can be chosen independently. Therefore, in any zone tariff with zone partition $\mathcal{Z}'$ and a price list P', we can replace $\mathcal{Z}'$ by $\mathcal{Z}$ and P' by P with $P(\sigma(W_d)) := P'(\sigma'(W_d))$ without changing the objective function values, where σ is the zone function with respect to $\mathcal{Z}$, which satisfies $\sigma(W_d) \neq \sigma(W_{d'})$ for $d \neq d'$, and σ' is the zone function with respect to $\mathcal{Z}'$. Thus, considering $\mathcal{Z}$ is not a restriction. For any efficient price list $p = (p_1, \ldots, p_K)$, we hence have $p_k \in \{w^1_{d_{k-1}}, w^2_{d_{k-1}}\} = \{1, 3\}$ for all $k \in [K]$ by Lemma 11.16. This yields for the number of attracted passengers of OD pair d_k:

$$\mathsf{pass}_{d_k}(p_{k+1}) = \begin{cases} 2 \cdot 2^{2k} = 2^{2k+1} & \text{if } p_{k+1} = 1, \\ 2^{2k} & \text{if } p_{k+1} = 3, \end{cases} \tag{11.10}$$

and for the contribution of revenue of OD pair d_k:

$$p_{k+1} \cdot \mathsf{pass}_{d_k}(p_{k+1}) = \begin{cases} 2^{2k+1} & \text{if } p_{k+1} = 1, \\ 3 \cdot 2^{2k} = \frac{3}{2} \cdot 2^{2k+1} & \text{if } p_{k+1} = 3. \end{cases} \tag{11.11}$$

By defining a vector $a \in \{0,1\}^{2K}$ corresponding to p by $a_{2k} := 1$ and $a_{2k+1} := 0$ if $p_{k+1} = w^2_{d_k}$, and $a_{2k} := 0$ and $a_{2k+1} := 1$ if $p_{k+1} = w^1_{d_k}$ for all $k \in [0, \ldots, K-1]$, we get that

$$\mathsf{pass}(p) = \sum_{k=0}^{K-1} \left(a_{2k}2^{2k} + a_{2k+1}2^{2k+1}\right) = \sum_{k=0}^{2K-1} a_k 2^k.$$

Hence, a is the binary representation of $\mathsf{pass}(p)$. Because price lists $\bar{p}, p' \in \{1,3\}^K$ with $\bar{p} \neq p'$ yield $\bar{a} \neq a'$, we have $|\{1,3\}|^K = 2^K$ different values for $\mathsf{pass}(p)$ with $p \in \{1,3\}^K$. Moreover, we show for $\bar{p}, p' \in \{1,3\}^K$ that if $\mathsf{pass}(\bar{p}) < \mathsf{pass}(p')$, then $\mathsf{rev}(\bar{p}) > \mathsf{rev}(p')$. For all $p = (p_1, \ldots, p_k) \in \{1,3\}^K$, it holds by formulas (11.10) and (11.11) that

$$\mathsf{pass}_{d_k}(p_{k+1}) + p_{k+1} \cdot \mathsf{pass}_{d_k}(p_{k+1}) = \begin{cases} 2 \cdot 2^{2k} + 2^{2k+1} = 4 \cdot 2^{2k} & \text{if } p_{k+1} = 1, \\ 2^{2k} + 3 \cdot (2^{2k}) = 4 \cdot 2^{2k} & \text{if } p_{k+1} = 3. \end{cases}$$

Therefore, for all $p \in \{1,3\}^K$, we have that $\mathsf{pass}(p) + \mathsf{rev}(p) = \sum_{k \in \{0,\ldots,K-1\}} 4 \cdot 2^{2k}$ is constant. For $\bar{p}, p' \in \{1,3\}^K$ with $\bar{p} \neq p'$ and $\mathsf{pass}(\bar{p}) < \mathsf{pass}(p')$, we thus have that $\mathsf{rev}(\bar{p})) > \mathsf{rev}(p')$.

Thus, the number of non-dominated points is $|\{1,3\}|^K = 2^K$, and Z-RPM is intractable. □

Corollary 11.20. *The price-setting subproblem of* Z-RPM *with multiple or single counting and without requiring the no-elongation property and without requiring the no-stopover property is intractable, even if the graph is linear, $G_d = 2$ for all $d \in D$ and $|D_k| = 1$ for all $k \in [K]$.*

Proof. The claim follows from the proof of Theorem 11.19 by fixing the considered zone partition $\mathcal{Z}$ beforehand. □

11.5.2 Specialized Solution Methods for the Price-Setting Subproblem of Z-RPM

We now develop two specialized solution methods for the price-setting subproblem of Z-RPM without requiring the no-elongation property and without requiring the no-stopover property. For simplicity, we assume that $D_k \neq \emptyset$ for all $k \in [K]$. This can be achieved by dropping the empty sets and renumbering the remaining ones. Because of the previous results, we cannot expect to find a polynomial time algorithm.

Enumeration The first approach makes use of Lemma 11.16. The idea is to compute the Pareto front Γ_k of F-RPM regarding D_k for each $k \in [K]$ and then to compose objective function values of Z-RPM as

$$\Gamma' = \left\{ \left(\sum_{k\in[K]} y_k, \sum_{k\in[K]} z_k \right) : (y_k, z_k) \in \Gamma_k \text{ for all } k \in [K] \right\}.$$

The set Γ' is a superset of the Pareto front Γ of Z-RPM and can be filtered for the non-dominated points by Algorithm 2.2. For each $k \in [K]$, the set Γ_k can be computed in $\mathcal{O}((\sum_{d\in D_k} G_d) \cdot \log(\sum_{d\in D_k} G_d))$ by Algorithm 11.2. However, the set Γ' can be quite large, which is important for the running time of filtering with Algorithm 2.2. We give an upper bound on the cardinality of Γ and Γ':

$$|\Gamma| \leq |\Gamma'| \leq \prod_{k\in[K]} |\Gamma_k| \leq \prod_{k\in[K]} \left(\sum_{d\in D_k} G_d \right) \leq \left(\frac{\sum_{d\in D} G_d}{K} \right)^K,$$

where the first inequality follows from the fact that the set Γ_k contains at most one point for each willingness to pay belonging to D_k for $k \in [K]$. The second inequality follows from the arithmetic mean-geometric mean (AM-GM) inequality [Ste04, Problem 2.1], meaning that the product is maximized if the number of points, and thus the number of demand groups G_d, is evenly distributed over the sets Γ_k with $k \in [K]$. Corollary 11.20 shows that this bound is tight because $|\Gamma| = 2^K$. Depending on the number of demand groups per OD pair and the number of demand groups per number of traversed zones, the running time of the first approach might hence be quite large.

Dynamic Program The second approach investigates the ϵ-constraint approach with an alternative IP formulation for the price-setting subproblem to obtain a dynamic program. We set $\bar{T} := \sum_{(d,g)\in S_{\text{dem}}} t_d^g$ as the total number of potential passengers, and

$$S_k := \left\{ (w_d^g, T) : d \in D_k, g \in [G_d], T = \sum_{d\in D_k} \sum_{g\in[G_d]:\, w_d^g \geq w} t_d^g \right\}, \ s_k := |S_k|,$$

which is the set of all pairs of a willingness to pay and the total demand that is attracted if the price for traversing k zones is set to this willingness to pay. For all $k \in [K]$, let $S_k = \{(w_k^1, T_k^1), \ldots, (w_k^{s_k}, T_k^{s_k})\}$ be an enumeration of S_k.

For $\epsilon \in \left[0, \bar{T}\right]$, we obtain the following IP modeling the ϵ-constraint problem $P_1(\epsilon)$ (2.2) for the price-setting subproblem of Z-RPM:

$$
\begin{aligned}
\max_{x_k^i} \quad & \sum_{k \in [K]} \sum_{i \in [s_k]} x_k^i \cdot w_k^i \cdot T_k^i \\
\text{s.t.} \quad & \sum_{k \in [K]} \sum_{i \in [s_k]} x_k^i \cdot T_k^i \geq \epsilon \\
& \sum_{i \in [s_k]} x_k^i = 1 \qquad \text{for all } k \in [K] \\
& x_k^i \in \{0,1\} \quad \text{for all } k \in [K], i \in [s_k].
\end{aligned}
\tag{11.12}
$$

For all $k \in [K]$ and $i \in [s_k]$, setting the binary variable x_k^i to 1 means $p_k = w_k^i$, which contributes T_k^i passengers and a revenue of $w_k^i \cdot T_k^i$. We reformulate the ϵ-constraint: Instead of setting a lower bound on the number of attracted passengers, we set an upper bound on the number of passengers that are not attracted and do not use public transport because their willingness to pay is smaller then the designated price. To do so, we define for all $k \in [K]$ and $i \in [s_k]$ a constant

$$
q_k^i := \sum_{d \in D_k} \sum_{g \in [G_d] \colon w_d^g < w_k^i} t_d^g,
$$

which is the number of passengers whose willingness to pay is smaller than w_k^i. We obtain:

$$
\begin{aligned}
& \sum_{k \in [K]} \sum_{i \in [s_k]} x_k^i \cdot T_k^i \geq \epsilon \\
\Leftrightarrow\ & \bar{T} - \sum_{k \in [K]} \sum_{i \in [s_k]} x_k^i \cdot q_k^i \geq \epsilon \\
\Leftrightarrow\ & \sum_{k \in [K]} \sum_{i \in [s_k]} x_k^i \cdot q_k^i \leq \bar{T} - \epsilon =: \lambda.
\end{aligned}
$$

Hence, IP (11.12) is equivalent to

$$
\begin{aligned}
\max_{x_k^i} \quad & \sum_{k \in [K]} \sum_{i \in [s_k]} x_k^i \cdot w_k^i \cdot T_k^i \\
\text{s.t.} \quad & \sum_{k \in [K]} \sum_{i \in [s_k]} x_k^i \cdot q_k^i \leq \lambda \\
& \sum_{i \in [s_k]} x_k^i = 1 \qquad \text{for all } k \in [K] \\
& x_k^i \in \{0,1\} \quad \text{for all } k \in [K], i \in [s_k]
\end{aligned}
\tag{11.13}
$$

	result	reference
F-RPM	tractable	Cor. 11.5
F-RPM	$\mathcal{O}(\|S\| \log(\|S\|))$ with $\|S\| \leq \sum_{d \in D} G_d$	Thm. 11.6
D-RPM	tractable	Cor. 11.11
D-RPM	$\mathcal{O}((\sum_{d \in D} G_d)^3)$	Thm. 11.12
Z-RPM	canonical decision problem is NP-complete	Thm. 11.17
Z-RPM price-setting	canonical decision problem is NP-complete	Cor. 11.18
Z-RPM	intractable	Thm. 11.19
Z-RPM price-setting	intractable	Cor. 11.20

Table 11.3: Overview of results for the revenue-passenger model. Note that the running time of F-RPM and D-RPM refers to finding the complete Pareto front.

with $\lambda \in [0, \bar{T}]$. Because x_k^i and q_k^i are integral for all $k \in [K]$ and $i \in [s_k]$, it is sufficient to consider $\lambda \in \{0, \dots, \bar{T}\}$. Note that IP (11.13) is a multiple-choice knapsack problem. Solving it for a total budget of $\bar{T}$ with a dynamic program yields optimal solutions for all problems of IP (11.13) with $\lambda \in \{0, \dots, \bar{T}\}$ in the dynamic programming table. By looking through the objective function values from $\lambda = 0$ to $\lambda = \bar{T}$ and keeping a solution whenever the revenue increases, we obtain the Pareto front of the price-setting subproblem of Z-RPM (see Section 2.2.2). The dynamic program can be performed in pseudo-polynomial time $\mathcal{O}(\bar{T} \cdot \sum_{k \in [K]} s_k)$ [DW87, Sec. 3.5]. Here, the running time depends on the total number of passengers $\bar{T}$, which may be quite large, and on $\sum_{k \in [K]} s_k \leq \sum_{k \in [K]} \sum_{d \in D_k} G_d = \sum_{d \in D} G_d$, i.e., the total number of demand groups.

11.6 Summary

We summarize the results of Chapter 11 in Table 11.3. An algorithm to compute the Pareto front of F-RPM in pseudo-linear time is given in Algorithm 11.1. The Pareto front of D-RPM is computed in cubic time by Algorithm 11.2, and a MILP is provided in Section 11.3.1. For the price-setting subproblem of Z-RPM, we present an enumeration method as well as a pseudo-polynomial dynamic programming method based on the multiple choice knapsack problem in Section 11.5.2.

Furthermore, an optimal fixed price of a flat tariff can always be chosen as the willingness to pay of a demand group (Lemma 11.4). For an optimal solution of an affine distance tariff, the willingness to pay of at least two demand groups with different distances is met exactly; or the willingness to pay of at least one demand group is met exactly and either the distance price is zero or the base amount is zero (Theorem 11.10).

Chapter 12

Outlook

In Part II of this thesis, we study the single-objective fare deviation model and the bi-objective revenue-passenger model. The aim is to determine flat, affine distance or zone tariffs, where we minimize the weighted sum of absolute deviations from given reference prices or consider the trade-off between revenue and number of passengers, respectively.

While we often obtain similar results for the two models (see Sections 10.5 and 11.6), we want to highlight the difference in the case of the price-setting subproblem for zone tariffs. For the fare deviation model, we develop a linear time algorithm if the no-elongation property and the no-stopover property are not required and it is still solvable in polynomial time if both are required. On the other hand, for the revenue-passenger model, we prove that the problem is intractable and that the canonical decision problem of the price-setting subproblem is NP-complete.

Furthermore, the bi-objective revenue-passenger model explicitly considers the number of passengers in one objective, while the single-objective fare deviation model assumes a fixed demand and takes into account the passenger's interests through an objective function that deteriorates the more the fares exceed the reference prices. Because of the relation of the fare deviation model for flat, affine distance and zone tariffs to median problems, we often obtain that the tariffs optimized by the fare deviation model satisfy that the price increases (decreases) for at most half of the passengers (see Section 10.2, formula (10.3) and Theorem 10.24).

As an addition to the models for affine distance tariffs, future work could consider affine distance tariffs with a maximum price. This a particularly interesting in practice because fares that are unlimited are not desirable for passengers. For comparison, the price function of a zone tariff is in practice usually given by a price list and therefore has a maximum price limit.

To model the passenger information, we use OD data with one path per OD pair and, in case of the fare deviation model, with one reference price per OD pair. We remark that both the fare deviation model and the revenue-passenger model can also be applied for OD data with several paths with a fixed number of passengers each as well as with several reference prices per OD pair. This is possible by allowing the same combination of an origin and a destination multiple times and thus splitting the OD pairs so that again each OD pair has one path and one reference price.

In this thesis, we assume that the route choice of passengers is fixed, which reflects that the main decision criterion is the travel time and not the price of a journey. However, changes in the fare structure could lead to changes in the preferred paths. In a flat or affine metric/beeline distance tariff, the fare is independent of the actual path of the OD pair, and, in an affine network distance tariff, a shortest path is also always a cheapest path [SU22, Cor. 1]. On the other hand, in a zone tariff, the fare of a path does not only depend on the price function but also the zone partition. Therefore, it is not possible to determine a cheapest path a priori. Integrating route choice into the fare deviation model for zone tariffs and the revenue-passenger model for zones would hence be interesting for future research to further concern the passenger perspective during fare planning.

The developed fare deviation models have been added to the open-source software library LinTim [Sch+24; Sch+] and can be solved with the implemented MILP formulations presented here. First tests show that the fare deviation model for flat and affine distance tariffs can be solved quickly even for large instances. For the fare deviation model for zone tariffs, the running time using the solver Gurobi [Gur24] grows quickly from few minutes for an instance with 15 stations to instances with 20 to 30 stations that cannot be solved in reasonable time. Therefore, in addition to solving the MILP with a solver like Gurobi, an interesting working direction is the design of an alternative solution method for the fare deviation model for zone tariffs, such as Benders decomposition or a branch-and-bound algorithm. Both methods are, for example, used in [AM18] for determining groups of service locations and prices for transportation with an application in air cargo. Furthermore, symmetry breaking [SS01; Mar10; PR19] might help to reduce the running time.

For the revenue-passenger model, the computational experiments for flat and affine distance tariffs show that the algorithms exploiting the structure of the tariffs perform better than the MILP-based ϵ-constraint method in most cases with more than one non-dominated point. Computational experiments investigating the running times of the MILP-based ϵ-constraint method and the specialized algorithms for the price-setting subproblem of the revenue-passenger model for zone tariffs would be interesting.

Chapter 13

Conclusion

In this thesis, we consider two different topics of public transport planning, namely the optimization of infrastructure in the context of a bus rapid transit (BRT) line in Part I and tariff optimization with a focus on flat, affine distance and zone tariffs in Part II.

The common idea of the BRT investment model, the fare deviation model and the revenue-passenger model is to take the perspective of the passengers as well as of the operators. This is realized in the BRT investment model and the revenue-passenger model by considering two objective functions that reflect, on the one hand, the number of attracted passengers and, on the other hand, the financial interests of the operator in terms of a budget or revenue. For these models, a Pareto front showing the trade-off between both objectives is computed. For the single-objective fare deviation model, the absolute deviation from reference prices is considered, which thus accounts for price increases as well as price reductions.

Because optimization models serve as a decision support tool in practice, the amount of information provided by the models about different options is crucial. Thus, the author of this thesis deems the bi-objective models developed here better suited to support informed decisions, in particular, because they allow more insights into the impact of a decision on passengers. As stated in the Introduction (Chapter 1), the transition towards sustainable transport modes is a major task for which it is important to involve people. By providing a reliable and comfortable public transport service as well as affordable fares, the attractiveness of using public transport is increased and may engage more people's attention.

Another important aspect for the applicability of optimization models in decision-making processes is the regular comparison of the model with reality. Within the project EASIER (sEAmless SustaInable EveRyday urban mobility) it was possible to develop the BRT investment model based on the development of a new BRT line in Greater Copenhagen. Also for fare planning, discussions with the German public transport association saarVV took place, which, for example, showed that affine distance tariffs with a fixed maximum price are of interest in practice. Moreover, the implementation of the fare deviation models in the open-source software library LinTim [Sch+; Sch+24] improves the visibility and the availability of tariff optimization models for researchers and practitioners.

The integration of planning steps in public transport, as it is for example studied in [Sch20; Gra24], takes more information into account and considers interactions between different planning steps. Therefore, it has the potential to yield models that better match real-world requirements. The downside of such models is that they quickly grow in size and are even harder to solve than the separate problems. Nevertheless, we may discuss the question as to how the optimization of infrastructure fits in with the optimization of tariffs and how these problems complement and enhance each other. We discuss this for the two bi-objective models, namely the BRT investment model and the revenue-passenger model:

- First, as we have just established, a common ground are the objectives, namely a high ridership and passenger satisfaction as well as low investment costs or a high revenue, respectively.
- Second, for the BRT investment model, we consider two passenger responses to infrastructure upgrades of which one attracts the full potential demand if a certain threshold of upgrades is reached, which is called MinImprov. A more detailed passenger response to fares based on extended OD data (Definition 11.1) is considered in the revenue-passenger model, where several demand groups per OD pair are allowed, which models a piecewise constant passenger response with multiple threshold levels for each OD pair. Thus, MinImprov can be seen as a special case with only one demand group per OD pair.
- Third, the travel time reductions realized by upgrades through infrastructure investments and the choice of tariffs and fares are both important criteria for route and mode choice.

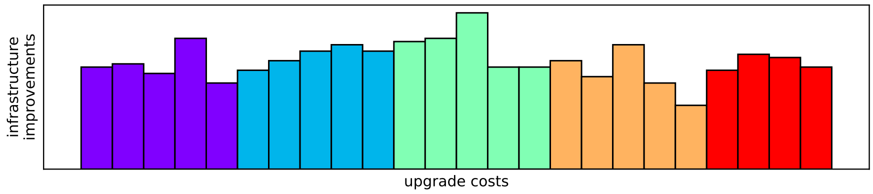
Based on the identified similarities, a combined model of infrastructure and tariff optimization could be interesting to consider, especially in a network context (instead of for a single line). A bi-objective model maximizing the number of passengers and maximizing the difference between revenue and investment costs would be a natural combination of the previous objectives. The demand could be modeled by extended OD data with a threshold for the acceptable travel time and a willingness to pay so that a demand group only uses public transport if the travel time and the fare are small enough. In this setting, an infrastructure upgrade might attract new passengers who in return pay fares so that the costs for the infrastructure are (partially) compensated by the newly acquired revenue. Especially interesting would be to also integrate the routing of the OD pairs based on the fares and the travel time. Overall, we see that there are interesting working directions that can be explored to develop models that reflect reality even better and contribute to improving public transport services.

Appendix

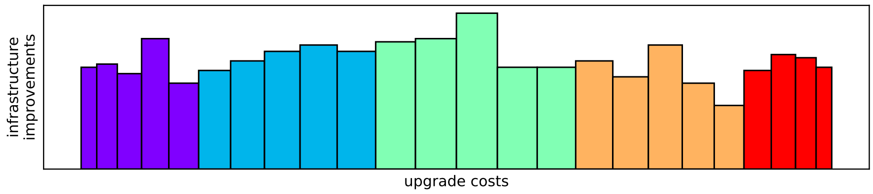
Appendix A

Computational Experiments: BRT Investment Model

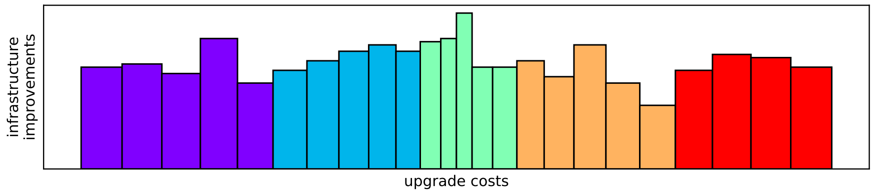
In Appendix A, additional information about the experiments on artificial instances (Section 6.1) and the Greater Copenhagen case study (Section 6.2) is provided. Figures A.1 and A.5 show the upgrade costs and the infrastructure improvements per segment as well as which segments belong to the same municipality for the artificial instances and the case study, respectively. Figure A.2 depicts the graphs of the artificial instances, marking the locations of stations with high demand within the demand patterns HUBS and TERMINI. Figure A.3 shows the corresponding histograms of the travel distances of the passengers. Moreover, as a supplement to Figure 6.6, Figure A.4 shows the evaluation of the non-dominated points of BRT($\star/Z \geq 1/|M| = 1$) for the artificial instances with the cost patterns ENDS and MIDDLE. Further plots that depict which segments are upgraded at certain investment budget levels are provided at `https://doi.org/10.11583/DTU.c.6805470`.



(a) UNIT



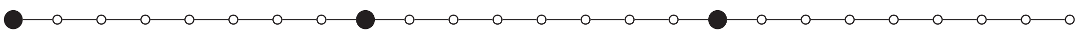
(b) MIDDLE



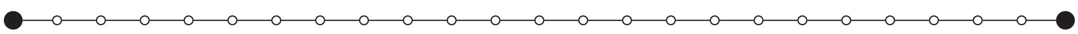
(c) ENDS

Figure A.1: Upgrade costs and infrastructure improvements per segment for the cost patterns UNIT, MIDDLE and ENDS for the artificial instances. Each bar represents a segment. The width of a bar represents the upgrade costs while the height reflects the infrastructure improvements. The colors indicate to which municipality a segment belongs.

(a) EVEN

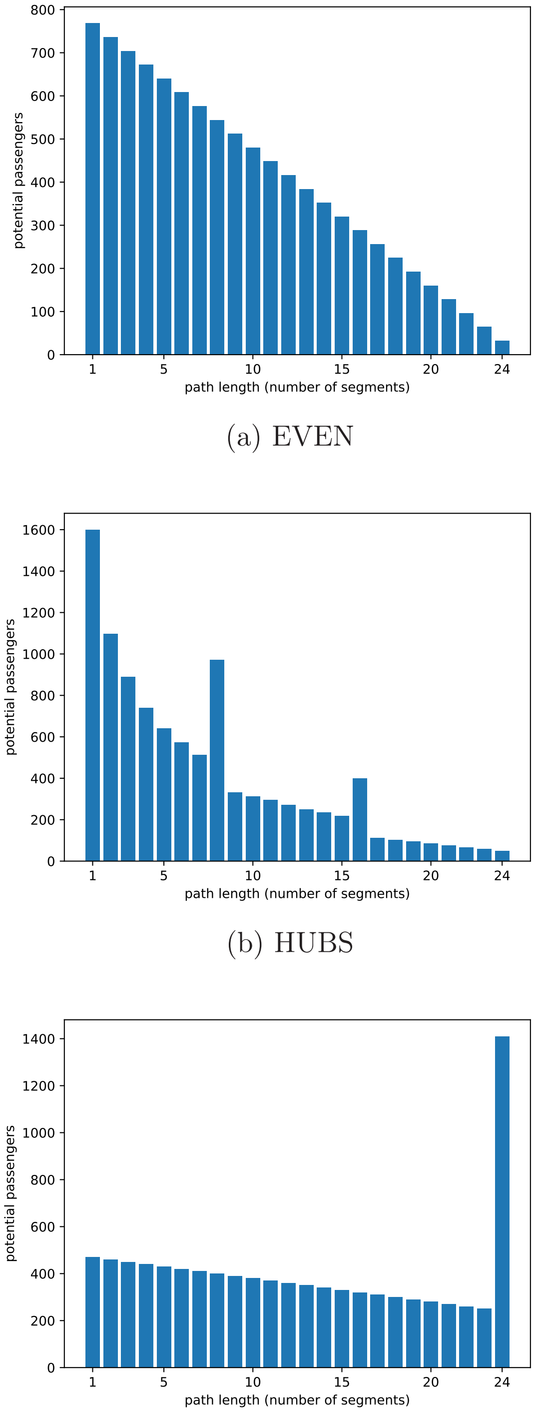


(b) HUBS



(c) TERMINI

Figure A.2: Line graph of the artificial instances. Stations with a high demand in the demand patterns HUBS and TERMINI are marked with filled black nodes. There are no large stations in demand pattern EVEN.

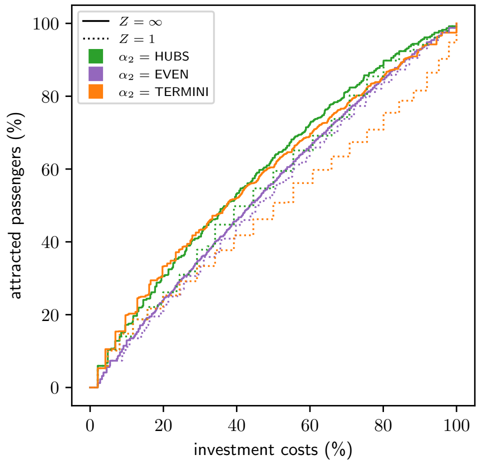


(a) EVEN

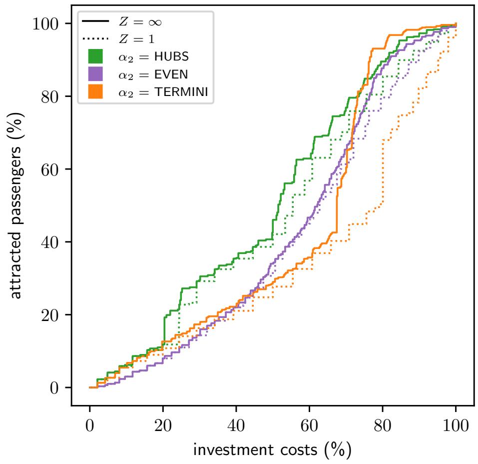
(b) HUBS

(c) TERMINI

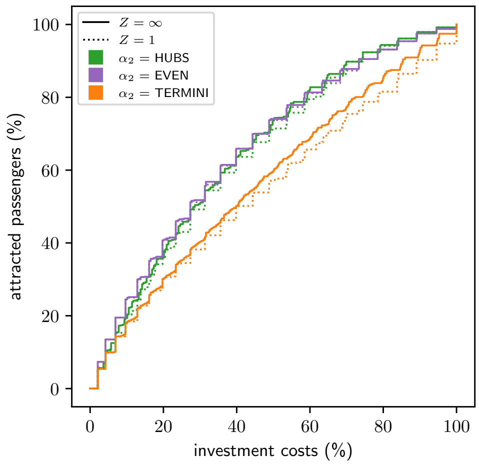
Figure A.3: Histogram of the travel distances of passengers. The height of a bar gives the demand of passengers traveling for a certain number of segments.



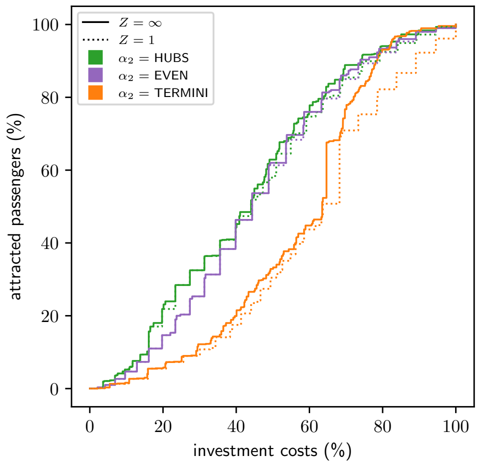
(a) LINEAR, α_1 = MIDDLE.



(b) MINIMPROV, α_1 = MIDDLE.

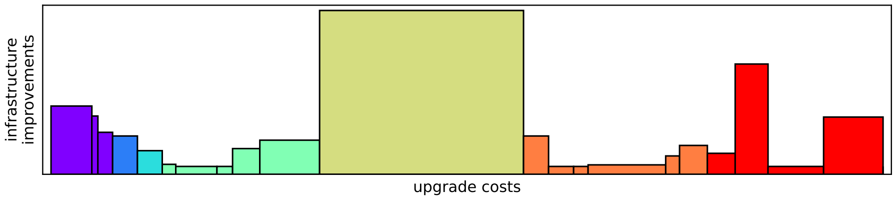


(c) LINEAR, α_1 = ENDS.

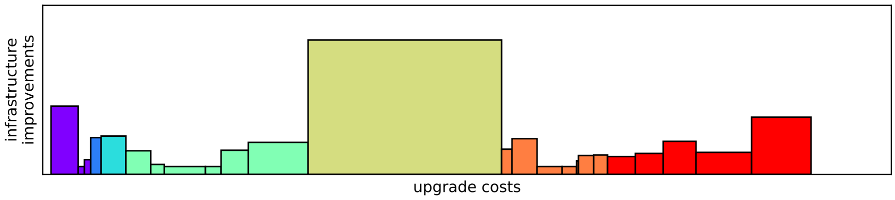


(d) MINIMPROV, α_1 = ENDS.

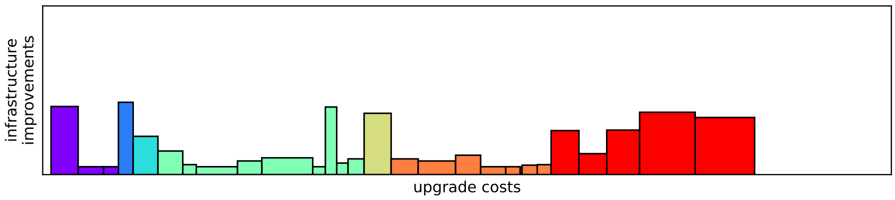
Figure A.4: Evaluation of the non-dominated points of BRT($\star/Z \geq 1/|M| = 1$) for artificial instances with cost pattern $\alpha_1 \in \{\text{ENDS}, \text{MIDDLE}\}$ and $Z \in \{1, \infty\}$. Both attracted passengers and investment costs are given as percentage of the total number of potential passengers and costs for upgrading all segments, respectively.



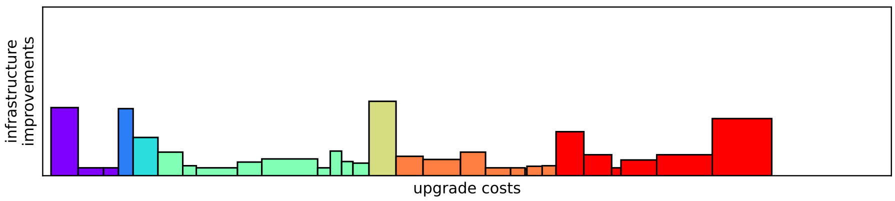
(a) Alternative 1



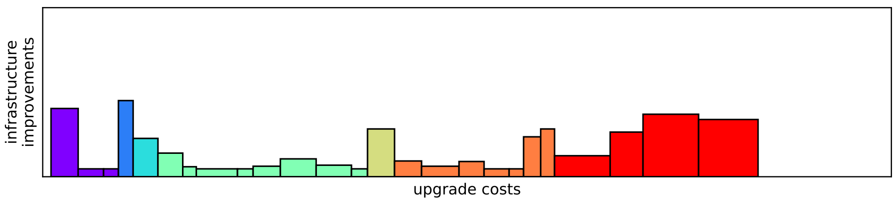
(b) Alternative 2



(c) Alternative 3



(d) Alternative 4



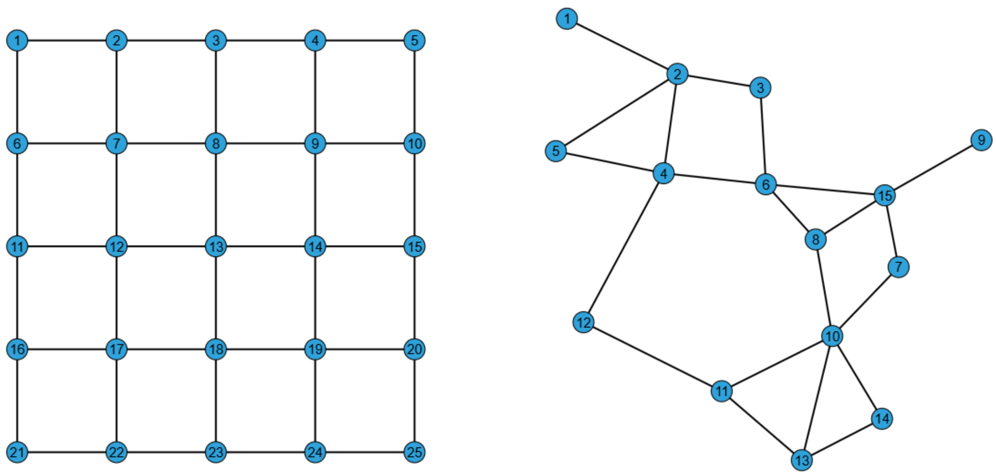
(e) Alternative 5

Figure A.5: Cost patterns and infrastructure improvements per segment for the five route alternatives from north (Aldershvilevej) to south (Ishøj St.). Each bar represents a segment. The width of a bar represents the upgrade costs while the height reflects the infrastructure improvements. The colors indicate to which municipality a segment belongs. Note that the two non-upgradable segments in Lyngby municipality are excluded.

Appendix B

Computational Experiments: Revenue-Passenger Model

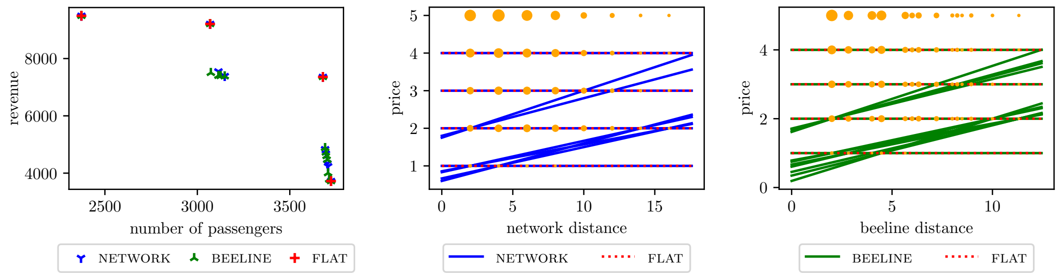
Appendix B contains supplementary material for the computational experiments conducted for F-RPM and D-RPM in Section 11.4. Figure B.1 shows the PTNs of the datasets `grid` and `mandl` used in the computational experiments in Section 11.4.



(a) `grid` (b) `mandl`

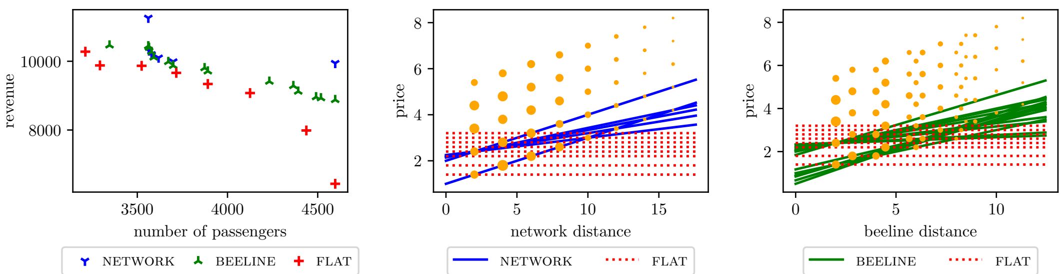
Figure B.1: PTNs of the LinTim datasets `grid` and `mandl`.

For investigating the Pareto fronts, efficient tariffs and the input data in Section 11.4, Figure B.2 to Figure B.5 show the Pareto fronts in (a) and corresponding efficient solutions in (b) and (c) for selected parameter settings for the `grid` instances. Additionally, (b) and (c) show the demand as points (l_d, w_d^g) weighted with the number of potential passengers t_d^g.



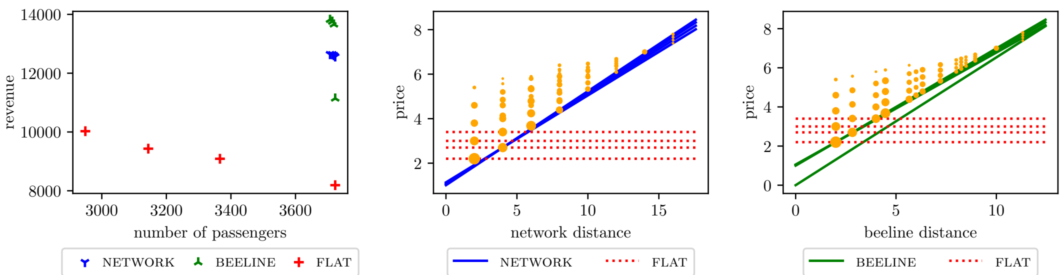
(a) Pareto fronts (b) NETWORK and FLAT (c) BEELINE and FLAT

Figure B.2: Instance of `grid` with parameters 5/INCREASING/w-FLAT/A.



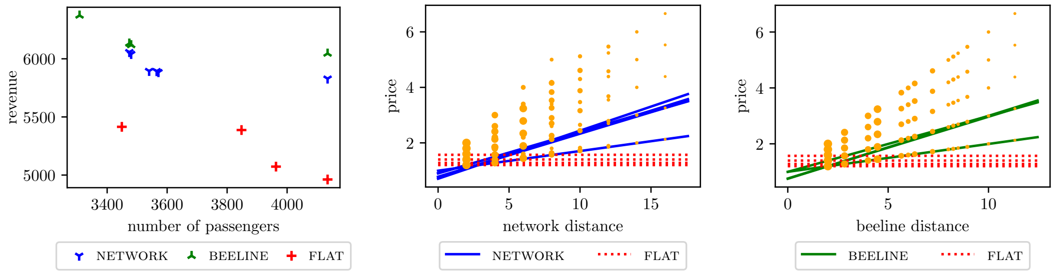
(a) Pareto fronts (b) NETWORK and FLAT (c) BEELINE and FLAT

Figure B.3: Instance of `grid` with parameters 5/RANDOM/w-NETWORK/A.



(a) Pareto fronts (b) NETWORK and FLAT (c) BEELINE and FLAT

Figure B.4: Instance of `grid` with parameters 5/DECREASING/w-BEELINE/B.



(a) Pareto fronts (b) NETWORK and FLAT (c) BEELINE and FLAT

Figure B.5: Instance of `grid` with parameters 5/EQUAL/w-BEELINE/C.

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