Dilute Instanton Gas of an O(3) Skyrme Model

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Abstract

The pure-Skyrme limit of a scale-breaking Skyrmed $O(3)$ sigma model in 1+1 dimensions is employed to study the effect of the Skyrme term on the semiclassical analysis of a field theory with instantons. The instantons of this model are self-dual and can be evaluated explicitly. They are also localised to an absolute scale, and their fluctuation action can be reduced to a scalar subsystem. This permits the explicit calculation of the fluctuation determinant and the shift in vacuum energy due to instantons. The model also illustrates the semiclassical quantisation of a Skyrmed field theory.

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1 Introduction

The physical motivation of the present work is to carry out the instanton analysis for a field theory described by a Lagrangian that includes Skyrme-like kinetic terms. Such systems are expected to arise as low energy effective theories, and are motivated by the physics at very high energy scale as described by superstring theory. The semiclassical analysis of physically interesting or relevant gauge field theories of this sort in 3+1 dimensions is very difficult to handle, so instead we consider an ungauged system in 1+1 dimensions as a toy model. We choose to work with such a model that supports exponentially localised instantons which can be evaluated analytically. Furthermore the eigenvalue problem for the fluctuation operator can be solved in closed form. This enables us to evaluate the (regularised) determinants exactly and hence to find the shift in vacuum energy due to instantons. Thus apart from affording us an example of the semiclassical quantisation of a theory with a Skyrme term, the model under consideration is remarkable in that it allows the complete semiclassical (instanton) analysis of a field theory. We think that this feature is of interest in its own right.

The model we employ, which was introduced in Ref. [1], is a Skyrmed version of a scale-breaking $O(3)$ sigma model [2]. While the latter system [2] does not support finite-sized instantons, our model [1] does. These instantons are not self-dual and are evaluated only numerically. In the present work, where we need to take the analysis as far as possible, it is desirable that the instantons be explicitly evaluated. This leads us to consider what we call the pure-Skyrme limit of the model in Ref. [1], described by the Lagrangian

$$\mathcal{L} = \frac{\lambda_2}{4\kappa^2}(\phi^{ab}_{\mu\nu})^2 + \lambda_1(\phi^a_\mu)^2 + \lambda_0\kappa^2 V(\phi^3) \tag{1}$$

in the $\lambda_1 \to 0$ limit. In (1) we have used the notation $\phi^a_\mu = \partial_\mu \phi^a$, $\phi^{ab}_{\mu\nu} = \phi^a_{[\mu}\phi^b_{\nu]}$, and of course the field ϕ is subject to $\phi^a\phi^a = 1$, with $a = 1, 2, 3$. κ is a constant with the dimension of inverse length, and the coupling constants λ_0, λ_1 and λ_2 are dimensionless. The choice of the potential $V(\phi^3)$ which breaks the $O(3)$ symmetry to $SO(2)$ specifies the model.

The radially symmetric vorticity N instanton solutions of this model were found numerically in Ref. [1]. Subsequently the self-dual solutions of the pure-Skyrme limit with $\lambda_1 \to 0$ in (1) were found analytically in Ref. [3], where it was also shown that these solutions were approximated by the numerically found solutions for small values of λ_1, continuously. This property was established *only for unit vorticity* $|N| = 1$.

In the present work, where we propose to employ the analytic solutions of the pure-Skyrme model

$$\mathcal{L} = \frac{\lambda_2}{4\kappa^2}(\phi^{ab}_{\mu\nu})^2 + \lambda_0\kappa^2 V(\phi^3) \tag{2}$$

as instantons, we need to give some justifications. The problem is that the vacuum field ϕ^a must be time independent, so in the context of our Euclidean space formulation, it must also be x_μ independent. This is assured by the vanishing of the quadratic kinetic term multiplying λ_1 in (1) in the asymptotic (vacuum) region. (Note that the vanishing of the quartic kinetic (Skyrme) term does not imply this.) Thus if we are to employ the self-dual solutions to (2) as instantons, we must consider the system (2) as an approximation to (1) with very small λ_1. This is justified by the fact that the topologically stable finite action solutions to (2) approximate those of (1) for very small λ_1 [3]. The only restriction is that this situation holds only for $|N| = 1$ instantons, but given that the latter are *absolutely* localised, it will be necessary to consider only unit topological charge instantons anyway. This justifies our use of the pure-Skyrme model (2) and its solutions as instantons, provided it is understood that (2) is seen as an approximation of (1).

Before we proceed, it may be relevant to note that in the limit of latter, model [4] being scale invariant, supports instantons featuring an arbitrary scale, while (1) and the model (2) employed here, are not scale invariant and hence support instantons localised to an absolute scale. Thus the instantons of both models (1) and (2) have *finite size* so that there is no need for us to constrain the size of instantons [9] as is necessary when the instantons are not fixed to an absolute scale. Another relevant remark is that the model (2) does not support either sphalerons, as in Ref. [2], or periodic instantons [6], unlike (1). The periodic instantons of (1) were found in Ref. [7], while the sphaleron induced thermal transitions in the model (1) were analysed in Ref. [8]. Here we will be concerned only with the (zero temperature) *finite size* instantons of model (2).

Accordingly, we have a 2 dimensional model at hand whose topologically stable finite action solutions can be considered to be the instantons of the theory. These instantons are known analytically, hence making the task of calculating the fluctuation determinant tractable, although their special symmetry properties result in an infinitely degenerate fluctuation operator, yielding a system of only one scalar fluctuation field instead of the two fluctuation parameter fields one expects for an $O(3)$–σ model.

Furthermore, the localisation of these instantons to an absolute scale prevents infrared divergences arising from the integration over the scale collective coordinate, with the result that it is possible to evaluate the vacuum energy by employing a dilute gas of unit topological charge instantons.

In Section 2 we present the instanton solutions and expose their symmetries. In Section 3 we derive the matrix fluctuation operator around the instanton solution using a convenient parametrisation and discuss the metric on the parameter space of fluctuations. In Section 4, we calculate the normalised determinant, first discussing the infinite degeneracy of the fluctuation operator. After the unphysical degrees of freedom related to the special symmetries of the model have been removed, we determine the spectrum of the resulting scalar fluctuation operator using zeta function regularisation [10, 11] and rescaling normalisation. Finally, we calculate the shift in vacuum energy due to a dilute instanton gas.

2 The instantons

The analytic evaluation of the instantons will be carried out for the family of models (2) specified by the potentials

$$V(\phi^3) = (1-\phi^3)^k, \qquad k = 2, 3, \ldots. \tag{3}$$

For the case $k = 1$ there does not exist a solution which is analytic at the origin, hence $k = 1$ in (3) is excluded.

The equations of motion of the models (2)–(3) are satisfied by the solutions of the following Bogomol'nyi equations

$$\sqrt{\lambda_2}\varepsilon_{\mu\nu}\varepsilon^{abc}\partial_\mu\phi^a\partial_\nu\phi^b\phi^c - \sqrt{\lambda_0}(1-\phi^3)^{\frac{k}{2}} = 0 \tag{4}$$

The topological charges, which are equal to the action, are given by the volume integrals of the densities

$$\varrho = 2\sqrt{\lambda_2\lambda_0}\varepsilon_{\mu\nu}\varepsilon^{abc}\partial_\mu\phi^a\partial_\nu\phi^b\phi^c(1-\phi^3)^{\frac{k}{2}}. \tag{5}$$

Before proceeding to solve (4), we note that the solutions of the self-duality equation (4) satisfy an infinite dimensional symmetry, which will be treated as an unphysical symmetry in Section 4. This is in contrast with the physical, finite dimensional, symmetries of (2) under spacetime translations and rotations. It is straightforward to check that (4) is invariant under

$$\phi^\alpha \to R^\alpha{}_\beta \phi^\beta \qquad \phi^3 \to \phi^3, \tag{6}$$

in which the $SO(2)$ rotation $R^\alpha{}_\beta = R^\alpha{}_\beta(|\phi^\gamma|)$ is a function of the modulus $|\phi^\alpha|$ of the functions ϕ^α, i.e. there is a local rotation symmetry in the inner symmetry space besides the usual global $O(3) \to SO(2)$ broken symmetry of the model which is equivalent to spacetime rotations. In Section 4, we shall find that this local symmetry results in a degenerate fluctuation operator around the solutions of equation (4).

We now proceed to solve (4) and to that end consider the radially symmetric field configuration of vorticity N

$$\phi^\alpha = \sin f(\rho)\, n^\alpha, \qquad \phi^3 = \cos f(\rho) \tag{7}$$

where we have used the dimensionless radial variable $\rho = \kappa r$ defined in terms of κ appearing in (1)–(2), and where $n^\alpha = (\cos N\theta, \sin N\theta)$ is a unit vector with winding number N. The Euler-Lagrange equations of this system for a particular choice of the potential $V(\cos f)$ were found in [3], subject to the asymptotic conditions

$$\lim_{\rho\to 0} f(\rho) = \pi, \qquad \lim_{\rho\to\infty} f(\rho) = 0. \tag{8}$$

In the radially symmetric field configuration (7), the reduced Bogomol'nyi (self– or antiself–dual) equation and the topological charge density of the resulting 1-dimensional subsystem corresponding to (4) and (5) respectively are

$$\frac{N}{\rho} \sin f \frac{df}{d\rho} = \pm\beta(1 - \cos f)^{\frac{k}{2}} \quad \text{with} \quad \beta = \sqrt{\frac{\lambda_0}{\lambda_2}} \tag{9}$$

$$\varsigma = 4\pi N\sqrt{\lambda_2\lambda_0}(1 - \cos f)^{\frac{k}{2}} \sin f \frac{df}{d\rho}. \tag{10}$$

For $N > 0$ we get solutions satisfying the boundary conditions (8) only if we choose the negative sign in (9). We call these solutions instantons although they solve the antiself–dual equation. The antiinstantons then, with $N<0$, are the solutions of the self–dual equation with the positive sign in (9). Instantons and antiinstantons hence differ in the direction of the vorticity n^α (7), but not in their radial behaviour $f(\rho)$.

The resulting solutions are

$$
\begin{aligned}
f(\rho) &= \arccos\left(1-2e^{-\frac{\beta}{2|N|}\rho^2}\right), \qquad k=2 & (11)\\
f(\rho) &= \arccos\left(1-\left[\frac{\beta}{2|N|}\frac{k-2}{2}\rho^2+2^{\frac{2-k}{2}}\right]^{\frac{2}{2-k}}\right), \qquad k>2 & (12)
\end{aligned}
$$

satisfying the asymptotic conditions (8) as required, and resulting in the topological charges N times the normalisation factor appearing in (10) multiplied further by $\frac{2^{\frac{k}{2}+1}}{\frac{k}{2}+1}$ for each $k>2$.

The functions (11)–(12) will be genuine solutions to the full equations (4) only if they describe fields ϕ^a that are singlevalued at the origin. It is easy to find the behaviour of $f(\rho)$ in the region $\rho \ll 1$ for the solutions (11)–(12). The conclusion is that for all these solutions we have the following behaviour for $\sin f(\rho)$

$$
\sin f \sim \rho, \qquad \rho \ll 1. \tag{13}
$$

This means that the solution field ϕ^a given by (7) is singlevalued at the origin *only for vorticity* $|N|=1$. We must therefore reject all solutions (11)–(12) *except those of unit vorticity*. This is perfectly consistent with our intention of considering a dilute gas of widely separated instantons and anti-instantons of *unit topological charge*, localised to an absolute scale $\kappa = \rho/r$.

We shall restrict henceforth to the $N=1$ instanton of the model (2) characterised by the potential (3) with $k=2$. The latter choice is made because the instantons (11) of that model are localised *exponentially* rather than by a *power* decay as in (12). The action of this instanton is readily calculated to be

$$
S_0(\lambda_2,\lambda_0) = \frac{8\pi}{\beta}\lambda_0. \tag{14}
$$

3 Fluctuations around the instanton and the vacuum–vacuum transition rate

To evaluate the contributions of quantum fluctuations around the background instanton field (11), we find it convenient to use a parametrisation of the fields ϕ^a which satisfies the constraint $\phi^a\phi^a = 1$ automatically, so that we do not have to take this constraint into account by introducing the Lagrange multiplier implicit in (1) and (2). Note that it would have been necessary to employ the Lagrange multiplier had we tried to solve the Euler-Lagrange equations of (2), but it was not necessary to do so when we solved the self-duality equation (4).

Such a parametrisation is

$$\phi^a = \begin{pmatrix} \phi^\alpha \\ \phi^3 \end{pmatrix} = \begin{pmatrix} \sin\Theta\cos\Phi \\ \sin\Theta\sin\Phi \\ \cos\Theta \end{pmatrix} \tag{15}$$

in terms of which the Lagrangian (2) specified by the potential (3) with $k = 2$ is expressed as

$$\mathcal{L} = \frac{\lambda_2}{4\kappa^2}|\partial_{[\mu}\Theta\partial_{\nu]}\Phi|^2\sin^2\Theta + \lambda_0\kappa^2(1-\cos\Theta)^2. \tag{16}$$

The instanton solution in whose background we calculate the second variation of (16) is

$$\Theta = f(\rho), \qquad \Phi = \theta, \tag{17}$$

where $f(\rho)$ is given by (11).

The Lagrangian (16) has all the invariances of (2), namely spacetime translations and rotations, as well as the infinite dimensional local rotation symmetry which in terms of the parameter fields is given by θ dependent translations of the field Φ. Whereas the former spacetime symmetries are broken by the solution (17), the Bogomol'nyi equation (9) and hence the solution itself is invariant under the latter local symmetry.

Fluctuations $\delta\phi^a$ around a configuration ϕ^a are related to fluctuations $\delta\Theta$, $\delta\Phi$ of the parameter fields by

$$\delta\phi^a = e^a_\Theta(\Theta,\Phi)\delta\Theta + e^a_\Phi(\Theta,\Phi)\delta\Phi \tag{18}$$

with

$$e_{\Theta}^{a}(\Theta,\Phi) := \begin{pmatrix} \cos\Theta\cos\Phi \\ \cos\Theta\sin\Phi \\ -\sin\Theta \end{pmatrix}, \quad e_{\Phi}^{a}(\Theta,\Phi) := \begin{pmatrix} -\sin\Theta\sin\Phi \\ +\sin\Theta\cos\Phi \\ 0 \end{pmatrix}. \tag{19}$$

Such fluctuations are elements of the tangent space to the inner symmetry space S^2 of the $O(3)$–σ model, $\delta\phi^a \in \mathrm{T}_{\phi^a}\mathrm{S}^2$. The parametrisation (15) induces a basis in this tangent space. With respect to this basis, the inner product of parameter tangent space vectors $\vec{\xi}_i$ is given by $\langle\vec{\xi}_1,\vec{\xi}_2\rangle = \vec{\xi}_1^T\hat{\mathcal{G}}\vec{\xi}_2$ with the induced metric

$$\hat{\mathcal{G}} = \begin{pmatrix} e_{\Theta}^{a}(f,\theta) \\ e_{\Phi}^{a}(f,\theta) \end{pmatrix} (e_{\Theta}^{a}(f,\theta), e_{\Phi}^{a}(f,\theta)) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 f \end{pmatrix} \tag{20}$$

The Skyrme term in the parameter field Lagrangian (16) is related to this inner product since it may be written as the square of the totally antisymmetric wedge product defined on the basis of this inner product.

It is now laborious but straightforward to calculate the second variation $\delta^{(2)}S$ of the action in the background of (17) which is the quadratic fluctuation contribution around the classical instanton solution. Denoting the variations of Θ and Φ by $\delta\Theta = u(\rho,\theta)$ and $\delta\Phi = v(\rho,\theta)$, we find

$$\delta^{(2)}S[u,v] = \int (u,v)\hat{\mathcal{M}}\begin{pmatrix} u \\ v \end{pmatrix}\rho d\rho d\theta \tag{21}$$

with the 2×2 matrix differential operator

$$\hat{\mathcal{M}} := \begin{pmatrix} \lambda_2\hat{L}^\dagger(1-\cos f(\rho))^2\hat{L} & -\sqrt{\lambda_0\lambda_2}\hat{L}^\dagger(1-\cos f(\rho))^2\frac{\partial}{\partial\theta} \\ \sqrt{\lambda_0\lambda_2}(1-\cos f(\rho))^2\hat{L}\frac{\partial}{\partial\theta} & -\lambda_0(1-\cos f(\rho))^2\frac{\partial^2}{\partial\theta^2} \end{pmatrix} \tag{22}$$

where $\hat{L}$ is a first–order differential operator,

$$\hat{L} := \frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{\sin f(\rho)}{1-\cos f(\rho)}, \qquad \hat{L}^\dagger = -\frac{\sin f(\rho)}{1-\cos f(\rho)}\frac{1}{\rho}\frac{\partial}{\partial\rho}. \tag{23}$$

The dagger denotes the adjoint in the corresponding Hilbert space. In particular, $\hat{\mathcal{M}}^\dagger = \hat{\mathcal{M}}$ is self–adjoint in the parameter space of the fluctuations with respect to the usual inner product which appears in eq. (21).

To calculate the one instanton vacuum–vacuum transition rate, one has to insert the fluctuation expansion of the Euclidean action around the instanton solution into the path integral,

$$\begin{aligned}\langle 0|0_V\rangle_1(\lambda_2,\lambda_0) &= \int_{|0\rangle \stackrel{1Inst}{\to} |0_V\rangle} \mathcal{D}\{\phi\} e^{-S[\phi]} \\ &= e^{-S_0(\lambda_2,\lambda_0)} \int \mathcal{D}\{u\}\mathcal{D}\{v\} e^{-\delta^{(2)} S[u,v]} = \frac{e^{-S_0(\lambda_2,\lambda_0)}}{\sqrt{\det \hat{\mathcal{M}}}} \qquad (24)\end{aligned}$$

where the Euclidean fluctuation path integral results in the determinant of $\hat{\mathcal{M}}$ in the usual way. V denotes the Euclidean spacetime volume of the instanton interpolating between the two vacuum states.

Substituting the fluctuation parameter fields u, v for the original field ϕ one has to take into account the symmetry properties of the parameter field measure $\mathcal{D}\{u\}\mathcal{D}\{v\}$ which is related to the induced metric $\hat{\mathcal{G}}$ on the space of the fluctuation fields. Hence expanding the fluctuations (u, v) in terms of the eigenfunctions $\vec{\xi}_i$ of $\hat{\mathcal{M}}$ with respect to the measure $\hat{\mathcal{G}}$, the determinant could be calculated as the product of the eigenvalues in the equation

$$\hat{\mathcal{M}}\vec{\xi}_i = \tilde{\omega}_i^2 \hat{\mathcal{G}}\vec{\xi}_i. \qquad (25)$$

The corresponding eigenfunctions can be orthonormalised with respect to the metric $\hat{\mathcal{G}}$.

In general, the eigenvalue problem (25) is difficult to solve. The only eigenfunctions which are easy to find are the physical zero modes related to the translational and rotational symmetries of the Lagrangian broken by the instanton solution $(\Theta, \Phi) = (f(r), \theta)$. In terms of this solution, the physical zero modes normalised with respect to the metric $\hat{\mathcal{G}}$ are given by

$$\vec{\xi}_x = \frac{1}{\kappa\sqrt{c}}\frac{\partial}{\partial x}\begin{pmatrix} f(r) \\ \theta \end{pmatrix}, \quad \vec{\xi}_y = \frac{1}{\kappa\sqrt{c}}\frac{\partial}{\partial y}\begin{pmatrix} f(r) \\ \theta \end{pmatrix}, \quad \vec{\xi}_\theta = \sqrt{\frac{\beta}{4\pi}}\frac{\partial}{\partial \theta}\begin{pmatrix} f(r) \\ \theta \end{pmatrix} \qquad (26)$$

where

$$c = 4\pi\left(\frac{\pi^2}{12} + \log 2\right), \qquad \beta = \sqrt{\frac{\lambda_0}{\lambda_2}}. \qquad (27)$$

These zero modes correspond to the fact that the instanton can be localised at an arbitrary spacetime point $x_0, y_0 \in V$ with angular direction $\theta_0 \in (0, 2\pi)$.

4 Evaluation of the fluctuation determinant: factorisation and regularisation

Instead of solving the eigenvalue equation (25) for the complete spectrum and calculating the determinant as product of all eigenvalues, we now proceed in a different way, exploiting the important feature that the matrix operator $\hat{\mathcal{M}}$ can be factorised,

$$\hat{\mathcal{M}} = \hat{\mathcal{N}}^\dagger \hat{\mathcal{N}}, \qquad \hat{\mathcal{N}} = (1 - \cos f(\rho)) \begin{pmatrix} \sqrt{\lambda_2}\hat{L} & -\sqrt{\lambda_0}\frac{\partial}{\partial\theta} \\ 0 & 0 \end{pmatrix}. \tag{28}$$

It is easy to check that $\hat{\mathcal{G}}\hat{\mathcal{N}} = \hat{\mathcal{N}}$, hence we can evaluate $\det \hat{\mathcal{M}}$ as product of the eigenvalues of the equation

$$\hat{\mathcal{M}}\vec{\chi}_i = \omega_i^2 \vec{\chi}_i. \tag{29}$$

Therefore the metric $\hat{\mathcal{G}}$ enters the final result only through the normalisation of the zero modes of $\hat{\mathcal{M}}$.

When the eigenvalue equation (29) is acted on from the left by the operator $\hat{\mathcal{N}}$ defined by (28), the resulting eigenvalue equation (for the new eigenfunction) has the same eigenvalues, now, however, with eigenfunctions $\vec{\psi}_i = \hat{\mathcal{N}}\vec{\chi}_i$. Since we are only interested in evaluating the determinant, i.e. in the eigenvalues but not in the eigenfunctions, we choose to work with the second eigenvalue equation

$$\hat{\mathcal{N}}\hat{\mathcal{N}}^\dagger \psi_n = \omega_n^2 \psi_n. \tag{30}$$

The new eigenvalue equation (30) has a much simpler structure than the previous one (29). This is because the operator

$$\hat{\mathcal{N}}\hat{\mathcal{N}}^\dagger = (1 - \cos f) \begin{pmatrix} \hat{H} & 0 \\ 0 & 0 \end{pmatrix} (1 - \cos f) \tag{31}$$

with

$$\hat{H} = \lambda_2 \hat{L}\hat{L}^\dagger - \lambda_0 \frac{\partial^2}{\partial\theta^2}, \tag{32}$$

($\hat{L}$ and $\hat{L}^\dagger$ given by (23)) is diagonal, and even more importantly, has only one nonvanishing diagonal element. This means the operator $\hat{\mathcal{N}}\hat{\mathcal{N}}^\dagger$ is infinitely degenerate due to the area (2 dimensional volume) preserving coordinate transformations resulting in the physical zero modes (26), and the transformations (6).

Hence for every eigenvalue we have an eigenspace rather than an eigenvalue, each element of this space arising from the action of the elements of the infinite dimensional symmetries on a definite eigenfunction in that eigenspace. We shall keep only one function from every such eigenspace, which amounts to the elimination of unphysical degrees of freedom. This amounts to rejecting the zero diagonal element in $\hat{\mathcal{N}}\hat{\mathcal{N}}^{\dagger}$, with the result that the determinant does not vanish.

As usual we shall treat the translational and rotational symmetries as physical, and will take their zero-modes into consideration, integrating over the collective cooridnates x_0, y_0 and θ_0 later on. Here, we exclude also these last zero eigenvalues from the operator $\hat{\mathcal{N}}\hat{\mathcal{N}}^{\dagger}$, and denote the resulting determinant by $\det'\hat{\mathcal{M}}$,

$$\det{}'\hat{\mathcal{M}} = \det\left[(1-\cos f)\hat{H}(1-\cos f)\right]. \tag{33}$$

In fact, this reduces the fluctuation analysis from two parameter fluctuation fields to only one scalar fluctuation field and hence simplifies the analysis considerably.

Next, there arises the question of the normalisation of the determinant. Since our theory (2) does not have a perturbative vacuum, we are forced to choose an arbitrary normalisation point. In terms of the two dimensionless coupling constants (λ_0, λ_2), we choose this point to be (λ_0', λ_2') satisfying

$$\frac{\lambda_0'}{\lambda_2'} = \frac{\lambda_0}{\lambda_2}, \tag{34}$$

amounting to a rescaling of the Lagrangian (2), but keeping $\beta^2 = \lambda_0/\lambda_2$ fixed. As a result, the β–dependent divergent contribution of $\det(1-\cos f)^2$ to $\det'\hat{\mathcal{M}}$ cancels in the normalised one instanton vacuum–vacuum transition rate

$$\frac{\langle 0|0_V\rangle_1(\lambda_2,\lambda_0)}{\langle 0|0_V\rangle_1(\lambda_2',\lambda_0')} = e^{-[S_0(\lambda_2,\lambda_0)-S_0(\lambda_2',\lambda_0')]}\left[\frac{\det'\hat{\mathcal{M}}(\lambda_2,\lambda_0)}{\det'\hat{\mathcal{M}}(\lambda_2',\lambda_0')}\right]^{-\frac{1}{2}}\mathcal{Z} \tag{35}$$

where $\mathcal{Z}$ denotes the contribution of the zero modes.

Hence the evaluation of $\det' \hat{\mathcal{M}}$ reduces further to the evaluation of the product of the eigenvalues of the following eigenvalue equation

$$\hat{H}\psi = \omega^2\psi$$

or

$$\Big(-\lambda_2 \frac{1}{\rho}\frac{\partial}{\partial\rho}\frac{1+\cos f}{1-\cos f}\frac{1}{\rho}\frac{\partial}{\partial\rho} - \lambda_0 \frac{\partial^2}{\partial\theta^2}\Big)\psi = \frac{\omega_n^2}{\kappa^2}\psi. \tag{36}$$

Using the notation $z = \cos f(\rho)$ and looking for separable solutions $\psi(z,\theta) = \mathrm{e}^{im\theta}P(z)$, with $m = 0, 1, 2, ...$, the eigenvalue equation (36) reduces to the ordinary diferential equation

$$\lambda_0\kappa^2(1-z)\frac{d}{dz}(1+z)\frac{d}{dz}P(z) = (\omega^2 - \lambda_0\kappa^2 m^2)P(z). \tag{37}$$

The solutions of this equation are the Jacobi polynomials, yielding the eigenvalue spectrum

$$\omega_{n,m}^2 = \lambda_0\kappa^2(n^2 + m^2), \tag{38}$$

with the integer $n = 1, 2, 3, \ldots$.

Since the spectrum of the operator $\hat{H}$ is discrete, we will use the method of Zeta function regularisation to evaluate its determinant. The generalised Zeta function for this operator is

$$\zeta_{\hat{H}}(s) = \sum_{n,m} \omega_{n,m}^{-s}. \tag{39}$$

This series converges for $\mathrm{Re}s \geq \frac{5}{2}$ (see (44) below) and can be analytically continued to a meromorphic function of s which has no singularity at $s = 0$. This allows us to define the determinant as follows

$$-\frac{1}{2}\log\det\hat{H} = \frac{d}{ds}\zeta_{\hat{H}}(s)|_{s=0}. \tag{40}$$

The summation over m in (39) yields

$$\zeta_{\hat{H}}(s) = (\lambda_0\kappa^2)^{-\frac{s}{2}}\frac{2\pi}{\Gamma(\frac{s}{2})}\sum_{n=1}^{\infty}(2n)^{\frac{1-s}{2}}\int_0^\infty \frac{t^{\frac{s}{2}-1}}{e^t - 1}J_{\frac{s-1}{2}}(nt)dt + (\lambda_0\kappa^2)^{-\frac{s}{2}}\zeta(s), \tag{41}$$

in which $J_{\frac{s-1}{2}}(nt)$ is a Bessel function and $\zeta(s)$ is the usual Zeta function. After summing over n in (41), we have

$$\zeta_{\hat{H}}(s) = (\lambda_0\kappa^2)^{-\frac{s}{2}}[2h_1(s) + 2h_2(s) + 2h_3(s) + \zeta(s)], \tag{42}$$

where

$$\begin{aligned}
h_1(s) &= \sqrt{\pi}\frac{2^{-s}}{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})}\int_0^\infty \frac{t^{s-\frac{3}{2}}}{e^t-1}dt && (43)\\
h_2(s) &= 2\pi\frac{2^{-s}}{\Gamma^2(\frac{s}{2})}\int_0^\infty \frac{t^{s-\frac{5}{2}}}{e^t-1}dt && (44)\\
h_3(s) &= 2\pi\frac{2^{-s}}{\Gamma^2(\frac{s}{2})}\sum_{n=1}^{\infty}\int_{2\pi n}^{2\pi(n+1)} \frac{(t^2-4\pi^2n^2)^{\frac{s}{2}-1}}{e^t-1}\frac{dt}{\sqrt{t}}. && (45)
\end{aligned}$$

The function $\zeta_{\hat{H}}(s)$ comprises radial fluctuation $\zeta(s)$, and angular fluctuation $h_i(s)$, $i = 1, 2, 3$, contributions. Since $\zeta'_{\hat{H}}(0)$ contains only $h_i(0)$ and $h'_i(0)$ terms, we will keep only constant terms and terms linear in s in $h_i(s)$. It is easy to see that $h_2(s)$ has no such terms and so in the limit $s \to 0$ we obtain

$$\begin{aligned}
h_1(s) &\approx -\sqrt{\pi}\zeta\left(-\frac{1}{2}\right)s && (46)\\
h_3(s) &\approx \frac{1}{2}\frac{h_0}{\sqrt{2\pi}}s && (47)
\end{aligned}$$

where in (47) h_0 is given by

$$h_0 = \sum_{n=1}^{\infty}\frac{n^{-\frac{3}{2}}}{e^{2\pi n}-1} = 0.00187217. \tag{48}$$

Because $h_i(0) = 0$, it follows that the angular fluctuation contributions to the determinant amount to a numerical factor only, while the parametric dependence is controlled by the contributions of the radial fluctuations.

The result of the foregoing analysis is that

$$\begin{aligned}
\zeta'_{\hat{H}}(0) &= \frac{1}{4}\log(\lambda_0\kappa^2) - \frac{1}{2}\log(2\pi) - 2\sqrt{\pi}\zeta\left(-\frac{1}{2}\right) + \frac{h_0}{\sqrt{2\pi}}\\
&= \frac{1}{4}\log(\lambda_0\kappa^2) - 0.181254, && (49)
\end{aligned}$$

and accordingly

$$\left[\det \hat{H}(\lambda_0)\right]^{-\frac{1}{2}} = \exp\left[\zeta'_{\hat{H}}(0)\right] = 0.8344223(\lambda_0\kappa^2)^{\frac{1}{4}}. \tag{50}$$

Finally, we have to take into account the zero modes, i.e. to integrate along the directions in which the action does not change in the path integral (24). These directions are just those of the collective coordinates, but the corresponding measures dc_x, dc_y, dc_θ involve the normalisation (27) of the zero modes (this is where the metric $\hat{\mathcal{G}}$ enters the calculation),

$$\frac{dc_x}{\sqrt{2\pi}} = \kappa\sqrt{c}\frac{dx_0}{\sqrt{2\pi}}, \quad \frac{dc_y}{\sqrt{2\pi}} = \kappa\sqrt{c}\frac{dy_0}{\sqrt{2\pi}}, \quad \frac{dc_\theta}{\sqrt{2\pi}} = \sqrt{\frac{4\pi}{\beta}}\frac{d\theta_0}{\sqrt{2\pi}}. \tag{51}$$

The zero mode factor $\mathcal{Z}$ in eq. (35) thus reads

$$\mathcal{Z} = \frac{2\kappa^2 c}{\pi\sqrt{2\beta}}\int dx_0 dy_0 d\theta_0 \tag{52}$$

where the θ–integration contributes a factor 2π and the x_0y_0–integration yields the instanton volume V.

5 Instanton density and vacuum energy shift

Collecting the results (14), (50), (52) of the previous sections, the one instanton contribution to the normalised vacuum–vacuum transition amplitude (35) reads

$$\frac{\langle 0|0_V\rangle_1(\lambda_2,\lambda_0)}{\langle 0|0_V\rangle_1(\lambda'_2,\lambda'_0)} = \mathrm{e}^{-[S_0(\lambda_2,\lambda_0)-S_0(\lambda'_2,\lambda'_0)]}\left[\frac{\det \hat{H}(\lambda_0)}{\det \hat{H}(\lambda'_0)}\right]^{-\frac{1}{2}}\frac{4\kappa^2 c}{\sqrt{2\beta}}V = RV, \tag{53}$$

where the quantity

$$R = 2\sqrt{2}\frac{\kappa^2 c}{\sqrt{\beta}}\left(\frac{\lambda_0}{\lambda'_0}\right)^{\frac{1}{4}} e^{-\frac{8\pi}{\beta}(\lambda_0-\lambda'_0)} \tag{54}$$

is recognised as the instanton density. ¿From this expression for R we deduce the condition for diluteness of the instanton gas, i.e.

$$\frac{8\pi}{\beta}(\lambda_0-\lambda'_0) \gg 1. \tag{55}$$

In the region of validity of the dilute gas approximation, the n instanton contribution will be

$$\frac{\langle 0|0_V\rangle_n(\lambda_2,\lambda_0)}{\langle 0|0_V\rangle_n(\lambda_2',\lambda_0')} = \frac{1}{n!}\,(RV)^n\,. \tag{56}$$

Then the degree of suppression of the instanton generated transition amplitude is obtained by summing over n,

$$\frac{\langle 0|0_V\rangle(\lambda_2,\lambda_0)}{\langle 0|0_V\rangle(\lambda_2',\lambda_0')} = \sum_{n=1}^{\infty}\frac{1}{n!}\,(RV)^n = e^{RV}. \tag{57}$$

In the limit $V\to\infty$ we obtain

$$\langle 0|0_V\rangle \approx e^{-EV} \tag{58}$$

where V is the space volume, and hence E can be treated as the vacuum energy density. Finally, substitution of the expression (58) in (57) yields the lowering shift in vacuum energy density due to the instantons (as evidenced by the dependence on λ_2)

$$E(\lambda_2,\lambda_0) - E(\lambda_2',\lambda_0') = -8\pi\sqrt{2}\left(\frac{\pi^2}{12}+\log 2\right)\frac{\kappa^2}{\sqrt{\beta}}\left(\frac{\lambda_0}{\lambda_0'}\right)^{\frac{1}{4}} e^{-\frac{8\pi}{\beta}(\lambda_0-\lambda_0')} \tag{59}$$

whose region of validity is controlled by condition (55).

6 Summary

In the above we performed and explicit evaluation of the normalised and regularised determinant of the fluctuation operator for an $O(3)$ Skyrme model in 2 dimensions. This calculation yielded the exact expressions for the dilute instanton gas density and the corresponding shift in vacuum energy. All this was made possible by three salient properties of the model used: first the fact that the instantons of this model were explicitly found, and second the special symmetry properties of the model which reduced the fluctuation action to a scalar system, both these features enabling the exact calculation of the fluctuation determinant. Thirdly and finally these instantons were localised to an absolute scale which allowed the construction of a dilute gas.

We have succeeded, in the context of the pure-Skyrme $O(3)$ model, to calculate the density of the (dilute) instanton gas (54) explicitly, which would not have been possible if our theory did not have an absolute scale. This would not have been achieved in practice if we could not have availed of explicit intantons and if we had not calculated the regularised determinants explicitly. The result (54) allowed the calculation of the shift in vacuum energy (59) due to the instantons, a quantity which to our knowledge has not previously been calculated explicitly for a field theory, but only in quantum mechanics [12]. In this sense, our results should be of interest, in addition to having achieved our main aim of tackling the semiclassical instanton analysis of a theory with Skyrme-like kinetic terms.

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