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Marie Roth

Character sheaves and modular representations of finite reductive groups

On the unitriangularity of their decomposition matrices

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Affidavit

I, undersigned, Roth Marie, hereby declare that the work presented in this manuscript is my own work, carried out under the scientific supervision of Olivier Dudas and Gunter Malle, in accordance with the principles of honesty, integrity and responsibility inherent to the research mission. The research work and the writing of this manuscript have been carried out in compliance with both the french national charter for Research Integrity and AMU charter on the fight against plagiarism.

This work has not been submitted previously either in this country or in another country in the same or in a similar version to any other examination body.

Kaiserslautern, December 4, 2024

List of publications and conferences, Academic CV

List of publications

- M. Roth. *Unitriangularity of decomposition matrices of the unipotent ℓ-blocks for simple adjoint exceptional groups.* 2024. arXiv: `2402.17616 [math]`

Attended conferences and summer schools

- May 2022: Workshop "Counting conjectures and beyond" in Cambridge, UK
- September 2022: "Young Group Theorists Workshop" in Les Diablerets, Switzerland
- September 2022: Annual Meeting of the SFB-TRR 195 in Blaubeuren (snapshot talk)
- September 2022: "Representation Theory at Villa Denis" near Kaiserslautern
- October 2022: "Darstellungstheorietage 2022" in Kaiserslautern
- December 2022: "Nikolaus Conference 2022" on Group and Representation Theory in Aachen (talk)
- March 2023: IRTG (Integrated Research Training Group of the SFB-TRR 195) Block Seminar at the Ebernburg, Bad Münster
- June 2023: Conference on "Representations of Reductive Groups" in Aachen
- July 2023: Summer School on Algebraic Groups and Their Representations in Birmingham, UK.
- August 2023: Summer School and Symposium on Geometric and Categorical Representation theory in Bath, UK.
- September 2023: IRTG Block seminar on Category Theory in Trippstadt

- September 2023: Annual Meeting of the SFB-TRR 195 in Saarbrücken (poster)
- October 2023: “Representation Theory at Villa Denis” near Kaiserslautern (talk)
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- July 2024: “Group Theory in Florence IV” (talk)
- July 2024: “Joint Block Seminar 2024 on Algebraic Number Theory” in Aachen
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Abstract

Abstract

In this thesis, we study finite reductive groups and their modular representations in non-defining characteristic.
In 1990, Geck stated a conjecture on the unitriangularity of decomposition matrices of these groups. Decomposition matrices encode the link between ordinary representations (over a field of characteristic zero) and modular representations (over a field of positive characteristic ℓ). In 2020, Brunat–Dudas–Taylor showed this conjecture for unipotent blocks for a very good prime number ℓ, introducing Kawanaka characters. Thanks to the Morita equivalence between unipotent blocks and non-isolated ones, Feng–Späth extended this result to non-isolated blocks in 2021. The aim of this thesis is to study possible generalisations of Brunat–Dudas–Taylor result.
Firstly, we extend this result for a bad prime ℓ in the case of simple groups for the unipotents blocks. Inspired by the Brunat–Dudas–Taylor method, we study the decomposition of some Kawanaka characters in terms of ordinary characters in the unipotent blocks. In order to do so, we compute the values of the characteristic functions of characters sheaves on mixed conjugacy classes, based on previous work of Lusztig.
Lastly, we show through the examples of G_2 and F_4 how the obtained method allows us to study the unitriangularity of isolated blocks for exceptional groups of adjoint types.

Keywords: Modular representations, finite reductive groups, decomposition matrices, character sheaves.

Résumé

L'objet de cette thèse est l'étude des groupes réductifs finis et plus particulièrement de leurs représentations modulaires en caractéristique transverse.
En 1990, Geck a énoncé une conjecture portant sur l'unitriangularité des matrices de décomposition de ces groupes. Les matrices de décomposition encodent le passage des représentations irréductibles dites ordinaires (sur un corps de caractéristique nulle) aux représentations modulaires (sur un corps de caractéristique positive un nombre premier ℓ). En 2020, Brunat–Dudas–Taylor ont démontré cette conjecture dans le cas des blocs unipotents pour un ℓ très bon avec l'introduction des caractères de Kawanaka. Grâce à l'équivalence de Morita entre les blocs unipotents et les blocs non-isolés, Feng–Späth ont étendu ce résultat aux blocs non-isolés en 2021. Le but de cette thèse est d'étudier des généralisations possibles du théorème de Brunat–Dudas–Taylor.
Dans un premier temps, on étend ce résultat pour ℓ mauvais dans le cas des groupes adjoints simples pour les blocs unipotents. En s'inspirant de la méthode de Brunat–Dudas–Taylor, on étudie la décomposition de certains caractères de Kawanaka. Pour ce faire, nous calculons les valeurs des fonctions caractéristiques des faisceaux caractères sur des classes de conjugaison mixtes. On se base sur les travaux de Lusztig.
Dans un second temps, on généralise la méthode obtenue afin d'étudier l'unitriangularité des blocs isolés pour les groupes exceptionnels de type adjoint. Nous traitons les cas des groupes simples adjoints de type G_2 et F_4.

Mots-clés: Représentations modulaires, groupes réductifs finis, matrices de décomposition, faisceaux characters.

Zusammenfassung

Der Gegenstand dieser Dissertation ist die Untersuchung endlicher reduktiver Gruppen und insbesondere ihrer modularen Darstellungen in transversaler Charakteristik.
Im Jahr 1990 stellte Geck eine Vermutung auf, die sich auf die Unitriangularität der Zerlegungsmatrizen dieser Gruppen bezog. Die Zerlegungsmatrizen kodieren den Übergang von gewöhnlichen irreduziblen Darstellungen (über einem Körper der Charakteristik Null) zu modularen Darstellungen (über einem Körper von positiver Charakteristik ℓ).
Im Jahr 2020 bewiesen Brunat–Dudas–Taylor diese Vermutung im Fall von unipotenten Blöcken für sehr gutes ℓ durch die Einführung von Kawanaka-Charakteren. Mit Hilfe geigneter Morita-Äquivalenzen zwischen unipotenten und nicht-isolierten Blöcken haben Feng–Späth dieses Ergebnis im Jahr 2021 auf nicht-isolierte Blöcke ausgeweitet. Das Ziel dieser Dissertation ist, mögliche Verallgemeinerungen zu untersuchen.
Zunächst erweitern wir dieses Ergebnis auf schlechte Primzahlen ℓ im Fall von einfachen adjungierten Gruppen für unipotente Blöcke. In Anlehnung an die Methode von Brunat–Dudas–Taylor wird die Zerlegung bestimmter Kawanaka-Charaktere untersucht. Dazu bestimmen wir die Werte der charakteristischen Funktionen der Charaktergarben auf gemischten Konjugationsklassen. Wir stützen uns dabei auf die Arbeit von Lusztig.
In einem zweiten Schritt hoffen wir, die erhaltene Methode zu verallgemeinern, um die Unitriangularität von isolierten Blöcken für exzeptionelle Gruppen vom adjungierten Typ untersuchen zu können. Wir behandeln die Fälle von G vom Typ G_2 und F_4.

Stichwörter: modulare Darstellungen, endliche reduktive Gruppe, Zerlegungsmatrizen, Charaktergarben.

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Contents

Introduction

Context

Groups as mathematical objects are very elementary to define. They satisfy a small list of axioms (existence of the neutral element, of inverses and associativity). However, this modest set of rules leaves room for a wide diversity of objects.
To restrict our field of study, we focus on the groups that are the "building blocks" of the other groups: the simple groups. The finite simple groups have been completely classified into three families:

- the cyclic groups of prime order,
- the alternating groups Alt_n for $n \geq 5$,
- and the finite groups of Lie type,

as well as 26 sporadic groups who do not belong to any of the previously listed families. The proof of the Classification of Finite Simple Groups (CFSG) is a monumental work spanning over at least 30 years. It was first announced in 1983 by Gorenstein, see [Asc04], [Sol18] for updates on the proof. This thesis concentrates on the last and in a sense most varied family, the finite groups of Lie type.

The majority of introductory books on group theory motivates their subject as the exploration of symmetry. A symmetry is an action that leaves the object we consider invariant, and groups formalise this notion. Therefore, it seems natural to investigate the action by linear maps of the elements of a group G on an $\mathbb{F}$-vector space V, for $\mathbb{F}$ a field. We say that V is an $\mathbb{F}[G]$**-module**. Note that the action of G defines a group homomorphism $\rho : G \to \mathrm{GL}(V)$. This is called an $\mathbb{F}$-**representation** of G. In other words, representation theory allows us to study an arbitrary finite group G by turning it into something we better comprehend, a subgroup of the invertible matrices $\mathrm{GL}(V)$. We then have the tools of linear algebra at our disposal. We often further assume that the field $\mathbb{F}$ is algebraically closed or at least contains all the $|G|$th roots of unity, in order to be able to triangularise the elements $\rho(g)$ for $g \in G$.

There are two distinct flavours to the representation theory of a finite group G; whether the characteristic of $\mathbb{F}$ divides the order of G or not. The second case is called

ordinary and is much better understood than the first one, referred to as **modular**. One reason is that when the characteristic of $\mathbb{F}$ is coprime to $|G|$ the $\mathbb{F}[G]$-modules are semisimple (they decompose into direct sums of irreducible submodules), whilst it is not true for modules over a field of characteristic ℓ dividing $|G|$. Nonetheless, both approaches consider the same group G and accordingly there should exist a link between them. Such a connection is encoded in the ℓ-decomposition matrices. Assuming we know everything about the ordinary representations of G and that we have computed the ℓ-decomposition matrix, we could extensively comprehend the irreducible representations of G over $\overline{\mathbb{F}}_\ell$. However, fully determining the ℓ-decomposition matrix for an arbitrary group G is in general an arduous task.
For cyclic groups of prime order, it is trivial. For sporadic groups, the ℓ-decomposition matrices can sometimes be explicitly computed, see for instance [HHM99]. However, already for symmetric groups, despite our very good knowledge of the ordinary representations, we do not yet know the ℓ-decomposition matrix.

One easier problem is to ascertain the unitriangularity of the ℓ-decomposition matrix. If this property holds, then we can label the irreducible $\overline{\mathbb{F}}_\ell$-representations of G. Moreover, it might help echelonise any set of projective characters. This yields valuable information in order to compute the rest of the decomposition matrix, as Dudas and Malle applied in [DM20b].
The unitriangularity of the ℓ-decomposition matrix has been established in the case of the symmetric groups [Jam78, Cor. 12.3], but it surprisingly fails for alternating groups, c.f. [BGJ23, Sect. 3.2]. In this work, we focus our attention on finite groups of Lie type.

Any ℓ-decomposition matrix has a decomposition into ℓ-blocks. For finite reductive groups, one union of blocks, called the **unipotent** ℓ-blocks, is of particular interest. Indeed, most other ℓ-blocks (the **non-isolated** blocks) are Morita equivalent to unipotent ℓ-blocks of smaller groups.

In 1985, Dipper showed the unitriangularity of the ℓ-decomposition matrix of $\mathrm{GL}_n(\mathbb{F}_q)$ for q a power of an odd prime $p \neq \ell$, [Dip85, Cor. 6.17], under certain conditions on ℓ, for instance $\ell \mid q-1$. His proof relies on the fact that the Weyl group of $\mathrm{GL}_n(q)$ is a symmetric group and hence has unitriangular decomposition matrix. Five years later, Geck made the following conjecture.

Conjecture ([Gec90, 2.5])**.** *Let G be a finite group of Lie type over $\mathbb{F}_q$ where q is a power of a prime p. For any prime $\ell \neq p$, the ℓ-decomposition matrix of the unipotent ℓ-blocks of G is lower-unitriangular.*

The next year, in [Gec91, Cor. B], he showed that the whole ℓ-decomposition matrix of the general unitary groups $\mathrm{GU}_n(\mathbb{F}_q)$ is unitriangular. Geck employed different tools than the ones used by Dipper for $\mathrm{GL}_n(\mathbb{F}_q)$. He combined the generalised Gelfand–Graev characters and the power of the theory of character sheaves developed by Lusztig.
The subject made a major step forward in 2020 when Brunat, Dudas and Taylor showed that Geck's conjecture holds under certain assumptions on p and ℓ, such as p and ℓ good

for G, see [BDT20, Thm. A]. They pushed further the techniques of Geck by considering summands of generalised Gelfand–Graev characters, called the Kawanaka characters.

The goal of this thesis is to remove some conditions in [BDT20, Thm. A] on the prime ℓ by extending the methods of Brunat–Dudas–Taylor. The case of the classical groups at the unique bad prime $\ell = 2$ was already treated by Geck in [Gec94] and Chaneb in [Cha21, Thm. 2.8], and we therefore focus on the exceptional groups. The main involved issue is to better understand the restriction of character sheaves to conjugacy classes. We show the following main theorem.

Theorem. *Let $\mathbf{G}$ be a simple exceptional group of adjoint type defined over k, an algebraically closed field of characteristic p with Frobenius endomorphism F. Assume that p is good for $\mathbf{G}$. Let ℓ be a bad prime for $\mathbf{G}$, then the decomposition matrix of the unipotent ℓ-blocks of $\mathbf{G}^F$ is lower-unitriangular.*

Combined with the previous results, the following statement holds true.

Theorem. *Let $\mathbf{G}$ be a connected reductive group defined over k, an algebraically closed field of characteristic p with Frobenius endomorphism F. We suppose that the derived subgroup of $\mathbf{G}$ is adjoint. Assume that p is good for $\mathbf{G}$. Let ℓ be a prime different from p, then the decomposition matrix of the unipotent ℓ-blocks of $\mathbf{G}^F$ is unitriangular.*

The proof of our main result leads us to develop methods that we believe are applicable to the remaining blocks. As a trial, we apply them to the isolated ℓ-blocks for groups of type G_2 and F_4.

Content of the thesis

This manuscript is divided into three parts, each of them consisting of two chapters. This description reflects the main elements we need to show the unitriangularity of the decomposition matrix of a union of blocks $\mathscr{B}$ for a finite reductive group G.

Our strategy relies on the fact that it is sufficient to show the unitriangularity of a decomposition matrix of modular projective modules (not necessarily indecomposable) into the irreducible ordinary modules in $\mathscr{B}$. To do so, we first need to compute the number n of projective modules needed, that is the number of irreducible $\overline{\mathbb{F}}_\ell[G]$-modules in the union of blocks $\mathscr{B}$. We then find n candidates for the irreducible ordinary modules and n candidates for the projective modular modules and lastly, check that the corresponding decomposition matrix is unitriangular. The last two chapters are dedicated to applying this strategy in our cases.

However, to find candidates for the ordinary modules, we first need to understand them. This is the aim of the first two chapters, where we present some known and significant results on the representation theory of finite reductive groups.

Nonetheless, our knowledge is not yet sufficient to be able to directly compute the decomposition matrix. To achieve our goal, we need to go to the other side of the mirror and look at the character sheaves of $\mathbf{G}$. There, we are able to compute the values of character sheaves at certain conjugacy classes, and conclude the proof of the unitriangularity of the decomposition matrix. This technical part is detailed in the third and fourth chapters.

Representation theory of finite groups of Lie type (Chapters 1 and 2)

The first two chapters are purely expository and gather some well-known results in the theory of finite reductive groups, statements that can be found in most textbooks. Our principal resource is the book by Geck and Malle [GM20].

Finite reductive groups

In Chapter 1, we define the finite reductive group $G \coloneqq \mathbf{G}^F$ as the fixed points under a Steinberg endomorphism F of a connected reductive algebraic group $\mathbf{G}$ defined over $k = \overline{\mathbb{F}}_p$ for p a prime number. This underlying infinite group will play an indispensable role throughout this thesis. For instance, algebraic groups come with some purely combinatorial data, known as the root datum which allows us to classify them. We also take advantage of this chapter to collect facts on the unipotent and semisimple conjugacy classes of $\mathbf{G}$.

Parameterisation of the ordinary characters

After laying out the general setup, we outline the ordinary representation theory of G in Chapter 2. The pivotal idea to treat the complex-valued representations of all the finite reductive groups at once came off the back of the work undertaken by Deligne and Lusztig [DL76]. They looked at certain G-equivariant varieties and then considered the alternating sum of their cohomologies with compact support. This construction gives a virtual character of G (a $\mathbb{Z}$-linear combination of irreducible ordinary characters).

This method enables us to partition the set $\mathrm{irr}_{\mathbb{C}}(G)$ of irreducible complex characters into rational series $\mathscr{E}(G, s)$ indexed by a set of representatives of the semisimple conjugacy classes in the dual group $(\mathbf{G}^*)^{F^*}$. The series indexed by the neutral element is called **unipotent** and is denoted by $\mathrm{Uch}(G)$. If the centre $Z(\mathbf{G})$ is connected, then there exists a bijection between $\mathscr{E}(G, s)$ and $\mathrm{Uch}(C_{\mathbf{G}^*}(s)^{F^*})$, thanks to the Jordan decomposition of characters [Lus84a, Thm. 4.23].

Moreover, Lusztig showed in [Lus84a] that each series can be further partitioned into families, themselves labelled in terms of a small finite group, the **ordinary canonical quotient**. To each family is also associated a family of characters of the Weyl group of $C^\circ_{\mathbf{G}^*}(s)$ and a unipotent conjugacy class of $\mathbf{G}$, called the **unipotent support**. This class gives information on the values of the characters in the corresponding family. In the case of the unipotent characters, to each family corresponds a different unipotent conjugacy class, called **special**. Furthermore, this class completely determines the ordinary canonical quotient.

On character sheaves (Chapters 3 and 4)

The character sheaves on $\mathbf{G}$ mirror in the geometric world the ordinary representations of G. When looking at the ordinary representations of G, we consider G-equivariant perverse sheaves on G-equivariant varieties. In a series of papers in the eighties, Lusztig developed a theory where he studied certain $\mathbf{G}$-equivariant perverse sheaves on $\mathbf{G}$-equivariant varieties: the **character sheaves**. This geometric approach is formidable to get information on the ordinary representation theory of G.

A geometric mirror of the ordinary representations

Firstly, characteristic functions of the F-stable character sheaves form a new basis for the class functions of G. On top of that, if the centre $Z(\mathbf{G})$ is connected, we understand the change of basis between characteristic functions of the F-stable character sheaves and ordinary irreducible characters of G ([Sho95b]).

Moreover, the set $\hat{\mathbf{G}}$ of character sheaves can be labelled in a similar way as $\mathrm{irr}_{\mathbb{C}}(G)$. Firstly, by the way they are constructed, each character sheaf belongs to a unique series $\hat{\mathbf{G}}_s$ where s runs over a set of representatives of the semisimple conjugacy classes in $\mathbf{G}^*$. Each series $\hat{\mathbf{G}}_s$ itself decomposes into families associated to families of the Weyl group of $C^\circ_{\mathbf{G}^*}(s)$ and parameterised using the same small finite group as for the ordinary irreducible characters.

Thus character sheaves are the counter part on the algebraic group $\mathbf{G}$ of the complex irreducible modules. We then have an easier access to the geometry of $\mathbf{G}$ and we can hope to deduce more information on the values of their characteristic functions than what we currently know on the values of ordinary characters.
In the fourth chapter, we use these properties to compute the values of character sheaves on certain conjugacy classes. To explain our methods, we first need to recall a second partition of $\hat{\mathbf{G}}$.

A labelling in terms of characters of relative Weyl groups

This different parameterisation is given in terms of **cuspidal induction data** of the form $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathscr{E}_0)$ where $\mathbf{L}$ is a Levi subgroup of $\mathbf{G}$, $\Sigma = D_0 Z^\circ(\mathbf{L})$ where D_0 is a conjugacy class of $\mathbf{L}$ whose semisimple part is isolated, and $\mathscr{E}_0$ is a local system on Σ.
The character sheaves in $\hat{\mathbf{G}}(\mathfrak{m})$ are labelled thanks to the irreducible characters of an algebra $\mathscr{A}_{\mathfrak{m}}$. In [Lus84b], Lusztig showed that this algebra is isomorphic to the group algebra of a certain relative Weyl group $W_{\mathfrak{m}}$ twisted by a cocycle. If D_0 is a unipotent conjugacy class, then he proved that the cocycle is trivial anf fully described this isomorphism. When $\mathbf{G}$ is simple and adjoint, Shoji confirmed that the cocycle is trivial in general ([Sho95a]). Assuming that p is good for $\mathbf{G}$ and that $\mathbf{G}$ is of adjoint type, we give a description of the isomorphism $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}] \cong \mathscr{A}_{\mathfrak{m}}$ in Lemmas 3.2.21 and 3.2.22. We base our reasoning on previous work by Bonnafé [Bon04] when D_0 is unipotent.

This partition of the character sheaves already gives us information on their values at a mixed conjugacy class $D = (su)_{\mathbf{G}}$ with $s \in \mathbf{G}$ a semisimple element and $u \in C^\circ_{\mathbf{G}}(s)$

unipotent. Again, let $\mathcal{A}$ be a character sheaf coming from the cuspidal induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}_0)$. Then $\mathcal{A}$ restricted to D is zero if the semisimple part of Σ does not contain a $\mathbf{G}$-conjugate of s.

Restriction to a unipotent conjugacy class

As a first step towards our main goal, we recall how to compute the restriction of character sheaves to a unipotent conjugacy class C. It is a consequence of the generalised Springer correspondence, a famous result due to Lusztig [Lus84b].
Let $\mathcal{A}$ be a character sheaf coming from the cuspidal induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}_0)$. Then the character sheaf $\mathcal{A}$ restricted to C is zero unless Σ is of the form $C_0 Z^\circ(\mathbf{L})$ where C_0 is a unipotent conjugacy class of $\mathbf{L}$. Assume this is the case, then the local system $\mathcal{E}_0$ comes from a local system $\mathcal{E}_{C_0}$ on C_0 and another one $\mathcal{E}_Z$ on $Z^\circ(\mathbf{L})$.
If $\mathcal{E}_Z$ is trivial, the generalised Springer correspondence tells us that $\mathcal{A}$ corresponds to a unique unipotent conjugacy class $C_{\mathcal{A}}$ of $\mathbf{G}$ and that $\mathcal{A}$ restricted to $C_{\mathcal{A}}$ is an irreducible local system on $C_{\mathcal{A}}$. Moreover, if $C \notin \overline{C_{\mathcal{A}}}$, then $\mathcal{A}$ restricted to C is zero as well.
On the other hand, if $\mathcal{E}_Z$ is not trivial, we get information on $\mathcal{A}$ restricted to C thanks to the isomorphism $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}] \cong \mathscr{A}_{\mathfrak{m}}$ fixed by Lusztig. Let $\mathfrak{m}' = (\mathbf{L}, \Sigma, \mathcal{E}_0')$ be the cuspidal induction datum where $\mathcal{E}_0'$ is constituted of $\mathcal{E}_{C_0}$ and the trivial local system on $Z^\circ(\mathbf{L})$. The group $W_{\mathfrak{m}}$ is a subgroup of $W_{\mathfrak{m}'}$. If the character sheaf $\mathcal{A}$ is labelled by a character ϕ of $W_{\mathfrak{m}}$, then the restriction of $\mathcal{A}$ to C comes from the restriction of the character sheaves in $\hat{\mathbf{G}}(\mathfrak{m}')$ labelled by characters ψ of $W_{\mathfrak{m}'}$ whose restriction contains ϕ.

Restriction to a mixed conjugacy class

When $D = (su)_{\mathbf{G}}$ is a mixed conjugacy class with $u \in C^\circ_{\mathbf{G}}(s)$ unipotent and $s \in \mathbf{G}$ semisimple, computing the stalk of $\mathcal{A}$ at su boils down to computing the stalk of $s^*\mathcal{A}$ at u. Here s^* denotes the pullback by the translation by s. Building on the previous work of Lusztig for the induction of a character sheaf, we decompose $(s^*\mathcal{A})_{(u)_{C^\circ_{\mathbf{G}}(s)}}$ (up to a shift) into a direct sum of characters sheaves of $\mathbf{H} := C^\circ_{\mathbf{G}}(s)$ restricted to the unipotent class $(u)_{\mathbf{H}}$. These character sheaves on $\mathbf{H}$ come from cuspidal induction data of $\mathbf{H}$ of the form $(\mathbf{L}_s, \Sigma_s, \mathcal{E})$ with Σ_s consisting of a unipotent conjugacy class of $\mathbf{H}$ times the centre of $\mathbf{L}_s$. We are thus back to the previous setting.
As before we want to use the labelling in terms of characters of the relative Weyl groups. That is why we needed to explicit the isomorphism $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}] \cong \mathscr{A}_{\mathfrak{m}}$. The details are laid out in Subsection 4.3.3 for the unipotently supported character sheaves and in Subsection 4.3.4 when $\mathbf{G}$ is simple of adjoint type and p is a good prime for $\mathbf{G}$.

The unitriangularity of the ℓ-decomposition matrix (Chapters 5 and 6)

In the last chapters, we focus on the main goal of this thesis. We fix a prime $\ell \neq p$ and an ℓ-modular system $(\mathbf{O}, \mathbf{K}, \overline{\mathbb{F}_\ell})$ for $\mathbf{G}$.

Thanks to Broué and Michel ([BM89]), the partition of the irreducible ordinary characters of G into ℓ-blocks is compatible with the partition into rational series: we can find a union of blocks $\mathscr{B}(G,t)$, for a semisimple ℓ'-element t of $(\mathbf{G}^*)^{F^*}$, such that the set of ordinary characters belonging to $\mathscr{B}(G,t)$ is a union of rational series of the form $\mathcal{E}(G,ts)$ where $s \in (\mathbf{G}^*)^{F^*}$ is an ℓ-element. The union of blocks indexed by the neutral element is called the **unipotent** ℓ-blocks and the ones indexed by isolated elements are said to be **isolated**. A semisimple element $t \in \mathbf{G}$ is said to be isolated if its connected centraliser is not contained in a proper Levi subgroup of $\mathbf{G}$.

Moreover, Bonnafé and Rouquier [BR03] showed a version of Jordan decomposition for the blocks: if $C_{\mathbf{G}^*}(t)$ is a Levi subgroup then the union of blocks $\mathscr{B}(G,t)$ is Morita equivalent to $\mathscr{B}(L,1)$ where $\mathbf{L}$ is the Levi subgroup of $\mathbf{G}$ in duality with $C_{\mathbf{G}^*}(t)$. Therefore, we concentrate on the unipotent and isolated ℓ-blocks.

Strategy of the proof

Let $\mathscr{B}$ be a union of ℓ-blocks of G. To show the unitriangularity of the decomposition matrix of $\mathscr{B}$, we apply the following strategy.

Step 1 Compute the number n of projective indecomposable modules in $\mathscr{B}$.

Step 2 Choose n ordinary irreducible modules $V_1, \ldots, V_n \in \mathrm{Irr}_{\mathbf{K}}(G)$ belonging to $\mathscr{B}$.

Step 3 Choose n projective modules $P_1, \ldots, P_n$ of $\overline{\mathbb{F}_\ell}[G]$.

Step 4 Check that the decomposition matrix D, given by $d_{ij} \coloneqq \langle V_i, P_j^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}\rangle$ for $1 \leq i, j \leq n$ is lower-unitriangular.

Note that it is sufficient to consider any projective modules of $\overline{\mathbb{F}_\ell}[G]$, not necessarily indecomposable ones.

Step 1 of the proof

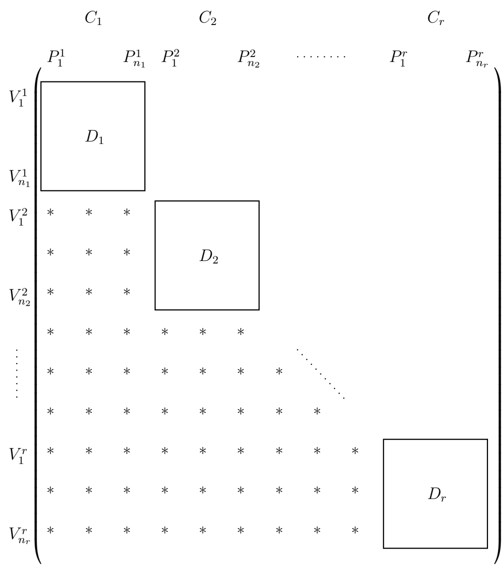
Concerning the first step, when ℓ is good and $Z(\mathbf{G})$ is connected, the number n is known for any union of blocks $\mathscr{B}(G,t)$ for a semisimple ℓ'-element t of $(\mathbf{G}^*)^{F^*}$. This is a result of Geck and Hiss [GH91]. When ℓ is bad and $\mathbf{G}$ of classical type or for the unipotent ℓ-blocks, it was also computed by Geck and Hiss in [Gec94] and [GH97]. In Proposition 5.1.14, we explain how one can use similar arguments to compute the number n of projective indecomposable modules in $\mathscr{B}(G,t)$ for an isolated semisimple ℓ'-element t of $(\mathbf{G}^*)^{F^*}$, when $\mathbf{G}$ is of exceptional type, simple modulo its centre, and p is good for $\mathbf{G}$. These numbers can be found in Appendix B.2.

Steps 2 and 3 of the proof

For Steps 2 and 3, we base our methods on the ones developed by Geck in [Gec91] and Brunat, Dudas and Taylor in [BDT20]. Fix a semisimple ℓ'-element t of $(\mathbf{G}^*)^{F^*}$. We determine the unipotent supports $C_1, \ldots, C_r$ of the characters belonging to $\mathscr{B}(G,t)$ with a total ordering $C_1 < \cdots < C_r$, such that $C_i < C_j$ if $\dim C_i \leq \dim C_j$ for all $1 \leq i < j \leq r$. Then, for each $1 \leq i \leq r$,

- we choose n_i irreducible modules $V_1^i, \ldots, V_{n_i}^i \in \mathscr{B}(G,t)$ with wave front set C_i (that is whose dual under Alvis–Curtis duality has unipotent support C_i),
- and n_i projective $\overline{\mathbb{F}_\ell}[G]$-modules $P_1^i, \ldots, P_{n_i}^i$.

We require $\sum_{1 \leq i \leq r} n_i = n$. In the unipotent case, the numbers n_i are conjectured by Chaneb [Cha19]. For the projective $\overline{\mathbb{F}_\ell}[G]$-modules, we choose the generalised Gelfand–Graev characters (GGGCs) or certain summands called the ℓ-Kawanaka modules. Thanks to their properties, the decomposition matrix D has the following shape.



Step 4 of the proof

The final chapter consists in verifying that the matrices D_i are lower-unitriangular for each unipotent class C_i we consider. Our current understanding of the values of ordinary characters as such is however not sufficient. Nonetheless, we know another basis for the class functions of G on which we hope to have more control: the characteristic functions of F-stable character sheaves. Therefore, we instead compute the decomposition of a Fourier transform of the Kawanaka characters into certain characteristic functions of F-stable character sheaves with unipotent support C_i.
This is when the restriction of character sheaves to mixed conjugacy classes of Chapter 4 becomes useful.

In the unipotent case, when ℓ is good for $\mathbf{G}$, the proof of Brunat–Dudas–Taylor uses a theorem of Lusztig [Lus15] which predicts the value of a character sheaf restricted to a conjugacy class whose unipotent part is its unipotent support. As we tried to reproduce a proof of this theorem, we found some counter-examples in the exceptional families of E_7 and E_8.

In our proof for simple exceptional groups of adjoint type for the unipotent blocks, we instead apply the formulas of Chapter 4 using CHEVIE [Mic15] to derive the information we need to show the unitriangularity. The general arguments are similar to the ones of Brunat–Dudas–Taylor but do not rely on [Lus15] and thus involve more case-by-case analysis. For ℓ bad, in order to avoid too many computations, we also use some properties of GGGCs given by Geck and Hézard in [GH08].
In the last section, we treat the cases of the isolated blocks of G_2 and F_4, applying similar methods.

Links between the chapters

The following figure summarises the various links between the chapters. An arrow from A to B means that B requires results stated in A.

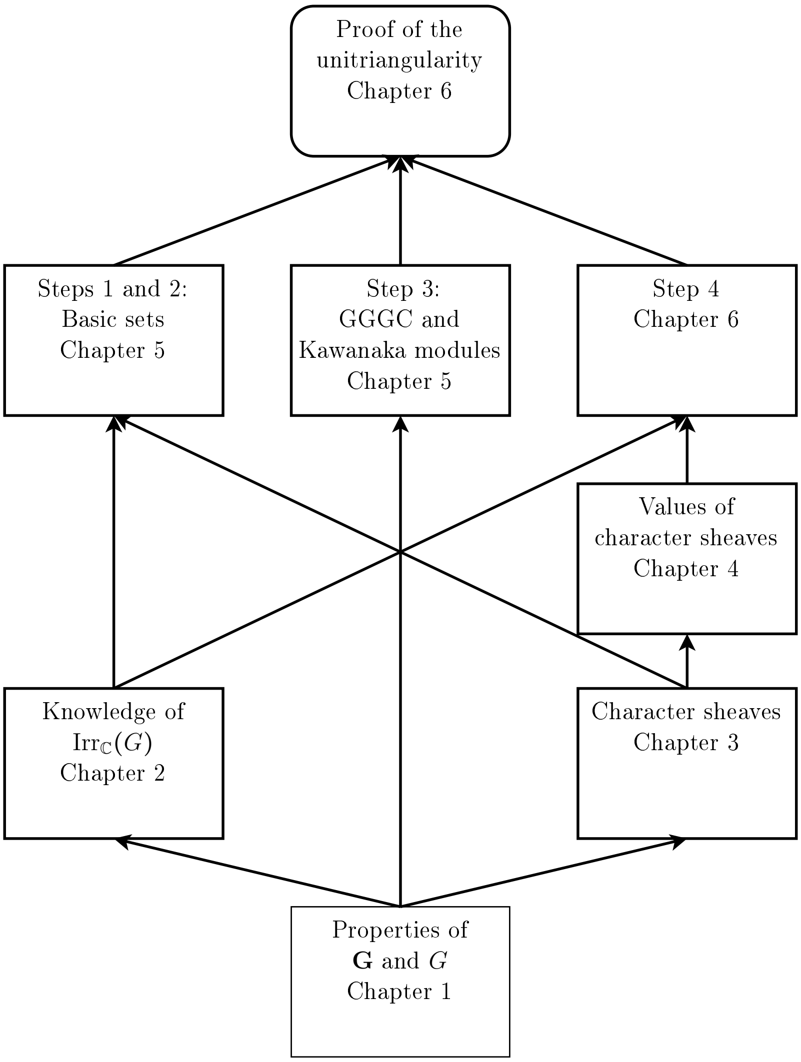


Figure 1: Links between the chapters

Perspectives

The themes discussed during this thesis open the door to new questions both about decomposition matrices of finite reductive groups, but also on the level of character sheaves.

About the unitriangularity of the ℓ-decomposition matrices

One innate problem is to conclude the proof of the unitriangularity of the ℓ-decomposition matrix of the finite reductive groups. Our results could be extended in different directions. Firstly, similar methods should be applicable to the isolated ℓ-blocks of the adjoint simple groups of type E_6, E_7, and E_8, assuming p is good. For the classical groups when the prime ℓ is good, we would need to understand the combinatorics behind the parameterisation of the ordinary characters instead of using CHEVIE [Mic15].
Another question would be to look at groups that are not necessarily simple of adjoint type. A first example to consider could be the isolated ℓ-blocks of $\mathrm{Sp}_n(\mathbb{F}_q)$, again assuming that the prime p is good.
Investigating the properties of the basic sets might also be worthwhile to group theorists. For instance, we could verify if they iare stable under group automorphisms.
Furthermore, we could ask how to remove the assumption on p. This seems a much more difficult question which requires the development of new tools. In particular, we would have to first define generalised Gelfand–Graev characters for a bad prime which satisfy the same properties as the GGGCs when p is good. So far, this has not been proven for the definition of GGGCs in bad characteristic given by Geck in [Gec21a].
At a more fundamental level, we could wonder if there is a conceptual reason behind the unitriangularity of the ℓ-decomposition matrices of finite groups of Lie type. In [CR17], Chuang and Rouquier explain that this result might be a consequence of a stronger version of Broué's abelian defect group conjecture ([Bro90, Sect. 6]). This local-global statement affirms the existence of a perverse equivalence between a block of the group algebra $\mathbf{K}[G]$ with abelian defect and its Brauer correspondent. Some progress has been made towards proving this conjecture, but contrary to other conjectures in the field, there is no reduction to simple groups, see the survey in [Mal17].

About character sheaves

Since we have a formula for the restriction of character sheaves to a mixed conjugacy class, it would be natural to try to derive a formula for the characteristic functions of the F-stable character sheaves. To do so, we would need to keep track of the isomorphism defining the characteristic function. If the resulting formula is relatively practical to use, this would have blatant applications to computing the character tables of finite reductive groups.
A second fascinating problem is to try to use our better understanding of the translation of character sheaves to understand their labelling in the same vein as Lusztig does in [Lus15]. If a character sheaf $\mathcal{A}$ on $\mathbf{G}$ is parameterised by an element a of the ordinary

canonical quotient, then the restriction of $\mathcal{A}$ to a conjugacy class $(su)_{\mathbf{G}}$ is zero unless the semisimple element s "corresponds" to a.
Finally, character sheaves and ordinary characters are closely related. One could inquire if a modular version of character sheaves, similarly connected to the modular characters, exists. A modular generalised Springer correspondence has already been identified by Achar, Henderson, Juteau and Riche, see [AHJR19]. Moreover, modular character sheaves on Lie algebras have been defined very recently by Sandvik in [San24] who extended ideas of Mirković in the ordinary case. It relies on properties of the Lie algebra which are however not available for connected reductive groups, such as the Fourier transform.

Notation

We list the basic notation and conventions taken in this thesis. Most of them have either been introduced in the introduction, can be found in the Appendix A or are very standard. The rest of the symbols introduced along the course of this manuscript can be found in the Index.
For the rest of this thesis, we fix p a prime number, q a power of p and ℓ another prime number. We will always assume that $p \neq \ell$.

Fields and rings

$\mathbf{\Lambda}$	any ring
$\mathbb{F}$	any field
$\overline{\mathbb{F}}$	the algebraic closure of the field $\mathbb{F}$
$\mathbb{C}$	field of complex numbers
$\mathbb{Q}$	field of rational numbers
$\mathbb{Z}$	ring of integers
$\mathbb{N}$	set of natural numbers
$\overline{\mathbb{Q}}_\ell$	algebraic closure of the field of ℓ-adic numbers.
$\mathbb{F}_q$	finite field of order q
$k \coloneqq \overline{\mathbb{F}}_p$	the algebraic closed field of characteristic p
$(\mathbb{Q}/\mathbb{Z})_{p'}$	the group of all elements in $\mathbb{Q}/\mathbb{Z}$ of order prime to p
$\mathbb{Z}_{(p)}$	the localisation of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$

The majority of the representation theory of finite reductive groups is defined over the ℓ-adic numbers. However, to compute the scalar product of characters, it is useful to consider complex conjugates. We therefore identify $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ via a fixed isomorphism.

Modules and characters

Let H be a finite group.

$\mathbf{\Lambda}[H]$ — the group algebra of H with coefficient in the ring $\mathbf{\Lambda}$

$\mathbf{\Lambda}[H]$-mod — category of left finite dimensional $\&|[\mathit{visibleon} =< 4- >]| * H]$-modules

$\mathrm{irr}_{\mathbb{F}}(H)$ — set of the irreducible characters of H over $\mathbb{F}$

$\mathbb{Z}\,\mathrm{irr}_{\mathbb{F}}(H)$ — set of $\mathbb{Z}$-linear combinations of the irreducible characters of H over $\mathbb{F}$ (if $\mathbb{Z} \subseteq \mathbb{F}$)

$\mathrm{Irr}_{\mathbb{F}}(H)$ — set of the isomorphisms classes of irreducible $\mathbb{F}[H]$-modules

$(\mathbf{O}, \mathbf{K}, \mathbf{k})$ — a splitting ℓ-modular system for H where $\mathbf{O}$ is a complete discrete valuation ring of characteristic 0 with maximal ideal M, the fraction field $\mathbf{K} = \mathrm{Frac}(\mathbf{O})$ has characteristic 0 and enough roots of unity (contains all the $|H|$th roots of unity) and $\mathbf{k} = \mathbf{O}/M$ is an algebraically closed field of characteristic ℓ, i.e. $\mathbf{k} = \overline{\mathbb{F}}_\ell$

$\mathrm{Proj}(H)$ — set of projective $\mathbf{k}[H]$-modules

χ_V — character associated to $V \in \mathbf{K}[H]$-mod

V_ϕ — a $\mathbf{K}[H]$-module with character $\phi \in \mathrm{irr}_{\mathbf{K}}(H)$

P_W — projective indecomposable module, projective cover of $W \in \mathrm{Irr}_k(H)$

$P^{\mathbf{O}}$ — the $\mathbf{O}[H]$-module (unique up to isomorphism) such that $P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{k} \cong P$ for $P \in \mathrm{Proj}(H)$

$\Psi_P \coloneqq \phi_{P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}}$ — character of the $\mathbf{K}[H]$-module $P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}$ for $P \in \mathrm{Proj}(H)$

$V_{\mathbf{O}}$ — free $\mathbf{O}[H]$-module such that $V_{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K} \cong V$ for V a $\mathbf{K}[H]$-module

$\langle \chi, \psi \rangle$ — scalar product of two characters χ and ψ of H,

$$\langle \chi, \psi \rangle \coloneqq \frac{1}{|H|} \sum_{h \in H} \chi(h)\overline{\psi(h)} = \langle V_\chi, V_\psi \rangle_{\mathbf{K}}$$

$\langle V, V' \rangle_{\mathbb{F}}$ — scalar product of two $\mathbb{F}[H]$-modules V, V',

$$\langle V, V' \rangle_{\mathbb{F}} \coloneqq \dim_{\mathbb{F}} \mathrm{Hom}_{\mathbb{F}H}(V, V')$$

$[P, V]$ — the decomposition number of V into $P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}$,

$$[P, V] \coloneqq \langle P, V_{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{k} \rangle_{\mathbf{k}} = \langle P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}, V \rangle_{\mathbf{K}} = \langle \Psi_P, \phi_V \rangle$$

for $P \in \mathrm{Proj}(H)$ and $V \in \mathbf{K}[H]$-mod

D^H — the decomposition matrix of H with entries indexed by $V \in \mathrm{Irr}_{\mathbf{K}}(H)$ and $W \in \mathrm{Irr}_{\mathbf{k}}(H)$

$$d^H_{V,W} = d^H_{\phi_V, \Psi_P} \coloneqq [P_W, V]$$

(Algebraic) group structure

For H a group, any elements $h, h' \in H$ and any subset $J \subseteq H$, we use the following notation.

$\lvert H \rvert$	order of H
$C_H(h), C_H(J)$	centraliser of h or J in H
$N_H(h), N_H(J)$	normaliser of h or J in H
$Z(H)$	centre of H
$[H, H]$	derived subgroup of H
$(h)_H$	conjugacy class of h in H
${}^h h'$, ${}^h J$	conjugation by h on the left, ${}^h h' \coloneqq h h' h^{-1}$, ${}^h J \coloneqq \{{}^h j \mid j \in J\}$
h'^h, J^h	conjugation by h on the right $h'^h \coloneqq h^{-1} h' h$, ${}^h J \coloneqq \{j^h \mid j \in J\}$

Let $\mathbf{G}$ be an algebraic group.

$\mathbf{G}^\circ$	connected component of $\mathbf{G}$ containing the identity
$\mathrm{ad}(g) : \mathbf{G} \to \mathbf{G}$	adjoint map sending $h \mapsto g h g^{-1}$ for $h \in \mathbf{G}$
$\mathrm{Lie}(\mathbf{G})$	the Lie algebra of $\mathbf{G}$, the group $\mathbf{G}$ acts on $\mathrm{Lie}(\mathbf{G})$ via the differential of the adjoint map

Chapter 1

Finite groups of Lie type

In this chapter, we gather all the principal notions and necessary results concerning the finite groups of Lie type. Roughly speaking, these groups are the fixed points under an endomorphism of a (connected reductive) linear algebraic group, i.e. an affine variety equipped with a group structure such that the group operation and the inversion are morphisms of varieties. This geometric aspect has crucial repercussions on the description of the finite groups of Lie type.

We recall the following vocabulary:

Definition 1.0.1. Let $\mathbf{G}$ be an algebraic group over $k \coloneqq \overline{\mathbb{F}}_p$, where p is a prime number. We say that $g \in \mathbf{G}$ is **unipotent** if g is a p-element. If g is a p'-element, g is said to be **semisimple**.
We denote by $R_u(\mathbf{G})$, the unipotent radical of $\mathbf{G}$, that is the maximal connected normal subgroup of $\mathbf{G}$ containing only unipotent elements. If $R_u(\mathbf{G}) = \{1\}$, we say that $\mathbf{G}$ is **reductive**. A connected reductive algebraic group is called **semisimple** if its center is finite. Lastly, if $\mathbf{G}$ is non-trivial and contains no proper, non-trivial closed connected normal subgroups, then $\mathbf{G}$ is said to be **simple**.

The main results of this thesis (Chapter 6) are concerned with simple or semisimple groups. However, we will often come across reductive groups, for instance as subgroups. We notice that connected reductive groups $\mathbf{G}$ are in some sense not too far from being semisimple. We have $\mathbf{G} = [\mathbf{G}, \mathbf{G}]Z^\circ(\mathbf{G})$ and the derived subgroup $[\mathbf{G}, \mathbf{G}]$ is semisimple (see [MT11, Cor. 8.22]).

In Section 1.1, we state the classification of connected reductive groups thanks to their root data. We review in the following section how these notions translate to finite groups after taking fixed points. It will allow us to study these groups and their representations in a generic way. As later on we will consider class functions, we use the last section to give an overview of the unipotent and semisimple conjugacy classes as well as their centralisers.
All the material exposed in this chapter can be found in greater detail in graduate textbooks. We mostly follow [GM20], [MT11] and [Car85].

1.1 Reductive groups

We assume that the reader is familiar with some basic notions concerning linear algebraic groups and algebraic geometry. If wanted, the books of Geck [Gec03] and Hartshorne [Har77] as well as Section 1.1 of [GM20] provide great introduction.

The main purpose of this section is to recall the definition of root datum and how it classifies the connected reductive algebraic groups. This combinatorial notion and some variants are powerful tools used to describe algebraic groups, Lie algebras, finite reflection groups, and other related concepts.

1.1.1 Root data

The notion of root datum was first introduced in [DG11, Exposé XXI]. We state here the definition and some basic properties, following [GM20, § 1.2] and [MT11, Appendix A].

Definition 1.1.1 ([GM20, § 1.2.1])**.** Let X and $\breve{X}$ be free abelian groups of the same finite rank such that there is a bilinear pairing $\langle\,,\,\rangle : X \times \breve{X} \to \mathbb{Z}$ which induces isomorphisms $\breve{X} \cong \mathrm{Hom}(X, \mathbb{Z})$ and $X \cong \mathrm{Hom}(\breve{X}, \mathbb{Z})$, i.e., a **perfect** pairing. Let $\Phi \subseteq X$ and $\breve{\Phi} \subseteq \breve{X}$ be finite subsets. The quadruple $(X, \Phi, \breve{X}, \breve{\Phi})$ is called a (reduced) **root datum** if the following conditions hold.

(Φ1) There is a bijection $\Phi \xrightarrow{\sim} \breve{\Phi}, \alpha \mapsto \breve{\alpha}$, such that $\langle \alpha, \breve{\alpha} \rangle = 2$ for all $\alpha \in \Phi$.

(Φ2) If $\alpha \in \Phi$, then $2\alpha \notin \Phi$.

(Φ3) For $\alpha \in \Phi$, we define endomorphisms

$$s_\alpha : X \to X \qquad\qquad \breve{s}_\alpha : \breve{X} \to \breve{X}$$
$$\lambda \mapsto \lambda - \langle \lambda, \breve{\alpha} \rangle \alpha \qquad\qquad \nu \mapsto \nu - \langle \alpha, \nu \rangle \breve{\alpha}$$

and we require that $s_\alpha(\Phi) = \Phi$ and $\breve{s}_\alpha(\breve{\Phi}) = \breve{\Phi}$ for all $\alpha \in \Phi$.

We call the elements in Φ the **roots** and the elements in $\breve{\Phi}$ the **co-roots**.
The sets $W := \langle s_\alpha \mid \alpha \in \Phi \rangle$ and $\breve{W} := \langle \breve{s}_\alpha \mid \alpha \in \Phi \rangle$ are the **Weyl groups** of Φ and $\breve{\Phi}$ respectively.

By [GM20, Lem. 1.2.3a], there is a unique group isomorphism $\delta : W \xrightarrow{\sim} \breve{W}$ such that $\delta(s_\alpha) = \breve{s}_\alpha$ for each $\alpha \in \Phi$. Moreover,

$$\langle w^{-1}(\lambda), \nu \rangle = \langle \lambda, \delta(w)\nu \rangle \quad \text{for all } w \in W,\ \lambda \in X,\ \nu \in \breve{X}.$$

From now on, we identify W with $\breve{W}$ using the isomorphism δ.

Since $X \cong \mathrm{Hom}(\breve{X}, \mathbb{Z})$, we can see X as a subgroup of $\mathrm{Hom}(\mathbb{Z}\breve{\Phi}, \mathbb{Z})$. If they are equal, we say that the root datum is **simply connected**. On the other hand, we always have $\mathbb{Z}\Phi \subseteq X$. If the two sets are equal, we say that the root datum is of **adjoint** type (c.f. [MT11, Def. 9.14]).

Remark that we can extend scalars from $\mathbb{Z}$ to $\mathbb{Q}$, setting $X_{\mathbb{Q}} \coloneqq X \otimes_{\mathbb{Z}} \mathbb{Q}$. In that case, Φ is a reduced crystallographic root system in the subspace $\mathbb{Q}\Phi$ of $X_{\mathbb{Q}}$ ([Bou68, Chap. VI, § 1, Déf. 1]). Therefore, there is a subset $\Delta \subseteq \Phi$ which is linearly independent in $\mathbb{Q}\Phi$ and such that every root can be written as either a $\mathbb{Q}_{\leq 0}$-linear combination or a $\mathbb{Q}_{\geq 0}$-linear combination of elements in Δ. We say that Δ is a **base** for Φ and we call its elements **simple** roots. If $\alpha \in \Phi$ is such that $\alpha = \sum_{\beta \in \Delta} x_\beta \beta$ with $x_\beta \in \mathbb{Q}_{\geq 0}$ for $\beta \in \Delta$, we call α a **positive root**. The set of positive roots is denoted by Φ^+ and we set $\Phi^- \coloneqq -\Phi^+$. We thus have $\Phi = \Phi^+ \sqcup \Phi^-$. In fact, it can be shown that every positive root is a $\mathbb{Z}_{\geq 0}$-linear combination of elements in Δ.
Moreover, W is a Weyl group with generators $\{s_\beta \mid \beta \in \Delta\}$ and relations $(s_\beta s_\gamma)^{m_{\beta\gamma}} = 1$ where $m_{\beta\gamma}$ denotes the order of $s_\beta s_\gamma$. To a Weyl group, one associates a **Dynkin diagram** defined as follows. Its vertices are labelled by Δ. For $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$, the corresponding vertices are joined by $|\langle \beta, \check{\alpha} \rangle|$ edges if $|\langle \beta, \check{\alpha} \rangle| \leq |\langle \alpha, \check{\beta} \rangle|$. If moreover, $|\langle \beta, \check{\alpha} \rangle| > 1$, the edge is oriented towards the vertex labelled α.
We say that a root system Φ is **indecomposable** if the underlying graph of its associated Dynkin diagram is connected. Weyl groups are determined by their Dynkin diagrams. Furthermore, connected Dynkin diagrams have been classified (see Table 1.1.1), and so have root systems [Hum78, Thm. 11.4]. We also fix a labelling of the simple roots of each indecomposable root system following the notation taken in CHEVIE [Mic15].

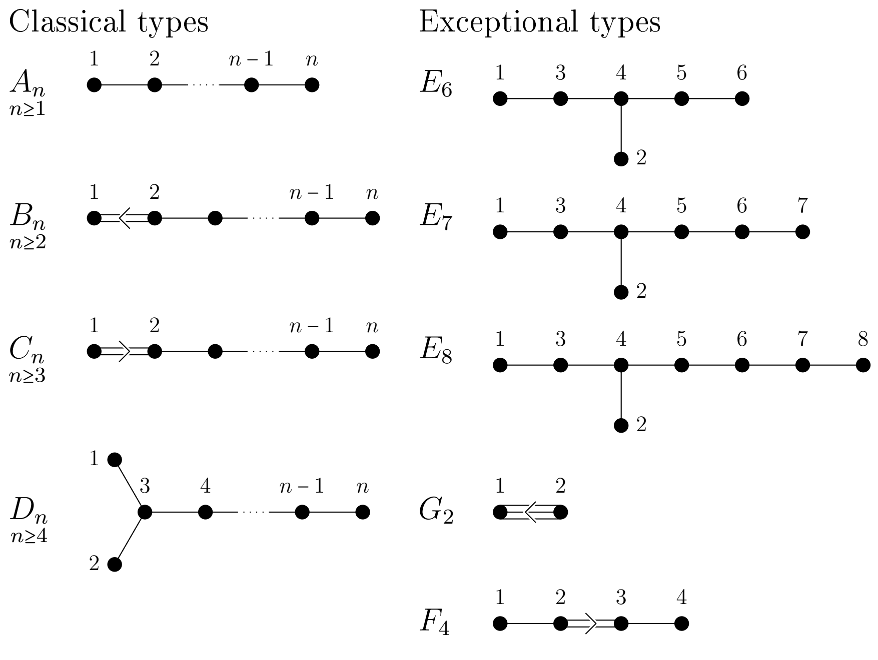


Table 1.1: Dynkin diagrams of the indecomposable crystallographic root systems

Lastly, we observe that if $(X, \Phi, \check{X}, \check{\Phi})$ is a root datum, then $(\check{X}, \check{\Phi}, X, \Phi)$ is also a root datum, called the **dual root datum** ([GM20, Lem. 1.2.3b]). The notion of dual comes

from the fact that the dual of $\mathbb{Q} \otimes_{\mathbb{Z}} X$ with respect to the pairing $\langle\, , \rangle$ can be identified with $\mathbb{Q} \otimes_{\mathbb{Z}} \check{X}$. In particular, we define for each simple root $\alpha \in \Delta$ a **fundamental co-weight** $\check{\omega}_\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} \check{X}$ such that $\langle \alpha, \check{\omega}_\beta \rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$.
Notice that the two root systems Φ and $\check{\Phi}$ are not always isomorphic. However, this is the case if Φ is of exceptional type since we easily check that the Dynkin diagrams are isomorphic. Moreover, if the root datum is adjoint (resp. simply connected) then its dual is simply connected (resp. adjoint), see [GM20, Ex. 1.5.20].

1.1.2 Root data of reductive groups

We can always associate a root datum to a connected reductive algebraic group, and thus deduce a classification. In this subsection, we explain how this process works. Let us consider $\mathbf{G}$, a connected reductive algebraic group over k. We also fix a maximal torus $\mathbf{T} \leq \mathbf{G}$, that is, an abelian algebraic subgroup of $\mathbf{G}$, isomorphic to a direct product of finite copies of $k^\times$ of maximal dimension.
We first define two free abelian groups of the same rank with a perfect pairing following [GM20, § 1.1.11].

Definition 1.1.2. A homomorphism of algebraic groups $\lambda : \mathbf{G} \to k^\times$ is called a **character** and the abelian group of all characters is denoted by $X(\mathbf{G})$. Symmetrically, a homomorphism of algebraic groups $\nu : k^\times \to \mathbf{G}$ is a **co-character** and it belongs to $\check{X}(\mathbf{G})$.

We consider $X(\mathbf{T})$ and $\check{X}(\mathbf{T})$. These two groups are free abelian, of finite rank the dimension of $\mathbf{T}$. We can also define a pairing $\langle\, , \rangle : X(\mathbf{T}) \times \check{X}(\mathbf{T}) \to \mathbb{Z}$ by the condition that $\lambda(\nu(\xi)) = \xi^{\langle \lambda, \nu \rangle}$ for all $\lambda \in X(\mathbf{T}), \nu \in \check{X}(\mathbf{T})$ and $\xi \in k^\times$.

We now define the roots, following [GM20, §1.1.12]. Let $\mathrm{Lie}(\mathbf{G})$ be the Lie algebra of $\mathbf{G}$. The maximal torus $\mathbf{T}$ acts on $\mathrm{Lie}(\mathbf{G})$ via the adjoint representation. To each character $\lambda \in X(\mathbf{T})$, we associate the weight subspace

$$\mathrm{Lie}(\mathbf{G})_\lambda := \{x \in \mathrm{Lie}(\mathbf{G}) \mid t.x = \lambda(t)x \quad \text{for all}\, t \in \mathbf{T}\}.$$

If $\mathrm{Lie}(\mathbf{G})_\lambda$ is not empty and $\lambda \neq 0$, we say that λ is a **root of G relative to T**. We denote by $\Phi(\mathbf{G}, \mathbf{T}) := \Phi(\mathbf{T})$ the set of all roots of $\mathbf{G}$ relative to $\mathbf{T}$.
Since $\mathbf{G}$ is reductive, we have $\dim \mathrm{Lie}(\mathbf{G})_\alpha = 1$ for all $\alpha \in \Phi(\mathbf{T})$ (see [GM20, §1.1.12]). We set $\mathbf{U}_\alpha(\mathbf{T})$ for the unique one-dimensional closed connected unipotent subgroup of $\mathbf{G}$ normalized by $\mathbf{T}$ with $\mathrm{Lie}(\mathbf{U}_\alpha(\mathbf{T})) = \mathrm{Lie}(\mathbf{G})_\alpha$, and we called it a **root subgroup**. Note that there is a canonical way to embed the Lie algebra $\mathrm{Lie}(\mathbf{U}_\alpha(\mathbf{T}))$ in $\mathrm{Lie}(\mathbf{G})$.
Associated to a maximal torus, there is a **Weyl** group $W^{\mathbf{G}}(\mathbf{T}) := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. This group acts via automorphisms on $X(\mathbf{T})$ and $\check{X}(\mathbf{T})$ as follows. For $w \in W^{\mathbf{G}}(\mathbf{T})$, we denote by $\dot{w}$ a representative of w in $N_{\mathbf{G}}(\mathbf{T})$. For $\lambda \in X(\mathbf{T})$ and $\nu \in \check{X}(\mathbf{T})$, we set

$$w.\lambda(t) := \lambda(\dot{w}^{-1} t \dot{w}) \quad \text{for}\, t \in \mathbf{T} \quad \text{and} \quad (w.\nu)(\xi) := \dot{w}\nu(\xi)\dot{w}^{-1} \quad \text{for}\, \xi \in k^\times.$$

Following [GM20, §1.3.1-1.3.2], we define for each $\alpha \in \Phi(\mathbf{T})$ a reflection $s_\alpha \in W^{\mathbf{G}}(\mathbf{T})$ and a co-root $\check{\alpha} \in \check{\Phi}(\mathbf{T})$ such that

$$s_\alpha.\lambda = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha \quad \text{for all } \lambda \in X(\mathbf{T}).$$

Then the quadruple $(X(\mathbf{T}), \Phi(\mathbf{T}), \check{X}(\mathbf{T}), \check{\Phi}(\mathbf{T}))$ is a root datum with Weyl group $W^{\mathbf{G}}(\mathbf{T})$.

There is a natural notion of isomorphisms of root data, see [GM20, §1.2.2]. If we choose another maximal torus $\mathbf{T}'$ of $\mathbf{G}$ then the root data $(X(\mathbf{T}), \Phi(\mathbf{T}), \check{X}(\mathbf{T}), \check{\Phi}(\mathbf{T}))$ and $(X(\mathbf{T}'), \Phi(\mathbf{T}'), \check{X}(\mathbf{T}'), \check{\Phi}(\mathbf{T}'))$ are isomorphic. Therefore, we might now speak of *the* root datum of $\mathbf{G}$. Moreover, the root data classify the connected reductive groups.

Theorem 1.1.3 (Chevalley Classification Theorem, [Spr09, Thm. 9.6.2, Thm. 10.1.1])**.** *Two connected reductive algebraic groups over k are isomorphic if and only if they have isomorphic root data. Furthermore, for each root datum there exists a connected reductive algebraic group which realises it. Lastly, a connected reductive group is simple if and only it is a semisimple group with an indecomposable root datum.*

Notation 1.1.4. If the context is clear, we might drop the symbol $\mathbf{T}$ and write

$$(X, \Phi, \check{X}, \check{\Phi}) \coloneqq (X(\mathbf{T}), \Phi(\mathbf{T}), \check{X}(\mathbf{T}), \check{\Phi}(\mathbf{T})) \quad \text{and} \quad W \coloneqq W^{\mathbf{G}} \coloneqq W^{\mathbf{G}}(\mathbf{T}).$$

As stated in Theorem 1.1.3, the root datum associated to $\mathbf{G}$ contains a lot of information on the structure of $\mathbf{G}$. We have

$$\mathbf{G} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi \rangle \quad \text{see [MT11, Thm. 8.17(g)].}$$

We could wonder what happens if we take a subset of Φ instead. Let us now fix Δ a base for Φ and Φ^+ the corresponding positive roots. For instance, we may consider

$$\mathbf{B} = \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \rangle.$$

This is a **Borel** subgroup of $\mathbf{G}$, i.e., a maximal closed connected solvable subgroup of $\mathbf{G}$. Recall that all Borel subgroups of $\mathbf{G}$ are conjugate ([MT11, Thm. 6.4]).

Remark 1.1.5. If we choose another base for Φ, then we get another Borel subgroup of $\mathbf{G}$. Conversely, if we fix a Borel subgroup $\mathbf{B}' \subseteq \mathbf{G}$ with $\mathbf{T} \subseteq \mathbf{B}'$, there is a unique base of $\mathbf{G}$ such that $\mathbf{B}'$ is generated by $\mathbf{T}$ and the positive root subgroups relative to this new base [GM20, Rmk. 1.3.4]. Therefore, we might sometimes speak of the root datum of $\mathbf{G}$ relative to $\mathbf{T}$ and $\mathbf{B}$ to indicate that we have fixed a base.

For a subset $I \subseteq \Delta$, we define

$$\Phi_I \coloneqq \Phi \cap \sum_{\alpha \in I} \mathbb{Z}\alpha.$$

Then Φ_I is a root system in $\mathbb{Q}\Phi_I$ with base I and Weyl group $W_I \coloneqq \langle s_\alpha \mid \alpha \in \Phi_I \rangle$, see [MT11, Prop. 12.1]. We define

$$\mathbf{P}_I \coloneqq \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \cup \Phi_I \rangle.$$

The subgroup $\mathbf{P}_I$ is closed, connected, self-normalising and contains $\mathbf{B}$. All overgroups of $\mathbf{B}$ in $\mathbf{G}$ arise in this way. Moreover, the subgroup $\mathbf{P}_I$ is isomorphic to $\mathbf{P}_J$ for $J \subseteq \Delta$ if and only if $I = J$ ([MT11, Prop. 12.2]). We call $\mathbf{P}_I$ a **standard parabolic subgroup** of $\mathbf{G}$ and a $\mathbf{G}$-conjugate of $\mathbf{P}_I$ is simply said to be **parabolic**.
Notice that the parabolic subgroups are not necessarily reductive. However, we can decompose $\mathbf{P}_I$ into its unipotent radical

$$R_u(\mathbf{P}_I) = \langle \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \backslash \Phi_I \rangle =: \mathbf{U}_I$$

and a complement group $\mathbf{L}_I := \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi_I \rangle$, which is connected reductive. We write $\mathbf{P}_I = \mathbf{U}_I \rtimes \mathbf{L}_I$ and this decomposition is called the **Levi decomposition** of $\mathbf{P}_I$, c.f. [MT11, Prop. 12.6, Def. 12.7]. We say that $\mathbf{L}_I$ is a (standard) **Levi** subgroup. It has root system Φ_I. More generally, for $J \subseteq \Phi$, we denote by Φ_J the root system generated by the roots in J, i.e.

$$\Phi_J := \Phi \cap \sum_{\alpha \in J} \mathbb{Z}\alpha.$$

and W_J the Weyl group generated by the reflections s_α for $\alpha \in \Phi_J$.

Another important way of rewriting $\mathbf{G}$ is through the Bruhat decomposition.

Theorem 1.1.6 (Bruhat decomposition, [MT11, Thm. 11.17])**.** *For $w \in W$, we fix a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T})$. The group $\mathbf{G}$ can be decomposed as follows:*

$$\mathbf{G} = \bigsqcup_{w \in W} \mathbf{B}\dot{w}\mathbf{B}.$$

More precisely, every $g \in \mathbf{G}$ can be written uniquely as $g = u\dot{w}b$ where $b \in \mathbf{B}$, $w \in W$ and $u \in \langle \mathbf{U}_\alpha \mid \alpha \in \Phi^+, w.\alpha \in \Phi^- \rangle$.

This result comes from the fact that $\mathbf{B}$ and $N_{\mathbf{G}}(\mathbf{T})$ form a BN-pair for $\mathbf{G}$ in the sense of [MT11, Def. 11.15].

Lastly, we consider the various possibilities for a semisimple group with a fixed root system Φ (but different root data). Recall that a root datum might be adjoint or simply connected or neither. If the root datum of a semisimple group is adjoint (resp. simply connected), we say that $\mathbf{G}$ is **adjoint** (resp. **simply connected**). If $\mathbf{G}$ is adjoint, then its centre is trivial. In general, we have $Z(\mathbf{G}) \cong \mathrm{Hom}(X/\mathbb{Z}\Phi, k^\times)$ ([GM20, Rmk. 1.3.5(b)]).

Proposition 1.1.7 ([GM20, Prop. 1.5.8])**.** *Let $\mathbf{G}$ be a semisimple group with root datum $(X, \Phi, \check{X}, \check{\Phi})$ with respect to a maximal torus $\mathbf{T}$ and some Borel subgroup $\mathbf{B} \geq \mathbf{T}$.*
There exists a surjective homomorphism $\tilde{f} : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ where $\mathbf{G}_{\mathrm{sc}}$ is simply connected semisimple with root datum $(X(\mathbf{T}_{\mathrm{sc}}), \Phi(\mathbf{T}_{\mathrm{sc}}), \check{X}(\mathbf{T}_{\mathrm{sc}}), \check{\Phi}(\mathbf{T}_{\mathrm{sc}}))$ relative to a maximal torus $\mathbf{T}_{\mathrm{sc}} \subseteq \mathbf{B}_{\mathrm{sc}}$, and $\Phi(\mathbf{T}_{\mathrm{sc}}) = \Phi$. Moreover, $\tilde{f}$ has finite central kernel, $\tilde{f}(\mathbf{T}_{\mathrm{sc}}) = \mathbf{T}$ and $\tilde{f}(\mathbf{B}_{\mathrm{sc}}) = \mathbf{B}$.
Symmetrically, there exists a surjective homomorphism $f' : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$ where $\mathbf{G}_{\mathrm{ad}}$ is adjoint semisimple with root datum $(X(\mathbf{T}_{\mathrm{ad}}), \Phi(\mathbf{T}_{\mathrm{ad}}), \check{X}(\mathbf{T}_{\mathrm{ad}}), \check{\Phi}(\mathbf{T}_{\mathrm{ad}}))$ relative to a maximal torus $\mathbf{T}_{\mathrm{ad}} \subseteq \mathbf{B}_{\mathrm{ad}}$ and $\Phi(\mathbf{T}_{\mathrm{ad}}) = \Phi$. Moreover, f' has finite central kernel, $f'(\mathbf{T}) = \mathbf{T}_{\mathrm{ad}}$ and $f'(\mathbf{B}) = \mathbf{B}_{\mathrm{ad}}$.

Simply connected or adjoint groups satisfy interesting properties that can be carried over to any semisimple group thanks to the above proposition. In the rest of this thesis, we will often consider adjoint groups in order to avoid a disconnected centre.

1.2 Finite reductive groups

Thanks to the previous section, we can now recall how the classification of reductive groups passes down to finite groups of Lie type. We start by stating some basic properties of Frobenius endomorphisms. We then use Section 1.1 to infer the definition of a complete root datum, as a way to classify our objects of study.

1.2.1 Definition and properties of the Frobenius

As stated in [MT11, Thm. 1.7], algebraic groups can be seen as matrix groups with coefficients over the infinite field $k = \overline{\mathbb{F}}_p$. We would like to study their finite counterparts, that is matrix groups defined over a finite field $\mathbb{F}_q$ for a p-power q.
Firstly, we can see $\mathbb{F}_q$ as the fixed points in k under the **standard Frobenius** map $F_q : k \to k, x \mapsto x^q$. More generally, we denote by F_q any map defined as follows

$$F_q : k^n \xrightarrow{\sim} k^n, \quad (x_1, \ldots, x_n) \mapsto (x_1^q, \ldots, x_n^q), \quad \text{for some}\, n \in \mathbb{Z}_{\geq 1}.$$

This bijection is a morphism of varieties with fixed point set equal to $\mathbb{F}_q^n$.
Moreover, we would like to keep track of the geometrical structure of algebraic groups. An affine variety V is **defined over** $\mathbb{F}_q$ or has an $\mathbb{F}_q$**-rational structure** if there is $n \in \mathbb{Z}_{\geq 1}$ and an isomorphism of affine varieties $\iota : V \to V'$ such that $V' \subseteq k^n$ is closed and stable under the standard Frobenius map F_q. Hence, $F := \iota^{-1} \circ F_q \circ \iota$ is a bijective endomorphism of V, and we call it the **Frobenius morphism of** V **with respect to the** $\mathbb{F}_q$**-structure**. We write V^F for the fixed points of V under F_q and in fact

$$V^F \cong V'^{F_q} = \{v \in V' \mid v \in \mathbb{F}_q^n\}.$$

In particular, this definition applies to any algebraic group $\mathbf{G}$ seen as an affine variety over k. Nonetheless, we doubtless also want to keep the group structure in mind. Thus, we additionally require that $F : \mathbf{G} \to \mathbf{G}$ is a group homomorphism. If this is the case, we say that $\mathbf{G}$ is **defined over** $\mathbb{F}_q$ as an algebraic group [GM20, §1.4.5]. Note that the group of fixed points $\mathbf{G}^F$ is a finite group. However, we do not get all the finite groups encompassed in the notion of finite reductive groups. For instance, we miss the Suzuki and Ree groups. Thus, we need to extend the definition.

Definition 1.2.1 ([GM20, Def. 1.4.7])**.** Let $\mathbf{G}$ be an algebraic group. An endomorphism of algebraic groups $F : \mathbf{G} \to \mathbf{G}$ is a **Steinberg** endomorphism if there exists $m \in \mathbb{N}$ such that F^m is the Frobenius morphism of the group $\mathbf{G}$ with respect to some $\mathbb{F}_q$-structure, for a p-power q.

The fixed point set of a Steinberg endomorphism is always finite, and if $\mathbf{G}$ is simple this property gives another characterisation of a Steinberg endomorphism, c.f. [Ste68, Thm. 10.13].

Definition 1.2.2 ([GM20, Def. 1.4.7])**.** Let $\mathbf{G}$ be a connected reductive group and let $F: \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism. We call $G \coloneqq \mathbf{G}^F$ a **finite group of Lie type** or a **finite reductive group**.

Notation 1.2.3. From now on, the bold script $\mathbf{G}$ always denotes the algebraic group while the normal script G is used for the finite group, provided that we have fixed a Steinberg endomorphism F. This applies to any algebraic group with a fixed Steinberg endomorphism. For instance, if $\mathbf{L}$ is an F-stable Levi subgroup of $\mathbf{G}$, we write $L \coloneqq \mathbf{L}^F$.

An indispensable tool to transfer information on algebraic groups to finite groups is the classical Lang–Steinberg theorem.

Theorem 1.2.4 (Lang–Steinberg Theorem, [Lan56],[Ste68, Thm. 10.1])**.** *Let* $\mathbf{G}$ *be a connected algebraic group and* $F: \mathbf{G} \to \mathbf{G}$ *a Steinberg endomorphism. Then the following map is surjective:*

$$\begin{aligned}\mathscr{L}: \mathbf{G} &\to \mathbf{G}\\ g &\mapsto g^{-1}F(g).\end{aligned}$$

Proof. For a proof, we refer the reader to [GM20, Thm. 1.4.8]. □

One application of this result is to understand how an F-stable $\mathbf{G}$-orbit splits into G-orbits.

Theorem 1.2.5 ([MT11, Thm. 21.11])**.** *Let* $\mathbf{G}$ *be a connected algebraic group and let* $F: \mathbf{G} \to \mathbf{G}$ *be a Steinberg endomorphism. Let* $V \neq \varnothing$ *be a set with a transitive* $\mathbf{G}$*-action and a compatible* F*-action* $F': V \to V$ *i.e. for all* $g \in \mathbf{G}, v \in V$*, we have* $F'(g.v) = F(g).F'(v)$*. Then*

(a) there exists $v \in V$ *such that* $F'(v) = v$*,*

(b) and if the stabiliser $Stab_{\mathbf{G}}(v)$ *is closed for some* $v \in V$*, then for any* $v_0 \in V^F$*, there is a natural* $1-1$ *correspondence:*

$$\{G\text{-orbits on} V^{F'}\} \overset{1-1}{\longleftrightarrow} \{F\text{-classes in} Stab_{\mathbf{G}}(v_0)/Stab^{\circ}_{\mathbf{G}}(v_0)\}.$$

Here the F-classes are the orbits of $\mathrm{Stab}_{\mathbf{G}}(v_0)/\mathrm{Stab}^{\circ}_{\mathbf{G}}(v_0)$ under F**-conjugation**. We say that two elements $g, g' \in \mathrm{Stab}_{\mathbf{G}}(v_0)/\mathrm{Stab}^{\circ}_{\mathbf{G}}(v_0)$ are F-conjugate if there exists an element $h \in \mathbf{G}$ such that $g = F(h)gh^{-1}$.

Remark 1.2.6. Let $\mathbf{G}$ be an algebraic group and $F: \mathbf{G} \to \mathbf{G}$ be a Steinberg endomorphism. Let $\mathbf{H}$ be a F-stable connected normal subgroup of $\mathbf{G}$. Then

$$\mathbf{G}^F/\mathbf{H}^F \cong (\mathbf{G}/\mathbf{H})^F.$$

Indeed, consider the map $f : \mathbf{G}^F \to (\mathbf{G}/\mathbf{H})^F$, $g \mapsto g\mathbf{H}$. It is well-defined. We check that it is surjective. Let $g \in \mathbf{G}$ such that $g\mathbf{H} = F(g\mathbf{H})$. Then, $\mathbf{H}$ acts transitively on $g\mathbf{H}$ and thus, by Theorem 1.2.5, there exists an element $h \in \mathbf{H}$ with $F(gh) = gh \in \mathbf{G}^F$. Now, $f(gh) = gh\mathbf{H} = g\mathbf{H}$ and thus the map f is surjective. Lastly, we observe that the kernel $\ker(f) = \mathbf{G}^F \cap \mathbf{H} = \mathbf{H}^F$ and we conclude that $\mathbf{G}^F/\mathbf{H}^F \cong (\mathbf{G}/\mathbf{H})^F$.

As a consequence of the previous theorem, we get the following result.

Corollary 1.2.7 ([MT11, Cor. 21.12]). *Let* $\mathbf{G}$ *be a connected algebraic group and let* $F : \mathbf{G} \to \mathbf{G}$ *be a Steinberg endomorphism. Up to G-conjugation, there exists a unique pair* $(\mathbf{T}, \mathbf{B})$ *consisting of an F-stable maximal torus* $\mathbf{T}$ *of* $\mathbf{G}$ *contained in an F-stable Borel subgroup* $\mathbf{B}$.

Definition 1.2.8. Let $\mathbf{G}$ be a connected algebraic group and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg endomorphism. An F-stable maximal torus of $\mathbf{G}$ contained in an F-stable Borel subgroup is said to be **maximally split**.

1.2.2 Classification of the finite reductive groups

We now come back to the notion of root datum and how it interacts with the Steinberg endomorphisms. Let $\mathbf{G}$ be a connected reductive group and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg map. We also fix a maximally split torus $\mathbf{T}_0$ of $\mathbf{G}$ contained in an F-stable Borel $\mathbf{B}_0$ and $(X, \Phi, \check{X}, \check{\Phi})$ the root datum of $\mathbf{G}$ relative to $\mathbf{T}_0$. By [GM20, Rmk. 1.3.4], there is a unique base Δ of Φ such that

$$\mathbf{B}_0 = \langle \mathbf{T}_0, \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \rangle.$$

Since both $\mathbf{B}_0$ and $\mathbf{T}_0$ are F-stable, the map F permutes the root subgroups $\mathbf{U}_\alpha$ for $\alpha \in \Phi^+$. Thus, F induces a permutation $\alpha \mapsto \alpha^\dagger$ on Φ^+ which must leave Δ invariant (see [MT11, Pf. of Lem. 11.10]). More precisely, $F : X \to X$ is a p-isogeny of root data in the sense of [GM20, Def. 1.2.9].

We describe the action of F on X. We set $X_\mathbb{R} := X \otimes_\mathbb{Z} \mathbb{R}$. There is $d \geq 0$ such that $F^d_{|X} = q^d \mathrm{id}_X$ and $F_{|X_\mathbb{R}} = qF_0$ for some q a fractional power of p and $F_0 \in \mathrm{Aut}(X_\mathbb{R})$ of order d ([MT11, Prop. 22.2]). If $d = 1$, we say that F is **split**.

Let us instead look at the map $F' = \mathrm{ad}(\dot{w}^{-1}) \circ F$ for some fixed $w \in W$ and $\dot{w} \in N_\mathbf{G}(\mathbf{T}_0)$ a representative of w. By [GM20, Lem. 1.4.14], the map F' is a Steinberg endomorphism and $\mathbf{G}^F \cong \mathbf{G}^{F'}$. Moreover, the torus $\mathbf{T}_0$ is also F'-stable and the map $F'_0 \in \mathrm{Aut}(X_\mathbb{R})$ defined as above for F' is such that $F'_0 = F_0 \circ w$ ([GM20, Rmk. 1.6.13]).

On the other hand, we could consider $\phi \in \mathrm{Aut}(X_\mathbb{R})$ a p-isogeny for some prime p such that ϕ can be written as $\phi = q\phi_0$ for some $\phi_0 \in \mathrm{Aut}(X_\mathbb{R})$ an invertible linear map of finite order which normalises W and $q \in \mathbb{R}_{\geq 0}$. Then, there exists a Steinberg endomorphism F_ϕ such that F_ϕ induces ϕ on $X_\mathbb{R}$ ([GM20, Thm. 1.3.12, Prop. 1.4.18]). These considerations lead to the following combinatorial definition.

Definition 1.2.9 ([GM20, Def. 1.6.10]). Let $(X, \Phi, \check{X}, \check{\Phi})$ be a root datum with Weyl group W. Let $\phi_0 \in \mathrm{Aut}(X_\mathbb{R})$ be an invertible map of finite order which normalises W.

Assume that $\mathscr{P}$, defined to be the set of $q \in \mathbb{R}_{\geq 0}$ such that $q\phi_0(X) \subseteq X$ and $q\phi_0$ is a p-isogeny of root data, is non-empty. We call the quintuple $\mathbb{G} := (X, \Phi, \check{X}, \check{\Phi}, \phi_0 W)$ a **complete root datum**. We set $\mathscr{P}_{\mathbb{G}} := \mathscr{P}$.

Therefore, we observe that to each complete root datum $\mathbb{G} = (X, \Phi, \check{X}, \check{\Phi}, \phi_0 W)$ and to each $q \in \mathscr{P}_{\mathbb{G}}$, we can associate a connected reductive group $\mathbf{G}$ (unique up to isomorphism) with root datum $(X, \Phi, \check{X}, \check{\Phi})$ and a Steinberg map $F_{q\phi_0}$. Writing $\mathbb{G}(q) := \mathbf{G}^{F_{q\phi_0}}$, we obtain a family of finite groups

$$\{\mathbb{G}(q) \mid q \in \mathscr{P}_{\mathbb{G}}\}$$

called the **series of finite groups of Lie type** defined by $\mathbb{G}$ ([GM20, Rmk. 1.6.12]).

Similarly as for root data, the **dual complete root datum** of $\mathbb{G} = (X, \Phi, \check{X}, \check{\Phi}, \phi_0 W)$ is the complete root datum $\mathbb{G}^* := (\check{X}, \check{\Phi}, X, \Phi, \phi_0^{\mathrm{tr}} W)$ (see [GM20, Ex. 1.6.19]). Here ϕ_0^{tr} is the transpose map defined through the perfect pairing $\langle\, ,\, \rangle : X \times \check{X} \to \mathbb{Z}$ extended to $\langle\, ,\, \rangle : X_{\mathbb{R}} \times \check{X}_{\mathbb{R}} \to \mathbb{R}$. We have $\mathscr{P}_{\mathbb{G}^*} = \mathscr{P}_{\mathbb{G}}$ and for each $q \in \mathscr{P}_{\mathbb{G}}$ we obtain two finite groups $\mathbb{G}(q)$ and $\mathbb{G}^*(q)$ coming respectively from $(\mathbf{G}, F)$ and $(\mathbf{G}^*, F^*)$, where $F = F_{q\phi_0}$ and $F^* = F_{q\phi_0^{\mathrm{tr}}}$. Those two pairs are in duality as in [GM20, Def 1.5.17]. In particular, if $\mathbf{T}_0$ (resp. $\mathbf{T}_0^*$) is a maximally split torus of $\mathbf{G}$ defining the root system $(X, \Phi, \check{X}, \check{\Phi})$ (resp. $(\check{X}, \check{\Phi}, X, \Phi)$) then

$$\lambda \circ F_{|\mathbf{T}_0} = F^*_{|\mathbf{T}_0^*} \circ \lambda \quad \text{for all } \lambda \in X.$$

Let us introduce a little more terminology. The Steinberg endomorphism $F : \mathbf{G} \to \mathbf{G}$ induces as well an automorphism on W, that by abuse of notation, we still denote by $F : W \to W$. In particular, for each $\alpha \in \Phi$, we have $F(s_\alpha) = s_{\alpha^\dagger}$, see [GM20, 1.6.1]. We distinguish between the following cases.

Definition 1.2.10 ([Lus84a, 3.1])**.** If for any $\alpha \neq \beta \in \Phi$ in the same $\dagger$-orbit, the order of the reflection $s_\alpha s_\beta$ is either 2 or 3, we say that F is **ordinary**.
If F induces the identity on W, we say that $G = \mathbf{G}^F$ is **untwisted**. If F is ordinary but not the identity, we say that G is **twisted**. Lastly, if F is not ordinary, the finite group G is called **very twisted**.

Notice that if F is a Frobenius map for $\mathbf{G}$, then F is always ordinary. In this thesis, we will most of the times assume F to be ordinary.

Hypothesis 1. From now on, we fix $\mathbf{G}$ a connected reductive group over k with Steinberg map $F : \mathbf{G} \to \mathbf{G}$. We also let $\mathbf{T}_0 \subseteq \mathbf{B}_0$ be a maximally split torus in an F-stable Borel subgroup $\mathbf{B}_0$ of $\mathbf{G}$ with associated root datum $(X, \Phi, \check{X}, \check{\Phi})$, base Δ of Φ and Weyl group W.

1.3 Interesting conjugacy classes and their centralisers

In the rest of this thesis, conjugacy classes will play a preponderant role as we will study different bases of the space of class functions. Notice that every **rational** conjugacy class $(g)_G$ is contained in a **geometric** conjugacy class $(g)_{\mathbf{G}}$ for $g \in G$. Moreover, we also have $C_G(g) = C_{\mathbf{G}}(g)^F$. We mainly concentrate on geometric conjugacy classes.
Clearly, for any $g \in \mathbf{G}$ there exists a unique semisimple element $g_s \in \mathbf{G}$ and a unique unipotent element $g_u \in \mathbf{G}$ such that $g = g_s g_u = g_u g_s$. This is called the **Jordan decomposition** of g and we will use this notation from now on[1]. In particular, we have $C_{\mathbf{G}}(g) = C_{C_{\mathbf{G}}(g_s)}(g_u)$. Thus, we focus on the conjugacy classes of semisimple and unipotent elements. For each case, we give a parameterisation of the conjugacy classes and a description of their centralisers.

1.3.1 Semisimple conjugacy classes

We start by giving a parameterisation of the semisimple conjugacy classes.

Proposition 1.3.1. *The set of semisimple conjugacy classes of* $\mathbf{G}$ *is in bijection with the orbits of* W *on* $\mathbf{T}_0$. *Moreover, the set of semisimple geometric conjugacy classes of* G *is in bijection with the* F*-stable orbits of* W *on* $\mathbf{T}_0$.

Proof. The proof relies on the Bruhat decomposition, see [Car85, Prop. 3.7.1, Cor. 3.7.2]. □

Centralisers of semisimple elements

Next, we consider the centraliser $C_{\mathbf{G}}(s)$ of a semisimple element $s \in \mathbf{G}$. The element s belongs to a maximal torus $\mathbf{T}$ of $\mathbf{G}$. Since $\mathbf{T}$ is abelian and connected, $\mathbf{T} \leq C^\circ_{\mathbf{G}}(s)$ if and only if $s \in \mathbf{T}$. Moreover, if a unipotent element $u \in \mathbf{G}$ belongs to $C_{\mathbf{G}}(s)$, then in fact $u \in C^\circ_{\mathbf{G}}(s)$ [MT11, Prop. 14.7]. As we can see, the connected centraliser of s already contains a maximal torus and all the unipotent elements of $C_{\mathbf{G}}(s)$. In fact, we sometimes have control on $C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$.

Theorem 1.3.2. *If* $[\mathbf{G}, \mathbf{G}]$ *is simply connected, then* $C_{\mathbf{G}}(s)$ *is connected.*
More generally, if $\mathbf{G}$ *is semisimple and* $\pi : \mathbf{G}_{\mathrm{sc}} \to \mathbf{G}$ *is a simply connected covering of* $\mathbf{G}$ *(as in Proposition 1.1.7), then* $C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$ *is isomorphic to a subgroup of* $\ker(\pi)$. *Moreover, if the order of* s *is prime to the order of* $\ker(\pi)$, *then* $C_{\mathbf{G}}(s)$ *is connected.*

Proof. For the proof of the first statement, see [Car85, Thm. 3.5.6].
For the second fact, let $g \in C_{\mathbf{G}}(s)$ and take $\tilde{s}, \tilde{g} \in \mathbf{G}_{\mathrm{sc}}$ such that $s = \pi(\tilde{s})$ and $g = \pi(\tilde{g})$. Then, $[\tilde{g}, \tilde{s}] = \tilde{g}\tilde{s}\tilde{g}^{-1}\tilde{s}^{-1} \in \ker(\pi)$ and we define a map

$$\nu : C_{\mathbf{G}}(s) \to \ker(\pi), \quad g \mapsto [\tilde{g}, \tilde{s}].$$

[1] The analogous result requires more work when $\mathbf{G}$ is defined over a field of characteristic zero, c.f. [MT11, Sect. 2].

This map is a group homomorphism since $\ker(\pi) \subseteq Z(\mathbf{G}_{\mathrm{sc}})$. Moreover, since $\ker(\pi)$ is finite, the image of $C^\circ_{\mathbf{G}}(s)$ under ν is trivial. On the other hand, if the commutator $[\tilde{g}, \tilde{s}] = 1$, then $\tilde{g} \in C_{\mathbf{G}_{\mathrm{sc}}}(\tilde{s}) = C^\circ_{\mathbf{G}_{\mathrm{sc}}}(\tilde{s})$, whence $g \in C^\circ_{\mathbf{G}}(s)$. Thus, the group homomorphism ν induces a bijective map from $C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$ to a subgroup of A of $\ker(\pi)$.
For the last statement, we show inductively that $[\tilde{g}, \tilde{s}^n] = [\tilde{g}, \tilde{s}]^n$ for any $n \in \mathbb{N}$ since

$$[\tilde{g}, \tilde{s}]^n = [\tilde{g}, \tilde{s}^{n-1}][\tilde{g}, \tilde{s}] = \tilde{g}\tilde{s}^n\tilde{s}^{-1}\tilde{g}^{-1}\tilde{s}\tilde{s}^{-n}[\tilde{g}, \tilde{s}] = \tilde{g}\tilde{s}^n(\tilde{s}^{-1}\tilde{g}^{-1}[\tilde{g}, \tilde{s}]\tilde{s})\tilde{s}^{-n} = [\tilde{g}, \tilde{s}^n].$$

Therefore, if s has order n coprime to the order of $\ker(\pi)$, then $\tilde{s}^n \in Z(\mathbf{G}_{\mathrm{sc}})$ and $[\tilde{g}, \tilde{s}]^n = 1$ for any $g \in C_{\mathbf{G}}(s)$. Thus, every element in A has order dividing n. Since n is coprime to the order of $\ker(\pi)$, it means that A is trivial. Hence, so is $C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$ and $C_{\mathbf{G}}(s)$ is connected. □

We now focus on the structure of $C_{\mathbf{G}}(s)$. Since all maximal tori of $\mathbf{G}$ are conjugate ([MT11, Cor. 6.5]), there is $h \in \mathbf{G}$, such that $s \in {}^h\mathbf{T}_0$. Without loss of generality, we may assume that $s \in \mathbf{T}_0$.

Theorem 1.3.3 ([MT11, Thm. 14.2]). *Let $s \in \mathbf{T}_0$. Let $\Phi(s) \coloneqq \{\alpha \in \Phi \mid \alpha(s) = 1\}$. Then*

$$C_{\mathbf{G}}(s) = \langle \mathbf{T}_0, \mathbf{U}_\alpha, \dot{w} \mid \alpha \in \Phi(s), w \in W \text{ with } s^w = s\rangle,$$

where $\dot{w}$ denotes a representative of W in $N_{\mathbf{G}}(\mathbf{T}_0)$. Moreover,

$$C^\circ_{\mathbf{G}}(s) = \langle \mathbf{T}_0, \mathbf{U}_\alpha \mid \alpha \in \Phi(s)\rangle.$$

Furthermore, the algebraic group $C^\circ_{\mathbf{G}}(s)$ is reductive, with root datum $(X, \Phi(s), \check{X}, \check{\Phi}(s))$ and Weyl group $W^\circ(s) \coloneqq \langle s_\alpha \mid \alpha \in \Phi(s)\rangle$ where $\check{\Phi}(s) \coloneqq \{\check{\alpha} \mid \alpha \in \Phi(s)\}$.

Proof. The proof relies on the Bruhat decomposition, see [MT11, Thm. 14.2]. □

Remark 1.3.4. We keep the notation of Theorem 1.3.3. Let $W(s) \coloneqq \{w \in W \mid s^w = s\}$. Then the quotient $W(s)/W^\circ(s)$ is isomorphic to the quotient $C_{\mathbf{G}}(s)/C^\circ_{\mathbf{G}}(s)$.

As a corollary of the above theorem, we notice that up to conjugation, there is only a finite number of centralisers of semisimple elements, even though there is an infinite number of semisimple conjugacy classes. Moreover, thanks to [Der81], we may always choose s (up to conjugation) such that $\Phi(s)$ is generated by a subset $\Delta(s)$ of $\tilde{\Delta} = \Delta \cup \{-\alpha_0\}$, where α_0 is the highest root of Φ, i.e. $\alpha_0 = \sum_{\alpha\in\Delta} n_\alpha \alpha$ and for any root $\beta = \sum_{\alpha\in\Delta} b_\alpha \alpha \in \Phi$, we have $n_\alpha \geq b_\alpha$ for all simple roots $\alpha \in \Delta$ (see [MT11, Prop. 13.10] for the existence). If $\Delta(s) \subseteq \Delta$, then $C^\circ_{\mathbf{G}}(s)$ is a Levi subgroup of $\mathbf{G}$. Consequently, a **pseudo-Levi** subgroup designates any subgroup of the form $C^\circ_{\mathbf{G}}(s)$ for a semisimple element $s \in \mathbf{G}$, as in [MS03]. This motivates the following definition.

Definition 1.3.5. We say that a semisimple element $s \in \mathbf{G}$ is **quasi-isolated** if $C_{\mathbf{G}}(s)$ is not included in a proper Levi subgroup. If $C^\circ_{\mathbf{G}}(s)$ itself is not contained in a proper Levi subgroup, we call the element s **isolated**.
An element $g \in \mathbf{G}$ with Jordan decomposition $g = g_s g_u$ is **isolated** if its semisimple part g_s is isolated.

Centralisers of isolated and quasi-isolated elements often stand out when studying algebraic groups because it is harder to apply inductive arguments. For instance, if $s \in \mathbf{G}$ is an isolated element, then $|\Delta(s)| = |\Delta|$ ([Bon05, Cor. 1.4]).

Quasi-isolated conjugacy classes

We now give a parameterisation of the semisimple quasi-isolated conjugacy classes, following [Bon05]. This time, we will see that the number of quasi-isolated classes is finite and that moreover, this number does not depend on p, if p is not too small. We first need to introduce some notation.
For the rest of this subsection, we assume that $\mathbf{G}$ is simple (see Remark 1.3.8 for a more general setting). We fix an isomorphism

$$\iota : (\mathbb{Q}/\mathbb{Z})_{p'} \xrightarrow{\sim} k^{\times}, \tag{1.1}$$

and $\tilde{\iota}$ the composition $\mathbb{Q} \to (\mathbb{Q}/\mathbb{Z})_{p'} \to k^{\times}$. We now consider

$$\begin{aligned} \tilde{\iota}_{\mathbf{T}_0} : \mathbb{Q} \otimes_{\mathbb{Z}} \check{X} &\to \mathbf{T}_0 \\ r \otimes \lambda &\mapsto \lambda(\tilde{\iota}(r)). \end{aligned} \tag{1.2}$$

We also fix some maximal tori $\mathbf{T}_{\mathrm{sc}} \subseteq \mathbf{G}_{\mathrm{sc}}$ and $\mathbf{T}_{\mathrm{ad}} \subseteq \mathbf{G}_{\mathrm{ad}}$ as in Proposition 1.1.7. We let

$$\tilde{\Delta}_{p'} \coloneqq \{\alpha \in \tilde{\Delta} \mid \check{\omega}_\alpha / n_\alpha \in \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} \check{X}(\mathbf{T}_{\mathrm{sc}})\},$$

where we set $n_{-\alpha_0} = 1$ and $\check{\omega}_{-\alpha_0} = 0$.
When $\mathbf{G}$ is simply connected, the parameterisation of the semisimple isolated classes is relatively easy to state.

Proposition 1.3.6 ([Bon05, Prop. 4.9]). *Assume that $\mathbf{G}$ is simply connected. The map $\tilde{\Delta}_{p'} \to \mathbf{G}$, $\alpha \mapsto t_\alpha \coloneqq \tilde{\iota}_{\mathbf{T}_0}(\check{\omega}_\alpha / n_\alpha)$ induces a bijection between $\tilde{\Delta}_{p'}$ and the set of conjugacy classes of semisimple isolated elements in $\mathbf{G}$. Moreover, $W^\circ(t_\alpha) = W_{\tilde{\Delta} - \alpha}$ and the order of t_α is equal to $n_\alpha o(\check{\omega}_\alpha)$, where $o(\check{\omega}_\alpha)$ denotes the order of the image of $\check{\omega}_\alpha$ in $\big(\mathbb{Q} \otimes_{\mathbb{Z}} \check{X}(\mathbf{T}_{\mathrm{sc}})\big)/\check{X}(\mathbf{T}_{\mathrm{sc}})$.*

In the other direction, when $\mathbf{G}$ is adjoint, we need to remove subsets of $\tilde{\Delta}$ of size bigger than 1 to accommodate for the quasi-isolated but not isolated elements. We let $\mathscr{Q}_{p'}$ be the set of subsets Q of $\tilde{\Delta}_{p'}$ of size not divisible by p and on which their stabiliser in $\mathscr{N} \coloneqq N_W(\tilde{\Delta})$ acts transitively. Let also $\mathscr{N}_{p'}$ be the set of p'-elements in $\mathscr{N}$.

Proposition 1.3.7 ([Bon05, Thm. 5.1]). *Assume that $\mathbf{G}$ is adjoint. Then a bijection between the set of orbits of $\mathscr{N}_{p'}$ on $\mathscr{Q}_{p'}$ to the set of conjugacy classes of quasi-isolated elements in $\mathbf{G}$ is induced by*

$$\begin{aligned} \mathscr{Q}_{p'} &\to \mathbf{G} \\ Q &\mapsto t_Q \coloneqq \tilde{\iota}_{\mathbf{T}_0}\left(\frac{1}{|Q|} \sum_{\alpha \in Q} \frac{\check{\omega}_\alpha}{n_\alpha}\right). \end{aligned}$$

Moreover, $W^\circ(t_Q) = W_{\tilde{\Delta} - Q}$ and the order of t_Q is equal to n_α for some $\alpha \in Q$.

Remark 1.3.8. For an arbitrary simple algebraic group, we need to replace $\mathscr{N}$ by the preimage of $\check{X}(\mathbf{T}_0)/\check{X}(\mathbf{T}_{\mathrm{sc}})$ under the bijection

$$\begin{aligned}\mathscr{N} &\tilde{\to} \check{X}(\mathbf{T}_{\mathrm{ad}})/\check{X}(\mathbf{T}_{\mathrm{sc}})\\ s_\alpha &\mapsto \check{\omega}_\alpha\end{aligned}$$

to get a parameterisation of the semisimple quasi-isolated conjugacy classes. We refer the reader to [Bon05, Thm. 4.6] for a general result for semisimple groups. Thanks to [Bon05, Prop. 2.3 and Rmk. 2.5], these results can be lifted to any connected reductive group.

If p does not divide any n_α for $\alpha \in \Delta$, then we in fact have $\tilde{\Delta}_{p'} = \tilde{\Delta}$. If for some root $\beta = \sum_{\alpha\in\Delta} b_\alpha \alpha$, the prime p divides some b_α, we say that p is **bad** for $\mathbf{G}$, [Car85, §1.14]. Otherwise, we say that p is **good**. The bad primes for the simple algebraic groups are listed in Table 1.2.

A_n	:	none
$B_n\,(n \geq 2), C_n\,(n \geq 3), D_n\,(n \geq 4)$	:	2
G_2, F_4, E_6, E_7	:	$2, 3$
E_8	:	$2, 3, 5$

Table 1.2: Bad primes for the simple algebraic groups

In the rest of this thesis, we will almost always assume that p is good for $\mathbf{G}$.

We recall a few more facts about isolated elements.

Lemma 1.3.9. *Let $\mathbf{G}$ be a connected reductive group and $s \in \mathbf{G}$ be a semisimple element. If s is isolated, then*

$$Z^\circ(C^\circ_{\mathbf{G}}(s)) = Z^\circ(\mathbf{G}).$$

Proof. The inclusion $Z^\circ(\mathbf{G}) \subseteq Z^\circ(C^\circ_{\mathbf{G}}(s))$ is clear. For the other direction, note that by [MT11, Prop. 12.10], $C_{\mathbf{G}}(Z^\circ(C^\circ_{\mathbf{G}}(s)))$ is a Levi subgroup of $\mathbf{G}$. However, the definition of the centre implies that

$$C^\circ_{\mathbf{G}}(s) \subseteq C_{\mathbf{G}}(Z^\circ(C^\circ_{\mathbf{G}}(s))).$$

Thus, since the semisimple element s is isolated, we must have $C_{\mathbf{G}}(Z^\circ(C^\circ_{\mathbf{G}}(s))) = \mathbf{G}$, hence $Z^\circ(C^\circ_{\mathbf{G}}(s)) \subseteq Z^\circ(\mathbf{G})$. □

Lemma 1.3.10. *Let $\mathbf{G}$ be a connected reductive group and $\mathbf{L}$ a Levi subgroup of $\mathbf{G}$. Fix an isolated semisimple element s of $\mathbf{L}$. Set $\mathbf{L}_s = C^\circ_{\mathbf{L}}(s)$ and $\mathbf{G}_s = C^\circ_{\mathbf{G}}(s)$. Then*

$$\mathbf{L} = C_{\mathbf{G}}(Z^\circ(\mathbf{L}_s)) \quad N_{\mathbf{G}_s}(\mathbf{L}) = N_{\mathbf{G}_s}(\mathbf{L}_s), \text{ and } N_{\mathbf{G}}(\mathbf{L}_s) \subseteq N_{\mathbf{G}}(\mathbf{L}).$$

Proof. Since $\mathbf{L}_s = \mathbf{L} \cap \mathbf{G}_s$, it follows that $N_{\mathbf{G}_s}(\mathbf{L}) \subseteq N_{\mathbf{G}_s}(\mathbf{L}_s)$.
Let us consider the other inclusion and the first assumption. Recall that $\mathbf{L} = C_{\mathbf{G}}(Z^\circ(\mathbf{L}))$ ([MT11, Prop. 12.6]). By the previous Lemma, we know that $Z^\circ(\mathbf{L}) = Z^\circ(\mathbf{L}_s)$, and thus $\mathbf{L} = C_{\mathbf{G}}(Z^\circ(\mathbf{L}_s))$. Now we can conclude since

$$N_{\mathbf{G}_s}(\mathbf{L}_s) \subseteq N_{\mathbf{G}_s}(Z^\circ(\mathbf{L}_s)) \subseteq N_{\mathbf{G}_s}(C_{\mathbf{G}}(Z^\circ(\mathbf{L}_s))) = N_{\mathbf{G}_s}(\mathbf{L}).$$

By a similar argument, we deduce that $N_{\mathbf{G}}(\mathbf{L}_s) \subseteq N_{\mathbf{G}}(\mathbf{L})$. □

1.3.2 Unipotent conjugacy classes

We now turn our gaze to the p-elements. We write $\mathrm{Ucl}(\mathbf{G})$ for the set of unipotent conjugacy classes of $\mathbf{G}$.
Firstly, we explain how a geometric unipotent conjugacy class splits into rational ones. Let $u \in G$ be a unipotent element, then the conjugacy class $C = (u)_{\mathbf{G}}$ is F-stable. Now, the set C^F might not be a unique rational G-conjugacy class. However, applying Theorem 1.2.5 since $C_{\mathbf{G}}(u)$ is closed, we have a bijection between the G-conjugacy classes in C^F and the F-classes in $C_{\mathbf{G}}(u)/C^{\circ}_{\mathbf{G}}(u)$.
The group $A_{\mathbf{G}}(u) \coloneqq C_{\mathbf{G}}(u)/C^{\circ}_{\mathbf{G}}(u)$ will play an important role in the rest of this thesis, for instance for the parameterisation of the unipotent characters. In some cases, the induced action of F on $A_{\mathbf{G}}(u)$ is trivial, and the G-conjugacy classes in C^F are in bijection with the conjugacy classes of $A_{\mathbf{G}}(u)$. We denote by u_C any F-stable element of C such that $A_{\mathbf{G}}(u_C) = A_{\mathbf{G}}(u_C)^F$. If the centre $Z(\mathbf{G})$ of $\mathbf{G}$ is connected and $\mathbf{G}/Z(\mathbf{G})$ is simple, such an element u_C always exists for any $C \in \mathrm{Ucl}(\mathbf{G})$ by [Tay13, Prop. 2.4].

We now give two ways of labelling the unipotent conjugacy classes of $\mathbf{G}$. The first method uses co-characters and Levi subgroups. The second one associates to each class a weighted Dynkin diagram. We make the following hypotheses.

Hypothesis 1.3.11. For the rest of this section, we assume that $\mathbf{G}$ is simple of adjoint type and p is good for $\mathbf{G}$.

Remark 1.3.12. Thanks to [Car85, Prop. 5.1], we may assume without loss of generality that $\mathbf{G}$ is simple of adjoint type.
We also assume that p is good for $\mathbf{G}$. In this case, the number and the structure of unipotent orbits is similar to the characteristic zero case. However, the parameterisation differs when p is bad. In both cases, the number of unipotent conjugacy classes is finite ([Ric67], [Lus76, Thm. 13]).

Both methods use a bijection between the unipotent conjugacy classes of $\mathbf{G}$ and the nilpotent orbits of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$ under the action of $\mathbf{G}$ by the adjoint map $\mathrm{Ad} : \mathbf{G} \times \mathfrak{g} \to \mathfrak{g}$. We denote by $\mathfrak{g}_{\mathrm{nil}}$ the variety of nilpotent elements of $\mathfrak{g}$ associated to $\mathbf{G}$ and by $\mathbf{G}_{\mathrm{uni}}$ the variety of unipotent elements of $\mathbf{G}$. In characteristic zero, the exponential map induces a $\mathbf{G}$-equivariant morphism between $\mathbf{G}_{\mathrm{uni}}$ and $\mathfrak{g}_{\mathrm{nil}}$. By [McN05, § 10], a similar result holds in positive characteristic.

Proposition 1.3.13 (Springer, Serre)**.** *Recall that* $\mathbf{G}$ *is simple of adjoint type and* p *is good for* $\mathbf{G}$*. There exists a homeomorphism of varieties* $\Psi_{spr} : \mathbf{G}_{\mathrm{uni}} \to \mathfrak{g}_{\mathrm{nil}}$ *such that for all elements* $g \in \mathbf{G}$ *and unipotent elements* $u \in \mathbf{G}_{\mathrm{uni}}$*,*

$$\Psi_{spr}({}^g u) = \mathrm{Ad}(g)(\Psi(u)).$$

The induced map between the unipotent conjugacy classes of $\mathbf{G}$ *and the nilpotent orbits of* $\mathfrak{g}$ *does not depend on the choice of* Ψ_{spr}*.*

Such maps are called **Springer homeomorphisms**. Note that they might not be isomorphisms of varieties. If it is the case, we say that Ψ_{spr} is a **Springer isomorphism**.

Lemma 1.3.14 ([MS03, Prop. 5]). *Recall that p is good for $\mathbf{G}$. If $\mathbf{G}$ does not contain any component of type A_n with $n \equiv -1 \mod p$, then there is a Springer homeomorphism which is an isomorphism of varieties.*

More generally, we can find another condition which applies to any Springer homeomorphism.

Definition 1.3.15. The group $\mathbf{G}$ is **proximate** if some simply connected covering of the derived subgroup of $\mathbf{G}$ is a separable morphism.

Lemma 1.3.16 ([Tay16, Lem. 3.4]). *If $\mathbf{G}$ is proximate, then any Springer homeomorphism is an isomorphism of varieties.*

We now fix a Springer homeomorphism Ψ_{spr} for $\mathbf{G}$.

Bala–Carter classification and some generalisations

Recall that a unipotent element $u \in \mathbf{G}$ is **distinguished** if $Z^\circ(\mathbf{G})$ is a maximal torus of $C_{\mathbf{G}}(u)$. Similarly, a nilpotent element in $\mathfrak{g}_{\text{nil}}$ is distinguished if it is the image by the Springer map of a distinguished unipotent element.
For any unipotent element $u \in \mathbf{G}$, there is a Levi subgroup $\mathbf{L}$ such that u is distinguished in $\mathbf{L}$. Indeed, let $\mathbf{T}$ be a maximal torus of $C_{\mathbf{G}}(u)$. Then, by [MT11, Prop. 12.10] the group $C_{\mathbf{G}}(\mathbf{T})$ is a Levi subgroup. By [Spr09, Prop. 6.4.2] the torus $\mathbf{T}$ is the unique maximal torus of $C^\circ_{C_{\mathbf{G}}(\mathbf{T})}(u)$, hence of $C_{C_{\mathbf{G}}(\mathbf{T})}(u)$. In particular, the torus $\mathbf{T}$ is central and u is distinguished in $C_{\mathbf{G}}(\mathbf{T})$.
Therefore, we obtain the map in the Bala–Carter classification, which parameterises unipotent conjugacy classes by conjugacy classes of pairs of a Levi subgroup $\mathbf{L}$ and a distinguished unipotent element in $\mathbf{L}$. McNinch and Sommers generalised this result to take into account the relative unipotent conjugacy classes, that is the conjugacy classes of $A_{\mathbf{G}}(u)$.

Theorem 1.3.17 ([MS03, Thm. 1]). *The $\mathbf{G}$-conjugacy orbits of the pairs $(u, tC^\circ_{\mathbf{G}}(u))$ with $u \in \mathbf{G}_{\text{uni}}$ and $t \in C_{\mathbf{G}}(u)$ a semisimple element are in bijection with the $\mathbf{G}$-conjugacy orbits of triples $(C^\circ_{\mathbf{G}}(s), tZ^\circ(C^\circ_{\mathbf{G}}(s)), u)$ where $s \in \mathbf{G}$ is semisimple, $u \in C^\circ_{\mathbf{G}}(s)_{\text{uni}}$ is distinguished in $C^\circ_{\mathbf{G}}(s)$ and $C^\circ_{\mathbf{G}}(tZ^\circ(C^\circ_{\mathbf{G}}(s))) = C^\circ_{\mathbf{G}}(s)$.*

Remark 1.3.18. The bijection here is given by taking the pseudo-Levi $C_{\mathbf{G}}(t, \mathbf{T})$ where $\mathbf{T}$ is a maximal torus of $C^\circ_{\mathbf{G}}(u, t)$.

A second step is then to parameterise the distinguished unipotent conjugacy classes in $\mathbf{G}$.

Definition 1.3.19. We say that $\lambda \in \check{X}$ is **associated** to $e \in \mathfrak{g}_{\text{nil}}$ if

1. for all $\xi \in \mathbf{k}^\times$, $\lambda(\xi).e = \xi^2 e$, and

2. there exists a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ such that $e \in \mathrm{Lie}(\mathbf{L})$ is distinguished and $\lambda(k^\times) \subseteq [\mathbf{L}, \mathbf{L}]$.

We then say that λ is associated to $u \in \mathbf{G}_{\mathrm{uni}}$ if it is associated to $\Psi_{spr}(u)$.

Lemma 1.3.20 ([Jan04, §5.3])**.** *For any unipotent element $u \in \mathbf{G}$, there exists a co-character associated to u and all such co-characters are conjugate under the action of $C^\circ_{\mathbf{G}}(u)$.*

To each co-character, we associate a parabolic subgroup following [Car85, Chap. 5] and [BDT20, §3.1].

Definition 1.3.21. Let $\lambda \in \check{X}$ be a co-character of $\mathbf{T}_0$. We define the following subgroups of $\mathbf{G}$:

$$\begin{aligned} \mathbf{P}_\lambda &\coloneqq \langle \mathbf{T}_0, \mathbf{U}_\alpha \mid \alpha \in \Phi \text{ with } \langle \alpha, \lambda \rangle \geq 0 \rangle, \\ \mathbf{L}_\lambda &\coloneqq \langle \mathbf{T}_0, \mathbf{U}_\alpha \mid \alpha \in \Phi \text{ with } \langle \alpha, \lambda \rangle = 0 \rangle, \\ \mathbf{U}_\lambda &\coloneqq \langle \mathbf{U}_\alpha \mid \alpha \in \Phi \text{ with } \langle \alpha, \lambda \rangle > 0 \rangle. \end{aligned}$$

The group $\mathbf{P}_\lambda$ is a parabolic subgroup of $\mathbf{G}$ with Levi subgroup $\mathbf{L}_\lambda$ and unipotent radical $\mathbf{U}_\lambda$. For any integer $i > 0$, we also define

$$\begin{aligned} \mathbf{U}_\lambda(i) &\coloneqq \langle \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \text{ with } \langle \alpha, \lambda \rangle \geq i \rangle, \\ \mathbf{U}_\lambda(-i) &\coloneqq \langle \mathbf{U}_\alpha \mid \alpha \in \Phi^+ \text{ with } \langle \alpha, \lambda \rangle \leq -i \rangle. \end{aligned}$$

Notice that $\mathbf{U}_\lambda(-i) = \mathbf{U}_{-\lambda}(i)$.

Remark that $\mathbf{L}_\lambda = C_{\mathbf{G}}(\lambda)$. There is another way of describing $\mathbf{P}_\lambda$ via the Lie algebra. The co-character λ induces a grading on the Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(\lambda, i)$ where

$$\mathfrak{g}(\lambda, i) = \{x \in \mathfrak{g} \mid \lambda(\xi).x = \xi^i x \text{ for all } \xi \in \mathbf{k}^\times\}.$$

Then

$$\mathrm{Lie}(\mathbf{P}_\lambda) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathfrak{g}(\lambda, i), \quad \mathrm{Lie}(\mathbf{L}_\lambda) = \mathfrak{g}(\lambda, 0) \quad \text{and} \quad \mathrm{Lie}(\mathbf{U}_\lambda) = \bigoplus_{i \in \mathbb{Z}_{>0}} \mathfrak{g}(\lambda, i).$$

Moreover for any integer i,

$$\mathrm{Lie}(\mathbf{U}_\lambda(i)) = \bigoplus_{j \geq i} \mathfrak{g}(\lambda, j) \quad \text{and} \quad \mathrm{Lie}(\mathbf{U}_\lambda(-i)) = \bigoplus_{j \geq i} \mathfrak{g}(\lambda, -j).$$

Proposition 1.3.22 ([Jan04, Prop. 5.9])**.** *Let $u \in \mathbf{G}$ be a unipotent element. If $\lambda, \mu \in \check{X}$ are associated to u, then the parabolic subgroups $\mathbf{P}_\lambda$ and $\mathbf{P}_\mu$ are equal.*
Moreover, $C_{\mathbf{G}}(u) \subseteq \mathbf{P}_\lambda$.

For $u \in \mathbf{G}$ unipotent and $\lambda \in X$ associated to u, we call $\mathbf{P}_\lambda$ the **canonical parabolic** associated with u.

The distinguished elements are characterised as follows.

Proposition 1.3.23 ([Car85, Cor. 5.7.5])**.** *Let $u \in \mathbf{G}$ be a unipotent element and $\lambda \in \check{X}$ be a co-character associated to u. Then, the element u is distinguished if and only if* $\dim(\mathfrak{g}(\lambda, 0)) = \dim(\mathfrak{g}(\lambda, 2))$.

Let $u \in \mathbf{G}_{\mathrm{uni}}$ and $\lambda \in X$ associated to u. There is a unique unipotent conjugacy class $C \in \mathrm{Ucl}(\mathbf{G})$ such that $C \cap \mathbf{U}_\lambda$ is open dense in $\mathbf{U}_\lambda$. Moreover, $C \cap \mathbf{U}_\lambda$ is a single $\mathbf{P}_\lambda$-conjugacy class ([Car85, Thm. 5.2.1]). If λ is associated to a distinguished unipotent element $u \in \mathbf{G}$, then $C = (u)_{\mathbf{G}}$, [Car85, Prop. 5.8.4]. The canonical parabolic subgroups associated to distinguished elements have been classified by Bala and Carter ([BC76a], [BC76b]). They are the **distinguished** parabolic subgroups $\mathbf{P} = \mathbf{U}_{\mathbf{P}} \rtimes \mathbf{L}_{\mathbf{P}}$ of $\mathbf{G}$, i.e. the parabolic subgroups such that

$$\dim(\mathrm{Lie}(\mathbf{L}_{\mathbf{P}})) = \dim(\mathbf{U}_{\mathbf{P}}/[\mathbf{U}_{\mathbf{P}}, \mathbf{U}_{\mathbf{P}}]).$$

A list can be found in Carter's book [Car85, after Thm. 5.9.6].

Remark 1.3.24. To summarise, the Bala–Carter parameterisation goes as follows. We consider $(\mathbf{L}, \mathbf{P}^{\mathbf{L}})$ where $\mathbf{P}^{\mathbf{L}}$ is a distinguished parabolic subgroup of $\mathbf{L}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}^{\mathbf{L}}$ and $C_{\mathbf{L}} \in \mathrm{Ucl}(\mathbf{L})$ be the unique unipotent conjugacy class such that $C_{\mathbf{L}} \cap \mathbf{U}$ is open dense in $\mathbf{U}$. Then, we associate the unipotent class $C = (C_{\mathbf{L}})_{\mathbf{G}}$ to the $\mathbf{G}$-conjugacy class of the pair $(\mathbf{L}, \mathbf{P}^{\mathbf{L}})$.
In the other direction, for $C \in \mathrm{Ucl}(\mathbf{G})$, the steps go as follows.

1. Choose $u \in C$.

2. Compute $\mathbf{T}$ a maximal torus of $C_{\mathbf{G}}(u)$ and set $\mathbf{L} = C_{\mathbf{G}}(\mathbf{T})$. We may assume without loss of generality that $\mathbf{L}$ is a standard Levi.

3. Find $\lambda \in \check{X}(\mathbf{L}, \mathbf{T}_0)$, a co-character associated to u.

4. Associate to C the $\mathbf{G}$-conjugacy class of the pair $(\mathbf{L}, \mathbf{P}^{\mathbf{L}}_\lambda)$, where $\mathbf{P}^{\mathbf{L}}_\lambda$ is the canonical parabolic of $\mathbf{L}$ associated to λ.

This parameterisation has the drawback that it is inductive. Moreover, it is not obvious which co-characters are associated to a unipotent element. This is rectified thanks to the weighted Dynkin diagrams.

Weighted Dynkin diagrams

We first concentrate on the characteristic zero case. We let $\mathbf{G}_{\mathbb{C}}$ be a reductive group defined over $\mathbb{C}$ with Borel subgroup $\mathbf{B}_{\mathbb{C}}$ and maximal torus $\mathbf{T}_{\mathbb{C}} \subseteq \mathbf{B}_{\mathbb{C}}$ such that it defines an isomorphic root datum $(\Phi(\mathbf{T}_{\mathbb{C}}), X(\mathbf{T}_{\mathbb{C}}), \check{\Phi}(\mathbf{T}_{\mathbb{C}}), \check{X}(\mathbf{T}_{\mathbb{C}}))$ with base $\Delta(\mathbf{T}_{\mathbb{C}})$ to the one associated to $(\mathbf{G}, \mathbf{B}_0, \mathbf{T}_0)$. We identify Δ with $\Delta(\mathbf{T}_{\mathbb{C}})$ as well as Φ with $\Phi(\mathbf{T}_{\mathbb{C}})$.
To each non-zero nilpotent orbit O, one associate an $\mathfrak{sl}_2$-triple $\{e, f, h\} \subseteq \mathfrak{g}_{\mathbb{C}} \coloneqq \mathrm{Lie}(\mathbf{G}_{\mathbb{C}})$ such that $e \in O$, by the Jacobson–Morozov Theorem ([Car85, Thm. 5.3.2]). We may further assume that $\alpha(h) \geq 0$ for all simple roots $\alpha \in \Delta$ after $\mathbf{G}_{\mathbb{C}}$-conjugation of the

triple $\{e, f, h\}$.
We then define the **weighted Dynkin diagram** associated to O as the map

$$d_O : \Delta \to \mathbb{Z}, \quad d_O(\alpha) = \alpha(h),$$

that we extend linearly to a map on all roots of $\mathbf{G}_\mathbb{C}$. By convention, we set $d_{\{0\}}(\alpha) = 0$ for all $\alpha \in \Phi$. The weighted Dynkin diagram d_O defined above does not depend on the choice of $\{e, f, h\}$ up to conjugation. Moreover, two nilpotent orbits O and O' have the same weighted Dynkin diagram if and only if they are the same ([Car85, Prop. 5.6.8]). We write $\mathscr{D}(\Delta, \mathbf{G})$ for the set of all the weighted Dynkin diagrams constructed as above.
We now come back to our algebraic group $\mathbf{G}$. Notice that for $d \in \mathscr{D}(\Delta, \Phi)$, the vector $\lambda_d = \sum_{\alpha \in \Delta} d(\alpha) \check{\varpi}_\alpha$ is such that for all roots $\alpha \in \Phi$

$$d(\alpha) = \langle \alpha, \lambda_d \rangle,$$

where $\langle\, ,\, \rangle$ is the pairing between $X(\mathbf{T}_\mathbb{C})$ and $\check{X}(\mathbf{T}_\mathbb{C})$. From the remark after [Car85, Lem. 5.6.5], we see that $\lambda_d \in \mathbb{Z}\Phi$, whence we obtain a co-character $\lambda_d \in \check{X}(\mathbf{T}_0)$.

This leads to the following bijection, see [Kaw86, Thm. 2.1.1], [Pre03, Thm. 2.7] and [Tay16, 3.22]. For $\lambda \in \check{X}$, we denote by $\mathfrak{g}(\lambda, 2)_{\mathrm{reg}}$ the unique open dense $\mathbf{L}_\lambda$-orbit of $\mathfrak{g}(\lambda, 2)$.

Theorem 1.3.25 (Kawanaka, Premet)**.** *Recall that p is good for $\mathbf{G}$. Then there is a bijection between the weighted Dynkin diagrams and the nilpotent $\mathbf{G}$-orbits, sending a diagram $d \in \mathscr{D}(\Delta, \Phi)$ to the orbit $\mathbf{G} \cdot \mathfrak{g}(\lambda_d, 2)_{reg}$.*

Remark 1.3.26. The weighted Dynkin diagrams can be found in CHEVIE [Mic15] as a field in the record of a unipotent class.

Lastly, we link the Bala–Carter classification to the one by weighted Dynkin diagrams. We define

$$\check{X}^{\mathbf{G}}_{\mathscr{D}} := \{{}^g\lambda_d \mid d \in \mathscr{D}(\Delta, \Phi),\, g \in \mathbf{G}\}.$$

For $u \in \mathbf{G}_{\mathrm{uni}}$ we define $\check{X}^{\mathbf{G}}_{\mathscr{D}}(u)$ as the subset of $\lambda \in \check{X}^{\mathbf{G}}_{\mathscr{D}}$ such that $\Psi_{\mathrm{spr}}(u)$ lies in $\mathfrak{g}(\lambda, 2)_{\mathrm{reg}}$.

Lemma 1.3.27 (Premet, [BDT20, Lem. 3.6])**.** *Let $u \in \mathbf{G}$ be a unipotent element. Then, the co-characters associated to u are exactly the co-characters in $\check{X}^{\mathbf{G}}_{\mathscr{D}}(u)$.*

Centralisers of unipotent elements

We shortly give some indications on the structure of the centraliser of a unipotent element $u \in \mathbf{G}$. The group $C_\mathbf{G}(u)$ is in general not reductive. However, its connected component decomposes into $C^\circ_\mathbf{G}(u) = \mathbf{L}(u) \cdot \mathbf{R}(u)$ where $\mathbf{R}(u)$ is the unipotent radical of $C_\mathbf{G}(u)$ and $\mathbf{L}(u)$ is reductive. The fact that $\mathbf{L}(u)$ is reductive is shown through case by case analysis, see for instance [Jan04, Prop. 5.11] for more references. These groups can be better described thanks to the canonical parabolic associated to u.

Proposition 1.3.28 ([Jan04, Prop. 5.10])**.** *Let $u \in \mathbf{G}_{\mathrm{uni}}$ and $\mathbf{P}$ the canonical parabolic associated to u, with Levi decomposition $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$. Then,*

$$\mathbf{L}(u) = \mathbf{L} \cap C_\mathbf{G}(u) \quad \textit{and} \quad \mathbf{R}(u) = \mathbf{U} \cap C_\mathbf{G}(u).$$

Chapter 2

Representation theory of finite groups of Lie type

In the previous chapter, we have seen that we can study finite reductive groups in a generic way by considering complete root data. We now bring our attention to their representations. In general, this is an extremely difficult problem, especially if we consider modular representations. Therefore, we will mostly focus on the ordinary characters. The decomposition matrix should then give us insight on the characteristic ℓ case. Firstly, we explain how to classify the complex-valued characters and then what is known about their values.
Around seventy years ago, Green completely determined the character table of $\mathrm{GL}_n(q)$ for any prime power q ([Gre55]). Since then, other series of finite reductive groups of low rank have been considered. To treat finite reductive groups as one, we present in Section 2.1 the theory developed by Deligne and Lusztig [DL76] and extended further by Lusztig. We mainly refer to [Lus84a] and [Lus76]. By making use of the geometric properties of $\mathbf{G}$, they constructed certain virtual characters, called the Deligne–Lusztig characters (see Definition 2.2.3).
From these, one deduces a partition of the irreducible characters of G into the so-called rational series, indexed by the G^*-conjugacy classes of F^*-stable semisimple elements in the dual group $\mathbf{G}^*$. The series corresponding to the neutral element $1 \in \mathbf{G}^*$ (called the unipotent series) is essential as all other series of G can be seen as unipotent series for smaller groups. This is the Jordan decomposition of characters (Theorem 2.2.16). Moreover, thanks to Lusztig we know a complete labelling of the unipotent characters, and hence of all irreducible ordinary characters. This classification is the main content of Section 2.2.
In the last section of this chapter, we present some of the well-known results on the values of the characters as well as a short summary on what has been done at the time of writing this thesis.

We keep the setting of Hypothesis 1: $\mathbf{G}$ is a connected reductive group over k with Steinberg map $F : \mathbf{G} \to \mathbf{G}$, $\mathbf{T}_0 \subseteq \mathbf{B}_0$ is a maximally split torus in an F-stable Borel $\mathbf{B}_0$ of $\mathbf{G}$ with associated root datum $(X, \Phi, \check{X}, \check{\Phi})$, base Δ and Weyl group W.

2.1 Deligne–Lusztig induction

We want to study the representation theory of finite groups of Lie type in a generic way. It means that we cannot rely on specific properties of a chosen finite group $\mathbf{G}^F$. On the other hand, all those groups come from the same algebraic group $\mathbf{G}$, and thus we would like to make use of the geometrical aspects. Grothendieck's fonctions-faisceaux dictionary brings us to look at G-equivariant (perverse) sheaves instead of class functions of G. We give a little more explanations, motivation and references in Subsection 2.1.1.
When one investigates the representations of a finite group H, a very practical tool is the induction process, which constructs representations of H from representations of its subgroups. A simple example is the regular representation $\mathbf{K}H$. It is the induction of the trivial representation of the trivial subgroup and contains all the irreducible representations of H as direct summands.[1] However, it is often a difficult problem to determine them.
In our case, the subgroups we want to consider are the F-stable Levi subgroups, one reason being that they are again connected reductive. We make this more explicit by defining Deligne–Lusztig induction in Subsection 2.1.2.
Lastly, we recall properties of a particular case of Deligne–Lusztig induction, called Harish-Chandra induction and which has the great advantage of working well with the modular representations.

2.1.1 Quick motivation for ℓ-adic cohomology

This subsection is not meant to give a detailed introduction to derived categories, equivariant sheaves or étale cohomology and we purposefully stay vague. Our goal is only to motivate the objects appearing in the representation theory of finite reductive groups.

Equivariant sheaves

We start by recalling the definition of equivariant sheaves. For more information, we refer the reader to [BL94]. Here all the sheaves we consider are sheaves of finite dimensional Λ-modules, for Λ a ring. For Y a variety, the category of sheaves on Y will be written $\mathrm{Sh}(Y)$.

Definition 2.1.1. Let $\mathbf{H}$ be an algebraic group over k (it can be finite) and Y an algebraic variety over k on which $\mathbf{H}$ acts continuously. Let $a : \mathbf{H} \times Y \to Y$ denote the action of $\mathbf{H}$ and $p : \mathbf{H} \times Y \to Y$ the projection on the second coordinate. We also denote by $p_{23} : \mathbf{H} \times \mathbf{H} \times Y \to \mathbf{H} \times Y$ the projection onto the second and third coordinates and $m : \mathbf{H} \times \mathbf{H} \to \mathbf{H}$ the multiplication in $\mathbf{H}$. An **$\mathbf{H}$-equivariant sheaf** on Y is the datum of a sheaf $\mathcal{F}$ on Y together with an isomorphism of sheaves $\phi : a^*\mathcal{F} \tilde{\to} p^*\mathcal{F}$ such that the co-cycle condition holds:

$$p_{23}^*\phi \circ (1 \times a)^*\phi = m^*\phi.$$

[1] Recall that $\mathbf{K}$ has characteristic zero and is "big enough" for H.

The category of $\mathbf{H}$-equivariant sheaves is denoted $\mathrm{Sh}_{\mathbf{H}}(Y)$.

The co-cycle condition implies at the level of stalks that for any $g, h \in \mathbf{H}$ and $x \in Y$, the isomorphism $\phi_{gh,x} : \mathcal{F}_{gh.x} \xrightarrow{\sim} \mathcal{F}_x$ is the same as the composition $\phi_{h,x} \circ \phi_{g,h.x}$. Moreover, for $h \in \mathbf{H}$, let $i_h : Y \to \{h\} \times Y \to \mathbf{H} \times Y$ be the inclusion. Then, $h^*\mathcal{F} := i_h^* a^* \mathcal{F}$ is isomorphic to $i_h^* p^* \mathcal{F} = \mathcal{F}$ via $i_h^* \phi$. So in some ways, the group $\mathbf{H}$ "acts on" $\mathcal{F}$. Taking global sections, we can see $\Gamma(Y, \mathcal{F}) := \mathcal{F}(Y)$ as a $\mathbf{\Lambda}[\mathbf{H}]$-module. Alternatively, if M is a $\mathbf{\Lambda}[\mathbf{H}]$-module, then we may see it as an $\mathbf{H}$-equivariant sheaf on Y by looking at the constant sheaf M on Y.

Derived categories and Grothendieck group

The functor of global sections $\Gamma(Y, -)$ is only a left exact functor from the $\mathbf{H}$-equivariant sheaves on Y to the category of $\mathbf{\Lambda}[\mathbf{H}]$-modules. So, we instead compare the two derived (bounded) categories.
The **derived category** $D(\mathscr{C})$ of a category $\mathscr{C}$ is a category whose objects are chain complexes of objects of $\mathscr{C}$. The subcategory whose objects are chains of complexes with finitely many non-zero cohomology groups is called **bounded** and we denote it $D^b(\mathscr{C})$. We get a total derived functor $R\Gamma(Y, -) : D(\mathrm{Sh}(Y)) \to D(\Lambda-\mathrm{mod})$, where we apply the global section functor $\Gamma(Y, -)$ to each sheaf in the chain complex. If $I^\bullet$ is an injective resolution of a sheaf $\mathcal{F}$ on Y, we have

$$H^n(R\Gamma(Y, I^\bullet)) := \frac{\ker(\Gamma(Y, I^n)) \to \Gamma(Y, I^{n+1}))}{\mathrm{im}(\Gamma(Y, I^{n-1}) \to \Gamma(Y, I^n))}.$$

We might sometimes abuse notation and write $R\Gamma(Y, \mathcal{F})$ for $R\Gamma(Y, I^\bullet)$. Note that there is also a way to define derived categories without taking injective resolutions. The precise definition and the properties of derived categories can be found in [Gor21] and [Aub10]. Now, we could wonder how to get back from $D^b(\mathbf{\Lambda}[G]\text{-mod})$ to $\mathbf{\Lambda}[G]$-mod. To do so, we need to look at the Grothendieck groups. The **Grothendieck group** $K_0(\mathscr{C})$ of an additive category $\mathscr{C}$ can be seen as the free abelian group whose elements are the isomorphism classes of objects in $\mathscr{C}$ and where the group law is given as follows: we write $[A] + [B] = [C]$ if there is a short exact sequence in $\mathscr{C}$

$$1 \to A \to C \to B \to 1.$$

For instance, the Grothendieck group of the category of $\mathbf{K}[G]$-modules is the group $\mathbb{Z}\,\mathrm{irr}_{\mathbf{K}}(G)$ of virtual ordinary characters of G.
For the Grothendieck group of the derived category $D^b(\mathscr{C})$, the notion of short exact sequence is replaced by the notion of distinguished triangle, i.e. any morphism of chain complexes $A^\bullet \to B^\bullet$ can be extended to a distinguished triangle

$$A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet.$$

Here $A[1]$ is the complex shifted by 1, that is $A[1]^n = A^{n+1}$. In that case, the Grothendieck group $K_0(D^b(\mathscr{C}))$ is the abelian group of isomorphism classes of objects in $D^b(\mathscr{C})$ where

the group law is given by distinguished triangles.
In general, there is an isomorphism ([KS94, Ex. I.27])

$$\begin{aligned} K_0(D^b(\mathscr{C})) &\xrightarrow{\sim} K_0(\mathscr{C}) \\ [A^\bullet] &\mapsto \sum_{i\in\mathbb{Z}} (-1)^i [H^i(A^\bullet)] \end{aligned}$$

with inverse sending the object $[A] \in K_0(\mathscr{C})$ to $[A^\bullet]$ where $A^0 = A$ and $A^i = 0$ for all $i \neq 0$.

Etale cohomology and ℓ-adic cohomology

To be exact, we do not exactly consider the derived category of G-equivariant sheaves for the Zariski topology. Indeed, this topology does not behave like complex topology for cohomology, because it is too coarse. We need to add open sets. We will therefore use étale sheaves and étale cohomology introduced by Grothendieck.
In the classical sense, sheaves on a topological space Y are functors from the category of open sets of Y (where the morphisms are the inclusions) to the category of finite dimensional Λ-modules. An **étale sheaf** on a scheme[2] Y is a functor from the category of étale morphisms of finite type from another scheme to Y. More details can be found in [Aub10] and in the appendix of [Car85]. We denote by $\mathrm{Sh}^{\text{ét}}(Y)$ the category of étale sheaves on Y.
If an algebraic group $\mathbf{H}$ over k acts on Y continuously, we could define $\mathbf{H}$-equivariant (étale) sheaves and then take the derived category. However, this is not necessarily well-defined ([BL94, 0.4]). Therefore, we will rather look at the $\mathbf{H}$-equivariant complexes in the bounded derived category of étale sheaves on a scheme Y and then take the derived functor of global sections.

Assume that our fixed group G (from Hypothesis 1) acts on Y continuously. Notice that to get different ΛG-modules, we could either change the sheaf for a fixed variety Y or change the variety itself. For the methods developed by Deligne and Lusztig, we will often change the scheme Y but "keep" the constant sheaf Λ.
For instance, we will look at the case where Λ is a field of characteristic zero. However, étale cohomology does not work well when Y is defined over a field whose characteristic divides the characteristic of Λ. We have to take direct limits of cohomologies of the constant sheaf $\mathbb{Z}/\ell^n\mathbb{Z}$ and then tensor over the ℓ-adic integers to get the cohomology with coefficients in $\mathbb{Q}_\ell$. We then extend scalars to get sheaves over the algebraic closure $\overline{\mathbb{Q}}_\ell$.
Lastly, we prefer to consider schemes satisfying nice properties (proper schemes). In general, we can embed any scheme Y in a proper scheme $\overline{Y}$. We then extend by zero the sheaf on Y to obtain a sheaf on $\overline{Y}$ and we look at its cohomology instead. For $\mathcal{F}$ an étale sheaf on Y, we write $H^n_c(Y,\mathcal{F})$ for this construction. This is called the (étale/ℓ-adic) cohomology with **compact support**. Similarly, we set $R\Gamma_c(Y,-) \coloneqq R\Gamma(Y,-)$. If Y is a nice enough scheme, that is separated and of finite type (for instance a variety)

[2]For simplication, we may think of scheme as a generalisation of varieties. In particular, any variety is a scheme.

then $H^{\bullet}\big(R\Gamma_c(Y,\overline{\mathbb{Q}}_\ell)\big)$ is a bounded complex ([Car85, Appendix (g)]). Moreover, the cohomologies $H^i(R\Gamma_c(Y,\overline{\mathbb{Q}}_\ell))$ are finite dimensional. For more accurate definitions, we advise the reader to read the appendix of [Car85].
To summarise, we first choose a variety Y on which G acts continuously. We apply the global sections functor and we get $R\Gamma_c(Y,\Lambda) \in D^b(\mathbf{\Lambda}[G]\text{-mod})$. The finite sum

$$\sum_{i\in\mathbb{Z}}(-1)^i[H^i(R\Gamma_c(Y,\Lambda))]$$

is then a virtual $\mathbf{\Lambda}[G]$-module in the Grothendieck group of $\mathbf{\Lambda}[G]$-mod.

Remark 2.1.2. So far, we still have not taken into account the topology of the algebraic group $\mathbf{G}$. The idea is the following. Since we want to study $\mathbf{\Lambda}[G]$-modules, we could look at the category of G-equivariant constant sheaves on a variety Y and take global sections. Alternatively, it suffices to consider the category of G-equivariant sheaves on a algebraic variety consisting of only one element. Instead of G-equivariant sheaves on the point, we look at $\mathbf{G}$-equivariant sheaves on $\mathbf{G}$, but where the action of $\mathbf{G}$ on $\mathbf{G}$ is given by F-conjugation ($g.h = ghF(g)^{-1}$ for $g,h \in \mathbf{G}$). These two categories are equivalent by the Lang–Steinberg theorem 1.2.4, see for instance [Éte23, below Cor. 1.2.2]. We give an intuitive argument on the level of stalks. If $\mathcal{F}$ is a $\mathbf{G}$-equivariant sheaf, then all its stalks are isomorphic since the Lang–Steinberg map is surjective. Moreover, on a given stalk, we also have an action of G. Consequently, by looking at $\mathcal{F}_1$ we get a G-equivariant sheaf on the point.

This closes our general imprecise remarks.

2.1.2 Definition and first properties of Deligne–Lusztig induction

In this section, we recall the powerful idea of Deligne and Lusztig on which a major part of the construction of representations of finite reductive groups relies. In their pioneering article [DL76], they generalised the process of induction by taking another representation than the regular one. Let $M \leq H$ be two finite groups and V a $\mathbf{\Lambda}[M]$-module. Then the induction is defined as follows:

$$\mathrm{Ind}_M^H(V) \coloneqq \mathbf{\Lambda}[H] \otimes_{\mathbf{\Lambda}[M]} V.$$

Instead of $\mathbf{\Lambda}[H]$, we could choose any $\mathbf{\Lambda}[H]$-module-$\mathbf{\Lambda}[M]$, that is a module with a left action of $\mathbf{\Lambda}[H]$ and a right action of $\mathbf{\Lambda}[M]$. In our case, the ambient group is G, our finite group of Lie type. Deligne and Lusztig chose for the subgroup the fixed points T of an F-stable maximal torus $\mathbf{T}$. For the $\mathbf{\Lambda}[G]$-module-$\mathbf{\Lambda}[T]$, they constructed a module coming from a variety on which both G and T act continuously. Later, in [Lus76], Lusztig extended this process to replace the torus $\mathbf{T}$ by any F-stable Levi subgroup.
We mostly follow [DM20, Chap. 9], see also [Dud18].

Definition 2.1.3. Let $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ be a Levi decomposition of a parabolic subgroup of $\mathbf{G}$, such that $\mathbf{L}$ is F-stable. The (generalised) **Deligne–Lusztig variety** associated to $\mathbf{L} \subseteq \mathbf{P}$ is defined as follows:

$$Y_{\mathbf{L}\subseteq\mathbf{P}} := \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}F(\mathbf{U})\}.$$

Remark 2.1.4. Observe that

$$Y_{\mathbf{L}\subseteq\mathbf{P}} \cong \{g \in \mathbf{G} \mid g^{-1}F(g) \in F(\mathbf{U})\}/(\mathbf{U} \cap F(\mathbf{U})),$$

see [DM20, Def. 9.1.1]. Moreover, if $\mathbf{P}$ is F-stable, then $F(\mathbf{U}) = \mathbf{U}$ and

$$\begin{aligned} Y_{\mathbf{L}\subseteq\mathbf{P}} &= \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}\} \\ &= \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid F(g\mathbf{U}) = g\mathbf{U}\} = (\mathbf{G}/\mathbf{U})^F \\ &= G/U, \end{aligned}$$

where we can apply Remark 1.2.6 for the last line since $\mathbf{U}$ is connected.

Note that $\mathbf{G}$ and respectively $\mathbf{L}$ act on $\mathbf{G}/\mathbf{U}$ by left (respectively right) multiplication and it induces an action of G and L on $Y_{\mathbf{L}\subseteq\mathbf{P}}$.

Definition 2.1.5. Let $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ be a Levi decomposition of a parabolic subgroup of $\mathbf{G}$, such that $\mathbf{L}$ is F-stable. The **Deligne–Lusztig induction functor** $\mathcal{I}^G_{\mathbf{L}\subseteq\mathbf{P}}$ is given by

$$\begin{aligned} \mathcal{I}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : D^b(\mathbf{\Lambda}[L]\text{-mod}) &\to D^b(\mathbf{\Lambda}[G]\text{-mod}) \\ C &\mapsto R\Gamma_c(Y_{\mathbf{L}\subseteq\mathbf{P}}, \Lambda) \overset{L}{\otimes}_{\mathbf{\Lambda}[L]} C. \end{aligned}$$

If the ambient group is clear, we may write $\mathcal{I}_{\mathbf{L}\subseteq\mathbf{P}}$.

Remark 2.1.6. The Deligne–Lusztig induction functor is usually denoted with an $\mathcal{R}$. However, to emphasize the similarities with induction of characters (and parabolic induction of character sheaves in Definition 3.2.1) we chose the letter $\mathcal{I}$.

Remark 2.1.7. We rewrite the Deligne–Lusztig induction functor in different ways.
Let V be a $\mathbf{\Lambda}[L]$-module, $V^\bullet$ an injective resolution corresponding to the constant sheaf M on $Y_{\mathbf{L}\subseteq\mathbf{P}}$ and $\pi : Y_{\mathbf{L}\subseteq\mathbf{P}} \to Y_{\mathbf{L}\subseteq\mathbf{P}}/L$ be the quotient map. Then by [BR03, Lem. 3.2],

$$\mathcal{I}_{\mathbf{L}\subseteq\mathbf{P}}(V^\bullet) \cong R\Gamma_c(Y_{\mathbf{L}\subseteq\mathbf{P}}/L, \pi_*\Lambda \otimes \pi_*V).$$

Alternatively, we could look at the induction of sheaves which are equivariant for the action given by F-conjugation, c.f. Remark 2.1.2 . We look at the following varieties

$$\mathbf{L} \xleftarrow{\alpha} \mathbf{P}F(\mathbf{P}) \xrightarrow{\beta} \mathbf{G}$$

where $\mathbf{L}$ acts on $\mathbf{L}$ by F-conjugation, $\mathbf{P}$ acts on $\mathbf{P}F(\mathbf{P})$ by F-conjugation and lastly $\mathbf{G}$ acts on itself also by F-conjugation. The map α is given by the Levi decomposition of the parabolic $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ and the map β by inclusion. Let V be a $\mathbf{\Lambda}[L]$-module, then we can consider the constant sheaf V on the point, and the corresponding $\mathbf{L}$-equivariant sheaf on $\mathbf{L}$ for the action by $\mathbf{L}$-conjugation by Remark 2.1.2. Let $\mathcal{V}$ be the corresponding $\mathbf{L}$-equivariant complex of sheaves in the bounded derived category of $\mathrm{Sh}(\mathbf{L})$. Then, the complex $\beta^*\alpha_*\mathcal{V}$ is $\mathbf{G}$-equivariant for the F-conjugation and corresponds to a complex $I(\mathcal{V})$ of G-equivariant sheaves on the point, hence a $\mathbf{\Lambda}[G]$-module. By [Éte23, Thm. 3.3.16], the above functor and the Deligne–Lusztig induction functor are equivalent and give isomorphic modules.

The induction functor has a left adjoint functor, called the **Deligne–Lusztig restriction functor**, that we denote by ${}^*\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$. It has an explicit description in terms of a derived Hom-functor, see [Dud18, § 3.1].

The case where $\mathbf{\Lambda} = \overline{\mathbb{Q}}_\ell$

We list a few properties when $\mathbf{\Lambda} = \overline{\mathbb{Q}}_\ell$. In that case, it is often sufficient to work with ordinary characters $\chi \in \mathrm{irr}_{\overline{\mathbb{Q}}_\ell}(G)$ to describe the representation theory of G, since the group algebra $\overline{\mathbb{Q}}_\ell[G]$ is semisimple. Using the Grothendieck group, we now define Deligne–Lusztig induction on characters.

Definition 2.1.8. Let $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ be a Levi decomposition of a parabolic subgroup of $\mathbf{G}$, such that $\mathbf{L}$ is F-stable. The **Deligne–Lusztig induction** of characters $I^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ is given by

$$\begin{aligned} I^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : \mathbb{Z}\,\mathrm{irr}_{\overline{\mathbb{Q}}_\ell}(L) &\to \mathbb{Z}\,\mathrm{irr}_{\overline{\mathbb{Q}}_\ell}(G) \\ \chi &\mapsto \left(g \mapsto \mathrm{Tr}\left(g, \sum_{i\in\mathbb{Z}} (-1)^i H^i_c(\mathcal{I}_{\mathbf{L}\subseteq\mathbf{P}}(V_\chi)) \right) \right). \end{aligned}$$

By [DM20, Prop. 9.1.6],

$$I^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\chi)(g) = |L|^{-1} \sum_{l\in L} \mathrm{Tr}\left((g,l), \sum_{i\in\mathbb{Z}} (-1)^i H^i_c(R\Gamma_c(Y_{\mathbf{L}\subseteq\mathbf{P}}, \overline{\mathbb{Q}}_\ell)) \right) \chi(l^{-1}).$$

The numbers

$$\mathscr{L}((g,l), Y_{\mathbf{L}\subseteq\mathbf{P}}) := \mathrm{Tr}\left((g,l), \sum_{i\in\mathbb{Z}} (-1)^i H^i_c(R\Gamma_c(Y_{\mathbf{L}\subseteq\mathbf{P}}, \overline{\mathbb{Q}}_\ell)) \right)$$

are called **Lefschetz numbers** and are integers. The corresponding **Deligne–Lusztig restriction** of characters is denoted by ${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$.

The Deligne–Lusztig induction functor is transitive.

Proposition 2.1.9 ([DM20, Prop. 9.18])**.** *Let* $\mathbf{Q} \subseteq \mathbf{P}$ *be two parabolic subgroups of* $\mathbf{G}$ *with respective Levi subgroups* $\mathbf{M} \subseteq \mathbf{L}$*. Assume that both* $\mathbf{M}$ *and* $\mathbf{L}$ *are* F*-stable. Then*

$$\mathcal{I}^{\mathbf{G}}_{\mathbf{M}\subseteq\mathbf{Q}} \cong \mathcal{I}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} \circ \mathcal{I}^{\mathbf{L}}_{\mathbf{M}\subseteq\mathbf{L}\cap\mathbf{Q}}.$$

Like for the usual induction in finite groups, we can write a Mackey-like formula for the Deligne–Lusztig induction and restriction of characters. If this formula holds, it implies that $I^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ is independent of the choice of the parabolic subgroup $\mathbf{P}$ (see above [GM20, Thm. 3.3.8] and [DM91, Thm. 5.3.1]). We might sometimes write $I^{\mathbf{G}}_{\mathbf{L}}$ instead of $I^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$. However, the Mackey formula is not proven in full generality, see [GM20, Thm. 3.3.7]. Nonetheless it holds for example if $\mathbf{P}$ is a Borel subgroup and $\mathbf{L}$ a maximal torus. We will focus on the other special properties of this case in Section 2.2.

2.1.3 Harish-Chandra induction

In this subsection, we consider the special case where not only the Levi subgroup but also the parabolic subgroups $\mathbf{P}$ are F-stable. We also assume that Λ is a field of characteristic different from p. In particular, we can choose $\Lambda \in \{\mathbf{K}, \mathbf{k}\}$. Historically, this construction was introduced before Deligne–Lusztig induction (in [Har70]) and the second one can be seen as a generalisation of the special case.

Definition 2.1.10. Let $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ be a Levi decomposition of an F-stable parabolic subgroup of $\mathbf{G}$. The **Harish-Chandra induction** is given as follows:

$$\mathcal{I}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} : \mathbf{\Lambda}[L]\text{-mod} \to \mathbf{\Lambda}[G]\text{-mod}$$
$$V \mapsto \mathrm{Ind}^G_P \circ \mathrm{Inf}^P_L(V).$$

Observe that by Remark 2.1.4, it makes sense to use the same symbol for the Harish-Chandra induction and the Deligne–Lusztig induction. Indeed, we can construct a canonical isomorphism between $\mathrm{Ind}^G_P \circ \mathrm{Inf}^P_L$ and $\mathbf{\Lambda}[G/U] \otimes_{\mathbf{\Lambda}[L]} -$. Similarly as for Deligne–Lusztig induction, we denote ${}^*\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ for the adjoint functor, given by $- \otimes_{\mathbf{\Lambda}[G]} \mathbf{\Lambda}[G/U]$, see [GM20, Def. 3.1.5].

We gather some results following [GM20, Sections 3.1, 3.2].

Since tensor product preserves projectivity we immediately get the following fact.

Corollary 2.1.11 ([GM20, Cor. 3.1.6])**.** *The Harish-Chandra functors $\mathcal{I}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ and ${}^*\mathcal{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ are exact and preserve projectives modules.*

In this case, there is a Mackey formula ([GM20, Thm. 3.1.11]). The Harish-Chandra functors are in fact independent of the parabolic containing a fixed Levi and we write simply $\mathcal{I}^{\mathbf{G}}_{\mathbf{L}}$ and ${}^*\mathcal{R}^{\mathbf{G}}_{\mathbf{L}}$.

By Proposition 2.1.9, the Harish-Chandra induction is transitive. The previous properties lead to the following definition.

Definition 2.1.12. Let $\mathbf{L}$ be a Levi subgroup of $\mathbf{G}$ contained in an F-stable parabolic subgroup and V an irreducible $\mathbf{\Lambda}[L]$-module. We say that the pair $(\mathbf{L}, V)$ is **cuspidal** if there is no F-stable Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ such that $\mathbf{M} \subseteq \mathbf{L}$ and no $\mathbf{\Lambda}[M]$-module V' such that

$$\langle V', {}^*\mathcal{R}^{\mathbf{L}}_{\mathbf{M}}(V)\rangle \neq 0.$$

For a cuspidal pair $(\mathbf{L}, V)$, we define the **Harish-Chandra series** $\mathscr{E}_\Lambda(\mathbf{G}, (\mathbf{L}, V))$ to be the set of all simple $\mathbf{\Lambda}[G]$-modules V' such that

1. the Levi subgroup $\mathbf{L}$ is minimal with ${}^*\mathcal{R}_{\mathbf{L}}^{\mathbf{G}}(V') \neq 0$ and
2. the module V is a composition factor of ${}^*\mathcal{R}_{\mathbf{L}}^{\mathbf{G}}(V')$.

Note that by [GM20, Prop. 3.1.16], these conditions are equivalent to V' being contained in the socle of $\mathcal{I}_{\mathbf{L}}^{\mathbf{G}}(V)$.
The Harish-Chandra series $\mathscr{E}_\Lambda(\mathbf{G}, (\mathbf{L}, V))$ depends only on the G-conjugacy class of $(\mathbf{L}, V)$. Moreover, it is non empty and it gives a partition of the isomorphism classes of simple $\mathbf{\Lambda}[G]$-modules, see [GM20, Cor. 3.1.17]. Lastly, the series $\mathscr{E}_\Lambda(\mathbf{G}, (\mathbf{L}, V))$ is in bijection with the set of simple modules of the Hecke algebra associated to $(\mathbf{L}, V)$ up to isomorphism [GM20, Thm. 3.1.18]. We make this more explicit in the case where Λ is of characteristic zero.

The case where $\mathbf{\Lambda}$ of characteristic zero

We detail a parameterisation of the Harish-Chandra series in terms of the group algebra of a (comparatively) small finite group associated to the cuspidal pair.

Definition 2.1.13 ([GM20, Def. 3.1.27])**.** Let $(\mathbf{L}_I, V)$ be a cuspidal pair for $\mathbf{G}$, with $I \subseteq \Delta$. The **relative Weyl group of $\mathbf{L}_I$** is given by

$$W^{\mathbf{G}}(\mathbf{L}_I) \coloneqq N_{\mathbf{G}}(\mathbf{L}_I)/\mathbf{L}_I.$$

The **relative Weyl group of L_I** is then

$$W^G(L_I) \coloneqq N_G(\mathbf{L}_I)/L_I.$$

The **relative Weyl group of $(\mathbf{L}_I, V)$** is

$$W^G(\mathbf{L}_I, V) \coloneqq \{n \in N_G(\mathbf{L}_I) \mid \mathrm{ad}(n)(V) \cong_{\mathbf{\Lambda}[L_I]} V\}/L_I \subseteq W^G(L_I).$$

We may omit the superscript $^{\mathbf{G}}$ when the ambient group is clear.

Note that despite the notation, the relative Weyl group $W^{\mathbf{G}}(\mathbf{L}_I, V)$ is not a Coxeter group in general. If $\mathbf{L}$ is any Levi subgroup, then there are $g \in G$ and $I \subseteq \Delta$ such that $\mathbf{L} = \mathbf{L}_I^g$. We define the relative Weyl groups of $\mathbf{L}$ by conjugating the one for $\mathbf{L}_I$ by g.

Theorem 2.1.14 (Howlett–Lehrer Comparison Theorem, [GM20, Thm. 3.2.5, Thm. 3.2.7])**.** *Assume $\mathbf{\Lambda}$ is a field of characteristic zero. Let $(\mathbf{L}, V)$ be a cuspidal pair for $\mathbf{G}$. For any Levi subgroup $\mathbf{L} \leq \mathbf{M} \leq \mathbf{G}$, there is a bijection $H_{\mathbf{L},V}^{\mathbf{M}}$ from $\mathrm{Irr}_\Lambda(W^M(\mathbf{L}, V))$ to $\mathscr{E}_\Lambda(\mathbf{M}, (\mathbf{L}, V))$. Moreover, the bijections can be chosen such that the following diagram commutes for each such $\mathbf{M}$. Here the left arrow is the usual induction in finite groups.*

$$\begin{array}{ccc} \mathbb{Z}\,\mathrm{Irr}_\Lambda(W^G(\mathbf{L}, V)) & \xrightarrow{H_{\mathbf{L},V}^{\mathbf{G}}} & \mathscr{E}_\Lambda(\mathbf{G}, (\mathbf{L}, V)) \\ {\scriptstyle \mathrm{Ind}}\uparrow & & \uparrow{\scriptstyle I_{\mathbf{M}}^{\mathbf{G}}} \\ \mathbb{Z}\,\mathrm{Irr}_\Lambda(W^M(\mathbf{L}, V)) & \xrightarrow[H_{\mathbf{L},V}^{\mathbf{M}}]{} & \mathscr{E}_\Lambda(\mathbf{M}, (\mathbf{L}, V)) \end{array}$$

Using the Harish-Chandra restriction and induction, we define a new self-adjoint map on the space of class functions.

Definition 2.1.15 ([GM20, Def. 3.4.1]). Let Δ' be the set of Coxeter generators of W^F (in bijection with the F-orbits of Δ). The **Alvis–Curtis duality** operator $D_{\mathbf{G}}$ is defined as follows:

$$D_{\mathbf{G}} : \mathbb{Z}\,\mathrm{irr}_{\overline{\mathbb{Q}}_\ell}(G) \to \mathbb{Z}\,\mathrm{irr}_{\overline{\mathbb{Q}}_\ell}(G), \quad D_{\mathbf{G}} \coloneqq \sum_{I \subseteq \Delta'} (-1)^{|I|} I_{\mathbf{L}_I}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}_I}^{\mathbf{G}}.$$

For any $\chi, \varphi \in \mathrm{irr}_{\overline{\mathbb{Q}}_\ell}(G)$, the operator $D_{\mathbf{G}}$ is self-adjoint:

$$\langle D_{\mathbf{G}}(\chi), \varphi \rangle = \langle \chi, D_{\mathbf{G}}(\varphi) \rangle.$$

Moreover $D_{\mathbf{G}} \circ D_{\mathbf{G}}$ is in fact the identity, [GM20, Prop. 3.4.2, Cor. 3.4.5]. Lastly, it permutes irreducible characters up to a sign.

Theorem 2.1.16 ([GM20, Prop. 3.4.7, Thm. 3.4.8]). *Let $\chi \in \mathscr{E}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}, (\mathbf{L}, V))$ for a cuspidal pair $(\mathbf{L}, V)$ of $\mathbf{G}$. Then $\epsilon_{\mathbf{L}} D_{\mathbf{G}}(\chi) \in \mathscr{E}_{\overline{\mathbb{Q}}_\ell}(\mathbf{G}, (\mathbf{L}, V))$ is irreducible. Here $\epsilon_{\mathbf{L}}$ is a sign depending only on $\mathbf{L}$, ([GM20, Def. 2.2.11]). Furthermore, if $W^{\mathbf{G}}(\mathbf{L}, V)$ is a Coxeter group (for instance if $Z(\mathbf{G})$ is connected), then for any $\psi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(W^{\mathbf{G}}(\mathbf{L}, V))$,*

$$D_{\mathbf{G}}(H_{\mathbf{L},V}^{\mathbf{G}}(\psi)) = \epsilon_{\mathbf{L}} H_{\mathbf{L},V}^{\mathbf{G}}(\psi \otimes \epsilon)$$

where ϵ is the sign character of $W^{\mathbf{G}}(\mathbf{L}, V)$.

2.2 Parameterisation of the ordinary characters

The Deligne–Lusztig induction enables us to construct many different $\mathbf{\Lambda}[G]$-modules. In fact, we can already deduce a lot of information on the irreducible representations by considering only the induction from an F-stable maximal torus. Similarly as for Harish-Chandra induction, we get a partition of $\mathrm{Irr}_{\mathbf{\Lambda}}(\mathbf{G})$ into the so-called Lusztig series, when $\mathbf{\Lambda}$ is a field of characteristic zero. One series, the unipotent series is of particular interest. Any other one is in bijection with the unipotent series of a smaller group. Furthermore, we see in Subsection 2.2.2 that the unipotent series can itself be partitioned and parameterised.

Hypothesis 2.2.1. For the rest of this chapter, we always assume that $\mathbf{\Lambda} = \overline{\mathbb{Q}}_\ell$.

Notation 2.2.2. For A a finite group, $\mathrm{irr}(A)$ denotes the set of irreducible characters of A over $\overline{\mathbb{Q}}_\ell$. For our purposes, we may identify the abstract field $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$ and might sometimes do so.

2.2.1 Lusztig series

In the characteristic zero case, it is sufficient to consider characters instead of modules by Maschke's theorem.

Definition 2.2.3 ([DL76], [Lus76])**.** Let $\mathbf{T}$ be an F-stable maximal torus and $\mathbf{B}$ a Borel subgroup containing $\mathbf{T}$. Let $\theta \in \operatorname{irr}(T)$. The virtual character $I_{\mathbf{T}\subseteq\mathbf{B}}^{\mathbf{G}}(\theta)$ is called a **Deligne–Lusztig character**.

Since the virtual character $I_{\mathbf{T}\subseteq\mathbf{B}}^{\mathbf{G}}(\theta)$ is independent of the Borel subgroup, we write $I_{\mathbf{T}}^{\mathbf{G}}(\theta)$. These virtual characters allow us to get all the irreducible characters of G.

Theorem 2.2.4 ([GM20, Cor. 2.2.19])**.** *For each $\chi \in \operatorname{irr}(G)$, there exists a maximal torus $\mathbf{T}$ of $\mathbf{G}$ and a character $\theta \in \operatorname{irr}(T)$, such that*

$$\langle \chi, I_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle \neq 0.$$

Moreover, we know when two Deligne–Lusztig characters share some constituents.

Proposition 2.2.5 ([GM20, Cor. 2.2.10])**.** *Let $\mathbf{T}$ and $\mathbf{T}'$ be F-stable maximal tori of $\mathbf{G}$ and $\theta \in \operatorname{irr}(T)$, $\theta' \in \operatorname{irr}(T')$. Then $I_{\mathbf{T}}^{\mathbf{G}}(\theta) = I_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ if and only if there is $g \in G$ such that $g\mathbf{T}g^{-1} = \mathbf{T}'$ and $\theta \circ \operatorname{ad}(g) = \theta'$. Moreover, if $I_{\mathbf{T}}^{\mathbf{G}}(\theta) \neq I_{\mathbf{T}'}^{\mathbf{G}}(\theta')$, then*

$$\langle I_{\mathbf{T}}^{\mathbf{G}}(\theta), I_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle = 0.$$

We now want to define the Deligne–Lusztig characters uniquely in terms of the data we have fixed in Hypothesis 1, that is the maximally split torus $\mathbf{T}_0$ of $\mathbf{G}$ and the corresponding Weyl group W.
Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$. There exists $g \in \mathbf{G}$ such that $g\mathbf{T}_0 g^{-1} = \mathbf{T}$. Notice that $g^{-1}F(g) \in N_{\mathbf{G}}(\mathbf{T}_0)$. If $t = gt_0g^{-1} \in \mathbf{T}$ is fixed by F, then $F(t_0) = F(g)^{-1}gt_0g^{-1}F(g)$. Let $w = g^{-1}F(g)\mathbf{T}_0$ and fix a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ of w. We write

$$\mathbf{T}_0[w] \coloneqq \{t \in \mathbf{T}_0 \mid F(t) = \dot{w}^{-1}t\dot{w}\}.$$

We clearly have $g^{-1}Tg = \mathbf{T}_0[w]$. We say that $\mathbf{T}$ is a **torus of type** w.
For any $\theta \in \operatorname{irr}(\mathbf{T}_0[w])$, we write

$$I_w^\theta \coloneqq I_{\mathbf{T}}^{\mathbf{G}}(\theta \circ \operatorname{ad}(g)),$$

see [GM20, Lem. 2.3.19] for more details. By Proposition 2.2.5, if $I_w^\theta = I_{w'}^{\theta'}$ then the pairs (w,θ) and (w',θ') are F-conjugate by an element of W. Observe that all Deligne–Lusztig characters can be written as I_w^θ, for some $w \in W$ and $\theta \in \operatorname{irr}(\mathbf{T}_0[w])$.

Geometric series

We would like to partition the set of irreducible characters of G using the Deligne–Lusztig characters. We first note the following.

Since we are considering virtual characters, there might exist a character $\chi \in \operatorname{irr}(G)$ such that $\langle I_{\mathbf{T}}^{\mathbf{G}}(\theta), \chi\rangle \neq 0$ and $\langle I_{\mathbf{T}'}^{\mathbf{G}}(\theta'), \chi\rangle \neq 0$ even if $\langle I_{\mathbf{T}}^{\mathbf{G}}(\theta), I_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle = 0$. Therefore, we cannot simply define a partition of $\operatorname{irr}(G)$ by looking at the constituents of each Deligne–Lusztig character.

Instead, we define a partition by saying that two characters χ, χ' are in the same equivalence class if there exist $\chi = \chi_0, \chi_1, \dots, \chi_n = \chi' \in \operatorname{irr}(G)$ and Deligne–Lusztig characters $I_{\mathbf{T}_i}^{\mathbf{G}}(\theta_i)$ for $1 \leq i \leq n$ such that $\langle \chi_{i-1}, I_{\mathbf{T}_i}^{\mathbf{G}}(\theta_i)\rangle \neq 0$ and $\langle \chi_i, I_{\mathbf{T}_i}^{\mathbf{G}}(\theta_i)\rangle \neq 0$ for $1 \leq i \leq n$. In other words, we construct a graph $\mathrm{DL}(G)$ with vertices $\operatorname{irr}(G)$, where two vertices are connected by an edge if they belong to the same character $I_{\mathbf{T}}^{\mathbf{G}}(\theta)$. Then the equivalence classes are the connected components of the graph.

There is another criterion to partially describe these equivalence classes, which generalises Proposition 2.2.5. We define the norm map for an F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$ and $d \in \mathbb{Z}_{\geq 0}$:

$$N_{F^d,F} : \mathbf{T} \to \mathbf{T}, \quad t \mapsto tF(t)\cdots F^{d-1}(t).$$

Theorem 2.2.6 (Exclusion Theorem [GM20, Thm. 2.3.2])**.** *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *be* F*-stable maximal tori of* $\mathbf{G}$ *and* $\theta \in \operatorname{irr}(T)$, $\theta' \in \operatorname{irr}(T')$. *Then if* $I_{\mathbf{T}}^{\mathbf{G}}(\theta)$ *and* $I_{\mathbf{T}'}^{\mathbf{G}}(\theta')$ *have an irreducible character of* G *in common, there exist* $d \in \mathbb{Z}_{\geq 0}$ *and* $g \in \mathbf{G}^{F^d}$ *such that* $g\mathbf{T}g^{-1} = \mathbf{T}'$ *and* $\theta \circ N_{F^d,F} \circ \operatorname{ad}(g) = \theta' \circ N_{F^d,F}$. *We then say that* $(\mathbf{T}, \theta)$ *and* $(\mathbf{T}', \theta')$ *are* ***geometrically conjugate****.*

Note that this result translates to a condition on the pairs (w, θ) for every $w \in W$ and $\theta \in \operatorname{irr}(\mathbf{T}_0[w])$.

Definition 2.2.7. We say two characters $\chi, \chi' \in \operatorname{irr}(G)$ are in the same **geometric series** if there exist two geometrically conjugate pairs $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$, where $\mathbf{T}$ (resp. $\mathbf{T}'$) is an F-stable maximal torus with $\theta \in \operatorname{irr}(T)$ (resp. $\theta' \in \operatorname{irr}(T')$) such that $\langle \chi, I_{\mathbf{T}}^{\mathbf{G}}(\theta)\rangle \neq 0$ and $\langle \chi', I_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle \neq 0$.

Remark 2.2.8. If F is the standard Frobenius, then since $k = \overline{\mathbb{F}}_p = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{F}_{p^n}$, it is clear for any $g \in \mathbf{G}$, there is $d \in \mathbb{Z}_{\geq 0}$ such that $g \in \mathbf{G}^{F^d}$. So in some ways we could say that the pairs $(\mathbf{T}, \theta)$ and $(\mathbf{T}', \theta')$ are conjugate over $\mathbf{G}$, hence the term *geometric.*

We now give another description of the geometric series. Let $\lambda \in X$ and n a positive integer prime to p. We define the following sets

$$\begin{aligned} \mathscr{Z}_{\lambda,n} &:= \{w \in W \mid \lambda \circ F - w.\lambda \in nX\}, \\ W_{\lambda,n} &:= \{w \in W \mid \lambda - w.\lambda \in nX\} \\ \text{and} \quad W_{\lambda,n}^{\circ} &:= \{w \in W \mid \lambda - w.\lambda \in n\mathbb{Z}\Phi\}. \end{aligned}$$

In general, the first and the second sets are not Coxeter groups. If the first one is not empty, then it is a coset of the second one which is a group ([GM20, Lem. 2.4.12]). However, the last group is indeed a Coxeter group with root system

$$\Phi_{\lambda,n} := \{\alpha \in \Phi \mid s_\alpha \in W_{\lambda,n}\},$$

see [GM20, 2.4.13]

Remark 2.2.9. If $Z(\mathbf{G})$ is connected, $W_{\lambda,n}$ is a Coxeter group and $W_{\lambda,n} = W^\circ_{\lambda,n}$, [DM20, 11.2.1].

Recall that we have fixed an isomorphism $\iota : (\mathbb{Q}/\mathbb{Z})_{p'} \tilde{\to} k^\times$ (1.1). Taking the exponential and identifying $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$, it gives a fixed isomorphism

$$\mathfrak{i} : k^\times \tilde{\to} \mu_{p'} := \{x \in \overline{\mathbb{Q}}_\ell \mid x^n = 1 \text{ for some } n \in \mathbb{Z}_{\geq 0}, p \nmid n\}. \tag{2.1}$$

Assume that $\mathscr{Z}_{\lambda,n} \neq \varnothing$ and let $w \in \mathscr{Z}_{\lambda,n}$. We consider $\lambda_w := \frac{1}{n}(\lambda \circ F - w.\lambda)$. It restricts to a character $\lambda_w : \mathbf{T}_0[w] \to k^\times$ and composing with ψ, we get $\theta_w := \mathfrak{i} \circ \lambda_w \in \mathrm{irr}(\mathbf{T}_0[w])$. Observe that for any $n' \geq 1$, we have $\mathscr{Z}_{\lambda,n} = \mathscr{Z}_{n'\lambda,n'n}$. Moreover, the pairs (λ, n) and $(n'\lambda, n'n)$ give rise to the same character of $\mathbf{T}_0[w]$, for a fixed $w \in \mathscr{Z}_{\lambda,n}$.
As in [Lus84a, 6.1], we say that the pair (λ, n) is **indivisible** if there is no integer $n' \geq 2$ such that $\lambda \in n'X$ and n' divides n.
We thus get another description of the geometric series.

Theorem 2.2.10 ([DL76, 10.1],[Lus84a, 6.5])**.** *Let $(\lambda, n) \in X \times \mathbb{N}$ be an indivisible pair such that $\mathscr{Z}_{\lambda,n} \neq \varnothing$. Then the set*

$$\mathscr{E}_{\lambda,n}(\mathbf{G}) := \{\chi \in \mathrm{irr}(G) \mid \langle I_w^{\theta_w}, \chi\rangle \neq 0 \textit{ for some } w \in \mathscr{Z}_{\lambda,n}\}$$

is a geometric series.
Moreover, if $(\lambda', n') \in X \times \mathbb{N}$ is another indivisible pair, we have $\mathscr{E}_{\lambda,n}(\mathbf{G}) = \mathscr{E}_{\lambda',n'}(\mathbf{G})$ if and only if $n = n'$ and there is $w \in W$ such that $\lambda' - w.\lambda \in nX$.

Let $w \in W$, $\theta \in \mathrm{irr}(\mathbf{T}_0[w])$ and n be an integer prime to p such that $\theta^n = 1_{\mathbf{T}_0[w]}$. Then by [GM20, Lem. 2.4.8], there is $\lambda \in X$ such that $w \in \mathscr{Z}_{\lambda,n}$ and $\mathfrak{i} \circ \lambda_w = \theta$. Moreover, if $\lambda' \in X$ also satisfies these conditions, then $\lambda' - w.\lambda \in nX$. As a result, all geometric series are of the form $\mathscr{E}_{\lambda,n}(\mathbf{G})$ for some indivisible pair (λ, n).
If we write $\Lambda(\mathbf{G}, F)$ for the set of indivisible pairs as in the above theorem 2.2.10, then

$$\mathrm{irr}(G) = \bigcup_{(\lambda,n) \in \Lambda(\mathbf{G},F)} \mathscr{E}_{\lambda,n}(\mathbf{G}).$$

If $Z(\mathbf{G})$ is connected, then we get an even better description of $\mathscr{E}_{\lambda,n}(\mathbf{G})$. We make it more explicit in Subsection 2.2.2.

We give yet another parameterisation of the geometric series, this time in terms of semisimple conjugacy classes of the dual group of $\mathbf{G}$. Let $(\mathbf{G}^*, F^*)$ be a connected reductive group with a Steinberg endomorphism in duality with $(\mathbf{G}, F)$ as in 1.2.2. Following [GM20, Sect. 2.5], we explain Lusztig's idea ([Lus84a, 8.4]) of associating to each pair $(\lambda, n) \in X \times \mathbb{N}$ a semisimple element of $\mathbf{G}^*$. Let $\mathbf{T}_0^*$ be the maximally split torus in $\mathbf{G}^*$ which is in duality with $\mathbf{T}_0$. Using the map 1.2, we set

$$\begin{aligned} X \times \mathbb{N} &\to \mathbf{T}_0^* \\ (\lambda, n) &\mapsto \tilde{i}_{\mathbf{T}_0^*}\left(\frac{1}{n} \otimes \lambda\right) =: t_{\lambda,n}. \end{aligned}$$

This map induces a surjection between $\Lambda(\mathbf{G}, F)$ and the F^*-stable $\mathbf{G}^*$-conjugacy classes of semisimple elements in $\mathbf{G}^*$. Moreover, two indivisible pairs (λ, n) and (λ', n') are sent to the same conjugacy class if and only if $n = n'$ and $\lambda' - w.\lambda \in nX$ for some $w \in W$, see [GM20, Prop. 2.5.5]. For $s \in \mathbf{T}_0^*$ such that $(s)_{\mathbf{G}^*}$ is F^*-stable, there is $(\lambda, n) \in \Lambda(\mathbf{G}, F)$ such that $(s)_{\mathbf{G}^*} = (t_{\lambda,n})_{\mathbf{G}^*}$. It is then well-defined to set $\mathscr{E}(\mathbf{G}, s) \coloneqq \mathscr{E}_{\lambda,n}(\mathbf{G})$, and we get

$$\mathrm{irr}(G) = \bigsqcup_s \mathscr{E}(\mathbf{G}, s),$$

where s runs over a set of representatives of F^*-stable semisimple conjugacy classes in $\mathbf{G}^*$.

Remark 2.2.11. The map above induces a map from the set of pairs (w, θ) where $w \in W$ and $\theta \in \mathrm{irr}(\mathbf{T}_0[w])$ to the F^*-stable conjugacy classes of semisimple elements in $\mathbf{G}^*$. As explained after Theorem 2.2.10, we associate a pair (λ, n) to (w, θ), whence a semisimple element $s_\theta \coloneqq t_{\lambda,n}$. By [GM20, Lem. 2.5.7], this map is well-defined.

Combining what we have seen after Proposition 2.2.5 and in Remark 2.2.11, we associate to each G-conjugacy class of pairs $(\mathbf{T}, \theta)$ (where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{irr}(\mathbf{T})$) a G^*-conjugacy class of pairs $(\mathbf{T}^*, s)$ where $\mathbf{T}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ and $s \in (\mathbf{T}^*)^{F^*}$, see [GM20, Cor. 2.5.14]. This map is in fact a bijection. Therefore, we can set

$$I_{\mathbf{T}^*}^{\mathbf{G}}(s) \coloneqq I_{\mathbf{T}}^{\mathbf{G}}(\theta).$$

Rational series

We now come back to our initial idea of partitioning $\mathrm{irr}(G)$ by the connected components of the graph $\mathrm{DL}(G)$. As a matter of fact, this partition is actually easier to state when we use the notation $I_{\mathbf{T}^*}^{\mathbf{G}}(s)$.

Definition 2.2.12. Let $s \in G^*$ be a semisimple element. The set $\mathscr{E}(G, s)$ consists of all $\chi \in \mathrm{irr}(G)$ such that $\langle I_{\mathbf{T}^*}^{\mathbf{G}}(s), \chi \rangle \neq 0$ for some F^*-stable maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$ containing s. It is called a **rational series** or a **Lusztig series** of characters of G.

Theorem 2.2.13 ([Lus77, 7.6])**.** *If $s_1, s_2 \in G^*$ are semisimple and conjugate over G^*, then $\mathscr{E}(G, s_1) = \mathscr{E}(G, s_2)$. Moreover,*

$$\mathrm{irr}(G) = \bigsqcup_s \mathscr{E}(G, s),$$

where s runs over a set of representatives of semisimple conjugacy classes in G^*. Moreover each rational series corresponds to a connected component of the graph $DL(G)$.

For a detailed proof, we refer the reader to [GM20, Thm.2.6.2 and Rmk. 2.6.19]. Note that each geometric series is a union of rational series. For a semisimple element $s \in G^*$, we set

$$\mathscr{E}(\mathbf{G}, s) \coloneqq \bigsqcup_t \mathscr{E}(G, t),$$

where t runs over a set of representatives of the semisimple G^*-conjugacy classes of F^*-stable elements in $(s)_{\mathbf{G}^*}$, see [GM20, Rmk. 2.6.3]. In particular, if $Z(\mathbf{G})$ is connected, then geometric and rational series coincide.

We explain how Deligne–Lusztig induction interacts with rational series.

Proposition 2.2.14 ([GM20, Prop.3.3.20])**.** *Deligne–Lusztig induction preserves rational series. Therefore, Harish-Chandra series are unions of rational series.*

Remark 2.2.15. Note that the Alvis–Curtis duality operator fixes rational series, c.f. [GM20, Cor. 3.4.6].

Jordan decomposition of characters

One astonishing result about the representation theory of finite reductive groups is that a lot of the information we want is concentrated in a unique rational series. It is summarised in Theorem 4.23 of Lusztig's book on characters of finite reductive groups [Lus84a].

Theorem 2.2.16 (Jordan decomposition of characters)**.** *Assume that $Z(\mathbf{G})$ is connected. Let $s \in G^*$ be a semisimple element. Let $\mathbf{H} = C_{\mathbf{G}^*}(s) (= C^\circ_{\mathbf{G}^*}(s))$. Then there is a bijection*

$$\mathscr{E}(G,s) \stackrel{1-1}{\longleftrightarrow} \mathscr{E}(H,1), \quad \chi \leftrightarrow \chi_u$$

such that for any F^*-stable maximal torus $\mathbf{T}^* \subseteq \mathbf{H}$, we have

$$\langle I^{\mathbf{G}}_{\mathbf{T}^*}(s), \chi\rangle = \epsilon_{\mathbf{G}}\epsilon_{\mathbf{H}} \langle I^{\mathbf{H}}_{\mathbf{T}^*}(1_{\mathbf{T}^*}), \chi_u\rangle.$$

Here $\epsilon_{\mathbf{G}}, \epsilon_{\mathbf{H}}$ are signs which can be read of the order of G, respectively H ([GM20, Def. 2.2.11]).

This bijection is not unique, but can be made so by requiring additional conditions ([DM90]). Moreover, a similar result holds when $Z(\mathbf{G})$ is not connected ([Lus88], [Lus08]).
We write $\mathrm{Uch}(G) \coloneqq \mathscr{E}(G,1)$ and we call it the set of **unipotent characters** of G.

2.2.2 Parameterisation of the unipotent characters

By Theorem 2.2.16, we can now focus our attention on the unipotent characters. From Theorem 2.1.14, we could envisage that the Weyl group W might play a role. We thus start by recalling various properties of the characters of W. In general, there should be more characters in $\mathrm{Uch}(G)$ than characters of W as there might be characters coming from $I^{\mathbf{G}}_{\mathbf{T}}(1_{\mathbf{T}})$ where $\mathbf{T}$ is not included in an F-stable Borel subgroup. Nevertheless, using the notions stated for the Weyl group, we define families of unipotent characters and state another prodigious aspect of [Lus84a, Theorem 4.23]. We will mostly follow [GM20, Chap. 4.1, Chap. 4.2].

Families of characters in the Weyl group

We describe a few notions that will help us split the irreducible characters of a Weyl group W into families.

Definition 2.2.17. We define the a-**invariant** inductively. If $W = \{1\}$, then we set $a_{1_W} = 0$. Assume $W \neq \{1\}$ and that the a-invariant is defined for any character of a proper parabolic subgroup of W. Then, for any $\psi \in \mathrm{irr}(W)$, we set

$$a'_\psi := \max\{a_\phi \mid \phi \in \mathrm{Irr}(W_I) \text{ for some } I \subsetneq \Delta \text{ and } \langle \mathrm{Ind}_{W_I}^W(\phi), \psi\rangle \neq 0\}$$

and

$$\omega_\psi := \sum_{s\in S} \frac{\psi(s)}{\psi(1)},$$

where S is the set of reflections of W. Lastly, we set

$$a_\psi := \max\{a'_\psi, a'_{\varepsilon\psi} - \omega_\psi\}.$$

Here, ε is the sign character of W.

Using the a-invariant, we recursively define a partial order on $\mathrm{irr}(W)$ which induces an equivalence relation.

Definition 2.2.18. We define inductively an order relation $\leq$ on $\mathrm{irr}(W)$. If $W = \{1\}$, then $1_W \leq 1_W$. Assume $W \neq \{1\}$ and the order $\leq$ is defined for any proper parabolic subgroup of W. Let $\psi, \psi' \in \mathrm{irr}(W)$. We write $\psi \leq \psi'$ if there is a sequence $\psi = \psi_0, \dots, \psi_m = \psi'$ such that for all $1 \leq i \leq m$, there are $I_i \subsetneq \Delta$ and $\phi_i, \phi'_i \in \mathrm{irr}(W_{I_i})$ with $\phi_i \leq \phi'_i$ such that either

$$\langle \mathrm{Ind}_{W_{I_i}}^W(\phi_i), \psi_{i-1}\rangle \neq 0, \quad \langle \mathrm{Ind}_{W_{I_i}}^W(\phi'_i), \psi_i\rangle \neq 0 \quad \text{and} \quad a_{\phi'_i} = a_{\psi_i},$$

or

$$\langle \mathrm{Ind}_{W_{I_i}}^W(\phi_i), \varepsilon\psi_i\rangle \neq 0, \quad \langle \mathrm{Ind}_{W_{I_i}}^W(\phi'_i), \varepsilon\psi_{i-1}\rangle \neq 0 \quad \text{and} \quad a_{\varepsilon\phi'_i} = a_{\varepsilon\psi_i}.$$

Each equivalence class is called a **family** of $\mathrm{irr}(W)$.

Note that if $\psi \leq \psi'$, then $a_\psi \leq a'_\psi$ ([GM20, Prop. 4.1.19]). As a result, a_ψ is constant on families. We now consider another invariant.

Definition 2.2.19. For $\psi \in \mathrm{irr}(W)$, the b-**invariant** b_ψ of ψ is defined as the smallest integer $n \in \mathbb{Z}_{\geq 1}$ such that ψ occurs in the character of the nth symmetric power of the natural representation of W.

As the a-invariant is defined inductively, we consider what happens to the b-invariant after induction.

Proposition 2.2.20 ([Lus92, 10.2a][Lus79, §3])**.** *Assume that W' is a subgroup of W generated by reflections. For each $\psi' \in \mathrm{irr}(W')$ and each $\phi \in \mathrm{irr}(W)$, if $\langle \phi, \mathrm{Ind}_{W'}^W(\psi')\rangle \neq 0$, then $b_\phi \geq b_{\psi'}$. Moreover, there exists a unique $\psi \in \mathrm{irr}(W)$ such that $\langle \psi, \mathrm{Ind}_{W'}^W(\psi')\rangle = 1$ and the b-invariants of ψ and ψ' agree. The character ψ is called the* ***j-induction*** *of ψ' and denoted $j_{W'}^W(\psi')$.*

We now link the a- and the b-invariants.

Proposition 2.2.21 ([GM20, Prop. 4.1.20], [Lus79, Prop. 4])**.** *Let $\psi \in \mathrm{irr}(W)$. We always have $a_\psi \leq b_\psi$. If $a_\psi = b_\psi$, we say that ψ is* ***special****. There is exactly one special character in each family. Moreover, if W' is a parabolic subgroup of W then the j-induction from W' to W of any special character is special.*

We lastly consider how the Steinberg endomorphism F of our group $\mathbf{G}$ interacts with the families of the Weyl group. The following discussion can be extended to any Coxeter group automorphism.
Recall that F acts on W, hence on the set of characters of W. We observe that F preserves the a- and the b-invariants, as well as the order $\leq$. In particular, it sends families to families and special characters to special characters. Moreover, the F-stable families are in fact F-fixed.

Proposition 2.2.22 ([Lus84a, 4.17])**.** *Assume that F is ordinary. If $\mathscr{F} \subseteq \mathrm{irr}(W)$ is an F-stable family, then all elements of $\mathscr{F}$ are F-stable.*

Remark 2.2.23. Lusztig determined explicitly the families of $\mathrm{irr}(W)$. From this, he was able to state and prove the two previous propositions. However, there is so far no proof which does not rely on case by case analysis.

Families of unipotent characters

We describe a classification of $\mathrm{Uch}(G)$ into families. Firstly, we can restrict our discussion to $\mathbf{G}$ simple of adjoint type, see [GM20, Rmk. 4.2.1]. For instance, there exists a Steinberg map $\overline{F} : \mathbf{G}/Z(\mathbf{G}) \to \mathbf{G}/Z(\mathbf{G})$ which commutes with F after taking the quotient map $\pi : \mathbf{G} \to \mathbf{G}/Z(\mathbf{G})$. Then by [DL76, 7.10] the following map

$$\mathrm{Uch}((\mathbf{G}/Z(\mathbf{G}))^{\overline{F}}) \longrightarrow \mathrm{Uch}(G), \quad \rho \mapsto \rho \circ \pi$$

is a bijection.

Hypothesis 2.2.24. For the rest of this subsection, we assume that $\mathbf{G}$ is simple adjoint.

We now describe a partition of $\mathrm{Uch}(G)$ into families. Recall that

$$\mathrm{Uch}(G) = \{\chi \in \mathrm{irr}(G) \mid \langle I_w^{1_{\mathbf{T}_0[w]}}, \chi \rangle \neq 0 \text{ for some } w \in W\}.$$

We then consider combinations of $I_w^{\theta_w}$ so that they are this time indexed by the irreducible characters of W or rather of $\tilde{W} := W \rtimes < F >$, to take into account the action of F on W. For $\psi \in \mathrm{irr}(W)$ which is F-stable, we choose an extension $\tilde{\psi}$ of ψ to $\tilde{W}$ such that $\tilde{\psi}(w) \in \mathbb{R}$ for all $w \in W$. Such an extension exists by [Lus84a, 3.2, 14.2]. We now define the **almost character** associated to ψ as

$$R_{\tilde{\psi}} := \frac{1}{|W|} \sum_{w \in W} \psi(w) I_w^{1_{\mathbf{T}_0[w]}}.$$

By [Lus84a, Prop. 3.9], for $\psi, \psi' \in \operatorname{irr}(W)^F$,

$$\langle R_{\tilde{\psi}}, R_{\tilde{\psi}'} \rangle = \langle \psi, \psi' \rangle = \delta_{\psi,\psi'}.$$

Now, similarly as with Deligne–Lusztig characters, we define a graph with vertices $\operatorname{Uch}(G)$, where two vertices are connected by an edge if they belong to the same almost character $R_{\tilde{\psi}}$. The connected components of the graph are called the **families** of $\operatorname{Uch}(G)$.

Remark 2.2.25. The Alvis–Curtis duality sends families to families of unipotent characters [GM20, Prop. 4.2.8].

Remark 2.2.26. We can define almost characters for any geometric series $\mathscr{E}_{\lambda,n}(\mathbf{G})$, by extending the definition. Fix $w_0 \in \mathscr{Z}_{\lambda,n}$ the unique element of $\mathscr{Z}_{\lambda,n}$ of minimal length ([GM20, Lem. 2.4.14]). For $\psi \in \operatorname{irr}(W^\circ_{\lambda,n})$ which is F-stable, we fix a chosen extension $\tilde{\psi}$ of ψ to $\tilde{W}_{\lambda,n}$ such that $\tilde{\psi}(w) \in \mathbb{R}$ for all $w \in W_{\lambda,n}$. The **almost character** is the following

$$R_{\tilde{\psi}} := \frac{1}{|W_{\lambda,n}|} \sum_{w \in W_{\lambda,n}} \psi(w) I_{w_0 w}^{\theta_{w_0 w}}.$$

The next question is to understand the numbers

$$\langle R_{\tilde{\psi}}, \chi \rangle$$

for any $\psi \in \operatorname{irr}(W)^F$ and $\chi \in \operatorname{Uch}(G)$. Note that they might depend on the choice of the extension $\tilde{\psi}$. There are at most two possibilities such that $\tilde{\psi}(w) \in \mathbb{R}$ for all $w \in W$, since F-extensions are unique up to a root of unity ([GM20, Prop.2.1.14]). We fix one, following [LuCS4, 17.2]. It is called the **preferred F-extension** of ψ.
To each family $\mathscr{F}$ of $\operatorname{irr}(W)$, we define the family $\tilde{\mathscr{F}}$ which consists of all irreducible characters of $W \rtimes < F > / < F^c >$ such that their restriction to W belongs to $\mathscr{F}$. Here c denotes the order of the action of F on W.

If F is ordinary, two characters $\psi, \psi' \in \operatorname{irr}(W)^F$ are in the same F-stable family if and only if the constituents of the almost characters $R_{\tilde{\psi}}$, $R_{\tilde{\psi}'}$ lie in the same family ([GM20, Prop. 4.2.3]). Thus, there is a bijection between the set of families of the Weyl group and the set of families of $\operatorname{Uch}(G)$. This means that for many questions, we can consider each family individually, in particular to compute the values $\langle R_{\tilde{\psi}}, \chi \rangle$.

In [Lus84a], Lusztig described in an extraordinary way each family $\mathscr{U}$ of $\operatorname{Uch}(G)$. To each family $\mathscr{U}$, he associates a finite group $\bar{A}_{\mathscr{U}}$, that he called the **canonical quotient**. With the help of these groups, he could label all the characters in a given family as well as compute the numbers $\langle R_{\tilde{\psi}}, \chi \rangle$. Before recalling his theorem, we need to introduce a little more notation.

Definition 2.2.27 (Lusztig's non abelian Fourier transform)**.** Let A be any finite group. We define $\mathcal{M}(A)$ as the set of A-conjugacy classes of pairs (a, ϕ) with $a \in A$

and $\phi \in \operatorname{irr}(C_A(a))$. We write $[a,\phi]$ for the conjugacy class of (a,ϕ) as above. We also define a pairing as in [Lus79, § 4]:

$$\begin{aligned}\{\,,\,\}: \mathcal{M}(A)\times\mathcal{M}(A) &\to \mathbb{C}\\ ([a,\phi],[b,\psi]) &\mapsto \frac{1}{|C_A(a)||C_A(b)|}\sum_{g\in A, a\in C_A(gbg^{-1})}\phi(gbg^{-1})\psi(g^{-1}a^{-1}g).\end{aligned}$$

Let $\tilde{A}$ be another finite group such that A is a normal subgroup of $\tilde{A}$ and $\tilde{A}/A$ is cyclic of order $c\in\mathbb{Z}_{\leq 0}$ with a generator $A'\subseteq\tilde{A}$, a coset of A. The set $\mathcal{M}(A\subseteq\tilde{A})$ consists of all $\tilde{A}$-conjugacy classes of pairs $(b,\psi)\in A\times\operatorname{irr}(C_{\tilde{A}}(b))$ such that $C_{\tilde{A}}(b)\cap A'\neq\varnothing$ and the restriction of ψ to $C_A(b)$ is irreducible.
Lastly, the set $\overline{\mathcal{M}}(A\subseteq\tilde{A})$ consists of all $\tilde{A}$-conjugacy classes of pairs $(a,\phi)\in A'\times\operatorname{irr}(C_A(a))$. We get a new pairing induced by the one in $\mathcal{M}(\tilde{A})$:

$$\begin{aligned}\{\,,\,\}: \overline{\mathcal{M}}(A\subseteq\tilde{A})\times\mathcal{M}(A\subseteq\tilde{A}) &\to \mathbb{C}\\ ([a,\phi],[b,\psi]) &\mapsto c\{[a,\tilde{\phi}],[b,\psi]\},\end{aligned}$$

where $\tilde{\phi}$ is the inflation of ϕ to $C_{\tilde{A}}(c)$.

Remark 2.2.28. Observe that $\mu_c := \{\xi\in\overline{\mathbb{Q}}_\ell^\times \mid \xi^c = 1\}$ acts on $\mathcal{M}(A\subseteq\tilde{A})$ as follows: for each $\xi\in\mu_c$, we let $\epsilon_\xi:\tilde{A}\to\mathbb{C}$ such that ϵ_ξ is the trivial character on A and $\epsilon_\xi(a) = x$ for each $a\in A'$. Then we consider the action of ξ sending $[b,\psi]$ to $[b,\psi\otimes\epsilon_\xi]$.

Theorem 2.2.29 (Lusztig, [Lus84a, Thm. 4,23])**.** *We assume that F is ordinary and let c be the order of F on W. To each family $\mathscr{U}$ of* Uch(G) *with corresponding F-stable family $\mathscr{F}\subseteq\operatorname{irr}(W)^F$, one can associate finite groups $\bar{A}_{\mathscr{F}}\trianglelefteq\tilde{A}_{\mathscr{F}}$ with $|\tilde{A}_{\mathscr{F}}:\bar{A}_{\mathscr{F}}| = c$ such that there exist an injection*

$$\tilde{\mathscr{F}}\hookrightarrow\mathcal{M}(\bar{A}_{\mathscr{F}}\subseteq\tilde{A}_{\mathscr{F}}),\quad \psi\mapsto x_\psi$$

and a bijection

$$\mathscr{U}\xrightarrow{\sim}\overline{\mathcal{M}}(\bar{A}_{\mathscr{F}}\subseteq\tilde{A}_{\mathscr{F}}),\quad \chi\mapsto x_\chi$$

such that, for all $\chi\in\mathscr{U}$ and $\psi\in\mathcal{F}$ with preferred extension $\tilde{\psi}$, we have

$$\langle R_{\tilde{\psi}},\chi\rangle = \Delta(x_\chi)\{x_\chi, x_{\tilde{\psi}}\},$$

where $\Delta(x_\chi)\in\{1,-1\}$ depends only on χ and can be computed explicitly.

This result is remarkable in many ways. First of all, the set of unipotent characters, the partition into families and the decomposition of the almost characters are generic and depend only on the complete root datum of $(\mathbf{G},F)$. Moreover, the groups $\bar{A}_{\mathscr{F}}$ depend only on the root system. They were at first defined in an ad-hoc way, even if Lusztig proposed another definition in [Lus84a, 13.1.3] that he proved in [Lus14]. To state this reinterpretation, we need the Springer correspondence (see Section 4.1 and

Definition 5.1.16).
In the setting above, we can rewrite

$$R_{\tilde{\psi}} = \sum_{\chi \in \mathscr{U}} \Delta(x_\chi)\{x_\chi, x_\psi\}\chi.$$

More generally, every element $x \in \mathcal{M}(\bar{A}_{\mathscr{U}})$ can be seen as an element $x \in \mathcal{M}(\bar{A}_{\mathscr{U}} \subseteq \tilde{A}_{\mathscr{U}})$ and we define the **unipotent almost character**

$$R_x = \sum_{\chi \in \mathscr{U}} \Delta(x_\chi)\{x_\chi, x\}\chi.$$

The unipotent almost characters have a geometric meaning in terms of characteristic functions of certain character sheaves, see Section 3.3.

Remark 2.2.30. The main Theorem 4.23 of Lusztig ([Lus84a, Thm. 4.23]) is stated for any series of characters and thus implies the Jordan decomposition of characters (Theorem 2.2.16). Assume that $Z(\mathbf{G})$ is connected and fix $(\lambda, n) \in \Lambda(\mathbf{G}, F)$. There is an element $w \in W$ such that $\mathscr{Z}_{\lambda,n} = wW_{\lambda,n} = wW^\circ_{\lambda,n}$. Moreover, there is a group homomorphism $\sigma : W_{\lambda,n} \to W_{\lambda,n}$ given by $\sigma = F \circ \mathrm{ad}(w)$. Assume furthermore that F is ordinary and σ has order $c \in \mathbb{N}$. In the same way as for W, we split $\mathrm{irr}(W_{\lambda,n})$ into families and associate to each family $\mathscr{F}$ a group $\bar{A}_{\mathscr{F}}$. To each σ-stable family $\mathscr{F} \subseteq \mathrm{irr}(W_{\lambda,n})^\sigma$, one can associate finite groups $\bar{A}_{\mathscr{F}} \trianglelefteq \tilde{A}_{\mathscr{F}}$ with $|\tilde{A}_{\mathscr{F}} : \bar{A}_{\mathscr{F}}| = c$ such that there exist injections

$$\tilde{\mathscr{F}} \hookrightarrow \mathcal{M}(\bar{A}_{\mathscr{F}} \subseteq \tilde{A}_{\mathscr{F}}), \quad \psi \mapsto x_\psi$$

and a bijection

$$\mathscr{E}_{\lambda,n}(\mathbf{G}) \xrightarrow{\sim} \bigsqcup_{\mathcal{F}} \overline{\mathcal{M}}(\bar{A}_{\mathscr{F}} \subseteq \tilde{A}_{\mathscr{F}}), \quad \chi \mapsto x_\chi,$$

where $\mathscr{F}$ runs over the F-stable families of $\mathrm{irr}(W_{\lambda,n})$, such that, for all $\chi \in \mathscr{E}_{\lambda,n}(\mathbf{G})$ and $\psi \in \mathscr{F}$ with preferred extension $\tilde{\psi}$, we have

$$\langle R_{\tilde{\psi}}, \chi \rangle = \Delta(x_\chi)\{x_\chi, x_{\tilde{\psi}}\},$$

where $\Delta(x_\chi) \in \{1, -1\}$ depends only on χ and can be computed explicitly. Similarly, one could define for any $x \in \mathcal{M}(\bar{A}_{\mathscr{F}} \subseteq \tilde{A}_{\mathscr{F}})$, the almost character

$$R_x = \sum_{\chi \in \mathscr{E}_{\lambda,n}} (-1)^{\ell(w)} \Delta(x_\chi)\{x_\chi, x\}\chi.$$

2.3 Computing ordinary characters

Thanks to the Jordan decomposition of characters (Theorem 2.2.16) and the discussion in the previous section, we now have a labelling of all ordinary irreducible characters of G. Notwithstanding, it is not obvious from the definitions how to compute their values, and it is undeniably a challenging question. In this section, we give some general results and an overview on the subject. However, the main current method reduces the problem to computing almost characters seen as characteristic functions of character sheaves. We will elaborate on this in Section 3.3.

2.3.1 Computing Deligne–Lusztig characters

The values of Deligne–Lusztig characters are not easy to compute. Nonetheless, some results are known from the theory. As we have seen the class of Deligne–Lusztig characters of the form $I_{\mathbf{T}}^{\mathbf{G}}(1_T)$ are of particular interest as they give rise to the unipotent characters. It transpires from the next theorem that they also play an eminent role to compute the character values.

Definition 2.3.1. Let $\mathbf{T} \subseteq \mathbf{G}$ be an F-stable maximal torus. The **Green function** $Q_{\mathbf{T}}^{\mathbf{G}} : \mathbf{G}_{\text{uni}}^F \to \overline{\mathbb{Q}}_\ell$ is defined by

$$Q_{\mathbf{T}}^{\mathbf{G}}(u) \coloneqq I_{\mathbf{T}}^{\mathbf{G}}(1_T)(u) \quad \text{for } u \in \mathbf{G}_{\text{uni}}^F.$$

The Green functions take values in the integers, and therefore $Q_{\mathbf{T}}^{\mathbf{G}}(u) = Q_{\mathbf{T}}^{\mathbf{G}}(u^{-1})$ for any $u \in \mathbf{G}_{\text{uni}}^F$ ([GM20, Def. 2.2.15]). Besides, they do not depend on the G-conjugacy class of $\mathbf{T}$ by Proposition 2.2.5. Each Deligne–Lusztig character can be expressed thanks to the Green functions.

Theorem 2.3.2 (Character formula [DL76, 4.2])**.** *Let $g \in G$ with Jordan decomposition $g = su = us$ where $s \in G$ is semisimple and $u \in \mathbf{G}_{uni}^F$. Let $\mathbf{H} \coloneqq C_{\mathbf{G}}^\circ(s)$. Then for any F-stable maximal torus $\mathbf{T} \subseteq \mathbf{G}$ and $\theta \in \operatorname{irr}(T)$,*

$$I_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = \frac{1}{|H|} \sum_{x \in G,\, s \in {}^x\mathbf{T}} Q_{{}^x\mathbf{T}}^{\mathbf{H}}(u)\theta(x^{-1}sx).$$

From the above theorem, we easily see that $I_{\mathbf{T}}^{\mathbf{G}}(\theta)(g) = 0$ if s is not G-conjugate to an element of T. Furthermore, for $u \in \mathbf{G}_{\text{uni}}^F$,

$$I_{\mathbf{T}}^{\mathbf{G}}(\theta)(u) = Q_{\mathbf{T}}^{\mathbf{G}}(u) \in \mathbb{Z}$$

for any $\theta \in \operatorname{irr}(\mathbf{T})$. The character formula reduces the computation of Deligne–Lusztig characters to two problems:

1. computing the values of the Green functions and
2. understanding the set of all $x \in G$ such that $s \in {}^x\mathbf{T}$.

We will discuss in more details how the first question can be tackled in Section 4.1. Note that the Green functions have been completely determined when p is good for $\mathbf{G}$ (for $\mathbf{G}$ of type F_4 in [Sho82], for $\mathbf{G}$ of classical type in [Sho83] and for type E_6, E_7 and E_8 by [BS84]). The bad characteristic case has been closed less than a year ago by Lübeck ([Lüb24]) who considered groups of type E_8 (see also [Gec20]).

2.3.2 Ordinary characters on semisimple or unipotent conjugacy classes

Using the previous character formula, Deligne and Lusztig inferred the values of a character at semisimple elements.

Proposition 2.3.3 ([DL76, 7.6])**.** *Let* $\chi \in \mathrm{irr}(G)$ *and* $s \in G$ *be semisimple. Let* $\mathbf{H} \coloneqq C^\circ_{\mathbf{G}}(s)$*. Then*

$$\chi(s) = \frac{1}{|H|} \sum_{(\mathbf{T},\theta)} \epsilon_{\mathbf{H}} \epsilon_{\mathbf{T}} \langle I_{\mathbf{T}}^{\mathbf{G}}(\theta), \chi \rangle \theta(s),$$

where the sum runs over the pairs $(\mathbf{T},\theta)$ *such that* $\mathbf{T}$ *is an* F*-stable maximal torus with* $s \in \mathbf{T}$ *and* $\theta \in \mathrm{irr}(T)$*. Here* $\epsilon_{\mathbf{H}}, \epsilon_{\mathbf{T}}$ *are signs which can be read of the order of* H*, respectively* T *([GM20, Def. 2.2.11]).*

For a unipotent character χ, we have $\langle I_{\mathbf{T}}^{\mathbf{G}}(\theta), \chi \rangle \neq 0$ implies that $\theta = 1_T$ and we obtain

$$\chi(s) = \frac{1}{|H|} \sum_{\mathbf{T}} \epsilon_{\mathbf{H}} \epsilon_{\mathbf{T}} \langle I_{\mathbf{T}}^{\mathbf{G}}(1_T), \chi \rangle,$$

where $\mathbf{T}$ runs over the F-stable maximal tori of $\mathbf{H}$. Thus, the value of $\chi(s)$ depends only on $\mathbf{H}$ and on the coefficients $\langle I_{\mathbf{T}}^{\mathbf{G}}(1_T), \chi \rangle$. Moreover, the numbers $\langle I_{\mathbf{T}}^{\mathbf{G}}(1_T), \chi \rangle$ are fully determined (see Theorem 2.2.29) and can be accessed in CHEVIE [Mic15].

We now consider the values of ordinary characters at unipotent elements. Again, this is a difficult problem. Nonetheless, we can to some extent determine if the value is zero or not.

Theorem 2.3.4 ([Lus92, Thm. 11.2], [Tay19, §9])**.** *Assume that* p *is good for* $\mathbf{G}$ *and* $Z(\mathbf{G})$ *is connected. Assume as well that* F *is a Frobenius map. Let* $\chi \in \mathrm{irr}(G)$*. There exists an* F*-stable unipotent class* $C \in \mathrm{Ucl}(\mathbf{G})$ *such that for any* $g \in G$ *with unipotent part* u*, we have*

$$\chi(g) \neq 0 \iff \dim(u)_{\mathbf{G}} < \dim C \text{ or } (u)_{\mathbf{G}} = C.$$

Moreover, there is $g \in G$ *with unipotent part* u *such that* $(u)_{\mathbf{G}} = C$ *and* $\chi(g) \neq 0$*. The unipotent class* C *is called the* ***unipotent support*** *of* χ*.*

In fact, in [GM00] Geck and Malle showed that the condition $\chi(g) \neq 0$ can be replaced by a condition on the average value of χ on the F-stable unipotent class C. The unipotent supports have all been determined for unipotent characters. We will explain a method to describe them in Section 4.1. It uses the Springer correspondence. The unipotent support gives us another way of characterising the families of unipotent characters.

Theorem 2.3.5 ([GM00, Prop. 4.2 and Cor. 5.2])**.** *We keep the same hypotheses as in Theorem 2.3.4. Two unipotent characters of* G *belong to the same family if and only they have the same unipotent support.*

2.3.3 Current state of knowledge

Thanks to Deligne–Lusztig theory, we now have a complete parameterisation of the ordinary characters of a finite group of Lie type. However, the description of the character

tables is far from being achieved.
For groups of classical type, no other series than $\mathrm{GL}_n(q)$ has been treated for each integer n. The most recent and complete results are from around thirty years ago with $\mathrm{SO}_8^+(q)$ by Geck and Pfeiffer [GP92] for q odd and Geck [Gec95] for q even as well as with $\mathrm{SO}_8^-(q)$ by Lübeck [GHLMP96]. Nonetheless, in 2020, Malle and Rotilio described in [MR20] and [Rot21] how to compute the generic character table of $\mathrm{Spin}_8^+(q)$, the 8-dimensional spin group in odd characteristic, whose centre is disconnected of order 4.
For exceptional groups, the case G_2 has been known for a long time. Indeed, the character tables of $G_2(q)$ were fully determined in 1974 by Chang and Ree [CR74] in good characteristic, in 1976 by Enomoto [Eno76] when $p = 3$ and ten years later by Enomoto and Yamada [EY86] for even q. Recently, Geck described a strategy to compute the character table of F_4, E_6 and ${}^2E_6$ in even characteristic, see [Gec23], [Gec24].
For a complete list of the determined character tables, we refer the reader to [GM20, Table 2.4].

Chapter 3

Character sheaves

In the previous chapter, to define Deligne–Lusztig characters, we have looked at varieties with an action of the finite group G and then considered G-equivariant (perverse) sheaves on them. Alternatively, we could have seen them as $\mathbf{G}$-equivariant sheaves on $\mathbf{G}$ for the twisted action by the Steinberg map F (see Remark 2.1.2). We could instead consider varieties which have an action of the algebraic group $\mathbf{G}$ and are moreover F-stable, or to be exact, varieties defined over $\mathbb{F}_q$ via the map F that we necessarily assume to be a Frobenius endomorphism. Similarly, we will look at $\mathbf{G}$-equivariant, (this time for the action by conjugation) and F-stable perverse sheaves on those varieties. This new approach has multiple benefits. Firstly, we notice that in some ways the conditions we require seem stronger, going from G- to $\mathbf{G}$-equivariance. In particular, we can have a better grasp on the $\mathbf{G}$-equivariant perverse sheaves and do not need to (mainly) restrict ourselves to the constant sheaf.
Moreover, it means that we may at first completely forget about the Frobenius map F. This is what we will do in the first two sections of this chapter. Nonetheless, we still want to gain information about the ordinary characters of G and we will imitate the construction of Deligne–Lusztig characters to define character sheaves (Definition 3.1.9). These are certain $\mathbf{G}$-equivariant irreducible perverse sheaves on $\mathbf{G}$. Additionally, we will see that we can also mimic the Harish-Chandra induction thanks to parabolic induction in Section 3.2.
In the third section, we finally add the Frobenius map F and bring our attention to the F-stable character sheaves. This allows us to describe a new basis of the space of class functions for G (Theorem 3.3.5). Furthermore, Lusztig conjectured in [Lus84a] that this new basis coincides with the set of almost characters. It appears to be true (at least in the connected centre case) thanks to the work of Shoji, c.f. [Sho95a], [Sho95b].

Most of this powerful theory was developed by Lusztig in a series of papers, [LuCS1] to [LuCS5], following his precursor article [Lus84b]. These are our main references for this chapter.

As in Hypothesis 1, we assume that $\mathbf{G}$ is a connected reductive group over k with Steinberg map $F : \mathbf{G} \to \mathbf{G}$, $\mathbf{T}_0 \subseteq \mathbf{B}_0$ is a maximally split torus in an F-stable Borel $\mathbf{B}_0$

of $\mathbf{G}$ with associated root datum $(X, \Phi, \check{X}, \check{\Phi})$, base Δ and Weyl group W.

3.1 Definition of character sheaves

Character sheaves on $\mathbf{G}$ are constructed to be the geometric analogue of characters of G. In this section, we define them following Lusztig and his paper [LuCS1]. As for characters, we start by introducing some notions on perverse sheaves that we need along this chapter. We then define character sheaves and explain how to partition their isomorphism classes. Along the way, we will try to convince the reader of the similarities with the characters.

3.1.1 Reminder on G-equivariant perverse sheaves

Recall that Deligne–Lusztig characters are defined thanks to an alternating sum of the cohomology groups with compact support of the derived global section functor

$$\sum_{i\in\mathbb{Z}}(-1)^i H^i(R\Gamma_c(Y, \overline{\mathbb{Q}}_\ell)),$$

for a Deligne–Lusztig variety Y (c.f. Definitions 2.1.3, 2.1.8). In this case, it happens that most terms of the sum are zero, except when $0 \leq i \leq 2\dim(Y)$, see [Car85, Property 7.1.1]. We make this property more formal and general by considering **perverse sheaves** introduced by Beilinson, Bernstein and Deligne in [BBD82]. Our very short (and thus incomplete) introduction follows [LuCS1, Sect. 1.1] and [MS89, Sect. 1]. For the general reference on perverse sheaves, we advise the reader to read [BBD82].

The bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves

On the character side, we were only looking at the constant étale sheaf $\overline{\mathbb{Q}}_\ell$ over a variety Y over k. To be able to change varieties or schemes, we often take the pushforward or pullback along morphisms of varieties. However, the pushforward of the constant sheaf is in general not the constant sheaf anymore. More generally, a **local system** is an étale sheaf which is locally constant, that is constant on an open neighborhood of any point (open for the étale cohomology). Again, the pushforward of a local system might not be locally constant anymore. Therefore, we look at **constructible sheaves**, i.e., sheaves on a scheme[1] Y such that there is a stratification into finitely many locally closed subsets of Y where the restriction is a finite locally constant sheaf, see [Del80, I.I]. We then consider the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves over a scheme Y over k, that we denote by $D_c^b(Y, \overline{\mathbb{Q}}_\ell)$.
If $f : Y \to Z$ is a morphism of algebraic varieties, then there exist functors between their respective bounded derived categories of constructible sheaves:

- the inverse image functor (or pullback) $f^* : D_c^b(Z, \overline{\mathbb{Q}}_\ell) \to D_c^b(Y, \overline{\mathbb{Q}}_\ell)$ with right adjoint (the pushforward) $f_* : D_c^b(Y, \overline{\mathbb{Q}}_\ell) \to D_c^b(Z, \overline{\mathbb{Q}}_\ell)$,

[1] In the rest of this chapter all the schemes will be algebraic varieties.

- the direct image functor with compact support (or proper pushforward) $f_! : D_c^b(Y,\overline{\mathbb{Q}}_\ell) \to D_c^b(Z,\overline{\mathbb{Q}}_\ell)$ with left adjoint (the proper pullback) $f^! : D_c^b(Z,\overline{\mathbb{Q}}_\ell) \to D_c^b(Y,\overline{\mathbb{Q}}_\ell)$.

Note that if f is proper, we have $f_! = f_*$. There is a self-equivalence duality functor

$$D : D_c^b(Y,\overline{\mathbb{Q}}_\ell) \to D_c^b(Y,\overline{\mathbb{Q}}_\ell)$$

such that $D^2 \cong \mathrm{Id}_{D_c^b(Y,\overline{\mathbb{Q}}_\ell)}$ called the **Verdier duality**. It comes from the proper pullback of the canonical morphism from Y to the point, see [Aub10, Def. 7.4] for the definition.

Perverse sheaves

We consider cohomologies and complexes where the cohomology vanishes in a "nice" way. Let $\mathcal{F} \in D_c^b(Y,\overline{\mathbb{Q}}_\ell)$ be a complex. We define the **support** of $H^i(\mathcal{F})$ for $i \in \mathbb{Z}$ as

$$\mathrm{supp}(H^i(\mathcal{F})) \coloneqq \overline{\{y \in Y \mid H^i(\mathcal{F})_y \neq 0\}}$$

and the **support** of $\mathcal{F}$ is then

$$\mathrm{supp}(\mathcal{F}) \coloneqq \overline{\{y \in Y \mid H^i(\mathcal{F})_y \neq 0 \text{ for some } i \in \mathbb{Z}\}}.$$

We then consider the category $D^{\leq 0}(Y)$ which is the full subcategory of $D_c^b(Y,\overline{\mathbb{Q}}_\ell)$ whose objects $\mathcal{F}$ are such that for all $i \in \mathbb{Z}$,

$$\dim \mathrm{supp}(H^i(\mathcal{F})) \leq -i.$$

Observe that in particular $\mathrm{supp}(H^i(\mathcal{F})) = \varnothing$ for any positive $i \in \mathbb{Z}_{\geq 0}$. We also define the dual of $D^{\leq 0}(Y)$, the category $D^{\geq 0}(Y)$ which is the full subcategory of $D_c^b(Y,\overline{\mathbb{Q}}_\ell)$ whose objects are of the form $D(\mathcal{F})$ for $\mathcal{F} \in D^{\leq 0}(Y)$.

The category of **perverse sheaves** on Y, denoted by $\mathscr{M}(Y)$, is then the full subcategory[2] whose objects belong to $D^{\leq 0}(Y)$ and $D^{\geq 0}(Y)$. This category is abelian and all objects have finite length ([BBD82, Thm. 1.3.6, Thm. 4.3.1]). Observe that this is also the case for the category of $\overline{\mathbb{Q}}_\ell[G]$-modules, whilst the category $D_c^b(Y,\overline{\mathbb{Q}}_\ell)$ is additive but not abelian. However, contrary to $\overline{\mathbb{Q}}_\ell[G]$-mod, the category $\mathscr{M}(Y)$ is not semisimple.

The irreducible perverse sheaves have been fully determined in [BBD82, Thm. 4.3.1 (ii)]. Let V be a locally closed, smooth, irreducible subvariety of Y and $\mathcal{L}$ an irreducible local system on V. We see $\mathcal{L}$ as a complex in $D_c^b(V,\overline{\mathbb{Q}}_\ell)$ by considering the chain complex whose cohomology groups are all trivial, except at $i = 0$ where it is equal to $\mathcal{L}$. Moreover, the shifted complex $\mathcal{L}[\dim(V)]$ is an irreducible perverse sheaf on V. There is a unique way to extend $\mathcal{L}[\dim(V)]$ to a perverse sheaf on $\overline{V}$. It is the shift by $\dim(V)$ of the **intersection cohomology complex** $IC(\overline{V},\mathcal{L})$, defined by Deligne, Goresky and MacPherson ([GM83]). The shifted intersection cohomology complex $\mathcal{F} = IC(\overline{V},\mathcal{L})[\dim V]$ is characterised by the following properties (as explained in [Sho88, 3.3]):

[2] We do not want to introduce t-structures in this thesis, but this construction can be made much more general. The category $\mathscr{M}(Y)$ is then the heart of $D_c^b(Y,\overline{\mathbb{Q}}_\ell)$.

1. $H^i(\mathcal{F}) = 0$ if $i < -\dim V$,
2. $H^{-\dim V}(\mathcal{F})_V = \mathcal{L}$,
3. $\dim \operatorname{supp}(H^i(\mathcal{F})) < -i$ if $i > -\dim V$, and
4. $\dim \operatorname{supp}(H^i(D(\mathcal{F}))) < -i$ if $i > -\dim V$.

Extending $\mathcal{F}$ by 0 to Y, we get an irreducible perverse sheaf on Y. Moreover, all simple objects in $\mathscr{M}(Y)$ arise in this way, [BBD82, 4.3.1].

From $D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ to $\mathscr{M}(Y)$

It turns out that if $f : Y \to Z$ is a morphism of algebraic varieties and $\mathcal{F} \in \mathscr{M}(Y)$, we do not necessarily have $f_*\mathcal{F} \in \mathscr{M}(Z)$. We thus would like to turn complexes of sheaves in $D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ into perverse sheaves. The inclusion of $D^{\leq 0}(Y)$ (resp. $D^{\geq 0}(Y)$) in $D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ has a right (resp. left) adjoint denoted by $\tau_{\leq 0}$ (resp. $\tau_{\geq 0}$). The functors $\tau_{\leq 0}\tau_{\geq 0}$ and $\tau_{\geq 0}\tau_{\leq 0}$ are canonically isomorphic [BBD82, Prop. 1.3.5]. Therefore, ${}^pH^0 \coloneqq \tau_{\leq 0}\tau_{\geq 0}$ is a functor from $D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ to $\mathscr{M}(Y)$. It is a cohomological[3] functor in the sense that if

$$\mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to \mathcal{F}[1]$$

is a distinguished triangle, then the sequence

$${}^pH^0(\mathcal{F}) \to {}^pH^0(\mathcal{F}') \to {}^pH^0(\mathcal{F}'')$$

is exact, by [BBD82, Thm. 1.3.6]. This sequence can be made into a long exact sequence. We set ${}^pH^i(\mathcal{F}) \coloneqq {}^pH^0(\mathcal{F}[i])$ for any $i \in \mathbb{Z}$. For the same distinguished triangle as above, we then have a long exact sequence

$$\cdots \to {}^pH^i(\mathcal{F}) \to {}^pH^i(\mathcal{F}') \to {}^pH^i(\mathcal{F}'') \to {}^pH^{i+1}(\mathcal{F}) \to \dots$$

Besides, ${}^pH^i(\mathcal{F}) = 0$ for all but finitely many integers $i \in \mathbb{Z}$.
We may now define semisimple objects in $D^b_c(Y, \overline{\mathbb{Q}}_\ell)$. A complex $\mathcal{F} \in D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ is **split** if $\mathcal{F}$ is isomorphic in $D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ to the direct sum $\bigoplus_{i \in \mathbb{Z}} {}^pH^i(\mathcal{F})[-i]$. If moreover all the ${}^pH^i(\mathcal{F})[-i]$ are semisimple, then $\mathcal{F}$ is said to be **semisimple**.
We now state the crucial decomposition theorem.

Theorem 3.1.1 (Decomposition Theorem, [BBD82, Thm. 6.2.5])**.** *Let $f : Y \to Z$ be a proper morphism. Let $\mathcal{F}$ be a simple perverse sheaf on Y such that there exists a finite étale covering $\pi : \tilde{Y} \to Y$ where $\pi^*\mathcal{F}$ is a constant sheaf (i.e. with* ***finite monodromy****). Then the pushforward $f_*\mathcal{F} \in D^b_c(Z, \overline{\mathbb{Q}}_\ell)$ is semisimple.*

[3] This is the notion corresponding to exact functors in triangulated categories

Equivariant perverse sheaves

Recall that in Subsection 2.1.1, we defined **G**-equivariant sheaves (Definition 2.1.1). We generalise this notion to perverse sheaves.

Definition 3.1.2. Let Y be an algebraic variety (over k) on which a connected algebraic group $\mathbf{H}$ (over k) acts. Let $a : \mathbf{H}\times Y \to Y$ denote the action of $\mathbf{H}$ and $p: \mathbf{H} \times Y \to Y$ the projection on the second coordinate. A perverse sheaf $\mathcal{F} \in \mathscr{M}(Y)$ is said to be **H-equivariant** if the perverse sheaves $a^*\mathcal{F}[\dim \mathbf{H}]$ and $p^*\mathcal{F}[\dim \mathbf{H}]$ are isomorphic in $\mathscr{M}(\mathbf{H}\times Y)$.
A split complex $\mathcal{F} \in D^b_c(Y,\overline{\mathbb{Q}}_\ell)$ is $\mathbf{H}$-equivariant if all ${}^pH^i(\mathcal{F})$ are $\mathbf{H}$-equivariant.

Note that $a^*\mathcal{F}[\dim \mathbf{H}]$ and $p^*\mathcal{F}[\dim \mathbf{H}]$ are indeed perverse sheaves due to the following fact: if $f : Y \to Z$ is smooth morphism of varieties with connected fibers of dimension d then $f^*[d]$ is a functor from $\mathscr{M}(Z)$ to $\mathscr{M}(Y)$ [LuCS1, 1.7.4]. Moreover, as in Definition 2.1.1, an $\mathbf{H}$-equivariant perverse sheaf also comes with a fixed isomorphism $\phi :\ a^*\mathcal{F}[\dim \mathbf{H}] \xrightarrow{\sim} p^*\mathcal{F}[\dim \mathbf{H}]$ satisfying a cocycle condition.

Remark 3.1.3. We consider the particular case of $\mathbf{H}$-equivariant local systems on Y where the action of $\mathbf{H}$ on Y is transitive, following [Sho88, 3.5]. Let $\mathcal{L}$ be a local system on Y. We see $\mathcal{L}$ in $D^b_c(Y,\overline{\mathbb{Q}}_\ell)$ and $\mathcal{L}[\dim(Y)] \in \mathscr{M}(Y)$. Assume $\mathcal{L}$ is $\mathbf{H}$-equivariant, that is $\mathcal{L}[\dim(Y)]$ is $\mathbf{H}$-equivariant. By definition, there is an isomorphism

$$\phi : a^*\mathcal{L}[\dim \mathbf{H} + \dim Y] \xrightarrow{\sim} p^*\mathcal{L}[\dim \mathbf{H} + \dim Y].$$

Fixing $y \in Y$, we get an isomorphism

$$\phi_{h,y} : \mathcal{L}_y \xrightarrow{\sim} \mathcal{L}_y,$$

for each $h \in \mathbf{H}$ such that $a(h,y) = y$. We write $\mathrm{Stab}_{\mathbf{H}}(y)$ for the set of such elements in $\mathbf{H}$ and $A_{\mathbf{H}}(y)$ for the component group $\mathrm{Stab}_{\mathbf{H}}(y)/\mathrm{Stab}^\circ_{\mathbf{H}}(y)$. The action of $\mathrm{Stab}^\circ_{\mathbf{H}}(y)$ on $\mathcal{L}_y$ is always trivial, and therefore we can see $\mathcal{L}_y$ as a $\overline{\mathbb{Q}}_\ell[A_{\mathbf{H}}(y)]$-module. If $\mathcal{L}$ is irreducible, then $\mathcal{L}_y$ is irreducible as a $\overline{\mathbb{Q}}_\ell[A_{\mathbf{H}}(y)]$-module.
On the other hand, let $\pi : \mathbf{H}/\mathrm{Stab}^\circ_{\mathbf{H}}(y) \to \mathbf{H}/\mathrm{Stab}_{\mathbf{H}}(y)$ be the quotient map. It is a finite étale covering with group $A_{\mathbf{H}}(y)$. Since the action of $\mathbf{H}$ on Y is transitive, there is an isomorphism $\mathbf{H}/\mathrm{Stab}_{\mathbf{H}}(y) \cong Y$ and we can consider the pushforward $\pi_*\overline{\mathbb{Q}}_\ell$. It is semisimple and decomposes as follows

$$\pi_*\overline{\mathbb{Q}}_\ell \cong \bigoplus_{V\in\mathrm{Irr}(\mathrm{End}(\pi_*\overline{\mathbb{Q}}_\ell))} \mathcal{L}_V \otimes V,$$

where $\mathcal{L}_V \coloneqq \mathrm{Hom}_{\mathrm{End}(\pi_*\overline{\mathbb{Q}}_\ell)}(V,\pi_*\overline{\mathbb{Q}}_\ell)$ is an irreducible local system on Y. By definition of the map π, the algebra $\mathrm{End}(\pi_*\overline{\mathbb{Q}}_\ell)$ is isomorphic to $\overline{\mathbb{Q}}_\ell[A_{\mathbf{H}}(y)]$. Moreover, $(\mathcal{L}_V)_y$ is isomorphic as a $\overline{\mathbb{Q}}_\ell[A_{\mathbf{H}}(y)]$-module to the dual module of V. This defines a bijection between the irreducible $\mathbf{H}$-equivariant local systems on Y and $\mathrm{Irr}(A_{\mathbf{H}}(y))$.

The notion of equivariance works well with respect to taking subquotients, pullback or proper pushforward. Let $\mathbf{H}$ be a connected algebraic group, Y, Z be two varieties on which $\mathbf{H}$ acts, and $f : Y \to Z$ be an $\mathbf{H}$-equivariant morphism of varieties. Let $\mathcal{F}$ be an $\mathbf{H}$-equivariant perverse sheaf on Y, and $\mathcal{F}'$ be an $\mathbf{H}$-equivariant perverse sheaf on Z. The following facts can be found in [LuCS1, §1.9].

Lemma 3.1.4. *Keeping the notation as given above, the following properties hold.*

1. *Any subquotient of $\mathcal{F}$ is $\mathbf{H}$-equivariant.*
2. *The perverse sheaves ${}^pH^i(f_!\mathcal{F})$ are $\mathbf{H}$-equivariant for all $i \in \mathbb{Z}$.*
3. *The perverse sheaves ${}^pH^i(f^*\mathcal{F})$ are $\mathbf{H}$-equivariant for all $i \in \mathbb{Z}$.*
4. *Assume that $\mathbf{H}$ acts trivially on Z and freely on Y. Suppose that for each $z \in Z$, there is an open neighborhood $V \subseteq Z$ with $z \in V$ and an $\mathbf{H}$-equivariant map $f_V : f^{-1}(V) \to \mathbf{H} \times V$ such that $p_2 \circ f_V = f$ where p_2 is the projection on the second coordinate. In other words f is a locally trivial principal fibration with group $\mathbf{H}$. Then $\mathcal{K} \in \mathscr{M}(Y)$ is $\mathbf{H}$-equivariant if and only if there is $\mathcal{K}' \in \mathscr{M}(Z)$ such that $\mathcal{K}$ is isomorphic to $f^*\mathcal{K}'[\dim \mathbf{H}]$.*

Note that if $Y = \mathbf{H} \times Z$ and f is the projection on the second coordinate, we then may write $\mathcal{K}' = i^*\mathcal{K}[-\dim \mathbf{H}]$ with $i : Z \to \mathbf{H} \times Z$, $z \mapsto (1, z)$ where $1 \in \mathbf{H}$ is the neutral element. Otherwise we use a glueing argument.

Characteristic functions

Lastly, we want to use $\mathbf{G}$-equivariant perverse sheaves to understand the representation theory of the finite group G. So we need to take into account the action of the Frobenius map.

Definition 3.1.5 ([LuCS2, §8.4])**.** Fix q a power of the prime p. Let Y be an algebraic variety defined over $\mathbb{F}_q$ via a Frobenius morphism F (c.f. above Definition 1.2.1) and $\mathcal{F} \in D^b_c(Y, \overline{\mathbb{Q}}_\ell)$. The complex $\mathcal{F}$ is said to be **F-stable** if there exists an isomorphism $\varphi :\ F^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. It induces isomorphisms for each $i \in \mathbb{Z}$ and each $y \in Y$,

$$\varphi_{i,y} : H^i(\mathcal{F})_{F(y)} \xrightarrow{\sim} H^i(\mathcal{F})_y.$$

The **characteristic function** of $\mathcal{F}$ (with respect to φ) is given by

$$\begin{aligned} \chi_{\mathcal{F},\varphi} : Y^F &\to \overline{\mathbb{Q}}_\ell \\ y &\mapsto \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}(\varphi_{i,y}, H^i(\mathcal{F})_y). \end{aligned}$$

Note that the sum is finite because we consider bounded complexes and the $H^i(\mathcal{F})_y$ are finite-dimensional vector spaces. Let $\mathbf{H}$ be a connected algebraic group defined over $\mathbb{F}_q$. If $\mathbf{H}$ acts on Y with an action defined over $\mathbb{F}_q$, then the characteristic function of an $\mathbf{H}$-equivariant perverse sheaf on Y is an $\mathbf{H}(\mathbb{F}_q)$-equivariant function on Y^F [Sho95a, 1.1]. Thus, we will consider **G**-equivariant perverse sheaves over **G** and we will have to restrict ourselves to the case where we assume that F is a Frobenius map for our fixed group **G**.

Remark 3.1.6. The characteristic functions play in some ways the same role as characters for representations. Keeping the notation of the above definition, we can also define the functions $\chi_{\mathcal{F},\varphi^n} : Y^{F^n} \to \overline{\mathbb{Q}}_\ell$ for $n \in \mathbb{N}_{\geq 1}$. If $\mathcal{F}$ is a semisimple perverse sheaf, the functions $(\chi_{\mathcal{F},\varphi^n})_{n\geq 1}$ determine $\mathcal{F}$ up to isomorphism, c.f. [MS89, 1.3.4].

3.1.2 Definition of character sheaves

We are now ready to define character sheaves. We will see that the construction is very similar to the one leading to the geometric series $\mathscr{E}_{\lambda,n}$ (Theorem 2.2.10). For $\lambda \in X$ and n a positive integer coprime to p, we want to define "Deligne–Lusztig character sheaves" in the way that $I_w^{\theta_w}$ was defined for $w \in \mathscr{Z}_{\lambda,n}$.

Kummer local systems

Firstly, we need to construct the equivalent of irreducible $\overline{\mathbb{Q}}_\ell$-modules of T_0 (= $\mathbf{T}_0^F$ as fixed in Hypothesis 1). If $V \in \mathrm{Irr}(T_0)$, then there is some integer m such that $V^{\otimes m}$ is the trivial module. In algebraic geometry, the sheaves on $\mathbf{T}_0$ with this property are called Kummer local systems.

Definition 3.1.7. We say that a $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}$ on $\mathbf{T}_0$ is **Kummer** (or **tame**) if there is $m \in \mathbb{N}$, coprime to p, such that $\mathcal{L}^{\otimes m} \cong \overline{\mathbb{Q}}_\ell$. We denote by $\mathcal{S}(\mathbf{T}_0)$ the set of isomorphism classes of Kummer $\overline{\mathbb{Q}}_\ell$-local systems on $\mathbf{T}_0$.

Kummer local systems on $\mathbf{T}_0$ are constructed as follows, see [LuCS1, 2.2]. Firstly, we fix for the rest of this thesis an injective group homomorphism

$$\mathfrak{j} : \{x \in k^\times \mid x^n = 1 \text{ for some } n \in \mathbb{N}\} \to \overline{\mathbb{Q}}_\ell^\times. \tag{3.1}$$

We may choose $\mathfrak{j}$ as the restriction of $\mathfrak{i}$ (2.1). A Kummer local system has the following form:

1. Let $n \in \mathbb{N}$ such that $(p,n) = 1$, and $\mu_n := \{x \in k^\times \mid x^n = 1\}$. Define $\rho_n : k \to k$, $x \mapsto x^n$. Then μ_n acts on the local system $(\rho_n)_*\overline{\mathbb{Q}}_\ell$.
2. Set $\mathcal{E}_{n,\mathfrak{j}}$ the summand of $(\rho_n)_*\overline{\mathbb{Q}}_\ell$ on which μ_n acts according to $\mathfrak{j}$.
3. Fix $\lambda \in X$ and consider the $\overline{\mathbb{Q}}_\ell$-local system on $\mathbf{T}_0$ of the form $\lambda^*\mathcal{E}_{n,\mathfrak{j}}$.

Note that for any $n' \in \mathbb{N}$ coprime to p the pairs (λ, n) and $(n'\lambda, n'n)$ give rise to isomorphic Kummer local systems. Therefore, we may assume that each local system comes from an indivisible pair (see before Theorem 2.2.10).

Let $w \in W$ and $\dot{w} \in N_{\mathbf{G}}(\mathbf{T}_0)$ a representative of w. Fix $\mathcal{L} = \lambda^*\mathcal{E}_{n,\mathfrak{j}}$ a Kummer local system for $(\lambda, n) \in X \times \mathbb{N}$ with n coprime to p. The action of w on $\mathbf{T}_0$ induces an action on $\mathcal{S}(\mathbf{T}_0)$ sending the isomorphism class of $\mathcal{L}$ to the isomorphism class of $\mathrm{ad}(\dot{w})^*\mathcal{L}$. Observe that $\mathbf{T}_0$ acts trivially on $\mathcal{L}$, since $\lambda \circ \mathrm{ad}(t) = \lambda$ for all $t \in \mathbf{T}_0$. We define

$$W_{\mathcal{L}} \coloneqq \{w \in W \mid \mathrm{ad}(w^{-1})^*\mathcal{L} \cong \mathcal{L}\}.$$

Observe that $W_{\mathcal{L}}$ is not always a Coxeter group. We set

$$\Phi_{\mathcal{L}} \coloneqq \{\alpha \in \Phi \mid s_\alpha \in W_{\mathcal{L}}\},$$

and $W_{\mathcal{L}}^\circ$ the Weyl group generated by $\{s_\alpha \mid \alpha \in \Phi_{\mathcal{L}}\}$. By [LuCS1, § 2.2.2], for each $w \in W_{\mathcal{L}}$, there exists a character $\lambda_w \in X$ such that $\mathrm{ad}(w^{-1})^*\mathcal{L} = \mathrm{ad}(w^{-1})^*\lambda^*\mathcal{E}_{n,\mathrm{j}} = (\lambda_w^n\lambda)^*\mathcal{E}_{n,\mathrm{j}}$. In other words, an element $w \in W$ belongs to $W_{\mathcal{L}}$ if and only if $\lambda - w.\lambda \in nX$. Thus,

$$W_{\mathcal{L}} = W_{\lambda,n}, \quad W_{\mathcal{L}}^\circ = W_{\lambda,n}^\circ \quad \text{and} \quad \Phi_{\mathcal{L}} = \Phi_{\lambda,n},$$

by comparing with the definitions for the above groups below Definition 2.2.7.

Remark 3.1.8. By [LuCS1, 2.2.2], an element $w \in W_{\mathcal{L}}$ if and only if $\mathcal{L}$ is equivariant for the action of $\mathbf{T}_0$ given by $\mathrm{ac}_w : \mathbf{T}_0 \times \mathbf{T}_0 \to \mathbf{T}_0, (t,t') \mapsto \dot{w}^{-1}t\dot{w}t't^{-1}$.

Character sheaves

We now give the definition of character sheaves. Recall that the Harish-Chandra induction from the torus goes as follows: we first inflate a character of T_0 to the Borel B_0 and then induce it to the whole group G. In the setting of perverse sheaves and character sheaves, the "inflation" of a Kummer local system $\mathcal{L}$ on $\mathbf{T}_0$ is simply the pullback $pr^*\mathcal{L}$ under the projection map $\mathbf{B}_0 = \mathbf{T}_0 \ltimes \mathbf{U}_0 \to \mathbf{T}_0$. Since we consider perverse sheaves, we in fact look at the intersection cohomology complex $IC(\mathbf{B}_0, pr^*\mathcal{L})$. It is a $\mathbf{B}_0$-equivariant perverse sheaf on $\mathbf{B}_0$. The analogue for induction should give us a $\mathbf{G}$-equivariant complex in $D_c^b(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$ from a $\mathbf{B}_0$-equivariant complex in $D_c^b(\mathbf{B}_0, \overline{\mathbb{Q}}_\ell)$. We first look at $IC(\mathbf{B}_0, pr^*\mathcal{L})$ extended by 0 to $\mathbf{G} - \mathbf{B}_0$ as a $\mathbf{B}_0$-equivariant complex of $\mathbf{G}$. For equivariant complexes, there is a usual induction in $D_c^b(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$: the inverse of the induction equivalence functor of [BL94, Def. 2.6.3] (from $D_c^b(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$ to $D_c^b(\mathbf{G} \times_{\mathbf{B}_0} \mathbf{G}, \overline{\mathbb{Q}}_\ell)$) followed by the pushforward to $\mathbf{G}$ via the action map (in our case, the conjugation).
To define Deligne–Lusztig characters, we somehow twist the Harish-Chandra induction by some $w \in W$. For character sheaves, it might even be easier to see how the twisting works as we only change $\mathbf{B}_0 = \mathbf{B}_0 1 \mathbf{B}_0$ to $\mathbf{B}_0 \dot{w} \mathbf{B}_0$. We make explicit this above discussion by following the construction given in [MS89, Def. 5.1.2].

Let $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$ and $w \in W_{\mathcal{L}}$. Fix $\dot{w}$ a representative of w in $N_{\mathbf{G}}(\mathbf{T}_0)$ and set

$$\mathbf{G}_w \coloneqq \mathbf{B}_0 \dot{w} \mathbf{B}_0.$$

Any element $g \in \mathbf{G}_w$ decomposes as $g = u\dot{w}tu'$ for some $u, u' \in \mathbf{U}_0$ and $t \in \mathbf{T}_0$. We consider the projection map $\mathrm{pr}_w : \mathbf{G}_w \to \mathbf{T}_0$ sending such $g = u\dot{w}tu'$ to $t \in \mathbf{T}_0$. We then set

$$\mathcal{A}_w^{\mathcal{L}} \coloneqq IC(\overline{\mathbf{G}_w}, \mathrm{pr}_w^*(\mathcal{L}))[\dim \mathbf{G}_w].$$

By [MS89, Lem. 4.1.2], this perverse sheaf is $\mathbf{B}_0$-equivariant for the action by conjugation. We apply the induction from $\mathbf{B}_0$-equivariant complexes to $\mathbf{G}$-equivariant complexes. We have the following diagram:

$$\mathbf{G} \longleftarrow \mathbf{G} \times \mathbf{G} \xrightarrow{\beta} \mathbf{G} \times_{\mathbf{B}_0} \mathbf{G} \xrightarrow{\gamma} \mathbf{G}$$

where

- the variety $\mathbf{G} \times_{\mathbf{B}_0} \mathbf{G}$ is the quotient of $\mathbf{G} \times \mathbf{G}$ by the action $b.(g, g') = (gb^{-1}, bg'b^{-1})$ for $g, g' \in \mathbf{G}$ and $b \in \mathbf{B}_0$,
- the map β is the quotient map,
- and the map γ is the conjugation map $\gamma : (g, g') \mapsto gg'g^{-1}$ for $g, g' \in \mathbf{G}$.

Since $\mathcal{A}_w^{\mathcal{L}}$ is $\mathbf{B}_0$-equivariant, by fact 4 of Lemma 3.1.4, there exists a canonical irreducible perverse sheaf $\tilde{\mathcal{A}}_w^{\mathcal{L}}$ on $\mathbf{G} \times_{\mathbf{B}_0} \mathbf{G}$ such that $\beta^* \tilde{\mathcal{A}}_w^{\mathcal{L}} \cong \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_w^{\mathcal{L}}[\dim \mathbf{G} - \dim \mathbf{B}_0]$. Here $\boxtimes$ denotes the external tensor product of perverse sheaves. This process so far is the description of the inverse of the induction equivalence functor as defined in [BL94, Def. 2.6.3]. We now put

$$\bar{\mathcal{K}}_w^{\mathcal{L}} \coloneqq (\gamma)_*(\tilde{\mathcal{A}}_w^{\mathcal{L}})[-\dim \mathbf{G} - \ell(w)].$$

This is a semisimple complex by Lusztig [LuCS3, Prop. 12.8]. Indeed, γ is proper and we can apply the Decomposition Theorem (Theorem 3.1.1). Here $\ell(w)$ is the value of the length function of W at $w \in W$. These $\bar{\mathcal{K}}_w^{\mathcal{L}}$ then play the role of the Deligne–Lusztig characters.

Definition 3.1.9. A **character sheaf** is an irreducible perverse sheaf which is an irreducible constituent of ${}^pH^i(\bar{\mathcal{K}}_w^{\mathcal{L}})$ for some $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$, $w \in W_{\mathcal{L}}$ and $i \in \mathbb{Z}$. We denote by $\hat{\mathbf{G}}_{\mathcal{L}}$ the set of isomorphism classes of character sheaves coming from the local system $\mathcal{L}$. We say that a character sheaf $\mathcal{A}$ is **unipotent** if its isomorphism class belongs to $\hat{\mathbf{G}}_{\overline{\mathbb{Q}}_\ell}$.

We will from now on often abuse notation and write $\mathcal{A}$ for the isomorphism class of a character sheaf $\mathcal{A}$ and write $\mathcal{A} \in \hat{\mathbf{G}}$. Note that it follows from the definition, and the fact that γ is proper, that character sheaves are $\mathbf{G}$-equivariant.

Remark 3.1.10. We explain why $\mathrm{pr}_w^*(\mathcal{L})$ is $\mathbf{B}_0$-equivariant. Let $w \in W_{\mathcal{L}}$ and $b = u_1 t_1 \in \mathbf{B}_0$. By Remark 3.1.8, $\mathcal{L}$ is equivariant under the action of $\mathbf{T}_0$ given by ac_w. In particular, we have $\mathrm{ac}_w(t_1, -)^* \mathcal{L} \cong \mathcal{L}$. Lastly, we observe that

$$\mathrm{pr}_w \circ \mathrm{ad}(b) = \mathrm{pr}_w \circ \mathrm{ac}_w(t_1, -)$$

which allows us to conclude that $\mathrm{ad}(b)^* \mathrm{pr}_w^* \mathcal{L} \cong \mathcal{L}$.

We make a few remarks concerning the definition of character sheaves. Firstly, since $\mathcal{A}_w^{\mathcal{L}}$ is zero outside $\overline{\mathbf{G}_w}$, we could replace the above diagram by its restriction to $\mathbf{G}_w$:

$$\overline{\mathbf{G}_w} \longleftarrow \mathbf{G} \times \overline{\mathbf{G}_w} \xrightarrow{\beta} \mathbf{G} \times_{\mathbf{B}_0} \overline{\mathbf{G}_w} \xrightarrow{\gamma} \mathbf{G}$$

Moreover, we could in fact first do everything at the level of local systems. That was the initial definition of Lusztig [LuCS1, §2.4 and Def. 2.10], that we recall here.
We let $\mathscr{B}$ be the variety of all Borel subgroups of $\mathbf{G}$. For each $w \in W$, we define

$$O(w) \coloneqq \{(\mathbf{B}_1, \mathbf{B}_2) \in \mathscr{B} \times \mathscr{B} \mid \exists g \in \mathbf{G} \text{ such that } {}^g\mathbf{B}_1 = \mathbf{B}_0, {}^g\mathbf{B}_2 = {}^{\dot{w}}\mathbf{B}_0\}.$$

For $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$ and $w \in W_{\mathcal{L}}$, we have the following diagram:

$$\mathbf{T}_0 \xleftarrow{pr} \dot{Y}_w \xrightarrow{i} Y_w \xrightarrow{\pi_w} \mathbf{G}$$

with

- the set $\dot{Y}_w \coloneqq \{(g, h\mathbf{U}) \in \mathbf{G} \times \mathbf{G}/\mathbf{U} \mid h^{-1}gh \in \mathbf{B}_0 w \mathbf{B}_0\}$,
- the map pr sending $(g, h\mathbf{U})$ to $pr_{\dot{w}}(h^{-1}gh) = pr_{\dot{w}}(u\dot{w}tu') = t$ for $u, u' \in \mathbf{U}$ and $t \in \mathbf{T}_0$ such that $h^{-1}gh = u\dot{w}tu'$ is the Bruhat decomposition,
- the set $Y_w \coloneqq \{(g, \mathbf{B}') \in \mathbf{G} \times \mathcal{B} \mid (\mathbf{B}', {}^g\mathbf{B}') \in O(w)\} = \mathbf{G} \times_{\mathbf{B}_0} \mathbf{G}_w$,
- the map $i : (g, h\mathbf{U}) \mapsto (g, {}^h\mathbf{B})$ for $g, h \in \mathbf{G}$,
- and the projection map $\pi_w : (g, \mathbf{B}') \mapsto g$ for $g \in \mathbf{G}, \mathbf{B}' \in \mathcal{B}$.

The inverse image $pr^*\mathcal{L}$ is $\mathbf{T}_0$-invariant. Thus, by fact 4 of Lemma 3.1.4, there exists a canonical $\overline{\mathbb{Q}}_\ell^*$-local system $\tilde{\mathcal{L}}$ on Y_w such that $pr^*\mathcal{L} \cong i^*(\tilde{\mathcal{L}})$. We put

$$\mathcal{K}_w^{\mathcal{L}} \coloneqq (\pi_w)_*(\tilde{\mathcal{L}}).$$

By [LuCS3, Prop. 12.7] a perverse sheaf $\mathcal{A}$ is an irreducible constituent of ${}^pH^i(\mathcal{K}_w^{\mathcal{L}})$ for some $i \in \mathbb{Z}$ and $w \in W_{\mathcal{L}}$ if and only if it is a constituent of ${}^pH^j(\bar{\mathcal{K}}_{w'}^{\mathcal{L}})$ for some $j \in \mathbb{Z}$ and $w' \in W_{\mathcal{L}}$, that is, if $\mathcal{A}$ is a character sheaf.
This method has the advantage that it is often easier to keep track of the local systems, but the downside is that the complex $\mathcal{K}_w^{\mathcal{L}}$ is in general not semisimple.

3.1.3 Series of character sheaves

The next step to mimic the case of ordinary characters is to consider the parameterisation of character sheaves. First, we need to check if the sets $\hat{\mathbf{G}}_{\mathcal{L}}$ define a similar partition as the geometric series in Theorem 2.2.10.

Proposition 3.1.11 ([LuCS3, Prop. 11.2]). *Let $\mathcal{L}$ and $\mathcal{L}'$ be two Kummer local systems of $\mathbf{T}_0$ coming from the indivisible pairs (λ, n) and (λ', n'). Then $\hat{\mathbf{G}}_{\mathcal{L}} \cap \hat{\mathbf{G}}_{\mathcal{L}'} \neq \varnothing$ if and only if $\mathcal{L}$ and $\mathcal{L}'$ are in the same W-orbit. Moreover, in that case $\hat{\mathbf{G}}_{\mathcal{L}} = \hat{\mathbf{G}}_{\mathcal{L}'}$.*

Note that $\mathcal{L}$ and $\mathcal{L}'$ are in the same W-orbit if and only $n = n'$ and $w.\lambda - \lambda' \in nX$ for some $w \in W$. From the above proposition, we can write

$$\hat{\mathbf{G}} = \bigsqcup_{\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)/W} \hat{\mathbf{G}}_{\mathcal{L}}.$$

Now, similarly as for the geometric series of characters, we could instead label the series of characters sheaves via the semisimple conjugacy classes in $\mathbf{G}^*$. Recall the map

$$\begin{aligned} X \times \mathbb{N} &\to \mathbf{T}_0^* \\ (\lambda, n) &\mapsto \tilde{i}_{\mathbf{T}_0^*}\left(\frac{1}{n} \otimes \lambda\right) =: t_{\lambda,n}. \end{aligned}$$

This map induces a surjection from the set of indivisible pairs (λ, n) to the set of conjugacy classes of semisimple elements in $\mathbf{G}^*$. Moreover, two indivisible pairs (λ, n) and (λ', n') are sent to the same conjugacy class if and only if $n = n'$ and $\lambda' - w.\lambda \in nX$ for some $w \in W$. Fora semisimple element $s \in \mathbf{T}_0^*$, there is an indivisible pair $(\lambda, n) \in X \times \mathbb{N}$ such that $(s)_{\mathbf{G}^*} = (t_{\lambda,n})_{\mathbf{G}^*}$. It is then well defined to set $\hat{\mathbf{G}}_s := \hat{\mathbf{G}}_{\lambda^* \mathcal{E}_{n,\mathfrak{j}}}$, and we get

$$\hat{\mathbf{G}} = \bigsqcup_{s} \hat{\mathbf{G}}_s,$$

where s runs over a set of representatives of the semisimple conjugacy classes in $\mathbf{G}^*$. The only difference with the geometric series case is that we do not require $\mathscr{Z}_{\lambda,n} \neq \varnothing$ nor the semisimple conjugacy classes to be F^*-stable.

To perfect the resemblance with characters, we define "almost character sheaves" and families of character sheaves. In order to achieve the first goal, we need to pass to the Grothendieck group $\mathbf{K}_0(\mathscr{M}(\mathbf{G}))$ of the perverse sheaves and more precisely to the subgroup $\hat{\mathbf{K}}_0(\mathscr{M}(\mathbf{G}))$ spanned by the isomorphic classes of character sheaves. We set $\langle -, - \rangle$ as the bilinear form on $\hat{\mathbf{K}}_0(\mathscr{M}(\mathbf{G})) \otimes \overline{\mathbb{Q}}_\ell$ defined by

$$\langle \mathcal{A}_1, \mathcal{A}_2 \rangle := \delta_{\mathcal{A}_1, \mathcal{A}_2},$$

for $\mathcal{A}_1, \mathcal{A}_2 \in \hat{\mathbf{G}}$ considered as elements of $\hat{\mathbf{K}}_0(\mathscr{M}(\mathbf{G}))$. Let $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$ and $\psi \in \operatorname{irr}(W_{\mathcal{L}})$. Following [LuCS3, 14.10], we set

$$\mathcal{R}_\psi := \frac{1}{|W_{\mathcal{L}}|} \sum_{w \in W_{\mathcal{L}}} \psi(w^{-1}) \sum_{i \in \mathbb{Z}} (-1)^{i + \dim \mathbf{G}} \, {}^p H^i(\mathcal{K}_w^{\mathcal{L}}).$$

We state the equivalent of Theorem 2.2.29.

Theorem 3.1.12 (Lusztig, [LuCS4, 17.8.3], [Lus12]). *Assume that $Z(\mathbf{G})$ is connected. To each family $\mathscr{F}$ of $\operatorname{irr}(W_{\mathcal{L}})$ (recall that $W_{\mathcal{L}} = W_{\mathcal{L}}^\circ$), one can associate a finite group $\bar{A}_{\mathscr{F}}$ such that there exist an injection*

$$\mathscr{F} \hookrightarrow \mathcal{M}(\bar{A}_{\mathscr{F}}), \quad \psi \mapsto x_\psi$$

and a bijection

$$\hat{\mathbf{G}}_{\mathcal{L}} \xrightarrow{\sim} \bigsqcup_{\mathscr{F}} \mathcal{M}(\bar{A}_{\mathscr{F}}), \quad \mathcal{A} \mapsto x_{\mathcal{A}},$$

where $\mathscr{F}$ *runs over the families of* $\operatorname{irr}(W_{\mathcal{L}})$, *such that for all* $\mathcal{A} \in \hat{\mathbf{G}}_{\mathcal{L}}$ *and* $\psi \in \mathscr{F}$,

$$\langle R_\psi, \mathcal{A} \rangle = \epsilon_{\mathcal{A}} \{x_{\mathcal{A}}, x_\psi\},$$

where $\epsilon_{\mathcal{A}} := (-1)^{codim(\operatorname{supp}\mathcal{A})} \in \{1,-1\}$ *depends only on* $\mathcal{A}$.

The result was first stated for p a good prime for $\mathbf{G}$ and then extended to any prime in [Lus12]. Note that amazingly, the groups $\bar{A}_{\mathscr{F}}$ are the same as the ones fixed by Lusztig in the setting of Theorem 2.2.29 for characters (so the notation is consistent). If $Z(\mathbf{G})$ is not connected, then the group $W_{\mathcal{L}}$ might not be a Weyl group. However, Lusztig still defined families of $W_{\mathcal{L}}$ and associated to them groups such that the same conclusion holds (c.f. [LuCS4, §17.8]).

The above result allows us to split the character sheaves of $\hat{\mathbf{G}}_{\mathcal{L}}$ into families. We say that $\mathcal{A}_1, \mathcal{A}_2 \in \hat{\mathbf{G}}_{\mathcal{L}}$ are in the same **family** $\mathscr{G}$ of $\hat{\mathbf{G}}_{\mathcal{L}}$ if there exist a family $\mathscr{F} \subseteq \operatorname{irr}(W_{\mathcal{L}})$ and $\psi_1, \psi_2 \in \mathscr{F}$ with

$$\langle R_{\psi_1}, \mathcal{A}_1 \rangle \neq 0 \quad \text{and} \quad \langle R_{\psi_2}, \mathcal{A}_2 \rangle \neq 0.$$

3.2 Parabolic induction of character sheaves

We continue on our path to describe various properties of character sheaves, based on what we use for ordinary characters. Like for representations of the finite group G, we would like to have some induction process for character sheaves. This is what Lusztig defined as parabolic induction and it resembles Harish-Chandra induction in many ways. In particular, we will see the independence from the parabolic, define cuspidal character sheaves and label the induction series associated to them thanks to some relative Weyl group.

3.2.1 Definition and first properties of parabolic induction

Following [MS89, §7.1] and [LuCS3, §3 and §4], we define parabolic induction. Similarly to Harish-Chandra induction, the functor is defined in two steps: inflation from a Levi subgroup to a parabolic subgroup of $\mathbf{G}$ followed by induction to the whole group $\mathbf{G}$. However, for this definition to work, for instance to have an adjoint functor, we need to work in the derived category of complexes $D^b_c(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$.

Parabolic induction and restriction

Definition 3.2.1. Let $\mathbf{P} := \mathbf{U} \rtimes \mathbf{L}$ be a Levi decomposition of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Consider the following diagram

$$\mathbf{L} \xleftarrow{\operatorname{pr}_{\mathbf{L}\subseteq\mathbf{P}}} \mathbf{P} \xleftarrow{\alpha} \mathbf{G} \times \mathbf{P} \xrightarrow{\beta} \mathbf{G} \times_{\mathbf{P}} \mathbf{P} \xrightarrow{\gamma} \mathbf{G}$$

with

- the projection map $\mathrm{pr}_{\mathbf{L}\subseteq\mathbf{P}} : \mathbf{P} \to \mathbf{L}$ sending $g = ul$ to $l \in \mathbf{L}$,
- the map $\alpha : (g,p) \mapsto p$ for $g \in \mathbf{G}$, $p \in \mathbf{P}$,
- the variety $\mathbf{G} \times_{\mathbf{P}} \mathbf{P}$ for the quotient of $\mathbf{G} \times \mathbf{P}$ by the $\mathbf{P}$-action $p.(g,q) = (gp^{-1}, pqp^{-1})$ for $p, q \in \mathbf{P}, g \in \mathbf{G}$,
- the quotient map β,
- and the conjugation map $\gamma : (g,p) \mapsto gpg^{-1}$ for $g \in \mathbf{G}, p \in \mathbf{P}$.

If $\mathcal{K}$ is an $\mathbf{L}$-equivariant perverse sheaf on $\mathbf{L}$, then $\mathrm{pr}^*_{\mathbf{L}\subseteq\mathbf{P}} \mathcal{K}[\dim \mathbf{U}]$ is a $\mathbf{P}$-invariant perverse sheaf on $\mathbf{P}$ (this is the inflation). Moreover,

$$\alpha^* \mathrm{pr}^*_{\mathbf{L}\subseteq\mathbf{G}} \mathcal{K}[\dim \mathbf{G} + \dim \mathbf{U}] \cong \overline{\mathbb{Q}}_\ell \boxtimes \mathrm{pr}^*_{\mathbf{L}\subseteq\mathbf{P}} \mathcal{K}[\dim \mathbf{U} + \dim \mathbf{G}]$$

is a $\mathbf{P}$-equivariant sheaf on $\mathbf{G} \times \mathbf{P}$. Thus by Lemma 3.1.4, there exists a canonical perverse sheaf $\tilde{\mathcal{K}}$ on $\mathbf{G} \times_{\mathbf{P}} \mathbf{P}$ such that $\alpha^* \mathrm{pr}^*_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K})[2\dim \mathbf{U}] \cong \beta^*(\tilde{\mathcal{K}})$. Moreover, $\tilde{\mathcal{K}}$ is $\mathbf{G}$-equivariant. We define the **parabolic induction** of $\mathcal{K}$ as

$$\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K}) \coloneqq \gamma_*(\tilde{\mathcal{K}}).$$

Thus $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ is a functor from the $\mathbf{L}$-equivariant perverse sheaves on $\mathbf{L}$ to $D^b_c(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$. Observe that the perverse sheaves ${}^pH^i(\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K}))$ are $\mathbf{G}$-equivariant for conjugation, since γ is proper. In fact if $\mathcal{K}$ is irreducible so is $\tilde{\mathcal{K}}$. From the decomposition theorem (Theorem 3.1.1), we conclude that $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K})$ is semisimple.

There is a functor from the $\mathbf{G}$-equivariant perverse sheaves on $\mathbf{G}$ to $D^b_c(\mathbf{L}, \overline{\mathbb{Q}}_\ell)$ called **parabolic restriction** ([LuCS1, 3.8]) and denoted $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$. For $\mathcal{F} \in D^b_c(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, we let $i_{\mathbf{P}\subseteq\mathbf{G}} : \mathbf{P} \to \mathbf{G}$ be the inclusion. Then

$$\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{F}) \coloneqq (\mathrm{pr}_{\mathbf{L}\subseteq\mathbf{P}})_! i^*_{\mathbf{P}\subseteq\mathbf{G}}(\mathcal{F})[\dim \mathbf{U}] \in D^b_c(\mathbf{L}, \overline{\mathbb{Q}}_\ell).$$

Properties of the parabolic induction of character sheaves

We now describe the properties of the parabolic induction and restriction functors when we apply them to character sheaves. Let $\mathbf{P} \coloneqq \mathbf{U} \rtimes \mathbf{L}$ be a Levi decomposition of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $\mathbf{T}_0 \subseteq \mathbf{L}$ and $\mathbf{B}_0 \subseteq \mathbf{P}$.
If $\mathcal{A} \in \hat{\mathbf{G}}$, then $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{A}) \in D^{\leq 0}(\mathbf{L})$ and is semisimple ([LuCS1, Thm. 4.4c, Thm. 3.9]). In fact, we may write $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{A})$ as a direct sum of (shifted) character sheaves.
Furthermore, if $\mathcal{A} \in \hat{\mathbf{L}}$, then $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{A}) \in \mathscr{M}(\mathbf{G})$ ([LuCS1, Thm. 4.4b]) and is semisimple since $\mathcal{A}$ is irreducible.

We now list a few properties of parabolic induction which are shared with Harish-Chandra induction. Firstly, the parabolic restriction behaves like a right adjoint functor.

Proposition 3.2.2 ([MS89, Prop. 7.1.3])**.** *Let* $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ *be a Levi decomposition of a parabolic subgroup* $\mathbf{P}$ *of* $\mathbf{G}$*. Let* $\mathcal{K}$ *be a* $\mathbf{G}$*-equivariant perverse sheaf on* $\mathbf{G}$ *and* $\mathcal{K}'$ *be an* $\mathbf{L}$*-equivariant perverse sheaf on* $\mathbf{L}$*. Assume that* $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K}) \in D^{\leq 0}(\mathbf{L})$*. Then*

$$\mathrm{Hom}_{D^b_c(\mathbf{G},\overline{\mathbb{Q}}_\ell)}(\mathcal{K}, \mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K}')) = \mathrm{Hom}_{D^b_c(\mathbf{L},\overline{\mathbb{Q}}_\ell)}(\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K}), \mathcal{K}').$$

Furthermore, the parabolic induction functor is transitive for character sheaves. More generally, it is transitive for any perverse sheaf under certain conditions.

Proposition 3.2.3 ([LuCS1, Prop. 4.2])**.** *Let* $\mathbf{Q} \subseteq \mathbf{P}$ *be two parabolic subgroups of* $\mathbf{G}$ *with respective Levi subgroups* $\mathbf{M} \subseteq \mathbf{L}$*. Let* $\mathcal{K} \in \mathscr{M}(\mathbf{M})$ *and assume that* $\mathrm{Ind}^{\mathbf{L}}_{\mathbf{M}\subseteq\mathbf{L}\cap\mathbf{Q}}(\mathcal{K})$ *lies in* $\mathscr{M}(\mathbf{L})$*. Then*

$$\mathrm{Ind}^{\mathbf{G}}_{\mathbf{M}\subseteq\mathbf{Q}}(\mathcal{K}) = \mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} \circ \mathrm{Ind}^{\mathbf{L}}_{\mathbf{M}\subseteq\mathbf{L}\cap\mathbf{Q}}(\mathcal{K}).$$

Moreover, there is a Mackey formula for character sheaves (see [MS89, Prop. 10.1.2] or [LuCS3, Prop. 15.2] for a different proof).

Lastly, parabolic induction preserves series of character sheaves.

Proposition 3.2.4 ([LuCS1, Prop. 4.8])**.** *Let* $\mathbf{P} \coloneqq \mathbf{U} \rtimes \mathbf{L}$ *be a Levi decomposition of a parabolic subgroup* $\mathbf{P}$ *of* $\mathbf{G}$ *such that* $\mathbf{T}_0 \subseteq \mathbf{L}$ *and* $\mathbf{B}_0 \subseteq \mathbf{P}$*. Let* $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$ *and* $\mathcal{A} \in \hat{\mathbf{L}}_{\mathcal{L}}$*. Then the irreducible components of* $\mathrm{Ind}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{A})$ *belong to* $\hat{\mathbf{G}}_{\mathcal{L}}$*.*

3.2.2 Cuspidal character sheaves and induction series

We now would like to define induction series of character sheaves in analogy with Harish-Chandra series and describe another partition of $\hat{\mathbf{G}}$.

Cuspidal character sheaves

Firstly, we define the cuspidal character sheaves. The initial definition of Lusztig ([LuCS1, Def. 3.10]) concerns perverse sheaves and goes as follows:

Definition 3.2.5. Let $\mathcal{K} \in \mathscr{M}(\mathbf{G})$ be $\mathbf{G}$-equivariant. We say that $\mathcal{K}$ is **cuspidal** if and only if it satisfies the two following conditions.

- There exists an integer $n \in \mathbb{Z}_{\geq 1}$, invertible in k, such that $\mathcal{K}$ is $\mathbf{G}\times Z^\circ(\mathbf{G})$-equivariant for the action of $\mathbf{G}\times Z^\circ(\mathbf{G})$ given by $(h,z).g \mapsto z^n hgh^{-1}$ for $z \in Z^\circ(\mathbf{G})$ and $h, g \in \mathbf{G}$.
- For any proper parabolic subgroup $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L} \neq \mathbf{G}$ such that $\mathbf{T}_0 \subseteq \mathbf{L}$ and $\mathbf{B}_0 \subseteq \mathbf{P}$ we have
$$\dim \mathrm{supp}\big(H^i\big(\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K})\big)\big) < -i.$$

We notice that any character sheaf satisfies the first condition by [LuCS1, Prop. 2.18b]. Moreover, thanks to [LuCS1, Thm. 6.9], the restriction of a character sheaf $\mathcal{A}$ is in fact a perverse sheaf, in particular $H^i\big(\mathrm{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{A})\big) = 0$ for all $i \neq 0$. We thus may take the following more intuitive definition for cuspidal character sheaves.

Definition 3.2.6. Let $\mathcal{A} \in \hat{\mathbf{G}}$. We say that $\mathcal{A}$ is **cuspidal** if and only if for any proper parabolic subgroup $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L} \neq \mathbf{G}$ such that $\mathbf{T}_0 \subseteq \mathbf{L}$ and $\mathbf{B}_0 \subseteq \mathbf{P}$ we have $\mathrm{Res}^{\mathbf{G}}_{\mathbf{L} \subseteq \mathbf{P}}(\mathcal{A}) = 0$.

In a tour de force, Lusztig showed through case-by-case analysis that every irreducible cuspidal perverse sheaf is a character sheaf [LuCS5, Thm. 23.1.b]. Therefore, we can forget the first definition of a cuspidal perverse sheaf (if it is irreducible). Moreover, we could set the definition of character sheaves as the irreducible constituents of the parabolic induction of irreducible cuspidal perverse sheaves. However, with that definition we would lose the partition into series $\hat{\mathbf{G}}_{\mathcal{L}}$ for $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$.

Character sheaves are irreducible perverse sheaves and as such can be written in terms of intersection cohomology complexes of irreducible local systems on some irreducible varieties. Thanks to [LuCS1, Prop. 3.12], we describe the structure of the intersection cohomology complex defining any cuspidal character sheaf.

Theorem 3.2.7. *Any cuspidal character sheaf on a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ with $\mathbf{T}_0 \subseteq \mathbf{L}$ is an intersection cohomology complex $IC(\bar{\Sigma}, \mathcal{E})[\dim \Sigma]$ where Σ is the inverse image under the map $\mathbf{L} \to \mathbf{L}/Z^\circ(\mathbf{L})$ of an isolated conjugacy class of $\mathbf{L}/Z^\circ(\mathbf{L})$ (see Definition 1.3.5) and $\mathcal{E}$ is a local system on Σ. Moreover, Σ and $\mathcal{E}$ are unique up to isomorphism.*

We say that any such triple $(\mathbf{L}, \Sigma, \mathcal{E})$ (with in particular $\mathbf{T}_0 \subseteq \mathbf{L}$) which gives rise to a cuspidal character sheaf is a **cuspidal induction datum** and we write $\mathfrak{M}^{\mathbf{G}}$ for the set of all cuspidal induction data of $\mathbf{G}$. If $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E})$ is a cuspidal induction datum, we write

$$\mathcal{A}_{\mathfrak{m}} \coloneqq IC(\overline{\Sigma}, \mathcal{E})[\dim(\Sigma)],$$

for the cuspidal character sheaf on $\mathbf{L}$.

Remark 3.2.8. Thanks to the proof of [LuCS1, Prop. 3.12], the pair $(\Sigma, \mathcal{E})$ is cuspidal for $\mathbf{L}$ in the sense of [Lus84b, Def. 2.4]. In particular, suppose that Σ contains unipotent elements. Let $C \in \mathrm{Ucl}(\mathbf{L})$ be the unipotent class of $\mathbf{L}$ such that $\Sigma = CZ^\circ(\mathbf{L})$. We canonically identify Σ with $C \times Z^\circ(\mathbf{L})$ via the map $i : CZ^\circ(\mathbf{L}) \to C \times Z^\circ(\mathbf{L})$. In this case, there exist $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$ and $\mathcal{E}_0$ a local system on C with $(\mathbf{L}, \Sigma, \mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell)$ being an induction datum for $\mathbf{G}$, such that

$$IC(\bar{\Sigma}, \mathcal{E})[\dim \Sigma] \cong IC(\bar{\Sigma}, i^*(\mathcal{E}_0 \boxtimes \mathcal{Z}))[\dim \Sigma].$$

Cuspidal character sheaves satisfy another interesting property.

Proposition 3.2.9 ([LuCS5, Thm. 23.1.a]). *For any cuspidal indution datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$, the cuspidal character sheaf $\mathcal{A}_{\mathfrak{m}}$ is **clean**, that is its restriction to $\overline{\Sigma} - \Sigma$ is zero.*

Another description of the parabolic induction

We now describe the induction of a cuspidal character sheaf following [LuCS2, 8.2]. As a matter of fact, we will see that it does not depend on the parabolic subgroup, but only

on the induction datum.
Let $\mathbf{P} \coloneqq \mathbf{U} \rtimes \mathbf{L}$ be a Levi decomposition of a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $\mathbf{T}_0 \subseteq \mathbf{L}$ and $\mathbf{B}_0 \subseteq \mathbf{P}$. We fix Σ the inverse image under the map $\mathbf{L} \to \mathbf{L}/Z^\circ(\mathbf{L})$ of an isolated conjugacy class and $\mathcal{E}$ an $\mathbf{L}$-equivariant local system on Σ. We consider the intersection cohomology complex

$$\mathcal{K} \coloneqq IC(\overline{\Sigma}, \mathcal{E})[\dim \Sigma].$$

We construct a perverse sheaf isomorphic to the induced perverse sheaf $\mathrm{Ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\mathcal{K})$.

We have the following diagram

$$\Sigma \xleftarrow{\alpha} \mathbf{G} \times \Sigma_{reg} \xrightarrow{\beta} \mathbf{G} \times_{\mathbf{L}} \Sigma_{reg} \xrightarrow{\gamma} Y_{\mathbf{L},\Sigma}$$

with

- the set $\Sigma_{reg} \coloneqq \{h \in \Sigma \mid C_{\mathbf{G}}^\circ(h_s) \subseteq \mathbf{L}\}$ and the set $Y_{\mathbf{L},\Sigma} \coloneqq \bigcup_{g\in\mathbf{G}} g\Sigma_{reg}g^{-1}$,
- the map α which is the projection on Σ of the second coordinate,
- the set $\mathbf{G} \times_{\mathbf{L}} \Sigma_{reg}$, quotient of $\mathbf{G} \times \Sigma_{reg}$ by the $\mathbf{L}$-action $l.(g,h) = (gl^{-1}, lhl^{-1})$ for $l \in \mathbf{L}, g \in \mathbf{G}$ and $h \in \Sigma_{reg}$,
- the quotient map β,
- and the conjugation map $\gamma : (g,h) \mapsto ghg^{-1}$ for $g \in \mathbf{G}, h \in \Sigma_{reg}$.

Since $\mathcal{E}$ is $\mathbf{L}$-equivariant, there exists a unique (up to isomorphism) local system $\tilde{\mathcal{E}}$ on $\mathbf{G} \times_{\mathbf{L}} \Sigma_{reg}$ such that $\alpha^*\mathcal{E} \cong \beta^*(\tilde{\mathcal{E}})$. Then, thanks to [Lus84b, Prop. 4.5], there is a canonical isomorphism

$$\mathrm{Ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\mathcal{K}) \cong IC(\overline{Y_{\mathbf{L},\Sigma}}, \gamma_*(\tilde{\mathcal{E}}))[\dim Y_{\mathbf{L},\Sigma}].$$

Now, for an induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E})$ as before, we write

$$\mathcal{K}_{\mathfrak{m}} \coloneqq IC(\overline{Y_{\mathbf{L},\Sigma}}, \gamma_*(\tilde{\mathcal{E}}))[\dim Y_{\mathbf{L},\Sigma}] \cong \mathrm{Ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\mathcal{A}_{\mathfrak{m}}).$$

Remark 3.2.10. Note that Σ_{reg} is open dense in $\bar{\Sigma}$, thus

$$IC(\overline{\Sigma}, \mathcal{E})[\dim \Sigma] = IC(\overline{\Sigma_{reg}}, (\mathcal{E})_{\Sigma_{reg}})[\dim \Sigma_{reg}].$$

Therefore, it makes sense to first consider the restriction of $\mathcal{E}$ to Σ_{reg}.
Moreover, the set $Y_{\mathbf{L},\Sigma}$ is a locally closed smooth irreducible subvariety of $\mathbf{G}$ of dimension equal to $\dim(\mathbf{G}) - \dim(\mathbf{L}) + \dim(\Sigma)$ which depends only on the $\mathbf{G}$-conjugacy class of $(\mathbf{L}, \Sigma)$. Lastly, the sets $Y_{\mathbf{L},\Sigma}$ for $(\mathbf{L}, \Sigma)$ as above define a finite partition of $\mathbf{G}$. Indeed for $g \in \mathbf{G}$ we may take $\mathbf{L}$ to be the minimal Levi subgroup containing $C_{\mathbf{G}}^\circ(g_s)$ and $\Sigma = (g)_{\mathbf{L}} Z^\circ(\mathbf{L})$, see [Lus84b, § 3.1] for all those facts.

Induction series

We finally define induction series and see how they induce a partition of $\hat{\mathbf{G}}$.

Definition 3.2.11. Let $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$ be a cuspidal induction datum. We define the **induction series** $\hat{\mathbf{G}}(\mathfrak{m})$ as the set of all character sheaves which are constituents of $\mathcal{K}_{\mathfrak{m}}$.

Note that by Proposition 3.2.4, all the constituents of $\mathcal{K}_{\mathfrak{m}}$ are character sheaves. By [LuCS1, Thm. 4.4a], every character sheaf belongs to some $\hat{\mathbf{G}}(\mathfrak{m})$ for some $\mathfrak{m} \in \mathfrak{M}^{\mathbf{G}}$. Thus, we can write

$$\hat{\mathbf{G}} = \bigcup_{\mathfrak{m} \in \mathfrak{M}^{\mathbf{G}}} \hat{\mathbf{G}}(\mathfrak{m}).$$

Furthermore, let $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}), \mathfrak{m}' = (\mathbf{L}', \Sigma', \mathcal{E}') \in \mathfrak{M}^{\mathbf{G}}$ be two cuspidal induction data. If $\mathcal{A} \in \hat{\mathbf{G}}$ is a component of both $\mathcal{K}_{\mathfrak{m}}$ and $\mathcal{K}_{\mathfrak{m}'}$, then there exists an element $g \in \mathbf{G}$ such that ${}^{g}\mathbf{L} = \mathbf{L}'$, ${}^{g}\Sigma = \Sigma'$ and $\mathrm{ad}(g^{-1})^{*}\mathcal{E} = \mathcal{E}'$, see [LuCS2, Cor. 7.6]. We say that $\mathfrak{m}$ and $\mathfrak{m}'$ are $\mathbf{G}$-conjugate. Therefore, we obtain

$$\hat{\mathbf{G}} = \bigsqcup_{\mathfrak{m}} \hat{\mathbf{G}}(\mathfrak{m}),$$

where $\mathfrak{m}$ runs over a set of representatives for the $\mathbf{G}$-orbits in $\mathfrak{M}^{\mathbf{G}}$. By [LuCS1, 4.3.1], all character sheaves in $\hat{\mathbf{G}}(\mathfrak{m})$ have support $\overline{Y}_{\mathbf{L},\Sigma}$.

Remark 3.2.12. This is very similar to the Harish-Chandra series of characters which depend only on the G-conjugacy classes of the cuspidal pairs (see below Definition 2.1.12).

We are left to discuss how to label the character sheaves in an induction series, as we did for characters in Harish-Chandra series in the Howlett–Lehrer Comparison Theorem (Theorem 2.1.14). That is what we do in the next subsection.

3.2.3 Decomposition of an induced cuspidal character sheaf

For the rest of this subsection we fix $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$ a cuspidal induction datum for $\mathbf{G}$. Recall that $\mathcal{K}_{\mathfrak{m}}$ is a semisimple perverse sheaf and thus decomposes into a direct sum of character sheaves. In fact, we can write

$$\mathcal{K}_{\mathfrak{m}} \cong \bigoplus_{V \in \mathrm{Irr}(\mathrm{End}(\mathcal{K}_{\mathfrak{m}}))} \mathcal{A}_V \otimes V,$$

where V runs over a set of representatives of isomorphism classes of irreducible $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$-modules. Here $\mathcal{A}_V := \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}})}(V, \mathcal{K}_{\mathfrak{m}})$ are the character sheaves in $\hat{\mathbf{G}}(\mathfrak{m})$. Based on our analogy with Harish-Chandra series, we would like to define a bijection between the algebra $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$ and the group algebra of some relative Weyl group. Lusztig showed that this idea works up to a twist by a 2-cocycle. We present some cases where one can show that this cocycle is trivial.

Relative Weyl groups

We first define the relative Weyl group associated to the cuspidal datum $\mathfrak{m}$.

Definition 3.2.13. The **relative Weyl group of** $(\mathbf{L}, \Sigma)$ is given by

$$W_{\mathbf{L},\Sigma} \coloneqq N_{\mathbf{G}}(\mathbf{L}, \Sigma)/\mathbf{L} \quad \text{with} \quad N_{\mathbf{G}}(\mathbf{L}, \Sigma) \coloneqq \{n \in \mathbf{G} \mid n\mathbf{L}n^{-1} = \mathbf{L}, n\Sigma n^{-1} = \Sigma\}.$$

The **relative Weyl group of** $\mathfrak{m}$ is then

$$W_{\mathfrak{m}} \coloneqq N_{\mathbf{G}}(\mathfrak{m})/\mathbf{L} \quad \text{with} \quad N_{\mathbf{G}}(\mathfrak{m}) \coloneqq \{n \in N_{\mathbf{G}}(\mathbf{L}, \Sigma) \mid \mathrm{ad}(n)^*\mathcal{E} \cong \mathcal{E}\}.$$

Notation 3.2.14. To simplify the notation, we also set $W_{\mathbf{L}} \coloneqq W^{\mathbf{G}}(\mathbf{L}) = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$. If we want to emphasise the ambient group, we might write it as a superscript, e.g., $W_{\mathbf{L}}^{\mathbf{G}}$ or $W_{\mathfrak{m}}^{\mathbf{G}}$.

Remark 3.2.15. In general, $W_{\mathfrak{m}}$ is not a Coxeter group but the semi-direct product of a Coxeter group with an abelian group. We describe it following Achar and Aubert [AA10, §4.1]. Assume that $\mathbf{L} = \mathbf{L}_I$ for $I \subseteq \Delta$. Let E be the real vector space on which W acts via its natural representation and E_I the subspace generated by all the $\alpha \in I$.
For any $\alpha \in \Phi$, if $w(I \cup \{\alpha\}) \subseteq \Delta$ for some $w \in W$, then $I \cup \{\alpha\}$ is a base for the root system it generates. If J is a base for the root system it generates, we write w_J for the longest element of the corresponding generated reflection group. We set

$$D_{\mathfrak{m}} \coloneqq \{\alpha \in \Phi \mid \exists w \in W \text{ such that } w(I \cup \{\alpha\}) \subseteq \Delta, w_{I\cup\{\alpha\}}w_I = w_I w_{I\cup\{\alpha\}} \in W_{\mathfrak{m}}\},$$

$$D_{\mathfrak{m}}^+ \coloneqq D_{\mathfrak{m}} \cap \Phi^+, \quad \Omega_{\mathfrak{m}} = \{w \in W_{\mathfrak{m}} \mid wD_{\mathfrak{m}}^+ \subseteq D_{\mathfrak{m}}^+\} \quad \text{and} \quad \Phi_{\mathfrak{m}} \coloneqq \{\alpha + E_I \mid \alpha \in D_{\mathfrak{m}}\}.$$

We define $W_{\mathfrak{m}}^\circ$ to be the Weyl group generated by the reflections s_α for $\alpha \in \Phi_{\mathfrak{m}}$. Then by [AA10, Prop. 4.1]

$$W_{\mathfrak{m}} = W_{\mathfrak{m}}^\circ \rtimes \Omega_{\mathfrak{m}}.$$

This description generalises the one given by Howlett for $W_{\mathbf{L}}$ ([How80]).

We now relabel the irreducible modules of $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$ using the relative Weyl group of $\mathfrak{m}$.

Theorem 3.2.16 ([LuCS2, § 10.2])**.** *The algebra* $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$ *is isomorphic to the group algebra* $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}]$ *twisted by a 2-cocycle.*

We make the isomorphism above more explicit, following [Lus84b, §3.4 and Prop. 3.5]. Recall the construction of $\mathcal{K}_{\mathfrak{m}}$. Let $\tilde{\mathcal{E}}$ be the canonical local system on $\mathbf{G} \times_{\mathbf{L}} \Sigma_{reg}$ such that $\alpha^*\mathcal{E} \cong \beta^*\tilde{\mathcal{E}}$. By the definition of intersection cohomology complexes, we have by [Lus84b, 4.4.1],

$$\mathrm{End}(\mathcal{K}_{\mathfrak{m}}) \cong \mathrm{End}(\gamma_*\tilde{\mathcal{E}}).$$

We thus have to define an isomorphism between $\mathrm{End}(\gamma_*\tilde{\mathcal{E}})$ and $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}]$ twisted by a 2-cocycle. Beforehand, for each $w \in W_{\mathfrak{m}}$, we fix a representative $\dot{w} \in N_{\mathbf{G}}(\mathbf{L})$ and define

$$\mathscr{A}_{\mathcal{E}} \coloneqq \bigoplus_{w \in W_{\mathfrak{m}}} \mathrm{Hom}(\mathrm{ad}(\dot{w})^*\mathcal{E}, \mathcal{E}).$$

We then follow the three next steps.

Step 1. Show that $\mathscr{A}_{\mathcal{E}}$ comes with a natural pairing which makes it isomorphic to the group algebra $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ twisted by a 2-cocycle.

Step 2. For each $w \in W(\mathfrak{m})$ and each isomorphism $\phi_{\dot{w}} : \mathrm{ad}(\dot{w})^*\mathcal{E} \tilde{\to} \mathcal{E}$, construct an endomorphism Φ_w of $\gamma_*\tilde{\mathcal{E}}$ independent of the choice of representatives $\dot{w}$.

Step 3. Check that the map $w \mapsto \Phi_w$ defines an algebra isomorphism from $\mathscr{A}_{\mathcal{E}}$ to $\mathrm{End}(\gamma_*\tilde{\mathcal{E}})$.

Let us describe the above steps in more details.
Step 1. Let $w, w' \in W_\mathfrak{m}$. We set

$$\mathscr{A}_{\mathcal{E},w} \coloneqq \mathrm{Hom}(\mathrm{ad}(\dot{w})^*\mathcal{E}, \mathcal{E}).$$

This vector space has dimension 1 since $\mathcal{E}$ is irreducible. There is a natural pairing

$$\begin{aligned} \mathscr{A}_{\mathcal{E},w} \times \mathscr{A}_{\mathcal{E},w'} &\to \mathscr{A}_{\mathcal{E},ww'} \\ (f,g) &\mapsto f \times g \coloneqq f \circ \mathrm{ad}(\dot{w})^*(g). \end{aligned}$$

Note that this pairing is associative. We fix basis elements $b_w \in \mathscr{A}_{\mathcal{E},w}$ for each $w \in W_\mathfrak{m}$. For all $w, w' \in W_\mathfrak{m}$, there exists a scalar $\lambda_{w,w'} \in \overline{\mathbb{Q}}_\ell$ such that

$$b_w \times b_{w'} = \lambda_{w,w'} b_{ww'}.$$

By associativity of the pairing, one can show that the map $W_\mathfrak{m} \times W_\mathfrak{m} \to \overline{\mathbb{Q}}_\ell$, $(w,w') \mapsto \lambda_{w,w'}$ is a 2-cocycle. Thus $\mathscr{A}_{\mathcal{E}}$ is isomorphic to $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ twisted by a 2-cocycle.

Step 2. Let $w \in W_\mathfrak{m}$. We have the following commutative diagram:

$$\begin{array}{ccccccc} \Sigma & \xleftarrow{\alpha} & \mathbf{G} \times \Sigma_{reg} & \xrightarrow{\beta} & \mathbf{G} \times_{\mathbf{L}} \Sigma_{reg} & \xrightarrow{\gamma} & Y_{\mathbf{L},\Sigma} \\ {\scriptstyle \mathrm{ad}(\dot{w})}\downarrow & & \downarrow{\scriptstyle \varphi_{\dot{w}}} & & \downarrow{\scriptstyle \bar{\varphi}_{\dot{w}}} & & \downarrow{\scriptstyle id} \\ \Sigma & \xleftarrow[\alpha]{} & \mathbf{G} \times \Sigma_{reg} & \xrightarrow[\beta]{} & \mathbf{G} \times_{\mathbf{L}} \Sigma_{reg} & \xrightarrow[\gamma]{} & Y_{\mathbf{L},\Sigma} \end{array}$$

with

- the $\mathbf{L}$-equivariant map $\varphi_{\dot{w}} : \mathbf{G} \times \Sigma_{reg} \to \mathbf{G} \times \Sigma_{reg}, (g,h) \mapsto (g\dot{w}^{-1}, \dot{w}h\dot{w}^{-1})$
- and $\bar{\varphi}_{\dot{w}} : \mathbf{G} \times_{\mathbf{L}} \Sigma_{reg} \to \mathbf{G} \times_{\mathbf{L}} \Sigma_{reg}, \beta((g,h)) \mapsto \beta((g\dot{w}^{-1}, \dot{w}h\dot{w}^{-1}))$.

Note that the map $\bar{\varphi}_{\dot{w}}$ is well defined since $\varphi_{\dot{w}}$ is $\mathbf{L}$-equivariant. Moreover, we have

$$\bar{\varphi}_{\dot{w}l} = \bar{\varphi}_{\dot{w}} \quad \text{for any } l \in \mathbf{L}.$$

Hence, we can write $\bar{\varphi}_w \coloneqq \bar{\varphi}_{\dot{w}}$.
Let us fix an isomorphism $\phi_{\dot{w}} : \mathrm{ad}(\dot{w})^*\mathcal{E} \tilde{\to} \mathcal{E}$. It induces a homomorphism

$$\alpha^*\phi_{\dot{w}} : \varphi_{\dot{w}}^*\alpha^*\mathcal{E} = \alpha^*\,\mathrm{ad}(\dot{w})^*\mathcal{E} \tilde{\to} \alpha^*\mathcal{E}.$$

From it, and from the isomorphism $\alpha^*\mathcal{E} \cong \beta^*\tilde{\mathcal{E}}$, we define a homomorphism

$$\beta^*\bar{\varphi}_w^*\tilde{\mathcal{E}} = \varphi_{\dot{w}}^*\beta^*\tilde{\mathcal{E}} \to \beta^*\tilde{\mathcal{E}}.$$

It gives rise to a homomorphism $\tilde{\phi}_{\dot{w}} : \bar{\varphi}_w^*\tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$. Lastly, precomposing $\gamma_*\tilde{\phi}_{\dot{w}}$ by the canonical isomorphism due to base change ($id^*\gamma_*\tilde{\mathcal{E}} \cong \gamma_*\bar{\varphi}_w^*\tilde{\mathcal{E}}$), we get an endomorphism

$$\Phi_{\dot{w}} : \gamma_*\tilde{\mathcal{E}} \to \gamma_*\bar{\varphi}_w^*\tilde{\mathcal{E}} \to \gamma_*\tilde{\mathcal{E}}.$$

We make a few observations.

- The morphism $\tilde{\phi}_{\dot{w}}$ (and thus $\Phi_{\dot{w}}$) depends only on $\phi_{\dot{w}}$, the local system $\tilde{\mathcal{E}}$ and the isomorphism $\alpha^*\mathcal{E} \cong \beta^*\tilde{\mathcal{E}}$. We did not make any choice in the construction.
- Since $\mathcal{E}$ is irreducible, $\phi_{\dot{w}}$ is unique up to multiplication by a scalar. Let $\phi'_{\dot{w}} = \lambda\phi_{\dot{w}}$ for $\lambda \in \overline{\mathbb{Q}}_\ell$, then $\tilde{\phi}'_{\dot{w}} = \lambda\tilde{\phi}_{\dot{w}}$.
- For $l \in \mathbf{L}$ and $\phi_l : \mathrm{ad}(l)^*\mathcal{E} \to \mathcal{E}$, the morphism $\tilde{\phi}_{\dot{w}l}$ coming from $\phi_{\dot{w}} \circ \mathrm{ad}(\dot{w})^*\phi_l$ is in fact equal to $\tilde{\phi}_{\dot{w}}$ (see [Lus84b, Proof of Prop. 3.5]). We choose ϕ_l to be the morphism coming from the $\mathbf{L}$-equivariance of $\mathcal{E}$. Therefore, we can set $\tilde{\phi}_w \coloneqq \tilde{\phi}_{\dot{w}}$ and $\Phi_w \coloneqq \Phi_{\dot{w}}$.

Therefore, after fixing a basis $\{\phi_{\dot{w}} \mid w \in W_\mathfrak{m}\}$ of $\mathscr{A}_\mathcal{E}$, we have defined a linear map

$$\mathscr{A}_\mathcal{E} = \bigoplus_{w \in W_\mathfrak{m}} \mathscr{A}_{\mathcal{E},w} \to \mathrm{End}(\gamma_*\tilde{\mathcal{E}}).$$

Step 3. We see that, by construction, the map above is injective. For a more detailed proof, we refer the reader to [Lus84b, Proof of Prop. 3.5]. There Lusztig also showed that the dimension of $\mathrm{End}(\gamma_*\tilde{\mathcal{E}})$ is at most $|W_\mathfrak{m}|$, which implies the bijectivity.

This concludes the description of the isomorphism

$$\mathscr{A}_\mathcal{E} \tilde{\to} \mathrm{End}(\gamma_*\tilde{\mathcal{E}}),$$

and hence of the isomorphism between $\mathrm{End}(\mathcal{K}_\mathfrak{m})$ and the group algebra $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ twisted by a 2-cocycle.
We now consider various cases where one can show that the 2-cocycle is in fact trivial.

Character sheaves with unipotent support

When the support of $\mathcal{K}_\mathfrak{m}$ contains unipotent elements, Lusztig made some choices for the basis elements of $\mathscr{A}_\mathcal{E}$ such that the cocycle is trivial.

Proposition 3.2.17 ([Lus84b, Thm. 9.2], [Lus86, Section 2])**.** *Let $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^\mathbf{G}$ be a cuspidal induction datum. Assume that Σ contains unipotent elements of $\mathbf{G}$. Then there is an isomorphism*

$$\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}] \cong \mathscr{A}_\mathcal{E}.$$

Proof. We are in the setting of Remark 3.2.8. Let $C \in \mathrm{Ucl}(\mathbf{L})$ be the unipotent class of $\mathbf{L}$ such that $\Sigma = CZ^\circ(\mathbf{L})$. Let $i : \Sigma \to C \times Z^\circ(\mathbf{L})$ be the canonical map and $\pi : CZ^\circ(\mathbf{L}) \to C, uz \mapsto u$. We write $\mathcal{E}$ as $i^*(\mathcal{E}_0 \boxtimes \mathcal{Z})$ where $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$ and $\mathcal{E}_0$ is a local system on C such that $(\mathbf{L}, \Sigma, \mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell)$ is an induction datum for $\mathbf{G}$.

We fix a basis of $\mathscr{A}_\mathcal{E}$ following [Lus86, Section 2], which will induce a trivial cocycle. In other words, for each $w \in W_\mathfrak{m}$ with representative $\dot{w}$, we fix $\theta_w \in \mathrm{Hom}(\mathrm{ad}(\dot{w})^*\mathcal{E}_0, \mathcal{E}_0)$ and $\phi_w^\mathcal{Z} \in \mathrm{Hom}(\mathrm{ad}(\dot{w})^*\mathcal{Z}, \mathcal{Z})$ and consider $b_w \coloneqq i^*(\theta_w \boxtimes \phi_w^\mathcal{Z})$. Observe that for $w, w' \in W_\mathfrak{m}$,

$$b_w \times b_{w'} = i^*(\theta_w \times \theta_{w'} \boxtimes \phi_w^\mathcal{Z} \times \phi_{w'}^\mathcal{Z}).$$

Alternatively, we could fix

$$\theta'_w \in \mathrm{Hom}(\mathrm{ad}(\dot{w})^*(\mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell), \mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell) \text{ and } \psi_w^\mathcal{Z} \in \mathrm{Hom}(\mathrm{ad}(\dot{w})^*(\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{Z}), \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{Z})$$

and consider $b_w \coloneqq i^*(\theta'_w \otimes \psi_w^\mathcal{Z})$, where $\otimes$ denotes the tensor product.

Firstly, we observe that by [Lus84b, Thm. 9.2b], $\mathrm{ad}(n)^*(\mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell) \cong \mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell$ for any $n \in N_\mathbf{G}(\mathbf{L})$. Thus,

$$W_\mathfrak{m} = \{n \in N_\mathbf{G}(\mathbf{L}) \mid \mathrm{ad}(n)^*\mathcal{Z} \cong \mathcal{Z}\}/\mathbf{L}.$$

Next, for any $w \in W_\mathbf{L}$, Lusztig fixed in [Lus84b, Thm. 9.2d] a homomorphism

$$\theta'_w : \mathrm{ad}(\dot{w})^*(\mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell) \to \mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell$$

by some condition on $\mathcal{K}_{\mathbf{L},\Sigma,\mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell}$. This basis satisfies that

$$\theta'_w \times \theta'_{w'} = \theta'_{ww'} \quad \text{for any } w, w' \in W_\mathbf{L}.$$

Finally, for $w \in W_\mathfrak{m}$, we fix the unique isomorphism $\phi_w^\mathcal{Z} : \mathrm{ad}(\dot{w}^{-1})^*\mathcal{Z} \xrightarrow{\sim} \mathcal{Z}$ such that $(\phi_w^\mathcal{Z})_1$ is the identity (as in [Lus86, §2.3]). Let $w, w' \in W_\mathfrak{m}$, then

$$\left(\phi_w^\mathcal{Z} \times \phi_{w'}^\mathcal{Z}\right)_1 = \left(\phi_w^\mathcal{Z}\right)_1 \circ \left(\mathrm{ad}(\dot{w})^*\phi_{w'}^\mathcal{Z}\right)_1 = id \circ \left(\phi_{w'}^\mathcal{Z}\right)_{\dot{w}1\dot{w}^{-1}} = id.$$

Thus, for any $w, w' \in W_\mathfrak{m}$,

$$\phi_w^\mathcal{Z} \times \phi_{w'}^\mathcal{Z} = \phi_{ww'}^\mathcal{Z}.$$

We then consider the basis

$$b_w^\mathcal{Z} \coloneqq i^*\left(\theta'_w \otimes \pi^*\phi_w^\mathcal{Z}\right) \quad \text{for} \quad w \in W_\mathfrak{m}.$$

The 2-cocycle defined by the natural pairing of this basis is then trivial. Hence, we have constructed an isomorphism of algebras

$$\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}] \xrightarrow{\sim} \mathscr{A}_\mathcal{E}.$$

□

The isomorphism above induces an isomorphism

$$\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}] \xrightarrow{\sim} \operatorname{End}(\gamma_*\tilde{\mathcal{E}}),$$

as we have described before, lifting each isomorphism $b_w^{\mathcal{Z}}$ to an endomorphism $B_w^{\mathcal{Z}}$ of $\gamma_*\tilde{\mathcal{E}}$.

Thanks to this construction, we can write

$$\mathcal{K}_\mathfrak{m} \cong \bigoplus_{V \in \operatorname{Irr}(W_\mathfrak{m})} \mathcal{A}_V^{\mathcal{Z}} \otimes V,$$

where V runs over a set of representatives of irreducible $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$-modules. Here the character sheaf $\mathcal{A}_V^{\mathcal{Z}} \coloneqq \operatorname{Hom}_{\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]}(V, \mathcal{K}_\mathfrak{m})$ is defined via the isomorphism $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}] \xrightarrow{\sim} \operatorname{End}(\mathcal{K}_\mathfrak{m})$ induced by the basis defined by θ'_w and $\phi_w^{\mathcal{Z}}$ for $w \in W_\mathfrak{m}$. The $\mathcal{A}_V^{\mathcal{Z}}$ are the character sheaves in $\hat{\mathbf{G}}(\mathfrak{m})$.

Remark 3.2.18. In [Bon04, §6.A], Bonnafé defined another isomorphism from $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ to $\operatorname{End}(\gamma_*\tilde{\mathcal{E}})$. He fixed $u \in C$ and showed that there exists a representative $\dot{w}$ of $w \in W_\mathfrak{m}$ which belongs to $C_\mathbf{G}(u)$ [Bon04, Eq. 5.4]. He chose for any $w \in W_\mathbf{L}$, the unique isomorphism $\sigma_w : \operatorname{ad}(\dot{w})^*\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}_0$ such that σ_w is the identity at the stalk at u. He then looked at the basis $b'_w \coloneqq \sigma_w \boxtimes \phi_w^{\mathcal{Z}}$. If $\mathbf{L} = \mathbf{T}_0$ or $\mathbf{L} = \mathbf{G}$, then in fact $\pi^*\sigma_w = \theta'_w$ [Bon04, Cor. 6.9].

Remark 3.2.19. Let $\mathcal{E}' \coloneqq i^*\big(\mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell\big)$ and $\mathfrak{m}' \coloneqq (\mathbf{L}, \Sigma, \mathcal{E}')$. We have an embedding of algebras

$$\mathscr{A}_\mathcal{E} = \bigoplus_{w \in W_\mathfrak{m}} \mathscr{A}_{\mathcal{E},w} \to \bigoplus_{w \in W_\mathbf{L}} \mathscr{A}_{\mathcal{E}',w} = \mathscr{A}_{\mathcal{E}'}$$
$$b_w^{\mathcal{Z}} = i^*\big(\theta'_w \otimes \pi^*\phi_w^{\mathcal{Z}}\big) \mapsto i^*\Big(\theta'_w \otimes \pi^*\phi_w^{\overline{\mathbb{Q}}_\ell}\Big) = b_w^{\overline{\mathbb{Q}}_\ell}.$$

In [Lus86, §2.6], Lusztig constructed an isomorphism between the restrictions $(\mathcal{K}_\mathfrak{m})_{\mathbf{G}_{\text{uni}}}$ and $(\mathcal{K}_{\mathfrak{m}'})_{\mathbf{G}_{\text{uni}}}$ compatible with the θ'_w for $w \in W_\mathbf{L}$. In other words, he defined a canonical isomorphism $(\mathcal{K}_\mathfrak{m})_{\mathbf{G}_{\text{uni}}} \xrightarrow{\sim} (\mathcal{K}_{\mathfrak{m}'})_{\mathbf{G}_{\text{uni}}}$ such that for any $w \in W_\mathfrak{m}$, the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{K}_\mathfrak{m})_{\mathbf{G}_{\text{uni}}} & \longrightarrow & (\mathcal{K}_{\mathfrak{m}'})_{\mathbf{G}_{\text{uni}}} \\
{\scriptstyle B_w^{\mathcal{Z}}}\big\downarrow & & \big\downarrow{\scriptstyle B_w^{\overline{\mathbb{Q}}_\ell}} \\
(\mathcal{K}_\mathfrak{m})_{\mathbf{G}_{\text{uni}}} & \longrightarrow & (\mathcal{K}_{\mathfrak{m}'})_{\mathbf{G}_{\text{uni}}}
\end{array}$$

Therefore, this isomorphism induces an isomorphism in $D_c^b(\mathbf{G}_{\text{uni}}, \overline{\mathbb{Q}}_\ell)$:

$$\big(\mathcal{A}_V^{\mathcal{Z}}\big)_{\mathbf{G}_{\text{uni}}} \cong \left(\mathcal{A}^{\overline{\mathbb{Q}}_\ell}_{\operatorname{Ind}_{W_\mathfrak{m}}^{W_\mathbf{L}}(V)}\right)_{\mathbf{G}_{\text{uni}}}$$

for any $V \in \operatorname{Irr}(W_\mathfrak{m})$, where $\operatorname{Ind}_{W_\mathfrak{m}}^{W_\mathbf{L}}(V)$ is the induced module.

Character sheaves in simple groups of adjoint type

If we strengthen the assumptions on $\mathbf{G}$, then one can show that the 2-cocycle is trivial for any induction series.

Proposition 3.2.20 ([Sho95a, Lem. 5.9])**.** *Assume that* $\mathbf{G}$ *is simple modulo its centre. Then, for any cuspidal datum* $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$*, there is an isomorphism*

$$\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}] \cong \mathscr{A}_\mathcal{E}.$$

In the proof, Shoji used the following argument of Lusztig ([Lus84b, §9.4]): assume that there exists a one-dimensional module V of $\mathscr{A}_\mathcal{E}$. Then, we choose for basis elements $a_w \in \mathscr{A}_{\mathcal{E},w}$ the isomorphisms acting as identity on V. To show that such a one-dimensional module of $\mathscr{A}_\mathcal{E}$ exists, Shoji considered the equivalent statement that there exists a character sheaf $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ such that $\langle \mathcal{A}, \mathcal{K}_\mathfrak{m} \rangle = 1$ and treated it by a case-by-case analysis.

We would like to have a better understanding of the basis elements. The idea is to construct from $\mathfrak{m}$ another induction datum $\mathfrak{m}' = (\mathbf{L}', \Sigma', \mathcal{E}')$ for a subgroup of $\mathbf{G}$ such that Σ' contains unipotent elements. We define basis elements for $\mathscr{A}_{\mathcal{E}'}$ that we could lift to basis elements of $\mathscr{A}_\mathcal{E}$.

Recall that Σ is the pullback under the quotient map $\mathbf{L} \to \mathbf{L}/Z^\circ(\mathbf{L})$ of an isolated conjugacy class in $\mathbf{L}/Z^\circ(\mathbf{L})$. In other words, there exist a semisimple element $s \in \mathbf{L}$ and a unipotent element $u \in C_\mathbf{L}(s)$ such that $\Sigma = (su)_\mathbf{L} Z^\circ(\mathbf{L})$ and $sZ^\circ(\mathbf{L})$ is isolated in $\mathbf{L}/Z^\circ(\mathbf{L})$. By [Bon04, Prop. 2.3b], s is isolated in $\mathbf{L}$.

We now fix $\mathbf{L}_s = C^\circ_\mathbf{L}(s)$ and $\mathbf{G}_s = C^\circ_\mathbf{G}(s)$. We also consider

$$C \coloneqq \{u \in \mathbf{L}_s \mid su \in \Sigma, u \text{ unipotent}\}.$$

By [LuCS2, Prop. 7.11c], the set C is in fact a single unipotent conjugacy class of $\mathbf{L}_s$, i.e.,

$$C = (u)_{\mathbf{L}_s}.$$

Recall that by Lemma 1.3.9, we have

$$Z^\circ(\mathbf{L}_s) = Z^\circ(\mathbf{L}).$$

We abuse notation and set s the translation map $s : (u)_{\mathbf{L}_s} Z^\circ(\mathbf{L}_s) \to \Sigma$, $x \mapsto sx$. Then by [LuCS2, Prop. 7.11a], the irreducible perverse sheaf $IC(\overline{CZ^\circ(\mathbf{L}_s)}, s^*\mathcal{E})[\dim CZ^\circ(\mathbf{L}_s)]$ is cuspidal, whence $\mathfrak{m}_s = (\mathbf{L}_s, CZ^\circ(\mathbf{L}), s^*\mathcal{E})$ is a cuspidal datum for $\mathbf{G}_s$.

Lemma 3.2.21. *We keep the notation above. Assume that* $N_{\mathbf{G}_s}(\mathbf{L}_s)/\mathbf{L}_s \cong N_\mathbf{G}(\mathbf{L})/\mathbf{L}$ *under the map* $g\mathbf{L}_s \mapsto g\mathbf{L}$*. Then, the algebra* $\operatorname{End}(\mathcal{K}_\mathfrak{m})$ *is isomorphic to the group algebra* $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ *(that is, the 2-cocycle is trivial).*

Proof. Thanks to Lemma 1.3.10, the map $N_{\mathbf{G}_s}(\mathbf{L}_s)/\mathbf{L}_s \to N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$, $g\mathbf{L}_s \mapsto g\mathbf{L}$ is well defined. For each $w \in W_{\mathfrak{m}}^{\mathbf{G}}$, we fix $\dot{w} \in N_{\mathbf{G}_s}(\mathbf{L}_s)$ such that $\dot{w}\mathbf{L} = w$. Note that if the representatives $\dot{w}, \dot{w}' \in N_{\mathbf{G}_s}(\mathbf{L}_s)$ satisfy $\dot{w}\mathbf{L} = \dot{w}'\mathbf{L}$, then $\dot{w}\mathbf{L}_s = \dot{w}'\mathbf{L}_s$. Moreover, by [Bon04, Eq. 5.4], we may even choose $\dot{w} \in C_{\mathbf{G}_s}(u)$. Thus, $\dot{w}$ belongs to $N_{\mathbf{G}_s}(\mathbf{L}_s, CZ^\circ(\mathbf{L}_s))$. Lastly, we observe that $\dot{w} \in N_{\mathbf{G}_s}(\mathfrak{m}_s)$ since

$$\mathrm{ad}(\dot{w}^{-1})^*(s^*\mathcal{E}) \cong s^* \,\mathrm{ad}(\dot{w}^{-1})^*(\mathcal{E}) \cong s^*(\mathcal{E}).$$

Therefore, setting $w_s = \dot{w}\mathbf{L}_s \in W_{\mathfrak{m}_s}^{\mathbf{G}_s}$, we have defined a group isomorphism

$$W_{\mathfrak{m}} \xrightarrow{\sim} W_{\mathfrak{m}_s}^{\mathbf{G}_s}, \quad w \mapsto w_s.$$

Finally, for each $w_s \in W_{\mathfrak{m}_s}^{\mathbf{G}_s}$, we fix basis elements $b_{w_s} \in \mathscr{A}_{s^*\mathcal{E}}$ as in the proof of Proposition 3.2.17. In particular, the map $w \mapsto b_{w_s}$ induces an isomorphism from $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}'}^{\mathbf{G}_s}]$ to $\mathscr{A}_{s^*\mathcal{E}}$. Now for each $w \in W_{\mathfrak{m}}$, we choose the unique isomorphism $a_w \in \mathscr{A}_{\mathcal{E},w} = (\mathrm{Hom}(\mathrm{ad}(\dot{w})^*\mathcal{E}, \mathcal{E}))$ such that

$$(a_w)_{su} = (b_{w_s})_u.$$

We check that the 2-cocycle is trivial. For $w, w' \in W_{\mathfrak{m}}$, we have

$$\begin{aligned}(a_w \times a_{w'})_{su} &= (a_w \circ \mathrm{ad}(\dot{w})^*(a_{w'}))_{su} = (a_w)_{su} \circ (a_{w'})_{su} \\ &= (b_{w_s})_u \circ (b_{w_s'})_u = (b_w)_u \circ (\mathrm{ad}(\dot{w}_s)^* b_{w_s'})_u \\ &= (b_{w_s w_s'})_u = (a_{ww'})_{su}.\end{aligned}$$

Thus we get an isomorphism between $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}]$ and $\mathscr{A}_{\mathfrak{m}}$, whence an isomorphism between $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}]$ and $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$. □

A similar result holds for classical groups.

Lemma 3.2.22. *Assume that* $\mathbf{G}$ *is simple of adjoint type and* p *is good for* $\mathbf{G}$*. Let* $(\Sigma, \mathcal{E})$ *be a cuspidal pair of a Levi subgroup* $\mathbf{L}$ *of* $\mathbf{G}$*. Then, there exists a semisimple element* $t \in \mathbf{L}$*, isolated in* $\mathbf{L}$*, such that the semisimple part of* Σ *is* $(t)_{\mathbf{L}} Z^\circ(\mathbf{L})$*, the element* t *is isolated in* $\mathbf{G}$*, and* $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L} = N_{C^\circ_{\mathbf{G}}(t)}(C^\circ_{\mathbf{L}}(t))/C^\circ_{\mathbf{L}}(t)$*.*

Proof. Firstly, we observe that the case where $\mathbf{L} = \mathbf{T}$ is trivial as we may choose $t = 1$. Similarly, for the cases where Σ is the preimage of a unipotent class of $\mathbf{L}/Z^\circ(\mathbf{L})$, we may assume that $t = 1$. The case where $\mathbf{L} = \mathbf{G}$ comes from [Lus84b, Prop. 2.7].
For the exceptional groups, we check the leftover cases (as listed in [AA10, Table 1]) using CHEVIE [Mic15]. We always have an embedding $N_{C^\circ_{\mathbf{G}}(t)}(C^\circ_{\mathbf{L}}(t))/C^\circ_{\mathbf{L}}(t) \to N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$ so we just check that these two finite groups have the same order.
We know focus on the classical groups. If $\mathbf{G}$ is of type A, then the Levi subgroups of the cuspidal induction data of $\mathbf{G}$ are maximal tori of $\mathbf{G}$. Thus, we now assume that $\mathbf{G}$ is of type B, C or D.
Firstly, we may assume without loss of generalities that $\mathbf{L}$ is a standard Levi subgroup, that is $\mathbf{L} = \mathbf{L}_I$ for some subset $\varnothing \neq I \subset \Delta$, $I \neq \Delta$. Note that $\mathbf{L}$ has connected centre ([Car85, Prop. 8.1.4]) and is a product of classical groups, of which at most one of type

different from A. We rephrase what we have just said in order to fix notation.
Let $n = |\Delta|$, then we set $S = [1, n]$. Moreover, we let $J_1, \ldots, J_k$ be disjoint intervals, included in S, such that

$$I = \bigsqcup_{i=1}^{k} I_k \quad \text{where} \quad I_i := \{\alpha_j \mid j \in J_i\} \text{ for } 1 \le i \le k.$$

Next, [Lus84b, Prop. 2.7] tells us that there exists a semisimple isolated element $t \in \mathbf{L}$ such that $\Sigma = (t)_{\mathbf{L}} Z^\circ(\mathbf{L})$.
We now recall some results of Subsection 1.3.1 on isolated elements. Let α_0^i be the highest short root of Φ_{I_i}. The connected reductive group $C^\circ_{\mathbf{L}}(t)$ has Weyl group $W_I(t)$. This Weyl group is generated by the reflections indexed by the root system $\Phi_I(t)$, which has base $I(t)$. Since t is isolated in $\mathbf{L}$, by [Bon05, Thm. 4.6] and since p is good for $\mathbf{G}$, we may assume that for each $1 \le i \le k$, there is $j_i \in J_i$ such that

$$I(t) = \bigsqcup_{i=1}^{k} I_k \backslash \{\alpha_{j_i}\} \cup \{-\alpha_0^i\}.$$

We write $\alpha_0^i = \sum_{\alpha \in I_i} n_\alpha^i \alpha$. By [Bon05, Table I], since $\mathbf{L}$ is a classical group, $n_\alpha^i \le 2$ for each $\alpha \in I_i$ and each $1 \le i \le k$. If W_I has only components of type A, $n_\alpha^i \le 1$ for each root $\alpha \in I_i$ and each $1 \le i \le k$. In this case, $W_I(t) = W_I$ and thus, $t \in Z^\circ(\mathbf{L})$, a case we have already treated. Thus, there is $1 \le d \le n$ such that $J_1 = [1, d]$ and $\mathbf{L}$ has exactly one component of the same type X as $\mathbf{G}$.
Moreover, we can rewrite, up to $\mathbf{L}$-conjugation,

$$I(t) = I_1 \backslash \{\alpha_{j_1}\} \cup \{-\alpha_0^1\} \sqcup \bigsqcup_{i=2}^{k} I_i.$$

Now, we fix

$$t' := \tilde{\iota}_{\mathbf{T}_0}\left(\frac{\breve{\omega}_{\alpha_{j_1}}}{n_{\alpha_{j_1}}}\right) \in \mathbf{T}_0.$$

Thanks to Proposition 1.3.7, the semisimple element t' is isolated in $\mathbf{G}$ and $W(t')$ has base $\Delta(t') = \Delta \backslash \{\alpha_{j_1}\} \cap \{-\alpha_0\}$ where α_0 is the highest short root of Φ. We claim that t' is $\mathbf{L}$-conjugate to t. For this, it suffices to observe that

$$t' = \tilde{\iota}_{\mathbf{T}_0}\left(\frac{\breve{\omega}_{\alpha_{j_1}}}{n_{\alpha_{j_1}}} + \sum_{i=2}^{k} \breve{\omega}_{-\alpha_0^i}\right)$$

and thus

$$I - I(t) = I - I(t')$$

and then apply [Bon05, Thm. 4.6].

From now on, we set $t = t'$. We need to show that $|N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}| \le |N_{C^\circ_{\mathbf{G}}(t)}(C^\circ_{\mathbf{L}}(t))/C^\circ_{\mathbf{L}}(t)|$. By [MT11, Cor. 12.11], it is equivalent to showing that

$$|N_W(W_I)/W_I| \le |N_{W(t)}(W_I(t))/W_I(t)|.$$

Now, thanks to [How80, Cor. 3], this reduces to showing that

$$|N_W(I)| \leq |N_{W(t)}(I(t))|.$$

We analyse each classical type individually.
First, we assume that $\mathbf{G}$ is of type B. To simplify the visualisation, we write down in Table 3.1 the shape of the Dynkin diagrams of the various involved Weyl groups. As

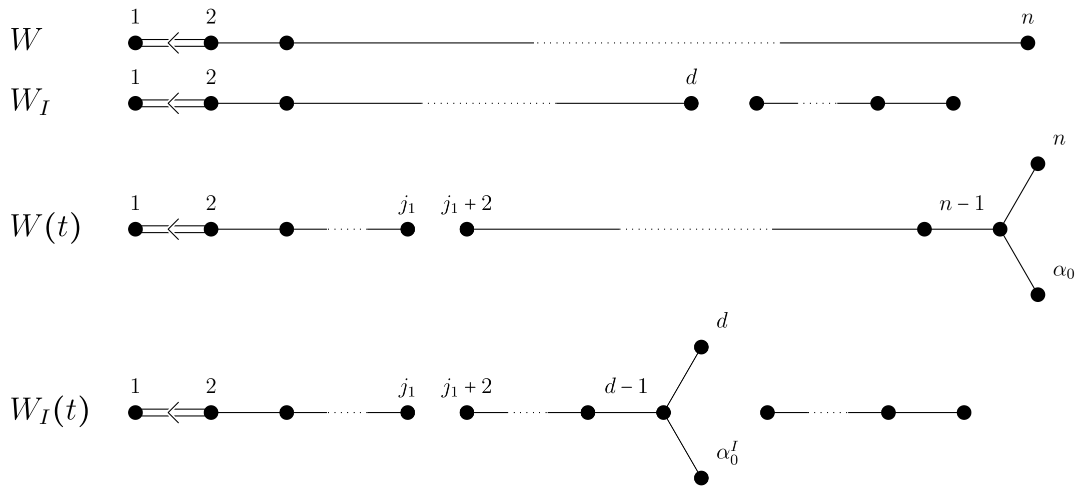


Table 3.1: Dynkin diagrams of the relative Weyl groups for type B

we explained before, W_I is a Coxeter group, which is a product of one group of type B_d and n_l groups of type A_l for $1 \leq l \leq s \coloneqq n-d-1$ for some $2 \leq d \leq n-1$. By [How80, after Lem. 10], we know that $N_W(I)$ is a Weyl group of type

$$B_{n_1} \times \cdots \times B_{n_s} \times B_k, \quad \text{where} \quad k = n - (d + \sum_{1 \leq i \leq s} (i+1)n_i).$$

On the other hand, we can read off from [Bon05, Table II] that $\Delta(t)$ yields a Weyl group of type $B_{j_1} \times D_{n-j_1}$ for some $1 \leq j_1 \leq d-1$. Similarly, $I(t)$ is the base of a Weyl group which is a product of groups of type B_{j_1}, D_{d-j_1} and n_l groups of type A_l for $1 \leq l \leq s$. Thanks to [How80, after Lem. 10], we observe that $N_{W(t)}(I(t))$ contains a Coxeter subgroup of type

$$B_{n_1} \times \cdots \times B_{n_s} \times B_{k'} \quad \text{where} \quad k' = (n - j_1) - ((d - j_1) + \sum_{1 \leq i \leq s} (i+1)n_i) = k.$$

Thus, $|N_W(I)| \leq |N_{W(t)}(I(t))|$ and we are done with the type B.

Now, we assume that $\mathbf{G}$ is simple of adjoint type C. In this case, W_I is product of one Coxeter group of type C_d and n_l groups of type A_l for $1 \leq l \leq s \coloneqq n-d-1$ for some $3 \leq d \leq n-1$. By [How80, after Lem. 10], the group $N_W(I)$ is of type

$$B_{n_1} \times \cdots \times B_{n_s} \times B_k, \quad \text{where} \quad k = n - (d + \sum_{1 \leq i \leq s} (i+1)n_i).$$

From [Bon05, Table II], we deduce that $\Delta(t)$ yields a Weyl group of type $C_{j_1} \times C_{n-j_1}$ for some $1 \leq j_1 \leq \lfloor d/2 \rfloor$. Moreover, $W_I(t)$ is a product of groups of type C_{j_1}, C_{d-j_1} and n_l groups of type A_l for $1 \leq l \leq s$. Finally, we conclude by observing that $N_{W(t)}(I(t))$ contains a Coxeter subgroup of type

$$B_{n_1} \times \cdots \times B_{n_s} \times B_{k'} \quad \text{where} \quad k' = (n - j_1) - ((d - j_1) + \sum_{1 \leq i \leq s} (i+1)n_i) = k.$$

Lastly, we assume that $\mathbf{G}$ is of type D. Then, W_I is product of one Coxeter group of type D_d and n_l groups of type A_l for $1 \leq l \leq s := n - d - 1$ for some $4 \leq d \leq n-1$. By [How80, after Lem. 10], the group $N_W(I)$ is again of type

$$B_{n_1} \times \cdots \times B_{n_s} \times B_k, \quad \text{where} \quad k = n - (d + \sum_{1 \leq i \leq s} (i+1)n_i).$$

Next, we observe from [Bon05, Table II] that $\Delta(t)$ yields a Weyl group of type $D_{j_1} \times D_{n-j_1}$ for some $1 \leq j_1 \leq \lfloor d/2 \rfloor$. Moreover, $W_I(t)$ is a product of groups of type D_{j_1}, D_{d-j_1} and n_l groups of type A_l for $1 \leq l \leq s$. As before, thanks to [How80, after Lem. 10], we observe that $N_{W(t)}(I(t))$ contains a Coxeter subgroup of type

$$B_{n_1} \times \cdots \times B_{n_s} \times B_{k'} \quad \text{where} \quad k' = (n - j_1) - ((d - j_1) + \sum_{1 \leq i \leq s} (i+1)n_i) = k.$$

This concludes our case by case analysis. □

3.3 Another basis for the space of class functions

From the previous two sections, we have established a clear parallel between character sheaves of a connected reductive group $\mathbf{G}$ and characters of a finite reductive group $G = \mathbf{G}^F$. In this section, we complete this analogy by finally bringing the Frobenius map F into play. We consider F-stable character sheaves, and see that their characteristic functions do not only define a basis for the space of class functions, but in fact agree in general with the almost characters (see below Theorem 2.2.29). This famous result, known as (one of) Lusztig's conjecture(s) was proven by Shoji in 1995 in two consecutive papers [Sho95a], [Sho95b] under certain assumptions such as $Z(\mathbf{G})$ connected.

Hypothesis 3.3.1. In this section, we always assume that the Steinberg map F fixed in Hypothesis 1 is a Frobenius map and gives $\mathbf{G}$ an $\mathbb{F}_q$-structure, for q a power of the prime p.

3.3.1 The F-stable character sheaves

Let $\mathcal{A}$ be an F-stable character sheaf on $\mathbf{G}$. We write that $\mathcal{A}$ belongs to $\hat{\mathbf{G}}^F$. By definition $\mathcal{A}$ is a constituent of some ${}^pH^i(\bar{\mathcal{K}}_w^{\mathcal{L}})$ for some local system $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$, $w \in W_{\mathcal{L}}$ and $i \in \mathbb{Z}$. Moreover, $\mathcal{A}$ also belongs to the induction series coming from some cuspidal datum $\mathfrak{m} \in \mathfrak{M}^{\mathbf{G}}$. In this subsection, we see that we may in fact choose $\mathfrak{m}$ such that it is F-stable as well.

F-stability and the definition of character sheaves

Let $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$ be a Kummer local system on $\mathbf{T}_0$ and assume that there exists an F-stable character sheaf $\mathcal{A}$ which is a constituent of ${}^pH^i(\bar{\mathcal{K}}_w^{\mathcal{L}})$ for some $w \in W_{\mathcal{L}}$ and $i \in \mathbb{Z}$. Then, $F^*\mathcal{A}$ is an irreducible constituent of $F^{*p}H^i(\bar{\mathcal{K}}_w^{\mathcal{L}})$. By [LuCS1, 1.8.1],

$$F^{*p}H^i(\bar{\mathcal{K}}_w^{\mathcal{L}}) = {}^pH^i(F^*\bar{\mathcal{K}}_w^{\mathcal{L}}).$$

We follow Definition 3.1.9 to compute $F^*\bar{\mathcal{K}}_w^{\mathcal{L}}$. Since $F^* \operatorname{pr}_w^* \mathcal{L} = \operatorname{pr}_{F^{-1}(w)}^* F^*\mathcal{L}$, we see that

$$F^*\mathcal{A}_w^{\mathcal{L}} \cong \mathcal{A}_{F^{-1}(w)}^{F^*\mathcal{L}},$$

whence

$$F^*\bar{\mathcal{K}}_w^{\mathcal{L}} \cong \bar{\mathcal{K}}_{F^{-1}(w)}^{F^*\mathcal{L}}.$$

By Proposition 3.1.11, since $\mathcal{A}$ is isomorphic to $F^*\mathcal{A}$, the local systems $\mathcal{L}$ and $F^*\mathcal{L}$ are in the same W-orbit, that is writing $\mathcal{L} = \lambda^*\mathcal{E}_{n,\mathrm{j}}$ for $\lambda \in X$ and $n \in \mathbb{N}$ coprime to p, there is some $w' \in W$ such that

$$\lambda \circ F - w'.\lambda \in nX (\iff w' \in \mathscr{Z}_{\lambda,n}).$$

Now, from Proposition 3.1.11 and the analogue for characters (see below Theorem 2.2.10), we can write

$$\hat{\mathbf{G}}^F = \bigsqcup_s \hat{\mathbf{G}}_s^F,$$

where s runs over a set of representatives of the F^*-stable semisimple conjugacy classes in $\mathbf{G}^*$.

Lastly, we consider the F-stable character sheaves in view of the parameterisation in Theorem 3.1.12. Let $\mathcal{L}$ be an F-stable Kummer local system on $\mathbf{T}_0$ and $\psi \in \operatorname{irr}(W_{\mathcal{L}})$, then

$$F^*\mathcal{R}_\psi = \mathcal{R}_{F.\psi}.$$

Therefore, for any F-stable character sheaf $\mathcal{A} \in \hat{\mathbf{G}}_{\mathcal{L}}^F$, the following holds:

$$\langle \mathcal{R}_\psi, \mathcal{A}\rangle = \langle \mathcal{R}_\psi, F^*\mathcal{A}\rangle = \langle F^*\mathcal{R}_\psi, \mathcal{A}\rangle = \langle \mathcal{R}_{F.\psi}, \mathcal{A}\rangle.$$

Thus, $\langle \mathcal{R}_\psi, \mathcal{A}\rangle \neq 0$ if and only if $\langle \mathcal{R}_{F.\psi}, \mathcal{A}\rangle \neq 0$. As a consequence, Theorem 3.1.12 tells us the character $F.\psi$ belongs to the same family as ψ, hence to an F-stable family. Since F is ordinary (because it is a Frobenius map), then in fact $F.\psi = \psi$ (Proposition 2.2.22). Generalising this result to any F-stable Kummer local system, Theorem 3.1.12 can then be rewritten as

Proposition 3.3.2 ([Sho95a, § 5]). *Assume that $Z(\mathbf{G})$ is connected. To each F-stable family $\mathscr{F}$ of* $\operatorname{irr}(W_{\mathcal{L}})$ *(recall that $W_{\mathcal{L}} = W_{\mathcal{L}}^\circ$), one can associate a finite group $\bar{A}_{\mathscr{F}}$ such that there exist an injection*

$$\mathscr{F} \hookrightarrow \mathcal{M}(\bar{A}_{\mathscr{F}}), \quad \psi \mapsto x_\psi,$$

and an injection

$$\hat{\mathbf{G}}^F_{\mathcal{L}} \hookrightarrow \bigsqcup_{\mathcal{F}} \mathcal{M}(\bar{A}_{\mathscr{F}}), \quad \mathcal{A} \mapsto x_{\mathcal{A}},$$

where $\mathscr{F}$ *runs over the* F*-stable families of* $\operatorname{irr}(W_{\mathcal{L}})$, *such that for all* $\mathcal{A} \in \hat{\mathbf{G}}^F_{\mathcal{L}}$ *and* $\psi \in \mathscr{F}$,

$$\langle R_\psi, \mathcal{A}\rangle = \epsilon_{\mathcal{A}}\{x_{\mathcal{A}}, x_\psi\},$$

where $\epsilon_{\mathcal{A}} \coloneqq (-1)^{codim(\operatorname{supp}\mathcal{A})} \in \{1,-1\}$ *depends only on* $\mathcal{A}$.

F-stability and parabolic induction

Let $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$ be a cuspidal induction datum and $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ be an F-stable character sheaf. Following [LuCS2, §10.5], we may assume that $\mathbf{L}$, Σ and $\mathcal{E}$ are F-stable. Indeed, $F^*\mathcal{A}$ is an irreducible component of $F^*\mathcal{K}_{\mathfrak{m}} = \mathcal{K}_{F^*\mathfrak{m}}$, where $F^*\mathfrak{m} = (F^{-1}(\mathbf{L}), F^{-1}(\Sigma), F^*\mathcal{E})$. Since $\mathcal{A} \cong F^*\mathcal{A}$, the two induction data $\mathfrak{m}$ and $F^*\mathfrak{m}$ must be conjugate by an element $g \in \mathbf{G}$ (see below Definition 3.2.11), i.e.,

$$F^{-1}(\mathbf{L}) = g\mathbf{L}g^{-1}, \quad F^{-1}(\Sigma) = g\Sigma g^{-1} \quad \text{and } F^*\mathcal{E} = \operatorname{ad}(g)^*\mathcal{E}.$$

We can then consider the induction datum $\mathfrak{m}'' = ({}^h\mathbf{L}, {}^h\Sigma, \operatorname{ad}(h)^*\mathcal{E})$ for some $h \in \mathbf{G}$ such that $F(g) = h^{-1}F(h)$. Such an element h exists by the Lang–Steinberg theorem 1.2.4. Note that here we define everything with respect to ${}^h\mathbf{T}_0 \subseteq {}^h\mathbf{L}$ instead of $\mathbf{T}_0$. Then, the character sheaf $\mathcal{A}$ is (isomorphic to) a component of $\mathcal{K}_{\mathfrak{m}''}$ and $\mathfrak{m}''$ is F-stable. By following the diagram defining $\mathcal{K}_{\mathfrak{m}}$, we observe that $\mathcal{K}_{\mathfrak{m}} \cong \mathcal{K}_{\mathfrak{m}''}$. Therefore, we may assume that if $\mathcal{A}$ is F-stable, it belongs to an induction series indexed by an F-stable induction datum. Observe that the perverse sheaf $\mathcal{K}_{\mathfrak{m}''}$ is also F-stable. Indeed, since $\mathcal{E}'' \coloneqq \operatorname{ad}(h)^*\mathcal{E}$ is F-stable, there is an isomorphism $\varphi_0 : F^*\mathcal{E}'' \xrightarrow{\sim} \mathcal{E}''$. By [LuCS2, 8.1.3], it lifts to an isomorphism $\varphi : F^*\mathcal{K}_{\mathfrak{m}''} \xrightarrow{\sim} \mathcal{K}_{\mathfrak{m}''}$.

3.3.2 Characteristic functions of character sheaves

Let $\mathcal{A} \in \hat{\mathbf{G}}$ be a character sheaf of $\mathbf{G}$. Assume that $\mathcal{A}$ is F-stable and fix an isomorphism $\varphi : F^*\mathcal{A} \xrightarrow{\sim} \mathcal{A}$. Since $\mathcal{A}$ is irreducible, any other isomorphism between $F^*\mathcal{A}$ and $\mathcal{A}$ is of the form $\lambda\varphi$ for some scalar $\lambda \in \overline{\mathbb{Q}}_\ell^\times$. In [LuCS5, §25.1], Lusztig gave guidelines to choose the isomorphism φ that we now describe. Let $\mathfrak{m} \in \mathfrak{M}^{\mathbf{G}}$ such that $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$, i.e. $\mathcal{A}$ is a constituent of $\mathcal{K}_{\mathfrak{m}}$, see Definition 3.2.11. Then $\operatorname{supp}(\mathcal{A}) = \overline{Y_{\mathbf{L},\Sigma}}$ and we let $d = \dim\operatorname{supp}\mathcal{A}$. By [LuCS3, Thm. 14.2a], there exists an isomorphism $\varphi_{\mathcal{A}}$ as above such that for any $n \in \mathbb{N}$ and any $y \in Y_{\mathbf{L},\Sigma}$ with $F^n(y) = y$, the eigenvalues of

$$(\varphi_{\mathcal{A}})^n_{d,y} : H^{-d}(\mathcal{A})_{F^n(y)} \to H^{-d}(\mathcal{A})_y$$

have norm $q^{n(\dim\mathbf{G}-d)/2}$. Such an isomorphism $\varphi_{\mathcal{A}}$ is determined up to a root of unity in $\overline{\mathbb{Q}}_\ell$.

Notation 3.3.3. Since we will refer it, we call † the condition determining $\varphi_{\mathcal{A}}$.

We explain another way of choosing an isomorphism $\phi_{\mathcal{A}} : F^*\mathcal{A} \tilde{\to} \mathcal{A}$ using the induction data. Assume that A belongs to $\hat{\mathbf{G}}(\mathfrak{m})$ for $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$. By the previous discussion, we may assume that $\mathfrak{m}$ is F-stable. In particular, we fix an isomorphism of local systems $\phi_0 : F^*\mathcal{E} \tilde{\to} \mathcal{E}$ over Σ which induces a map of finite order on the stalk of $\mathcal{E}$ at any F-rational point of Σ. Recall the definition of parabolic induction defining $\mathcal{K}_{\mathfrak{m}}$. By [LuCS2, 8.1.3], the varieties $\mathbf{G} \times \Sigma_{reg}$, $\mathbf{G} \times_{\mathbf{L}} \Sigma_{reg}$ and $Y_{\mathbf{L},\Sigma}$ have a natural $\mathbb{F}_q$-structure. The isomorphism ϕ_0 gives rise to an isomorphism of local systems $\tilde{\phi}_0 : F^*\tilde{\mathcal{E}} \tilde{\to} \tilde{\mathcal{E}}$ and thus to an isomorphism $F^*\gamma_*\tilde{\mathcal{E}} \to \gamma_*\tilde{\mathcal{E}}$. Hence we have constructed an isomorphism $\phi_{\mathfrak{m}} : F^*\mathcal{K}_{\mathfrak{m}} \tilde{\to} \mathcal{K}_{\mathfrak{m}}$. By the definition of $\mathcal{K}_{\mathfrak{m}}$ as an intersection cohomology complex coming from a local system on $Y_{\mathbf{L},\Sigma}$, the map $\phi_{\mathfrak{m}}$ acts on $H^{-d}(\mathcal{K}_{\mathfrak{m}})_y$ as a map of finite order.
Let $\phi_{\mathcal{A}} : F^*\mathcal{A} \to \mathcal{A}$ be any isomorphism. Thanks to [LuCS2, 10.4], the following map

$$\begin{aligned} \sigma_{\mathcal{A}} : \mathrm{Hom}(\mathcal{A}, \mathcal{K}_{\mathfrak{m}}) &\to \mathrm{Hom}(\mathcal{A}, \mathcal{K}_{\mathfrak{m}}) \\ v &\mapsto \phi_{\mathfrak{m}} \circ F^*(v) \circ \phi_{\mathcal{A}}^{-1} \end{aligned}$$

is an isomorphism of $\overline{\mathbb{Q}}_\ell$-vector spaces which is $\mathscr{A}_{\mathcal{E}}$-semilinear, i.e. such that for all $\theta \in \mathscr{A}_{\mathcal{E}}$ and $v \in \mathrm{Hom}(\mathcal{A}, \mathcal{K}_{\mathfrak{m}})$, we have $\sigma_{\mathcal{A}}(\theta.v) = (\phi_{\mathfrak{m}} \circ F^*(\theta) \circ \phi_{\mathfrak{m}}^{-1}).\sigma_{\mathcal{A}}(v)$.
The decomposition

$$\bigoplus_{\mathcal{A}} \mathcal{A} \otimes V_{\mathcal{A}} \cong \mathcal{K}_{\mathfrak{m}},$$

where $\mathcal{A}$ runs over the set of irreducible components of $\mathcal{K}_{\mathfrak{m}}$ gives rise for any $i \in \mathbb{Z}$ and any $g \in \mathbf{G}$, to an isomorphism

$$\bigoplus_{\mathcal{A}} H^i(\mathcal{A})_g \otimes V_{\mathcal{A}} \cong H^i(\mathcal{K}_{\mathfrak{m}})_g.$$

In particular, the endomorphisms $\phi_{\mathcal{A}} \otimes \sigma_{\mathcal{A}}$ are compatible with $\phi_{\mathfrak{m}}$ under this isomorphism ([LuCS2, 10.4.1]).
We fix a particular choice of $\phi_{\mathcal{A}}$ as follows. Recall that $V_{\mathcal{A}} \coloneqq \mathrm{Hom}(\mathcal{A}, \mathcal{K}_{\mathfrak{m}})$ is an irreducible $\mathscr{A}_{\mathcal{E}}$-module. Since $\sigma_{\mathcal{A}}$ is $\mathscr{A}_{\mathcal{E}}$-semilinear, there is a certain power m of $\sigma_{\mathcal{A}}$ such that $\sigma_{\mathcal{A}}^m$ acts as an automorphism on the irreducible $\mathscr{A}_{\mathcal{E}}$-module $V_{\mathcal{A}}$, hence as multiplication by a scalar. Thus, we may now choose $\phi_{\mathcal{A}}$ such that $\sigma_{\mathcal{A}}$ is of finite order. Note that this determines $\phi_{\mathcal{A}}$ up to a root of unity.

We now would like to relate the isomorphisms $\varphi_{\mathcal{A}}$ and $\phi_{\mathcal{A}}$, following [Het23a, 3.2.25] in our more general case. Since $\mathcal{A}$ is irreducible, there exists a scalar $\xi \in \overline{\mathbb{Q}}_\ell^\times$, such that $\varphi_{\mathcal{A}} = \xi\phi_{\mathcal{A}}$. We set

$$\begin{aligned} \theta_{\mathcal{A}} : \mathrm{Hom}(\mathcal{A}, \mathcal{K}_{\mathfrak{m}}) &\to \mathrm{Hom}(\mathcal{A}, \mathcal{K}_{\mathfrak{m}}) \\ v &\mapsto \phi_{\mathfrak{m}} \circ F^*(v) \circ \varphi_{\mathcal{A}}^{-1}. \end{aligned}$$

Note that $\theta_{\mathcal{A}} = \xi^{-1}\sigma_{\mathcal{A}}$. Moreover, for any $y \in Y_{\mathbf{L},\Sigma}^F$, we can identify $\varphi_{\mathcal{A}} \otimes \theta_{\mathcal{A}}$ with $\phi_{\mathfrak{m}}$. By definition of $\varphi_{\mathcal{A}}$, the eigenvalues of

$$(\varphi_{\mathcal{A}})_{d,y} : H^{-d}(\mathcal{A})_y \to H^{-d}(\mathcal{F})_y$$

have norm $q^{(\dim \mathbf{G}-d)/2}$. Since $\phi_{\mathfrak{m}}$ acts on $H^{-d}(\mathcal{K}_{\mathfrak{m}})_y$ as a map of finite order, the eigenvalues of $\theta_{\mathcal{A}}$ have norm $q^{-(\dim \mathbf{G}-d)/2}$. Hence, the map $q^{(\dim \mathbf{G}-d)/2}\theta_{\mathcal{A}} = q^{(\dim \mathbf{G}-d)/2}\xi^{-1}\sigma_{\mathcal{A}}$ is a map of finite order, like $\sigma_{\mathcal{A}}$. In particular, we could fix $\phi_{\mathcal{A}}$ as $q^{-(\dim \mathbf{G}-d)/2}\varphi_{\mathcal{A}}$. In that case, we have

$$\chi_{\mathcal{A},\varphi_{\mathcal{A}}} = q^{(\dim \mathbf{G}-d)/2}\chi_{\mathcal{A},\phi_{\mathcal{A}}}.$$

The isomorphism $\phi_{\mathcal{A}}$ satisfies a nice property that we will need later on.

Lemma 3.3.4. *Let D be an F-stable conjugacy class of $\mathbf{G}$ and $\mathcal{A}$ an F-stable character sheaf of $\mathbf{G}(\mathfrak{m})$ for an F-stable induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E})$. Then, for any $h \in D^F$, the map $\phi_{\mathcal{A}}$ has defined above induces a map $(\phi_{\mathcal{A}})_{a_0,h}$ which acts on $H^{a_0}(\mathcal{A})_h$ as $q^{(a_0+d)/2}$ times a map of finite order, where $a_0 \coloneqq -\dim(D) - \dim(Z^\circ(\mathbf{L}))$.*

Proof. The proof of this lemma is a slight modification of [LuCS5, 24.2.4] to our more general case. We keep the notation fixed before, in particular let $h \in D^F$. We fix an F-stable parabolic subgroup $\mathbf{P}$ with Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U_P}$ and $\mathbf{B}_0$. We define the following sets

$$X \coloneqq \{(g, xP) \in \mathbf{G} \times \mathbf{G}/\mathbf{P} \mid x^{-1}gx \in \Sigma\mathbf{U_P}\},$$
$$\overline{X} \coloneqq \{(g, xP) \in \mathbf{G} \times \mathbf{G}/\mathbf{P} \mid x^{-1}gx \in \overline{\Sigma}\mathbf{U_P}\},$$

and

$$\hat{X} \coloneqq \{(g, x) \in \mathbf{G} \times \mathbf{G} \mid x^{-1}gx \in \overline{\Sigma}\mathbf{U_P}\}.$$

Let $\mathcal{A}_{\mathfrak{m}} = IC(\overline{\Sigma}, \mathcal{E})[\dim \Sigma]$. Then $\mathrm{Ind}_{\mathbf{L}\subseteq\mathbf{P}}^{\mathbf{G}}(\mathcal{A}_{\mathfrak{m}})$ may be defined as follows. Consider the following diagram

$$\overline{\Sigma} \xleftarrow{\mathrm{pr}_{\mathbf{L}\subseteq\mathbf{P}}} \mathbf{P} \xleftarrow{\hat{\gamma}} \hat{X} \xrightarrow{\beta} \overline{X} \xrightarrow{\hat{\alpha}} \overline{Y}_{,\Sigma}.$$

with

- the projection map $\mathrm{pr}_{\mathbf{L}\subseteq\mathbf{P}} : \mathbf{P} \to \mathbf{L}$ sending $g = ul$ to $l \in \mathbf{L}$,
- the conjugation map $\hat{\gamma} : (g, x) \mapsto x^{-1}gx$ for $g \in \mathbf{G}$, $x \in \mathbf{G}$,
- the quotient map β,
- and the projection map $\hat{\alpha} : (g, xP) \mapsto g$ for $g \in \mathbf{G}, x \in \mathbf{G}$.

Note that it is a reformulation of the diagram in Definition 3.2.1. We let $\overline{\mathcal{A}_{\mathfrak{m}}} \coloneqq IC(\overline{X}, \overline{\mathcal{E}})$ where $\overline{\mathcal{E}}$ is the canonical local system on X such that $\beta^*\overline{\mathcal{E}} \cong \hat{\gamma}^* \mathrm{pr}_{\mathbf{L}\subseteq\mathbf{P}}^* \mathcal{E}$. Moreover, we have $\hat{\gamma}^* \mathrm{pr}_{\mathbf{L}\subseteq\mathbf{P}}^* \mathcal{A}_{\mathfrak{m}}[\dim \mathbf{G} + \dim \mathbf{U_P}] \cong \beta^*\overline{\mathcal{A}_{\mathfrak{m}}}[\dim \mathbf{P}]$. As we have already seen, by [Lus84b, Prop. 4.5], $\mathcal{K}_{\mathfrak{m}}$ is canonically isomorphic to $\hat{\alpha}^*\overline{\mathcal{A}_{\mathfrak{m}}}$.
Therefore, for any $a \in \mathbb{Z}$,

$$H^a(\mathcal{K}_{\mathfrak{m}})_h = H^a_c(\hat{\alpha}^{-1}(h) \cap \overline{X}, \overline{\mathcal{A}_{\mathfrak{m}}}).$$

However, by Lemma 3.2.9, the perverse sheaf $\mathcal{A}_{\mathfrak{m}}$ is clean and thus $\mathcal{A}_{\mathfrak{m}} = \mathcal{E}[\dim \Sigma]$ and

$$\overline{\mathcal{A}_{\mathfrak{m}}} = \overline{\mathcal{E}}[\dim(\mathbf{G}) + \dim \mathbf{U_P} - \dim \mathbf{P} + \dim \Sigma].$$

Consequently, $H^a(\mathcal{K}_\mathfrak{m})_h = H^a_c(\hat{\alpha}^{-1}(h)\cap X, \overline{\mathcal{E}}[d]) = H^{a+d}_c(\hat{\alpha}^{-1}(h)\cap X, \overline{\mathcal{E}})$.
Now the map ϕ_0 gives rise to an isomorphism $\overline{\phi}_0 : F^*\overline{\mathcal{E}} \xrightarrow{\sim} \overline{\mathcal{E}}$ which induces a map of finite order on the stalk of $\overline{\mathcal{E}}$ at any F-rational point of X. If we can show that $\dim \hat{\alpha}^{-1}(h)\cap X \leq \frac{1}{2}(a_0+d)$, then it follows that $\overline{\phi}_0$ acts on $H^{a_0+\dim \operatorname{supp}(\mathcal{A})}(\hat{\alpha}^{-1}(h)\cap X, \overline{\mathcal{E}})$ as $q^{(a_0+\dim\operatorname{supp}(\mathcal{A}))/2}$ times a map of finite order, and hence, so does $\phi_\mathfrak{m}$ on $H^{a_0}(\mathcal{K}_\mathfrak{m})_h$. According to the isomorphism $\bigoplus_\mathcal{A} H^i(\mathcal{A})_g \otimes V_\mathcal{A} \cong H^i(\mathcal{K}_\mathfrak{m})_g$, the map $\phi_\mathfrak{m}$ on $H^{a_0}(\mathcal{K}_\mathfrak{m})_g$ corresponds to the map $\phi_\mathcal{A}\otimes\sigma_\mathcal{A}$ on $H^{a_0}(\mathcal{A})_g\otimes V_\mathcal{A}$. We conclude that $\phi_\mathcal{A}$ acts on $H^{a_0}(\mathcal{A})_g$ as $q^{(a_0+d)/2}$ times a map of finite order.
We are left to show that $\dim \hat{\alpha}^{-1}(h)\cap X \leq \frac{1}{2}(a_0 + \dim\operatorname{supp}(\mathcal{A}))$. We first observe that

$$\hat{\alpha}^{-1}(h)\cap X = \{x\mathbf{P}\in \mathbf{G}/\mathbf{P} \mid x^{-1}hx \in \Sigma\mathbf{U_P}\} := X_h.$$

We rewrite $\Sigma = D_0 Z^\circ(\mathbf{L})$ where D_0 is a conjugacy class of $\mathbf{L}$ and we obtain, since $\mathbf{L}$ acts normally on $\mathbf{U_P}$,

$$X_h = \bigsqcup_{z\in Z^\circ(\mathbf{L})} \{x\mathbf{P}\in \mathbf{G}/\mathbf{P} \mid x^{-1}hx \in D_0 z\mathbf{U_P}\}.$$

Next, we verify that there is a finite number of $z\in Z^\circ(\mathbf{L})$ such that

$$X_{h,z} := \{x\mathbf{P}\in \mathbf{G}/\mathbf{P} \mid x^{-1}hx \in D_0 z\mathbf{U_P}\} \neq \varnothing.$$

Without loss of generality, we may assume that the semisimple part h_s of h is such that $D_0 = (h_s v)_\mathbf{L}$ for some unipotent element $v \in C_\mathbf{L}(h_s)$. Thus the number of $z\in Z^\circ(\mathbf{L})$ such that $X_{h,z}\neq\varnothing$ is smaller than $|Z^\circ(\mathbf{L})\cap h_s^{-1}(h_s)_\mathbf{G}|$. Let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$, we then have

$$Z^\circ(\mathbf{L})\cap h_s^{-1}(h_s)_\mathbf{G} \subseteq (h_s)_\mathbf{G}\cap \mathbf{T}.$$

The right-hand-side set is finite by a standard linear algebra argument: we may see everything sitting inside a general linear group and $\mathbf{T}$ as subgroup of the diagonal matrices. Conjugation preserves the eigenvalues and we can conclude.

By [Lus84b, Prop. 1.2b] or the rewriting in [Sho88, Thm. 1.4i], we know that for each $z\in Z^\circ(\mathbf{L})$,

$$\dim(\{y\mathbf{P} \mid y^{-1}hy\in D_0 z_i \mathbf{U_P}\}) \leq \frac{1}{2}(\dim\mathbf{G} - \dim\mathbf{L} + \dim(D_0) - \dim(D)).$$

Thus, since $\dim(\Sigma) = \dim D_0 + \dim Z^\circ(\mathbf{L})$, $\dim(\hat{\alpha}^{-1}(h)\cap X) \leq \frac{1}{2}(a_0+d)$ and this ends the proof of this lemma. ☐

If we further assume that $\mathcal{A}_D \cong \mathcal{L}[-a_0]$, then we define an isomorphism $\psi : F^*\mathcal{E}\xrightarrow{\sim}\mathcal{E}$ by the requirement that $q^{(a_0+d)/2}\psi$ corresponds to the map $\phi_\mathcal{A} : H^{a_0}(\mathcal{A}) \to H^{a_0}(\mathcal{A})$. Thanks to the above lemma, $\psi_h : \mathcal{E}_h \to \mathcal{E}_h$ is of finite order for any $h\in D^F$.
In particular, for any $h\in D^F$,

$$\chi_{\mathcal{A},\varphi_\mathcal{A}}(h) = q^{(a_0+d)/2} q^{(\dim\mathbf{G}-d)/2}\chi_{\mathcal{L},\psi}(h) = q^{(\dim(\mathbf{G})-\dim(D)-\dim(Z^\circ(\mathbf{L})))/2}\chi_{\mathcal{L},\psi}(h). \tag{3.2}$$

3.3.3 F-stable character sheaves and representation theory of finite reductive groups

Recall from Subsection 3.1.1 that for an F-stable $\mathbf{G}$-equivariant complex in $D^b_c(\mathbf{G}, \overline{\mathbb{Q}}_\ell)$, the characteristic function is a G-equivariant function, that is a class function. In this subsection, we finally see how characteristic functions of F-stable character sheaves relate to the ordinary characters of G.

From now on, we fix for each F-stable character sheaf an isomorphism $\varphi^{\mathcal{A}}$ satisfying the condition † (Notation 3.3.3). This might lead us to abuse notation and write *the* characteristic function of $\mathcal{A}$ and denote it by $\chi_{\mathcal{A}}$ for $\chi_{\mathcal{A},\varphi^{\mathcal{A}}}$ as in Definition 3.1.5.
Since $\mathcal{A}$ is $\mathbf{G}$-equivariant, the map $\chi_{\mathcal{A},\varphi^{\mathcal{A}}}$ is in fact a class function on G.

Theorem 3.3.5 ([LuCS5, Thm. 25.2], [Lus12, 3.10]). *The set of characteristic functions*

$$\{\chi_{\mathcal{A},\varphi^{\mathcal{A}}} \mid \mathcal{A} \in \hat{\mathbf{G}},\ \mathcal{A}\ F\text{-stable}\}$$

is an orthonormal basis for the space of class functions of G.

As a consequence, there are as many isomorphism classes of F-stable character sheaves of $\mathbf{G}$ as ordinary characters of G.

Therefore, Theorem 3.3.5 tells us that to compute ordinary characters, it suffices to solve the two following problems:

1. understand the change of basis from the set of characteristic functions of the F-stable character sheaves to the set of irreducible ordinary characters,
2. and compute the characteristic functions of the F-stable character sheaves.

Lusztig's conjecture and Shoji's theorem

We now consider the first problem. In his book about ordinary characters of finite reductive groups, Lusztig conjectured that the almost characters (see Remark 2.2.26) are in fact characteristic functions of certain F-stable perverse sheaves [Lus84a, 13.6]. Around ten years later, Shoji provided a proof of this conjecture in the case where the centre of $\mathbf{G}$ is connected.

Theorem 3.3.6 ([Sho95a, Thm. 5.7], [Sho95b, Thm. 3.2, Thm. 4.1]). *Assume that $Z(\mathbf{G})$ is connected. Let $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$ be a Kummer local system and $\mathcal{A} \in \hat{\mathbf{G}}^F_{\mathcal{L}}$. We may assume that $\mathcal{L}$ is F-stable. Let $\mathscr{F}$ be an F-stable family of* $\mathrm{irr}(W_{\mathcal{L}})$ *such that $x_{\mathcal{A}} \in \mathcal{M}(\bar{A}_{\mathscr{F}})$ under the injection of Proposition 3.3.2. Then, in the setting of Theorem 2.2.29 and Remark 2.2.30, there exists $x \in \mathcal{M}(\bar{A}_{\mathscr{F}} \subseteq \tilde{A}_{\mathscr{F}})$ and a root of unity ζ_x such that*

$$R_x = \zeta_x \chi_{\mathcal{A},\varphi_{\mathcal{A}}}.$$

Symmetrically, for every $x \in \mathcal{M}(\bar{A}_{\mathscr{F}} \subseteq \tilde{A}_{\mathscr{F}})$, there exists $\mathcal{A} \in \hat{\mathbf{G}}^F_{\mathcal{L}}$ such that $x_{\mathcal{A}} \in \mathcal{M}(\bar{A}_{\mathscr{F}})$ and

$$R_x = \zeta_x \chi_{\mathcal{A},\varphi_{\mathcal{A}}},$$

where ζ_x is a root of unity.

Therefore, thanks to Remark 2.2.30, to solve the first problem, one needs to determine the scalar ζ_x. Note that ζ_x depends on the choice of isomorphism $\varphi_{\mathcal{A}}$. This question has been settled by Shoji for classical groups in [Sho97] in good characteristic and for even characteristic in [Sho09]. For exceptional groups, this work spans over various articles by different authors. For type F_4, it is due to Marcelo–Shinoda [MS95], completed by Geck in [Gec19] and [Gec21b]. In these two papers, Geck also treated the groups of type E_6 ($p \neq 3$) and E_7 ($p \neq 2$). This was completed by Hetz in [Het19] and [Het22] who also considered groups of type E_8 and ${}^2E_6$ in [Het24].

Concerning the second problem, the values of the characteristic functions of F-stable character sheaves are known in principle thanks to a strategy presented by Lusztig. We advise the reader to read [Het23a, Section 3.2] for an exhaustive exposition. In the next chapter, we will focus on describing the character sheaves when restricted to a conjugacy class. To deduce results on ordinary characters, one still needs to understand how to keep track of the isomorphisms $\varphi_{\mathcal{A}}$ and how to compute the characteristic functions afterwards.

Chapter 4

Restricting character sheaves

In this chapter, we analyse the restriction of a character sheaf to a conjugacy class. This is a first step towards the computation of characteristic functions of character sheaves. This question was raised along the course of this PhD thesis in order to generalise methods developed by Brunat–Dudas–Taylor [BDT20] for unipotent characters to non-unipotent characters.

In the first section, we will focus on the restriction of character sheaves to unipotent conjugacy classes. In this case, the generalised Springer correspondence described by Lusztig [Lus84b] gives us a full and complete answer.
In the next two sections, we will treat the general case of a conjugacy class $(su)_{\mathbf{G}}$ with $s \in \mathbf{G}$ semisimple and $u \in C_{\mathbf{G}}(s)_{\text{uni}}$ by considering the translation of character sheaves by the element s and then restricting to $(u)_{C^\circ_{\mathbf{G}}(s)}$. We will first assume that s is central and then move on to an arbitrary semisimple element.
In both situations, we start by considering how the translation impacts the complex $\mathcal{K}_{\mathfrak{m}}$ for some cuspidal induction datum $\mathfrak{m} \in \mathfrak{M}^{\mathbf{G}}$. We will construct an isomorphism from the complex $(s^*(\mathcal{K}_{\mathfrak{m}}))_{(u)_{C^\circ_{\mathbf{G}}(s)}}$ to a direct sum of induction complexes $\mathcal{K}_{\mathfrak{m}'}$ of $C^\circ_{\mathbf{G}}(s)$ as done in [LuCS2, §8]. For the last step, we will study how this isomorphism behaves when we restrict it to a character sheaf $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$.

We keep the notation introduced in Chapter 1 and Chapter 3. So in particular, we assume that $\mathbf{G}$ is a connected reductive group with an $\mathbb{F}_q$-structure given by a Frobenius map F.

4.1 The importance of the unipotent variety: generalised Springer correspondence

Thanks to Proposition 2.3.3, we know the values of the ordinary characters of G at semisimple elements. For character sheaves, the theory developed by Lusztig gives us information about their values at unipotent conjugacy classes.

First and foremost, the induction series determines whether the restriction of a character sheaf to $\mathbf{G}_{\text{uni}}$ is zero or not. Indeed, let $\mathcal{A} \in \hat{\mathbf{G}}$ be a character sheaf in the induction series $\hat{\mathbf{G}}(\mathfrak{m})$, where $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$. We write $\mathcal{A}_{\mathfrak{m}} \coloneqq IC(\overline{\Sigma}, \mathcal{E})[\dim \Sigma]$ for the cuspidal sheaf on $\mathbf{L}$. By definition, there exists a parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$ with Levi decomposition $\mathbf{P} = \mathbf{U_P} \rtimes \mathbf{L}$ such that $\operatorname{Ind}_{\mathbf{L} \subseteq \mathbf{P}}^{\mathbf{G}}(\mathcal{A}_{\mathfrak{m}}) \cong \mathcal{K}_{\mathfrak{m}}$ and $\mathcal{A}$ is direct summand of $\mathcal{K}_{\mathfrak{m}}$. By [Lus86, 2.9], the support of $\mathcal{A}$ is completely determined by $\mathbf{P}$ and $\mathfrak{m}$:

$$\operatorname{supp}(\mathcal{A}) = \bigcup_{g \in \mathbf{G}} g \operatorname{supp}(\mathcal{A}_{\mathfrak{m}}) \mathbf{U_P} g^{-1}.$$

By Proposition 3.2.9, this means that

$$\{y \in Y \mid H^i(\mathcal{A})_y \neq 0 \text{ for some } i \in \mathbb{Z}\} \subseteq \bigcup_{g \in \mathbf{G}} g \Sigma \mathbf{U_P} g^{-1}. \tag{4.1}$$

Therefore,

$$\{y \in Y \mid H^i(\mathcal{A})_y \neq 0 \text{ for some } i \in \mathbb{Z}\} \cap \mathbf{G}_{\text{uni}} \subseteq \bigcup_{g \in \mathbf{G}} g \Sigma \mathbf{U_P} \cap \mathbf{G}_{\text{uni}} g^{-1}$$

and

$$\{y \in Y \mid H^i(\mathcal{A})_y \neq 0 \text{ for some } i \in \mathbb{Z}\} \cap \mathbf{G}_{\text{uni}} \neq \varnothing \iff \Sigma \cap \mathbf{G}_{\text{uni}} = \Sigma \cap \mathbf{L}_{\text{uni}} \neq \varnothing.$$

Therefore, to study the restriction of character sheaves to the unipotent varieties, we place ourselves in the setting of Remark 3.2.8 and assume that $\Sigma = CZ^\circ(\mathbf{L})$ with C a unipotent conjugacy class of $\mathbf{L}$ and $\mathcal{E} = \mathcal{E}_0 \boxtimes \mathcal{Z}$ with $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$ and $\mathcal{E}_0$ is an $\mathbf{L}$-equivariant irreducible local system on C.

4.1.1 Character sheaves restricted to unipotent conjugacy classes

In this subsection, we discuss the restriction of character sheaves to the unipotent variety $\mathbf{G}_{\text{uni}}$. We now fix a cuspidal induction datum $\mathfrak{m} = (\mathbf{L}, CZ^\circ(\mathbf{L}), \mathcal{E}_0 \boxtimes \mathcal{Z})$ where C is a unipotent class of $\mathbf{L}$, $\mathcal{E}_0$ is an irreducible $\mathbf{L}$-equivariant local system on C and $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$. Recall that $\mathcal{K}_{\mathfrak{m}}$ is semisimple and

$$\mathcal{K}_{\mathfrak{m}} \cong \bigoplus_{V \in \operatorname{Irr}(\operatorname{End}(\mathcal{K}_{\mathfrak{m}}))} \mathcal{A}_V \otimes V,$$

where V runs over a set of representatives of the isomorphism classes of irreducible $\operatorname{End}(\mathcal{K}_{\mathfrak{m}})$-modules and $\mathcal{A}_V \coloneqq \operatorname{Hom}_{\operatorname{End}(\mathcal{K}_{\mathfrak{m}})}(V, \mathcal{K}_{\mathfrak{m}})$ are the character sheaves in $\hat{\mathbf{G}}(\mathfrak{m})$. Thus,

$$(\mathcal{K}_{\mathfrak{m}}[-\dim Z^\circ(\mathbf{L})])_{\mathbf{G}_{\text{uni}}} \cong \bigoplus_{V \in \operatorname{Irr}(\operatorname{End}(\mathcal{K}_{\mathfrak{m}}))} (\mathcal{A}_V[-\dim Z^\circ(\mathbf{L})])_{\mathbf{G}_{\text{uni}}} \otimes V.$$

We want to understand the complexes $(\mathcal{A}_V[-\dim Z^\circ(\mathbf{L})])_{\mathbf{G}_{\text{uni}}}$ for $V \in \operatorname{Irr}(\operatorname{End}(\mathcal{K}_{\mathfrak{m}}))$. Note that it is not a priori clear if the complexes $(\mathcal{A}_V[-\dim Z^\circ(\mathbf{L})])_{\mathbf{G}_{\text{uni}}}$ are semisimple.

The generalised Springer correspondence (or the case when $\mathcal{Z} = \overline{\mathbb{Q}}_\ell$)

We assume that $\mathcal{Z} = \overline{\mathbb{Q}}_\ell$. By [Lus84b, 6.6.1], the complex $\mathcal{K}_\mathfrak{m}[-\dim Z^\circ(\mathbf{L})]_{\mathbf{G}_{\mathrm{uni}}} =: \mathcal{K}_1$ is also semisimple and therefore decomposes (see also [Sho88, 11.1.1]). Moreover, the natural map $\mathrm{End}(\mathcal{K}_\mathfrak{m}) \to \mathrm{End}(\mathcal{K}_1)$ is an isomorphism ([Lus84b, 6.8] or [Sho88, 11.3]). Therefore, we conclude that

$$\mathcal{K}_1 \cong \bigoplus_{V \in \mathrm{Irr}(\mathrm{End}(\mathcal{K}_\mathfrak{m}))} \mathcal{A}'_V \otimes V$$

where $\mathcal{A}'_V \cong (\mathcal{A}_V[-\dim Z^\circ(\mathbf{L})])_{\mathbf{G}_{\mathrm{uni}}}$ is an irreducible $\mathbf{G}$-equivariant perverse sheaf in the category $\mathscr{M}(\mathbf{G}_{\mathrm{uni}})$. The irreducible $\mathbf{G}$-equivariant perverse sheaves in $\mathscr{M}(\mathbf{G}_{\mathrm{uni}})$ are shifted intersection cohomology complexes over $\mathbf{G}$-stable locally closed smooth irreducible subvarieties of $\mathbf{G}_{\mathrm{uni}}$. Therefore, those varieties are unions of unipotent conjugacy classes of $\mathbf{G}$ and by irreducibility, they are simply unipotent conjugacy classes (since there are finitely many unipotent conjugacy classes). Hence, for each V in $\mathrm{Irr}(\mathrm{End}(\mathcal{K}_\mathfrak{m}))$,

$$(\mathcal{A}_V[-\dim Z^\circ(\mathbf{L})])_{\mathbf{G}_{\mathrm{uni}}} \cong \mathcal{A}'_V = IC(\overline{C}_V, \mathcal{E}_V)[\dim C_V],$$

where $C_V \in \mathrm{Ucl}(\mathbf{G})$ and $\mathcal{E}_V$ is an irreducible $\mathbf{G}$-equivariant local system on C_V.
Recall that $\mathrm{End}(\mathcal{K}_\mathfrak{m}) \cong \overline{\mathbb{Q}}_\ell[W_\mathfrak{m}] = \overline{\mathbb{Q}}_\ell[W_\mathbf{L}]$, c.f. Proposition 3.2.17 and its proof. Thus, we have defined an injective map

$$\mathfrak{Spr}_{C,\mathcal{E}_0} : \mathrm{Irr}(W_\mathbf{L}) \to \{(C', \mathcal{E}') \mid C' \in \mathrm{Ucl}(\mathbf{G}),\ \mathcal{E}' \text{ irreducible } \mathbf{G}\text{-equivariant local system on } C'\}.$$

Note that this map depends on C and $\mathcal{E}_0$.
Let us denote by $\mathfrak{N}^\mathbf{G}$ a set of representatives of the $\mathbf{G}$-conjugacy classes of all pairs $(C', \mathcal{E}')$ where C' is a unipotent class on $\mathbf{G}$ and $\mathcal{E}'$ is an irreducible local system on C'. We write $\mathfrak{N}_0^\mathbf{G}$ for the subset of $\mathfrak{N}^\mathbf{G}$ consisting of pairs $(C', \mathcal{E}')$ such that the induction datum$(\mathbf{G}, C'Z^\circ(\mathbf{G}), \mathcal{E}' \boxtimes \overline{\mathbb{Q}}_\ell)$ is cuspidal for $\mathbf{G}$. So in other words, for each Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ and each pair $(C, \mathcal{E}_0) \in \mathfrak{N}_0^\mathbf{L}$, there is an injective map

$$\mathfrak{Spr}_{C,\mathcal{E}_0} : \mathrm{Irr}(W_\mathbf{L}) \hookrightarrow \mathfrak{N}^\mathbf{G}.$$

In the other direction, if $(C', \mathcal{E}') \in \mathfrak{N}^\mathbf{G}$, then there are a unique Levi subgroup $\mathbf{L}$ and a unique pair $(C, \mathcal{E}_0) \in \mathfrak{N}_0^\mathbf{L}$ (up to $\mathbf{G}$-conjugation) such that $(C', \mathcal{E}')$ belongs to the image of $\mathfrak{Spr}_{C,\mathcal{E}_0}$, see [Lus84b, Prop. 6.3]. In this way, Lusztig has constructed a bijective map

$$\mathfrak{Spr} : \bigsqcup_{\mathbf{L}} \bigsqcup_{\mathfrak{n} \in \mathfrak{N}_0^\mathbf{L}} \mathrm{Irr}(W_\mathbf{L}) \xrightarrow{\sim} \mathfrak{N}^\mathbf{G},$$

where $\mathbf{L}$ runs over the Levi subgroup of $\mathbf{G}$ up to conjugation ([Lus84b, Thm. 6.5]). We call this map the **generalised Springer correspondence**. This theorem was inspired by results of [BM81] and [Spr76] who only considered the special case where the Levi subgroup is the maximal torus $\mathbf{T}_0$. In that case it is called the (ordinary) **Springer correspondence**.

Remark 4.1.1. In its original definition by Springer, the map differs from the one defined by Lusztig by tensoring by the sign character of W.

The generalised Springer correspondence has been completely determined. Thanks to [Lus84b, 10.1], it suffices to consider the case where $\mathbf{G}$ is simple and simply connected and then to proceed inductively, assuming that the cuspidal pairs for all proper Levi subgroups are known. This work was started by Lusztig in [Lus84b], carried on by Lusztig–Spaltenstein for classical groups [LS85] and Spaltenstein for most exceptional groups [Spa85], and concluded by Lusztig in [Lus19] and Hetz [Het23b] for the leftover cases in E_8.
These results are accessible in CHEVIE [Mic15].

Remark 4.1.2. Let us now briefly consider the characteristic functions. Assume that the character sheaf $\mathcal{A} = \mathcal{A}_V$ is the direct summand of the semisimple sheaf $\mathcal{K}_\mathfrak{m}$ corresponding to the pair $(C', \mathcal{E}') \in \mathfrak{N}^{\mathbf{G}}$ under the generalised Springer correspondence. Recall that we may choose a representative $u_{C'}$ of C' such that F acts trivially on $A_{\mathbf{G}}(u_{C'})$. Moreover, by Remark 3.1.3, we assume that $\mathcal{E}'$ corresponds to the irreducible character $\phi \in \operatorname{irr}(A_G(u_{C'}))$. We can always choose an isomorphism $\varphi : F^*\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ such that for any $g \in \mathbf{G}$ with ${}^g u_{C'} \in G$, the characteristic function takes values

$$\chi_{\mathcal{F},\varphi}({}^g u_{C'}) = q^{\frac{1}{2}(\dim \mathbf{G} - \dim(C') - \dim Z^\circ(\mathbf{L}))} \phi(g^{-1}F(g) C^\circ_{\mathbf{G}}(u_{C'})).$$

The normalisation follows from Subsection 3.3.2 and in particular Equation 3.2.

The case when $\mathcal{Z}$ is arbitrary

The generalised Springer correspondence combined with Remark 3.2.19 allows us to describe the restriction of any character sheaf to the unipotent variety $\mathbf{G}_{\text{uni}}$.
We now assume that $\mathcal{Z}$ is not necessarily the constant perverse sheaf, so the cuspidal induction datum $\mathfrak{m}$ has the following shape: $\mathfrak{m} = (\mathbf{L}, CZ^\circ(\mathbf{L}), \mathcal{E}_0 \boxtimes \mathcal{Z})$ where C is a unipotent class of the Levi subgroup $\mathbf{L}$, $\mathcal{E}_0$ is an irreducible $\mathbf{L}$-equivariant local system on C and $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$. From the isomorphism in Remark 3.2.19, we get for any $V \in \operatorname{Irr}(W_\mathfrak{m})$,

$$\left(\mathcal{A}_V^{\mathcal{Z}}\right)_{\mathbf{G}_{\text{uni}}} \cong \left(\mathcal{A}^{\overline{\mathbb{Q}}_\ell}_{\operatorname{Ind}_{W_\mathfrak{m}}^{W_\mathbf{L}}(V)}\right)_{\mathbf{G}_{\text{uni}}}.$$

Here $\mathcal{A}_V^{\mathcal{Z}}$ is a constituent of $\mathcal{K}_\mathfrak{m}$. Using the generalised Springer correspondence, we then conclude:

$$\left(\mathcal{A}_V^{\mathcal{Z}}\right)_{\mathbf{G}_{\text{uni}}}[-\dim Z^\circ(\mathbf{L})] \cong \bigoplus_{V' \in \operatorname{Irr}(W_\mathbf{L})} \langle \operatorname{Ind}_{W_\mathfrak{m}}^{W_\mathbf{L}}(V), V' \rangle IC(\overline{C}_{V'}, \mathcal{E}_{V'})[\dim C_{V'}].$$

Remark 4.1.3. Note that this correspondence depends on C and $\mathcal{E}_0$ as well as the choice of the isomorphism fixed in Remark 3.2.19.

As a particular case, we consider the restriction of the character sheaf to a certain unipotent class $C_0 \in \operatorname{Ucl}(\mathbf{G})$. We decompose the previous sum into three parts: the first one with the $V' \in \operatorname{Irr}(W_\mathbf{L})$ such that $C_{V'} = C_0$, the second one with the $V' \in \operatorname{Irr}(W_\mathbf{L})$

such that $C_{V'} \neq C_0$ and $C_0 \subseteq \overline{C_{V'}}$, and the rest of the sum. To rephrase it, we define the natural partial order on the unipotent classes. For $C, C' \in \mathrm{Ucl}(\mathbf{G})$, we write

$$C \leq C' \iff C \subseteq \overline{C'}.$$

The previous isomorphism yields

$$\begin{aligned}
\left(\mathcal{A}_V^{\mathcal{Z}}\right)_{C_0}[-\dim Z^\circ(\mathbf{L})] &\cong \bigoplus_{V' \in \mathrm{Irr}(W_{\mathbf{L}})} \langle \mathrm{Ind}_{W_{\mathfrak{m}}}^{W_{\mathbf{L}}}(V), V' \rangle \left(IC(\overline{C}_{V'}, \mathcal{E}_{V'})[\dim C_{V'}]\right)_{C_0} \\
&\cong \bigoplus_{\substack{V' \in \mathrm{Irr}(W_{\mathbf{L}}),\\ C_{V'} = C_0}} \langle \mathrm{Ind}_{W_{\mathfrak{m}}}^{W_{\mathbf{L}}}(V), V' \rangle \left(IC(\overline{C}_{V'}, \mathcal{E}_{V'})[\dim C_{V'}]\right)_{C_0} \\
&\bigoplus \bigoplus_{\substack{V' \in \mathrm{Irr}(W_{\mathbf{L}}),\\ C_0 \not\leq C_{V'}}} \langle \mathrm{Ind}_{W_{\mathfrak{m}}}^{W_{\mathbf{L}}}(V), V' \rangle \left(IC(\overline{C}_{V'}, \mathcal{E}_{V'})[\dim C_{V'}]\right)_{C_0} \\
&\bigoplus \bigoplus_{\substack{V' \in \mathrm{Irr}(W_{\mathbf{L}}),\\ C_0 \neq C_{V'}, C_0 \leq C_{V'}}} \langle \mathrm{Ind}_{W_{\mathfrak{m}}}^{W_{\mathbf{L}}}(V), V' \rangle \left(IC(\overline{C}_{V'}, \mathcal{E}_{V'})[\dim C_{V'}]\right)_{C_0}.
\end{aligned}$$

The description of the intersection cohomology complex (see Subsection 3.1.1) implies that

$$\left(IC(\overline{C}_{V'}, \mathcal{E}_{V'})[\dim C_{V'}]\right)_{C_0} = 0$$

if $C_0 \not\leq C_{V'}$. Moreover, if $C_{V'} = C_0$,

$$\left(IC(\overline{C}_{V'}, \mathcal{E}_{V'})[\dim C_{V'}]\right)_{C_0} = \mathcal{E}_{V'}[\dim C_{V'}].$$

Therefore, we rewrite

$$\begin{aligned}
\left(\mathcal{A}_V^{\mathcal{Z}}\right)_{C_0}[-\dim Z^\circ(\mathbf{L})] &\cong \bigoplus_{\substack{V' \in \mathrm{Irr}(W_{\mathbf{L}}),\\ C_{V'} = C_0}} \langle \mathrm{Ind}_{W_{\mathfrak{m}}}^{W_{\mathbf{L}}}(V), V' \rangle \mathcal{E}_{V'}[\dim C_{V'}] \\
&\bigoplus \bigoplus_{\substack{V' \in \mathrm{Irr}(W_{\mathbf{L}}),\\ C_0 \neq C_{V'}, C_0 \leq C_{V'}}} \langle \mathrm{Ind}_{W_{\mathfrak{m}}}^{W_{\mathbf{L}}}(V), V' \rangle \left(IC(\overline{C}_{V'}, \mathcal{E}_{V'})[\dim C_{V'}]\right)_{C_0}.
\end{aligned}$$

Thus, if C_0 is "big enough", the restriction of the character sheaf $\mathcal{A}_V^{\mathcal{Z}}$ to C_0 is zero or a (non necessarily irreducible) local system, i.e., something relatively easy to understand. We will see in the next subsection what "big enough" means.

Deducing information about the characteristic function from the structure of a character sheaf is not necessarily clear or natural. When F is split and $\mathbf{L}$ is contained in an F-stable parabolic subgroup of $\mathbf{G}$, Lusztig gives an answer in [Lus86]. It was later extended independently by Taylor in [Tay14] and Digne–Lehrer–Michel in [DLM14].

4.1.2 The unipotent support of character sheaves

For a given character sheaf $\mathcal{A} \in \hat{\mathbf{G}}$, we would like to define in some way "the biggest" conjugacy class D of $\mathbf{G}$ such that $\mathcal{A}_{|D} \neq 0$. Since so far in this thesis, we only really know the restriction of a character sheaf to a unipotent class, this will have to appear in the definition. Hence, the "biggest" must be an adjective concerning the unipotent part of D. On the other hand, we cannot blindly copy the definition of unipotent support of ordinary characters as stated in Theorem 2.3.4. Indeed, it might happen that $\mathcal{A}_{\mathbf{G}_{\mathrm{uni}}} = 0$, and thus that there is no $C \in \mathrm{Ucl}(\mathbf{G})$ such that $\mathcal{A}_C \neq 0$. This leads us to the following definition.

Definition 4.1.4 ([Lus92, 10.6])**.** Let $\mathscr{G}$ be a family of $\hat{\mathbf{G}}_{\mathcal{L}}$ for $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$. The **unipotent support** $C_{\mathscr{G}}$ of $\mathscr{G}$ is the unique unipotent class of $\mathbf{G}$ satisfying the following properties:

1. for any character sheaf $\mathcal{A} \in \mathscr{G}$ and for any conjugacy class D of $\mathbf{G}$ with unipotent part $C' \in \mathrm{Ucl}(\mathbf{G})$ such that $C_{\mathscr{G}} \neq C'$ and $\dim C_{\mathscr{G}} \leq \dim C'$, the restriction $\mathcal{A}_{|_D} = 0$, and

2. there exists a conjugacy class D of $\mathbf{G}$ and a character sheaf $\mathcal{A} \in \mathscr{G}$ such that the unipotent part of D is $C_{\mathscr{G}}$ and $\mathcal{A}_{|_D} \neq 0$.

We also say that $C_{\mathscr{G}}$ is the unipotent support of any character sheaf $\mathcal{A} \in \mathscr{G}$.

Since character sheaves are $\mathbf{G}$-equivariant, it is clear that such a unipotent class exists for each character sheaf of a family $\mathscr{G}$. Lusztig showed in [Lus92, Thm. 10.7] that such a unipotent class is unique and gave another characterisation assuming some conditions on p.

Description of the unipotent support

We unravel this description under the assumption that $Z(\mathbf{G})$ is connected and that p is acceptable for $\mathbf{G}$ (see [Tay16, Def. 6.1]). If $\mathbf{G}$ is a simple exceptional group of adjoint type, then any good prime is acceptable. Let $\mathscr{G}$ be a family of $\hat{\mathbf{G}}_{\mathcal{L}}$ for $\mathcal{L}$ an irreducible Kummer local system on $\mathbf{T}_0$. Theorem 3.1.12 allows us to associate to $\mathscr{G}$ a unique family $\mathscr{F}$ of the Weyl group $W_{\mathcal{L}} = W_{\mathcal{L}}^{\circ}$. We fix $\psi \in \mathrm{irr}(W_{\mathcal{L}})$, the unique special character of $\mathscr{F}$ (see Proposition 2.2.21). We then consider the j-induction $\psi' = j_{W_{\mathcal{L}}}^{W}(\psi)$ of ψ as in Proposition 2.2.20. Lastly, we let $(C', \mathcal{E}') \in \mathfrak{N}^{\mathbf{G}}$ be the image of ψ' under the Springer correspondence. Lusztig showed in [Lus92, Thm. 10.7] that C' is then the unipotent support of $\mathscr{G}$. Moreover, the local system $\mathcal{E}'$ is trivial. This was later generalised to p good by Taylor [Tay13]
If $Z(\mathbf{G})$ is not connected and $W_{\mathcal{L}}$ is not necessarily a Weyl group, then this construction still works, but one needs at first to extend the definition of special character to $W_{\mathcal{L}}$.

We make a few remarks concerning the unipotent support. Firstly, we come back to the setting of the previous subsection. We assume that $\mathcal{A}$ belongs to the induction series $\hat{\mathbf{G}}(\mathfrak{m})$ with $\mathfrak{m} = (\mathbf{L}, CZ^{\circ}(\mathbf{L}), \mathcal{E}_0 \boxtimes \mathcal{Z}) \in \mathfrak{M}^{\mathbf{G}}$ where C is a unipotent class of the Levi subgroup $\mathbf{L}$, $\mathcal{E}_0$ is an irreducible local system on C and $\mathcal{Z} \in \mathcal{S}(Z^{\circ}(\mathbf{L}))$. Then $\mathcal{A} = \mathcal{A}_V^{\mathcal{Z}}$ for some $V \in \mathrm{Irr}(W_{\mathfrak{m}})$. Let $C_{\mathcal{A}}$ be the unipotent support of $\mathcal{A}$. The definition of the unipotent support implies that the restriction of $\mathcal{A}$ to $C_{\mathcal{A}}$ is a sum of irreducible local systems:

$$\left(\mathcal{A}_V^{\mathcal{Z}}\right)_{C_{\mathcal{A}}}[-\dim Z^\circ(\mathbf{L})] \cong \bigoplus_{\substack{V' \in \operatorname{Irr}(W_{\mathbf{L}}),\\ C_{V'}=C_{\mathcal{A}}}} \langle \operatorname{Ind}_{W_{\mathrm{m}}}^{W_{\mathbf{L}}}(V), V'\rangle \mathcal{E}_{V'}[\dim C_{\mathcal{A}}].$$

In the **principal series** case, that is if $\mathbf{L} = \mathbf{T}_0$, and hence $\mathcal{Z} \in \mathcal{S}(\mathbf{T}_0)$, we conclude that

$$\left(\mathcal{A}_V^{\mathcal{Z}}\right)_{C_{\mathcal{A}}}[-\dim \mathbf{T}_0] \cong \bigoplus_{\substack{V' \in \operatorname{Irr}(W),\\ C_{V'}=C_{\mathcal{A}}}} \langle \operatorname{Ind}_{W_{\mathcal{Z}}}^{W}(V), V'\rangle \mathcal{E}_{V'}[\dim C_{\mathcal{A}}].$$

Here $W_{\mathcal{Z}}$ is the relative Weyl group of the Kummer local system $\mathcal{Z} \in \mathcal{S}(\mathbf{T}_0)$ (see below Definition 3.1.7). Therefore, if $V \in \operatorname{Irr}(W_{\mathcal{Z}})$ is a module corresponding to a special character of $W_{\mathcal{Z}}$, then the trivial local system $\overline{\mathbb{Q}}_\ell$ appears in the sum and $(\mathcal{A}_V^{\mathcal{Z}})_{C_{\mathcal{A}}} \neq 0$.

Lastly, let us consider the particular case when $\mathscr{G}$ is a family of unipotent character sheaves. Since the Springer correspondence is injective, we observe that two distinct families of unipotent character sheaves have different unipotent supports. Therefore, we obtain an analogous statement as Theorem 2.3.5 for ordinary unipotent characters.

Remark 4.1.5. The similarities between the unipotent support of character sheaves and the unipotent support of characters is not due to chance. In fact, the above description through the Springer correspondence also holds for the unipotent support of characters (see for instance [GM00, Thm. 3.7]) and leads to Theorem 2.3.5.

Special conjugacy classes

By a semantic shift, we say that a unipotent class $C \in \operatorname{Ucl}(\mathbf{G})$ is **special** if there is special character $\phi \in \operatorname{irr}(W)$ such that $\mathfrak{Spr}(\phi) = (C, \overline{\mathbb{Q}}_\ell)$. In other words, a unipotent class is special if and only if it is the unipotent support of a unipotent character sheaf. Moreover, we get a bijection between the families of $\operatorname{irr}(W)$ and the special unipotent classes of $\mathbf{G}$.

More generally, we say that an element $g \in \mathbf{G}$ with Jordan decomposition $g = su = us$ (with $s \in \mathbf{G}$ semisimple, $u \in \mathbf{G}_{\mathrm{uni}}$) is **special** if u is an element of a special unipotent class of $C^\circ_{\mathbf{G}}(s)$. As a result, we obtain a new parameterisation of the character sheaves. To begin with, recall that the Kummer local system $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$ is defined thanks to an indivisible pair $(\lambda, n) \in X \times \mathbb{N}$ (see below Definition 3.1.7) which itself corresponds to a semisimple conjugacy class $(s)_{\mathbf{G}^*}$ in the dual group $\mathbf{G}^*$ of $\mathbf{G}$. Furthermore, we partition $\hat{\mathbf{G}}_s = \hat{\mathbf{G}}_{\mathcal{L}}$ into families indexed by the families of $\operatorname{irr}(W_{\mathcal{L}})$ ($W_{\mathcal{L}} \cong W_s$), c.f. Theorem 3.1.12. Here W_s is the Weyl group of $C^\circ_{\mathbf{G}^*}(s)$. Thus we obtain

$$\hat{\mathbf{G}}_s = \bigsqcup_{\mathscr{F}} \hat{\mathbf{G}}_{s,\mathscr{F}}$$

where $\mathscr{F}$ runs over the families of $\operatorname{irr}(W_s)$ and $\hat{\mathbf{G}}_{s,\mathscr{F}}$ is the family of $\hat{\mathbf{G}}_s$ corresponding to $\mathscr{F}$. Lastly, the Springer correspondence associates to each family $\mathscr{F} \in \operatorname{irr}(W_s)$ a special unipotent class $(u)_{C^\circ_{\mathbf{G}^*}(s)}$, whence a special element $g_{\mathscr{F}} = su \in \mathbf{G}^*$. Therefore, setting $\hat{\mathbf{G}}_{g_{\mathscr{F}}} \coloneqq \hat{\mathbf{G}}_{s,\mathscr{F}}$ the partition of character sheaves becomes

$$\hat{\mathbf{G}}_s = \bigsqcup_{\mathscr{F}} \hat{\mathbf{G}}_{g_{\mathscr{F}}}$$

where $\mathscr{F}$ runs over the families of $\mathrm{irr}(W_s)$.
On the other hand, let us consider a special element $g = su \in \mathbf{G}^*$ with $s \in \mathbf{G}^*$ semisimple and $u \in C_{\mathbf{G}^*}(s)$ unipotent. To the special unipotent class $(u)_{C^\circ_{\mathbf{G}^*}(s)}$ is associated a unique family $\mathscr{F}_g$ of $\mathrm{irr}(W_s)$ and it satisfies $g = g_{\mathscr{F}_g}$ for $g_{\mathscr{F}_g}$ as constructed above. Thus, setting $\hat{\mathbf{G}}_g \coloneqq \hat{\mathbf{G}}_{s,\mathscr{F}_g}$ allows us to write

$$\hat{\mathbf{G}} = \bigsqcup_{g} \hat{\mathbf{G}}_g,$$

where g runs over a set of representatives of the conjugacy classes of special elements in $\mathbf{G}^*$.

Notation 4.1.6. Let $g \in \mathbf{G}^*$ be special, we denote by C_g the unipotent support of the characters in the family $\mathcal{F}_g$.

Remark 4.1.7. We obtain a similar decomposition for ordinary characters assuming that the centre $Z(\mathbf{G})$ is connected and p is good for $\mathbf{G}$. For $g = su \in \mathbf{G}^*$ as before, if g is F^*-stable, then we write $\mathrm{irr}(G)_g$ for the family of ordinary characters in $\mathscr{E}(G,s)$ corresponding to the family $\mathscr{F}$ of $\mathrm{irr}(W_s)$ and we get

$$\mathrm{irr}(G) = \bigsqcup_{g} \mathrm{irr}(G)_g,$$

where g runs over a set of representatives of the F^*-stable conjugacy classes of special elements in $\mathbf{G}^*$.

To summarise, the generalised Springer correspondence allows us to get a good understanding of character sheaves when restricted to the unipotent variety $\mathbf{G}_{\mathrm{uni}}$ and an excellent one when restricted to their unipotent support. In the next sections, we will consider the restriction of character sheaves to any conjugacy class.

4.2 Translation of character sheaves

Let $\mathcal{A} \in \hat{\mathbf{G}}$ be a character sheaf, $s \in \mathbf{G}$ be a semisimple element and $u \in C_{\mathbf{G}}(s)$ a unipotent element. To compute the characteristic function of $\mathcal{A}$ at $su \in \mathbf{G}$, we need to understand the stalks $H^i(\mathcal{A})_{su}$ for $i \in \mathbb{Z}$. Equivalently, we could look at $H^i(s^*\mathcal{A})_u$ where we abuse notation and write $s : \mathbf{G} \to \mathbf{G}$, $g \mapsto sg$, for the translation by s on the left. This brings us back to looking at the restriction of $s^*\mathcal{A}$ to the unipotent elements.

For the rest of this section, unless precised otherwise, we fix an element $s \in \mathbf{T}_0$ as well as the translation $s : \mathbf{G} \to \mathbf{G}$, $g \mapsto sg$, for $g \in \mathbf{G}$.

4.2.1 Translation and families of character sheaves

For any $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$, we show in this subsection that the families of $\hat{\mathbf{G}}_{\mathcal{L}}$ are stable for the translation by s, as stated in [LuCS4, 17.17]. We first consider the translation of local systems.

Lemma 4.2.1. *Let $\mathbf{T} \subseteq \mathbf{G}$ be any torus and consider $\mathcal{L} \in \mathcal{S}(\mathbf{T})$. Then for any $s \in \mathbf{T}$,*

$$s^*\mathcal{L} \cong \mathcal{L}.$$

Proof. Fix $s \in \mathbf{T}$. Let $\lambda \in \mathrm{Hom}(\mathbf{T}, k^\times)$ be a character of $\mathbf{T}$ and $n \in \mathbb{N}$ an integer coprime to p such that $\mathcal{L} = \lambda^*\mathcal{E}_{n,\mathrm{j}}$. For any $c \in \overline{\mathbb{Q}}_\ell$, we write $m_c : \overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell, x \mapsto cx$ for the multiplication by c. Then, $\lambda \circ s = m_{\lambda(s)} \circ \lambda$, whence

$$s^*\lambda^*\mathcal{E}_{n,\mathrm{j}} = \lambda^* m^*_{\lambda(s)}\mathcal{E}_{n,\mathrm{j}}.$$

Recall that for any $m \in \mathbb{N}$, the Kummer local system $(\lambda^m)^*\mathcal{E}_{nm,\mathrm{j}}$ is isomorphic to $\mathcal{L}$. In particular, if m denotes the order of s, our claim holds since

$$s^*\mathcal{L} = s^*\lambda^*\mathcal{E}_{n,\mathrm{j}} \cong s^*(\lambda^m)^*\mathcal{E}_{nm,\mathrm{j}} = (\lambda^m)^* m^*_{\lambda^m(s)}\mathcal{E}_{nm,\mathrm{j}} = (\lambda^m)^*\mathcal{E}_{nm,\mathrm{j}} \cong \mathcal{L}. \qquad \square$$

We now consider the induction series.

Lemma 4.2.2. *Let $\mathcal{L} \in \mathcal{S}(\mathbf{T}_0)$. For any $w \in W_{\mathcal{L}}$, and any $s \in Z(\mathbf{G})$,*

$$s^*\bar{\mathcal{K}}^{\mathcal{L}}_w \cong \bar{\mathcal{K}}^{\mathcal{L}}_w.$$

Therefore, if $\mathcal{A} \in \hat{\mathbf{G}}_{\mathcal{L}}$, then $s^*\mathcal{A} \in \hat{\mathbf{G}}_{\mathcal{L}}$. Moreover, the families of character sheaves in $\hat{\mathbf{G}}_{\mathcal{L}}$ are stable under translation by s.

Proof. Fix $w \in W_{\mathcal{L}}$ and $s \in \mathbf{T}_0$. We show that $s^*\bar{\mathcal{K}}^{\mathcal{L}}_w \cong \bar{\mathcal{K}}^{(w^{-1}sw)^*\mathcal{L}}_w$ and then conclude using Lemma 4.2.1. Recall the definition of $\bar{\mathcal{K}}^{\mathcal{L}}_w$. We have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbf{G} & \xleftarrow[\alpha]{} & \mathbf{G}\times\mathbf{G} & \xrightarrow{\beta} & \mathbf{G}\times_{\mathbf{B}_0}\mathbf{G} & \xrightarrow{\gamma} & \mathbf{G} \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \mathrm{id}\times s} & & \downarrow{\scriptstyle \mathrm{id}\times_{\mathbf{B}_0} s} & & \downarrow{\scriptstyle s} \\
\mathbf{G} & \xleftarrow[\alpha]{} & \mathbf{G}\times\mathbf{G} & \xrightarrow[\beta]{} & \mathbf{G}\times_{\mathbf{B}_0}\mathbf{G} & \xrightarrow[\gamma]{} & \mathbf{G}
\end{array}$$

By definition, $\bar{\mathcal{K}}^{\mathcal{L}}_w = \gamma_*\tilde{\mathcal{A}}^{\mathcal{L}}_w$. By base change,

$$s^*\bar{\mathcal{K}}^{\mathcal{L}}_w = s^*\gamma_*\tilde{\mathcal{A}}^{\mathcal{L}}_w \cong \gamma_*(\mathrm{id}\times_{\mathbf{B}_0}s)^*\tilde{\mathcal{A}}^{\mathcal{L}}_w.$$

Showing the existence of an isomorphism between $s^*\bar{\mathcal{K}}^{\mathcal{L}}_w$ and $\bar{\mathcal{K}}^{(w^{-1}sw)^*\mathcal{L}}_w$ reduces then to exhibiting an isomorphism between $(\mathrm{id}\times_{\mathbf{B}_0}s)^*\tilde{\mathcal{A}}^{\mathcal{L}}_w$ and $\tilde{\mathcal{A}}^{(w^{-1}sw)^*\mathcal{L}}_w$.
By definition, $\tilde{\mathcal{A}}^{(w^{-1}sw)^*\mathcal{L}}_w$ is the only irreducible perverse sheaf on $\mathbf{G}\times_{\mathbf{B}_0}\mathbf{G}$ up to isomorphism such that $\beta^*\tilde{\mathcal{A}}^{(w^{-1}sw)^*\mathcal{L}}_w \cong \overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}^{(w^{-1}sw)^*\mathcal{L}}_w$. Thus, we need to show that the

complex $\beta^*(\mathrm{id}\times_{\mathbf{B}_0} s)^*\tilde{\mathcal{A}}_w^{\mathcal{L}}$ is isomorphic to $\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_w^{(w^{-1}sw)^*\mathcal{L}}$. To do so, we consider the following sequence of isomorphisms

$$\beta^*(\mathrm{id}\times_{\mathbf{B}_0} s)^*\tilde{\mathcal{A}}_w^{\mathcal{L}} = (\mathrm{id}\times s)^*\beta^*\tilde{\mathcal{A}}_w^{\mathcal{L}} \cong (\mathrm{id}\times s)^*\left(\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_w^{\mathcal{L}}\right) = \overline{\mathbb{Q}}_\ell \boxtimes s^*\mathcal{A}_w^{\mathcal{L}}.$$

Now, by the characterisation of shifted intersection cohomology complexes, we observe that $s^*\mathcal{A}_w^{\mathcal{L}} = IC(\overline{\mathbf{G}_w}, s^*\operatorname{pr}_w^*(\mathcal{L}))[\dim \mathbf{G}_w] = \mathcal{A}_w^{(w^{-1}sw)^*\mathcal{L}}$ since $s^*\operatorname{pr}_w^*(\mathcal{L}) = \operatorname{pr}_w^*(w^{-1}sw)^*\mathcal{L}$. Therefore, there exists an isomorphism between $s^*\bar{\mathcal{K}}_w^{\mathcal{L}}$ and $\bar{\mathcal{K}}_w^{(w^{-1}sw)^*\mathcal{L}}$. Since $(w^{-1}sw)^*\mathcal{L} \cong \mathcal{L}$ by Lemma 4.2.1, we deduce that $s^*\bar{\mathcal{K}}_w^{\mathcal{L}} \cong \bar{\mathcal{K}}_w^{\mathcal{L}}$.
The last statement follows from Theorem 3.1.12. □

Even though the translation preserves the families of character sheaves, it might not fix them individually. It is in general not clear how the labelling of the families is impacted by the translation. However, Lusztig described this phenomenon in the particular case when the translation is by a central element.
Let $\mathcal{A} \in \hat{\mathbf{G}}_{\mathcal{L}}$ and consider $z^*\mathcal{A}$ when $z \in Z(\mathbf{G})$. Theorem 3.1.12 might also be stated when the centre $Z(\mathbf{G})$ is not connected, see [LuCS5, Thm. 23.1]. In particular, the character sheaf $\mathcal{A}$ belongs to a family $\mathscr{G}$ of $\hat{\mathbf{G}}_{\mathcal{L}}$, to which we associate a finite group $\bar{A}_{\mathscr{G}}$ such that there is a bijection between $\mathscr{G}$ and $\mathcal{M}(\bar{A}_{\mathscr{G}})$. If $\mathcal{A}$ is sent to the class $[x,\sigma] \in \mathcal{M}(\bar{A}_{\mathscr{G}})$, then $z^*\mathcal{A}$ is sent to $[x, \sigma\otimes\sigma_z] \in \mathcal{M}(\bar{A}_{\mathscr{G}})$ where σ_z is a character of $\bar{A}_{\mathscr{G}}$ depending only on z. Moreover, for any $z' \in Z^\circ(\mathbf{G})$, $\sigma_z = \sigma_{zz'}$ and $\sigma_{z'}$ is trivial.

4.2.2 Translation and induction series

Character sheaves are also partitioned into induction series. Let $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ for $\mathfrak{m} \in \mathfrak{M}^{\mathbf{G}}$, then $\mathcal{A} = \mathcal{A}_V$ where V is an irreducible module of $\operatorname{End}(\mathcal{K}_{\mathfrak{m}})$. In this subsection, we explain to which induction series $s^*\mathcal{A}_V$ belongs.

First recall that we have assumed that $\mathbf{T}_0 \subseteq \mathbf{L}$ and that we have fixed $s \in \mathbf{T}_0$, whence $s^{-1}\mathbf{L} = \mathbf{L}$. The first problem we encounter is that $s^*\mathcal{E}$ might not be $\mathbf{L}$-equivariant. Thus, $s^*\mathfrak{m} = (\mathbf{L}, s^{-1}\Sigma, s^*\mathcal{E})$ does not define an induction datum and we cannot hope for an isomorphism between $s^*\mathcal{K}_{\mathfrak{m}}$ and $\mathcal{K}_{s^*\mathfrak{m}}$ in general. However, it does work if $s \in Z(\mathbf{L})$ and this is the case we consider now. In the next section, we will investigate what happens when we translate by an arbitrary element $s \in \mathbf{T}_0$.

Lemma 4.2.3. *Let $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$. For any $z \in Z(\mathbf{G})$,*

$$z^*\mathcal{K}_{\mathfrak{m}} \cong \mathcal{K}_{z^*\mathfrak{m}},$$

where $z^*\mathfrak{m} = (\mathbf{L}, z^{-1}\Sigma, z^*\mathcal{E})$.

Proof. The proof goes along exactly the same lines as in Lemma 4.2.2, where we instead consider the following commutative diagram, coming from 3.2.2:

$$\begin{array}{ccccccc}
z^{-1}\Sigma & \longleftarrow & \mathbf{G}\times z^{-1}\Sigma_{reg} & \xrightarrow{\beta} & \mathbf{G}\times_{\mathbf{L}} z^{-1}\Sigma_{reg} & \xrightarrow{\gamma} & Y_{\mathbf{L},z^{-1}\Sigma} \\
\downarrow{\scriptstyle z} & & \downarrow{\scriptstyle id\times z} & & \downarrow{\scriptstyle id\times_{\mathbf{L}} z} & & \downarrow{\scriptstyle z} \\
\Sigma & \xleftarrow[\alpha]{} & \mathbf{G}\times\Sigma_{reg} & \xrightarrow[\beta]{} & \mathbf{G}\times_{\mathbf{L}}\Sigma_{reg} & \xrightarrow[\gamma]{} & Y_{\mathbf{L},\Sigma}
\end{array}$$

We first observe that $z^*\mathcal{K}_{\mathfrak{m}} = IC(\overline{z^{-1}Y_{\mathbf{L},\Sigma}}, z^*\gamma_*(\tilde{\mathcal{E}}))[\dim z^{-1}Y_{\mathbf{L},\Sigma}]$ and since $z^{-1}Y_{\mathbf{L},\Sigma} = Y_{\mathbf{L},z^{-1}\Sigma}$,

$$z^*\mathcal{K}_{\mathfrak{m}} = IC(\overline{Y_{\mathbf{L},z^{-1}\Sigma}}, z^*\gamma_*(\tilde{\mathcal{E}}))[\dim Y_{\mathbf{L},z^{-1}\Sigma}].$$

We then study $z^*\gamma_*(\tilde{\mathcal{E}})$ and observe that it is isomorphic to $\gamma_*(\widetilde{z^*\mathcal{E}})$. □

Observe that by definition of the parabolic restriction, for any proper parabolic subgroup $\mathbf{P} = \mathbf{U} \rtimes \mathbf{L}$ of $\mathbf{G}$ such that $\mathbf{T}_0 \subseteq \mathbf{L}$ and $\mathbf{B}_0 \subseteq \mathbf{P}$, $\operatorname{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(z^*\mathcal{K}_{\mathfrak{m}}) = z^* \operatorname{Res}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(\mathcal{K}_{\mathfrak{m}})$. Thus, if $\mathcal{K}_{\mathfrak{m}}$ is cuspidal, so is $z^*\mathcal{K}_{\mathfrak{m}}$.

4.2.3 Central translation of unipotently supported character sheaves.

From now on, we consider the particular case where the character sheaves are unipotently supported. We are in the setting of Remark 3.2.8. We fix a cuspidal induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$ with $C \in \mathrm{Ucl}(\mathbf{L})$ such that $\Sigma = CZ^\circ(\mathbf{L})$. We describe $\mathcal{E} = i^*(\mathcal{E}_0 \boxtimes \mathcal{Z})$ with $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$, $\mathcal{E}_0$ an irreducible local system on C, and $i : \Sigma \to C \times Z^\circ(\mathbf{L})$ the canonical map.
We also fix $z \in Z^\circ(\mathbf{L})$. Observe that

$$z^*\mathfrak{m} = (\mathbf{L}, \Sigma, z^*\mathcal{E}) = (\mathbf{L}, \Sigma, i^*(\mathcal{E}_0 \boxtimes z^*\mathcal{Z})).$$

Therefore, by Lemma 4.2.1, there is an isomorphism between $z^*\mathcal{Z}$ and $\mathcal{Z}$. Hence, if $z \in Z(\mathbf{G})$, we have $z^*\mathcal{K}_{\mathfrak{m}}$ and $\mathcal{K}_{\mathfrak{m}}$ are isomorphic.

Thanks to the isomorphism $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}] \xrightarrow{\sim} \mathrm{End}(\mathcal{K}_{\mathfrak{m}})$ defined in Proposition 3.2.17, we label the character sheaves of $\hat{\mathbf{G}}(\mathfrak{m})$ by the irreducible $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}]$-modules. Recall that this isomorphism is the composition of two isomorphisms: one between $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}]$ and $\mathscr{A}_{\mathcal{E}}$ and a second between $\mathscr{A}_{\mathcal{E}}$ and $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$. The latter, that we call Lift, consists in lifting the isomorphisms $\mathrm{ad}(w)^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ to endomorphisms of $\mathcal{K}_{\mathfrak{m}}$. The first one was fixed as follows:

1. For each $w \in W_{\mathfrak{m}}$, choose a representative $\dot{w}$.
2. For each $w \in W_{\mathfrak{m}}$, fix an isomorphism $\theta'_w : \mathrm{ad}(\dot{w})^*(\mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell$ following Lusztig [Lus84b, Thm. 9.2d]
3. Choose the unique isomorphism $\phi^{\mathcal{Z}}_w : \mathrm{ad}(\dot{w})^*\mathcal{Z} \xrightarrow{\sim} \mathcal{Z}$ such that $(\phi^{\mathcal{Z}}_w)_1$ is the identity, for $w \in W_{\mathfrak{m}}$.
4. Construct the basis elements $b^{\mathcal{Z}}_w : \theta'_w \otimes (id \boxtimes \phi^{\mathcal{Z}}_w)$ for $w \in W_{\mathfrak{m}}$ and consider the isomorphism

$$\begin{aligned} b^{\mathcal{Z}} : \overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}] &\xrightarrow{\sim} \mathscr{A}_{\mathcal{E}} \\ w &\mapsto b^{\mathcal{Z}}_w. \end{aligned}$$

Let $V \in \mathrm{Irr}(W_{\mathfrak{m}})$, then we set

$$\mathcal{A}_V \coloneqq \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}})}\big(\mathrm{Lift}(b^{\mathcal{Z}}(V)), \mathcal{K}_{\mathfrak{m}}\big).$$

Here by $\mathrm{Lift}(b^{\mathcal{Z}}(V))$ we mean the module V seen as an $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$-module via the action induced by the isomorphism $\mathrm{Lift} \circ b^{\mathcal{Z}}$.
Therefore,

$$z^*\mathcal{A}_V = \mathrm{Hom}_{\mathrm{End}(z^*\mathcal{K}_{\mathfrak{m}})}\big(z^*(\mathrm{Lift}(b^{\mathcal{Z}}(V))), z^*\mathcal{K}_{\mathfrak{m}}\big).$$

The isomorphism $z^*(\mathrm{Lift} \circ b^{\mathcal{Z}}) : \overline{\mathbb{Q}}_\ell[W_{z^*\mathfrak{m}}] \xrightarrow{\sim} \mathrm{End}(\mathcal{K}_{z^*\mathfrak{m}})$ sends the element $w \in W_{\mathfrak{m}}$ to $z^*b_w^{\mathcal{Z}} : z^*\mathrm{ad}(w)^*\mathcal{E} \to z^*\mathcal{E}$ and then lifts $z^*b_w^{\mathcal{Z}}$ to $\mathrm{End}(z^*\mathcal{K}_{\mathfrak{m}})$. In general, it is not true that $z^*b_w^{\mathcal{Z}} \in \mathscr{A}_{z^*\mathcal{E}}$.

From now on, we assume furthermore that $z \in Z(\mathbf{G}) \cap Z^\circ(\mathbf{L})$. In this case, the translation by z commutes with the conjugation by w and $z^*\mathrm{ad}(w)^*\mathcal{E} = \mathrm{ad}(w)^*z^*\mathcal{E}$, hence $z^*b_w^{\mathcal{Z}} \in \mathscr{A}_{z^*\mathcal{E}}$. Thanks to the isomorphism $z^*\mathcal{K}_{\mathfrak{m}} \cong \mathcal{K}_{z^*\mathfrak{m}}$ of Lemma 4.2.3, we obtain

$$z^*\mathcal{A}_V \cong \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{z^*\mathfrak{m}})}\big(\mathrm{Lift}(z^*b^{\mathcal{Z}}(V)), \mathcal{K}_{z^*\mathfrak{m}}\big).$$

However, the isomorphism $z^*b^{\mathcal{Z}}$ differs from the isomorphism $b^{z^*\mathcal{Z}}$ and thus we cannot write $z^*\mathcal{A}_V \cong \mathcal{A}_V$. To overcome this problem, we need to further investigate the isomorphism $b^{\mathcal{Z}}$.
When considering the translation by $z \in Z(\mathbf{G}) \cap Z^\circ(\mathbf{L})$ and the local system $\mathcal{E}_0 \boxtimes z^*\mathcal{L}$, the first two steps in the definition of $b^{\mathcal{Z}}$ stay exactly the same. We thus turn our attention to $\phi_w^{\mathcal{Z}}$ with $w \in W_{\mathfrak{m}}$.

Description of the isomorphism $\phi_w^{\mathcal{Z}}$

We want to get a better understanding of the isomorphism $\phi_w^{\mathcal{Z}} : \mathrm{ad}(\dot{w})^*\mathcal{Z} \xrightarrow{\sim} \mathcal{Z}$ for a Kummer local system $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$ and $w \in W_{\mathfrak{m}}$.
By the description of Kummer local systems, there is $\lambda \in \mathrm{Hom}(Z^\circ(\mathbf{L}), k^\times) = X(Z^\circ(\mathbf{L}))$ and $n \in \mathbb{N}$ coprime to p such that $\mathcal{Z} = \lambda^*\mathcal{E}_{n,\mathrm{j}}$. For $w \in W_{\mathfrak{m}}$, the isomorphism $\mathrm{ad}(\dot{w})^*\mathcal{Z} \cong \mathcal{Z}$ implies that $\lambda \circ \mathrm{ad}(\dot{w}) - \lambda \in nX(Z^\circ(\mathbf{L}))$. Namely, there exists a character $\lambda_w \in X(Z^\circ(\mathbf{L}))$ such that $\lambda \circ \mathrm{ad}(\dot{w}) = \lambda\lambda_w^n$. This character does not depend on the choice of representative $\dot{w}$ since for any $l \in \mathbf{L}$, $\lambda \circ \mathrm{ad}(l) = \lambda$.
The isomorphism $\phi_w^{\mathcal{Z}} : \mathrm{ad}(\dot{w})^*\mathcal{Z} \to \mathcal{Z}$ is thus an isomorphism $\phi_w^{\mathcal{Z}} : (\lambda\lambda_w^n)^*\mathcal{E}_{n,\mathrm{j}} \to \lambda^*\mathcal{E}_{n,\mathrm{j}}$.
More generally, for any $\lambda, \gamma \in X(Z^\circ(\mathbf{L}))$ and any $n \in \mathbb{N}$ coprime to p, we will describe the isomorphisms

$$\phi_{\lambda,n,\gamma} : (\lambda\gamma^n)^*\mathcal{E}_{n,\mathrm{j}} \xrightarrow{\sim} \lambda^*\mathcal{E}_{n,\mathrm{j}}$$

such that $(\phi_{\lambda,n,\gamma})_1$ is the identity.

We fix such $n \in \mathbb{N}$ and $\lambda, \gamma \in X(Z^\circ(\mathbf{L}))$. We start with a few observations, defining some morphisms and keeping track of their restriction to the stalks.

In the first place, we recall that $\mathcal{E}_{n,\mathfrak{j}}$ is the summand of $(\rho_n)_*\overline{\mathbb{Q}}_\ell$ on which μ_n acts according to $\mathfrak{j}$. Therefore, to understand the isomorphism $\phi_{\lambda,n,\gamma}$ it suffices to define a μ_n-equivariant isomorphism

$$\Phi_{\lambda,n,\gamma} : (\lambda\gamma^n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \tilde{\to} (\rho_n)_*\overline{\mathbb{Q}}_\ell.$$

We do this in three steps.

Step 1. Define a μ_n-equivariant isomorphism

$$(\lambda\gamma^n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \tilde{\to} (\lambda)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \otimes_{\mu_n} (\gamma^n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell.$$

Step 2. Define a μ_n-equivariant isomorphism

$$(\gamma^n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \tilde{\to} \mathcal{C},$$

where $\mathcal{C}$ is the constant sheaf on $Z^\circ(\mathbf{L})$ which takes value $\overline{\mathbb{Q}}_\ell[\mu_n]$.

Step 3. Combine the two previous isomorphisms to get a μ_n-equivariant isomorphism:

$$\Phi_{\lambda,n,\gamma} : (\lambda\gamma^n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \tilde{\to} (\rho_n)_*\overline{\mathbb{Q}}_\ell.$$

Beforehand, let us describe the stalks of $\mathcal{E}_{n,\mathfrak{j}}$. For $c \in k^\times$, the stalk $\left((\rho_n)_*\overline{\mathbb{Q}}_\ell\right)_c$ can be seen as the n-dimensional $\overline{\mathbb{Q}}_\ell$-vector space $\overline{\mathbb{Q}}_\ell[\rho_n^{-1}(c)]$, with action of μ_n on $\rho_n^{-1}(c)$ by multiplication. In that setting, $(\mathcal{E}_{n,\mathfrak{j}})_c$ is the $\overline{\mathbb{Q}}_\ell$-vector subspace of dimension one on which the action of $x \in \mu_n$ is simply multiplication by $\mathfrak{j}(x)$.

Step 1. As stated in [MS89, 2.1.2], there is a μ_n-equivariant isomorphism:

$$(\lambda\gamma)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \to (\lambda)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \otimes_{\mu_n} (\gamma)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell.$$

On the stalk at $t \in Z^\circ(\mathbf{L})$, we get a morphism of μ_n-modules

$$\overline{\mathbb{Q}}_\ell[\rho_n^{-1}(\lambda(t)\gamma(t))] \to \overline{\mathbb{Q}}_\ell[\rho_n^{-1}(\lambda(t))] \otimes_{\mu_n} \overline{\mathbb{Q}}_\ell[\rho_n^{-1}(\gamma(t))].$$

Step 2. Let us write $\mathcal{C}_k$ for the constant sheaf on $k^\times$ which takes value $\overline{\mathbb{Q}}_\ell[\mu_n]$.
The adjunction $\varepsilon_n : (\rho_n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$ is given by the μ_n-equivariant isomorphism

$$(\rho_n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \tilde{\to} \mathcal{C}_k.$$

Taking the pullback by γ, we get a μ_n-equivariant isomorphism

$$(\gamma^n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell = \gamma^*(\rho_n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \tilde{\to} \gamma^*\mathcal{C}_k.$$

By definitions of the pullback and of constant sheaves, $\gamma^*\mathcal{C}_k = \mathcal{C}$. On the stalk at $t \in Z^\circ(\mathbf{L})$, we get an isomorphism of μ_n-modules

$$\overline{\mathbb{Q}}_\ell[\rho_n^{-1}(\gamma^n(t))] = \overline{\mathbb{Q}}_\ell[\gamma(t)\mu_n] \tilde{\to} \overline{\mathbb{Q}}_\ell[\mu_n]$$

by multiplication by $\gamma(t)^{-1}$.

Step 3. Combining the two previous μ_n-equivariant morphisms, we get an isomorphism

$$\Phi_{\lambda,n,\gamma} : (\lambda\gamma^n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \tilde{\to} (\lambda)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \otimes_{\mu_n} (\gamma^n)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \tilde{\to} (\lambda)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell \otimes_{\mu_n} \mathcal{C} \tilde{\to} (\lambda)^*(\rho_n)_*\overline{\mathbb{Q}}_\ell.$$

On the stalk at $t \in Z^\circ(\mathbf{L})$, we get an isomorphism of μ_n-modules

$$\overline{\mathbb{Q}}_\ell[\rho_n^{-1}(\lambda(t)\gamma^n(t))] = \overline{\mathbb{Q}}_\ell[\gamma(t)\rho_n^{-1}(\lambda(t))] \tilde{\to} \overline{\mathbb{Q}}_\ell[\rho_n^{-1}(\lambda(t))],$$

given by multiplication by $\gamma(t)^{-1}$. In particular, if $t^n = 1$, then $\gamma(t)^{-1} \in \mu_n$ and so this morphism is simply the action of $\gamma(t)^{-1}$.
This μ_n-equivariant morphism $\Phi_{\lambda,n,\gamma}$ restricts to an isomorphism:

$$\phi_{\lambda,n,\gamma} : (\lambda\gamma^n)^*\mathcal{E}_{n,\mathfrak{j}} \tilde{\to} \lambda^*\mathcal{E}_{n,\mathfrak{j}}.$$

On the stalk at $t \in Z^\circ(\mathbf{L})$ such that $t^n = 1$, we get an automorphism of μ_n-modules

$$(\mathcal{E}_{n,\mathfrak{j}})_{\lambda(t)\gamma^n(t)} \to (\mathcal{E}_{n,\mathfrak{j}})_{\lambda(t)},$$

given by the action of $\gamma(t)^{-1} \in \mu_n$, that is, multiplication by $\mathfrak{j}(\gamma(t)^{-1})$. In particular, at the stalk $t = 1$, the isomorphism is simply the identity.
The above discussion leads to the following result.

Lemma 4.2.4. *Let $\mathcal{Z} = \lambda^*\mathcal{E}_{n,\psi} \in \mathcal{S}(Z^\circ(\mathbf{L}))$ for $n \in \mathbb{N}$ and $\lambda \in X(Z^\circ(\mathbf{L}))$. Let $w \in W_{\mathbf{L}}$ such that $\mathrm{ad}(w)^*\mathcal{Z} \cong \mathcal{Z}$. Recall that there is $\lambda_w \in X(Z^\circ(\mathbf{L}))$ such that $\lambda \circ \mathrm{ad}(\dot{w}) = \lambda\lambda_w^n$. Then*

$$\phi_w^{\mathcal{Z}} = \phi_{\lambda,n,\lambda_w}.$$

Proof. Since $\mathcal{Z}$ is irreducible, it suffices to check that $(\phi_{\lambda,n,\lambda_w})_1$ is equal to $(\phi_w^{\mathcal{Z}})_1$, which is the identity by definition. The claim follows from the previous discussion. □

Central translation of a unipotently supported character sheaf.

Lemma 4.2.5. *Let $z \in Z(\mathbf{G})$ and $\mathfrak{m} = (\mathbf{L}, CZ^\circ(\mathbf{L}), \mathcal{E}_0 \boxtimes \mathcal{Z})$ be a cuspidal induction datum, with $C \in \mathrm{Ucl}(\mathbf{L})$, $\mathcal{E}_0$ a local system on C, and $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$ such that $\mathcal{Z} = \lambda^*\mathcal{E}_{n,\mathfrak{j}}$ where $\lambda \in X(Z^\circ(\mathbf{L}))$. Let $V \in \mathrm{Irr}(W_\mathfrak{m})$ and $\mathcal{A}_V$ be the summand of $\mathcal{K}_\mathfrak{m}$ corresponding to V under the isomorphism fixed in Proposition 3.2.17. Assume that $z \in Z(\mathbf{G}) \cap Z^\circ(\mathbf{L})$, then*

$$z^* A_V^{\mathcal{Z}} \cong A_{V\otimes X^z}^{z^*\mathcal{Z}},$$

where X^z is the one-dimensional module of $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ whose character is $\chi_z : w \mapsto \mathfrak{j}(\lambda_w(z))$.

Proof. As we discussed before, we need to compare the two isomorphisms $z^*b^{\mathcal{Z}}$ and $b^{z^*\mathcal{Z}}$. Firstly, we claim that $z^*\phi_w^{\mathcal{Z}} = \mathfrak{j}(\lambda_w(z)^{-1})\phi_w^{z^*\mathcal{Z}}$ for any $w \in W_\mathfrak{m}$. Since $z^*\mathcal{Z}$ is an irreducible local system, the vector space $\mathrm{Hom}(\mathrm{ad}(w)^*z^*\mathcal{Z}, z^*\mathcal{Z})$ is one-dimensional. Thus, the two isomorphisms $z^*(\phi_w^{\mathcal{Z}})$ and $\phi_w^{z^*\mathcal{Z}}$ differ by a scalar. To determine this scalar, it suffices to consider the stalks at 1. On one hand, by definition, $\left(\phi_w^{z^*\mathcal{Z}}\right)_1$ is the identity. On the other hand, by Lemma 4.2.4, $(z^*(\phi_w^{\mathcal{Z}}))_1 = (\phi_{\lambda,n,\lambda_w})_z$. The latter is given by multiplication by the scalar $\mathfrak{j}(\lambda_w(z)^{-1})$ as $\lambda_w(z) \in \mu_n$. Indeed, by definition $\lambda_w^n = \lambda \circ \mathrm{ad}(\dot{w}) - \lambda$ and $\mathrm{ad}(w)(z) = z$, so $\lambda_w(z)^n = 1$. We conclude that

$$z^*\phi_w^{\mathcal{Z}} = \mathfrak{j}(\lambda_w(z)^{-1})\phi_w^{z^*\mathcal{Z}}.$$

Therefore, $z^*b^{\mathcal{Z}} = \chi_z \otimes b^{z^*\mathcal{Z}}$. We now observe that $z^*b^{\mathcal{Z}}(V) = (b^{z^*\mathcal{Z}})^*(V \otimes X_z)$ where X_z is the one-dimensional module with action of $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ given by multiplication by $\mathfrak{j}(\lambda_w(z))$. □

To conclude this section, we would like to describe $\mathfrak{j}(\lambda_w(z))$ more precisely. We generalise different facts due to Lusztig [LuCS3, § 11.8] when $\mathbf{L} = \mathbf{T}_0$.

Lemma 4.2.6. *Let $z \in Z(\mathbf{G})$, $\mathcal{L} = \lambda^*(\mathcal{E}_{n,\mathfrak{j}}) \in \mathcal{S}(Z^\circ(\mathbf{L}))$. The following hold:*

1. *For any $w \in W_\mathfrak{m}$ and any $z \in Z^\circ(\mathbf{G})$, $\mathfrak{j}(\lambda_w(z)) = 1$.*
2. *For any $w \in W_\mathfrak{m}^\circ$, any $z \in Z(\mathbf{G})$, $\mathfrak{j}(\lambda_w(z)) = 1$.*
3. *If $Z(\mathbf{G}) \subseteq Z^\circ(\mathbf{L})$, the map $W_\mathfrak{m}/W_\mathfrak{m}^\circ \to \mathrm{Hom}(Z(\mathbf{G})/Z^\circ(\mathbf{G}), \overline{\mathbb{Q}}_\ell^\times)$, $w \mapsto (z \mapsto \mathfrak{j}(\lambda_w(z)))$, is injective.*

Proof. Fact 1. As we have seen in the proof of Lemma 4.2.5, $\lambda_w(z) \in \mu_n$ for each element $w \in W_\mathfrak{m}$ and $z \in Z(\mathbf{G})$. Besides, λ_w induces a continuous map from $Z(\mathbf{G})$ to μ_n, hence $\lambda_w(Z^\circ(\mathbf{G})) = 1$.
Fact 2. If $w \in W_\mathfrak{m}^\circ$, then w is a product of s_α for $\alpha \in \Phi_\mathfrak{m}$. Let $\alpha \in \Phi$, then $s_\alpha.\lambda = \lambda - \langle \lambda, \check{\alpha}\rangle\alpha$. On the other hand, if $\alpha \in \Phi_\mathfrak{m}$, then $s_\alpha \in W_\mathfrak{m}$. In particular, there is $\lambda_{s_\alpha} \in X$ such that $s_\alpha.\lambda - \lambda = n\lambda_{s_\alpha}$. Hence, $\langle \lambda, \check{\alpha}\rangle \equiv 0 \mod n$ and $\lambda_{s_\alpha} = n_\alpha \alpha$ for some $n_\alpha \in \mathbb{N}$. Therefore, for any $w \in W_\mathfrak{m}^\circ$, the character λ_w is in the root lattice $\Phi_\mathfrak{m}$. In particular, it means that $\lambda_w(Z(\mathbf{G})) = 1$ ([MT11, Thm. 8.17(h)]).
Fact 3. If $Z(\mathbf{G}) = Z^\circ(\mathbf{G})$, then by [AA10, Prop.4.4] the group $W_\mathfrak{m}$ is a Coxeter group and $W_\mathfrak{m} = W_\mathfrak{m}^\circ$. We assume now that $Z(\mathbf{G})$ is not connected and let $w \in W_\mathfrak{m} \subseteq W_\mathbf{L}^\mathbf{G}$ such that $\lambda_w(Z(\mathbf{G})) = \{1\}$ and set $\tilde{\mathbf{G}} := (\mathbf{G} \times Z^\circ(\mathbf{L}))/Z(\mathbf{G})$ where $Z(\mathbf{G})$ is embedded diagonally. The group $\tilde{\mathbf{G}}$ has a connected centre. We set $\tilde{\mathbf{L}} = (\mathbf{L} \times Z^\circ(\mathbf{L}))/Z(\mathbf{G})$ and we note that $Z^\circ(\tilde{\mathbf{L}}) = (Z^\circ(\mathbf{L}) \times Z^\circ(\mathbf{L}))/Z(\mathbf{G})$. Furthermore, $W_{\tilde{\mathbf{L}}}^{\tilde{\mathbf{G}}}$ can be identified with $W_\mathbf{L}^\mathbf{G}$. As in [LuCS3, § 11.8], we extend λ to a character

$$\tilde{\lambda} : Z^\circ(\tilde{\mathbf{L}}) \to k^\times, \quad (z, z')Z(\mathbf{G}) \mapsto \lambda(z)\lambda(w^{-1}z'w)^{-1}.$$

Observe that for $(z, z')Z(\mathbf{G}) \in Z^\circ(\tilde{\mathbf{L}})$,

$$\begin{aligned} w.\tilde{\lambda}((z, z')Z(\mathbf{G})) &= \lambda(wzw^{-1})\lambda(wz'w^{-1})^{-1} \\ &= \lambda(z)\lambda(wz'w^{-1})^{-1}\lambda_w^n(z) \\ &= \tilde{\lambda}((z, z')Z(\mathbf{G}))\lambda_w^n(z). \end{aligned}$$

We set
$$\tilde{\lambda}_w : Z^\circ(\tilde{\mathbf{L}}) \to k^\times, (z, z')Z(\mathbf{G}) \mapsto \lambda_w(z).$$
By hypothesis ($\lambda_w(Z(\mathbf{G})) = 1$), this character is well-defined and $w.\tilde{\lambda} = \tilde{\lambda}\tilde{\lambda}_w^n$. Since $\tilde{\mathbf{G}}$ has connected centre, w is a product of reflections s_α for $\alpha \in \Phi$ such that $s_\alpha.\tilde{\lambda} = \tilde{\lambda}\tilde{\lambda}_{s_\alpha}^n$. Restricting to $Z^\circ(\mathbf{L})$, we conclude that $s_\alpha \in W_{\mathfrak{m}}^\circ$ and thus $w \in W_{\mathfrak{m}}^\circ$. □

Remark 4.2.7. This lemma does not allow us to completely describe the character χ_z of Lemma 4.2.5. However, it does gives us some information. For instance, if $W_{\mathfrak{m}}/W_{\mathfrak{m}}^\circ$ is a cyclic group of order 2 generated by $wW_{\mathfrak{m}}^\circ$, and $Z(\mathbf{G})/Z^\circ(\mathbf{G})$ is also cyclic of order 2, then χ_z is the trivial character if $z \in Z^\circ(\mathbf{G})$. If $z \notin Z^\circ(\mathbf{G})$, then the character χ_z takes value -1 on $wW_{\mathfrak{m}}^\circ$ and 1 on $W_{\mathfrak{m}}^\circ$.

4.3 Restriction of a character sheaf to a mixed conjugacy class

We now come back to our initial goal of understanding the restriction of a character sheaf $\mathcal{A} \in \hat{\mathbf{G}}$ to any conjugacy class $(su)_{\mathbf{G}}$ where $s \in \mathbf{G}$ is semisimple and $u \in C_{\mathbf{G}}(s)$ is unipotent. As we have discussed at the beginning of the previous section, in order to compute the cohomology $H^i(\mathcal{A})_{su}$ for $i \in \mathbb{Z}$, we could instead focus on $(s^*\mathcal{A})_{(u)_{C^\circ_{\mathbf{G}}(s)}}$.

We will proceed in a similar way to our discussion in Subsection 4.2.2. For the rest of this section, we fix an induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$ where $\mathbf{L}$ is a Levi subgroup of a parabolic subgroup $\mathbf{P} \subseteq \mathbf{G}$, Σ is the inverse image under the map $\mathbf{L} \to \mathbf{L}/Z^\circ(\mathbf{L})$ of an isolated conjugacy class, and $\mathcal{E}$ is a local system on Σ. We will study the character sheaves in $\hat{\mathbf{G}}(\mathfrak{m})$ restricted to the mixed conjugacy class $(su)_{\mathbf{G}}$. As we have argued before, $\mathcal{A}_{(su)_{\mathbf{G}}} \neq 0$ implies that up to $\mathbf{G}$-conjugation we may assume that $s \in \mathbf{L}$ and that $\Sigma = (sv)_{\mathbf{L}}Z^\circ(\mathbf{L})$ for some $v \in C_{\mathbf{G}}(s)$.
We start by studying the complex $(s^*\mathcal{K}_{\mathfrak{m}})_{(u)_{C^\circ_{\mathbf{G}}(s)}}$ and show that it is isomorphic to a direct sum of some $\mathcal{K}_{\mathfrak{m}'}$ for some different cuspidal data $\mathfrak{m}'$ of $C^\circ_{\mathbf{G}}(s)$. In a second step, we will see how to go down to the constituents of $\mathcal{K}_{\mathfrak{m}}$, that is we will study how the isomorphism behaves with respect to the action of $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$. As before, we focus on the cases where we know that $\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$ is isomorphic to $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}]$.

To simplify notation, we write $\mathbf{G}_s = C^\circ_{\mathbf{G}}(s)$.

4.3.1 Restriction of an induced cuspidal perverse sheaf to the centraliser of a semisimple element

Following [MS89, Section 8] and [LuCS2, § 8], we decompose $(s^*\mathcal{K}_{\mathfrak{m}})_{(\mathbf{G}_s)_{\text{uni}}}$ into a direct sum of semisimple complexes on $\mathbf{G}_s$. To do so, recall that $\mathcal{K}_{\mathfrak{m}} = IC(\overline{Y_{\mathbf{L},\Sigma}}, \gamma_*(\tilde{\mathcal{E}}))[\dim Y_{\mathbf{L},\Sigma}]$. Therefore as a first approach, we need to understand the set $s^{-1}\,\mathrm{supp}(\mathcal{K}_{\mathfrak{m}}) \cap (\mathbf{G}_s)_{\text{uni}}$ and the restriction of $s^*\mu_*(\tilde{\mathcal{E}})$ to this set. By Remark 3.2.10, the variety $\mathbf{G}_s$ is partitioned into varieties $Y_{\mathbf{L}',\Sigma'}$ where $\mathbf{L}'$ is a Levi subgroup of $\mathbf{G}_s$ and Σ' is the preimage of an isolated class in $\mathbf{L}'/Z^\circ(\mathbf{L}')$. Thus

$$s^{-1}\operatorname{supp}(\mathcal{K}_\mathfrak{m}) \cap (\mathbf{G}_s)_{\mathrm{uni}} = \bigsqcup_{(\mathbf{L}',\Sigma')} Y_{\mathbf{L}',\Sigma'} \cap s^{-1}\operatorname{supp}(\mathcal{K}_\mathfrak{m}) \cap (\mathbf{G}_s)_{\mathrm{uni}},$$

where $(\mathbf{L}',\Sigma')$ runs over the pairs of $\mathbf{G}_s$ as defined above. We make two observations.

1. The condition $Y_{\mathbf{L}',\Sigma'} \cap (\mathbf{G}_s)_{\mathrm{uni}} \neq \varnothing$ means that $\Sigma' = C'Z^\circ(\mathbf{L}')$ for some $C \in \mathrm{Ucl}(\mathbf{L}')$.
2. By Equation 4.1, this condition implies that there exist $g \in \mathbf{G}$ and $z \in Z^\circ(\mathbf{L})$ such that $s^{-1}gszg^{-1}$ belongs to the semisimple part of $\Sigma'_{reg} = \{h \in \Sigma' \mid C^\circ_{\mathbf{G}}(h_s) \subseteq \mathbf{L}\}$. In particular, we may assume that $\mathbf{L}' = C^\circ_{\mathbf{G}_s}(s^{-1}gszg^{-1}) = C^\circ_{\mathbf{G}_s}(gszg^{-1})$; see Remark 3.2.10.

We thus define the two following sets

$$M := \{m \in \mathbf{G} \mid m^{-1}sm \in (s)_\mathbf{L} Z^\circ(\mathbf{L})\}$$

and

$$\bar{M} := \mathbf{G}_s \backslash M / \mathbf{L}.$$

Remark 4.3.1. Let $m \in M$. There are $l \in \mathbf{L}$ and $z \in Z^\circ(\mathbf{L})$ such that $m^{-1}sm = ltl^{-1}z$. Then $ml \in M$ and $C^\circ_G(s)m\mathbf{L} = C^\circ_G(s)ml\mathbf{L}$. Therefore, for each $\mu \in \bar{M}$, we may and will fix a representative $\dot{\mu} \in M$ such that $\dot{\mu}^{-1}s\dot{\mu} = sz_\mu$ for some $z_\mu \in Z^\circ(\mathbf{L})$. We will often abuse notation and write only μ for $\dot{\mu}$ for any $\mu \in \bar{M}$.

To each $\mu \in \bar{M}$, we associate a cuspidal induction datum $\mathfrak{m}_\mu = (\mathbf{L}_\mu, \Sigma_\mu, \mathcal{E}_\mu)$ of $\mathbf{G}_s$ where

$$\mathbf{L}_\mu := \dot{\mu}\mathbf{L}\dot{\mu}^{-1} \cap \mathbf{G}_s,$$

$$\Sigma_\mu := Z^\circ(\mathbf{L}_\mu)C_\mu \text{ with } C_\mu := \{u \in \mathbf{G}_s \mid u \text{ unipotent}, \dot{\mu}^{-1}su\dot{\mu} \in \Sigma\},$$

and

$$\mathcal{E}_\mu := \tau_\mu^*\mathcal{E} \text{ for } \tau_\mu : \Sigma_\mu \to \Sigma,\ g \mapsto \dot{\mu}^{-1}sg\dot{\mu}.$$

By [LuCS2, Prop. 7.11], the complex

$$\mathcal{A}_\mu := \mathcal{A}_{\mathfrak{m}_\mu} = IC(\overline{\Sigma_\mu}, \mathcal{E}_\mu)[\dim \Sigma_\mu]$$

is indeed an irreducible cuspidal character sheaf of $\mathbf{L}_\mu$. Moreover, C_μ is a unipotent conjugacy class of $\mathbf{L}_\mu$, so $C_\mu = (\dot{\mu}v\dot{\mu}^{-1})_{\mathbf{L}_\mu}$. Lastly, we set $\mathcal{K}_\mu := \mathcal{K}_{\mathfrak{m}_\mu}$ as a semisimple complex on $\mathbf{G}_s$.

Remark 4.3.2. Alternatively, to each element $\mu \in \bar{M}$, we associate a cuspidal induction datum $\mathfrak{m}_{0,\mu} = (\mathbf{L}_0, C_0Z^\circ(\mathbf{L}_0), \mathcal{E}_{0,\mu})$ of $C^\circ_{\mathbf{G}}(sz_\mu)$, where

$$\mathbf{L}_0 := C^\circ_{\mathbf{L}}(s) = \dot{\mu}^{-1}\mathbf{L}_\mu\dot{\mu}, \quad C_0 := (v)_{\mathbf{L}_0}$$

and $\mathcal{E}_{0,\mu}$ is the local system on $\Sigma_0 := C_0Z^\circ(\mathbf{L}_0)$ obtained as the inverse image of $\mathcal{E}$ under the map $\Sigma_0 \to \Sigma, g \mapsto sz_\mu g$. Then setting $\mathcal{A}_{0,\mu} := \mathcal{A}_{\mathfrak{m}_{0,\mu}}$, we have

$$\mathcal{A}_\mu = \mathrm{ad}(\dot{\mu}^{-1})^*\mathcal{A}_{0,\mu}.$$

Using the previous notation, we are finally able to state the following theorem.

Proposition 4.3.3 ([LuCS2, §8]). *There is an open neighborhood sU of s in $\mathbf{G}_s$, such that $(\mathbf{G}_s)_{\mathrm{uni}} \subseteq U$ and there is an isomorphism*

$$\mathcal{T} : s^*((\mathcal{K}_{\mathfrak{m}})_{sU}) \xrightarrow{\sim} \bigoplus_{\mu \in \bar{M}} (\mathcal{K}_\mu)_U[\dim(\mathbf{G}) - \dim(\mathbf{G}_s)].$$

Proof. We give a sketch of the proof by firstly describing the isomorphism on the level of local systems thanks to the proof of [MS89, Prop. 8.2.3] and the discussion following it. The definition of parabolic induction at the level of local systems leads to the following commutative diagram for each $\mu \in \bar{M}$:

$$\begin{array}{ccccccc}
\Sigma_\mu & \xleftarrow{\alpha_\mu} & \mathbf{G}_s \times \Sigma_{\mu\, reg} & \xrightarrow{\beta_\mu} & \mathbf{G}_s \times_{\mathbf{L}_\mu} \Sigma_{\mu\, reg} & \xrightarrow{\gamma_\mu} & Y_{\mathbf{L}_\mu, \Sigma_\mu} \\
\downarrow{\scriptstyle \tau_\mu} & & \downarrow{\scriptstyle s_\mu} & & \downarrow{\scriptstyle s_\mu} & & \downarrow{\scriptstyle s} \\
\Sigma & \xleftarrow[\alpha]{} & \mathbf{G} \times \Sigma_{reg} & \xrightarrow[\beta]{} & \mathbf{G} \times_{\mathbf{L}} \Sigma_{reg} & \xrightarrow[\gamma]{} & Y_{\mathbf{L}, \Sigma}
\end{array}$$

with

- the map $\tau_\mu : g \mapsto \dot{\mu}^{-1} s g \dot{\mu}$,
- the map $s_\mu : (h, g) \mapsto (h\dot{\mu}, \dot{\mu}^{-1} s g \dot{\mu})$, for $h \in \mathbf{G}_s$ and $g \in \Sigma_{\mu, reg}$,
- and the map $s : g \mapsto sg$ for $g \in Y_{\mathbf{L}_\mu, \Sigma_\mu}$.

To be able to navigate the diagram, we define a few more sets:

$$S \coloneqq \gamma^{-1}(sU \cap Y_{\mathbf{L},\Sigma}),\ S_\mu \coloneqq \gamma_\mu^{-1}(U \cap Y_{\mathbf{L}_\mu, \Sigma_\mu}) \text{ and lastly } T_\mu = s_\mu(S_\mu).$$

Since $\mathcal{K}_{\mathfrak{m}} = IC(\overline{Y_{\mathbf{L},\Sigma}}, \gamma_*(\tilde{\mathcal{E}}))[\dim Y_{\mathbf{L},\Sigma}]$, we study $s^*(\gamma_*(\tilde{\mathcal{E}})_{sU \cap Y_{\mathbf{L},\Sigma}}) = (s^* \gamma_*(\tilde{\mathcal{E}}))_{U \cap s^{-1} Y_{\mathbf{L},\Sigma}}$. By [LuCS2, below 8.7.12], we have $(sU \cap Y_{\mathbf{L},\Sigma}) = \bigcup_{\mu \in \bar{M}} \gamma(T_\mu)$ and thus

$$s^*(\gamma_*(\tilde{\mathcal{E}})_{sU \cap Y_{\mathbf{L},\Sigma}}) \cong \bigoplus_{\mu \in \bar{M}} s^*(\gamma_*(\tilde{\mathcal{E}}))_{\gamma(T_\mu)}.$$

By the change of basis theorem,

$$s^*(\gamma^*(\tilde{\mathcal{E}}_{T_\mu})) \cong (\gamma_\mu)_* s_\mu^*(\tilde{\mathcal{E}}_{T_\mu}) = (\gamma_\mu)_*((s_\mu^* \tilde{\mathcal{E}})_{S_\mu}).$$

We check that $(s_\mu^* \tilde{\mathcal{E}})_{S_\mu} \cong (\tilde{\mathcal{E}}_\mu)_{S_\mu}$. By definition of $\tilde{\mathcal{E}}_\mu$ as the unique local system up to isomorphism such that $\alpha_\mu^* \mathcal{E}_\mu \cong \beta_\mu^* \tilde{\mathcal{E}}_\mu$, it suffices to check that $\beta_\mu^* s_\mu^* \tilde{\mathcal{E}} \cong \alpha_\mu^* \mathcal{E}_\mu$. Following the diagram, we see

$$\beta_\mu^* s_\mu^* \tilde{\mathcal{E}} = s_\mu^* \beta^* \tilde{\mathcal{E}} \cong s_\mu^* \alpha^* \mathcal{E} = \alpha_\mu^* \tau_\mu^* \mathcal{E} = \alpha_\mu^* \mathcal{E}_\mu$$

whence $s_\mu^* \tilde{\mathcal{E}} \cong \tilde{\mathcal{E}}_\mu$. Therefore, we have defined an isomorphism

$$\mathcal{T} : s^*(\gamma_*(\tilde{\mathcal{E}})_{sU \cap Y_{\mathbf{L},\Sigma}}) \xrightarrow{\sim} \bigoplus_{\mu \in \bar{M}} (\gamma_\mu)_*(\tilde{\mathcal{E}}_{\mu S_\mu}) = \bigoplus_{\mu \in \bar{M}} ((\gamma_\mu)_* \tilde{\mathcal{E}}_\mu)_{U \cap Y_{\mathbf{L}_\mu, \Sigma_\mu}}.$$

By definition of intersection cohomology complexes, the above discussion gives rise to an isomorphism

$$\mathcal{T} : s^*((\mathcal{K}_{\mathfrak{m}})_{sU\cap Y_{\mathbf{L},\Sigma}}) \xrightarrow{\sim} \bigoplus_{\mu\in\bar{M}} (\mathcal{K}_\mu)_{U\cap Y_{\mathbf{L}_\mu,\Sigma_\mu}}[\dim Y_{\mathbf{L},\Sigma} - \dim Y_{\mathbf{L}_\mu,\Sigma_\mu}].$$

By [LuCS2, 8.8.4-8.8.7], this isomorphism can be uniquely extended to an isomorphism

$$\mathcal{T} : s^*((\mathcal{K}_{\mathfrak{m}})_{sU}) \xrightarrow{\sim} \bigoplus_{\mu\in\bar{M}} (\mathcal{K}_\mu)_{U}[\dim Y_{\mathbf{L},\Sigma} - \dim Y_{\mathbf{L}_\mu,\Sigma_\mu}].$$

We conclude thanks to [MS89, Lem. 8.2.6 ii], from which we know that

$$\dim Y_{\mathbf{L}_\mu,\Sigma_\mu} = \dim \mathbf{G}_s - \dim \mathbf{L} + \dim \Sigma \text{ and } \dim Y_{\mathbf{L},\Sigma} = \dim \mathbf{G} - \dim \mathbf{L} + \dim \Sigma. \quad \square$$

Remark 4.3.4. Observe that $s^*((\mathrm{Ind}_{\mathbf{B}}^{\mathbf{G}}(K))_{|U})$ might not be a semisimple perverse sheaf since $(\mathcal{K}_\mu)_{s^{-1}U}[\dim \mathbf{G} - \dim \mathbf{G}_s]$ are not necessarily semisimple perverse sheaves. However, Lusztig showed in [Lus15, Prop. 1.4] that

$$s^*((\mathcal{K}_{\mathfrak{m}})_{s(\mathbf{G}_s)_{\mathrm{uni}}})[-\dim(\mathbf{G}) + \dim(\mathbf{G}_s) - \dim(Z^\circ(\mathbf{L}))] \cong \bigoplus_{\mu\in\bar{M}} (\mathcal{K}_\mu)_{(\mathbf{G}_s)_{\mathrm{uni}}}[-\dim(Z^\circ(\mathbf{L}))],$$

is indeed semisimple.

Notation 4.3.5. For the rest of this chapter, we set

$$d := -\dim(\mathbf{G}) + \dim(\mathbf{G}_s) - \dim(Z^\circ(\mathbf{L})) \text{ and } e := -\dim(Z^\circ(\mathbf{L})).$$

We would like to use this isomorphism to deduce the decomposition of $s^*(\mathcal{A}_{s(\mathbf{G}_s)_{\mathrm{uni}}})$ for any character sheaf $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$. Firstly, we consider the restriction of $\mathcal{A}$ to a conjugacy class whose unipotent part is the unipotent support of $\mathcal{A}$.

Proposition 4.3.6 ([Lus15, Thm. 1.2])**.** *Assume that p is good for $\mathbf{G}$. Let $\mathcal{A} \in \hat{\mathbf{G}}$ with unipotent support C. Let D be any conjugacy class of $\mathbf{G}$ such that its unipotent part is equal to C. Then $\mathcal{A}_D \cong \mathcal{L}[\dim(D) + \dim(Z^\circ(\mathbf{L}))]$, where $\mathcal{L}$ is a local system on D.*

Proof. We give the outline of the proof given by Lusztig in [Lus15, § 1.7]. By assumption there is a semisimple element $s \in \mathbf{G}$ and a unipotent element $u \in C$ with $su = us$ such that D is the conjugacy class of su. Assume that $\mathcal{A}$ belongs to the induction series indexed by $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E})$. Using Proposition 4.3.3 and the remark following it, we deduce that the complex $s^*(\mathcal{A}_{s(\mathbf{G}_s)_{\mathrm{uni}}})[d]$ decomposes into a direct sum of irreducible $\mathbf{G}_s$-equivariant perverse sheaves $\mathcal{A}_1, \dots, \mathcal{A}_n$

$$s^*(\mathcal{A}_{s(\mathbf{G}_s)_{\mathrm{uni}}})[d] = (\mathcal{A}_1)_{(\mathbf{G}_s)_{\mathrm{uni}}} \oplus \cdots \oplus (\mathcal{A}_n)_{(\mathbf{G}_s)_{\mathrm{uni}}}.$$

For each $1 \leq i \leq n$, there is a unique unipotent class C_i of $\mathbf{G}_s$ such that $(\mathcal{A}_i)_C$ is a local system $\mathcal{L}_i[\dim C_i]$ and for any unipotent class C' of $\mathbf{G}_s$, $(\mathcal{A}_i)'_C = 0$ if $C' \notin \overline{C_i}$. This is the same argument as for the generalised Springer correspondence in Subsection 4.1.1

applied to $(\mathcal{K}_\mu)_{(\mathbf{G}_s)_{\mathrm{uni}}}[-\dim(Z^\circ(\mathbf{L}))]$ for each $\mu \in \bar{M}$. We consider the restriction to the class $(su)_\mathbf{G}$. We write $C_0 = (u)_{\mathbf{G}_s}$. The decomposition becomes then

$$s^*(\mathcal{A}_{s(u)_{\mathbf{G}_s}})[d] = \bigoplus_{C_0 \subseteq \overline{C_i}} (\mathcal{A}_i)_{(u)_{\mathbf{G}_s}}.$$

On the other hand, for any $v \in \mathbf{G}_s$ unipotent,

$$(u)_{\mathbf{G}_s} \subseteq \overline{(v)_{\mathbf{G}_s}} - (v)_{\mathbf{G}_s} \text{ implies } (u)_\mathbf{G} \subseteq \overline{(v)_\mathbf{G}} - (v)_\mathbf{G}.$$

Indeed, $\overline{(v)_{\mathbf{G}_s}} \subseteq \overline{(v)_\mathbf{G}}$ and if $(u)_\mathbf{G} = (v)_\mathbf{G}$ then $\dim(u)_{\mathbf{G}_s} = \dim(v)_{\mathbf{G}_s}$ and we conclude since the set $\overline{(v)_{\mathbf{G}_s}} - (v)_{\mathbf{G}_s}$ consists of unipotent conjugacy classes of dimension strictly smaller than $\dim(v)_{\mathbf{G}_s}$.
By definition of the unipotent support, we must have $s^*((\mathcal{A})_{|s(v)_{\mathbf{G}_s}}) = 0$ if $(u)_{\mathbf{G}_s} \subseteq \overline{(v)_{\mathbf{G}_s}} - (v)_{\mathbf{G}_s}$. Therefore,

$$s^*(\mathcal{A}_{sC_0})[d-e] = \bigoplus_{C_0 = C_i} (\mathcal{A}_i)_{C_0}.$$

Thus, the restriction $s^*(\mathcal{A}_{s(u)_{\mathbf{G}_s}})[d] = \mathcal{L}[\dim(C_0)]$ for some local system $\mathcal{L}$ on C_0, whence $\mathcal{A}_{s(u)_{\mathbf{G}_s}}[d - \dim(C_0)]$ is a local system.
We decompose $s(u)_\mathbf{G} = \bigcup_{i=1}^r sC_i'$ where C_i' are unipotent conjugacy classes of $\mathbf{G}_s$. Notice that $\dim(C_i') = \dim(C_0)$ for each $1 \leq i \leq r$. Applying the same argument to each C_i', we conclude that $\mathcal{A}_D[d - \dim(C_0)]$ is a local system. The proof is closed by observing that

$$\begin{aligned}\dim(D) &= \dim(\mathbf{G}) - \dim(C_\mathbf{G}(su)) \\ &= \dim(\mathbf{G}) - \dim(C_\mathbf{G}(su)) \\ &= \dim(\mathbf{G}) - (\dim(C_\mathbf{G}(s)) - \dim(u)_{C_\mathbf{G}(s)}) \\ &= \dim(\mathbf{G}) - \dim(\mathbf{G}_s) + \dim(C_0),\end{aligned}$$

whence $d - \dim(C_0) = -\dim(\mathbf{G}) + \dim(\mathbf{G}_s) - \dim(Z^\circ(\mathbf{L})) - \dim(C_0) = -\dim(D) - \dim(Z^\circ(\mathbf{L}))$. □

We want to get a better description of the local system $\mathcal{L}$. Before that, we describe the set $\bar{M}$ in greater detail.

Action of $W_\mathfrak{m}$ on $\bar{M}$

As in [Lus15], we define an action of $W_\mathfrak{m}$ on the set $\bar{M}$. For each $w \in W_\mathfrak{m}$, we fix a representative $\dot{w} \in N_\mathbf{G}(\mathfrak{m})$.

Definition 4.3.7. We define the action of $N_\mathbf{G}(\mathfrak{m})$ by $nm := mn^{-1}$ for all $m \in M$ and all $n \in N_\mathbf{G}(\mathfrak{m})$. It induces a well defined action of $W_\mathfrak{m}$ on the finite set $\bar{M}$ by

$$w.\mu := w.\mathbf{G}_s\dot{\mu}\mathbf{L} := \mathbf{G}_s\dot{\mu}\dot{w}^{-1}\mathbf{L}$$

for all $\mu \in \bar{M}$, $w \in W_\mathfrak{m}$.
We fix a set Λ of orbit representatives for the action of $W_\mathfrak{m}$ on $\bar{M}$,

$$\bar{M} = \bigsqcup_{\lambda \in \Lambda} W_\mathfrak{m}.\lambda.$$

We may then write

$$s^*(\gamma_*(\tilde{\mathcal{E}})_{sU\cap Y_{\mathbf{L},\Sigma}}) \cong \bigoplus_{\lambda\in\Lambda}\bigoplus_{\mu\in W_{\mathfrak{m}}.\lambda} ((\gamma_\mu)_*\tilde{\mathcal{E}_\mu})_{U\cap Y_{\mathbf{L}_\mu,\Sigma_\mu}}.$$

We want to understand the set $W_{\mathfrak{m}}.\lambda$ for $\lambda\in\Lambda$.
Firstly, we observe that for any $\mu\in\bar{M}$, the stabilizer of $\dot{\mu}$ by the action of $N_{\mathbf{G}}(\mathfrak{m})$ is given by $\mathrm{Stab}_{N_{\mathbf{G}}(\mathfrak{m})}(\dot{\mu}) = N_{\mathbf{G}}(\mathfrak{m})\cap \mathbf{L}\dot{\mu}^{-1}\mathbf{G}_s\dot{\mu}$. Since $\mathbf{L}\subseteq N_{\mathbf{G}}(\mathfrak{m})$,

$$\mathrm{Stab}_{N_{\mathbf{G}}(\mathfrak{m})}(\dot{\mu}) = \mathbf{L}N_{\dot{\mu}^{-1}\mathbf{G}_s\dot{\mu}}(\mathfrak{m}) = N_{\dot{\mu}^{-1}\mathbf{G}_s\dot{\mu}}(\mathfrak{m})\mathbf{L}.$$

Therefore, the stabiliser of $\mu\in\bar{M}$ under the action of $W_{\mathfrak{m}}$ is

$$W_{\mathfrak{m}}^{\mu} \coloneqq \mathrm{Stab}_{W_{\mathfrak{m}}}(\mu) = N_{\dot{\mu}^{-1}\mathbf{G}_s\dot{\mu}}(\mathfrak{m})/(\mathbf{L}\cap\dot{\mu}^{-1}\mathbf{G}_s\dot{\mu}) = N_{\dot{\mu}^{-1}\mathbf{G}_s\dot{\mu}}(\mathfrak{m})/\mathbf{L}_0.$$

We describe this stabiliser more precisely.

Lemma 4.3.8. *Let $\mu\in\bar{M}$ and write $\mathbf{G}_s^{\mu}\coloneqq\dot{\mu}^{-1}\mathbf{G}_s\dot{\mu}$, then*

$$N_{\mathbf{G}_s^{\mu}}(\mathbf{L},\Sigma)/\mathbf{L}_0 = W_{\mathbf{L}_0,\Sigma_0}^{\mathbf{G}_s^{\mu}} \text{ and } W_{\mathfrak{m}}^{\mu}\subseteq W_{\mathfrak{m}_{0,\mu}}^{\mathbf{G}_s^{\mu}}.$$

Moreover, if $s\in Z(\mathbf{L})$ or $W_{\mathfrak{m}} = W_{\mathbf{L},\Sigma}$, then

$$W_{\mathfrak{m}}^{\mu} = W_{\mathfrak{m}_{0,\mu}}^{\mathbf{G}_s^{\mu}}.$$

Proof. Firstly, we note that $\mathbf{L}_0 = C_{\mathbf{L}}^{\circ}(s) = C_{\mathbf{L}}^{\circ}(sz_\mu)$ and $\mathbf{G}_s^{\mu} = C_{\mathbf{G}}^{\circ}(\dot{\mu}^{-1}s\dot{\mu}) = C_{\mathbf{G}}^{\circ}(sz_\mu)$. Since sz_μ is isolated in $\mathbf{L}$, by Lemma 1.3.10,

$$N_{\mathbf{G}_s^{\mu}}(\mathbf{L}) = N_{\mathbf{G}_s^{\mu}}(\mathbf{L}_0).$$

Next, we claim that $N_{\mathbf{G}_s^{\mu}}(\mathbf{L},\Sigma) = N_{\mathbf{G}_s^{\mu}}(\mathbf{L}_0,\Sigma_0)$. We first show that $N_{\mathbf{G}_s^{\mu}}(\mathbf{L},\Sigma)\subseteq N_{\mathbf{G}_s^{\mu}}(\mathbf{L}_0,\Sigma_0)$. We notice that the support $\Sigma_0 = C_0Z^{\circ}(\mathbf{L}_0)$ and that $C_0 = \dot{\mu}^{-1}C_\mu\dot{\mu}$ is the set of unipotent elements $u\in\mathbf{G}_s^{\mu}$ such that $\dot{\mu}^{-1}s\dot{\mu}u\dot{\mu}^{-1}\dot{\mu} = sz_\mu u\in\Sigma$. Thus, since any $n\in N_{\mathbf{G}_s^{\mu}}(\mathbf{L},\Sigma)$ fixes sz_μ we can conclude.
For the other direction, let $n\in N_{\mathbf{G}_s^{\mu}}(\mathbf{L}_0,\Sigma_0)$. Then, there is $l\in\mathbf{L}_0$ such that $nvn^{-1} = lvl^{-1}$, whence

$$nsvn^{-1} = nsz_\mu z_\mu^{-1}vn^{-1} = slvl^{-1}z_\mu nz_\mu^{-1}n^{-1} = lsvl^{-1}z_\mu nz_\mu^{-1}n^{-1}.$$

Therefore, $n\Sigma n^{-1} = (lsvl^{-1}z_\mu nz_\mu^{-1}n^{-1})_{\mathbf{L}}Z^{\circ}(\mathbf{L}) = \Sigma$.
For the last statements, we recall that the local system $\mathcal{E}_{0,\mu}$ is the inverse image under the map $sz_\mu : \Sigma_0\to\Sigma, g\mapsto sz_\mu g$. Now if $n\in N_{\mathbf{G}_s^{\mu}}(\mathfrak{m})$, then $n\in N_{\mathbf{G}_s^{\mu}}(\mathbf{L}_0,\Sigma_0)$ and $\mathrm{ad}(n)$ commutes with the translation by sz_μ. Since $\mathrm{ad}(n)^*\mathcal{E}\cong\mathcal{E}$, we conclude that $N_{\mathbf{G}_s^{\mu}}(\mathfrak{m})\subseteq N_{\mathbf{G}_s^{\mu}}(\mathfrak{m}_{0,\mu})$.
Assume now that $s\in Z(\mathbf{L})$, then $\mathbf{L} = \mathbf{L}_0$ and $\Sigma_0 = s^{-1}\Sigma$. The map $sz_\mu : \Sigma_0\to\Sigma$ is therefore a bijection and $\mathcal{E}$ is the preimage under the map $z_\mu^{-1}s^{-1} : \Sigma\to\Sigma_0$ of $\mathcal{E}_{0,\mu}$. A similar argument as before allows us to deduce that $N_{\mathbf{G}_s^{\mu}}(\mathfrak{m}_{0,\mu}) = N_{\mathbf{G}_s^{\mu}}(\mathfrak{m})$.
Finally, assume that $W_{\mathfrak{m}} = W_{\mathbf{L},\Sigma}$. For any $n\in N_{\mathbf{G}_s^{\mu}}(\mathfrak{m}_{0,\mu})\subseteq N_{\mathbf{G}_s^{\mu}}(\mathbf{L},\Sigma)$, there is an element $n'\in N_{\mathbf{G}}(\mathfrak{m})$ and $l\in\mathbf{L}$ such that $n = n'l$. Then

$$\mathrm{ad}(n)^*\mathcal{E} = \mathrm{ad}(n'l)^*\mathcal{E} = \mathrm{ad}(l)^*\,\mathrm{ad}(n')^*\mathcal{E}\cong\mathcal{E},$$

since $\mathcal{E}$ is $\mathbf{L}$-equivariant. □

Now, we fix a set of representatives $V^\lambda \subseteq W_{\mathfrak{m}}$ such that $W_{\mathfrak{m}} \coloneqq \bigsqcup_{v\in V^\lambda} vW_{\mathfrak{m}}^\lambda$. In other words, we write

$$W_{\mathfrak{m}}.\lambda = \{v.\lambda \mid v \in V^\lambda\},$$

and the isomorphism $\mathcal{T}$ becomes

$$s^*(\gamma_*(\tilde{\mathcal{E}})_{sU\cap Y_{\mathbf{L},\Sigma}}) \cong \bigoplus_{\lambda\in\Lambda}\bigoplus_{v\in V^\lambda}((\gamma_{v.\lambda})_*\tilde{\mathcal{E}}_{v.\lambda})_{U\cap Y_{\mathbf{L}_{v.\lambda},\Sigma_{v.\lambda}}}.$$

Let us rewrite $\tilde{\mathcal{E}}_{v.\lambda}$ for $\lambda\in\Lambda$ and $v\in V^\lambda$.

Lemma 4.3.9. *Let $\lambda\in\Lambda$ and $v\in V^\lambda$. Then*

$$\mathbf{L}_{v.\lambda} = \mathbf{L}_\lambda, \quad C_{v.\lambda} = C_\lambda, \quad \Sigma_{v.\lambda} = \Sigma_\lambda.$$

Moreover,

$$\tilde{\mathcal{E}}_{v.\lambda} \cong s_\lambda^*\bar{\varphi}_v^*\tilde{\mathcal{E}},$$

where $\bar{\varphi}_v : \mathbf{G}\times_{\mathbf{L}}\Sigma_{reg} \to \mathbf{G}\times_{\mathbf{L}}\Sigma_{reg}, \beta((g,h)) \mapsto \beta((g\dot{v}^{-1}, \dot{v}h\dot{v}^{-1}))$.

Proof. By definition, we have $\mathbf{L}_{v.\lambda} = \lambda\dot{v}^{-1}\mathbf{L}\dot{v}\lambda^{-1}\cap\mathbf{G}_s$, $\Sigma_{v.\lambda} = Z^\circ(\mathbf{L}_{v.\lambda})C_{\lambda\dot{v}^{-1}}$. Now, by definition of $W_{\mathfrak{m}}$, $\dot{v}\in N_{\mathbf{G}}(\mathbf{L})$, whence $\mathbf{L}_{v.\lambda} = \mathbf{L}_\lambda$. Moreover since the conjugation by $\dot{v}$ stabilises Σ, $C_{\lambda\dot{v}^{-1}} = C_\lambda$.
From the proof of Proposition 4.3.3, we recall that $\tilde{\mathcal{E}}_{v.\lambda} \cong s_{v.\lambda}^*\tilde{\mathcal{E}}$. Noting that $s_{v.\lambda} = \bar{\varphi}_v\circ s_\lambda$, we conclude that

$$\tilde{\mathcal{E}}_{v.\lambda} \cong s_\lambda^*\bar{\varphi}_v^*\tilde{\mathcal{E}}.$$

□

In particular, the isomorphism of Proposition 4.3.3 can be rewritten as

$$\mathcal{T} : s^*(\gamma_*(\tilde{\mathcal{E}})_{sU\cap Y_{\mathbf{L},\Sigma}}) \xrightarrow{\sim} \bigoplus_{\lambda\in\Lambda}\bigoplus_{v\in V^\lambda}((\gamma_\lambda)_* s_\lambda^*\bar{\varphi}_v^*\tilde{\mathcal{E}})_{U\cap Y_{\mathbf{L}_\lambda,\Sigma_\lambda}}.$$

We will use this isomorphism to understand the restriction of a character sheaf to a mixed conjugacy class.

4.3.2 Restriction of a character sheaf to a mixed conjugacy class

Let $\mathcal{A}\in\hat{\mathbf{G}}(\mathfrak{m})$ be a character sheaf. There exists a unique $V\in\mathrm{Irr}(\mathrm{End}(\mathcal{K}_{\mathfrak{m}}))$ such that the character sheaf $\mathcal{A} = \mathcal{A}_V = \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}})}(V,\mathcal{K}_{\mathfrak{m}})$. Then

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] = \mathrm{Hom}_{\mathrm{End}(s^*(\mathcal{K}_{\mathfrak{m}})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d])}(V, s^*(\mathcal{K}_{\mathfrak{m}})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d]),$$

where V now denotes the (not necessarily irreducible) $\mathrm{End}(s^*(\mathcal{K}_{\mathfrak{m}})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d])$-module with underlying vector space V and action of $\theta\in\mathrm{End}(s^*(\mathcal{K}_{\mathfrak{m}})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d])$ given by the action of $\phi\in\mathrm{End}(\mathcal{K}_{\mathfrak{m}})$ when $\theta = s^*\phi_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d]$. Using the isomorphism $\mathcal{T}$ of Proposition 4.3.3, we get

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] \cong \mathrm{Hom}_{\mathrm{End}(\bigoplus_{\mu\in\bar{M}}(\mathcal{K}_\mu)_{(\mathbf{G}_s)_{\mathrm{uni}}}[e])}(V, \bigoplus_{\mu\in\bar{M}}(\mathcal{K}_\mu)_{(\mathbf{G}_s)_{\mathrm{uni}}}[e]),$$

where, this time, V is viewed as an $\mathrm{End}(\bigoplus_{\mu\in\bar{M}}(\mathcal{K}_\mu)_{(\mathbf{G}_s)_{\mathrm{uni}}}[e])$-module with the action of $\mathcal{T}\circ\theta\circ\mathcal{T}^{-1} \in \mathrm{End}(\bigoplus_{\mu\in\bar{M}}(\mathcal{K}_\mu)_{(\mathbf{G}_s)_{\mathrm{uni}}}[e])$ given by the action of $\theta\in\mathrm{End}(s^*(\mathcal{K}_\mathfrak{m})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d])$. As for the central translation, it is not very clear what the $\mathrm{End}(\bigoplus_{\mu\in\bar{M}}(\mathcal{K}_\mu)_{(\mathbf{G}_s)_{\mathrm{uni}}}[e])$-module V actually is, firstly because we do not know the $\mathrm{End}(\mathcal{K}_\mathfrak{m})$-module V. To be able to discuss this further, we make the following hypothesis.

Hypothesis 4.3.10. For the rest of this section, we assume that $\mathrm{End}(\mathcal{K}_\mathfrak{m})$ is isomorphic to the group algebra $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$.

We also fix an isomorphism from $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ to $\mathrm{End}(\mathcal{K}_\mathfrak{m})$ as we did in Subsection 3.2.3:

1. For each $w\in W_\mathfrak{m}$, we choose a representative $\dot{w}\in N_\mathbf{G}(\mathbf{L})$.
2. We fix basis elements $a_w:\mathrm{ad}(\dot{w})^*\mathcal{E}\to\mathcal{E}$ for $w\in W_\mathfrak{m}$ and consider the isomorphism
$$\begin{aligned} a:\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}] &\tilde{\to} \mathscr{A}_\mathcal{E} \\ w &\mapsto a_w. \end{aligned}$$
3. We lift each isomorphism a_w to $\tilde{a}_w:\bar{\varphi}_w^*\tilde{\mathcal{E}}\tilde{\to}\tilde{\mathcal{E}}$.
4. We precompose the isomorphism $\gamma_*\tilde{a}_w$ by the isomorphism $bc_w:\gamma_*\tilde{\mathcal{E}}\tilde{\to}\gamma_*\bar{\varphi}_w^*\tilde{\mathcal{E}}$ (due to base change) to get an endomorphism $A_w:\gamma_*\tilde{\mathcal{E}}\to\gamma_*\tilde{\mathcal{E}}$.
5. Lastly, we use the isomorphism $Ic:\mathrm{End}(\gamma_*\tilde{E})\tilde{\to}\mathrm{End}(\mathcal{K}_\mathfrak{m})$ given by the definition of $\mathcal{K}_\mathfrak{m}$.

Therefore, for a character sheaf $\mathcal{A}\in\hat{\mathbf{G}}(\mathfrak{m})$ there is a unique $V\in\mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}])$ such that $\mathcal{A}\cong\mathrm{Hom}_{\mathrm{End}(\mathcal{K}_\mathfrak{m})}(V,\mathcal{K}_\mathfrak{m})$, where we see V as an $\mathrm{End}(\mathcal{K}_\mathfrak{m})$-module via the action of $Ic(A_w)\in\mathrm{End}(\mathcal{K}_\mathfrak{m})$ given by the action of $w\in W_\mathfrak{m}$.
We now would like to understand V seen as an $\mathrm{End}(\bigoplus_{\mu\in\bar{M}}(\mathcal{K}_\mu)_{(\mathbf{G}_s)_{\mathrm{uni}}}[e])$-module. Namely, for each $w\in W_\mathfrak{m}$, we want to describe the isomorphism $\mathcal{T}\circ s^*(Ic(A_w)_{sU\cap Y_{\mathbf{L},\Sigma}})\circ\mathcal{T}^{-1}$, or rather the isomorphism $\mathcal{T}\circ s^*((A_w)_{sU\cap Y_{\mathbf{L},\Sigma}})\circ\mathcal{T}^{-1}\in\mathrm{End}(\bigoplus_{\mu\in\bar{M}}((\gamma_\mu)_*\tilde{\mathcal{E}}_\mu)_{U\cap Y_{\mathbf{L}_\mu,\Sigma_\mu}})$. Alternatively, we will instead describe $\bigoplus_{\mu\in\bar{M}}((\gamma_\mu)_*\tilde{\mathcal{E}}_\mu)_{U\cap Y_{\mathbf{L}_\mu,\Sigma_\mu}})$ seen as a $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$-module via the action of $w\in W_\mathfrak{m}$ given by

$$B_w := \mathcal{T}\circ s^*((A_w)_{(sU\cap Y_{\mathbf{L},\Sigma})})\circ\mathcal{T}^{-1}.$$

By [LuCS2, 8.7.13], the set $sU\cap Y_{\mathbf{L},\Sigma}=\bigsqcup_{\lambda\in\Lambda}s(U\cap Y_{\mathbf{L}_\lambda,\Sigma_\lambda})$, whence

$$s^*((A_w)_{(sU\cap Y_{\mathbf{L},\Sigma})})=\bigoplus_{\lambda\in\Lambda}s^*((A_w)_{s(U\cap Y_{\mathbf{L}_\lambda,\Sigma_\lambda})}).$$

We now decompose B_w into $B_w=\bigoplus_{\lambda\in\Lambda}B_w^\lambda$ where

$$B_w^\lambda=\mathcal{T}\circ s^*((A_w)_{s(U\cap Y_{\mathbf{L}_\lambda,\Sigma_\lambda})})\circ\mathcal{T}^{-1}.$$

In particular, each B_w^λ belongs to $\mathrm{End}\big(\bigoplus_{v\in V^\lambda}((\gamma_\lambda)_*s_\lambda^*\bar{\varphi}_v^*\tilde{\mathcal{E}}_\lambda)_{U\cap Y_{\mathbf{L}_\lambda,\Sigma_\lambda}}\big)$. We now describe for each $\lambda\in\Lambda$ the $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$-module $\bigoplus_{v\in V^\lambda}((\gamma_\lambda)_*s_\lambda^*\bar{\varphi}_v^*\tilde{\mathcal{E}})_{U\cap Y_{\mathbf{L}_\lambda,\Sigma_\lambda}}$ where the action of $w\in W_\mathfrak{m}$ is given by B_w^λ.

Lemma 4.3.11. *For each* $\lambda \in \Lambda$ *and each* $w \in W_{\mathfrak{m}}^{\lambda}$, *the map* B_w^{λ} *induces an endomorphism* $b_w^{\lambda} \in \operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$, *which gives* $(\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}}$ *the structure of a* $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}^{\lambda}]$*-module. Furthermore, there is an isomorphism of* $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}]$*-modules*

$$\bigoplus_{v \in V_\lambda} ((\gamma_\lambda)_* s_\lambda^* \bar{\varphi}_v^* \tilde{\mathcal{E}})_{U \cap Y_{\mathbf{L}_\lambda, \Sigma_\lambda}} \cong \operatorname{Ind}_{W_{\mathfrak{m}}^{\lambda}}^{W_{\mathfrak{m}}} \big(((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})_{U \cap Y_{\mathbf{L}_\lambda, \Sigma_\lambda}} \big),$$

where $W_{\mathfrak{m}}$ *acts on the left-hand side via* B_w^{λ} *for* $w \in W_{\mathfrak{m}}$ *and* $W_{\mathfrak{m}}^{\lambda}$ *acts on* $((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})_{U \cap Y_{\mathbf{L}_\lambda, \Sigma_\lambda}}$ *via* $(b_w^{\lambda})_{U \cap Y_{\mathbf{L}_\lambda, \Sigma_\lambda}}$ *for* $w \in W_{\mathfrak{m}}^{\lambda}$.

Proof. We fix $w \in W_{\mathfrak{m}}$. To simplify notation, we denote by i the inclusion $U \cap Y_{\mathbf{L}_\lambda, \Sigma_\lambda} \subseteq Y_{\mathbf{L}_\lambda, \Sigma_\lambda}$. By construction of A_w,

$$B_w^{\lambda} = \mathcal{T} \circ i^* s^* \gamma_* \tilde{a}_w \circ \mathcal{T}^{-1} \circ \mathcal{T} \circ i^* s^* bc_w \circ \mathcal{T}^{-1}.$$

Using the proof of Proposition 4.3.3, we make explicit $\mathcal{T}$. We set

$$bc_\lambda(\tilde{\mathcal{E}}) : i^* s^* \gamma_* \tilde{\mathcal{E}} \to i^* (\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}}$$

to be the isomorphism due to base change. For each $v \in V_\lambda$, we write $bc_v(\tilde{\mathcal{E}}) : \gamma_* \tilde{\mathcal{E}} \to \gamma_* \bar{\varphi}_v^* \tilde{\mathcal{E}}$ for the isomorphism induced by base change. Then

$$\mathcal{T} = \bigoplus_{v \in V_\lambda} bc_\lambda(\bar{\varphi}_v^* \tilde{\mathcal{E}}) \circ i^* s^* bc_v(\tilde{\mathcal{E}}).$$

We first describe $\mathcal{T} \circ i^* s^* bc_w \circ \mathcal{T}^{-1}$. For each $v \in V_\lambda$ there exists $w_0 \in W_{\mathfrak{m}}^{\lambda}$ and $v' \in V_\lambda$ such that $wv = v' w_0$. Then, we have

$$\begin{aligned}
\mathcal{T} \circ i^* s^* bc_w(\tilde{\mathcal{E}}) &= \bigoplus_{v \in V_\lambda} bc_\lambda(\bar{\varphi}_v^* \bar{\varphi}_w^* \tilde{\mathcal{E}}) \circ i^* s^* bc_v(\bar{\varphi}_w^* \tilde{\mathcal{E}}) \circ i^* s^* bc_w(\tilde{\mathcal{E}}) \\
&= \bigoplus_{v \in V_\lambda} bc_\lambda(\bar{\varphi}_{wv}^* \tilde{\mathcal{E}}) \circ i^* s^* bc_{wv}(\tilde{\mathcal{E}}) \\
&= \bigoplus_{v \in V_\lambda} bc_\lambda(\bar{\varphi}_{v' w_0}^* \tilde{\mathcal{E}}) \circ i^* s^* bc_{v' w_0}(\tilde{\mathcal{E}}) \\
&= \bigoplus_{v \in V_\lambda} bc_\lambda(\bar{\varphi}_{w_0}^* \bar{\varphi}_{v'}^* \tilde{\mathcal{E}}) \circ i^* s^* bc_{w_0}(\bar{\varphi}_{v'}^* \tilde{\mathcal{E}}) \circ i^* s^* bc_{v'}(\tilde{\mathcal{E}}).
\end{aligned}$$

By Lemma 4.3.8, $\lambda w_0 \lambda^{-1} \in W_{\mathfrak{m}_\lambda}^{\mathbf{G}_s}$. In particular, the isomorphism $\bar{\varphi}_{\lambda w_0 \lambda^{-1}} : \mathbf{G}_s \times_{\mathbf{L}_\lambda} \Sigma_{\lambda, reg}$ is well defined. We observe that $s_\lambda \circ \bar{\varphi}_{\lambda w_0 \lambda^{-1}} = \bar{\varphi}_{w_0} \circ s_\lambda$. Therefore,

$$bc_\lambda(\bar{\varphi}_{w_0}^* \bar{\varphi}_{v'}^* \tilde{\mathcal{E}}) \circ i^* s^* bc_{w_0}(\bar{\varphi}_{v'}^* \tilde{\mathcal{E}}) = i^* bc_{\lambda w_0 \lambda^{-1}}(s_\lambda^* \bar{\varphi}_{v'}^* \tilde{\mathcal{E}}) \circ bc_\lambda(\bar{\varphi}_{v'}^* \tilde{\mathcal{E}}),$$

where $bc_{\lambda w_0 \lambda^{-1}}(s_\lambda^* \bar{\varphi}_{v'}^* \tilde{\mathcal{E}})$ denotes the isomorphism $(\gamma_\lambda)_*(s_\lambda^* \bar{\varphi}_{v'}^* \tilde{\mathcal{E}}) \to (\gamma_\lambda)_* \bar{\varphi}_{\lambda w_0 \lambda^{-1}}^* (s_\lambda^* \bar{\varphi}_{v'}^* \tilde{\mathcal{E}})$ due to base change. We conclude that

$$\mathcal{T} \circ i^* s^* bc_w = \bigoplus_{v \in V_\lambda} i^* bc_{\lambda w_0 \lambda^{-1}}(s_\lambda^* \bar{\varphi}_{v'}^* \tilde{\mathcal{E}}) \circ bc_\lambda(\bar{\varphi}_{v'}^* \tilde{\mathcal{E}}) \circ i^* s^* bc_{v'}(\tilde{\mathcal{E}}).$$

Thus, $\mathcal{T} \circ i^* s^* bc_w \circ \mathcal{T}^{-1}$ is the morphism which consists of first rearranging the terms of the sum, sending v to v', and then acting on each summand via $i^* bc_{\lambda w_0 \lambda^{-1}}(s_\lambda^* \bar{\varphi}_{v'}^* \tilde{\mathcal{E}})$.

We now consider $\mathcal{T} \circ i^* s^* \gamma_* \tilde{a}_w \circ \mathcal{T}^{-1}$. By definition, the isomorphisms induced by base change are natural, and therefore,

$$\mathcal{T} \circ i^* s^* \gamma_* \tilde{a}_w \circ \mathcal{T}^{-1} = \bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \bar{\varphi}_v^* \tilde{a}_w.$$

To conclude, we consider the following isomorphism

$$\mathcal{X} \coloneqq \bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \tilde{a}_v : \bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \bar{\varphi}_v^* \tilde{\mathcal{E}} \xrightarrow{\sim} \bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}}.$$

It gives $\bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}}$ the structure of $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$-module via the action of $w \in W_\mathfrak{m}$ by $\mathcal{X} \circ B_w^\lambda \circ \mathcal{X}^{-1}$. The morphism $\mathcal{X} \circ \mathcal{T} \circ i^* s^* bc_w \circ \mathcal{T}^{-1} \circ \mathcal{X}^{-1}$ consists of rearranging the terms of the sum and acting on each of the terms by $i^* bc_{\lambda w_0 \lambda^{-1}}(s_\lambda^* \tilde{\mathcal{E}})$. Moreover, by definition of $\tilde{a}_w$, and thanks to the hypothesis 4.3.10 of a trivial cocycle,

$$\mathcal{X} \circ \bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \bar{\varphi}_v^* \tilde{a}_w = \bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \tilde{a}_{wv} = \bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \bar{\varphi}_{w_0}^* \tilde{a}_{w_0} \circ \bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \tilde{a}_{v'}.$$

Therefore, $w \in W_\mathfrak{m}$ acts on $\bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}}$ by first rearranging the terms, then acting via $i^*(\gamma_\lambda)_* s_\lambda^* \bar{\varphi}_{w_0}^* \tilde{a}_{w_0} \circ i^* bc_{\lambda w_0 \lambda^{-1}}(s_\lambda^* \tilde{\mathcal{E}})$. For each $w_0 \in W_\mathfrak{m}^\lambda$, we set

$$b_{w_0}^\lambda \coloneqq (\gamma_\lambda)_* s_\lambda^* \tilde{a}_{w_0} \circ bc_{\lambda w_0 \lambda^{-1}}(s_\lambda^* \tilde{\mathcal{E}}) \in \operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}}).$$

By definition of the induction of modules, we conclude

$$\bigoplus_{v \in V_\lambda} i^*(\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}} = \operatorname{Ind}_{W_\mathfrak{m}^\lambda}^{W_\mathfrak{m}} \left(i^*(\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}} \right)$$

for the action of $W_\mathfrak{m}^\lambda$ on $i^*(\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}}$ given by $i^* b_{w_0}^\lambda$. This proves the lemma. □

Corollary 4.3.12. *Let* $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ *and* $V \in \operatorname{Irr}(\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}])$ *such that* $\mathcal{A} \cong \operatorname{Hom}_{\operatorname{End}(\mathcal{K}_\mathfrak{m})}(V, \mathcal{K}_\mathfrak{m})$. *Then*

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] \cong \bigoplus_{\lambda \in \Lambda} (\mathcal{A}'_{\operatorname{Res}_{W_\mathfrak{m}^\lambda}^{W_\mathfrak{m}} V})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}},$$

where $\mathcal{A}'_{\operatorname{Res}_{W_\mathfrak{m}^\lambda}^{W_\mathfrak{m}} V} = \operatorname{Hom}_{\operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})}(\operatorname{Res}_{W_\mathfrak{m}^\lambda}^{W_\mathfrak{m}} V, (\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$ *with the restriction* $\operatorname{Res}_{W_\mathfrak{m}^\lambda}^{W_\mathfrak{m}} V$ *viewed as an* $\operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$*-module under the isomorphism given by* b_w^λ *for* $w \in W_\mathfrak{m}^\lambda$.

Proof. By definition of the action of $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$ on $\bigoplus_{\lambda \in \Lambda} \bigoplus_{v \in V_\lambda} ((\gamma_\lambda)_* s_\lambda^* \bar{\varphi}_v^* \tilde{\mathcal{E}})_{U \cap Y_{\mathbf{L}_\lambda, \Sigma_\lambda}}$, the isomorphism $\mathcal{T}$ commutes with the action of $\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}]$. Therefore, by Lemma 4.3.11 and the adjunction between restriction and induction, we obtain

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] \cong \bigoplus_{\lambda \in \Lambda} (\mathcal{A}'_{\operatorname{Res}_{W_\mathfrak{m}^\lambda}^{W_\mathfrak{m}} V})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}}.$$

□

Remark 4.3.13. Let $\lambda \in \Lambda$ and $w \in \lambda W_{\mathfrak{m}}^{\lambda}\lambda^{-1}$. Then $w \in W_{\mathfrak{m}_\lambda}^{\mathbf{G}_s}$ by Lemma 4.3.8. Let $w_0 = \lambda^{-1}w\lambda \in W_{\mathfrak{m}}^{\lambda}$. The $\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}^{\lambda}]$-module $(\gamma_\lambda)_* s_\lambda^* \bar{\varphi}_v^* \tilde{\mathcal{E}}$ may be seen as a $\overline{\mathbb{Q}}_\ell[\lambda W_{\mathfrak{m}}^{\lambda}\lambda^{-1}]$-module via the action of $w \in \lambda W_{\mathfrak{m}}^{\lambda}\lambda^{-1}$ given by $b_w^\lambda := b_{w_0}^\lambda$. By definition

$$b_w^\lambda = (\gamma_\lambda)_* s_\lambda^* \tilde{a}_{\lambda^{-1}w\lambda} \circ bc_w(s_\lambda^* \tilde{\mathcal{E}}).$$

By construction of $\tilde{a}_{\lambda^{-1}w\lambda}$ and diagram chasing, we see that the map $s_\lambda^* \tilde{a}_{\lambda^{-1}w\lambda}$ is the lift of $\tau_\lambda^*(a_{\lambda^{-1}w\lambda}) \in \mathscr{A}_{\mathcal{E}_\lambda}$.

Note that the action of $\lambda W_{\mathfrak{m}}^{\lambda}\lambda^{-1} \subseteq W_{\mathfrak{m}_\lambda}^{\mathbf{G}_s}$ we have fixed might differ from the action given by the isomorphism in Proposition 3.2.17. In particular, we cannot directly apply Remark 3.2.19 to compute $\mathcal{A}'_{\mathrm{Res}^{W_{\mathfrak{m}}}_{W_{\mathfrak{m}}^{\lambda}} V}[e]_{(\mathbf{G}_s)_{\mathrm{uni}}}$. To compare the two actions, it suffices to look at the two isomorphisms b_w and $\tau_\lambda^*(a_{\lambda^{-1}w\lambda})$ as defined in Proposition 3.2.17 for each $w \in \lambda W_{\mathfrak{m}}^{\lambda}\lambda^{-1}$. In the next subsections, we compare these two actions in different scenarios.
Before that, we first describe the set Λ.

A description of the set Λ

As in [AA10, § 6.2], we fix a set $R \subset \bar{M}$ such that the groups $\mathbf{L}_r$ for $r \in R$ are not conjugate under $\mathbf{G}_s$, but for all $\mu \in \bar{M}$, there is $h \in \mathbf{G}_s$ and $r \in R$ such that $\mathbf{L}_\mu = {}^h\mathbf{L}_r$. We set for all $r \in R$,

$$\bar{M}_r := \{\mu \in \bar{M} \mid \mathbf{L}_\mu = {}^h\mathbf{L}_r \text{ for some } h \in \mathbf{G}_s\},$$

and we have

$$\bar{M} = \bigsqcup_{r \in R} \bar{M}_r.$$

Observe that since $\mathbf{L}_\mu = \mathbf{L}_{w.\mu}$ for all $w \in W_{\mathfrak{m}}$, the set $\bar{M}_r$ is $W_{\mathfrak{m}}$-invariant for all $r \in R$.

Lemma 4.3.14 ([AA10, Thm. 7.2])**.** *If* $\mathbf{G}$ *is semisimple, quasi-simple and different from the projective symplectic groups* PSp_{2n}*, the projective simply orthogonal groups* PSO_{2n}*, the half-spin groups* $1/2\,\mathrm{Spin}_{2n}$ *and* E_7 *simply-connected, then* $|R| = 1$.

Lemma 4.3.15. *Assume that* $W_{\mathbf{L},\Sigma} = W_{\mathbf{L}}$*. Then for all* $r \in R$*, the map*

$$\begin{aligned} \omega : W_{\mathbf{L}}/W_{\mathbf{L}_0}^{\mathbf{G}_s^r} &\to \bar{M}_r \\ wW_{\mathbf{L}_0}^{\mathbf{G}_s^r} &\mapsto \mathbf{G}_s \dot{r} \dot{w}^{-1} \mathbf{L} \end{aligned}$$

is a $W_{\mathfrak{m}}$*-equivariant bijection.*

Proof. We first show that the map ω is well-defined and $W_{\mathfrak{m}}$-equivariant.
Fix $wW_{\mathbf{L}_0}^{\mathbf{G}_s^r} \in W_{\mathbf{L}}/W_{\mathbf{L}_0}^{\mathbf{G}_s^r}$. Since $W_{\mathbf{L},\Sigma} = W_{\mathbf{L}}$, we have $\dot{r}\dot{w}^{-1} \in M$ and $\mathbf{G}_s\dot{r}\dot{w}^{-1}\mathbf{L} \in \bar{M}_r$ for all $w \in W_{\mathbf{L}}$. Furthermore, let $n \in N_{\mathbf{G}}(\mathbf{L})$ and $l \in \mathbf{L}$ such that $nl = \dot{w}$. Since $\dot{w} \in N_{\mathbf{G}}(\mathbf{L})$,

$$\mathbf{G}_s \dot{r} n \mathbf{L} = \mathbf{G}_s \dot{r} l \dot{w}^{-1} \mathbf{L} = \mathbf{G}_s \dot{r} \dot{w}^{-1} \mathbf{L}.$$

Now, let $w_0 \in W_{\mathbf{L}_0}^{\mathbf{G}_s^r}$. Since $\dot{w}_0 \in \mathbf{G}_s^r$,

$$\mathbf{G}_s \dot{r} \dot{w}_0^{-1} \dot{w}^{-1} \mathbf{L} = \mathbf{G}_s \dot{r} \dot{w}_0^{-1} \dot{r}^{-1} \dot{r} \dot{w}^{-1} \mathbf{L} = \mathbf{G}_s \dot{r} \dot{w}^{-1} \mathbf{L}.$$

Lastly, we fix $v \in W_{\mathfrak{m}}$. Observe that

$$\omega(vwW_{\mathbf{L}_0}^{\mathbf{G}_s{}^r}) = \mathbf{G}_s \dot{r} \dot{w}^{-1} \dot{v}^{-1} \mathbf{L} = v.\mathbf{G}_s \dot{r} \dot{w}^{-1} \mathbf{L} = v.\omega(wW_{\mathbf{L}_0}^{\mathbf{G}_{s^r}}).$$

Therefore, ω is a well-defined $W_{\mathfrak{m}}$-equivariant map.
We show that the map ω is surjective. Let $\mu \in \bar{M}_r$. By definition of $\bar{M}_r$, there is $h \in \mathbf{G}_s$ such that ${}^{\mu}\mathbf{L}_0 = \mathbf{L}_{\mu} = {}^{h}\mathbf{L}_r = {}^{h\dot{r}}\mathbf{L}_0$. In particular, it means that $\mu^{-1} h \dot{r} \in N_{\mathbf{G}}(\mathbf{L}_0)$, whence $\mu^{-1} h \dot{r} \in N_{\mathbf{G}}(\mathbf{L})$ by Lemma 1.3.10. We write $n = \mu^{-1} h \dot{r}$. We have

$$\omega(n\mathbf{L}W_{\mathbf{L}_0}^{\mathbf{G}_s{}^r}) = \mathbf{G}_s \dot{r} n^{-1} \mathbf{L} = \mathbf{G}_s \dot{r} \dot{r}^{-1} h^{-1} \mu \mathbf{L} = \mathbf{G}_s \mu \mathbf{L}$$

and the map ω is surjective.
We are left to show that the map ω is injective. Let $w_1, w_2 \in W_{\mathbf{L}}$ be two elements such that $\omega(w_1 W_{\mathbf{L}_0}^{\mathbf{G}_s^r}) = \omega(w_2 W_{\mathbf{L}_0}^{\mathbf{G}_s^r})$. Then there is $h \in \mathbf{G}_s$, $l \in \mathbf{L}$ such that $h \dot{r} \dot{w}_1^{-1} l = \dot{r} \dot{w}_2^{-1}$. Thus,

$$\dot{r}^{-1} h \dot{r} = \dot{w}_2^{-1} l^{-1} \dot{w}_1 \in N_{\mathbf{G}}(\mathbf{L}) \cap \mathbf{G}_s^r \subseteq N_{\mathbf{G}_s^r}(\mathbf{L}) \subseteq N_{\mathbf{G}_s^r}(\mathbf{L}_0),$$

whence $w_1 W_{\mathbf{L}_0}^{\mathbf{G}_s^r} = w_2 W_{\mathbf{L}_0}^{\mathbf{G}_s^r}$ and ω is injective. □

Corollary 4.3.16. *If $|R| = 1$ and $W_{\mathbf{L},\Sigma} = W_{\mathbf{L}}$, the map ω induces a bijection from the set of double cosets $W_{\mathfrak{m}} \backslash W_{\mathbf{L}} / W_{\mathbf{L}_0}^{\mathbf{G}_s}$ to Λ.*

Remark 4.3.17. The assumption $W_{\mathbf{L},\Sigma} = W_{\mathbf{L}}$ is satisfied in particular when:

- $\mathbf{L} = \mathbf{G}$ (then $W_{\mathbf{G},\Sigma} = W_{\mathbf{G}} = 1$),
- $\mathbf{L} = \mathbf{T}$ (then $\Sigma = \mathbf{T}$),
- Σ is the preimage of a unipotent class ([Lus84b, Thm. 9.2.b]),
- Σ is the unique preimage of an isolated class which belongs to a cuspidal pair of $\mathbf{L}$ (up to $\mathbf{L}$-conjugation),
- or $K_{\mathfrak{m}}$ is unipotent and $Z(\mathbf{G})$ is connected, since in this case $W_{\mathfrak{m}} = W_{\mathbf{L}}$ by [AA10, Prop. 4.4].

4.3.3 Restriction of a unipotently supported character sheaf to a mixed conjugacy class

We suppose that Σ contains unipotent elements. In other words $\Sigma = (v)_{\mathbf{L}} Z^{\circ}(\mathbf{L})$ and we assume that $s \in Z^{\circ}(\mathbf{L})$. By Remark 3.2.8, we write $\mathcal{E} = i^*(\mathcal{E}_0 \boxtimes \mathcal{Z})$ where $\mathcal{E}_0$ is an irreducible local system on $(v)_{\mathbf{L}}$, $\mathcal{Z} = \mu^* \mathcal{E}_{n,\mathfrak{j}}$ is a Kummer local system on $Z^{\circ}(\mathbf{L})$

with $\mu \in X(Z^\circ(\mathbf{L}))$ and $n \in \mathbb{N}$ coprime to p, and $i : \Sigma \to (v)_{\mathbf{L}} \times Z^\circ(\mathbf{L})$ is the canonical map. For $\lambda \in \Lambda$, the local system $\mathcal{E}_\lambda$ on $\Sigma_\lambda = \lambda\Sigma\lambda^{-1}$ is

$$\begin{aligned} s^* \operatorname{ad}(\lambda)^* i^*(\mathcal{E}_0 \boxtimes \mathcal{Z}) &= \operatorname{ad}(\lambda)^*(\lambda s \lambda^{-1})^* i^*(\mathcal{E}_0 \boxtimes \mathcal{Z}) \\ &= \operatorname{ad}(\lambda)^*(s z_\lambda)^* i^*(\mathcal{E}_0 \boxtimes \mathcal{Z}) \\ &= \operatorname{ad}(\lambda)^* i^*(\mathcal{E}_0 \boxtimes (s z_\lambda)^* \mathcal{Z}). \end{aligned}$$

We make a specific choice of a_w for $w \in W_{\mathfrak{m}}$ following Proposition 3.2.17.
We set $a_w = b_w^{\mathcal{Z}} : \theta'_w \otimes (id \boxtimes \phi_w^{\mathcal{Z}})$ for θ'_w as given by Lusztig in [Lus84b, Thm. 9.2d] and the morphism $\phi_w^{\mathcal{Z}} : \operatorname{ad}(w)^* \mathcal{Z} \to \mathcal{Z}$ is determined by the condition that $(\phi_w^{\mathcal{Z}})_1$ is the identity. Similarly, for $w \in W_{\mathfrak{m}_\lambda}^{\mathbf{G}_s}$, the choice of $b_w^{\mathcal{E}_\lambda} : \operatorname{ad}(w)^* \mathcal{E}_\lambda \to \mathcal{E}_\lambda$ is fixed in Proposition 3.2.17, with $b_w^{\mathcal{E}_\lambda} = \operatorname{ad}(\lambda)^* b_{\lambda w \lambda^{-1}}^{(s z_\lambda)^* \mathcal{Z}}$. Observe that $\tau_\lambda^* a_{\lambda w \lambda^{-1}} = \operatorname{ad}(\lambda)^*(s z_\lambda)^* b_{\lambda w \lambda^{-1}}^{\mathcal{Z}}$, since the map τ_λ sends $g \in \Sigma_\lambda$ to $\dot\lambda^{-1} s g \dot\lambda$. We then obtain the following result.

Lemma 4.3.18. *Let $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ and $V \in \operatorname{Irr}(\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}])$ such that $\mathcal{A} \cong \operatorname{Hom}_{\operatorname{End}(\mathcal{K}_{\mathfrak{m}})}(V, \mathcal{K}_{\mathfrak{m}})$. Then for any $s \in Z^\circ(\mathbf{L})$,*

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] \cong \bigoplus_{\lambda \in \Lambda} (\mathcal{A}'_{\operatorname{Res}^{W_{\mathfrak{m}}}_{W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}}(V) \otimes X^s_\lambda})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}},$$

where $\mathcal{A}'_{\operatorname{Res}^{W_{\mathfrak{m}}}_{W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}}(V) \otimes X^s_\lambda} = \operatorname{Hom}_{\operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})}(\operatorname{Res}^{W_{\mathfrak{m}}}_{W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}}(V) \otimes X^s_\lambda, (\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$.
We see $\operatorname{Res}^{W_{\mathfrak{m}}}_{W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}}(V) \otimes X^s_\lambda$ as an $\operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$-module under the isomorphism given by $b_w^{\mathcal{E}_\lambda}$ for $w \in W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}$ and X^s_λ is the $\overline{\mathbb{Q}}_\ell[W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}]$-module with character $\chi^s_\lambda : w \mapsto \mathfrak{j}(\mu_{\lambda^{-1} w \lambda}(\lambda s \lambda^{-1}))$.

Proof. By Corollary 4.3.12 and Remark 4.3.13,

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] \cong \bigoplus_{\lambda \in \Lambda} (\mathcal{A}'_{\operatorname{Res}^{W_{\mathfrak{m}}}_{\lambda W^\lambda_{\mathfrak{m}} \lambda^{-1}} V})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}},$$

where $\mathcal{A}'_{\operatorname{Res}^{W_{\mathfrak{m}}}_{\lambda W^\lambda_{\mathfrak{m}} \lambda^{-1}} V} = \operatorname{Hom}_{\operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})}(\operatorname{Res}^{W_{\mathfrak{m}}}_{\lambda W^\lambda_{\mathfrak{m}} \lambda^{-1}} V, (\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$ with $\operatorname{Res}^{W_{\mathfrak{m}}}_{\lambda W^\lambda_{\mathfrak{m}} \lambda^{-1}} V$ seen as an $\operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$-module under the isomorphism given by b_w^λ for each $w \in W^\lambda_{\mathfrak{m}}$. By Lemma 4.3.8, $\lambda W^\lambda_{\mathfrak{m}} \lambda^{-1} = W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}$.
We need to compare $b_w^{\mathcal{E}_\lambda}$ and b_w^λ, or alternatively $b_{\lambda w \lambda^{-1}}^{(s z_\lambda)^* \mathcal{Z}}$ and $(s z_\lambda)^* b_{\lambda w \lambda^{-1}}^{\mathcal{Z}}$. Applying Lemma 4.2.5 and its proof, we obtain $b_w^\lambda = \mathfrak{j}(\mu_{\lambda^{-1} w \lambda}(\lambda s \lambda^{-1})^{-1}) b_w^{\mathcal{E}_\lambda}$.
Any $\operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$-module V under the isomorphism given by b_w^λ for $w \in W^\lambda_{\mathfrak{m}}$ is then isomorphic to the $\operatorname{End}((\gamma_\lambda)_* s_\lambda^* \tilde{\mathcal{E}})$-module $V \otimes \chi^s_\lambda$ under the isomorphism given by $b_w^{\mathcal{E}_\lambda}$ for $w \in W^\lambda_{\mathfrak{m}}$. Therefore,

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] \cong \bigoplus_{\lambda \in \Lambda} (\mathcal{A}'_{\operatorname{Res}^{W_{\mathfrak{m}}}_{W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}}(V) \otimes X^s_\lambda})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}}$$

under the assumptions of the lemma. ∎

Corollary 4.3.19. *Let* $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ *and* $V \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}])$ *such that* $\mathcal{A} \cong \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_\mathfrak{m})}(V, \mathcal{K}_\mathfrak{m})$. *For any* $s \in Z^\circ(\mathbf{L})$, $s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d]$ *is isomorphic to*

$$\bigoplus_{V' \in \mathrm{Irr}(W_{\mathbf{L}_0}^{\mathbf{G}_s})} \bigoplus_{w \in W_\mathfrak{m} \backslash W_\mathbf{L} / W_{\mathbf{L}_0}^{\mathbf{G}_s}} \langle \mathrm{Res}^{W_\mathfrak{m}}_{W^{\mathbf{G}_s}_{\mathfrak{m}_{w^{-1}}}}(V) \otimes X^s_w, \mathrm{Res}^{W^{\mathbf{G}_s}_{\mathbf{L}_0}}_{W^{\mathbf{G}_s}_{\mathfrak{m}_{w^{-1}}}}(V' \circ \mathrm{ad}(w^{-1})) \rangle (\mathrm{ad}(w^{-1})^* \mathcal{A}'_{V'})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}},$$

where $\mathcal{A}'_{V'} = \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}_0})}(V', \mathcal{K}_{\mathfrak{m}_0})$ *with* $\mathfrak{m}_0 = (\mathbf{L}, \Sigma, \mathcal{E}_0 \boxtimes \overline{\mathbb{Q}}_\ell)$. *We see* V' *as an* $\mathrm{End}(\mathcal{K}_{(\mathfrak{m}_0)})$*-module under the isomorphism defined in Proposition 3.2.17 and* X^s_w *is the module of* $\overline{\mathbb{Q}}_\ell[W^{\mathbf{G}_s}_{\mathfrak{m}_{w^{-1}}}]$ *whose character is* $\chi^s_w : w_0 \mapsto \mathfrak{j}(\mu_{w w_0 w^{-1}}(wsw^{-1}))$.

Proof. This is a consequence of Lemma 4.3.18, Remark 3.2.19 and Corollary 4.3.16 (to rewrite the indexes of the sum). □

Corollary 4.3.20. *Let* $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ *and* $V \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}])$ *such that* $\mathcal{A} \cong \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_\mathfrak{m})}(V, \mathcal{K}_\mathfrak{m})$. *Assume that* $\mathcal{A}$ *is a unipotent character sheaf in the principal series and that* $Z(\mathbf{G})$ *is connected. Then,* $s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[-\dim(\mathbf{G}) + \dim(\mathbf{G}_s) - \dim(\mathbf{T}_0)]$ *is isomorphic to*

$$\bigoplus_{V' \in \mathrm{Irr}(W^{\mathbf{G}_s})} \langle \mathrm{Res}^W_{W^{\mathbf{G}_s}}(V), V' \rangle (\mathcal{A}'_{V'})[-\dim \mathbf{T}_0]_{(\mathbf{G}_s)_{\mathrm{uni}}},$$

where $\mathcal{A}'_{V'} = \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}_0})}(V', \mathcal{K}_{\mathfrak{m}_0})$ *with* $\mathfrak{m}_0 = (\mathbf{T}_0, \mathbf{T}_0, \overline{\mathbb{Q}}_\ell) \in \mathfrak{M}^{\mathbf{G}_s}$. *We see* V' *as an* $\mathrm{End}(\mathcal{K}_{\mathfrak{m}_0})$*-module under the isomorphism defined in Proposition 3.2.17.*

Proof. Since $\mathcal{A}$ is unipotent, $W_\mathfrak{m} = W$ and $N_\mathbf{G}(\mathfrak{m}) = N_\mathbf{G}(\mathbf{T}_0)$. Moreover,

$$W^1_\mathfrak{m} = \mathrm{Stab}_{W_\mathfrak{m}}(1) = W^{\mathbf{G}_s}$$

which is a Weyl group, because it is the Weyl group of the connected reductive group $\mathbf{G}_s$. Applying Lemma 4.2.6, we conclude that the character χ^s_w in the previous corollary is trivial. □

4.3.4 Restriction of a character sheaf from a simple group of adjoint type

In this subsection, we focus on the particular case when $\mathbf{G}$ is a simple group of adjoint type. Moreover we assume that $s \notin Z^\circ(\mathbf{L})$. Otherwise we are in the case of the previous section.

Thanks to [Lus84b, 2.3], we may write $\mathcal{E} = i^*(\mathcal{F} \otimes \mathcal{L})$ with

- $i : \mathbf{L} \to \mathbf{L}/Z^\circ(\mathbf{L}) \times \mathbf{L}/[\mathbf{L}, \mathbf{L}]$,
- $\mathcal{F}$ a local system on $\mathbf{L}/Z^\circ(\mathbf{L})$, and
- $\mathcal{Z} = \mu^* \mathcal{E}_{n,\phi}$ a Kummer local system on $\mathbf{L}/[\mathbf{L}, \mathbf{L}]$ with $\mu \in X(\mathbf{L}/[\mathbf{L}, \mathbf{L}])$ whose inverse image under $j : Z^\circ(\mathbf{L}) \to \mathbf{L}/[\mathbf{L}, \mathbf{L}]$ is a Kummer local system $\mathcal{Z}' = (\mu \circ j)^* \mathcal{E}_{n,\phi}$ on $Z^\circ(\mathbf{L})$.

We now fix the isomorphisms a_w for each $w \in W_{\mathfrak{m}}$ following Lemma 3.2.21. By Lemma 3.2.22, we choose $t \in \mathbf{L}$ such that $t = lsl^{-1}z$ for some $l \in \mathbf{L}$ and $z \in Z^\circ(\mathbf{L})$ and $N_{\mathbf{G}}(\mathbf{L})/\mathbf{L} = N_{C^\circ_{\mathbf{G}}(t)}(C^\circ_{\mathbf{L}}(t))/C^\circ_{\mathbf{L}}(t)$. Note that without loss of generality, we may assume that $l = 1$. Recall from the proof of Lemma 3.2.21 that we have defined a group isomorphism

$$W_{\mathfrak{m}} \to W^{\mathbf{G}_t}_{\mathfrak{m}_t}, \quad w \mapsto w_t = \dot{w}\mathbf{L}_t,$$

where $\mathbf{G}_t = C^\circ_{\mathbf{G}}(t)$, and $\mathfrak{m}_t = (\mathbf{L}_t, \Sigma_t, \mathcal{E}_t)$ with $\mathbf{L}_t = C^\circ_{\mathbf{L}}(t) = \mathbf{L}_0$, $\Sigma_t = (v)_{\mathbf{L}_t} Z^\circ(\mathbf{L}_t) = \Sigma_0$, and the local system $\mathcal{E}_t$ is the inverse image of $\mathcal{E}$ under the map $t : \Sigma_t \to \Sigma, g \to tg$. In particular, we can write $\mathcal{E}_t = i^*(t^*\mathcal{F} \boxtimes \mathcal{Z}_t)$ where the inverse image of $\mathcal{Z}_t = z^*\mathcal{Z}$ under the map $Z^\circ(\mathbf{L}) = Z^\circ(\mathbf{L}_t) \to \mathbf{L}_t/[\mathbf{L}_t, \mathbf{L}_t]$ is the Kummer local system $z^*\mathcal{Z}'$.
Finally, for each $w \in W^{\mathbf{G}_t}_{\mathfrak{m}_t}$, we fix basis elements $b^{\mathcal{Z}_t}_{w_t} \in \mathscr{A}_{t^*\mathcal{E}}$ as in the proof of Proposition 3.2.17. For each $w \in W_{\mathfrak{m}}$, we choose the unique isomorphism a_w such that

$$(a_w)_{tu} = \left(b^{\mathcal{Z}_t}_{w_t}\right)_u.$$

For $\lambda \in \Lambda$, the local system $\mathcal{E}_\lambda$ on Σ_λ is in fact $\mathrm{ad}(\lambda)^*(tz_\lambda z^{-1})^*\mathcal{E} = \mathrm{ad}(\lambda)^*(z_\lambda z^{-1})^*\mathcal{E}_t$. For each $w \in W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}$, the choice of $b^{\mathcal{E}_\lambda}_w : \mathrm{ad}(w)^*\mathcal{E}_\lambda \to \mathcal{E}_\lambda$ is fixed in Proposition 3.2.17, with $b^{\mathcal{E}_\lambda}_w = \mathrm{ad}(\lambda)^* b^{(z_\lambda z^{-1})^*\mathcal{Z}_t}_{\lambda^{-1}w\lambda}$.

Now, exactly as in the unipotently supported case we obtain the following description of the restriction of a character sheaf.

Lemma 4.3.21. *Let $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ and $V \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}])$ such that $\mathcal{A} \cong \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}})}(V, \mathcal{K}_{\mathfrak{m}})$. Then*

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] \cong \bigoplus_{\lambda\in\Lambda} (\mathcal{A}'_{\mathrm{Res}^{W_{\mathfrak{m}}}_{\lambda W^\lambda_{\mathfrak{m}}\lambda^{-1}}(V)\otimes X^s_\lambda})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}},$$

where

$$\mathcal{A}'_{\mathrm{Res}^{W_{\mathfrak{m}}}_{\lambda W^\lambda_{\mathfrak{m}}\lambda^{-1}}(V)\otimes X^s_\lambda} = \mathrm{Hom}_{\mathrm{End}((\gamma_\lambda)_* s^*_\lambda \tilde{\mathcal{E}})}(\mathrm{Res}^{W_{\mathfrak{m}}}_{\lambda W^\lambda_{\mathfrak{m}}\lambda^{-1}}(V) \otimes X^s_\lambda, (\gamma_\lambda)_* s^*_\lambda \tilde{\mathcal{E}})$$

with $\mathrm{Res}^{W_{\mathfrak{m}}}_{\lambda W^\lambda_{\mathfrak{m}}\lambda^{-1}}(V) \otimes X^s_\lambda$ viewed as an $\mathrm{End}((\gamma_\lambda)_* s^*_\lambda \tilde{\mathcal{E}})$-module under the isomorphism given by $b^{\mathcal{E}_\lambda}_w$ for $w \in \lambda W^\lambda_{\mathfrak{m}}\lambda^{-1}$ and X^s_λ is the module of $\overline{\mathbb{Q}}_\ell[\lambda W^\lambda_{\mathfrak{m}}\lambda^{-1}]$ whose character is given by $\chi^s_\lambda : w \mapsto \mathfrak{j}(\mu_{\lambda^{-1}w\lambda}(\lambda z_\lambda^{-1} z \lambda^{-1}))$.

Proof. As for the proof of the unipotently supported case (see Lemma 4.3.18), we need to compare $b^{\mathcal{E}_\lambda}_w$ and b^λ_w, or alternatively $(b^{(z_\lambda z^{-1})^*\mathcal{Z}_t}_{\lambda^{-1}w\lambda})_{z_\lambda^{-1}zu}$ and $\left(b^{\mathcal{Z}_t}_{w_t}\right)_u$. Applying Lemma 4.2.5 and its proof, we obtain $b^\lambda_w = \mathfrak{j}(\mu_{\lambda^{-1}w\lambda}(\lambda z_\lambda^{-1} z \lambda^{-1})^{-1}) b^{\mathcal{E}_\lambda}_w$.
Any $\mathrm{End}((\gamma_\lambda)_* s^*_\lambda \tilde{\mathcal{E}})$-module V' under the isomorphism given by b^λ_w for $w \in W^\lambda_{\mathfrak{m}}$ is then isomorphic to the $\mathrm{End}((\gamma_\lambda)_* s^*_\lambda \tilde{\mathcal{E}})$-module $V \otimes \chi^s$ under the isomorphism given by $b^{\mathcal{E}_\lambda}_w$ for $w \in W^\lambda_{\mathfrak{m}}$. Therefore,

$$s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d] \cong \bigoplus_{\lambda\in\Lambda} (\mathcal{A}'_{\mathrm{Res}^{W_{\mathfrak{m}}}_{W^{\mathbf{G}_s}_{\mathfrak{m}_\lambda}}(V)\otimes X^s})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}}$$

under the assumptions of the lemma. □

Corollary 4.3.22. *Let* $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ *and* $V \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_\mathfrak{m}])$ *such that* $\mathcal{A} \cong \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_\mathfrak{m})}(V, \mathcal{K}_\mathfrak{m})$. *Then* $s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d]$ *is isomorphic to*

$$\bigoplus_{V' \in \mathrm{Irr}(W_{\mathbf{L}_0}^{\mathbf{G}_s})} \bigoplus_{w \in W_\mathfrak{m} \backslash W_\mathbf{L} / W_{\mathbf{L}_0}^{\mathbf{G}_s}} \langle \mathrm{Res}^{W_\mathfrak{m}}_{w^{-1} W_\mathfrak{m}^{w^{-1}} w}(V) \otimes X_w^s, \mathrm{Res}^{W_{\mathbf{L}_0}^{\mathbf{G}_s}}_{w^{-1} W_\mathfrak{m}^{w^{-1}} w}(V' \circ \mathrm{ad}(w^{-1})) \rangle (\mathrm{ad}(w^{-1})^* \mathcal{A}'_{V'})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}},$$

where $\mathcal{A}'_{V'} = \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}_0})}(V', \mathcal{K}_{\mathfrak{m}_0})$ *with* $\mathfrak{m}_0 = (\mathbf{L}_0, \Sigma_0, s^*\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell)$. *We see* V' *as an* $\mathrm{End}(\mathcal{K}_{\mathfrak{m}_0})$-*module under the isomorphism defined in Proposition 3.2.17 and* X_w^s *is the module of* $\overline{\mathbb{Q}}_\ell[W^{\mathbf{G}_s}_{\mathfrak{m}_{w^{-1}}}]$ *whose character is* $\chi_w^s : w_0 \mapsto \mathfrak{j}(\mu_{w w_0 w^{-1}}(w z z_\lambda^{-1} w^{-1}))$.

Proof. This is a consequence of Lemma 4.3.21, Remark 3.2.19 and 4.3.16. □

Chapter 5

Ordinary and projective representations in blocks of Brauer characters

After introducing the finite groups of Lie type and their representation theory, both in terms of ordinary characters and of character sheaves, we are finally ready to approach the main theme of this thesis: the unitriangularity of the decomposition matrices.
To do so, we treat this question block of $\mathbf{k}[G]$ by block of $\mathbf{k}[G]$, or to be precise, union of blocks by union of blocks. Let us describe a first strategy to show that $\mathscr{B}$, a union of ℓ-blocks of $\mathbf{k}[G]$, has a lower-unitriangular decomposition matrix.

Step 1 Compute the number n of projective indecomposable modules in $\mathscr{B}$.

Step 2 Choose n ordinary irreducible modules $V_1, \dots, V_n \in \mathrm{Irr}_{\mathbf{K}}(G)$ belonging to $\mathscr{B}$.

Step 3 Find the n projective indecomposable modules $P_1, \dots, P_n$ of $\mathbf{k}[G]$ belonging to the union of blocks $\mathscr{B}$.

Step 4 Check that the decomposition matrix given by $\langle V_i, P_j^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K} \rangle$ for $1 \leq i, j \leq n$ is lower-unitriangular.

The obvious problem with this method is that we would like to use the decomposition matrix to get information about the PIMs of $\mathbf{k}[G]$ and not the other way around. If we could do Step 3, then computing the decomposition matrix would be a much easier task. Fortunately, the following result allows us to simply look at projective $\mathbf{k}[G]$-modules, not necessarily indecomposable ones.

Proposition 5.0.1 ([Gec94, Lem. 2.6])**.** *Let A be a finite group. Let $\mathscr{B}$ be a union of ℓ-blocks of the group A and $n \coloneqq |\mathrm{irr}_{\mathbf{k}}(\mathscr{B})|$. Assume that there exist irreducible $\mathbf{K}[A]$-modules $V_1, \dots, V_n$ in $\mathscr{B}$ and projective $\mathbf{k}[A]$-modules $P_1, \dots, P_n$ such that the decomposition matrix $([V_i, P_j])_{1 \leq i,j \leq n}$ is lower unitriangular. Then the ℓ-decomposition matrix of $\mathscr{B}$ is unitriangular.*

Therefore, in our plan, **Step 3** and **Step 4** become

Step 3 Choose n projective modules $P_1, \dots, P_n$ of $\mathbf{k}[G]$.

Step 4 Check that the decomposition matrix $\langle V_i, P_j^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}\rangle$ for $1 \leq i, j \leq n$ is lower-unitriangular.

This chapter focus on the first three steps of our plan whilst the following one is dedicated to showing the last step in some cases. In both chapters, our reasoning and ideas are very much inspired by the notions developed by Brunat, Dudas and Taylor in [BDT20].
In Section 5.1, we will define the union of blocks we will focus on and give some indication about a basic set, slightly generalising results of Geck and Hiss from [GH91], [Gec94]. We will describe some projective modules of $\mathbf{k}[G]$, called the Kawanaka modules, in Section 5.2.

We ecall that in Hypothesis 1, we have fixed $\mathbf{G}$ a connected reductive group defined over $k = \overline{\mathbb{F}}_p$. We look at the ℓ-decomposition matrices of $G = \mathbf{G}^F$ for a prime $\ell \neq p$. Some arguments in this chapter and the next one require us to make use of characteristic functions of character sheaves. We thus make the following hypothesis.

Hypothesis 2. For the rest of this thesis, we assume that the Steinberg endomorphism F in Hypothesis 1 is a Frobenius map and gives the group $\mathbf{G}$ an $\mathbb{F}_q$-structure.

5.1 Counting modular representations

This section is concerned with the first two steps of our strategy to show the unitriangularity of the ℓ-decomposition matrix of G. After partitioning the decomposition matrix into a union of ℓ-blocks compatible with Lusztig series thanks to Broué–Michel ([BM89]), we will count the number of irreducible modular representations in a union of ℓ-blocks. We will also find a labelling of the characters in it.
For the unipotent ℓ-blocks, a basic set was found by Geck and Hiss [GH91] when ℓ is good and $Z(\mathbf{G})$ is connected. Its parameterisation is a consequence of Lusztig's results. When $\mathbf{G}$ is simple modulo its centre and ℓ bad, Geck–Hiss established the number of irreducible Brauer characters in [GH97] and a labelling was determined by Chaneb. We generalise the results of Geck–Hiss to isolated blocks when ℓ is bad.

5.1.1 The ℓ-blocks of the decomposition matrix

As we have seen in the introduction, the group algebra $\mathbf{k}[G]$ is partitioned into ℓ-blocks

$$\mathbf{k}[G] = \mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_n.$$

In particular, we can split the irreducible modules according to the block to which they belong.

On the other hand, the ordinary characters are also partitioned into geometric series indexed by the F^*-stable semisimple conjugacy classes of the dual group $\mathbf{G}^*$, see Theorem 2.2.10.
Thanks to Broué–Michel [BM89], these two partitions are compatible.

Theorem 5.1.1 ([BM89, Thm. 2.2])**.** *Let $t \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of order prime to ℓ. Define*

$$\mathscr{E}_\ell(G,t) := \bigsqcup_s \mathscr{E}(\mathbf{G}, st),$$

where s runs over a set of representatives of F^*-stable geometric conjugacy classes of semisimple ℓ-elements of $\mathbf{G}^*$ which commute with t. There exists a union of ℓ-blocks $\mathscr{B}(G,t)$ of G such that

$$\mathscr{E}_\ell(G,t) = \mathrm{irr}(\mathscr{B}(G,t)).$$

Remark 5.1.2. Since geometric series are a union of rational series,

$$\mathscr{E}_\ell(G,t) = \bigsqcup_s \bigsqcup_{s'} \mathscr{E}(G, s'),$$

where s runs over a set of representatives of F^*-stable conjugacy classes of semisimple ℓ-elements of $\mathbf{G}^*$ which commute with t, and s' runs over a set of representatives of the semisimple G^*-conjugacy classes of F^*-stable elements in $(st)_{\mathbf{G}^*}$ If $Z(\mathbf{G})$ is connected, rational series and geometric series coincide. In particular,

$$\mathscr{E}_\ell(G,t) = \bigsqcup_s \mathscr{E}(G, ts),$$

where s runs over a set of representatives of conjugacy classes of semisimple ℓ-elements of $(\mathbf{G}^*)^{F^*}$ which commute with t.

Definition 5.1.3. We call the union of blocks $\mathscr{B}(G,1)$ the **unipotent ℓ-blocks**. If t is isolated in $\mathbf{G}^*$, we say the the union of ℓ-blocks $\mathscr{B}(G,t)$ is **isolated**.

These unipotent ℓ-blocks are particularly important as all non-isolated unions of blocks are Morita equivalent to a union of unipotent ℓ-blocks of a smaller connected reductive group, thanks to Bonnafé and Rouquier.

Theorem 5.1.4 ([BR03, Thm. 11.8])**.** *Let $t \in (\mathbf{G}^*)^{F^*}$ be a semisimple element of order prime to ℓ. Assume that $C_{\mathbf{G}^*}(t)$ is contained in a Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$. Let $\mathbf{L}$ be the Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{L}$. Then, $\mathscr{B}(L,1)$ and $\mathscr{B}(G,t)$ are Morita equivalent.*

Therefore, in this thesis, our priority will be the unipotent ℓ-blocks, followed by the isolated ℓ-blocks.

5.1.2 The number of modular representations

We fix $t \in \mathbf{G}^*$ an isolated ℓ'-element. We want to know the number of Brauer characters in the union of blocks $\mathscr{B}(G,t)$. More precisely, we want to find a basic set for $\mathscr{B}(G,t)$.

Definition 5.1.5. Let $\mathscr{B}$ be a union of blocks of G. A set of Brauer characters is a **basic set** for $\mathscr{B}$ if it is a $\mathbb{Z}$-basis for the set of Brauer characters in $\mathscr{B}$.

The set of Brauer characters corresponding to the irreducible $\mathbf{k}[G]$-modules in $\mathscr{B}$ is always a basic set. However, this is obviously not the one we are looking for.
If χ is a virtual ordinary character of G, then its restriction $\breve{\chi}$ to the ℓ'-elements is a virtual Brauer character. We would rather find a basic set consisting of the restriction to the ℓ'-elements of virtual ordinary characters or even of ordinary irreducible characters. If the basic set comes from ordinary characters, we say that it is **ordinary**.

In order to do so, we follow Geck–Hiss and require some conditions on the centre of $\mathbf{G}$.

Hypothesis 5.1.6. For the rest of this section, we assume $Z(\mathbf{G})$ is connected.

When ℓ is good for $\mathbf{G}$, the problem of finding a basic set has been solved for any union of blocks $\mathscr{B}(G,t)$, for $t \in \mathbf{G}^*$ an ℓ'-element.

Theorem 5.1.7 ([GH91, Thm. 5.1])**.** *Assume that ℓ is good for $\mathbf{G}$. Let $t \in \mathbf{G}^*$ be an isolated ℓ'-element. Then the set $\breve{\mathscr{E}}(G,t) \coloneqq \{\breve{\chi} \mid \chi \in \mathscr{E}(G,t)\}$ is an ordinary basic set of the union of blocks $\mathscr{B}(G,t)$.*

When ℓ is bad, the situation is less neat and requires some more analysis to be stated.

A basic set when ℓ is bad

From now on, we assume that ℓ is bad for $\mathbf{G}$.
Firstly, we suppose that $\mathbf{G}$ is of classical type. In that case, the prime ℓ is equal to 2 and all isolated elements of $\mathbf{G}^*$ are 2-elements ([GH91, Prop. 2.1]). Therefore, we only consider the unipotent ℓ-blocks.

Theorem 5.1.8 ([Gec94, Prop. 2.4 and Thm. 2.5])**.** *Assume that $\mathbf{G}$ has only simple components of classical type. Then $|\mathscr{B}(G,1)|$ equals the number of unipotent conjugacy classes of G. Moreover, there exists an ordinary basic set for $\mathscr{B}(G,1)$.*

We now focus on the case where $\mathbf{G}$ is of exceptional type, applying the same methods as the ones developed by Geck–Hiss in [GH97] to compute the number of Brauer characters in the unipotent blocks. We follow their reasoning, adapting their chapters 5 and 6 to our case.

Hypothesis 5.1.9. For the rest of this section, we assume that $\mathbf{G}$ is of exceptional type, simple modulo its centre, and that p is good for $\mathbf{G}$. We also suppose that ℓ is bad for $\mathbf{G}$.

The idea is that instead of considering ordinary characters, we could look at the other basis for the class functions given by (a subset) of the almost characters, or thanks to Theorem 3.3.6 the basis given by the characteristic functions of character sheaves.

For each F-stable character sheaf $\mathcal{A} \in \hat{\mathbf{G}}$, we have fixed in Subsection 3.3.3 an isomorphism $\varphi_{\mathcal{A}} : F^*\mathcal{A} \to \mathcal{A}$ which leads to the characteristic function $\chi_{\mathcal{A}}$. For $s \in \mathbf{G}^*$, we write

$$\Xi_s(G) := \{\chi_{\mathcal{A}} \mid \mathcal{A} \in \hat{\mathbf{G}}_s, \text{ where } \mathcal{A} \text{ is } F\text{-stable}\}.$$

Recall that if $\mathcal{A} \in \hat{\mathbf{G}}_s$, then there is an induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$ such that $\mathcal{A}$ appears as a composition factor of $\mathcal{K}_{\mathfrak{m}}$, induced from $\mathcal{A}_{\mathfrak{m}} = IC(\overline{\Sigma}, \mathcal{E})[\dim \Sigma]$ and $\mathcal{A}_{\mathfrak{m}} \in \hat{\mathbf{L}}_s$. Moreover, $\mathfrak{m}$ is unique up to $\mathbf{G}$-conjugation. By Proposition 3.2.20

$$|\hat{\mathbf{G}}(\mathfrak{m})| = |\operatorname{Irr}(\overline{\mathbb{Q}}_\ell[W_{\mathfrak{m}}])|.$$

Now if $\mathcal{A}$ is F-stable, we may choose $\mathfrak{m}$ to be F-stable as well (see Subsection 3.3.1). We write

$$\Xi_{\mathfrak{m}}(G) := \{\chi_{\mathcal{A}} \mid \mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m}), \text{ where } \mathcal{A} \text{ is } F\text{-stable}\}.$$

Moreover, by [Lus90, 9.2] and [Sho96, Thm. 4.2],

$$I_{\mathbf{L}}^{\mathbf{G}}(\chi_{\mathcal{A}_{\mathfrak{m}}}) = \xi \chi_{\mathcal{K}_{\mathfrak{m}}}$$

for some $\xi \in \overline{\mathbb{Q}}_\ell^\times$. Note that $\mathcal{A}_{\mathfrak{m}}$ is a character sheaf of $\mathbf{L}$ and $\chi_{\mathcal{A}_{\mathfrak{m}}}$ is then an almost character of L up to a sign. Lastly, we say that $\chi_{\mathcal{A}}$ is **cuspidal** if and only if $\mathcal{A}$ is cuspidal.

We now slightly adapt the proof of [GH97, Thm. 6.3] to obtain the following result.

Proposition 5.1.10. *Every Brauer character in $\mathscr{B}(G,t)$ is a $\mathbf{K}$-linear combination of elements in $\breve{\mathscr{E}}(G,t) = \{\breve{\chi} \mid \chi \in \mathscr{E}(G,t)\}$.*

Proof. By Theorem 5.1.1, the restriction of the ordinary characters in $\mathscr{E}_\ell(G,t)$ to the ℓ'-elements generate the Brauer characters in $\mathscr{B}(G,t)$.
Instead of ordinary characters, we may consider almost characters or the characteristic functions of character sheaves. In other words, we want to show that for each ℓ-element $s \in C_{\mathbf{G}^*}(t)$ and each $\chi \in \Xi_{ts}(G)$, the restriction $\breve{\chi}$ can be written as a $\mathbf{K}$-linear combination of elements in $\{\breve{\psi} \mid \psi \in \Xi_t(G)\}$. We fix $s \in \mathbf{G}^*$ a semisimple ℓ-element and $\chi \in \Xi_{ts}(G)$.
There is a cuspidal induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$ such that $\chi \in \Xi_{\mathfrak{m}}(G)$ where $\mathbf{L}$ is a non-necessarily proper Levi subgroup of $\mathbf{G}$. In particular, $\psi := \chi_{\mathcal{A}_{\mathfrak{m}}} \in \Xi_{ts}(L)$. If χ is cuspidal, $\chi = \psi$.

Claim: *There is $\psi' \in \Xi_t(L)$ such that $\breve{\psi} = \breve{\psi}'$.*
Firstly, we observe that $ts \in Z(\mathbf{L}^*)$. Indeed, it is obvious if $\mathbf{L}^*$ is a maximal torus, since then $Z(\mathbf{L}^*) = \mathbf{L}^*$. If χ is cuspidal (i.e. $\mathbf{L} = \mathbf{G}$), $ts \in Z(\mathbf{G}^*)$ since all cuspidal character sheaves of an adjoint exceptional group belong to a central series (see for instance [DLM14, Appendix C.]). If $\mathbf{L}$ is not a maximal torus nor the whole group $\mathbf{G}$, then $\mathbf{L}$ is of type B_2, D_4, E_6 or E_7. In each case, the only possibility for ψ to be cuspidal is if $st \in Z(\mathbf{L}^*)$, thanks to [LuCS4, Proof of Prop. 19.3].
We claim that $C_{\mathbf{L}^*}(s) = \mathbf{L}^*$. Indeed, let $x \in \mathbf{L}^*$, then $xtsx^{-1} = ts$. Let $o(t)$ and $o(s)$ denote the order of t and s respectively. Since s and t commute, $xs^{o(t)}x^{-1} = s^{o(t)}$. Now the orders $o(t)$ and $o(s)$ are coprime. Thus, there exists an integer a such that $ao(t)$ is congruent to 1 modulo $o(s)$. Thus, $(s^{o(t)})^a = s$ and we conclude that $xsx^{-1} = s$. In particular, the element s is central in $\mathbf{L}^*$.
Let λ_s be the linear character of L "dual" to s: $(\lambda_s)_{\mathbf{T}_0}$ is sent to s under the map from $\mathrm{irr}(\mathbf{T}_0)$ to $\mathbf{T}_0^*$ ([GM20, Prop. 2.5.20], [Lus77, 7.4.2]). It has order a power of ℓ. The character ψ' defined by $\psi = \psi'\lambda_s$ belongs to $\Xi_t(L)$ by the Jordan decomposition of characters ([GM20, Thm. 4.7.1(3)]) and $\breve{\psi} = \breve{\psi}'$.

The proof then follows by the exact same argument as in [GH97, Thm. 6.3]. By [LuCS2, 10.4.5 and 10.6.1], there are $a_w \in \mathbf{K}$ for each $w \in W_{\mathfrak{m}}$ such that

$$\chi = \sum_{w\in W_{\mathfrak{m}}} a_w \chi_{\mathcal{A}_{\mathfrak{m}^w},\varphi^w_{\mathcal{A}_{\mathfrak{m}^w}}},$$

where $\mathfrak{m}^w = (w\mathbf{L}w^{-1}, w\Sigma w^{-1}, \mathrm{ad}(w^{-1})\mathcal{E})$ and $\varphi_w : F^*\mathrm{ad}(w^{-1})\mathcal{E} \to \mathrm{ad}(w^{-1})\mathcal{E}$ is a fixed isomorphism. In particular, $\mathcal{A}_{\mathfrak{m}^w} \in \widehat{w\mathbf{L}w^{-1}}_{wtsw^{-1}}$. In other words, there are $a'_w \in \mathbf{K}$ for $w \in W_{\mathfrak{m}}$ such that

$$\chi = \sum_{w\in W_{\mathfrak{m}}} a'_w I^{\mathbf{G}}_{w\mathbf{L}w^{-1}}(\chi_{\mathcal{A}_{\mathfrak{m}^w}}),$$

with $\chi_{\mathcal{A}_{\mathfrak{m}^w}} \in \Xi_{wstw^{-1}}((w\mathbf{L}w^{-1})^F)$ cuspidal. Now, Deligne–Lusztig induction commutes with restriction to ℓ'-elements ([DM20, Prop. 10.1.6]). Therefore, thanks to the claim, the character $\breve{\chi}$ is a $\mathbf{K}$-linear combination of restrictions of elements in Ξ_t. □

Remark 5.1.11. Observe that the proof of the claim shows that $t \in Z(\mathbf{L}^*)$.

In fact, following the proof of [GH91, Thm. 3.1], we could even prove a stronger result.

Proposition 5.1.12. *Assume that $t \neq 1$. Suppose as well that F acts trivially on W. Every Brauer character in $\mathscr{B}(G,t)$ is a $\mathbb{Z}$-linear combination of elements in $\breve{\mathscr{E}}(G,t)$ except possibly if $\mathbf{G}$ is of type E_8 and $\ell \in \{2,3\}$.*

Proof. Let s be a semisimple ℓ-element of $(\mathbf{G}^*)^{F^*}$ commuting with t and $\chi \in \mathcal{E}(G,ts)$. We want to show that $\breve{\chi}$ is a $\mathbb{Z}$-linear combination of elements in $\breve{\mathscr{E}}(G,t)$. If $s \in Z(\mathbf{G}^*)$, similarly as in the previous proof, we can show that there is $\chi' \in \mathcal{E}(G,t)$ such that $\breve{\chi} = \breve{\chi}'$. Otherwise, by the description of the isolated semisimple elements, we know that all the semisimple elements have order a prime power (expect one conjugacy class in E_8 of order 6). Thus, the element ts is not isolated and there is a Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ such that its corresponding dual $\mathbf{L}^*$ in $\mathbf{G}^*$ contains $C_{\mathbf{G}^*}(ts)$. Therefore, there is $\psi \in \mathcal{E}(L, ts)$ and $\epsilon \in \{-1, 1\}$ such that

$$\chi = \epsilon I_{\mathbf{L}}^{\mathbf{G}}(\psi)$$

by [DM87, Prop. 6.6].
After choosing the Levi subgroup $\mathbf{L}^*$ minimal such that $C_{\mathbf{G}^*}(ts) \subseteq \mathbf{L}^*$, we conclude that $C_{\mathbf{G}^*}(ts) = C_{\mathbf{L}^*}(ts)$ is not contained in a proper Levi subgroup of $\mathbf{L}^*$. Note that since $Z(\mathbf{G})$ is connected, $\mathbf{G}^*$ is simply connected and $C_{\mathbf{G}^*}(ts)$ is a connected reductive group (Theorem 1.3.2). If $\mathbf{G}$ is not of type E_8, then the isolated elements of $\mathbf{L}^*$ also have order a prime power and hence $\mathbf{L}^* = C_{\mathbf{L}^*}(ts)$. If $\mathbf{G}$ is of type E_8 and $\ell = 5$, then the order of st is divisible by 5 and st is not isolated in $\mathbf{L}^*$. Thus $st \in Z(\mathbf{L}^*)$.
By the same argument as in the claim, we conclude that there is $\psi' \in \mathcal{E}(L, t)$ such that $\breve{\psi} = \breve{\psi}'$. Since Deligne–Lusztig induction commutes with restriction to ℓ'-elements ([DM20, Prop. 10.1.6]) and preserves Lusztig series, we conclude that $\breve{\chi}$ is a $\mathbb{Z}$-linear combination of elements in $\breve{\mathscr{E}}(G, t)$. □

Remark 5.1.13. The condition that F acts trivially on W ensures that the Levi subgroup $\mathbf{L}$ is F-stable.

Another small modification of the proof of [GH97, Thm. 6.4] for the unipotent ℓ-blocks ($t = 1$) leads us to find a basis for the lattice of the Brauer characters in $\mathscr{B}(G, t)$.

Proposition 5.1.14. *The following set is a* $\mathbf{K}$*-basis for the Brauer characters in* $\mathscr{B}(G, t)$*:*

$$\breve{\Xi}'_t \coloneqq \{\breve{I}_{\mathbf{L}}^{\mathbf{G}}(\psi) \mid (\mathbf{L}, \psi) \in \Xi'_t\}$$

where

$$\Xi'_t \coloneqq \{(\mathbf{L}, \psi) \mid \psi = \xi \chi_{\mathcal{A}_{\mathfrak{m}}} \text{ is an almost character of } L \text{ for } \mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$$
$$\text{and some } \xi \in \mathbf{K}^\times,\ \mathcal{A}_{\mathfrak{m}} \in \hat{\mathbf{L}}_t\ F\text{-stable},\ \breve{\psi} \neq 0\}.$$

Moreover, $(\mathbf{L}, \psi), (\mathbf{M}, \theta) \in \Xi'_t$ *are such that* $\breve{I}_{\mathbf{L}}^{\mathbf{G}}(\psi) = \breve{I}_{\mathbf{M}}^{\mathbf{G}}(\theta)$ *if and only if* $(\mathbf{L}, \psi)$ *and* $(\mathbf{M}, \theta)$ *are conjugate in* G*.*

Proof. We follow the steps of the proof of [GH97, Thm. 6.4]. Firstly, by Proposition 5.1.10, the above set generates the space of Brauer characters in $\mathscr{B}(G, t)$. We now check that they are linearly independent.

(a) Let us fix $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$ such that $\psi = \xi \chi_{\mathcal{A}_{\mathfrak{m}}} \in \Xi_t(\mathbf{L})$ for some root of unity $\xi \in \mathbf{K}$ and $\breve{\psi} \neq 0$. Since every cuspidal character sheaf is clean by Proposition 3.2.9, ψ has support $(sv)_{\mathbf{L}} Z^\circ(\mathbf{L})$ where $v \in \mathbf{L}$ is unipotent and $sZ^\circ(\mathbf{L})$ is a semisimple isolated element in $\mathbf{L}/Z^\circ(\mathbf{L})$. Since $\breve{\psi} \neq 0$ and $\ell \neq p$, there must be $z \in Z^\circ(\mathbf{L})$ such that sz is an F-stable ℓ'-element. Therefore, we may assume that s is an F-stable ℓ'-element and $\Sigma = (sv) Z^\circ(\mathbf{L})$.

(b) We now define $\psi' := \sum_z \psi \cdot \lambda_z$ where z runs over the ℓ-elements of $Z(L^*)$ and λ_z is the character of L dual to z. Clearly, we have $\breve{\psi}' = a\breve{\psi}$ where a is the number of elements z in the sum.
Moreover, as in the proof of [GH97, Thm. 6.4], we claim that for $x \in L$ if $\psi'(x) \neq 0$ then x is ℓ-regular. We write $x = x_{\ell'}x_\ell$, where x_ℓ is an ℓ-element commuting with $x_{\ell'}$ which is ℓ-regular. If $\psi'(x) \neq 0$, then $\psi(x) \neq 0$ and $x \in (sv)Z^\circ(\mathbf{L})$. Hence by (a), we have $x_\ell \in Z^\circ(\mathbf{L})$. On the other hand, $\psi'(x) \neq 0$ also implies that $\sum_z \lambda_z(x) \neq 0$. Therefore, by the orthogonality relations of the characters of a finite abelian ℓ-group, we must have $x_\ell \in \bigcap_z \ker(\lambda_z)$, whence $x_\ell \in [L, L]$. Consequently, $x_\ell \in Z([L, L])$. Now, in the proof of [GH97, Thm. 6.4], Geck and Hiss recall that $|Z([L, L])|$ divides the determinant d of the Cartan matrix of $\mathbf{L}$. We list for each possible $\mathbf{L}$ the value of d. Thanks to [GH97, 5.5], we notice that if $\mathbf{L}$ is of type B_2, D_4 or E_7, then Σ is the

$\mathbf{L}$	$\mathbf{T}$	B_2	D_4	E_6	E_7	G_2	F_4	E_8
d	1	2	4	3	2	1	1	1

preimage of a class of some 2-elements. If $\mathbf{L}$ is of type E_6, then Σ is the preimage of a conjugacy class of some 3-elements. By (a), we must have $\ell \neq o(s)$. Thus, ℓ does not divide d. We conclude that $x_\ell = 1$.
Therefore, $\psi' = \breve{\psi}'$.

(c) Observe that $\psi \cdot \lambda_z \in \Xi_{sz}(L)$ for each ℓ-element $z \in Z(L^*)$ by [DM91, Thm. 13.30]. Then, thanks to [GH97, Prop. 5.4c], we conclude that $I_{\mathbf{L}}^{\mathbf{G}}(\psi\lambda_z) \neq 0$ for each ℓ-element $z \in Z(L^*)$. Thus, since Deligne–Lusztig induction preserves geometric series by Proposition 2.2.14,

$$\langle I_{\mathbf{L}}^{\mathbf{G}}(\psi'), I_{\mathbf{L}}^{\mathbf{G}}(\psi')\rangle = \sum_z \langle I_{\mathbf{L}}^{\mathbf{G}}(\psi\lambda_z), I_{\mathbf{L}}^{\mathbf{G}}(\psi\lambda_z)\rangle \neq 0.$$

In particular,

$$\breve{I}_{\mathbf{L}}^{\mathbf{G}}(\psi) = I_{\mathbf{L}}^{\mathbf{G}}(\breve{\psi}) = I_{\mathbf{L}}^{\mathbf{G}}(1/a\breve{\psi}') = 1/aI_{\mathbf{L}}^{\mathbf{G}}(\psi') \neq 0.$$

Thus no virtual character in $\breve{\Xi}'_t$ is zero.
The last step is exactly the same as in [GH97].

(d) We let $\mathfrak{m}_i = (\mathbf{L}_i, \Sigma_i, \mathcal{E}_i)$ be cuspidal induction data such that $\psi_i := \chi_{\mathcal{A}_{\mathfrak{m}_i}} \in \Xi_t(\mathbf{L})$ and $\breve{\psi}_i \neq 0$ for $1 \leq i \leq n$. Assume furthermore that for $1 \leq i, j \leq n$, if $i \neq j$ then $\breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i) \neq \breve{I}_{\mathbf{L}_j}^{\mathbf{G}}(\psi_j)$. We show that $(\breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i))_{1\leq i\leq n}$ are linearly independent. Suppose there are $a_i \in \mathbf{K}$ for $1 \leq i \leq n$ such that $\sum_{i=1}^n a_i\breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i) = 0$. Using the same construction as in step (b), with $\psi'_i = a'_i\breve{\psi}_i$, we see that $\sum_{i=1}^n a_i/a'_i\breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi'_i) = 0$.
Fix $1 \leq i \leq n$. We claim that $\langle \breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi'_i), \breve{I}_{\mathbf{L}_j}^{\mathbf{G}}(\psi'_j)\rangle \neq 0$ if and only if $i = j$. One direction is given by (c). Assume that $\langle \breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi'_i), \breve{I}_{\mathbf{L}_j}^{\mathbf{G}}(\psi'_j)\rangle \neq 0$ for some j. By definition, there are $z \in Z(L_i^*)$ and $z' \in Z(L_j^*)$, both ℓ-elements, such that $\langle \breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i\lambda_z), \breve{I}_{\mathbf{L}_j}^{\mathbf{G}}(\psi_j\lambda_{z'})\rangle \neq 0$. The class functions $\psi_i\lambda_z$ and $\psi_j\lambda_{z'}$ are cuspidal almost characters and thus there

are $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}), \mathfrak{m}' = (\mathbf{L}', \Sigma', \mathcal{E}') \in \mathfrak{M}^{\mathbf{G}}$ and $\xi, \xi' \in \mathbf{K}$ such that $\breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i \lambda_z) = \xi \chi_{\mathcal{K}_{\mathfrak{m}}}$ and $\breve{I}_{\mathbf{L}_j}^{\mathbf{G}}(\psi_j \lambda_{z'}) = \xi' \chi_{\mathcal{K}_{\mathfrak{m}'}}$. In particular, $\langle \chi_{\mathcal{K}_{\mathfrak{m}}}, \chi_{\mathcal{K}_{\mathfrak{m}'}} \rangle \neq 0$. By [LuCS2, Cor. 9.9] and [LuCS5, Thm. 25.6], it means that there is $g \in G$ such that $\mathbf{L} = g\mathbf{L}'g^{-1}$, $\Sigma = g\Sigma' g^{-1}$ and $\mathcal{E} = \mathrm{ad}(g)^* \mathcal{E}'$, whence $\chi_{\mathcal{A}_{\mathfrak{m}}} = \chi_{\mathcal{A}_{\mathfrak{m}'}} \circ \mathrm{ad}(g)$. By a character formula similar to Theorem 2.3.2 (see [DM20, Prop. 10.1.2]), we get that $I_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i \lambda_z) = I_{\mathbf{L}_j}^{\mathbf{G}}(\psi_j \lambda_{z'})$, hence

$$\breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i) = I_{\mathbf{L}_i}^{\mathbf{G}}(\breve{\psi}_i) = \breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i \lambda_z) = \breve{I}_{\mathbf{L}_j}^{\mathbf{G}}(\psi_j \lambda_{z'}) = I_{\mathbf{L}_j}^{\mathbf{G}}(\breve{\psi}_j) = \breve{I}_{\mathbf{L}_j}^{\mathbf{G}}(\psi_j)$$

and $i = j$ by assumption. We conclude that $a_i/a_i' = \langle \breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i'), \sum_{i=1}^n a_i/a_i' \breve{I}_{\mathbf{L}_i}^{\mathbf{G}}(\psi_i') \rangle = 0$. Therefore $a_i = 0$ for each $1 \leq i \leq n$ and $\breve{\Xi}_t'$ is a free set.

Lastly, we show that if $(\mathbf{L}, \psi), (\mathbf{M}, \theta) \in \Xi_t'$ are such that $\breve{I}_{\mathbf{L}}^{\mathbf{G}}(\psi) = \breve{I}_{\mathbf{M}}^{\mathbf{G}}(\theta)$ then $(\mathbf{L}, \psi)$ and $(\mathbf{M}, \theta)$ are conjugate under G. The arguments are very similar to steps (c) and (d). We use the same construction as in step (b) with $\psi' = a\breve{\psi}$ and $\theta' = b\breve{\theta}$. Then, the virtual character $I_{\mathbf{L}}^{\mathbf{G}}(\psi')$ is a scalar multiple of $I_{\mathbf{M}}^{\mathbf{G}}(\theta')$ and $\langle I_{\mathbf{L}}^{\mathbf{G}}(\psi), I_{\mathbf{M}}^{\mathbf{G}}(\theta') \rangle \neq 0$. In particular, there is an ℓ-element $z \in Z(M^*)$ such that $\langle I_{\mathbf{L}}^{\mathbf{G}}(\psi), I_{\mathbf{M}}^{\mathbf{G}}(\theta\lambda_z) \rangle \neq 0$. By a similar argument as in step (d), we conclude that $I_{\mathbf{L}}^{\mathbf{G}}(\psi) = I_{\mathbf{M}}^{\mathbf{G}}(\theta\lambda_z)$. Since induction preserves the geometric series, the semisimple elements t and tz are conjugate in G. Now since t is an ℓ'-element which commutes with the ℓ-element z, we must have $z = 1$. Thus, the two virtual characters are equal, $I_{\mathbf{L}}^{\mathbf{G}}(\psi) = I_{\mathbf{M}}^{\mathbf{G}}(\theta)$. Now, by a similar argument as in step (d), we conclude that $(\mathbf{L}, \psi)$ and $(\mathbf{M}, \theta)$ are G-conjugate. □

Using this, we can derive a basic set for the non-unipotent isolated series when ℓ is bad.

Proposition 5.1.15. *Assume that $t \neq 1$. Suppose as well that F acts trivially on W. The set $\breve{\mathscr{E}}(G, t)$ is a basic set for the Brauer characters in $\mathscr{B}(G, t)$ except possibly if* $\mathbf{G}$ *is of type E_8 and $\ell \in \{2, 3\}$.*

Proof. Thanks to Proposition 5.1.12, we know that $\breve{\mathscr{E}}(G, t)$ is a generating set. Thus, we need to check that $|\mathscr{E}(G, t)|$ is smaller than the number of Brauer characters in $\mathscr{B}(G, t)$. To compute this number, we compute the size of the $\mathbf{K}$-basis defined in Proposition 5.1.14. To do so, we find a set of representatives $(\mathbf{L}, \psi)$ for all the cuspidal pairs in Ξ_t' up to $\mathbf{G}$-conjugation and then for each representative $(\mathbf{L}, \psi)$ compute the number of G-conjugacy classes in

$$\{(g\mathbf{L}g^{-1}, \psi \circ \mathrm{ad}(g)) \mid g \in \mathbf{G}, g\mathbf{L}g^{-1}, \psi \circ \mathrm{ad}(g) \text{ both } F\text{-stable}\}.$$

In other words, we find the number of F-conjugacy classes in $N_{\mathbf{G}}(\mathbf{L}, \psi)/\mathbf{L}$ or if $(\mathbf{L}, \psi)$ corresponds to the cuspidal induction datum $\mathfrak{m} \in \mathfrak{M}^{\mathbf{G}}$, the number of F-conjugacy classes in $W_{\mathfrak{m}}$. By [AA10, Prop. 4.4], this number equals the number of F^*-conjugacy classes in $N_{C_{\mathbf{G}^*}(t)}(C_{\mathbf{L}^*}(t))/C_{\mathbf{L}^*}(t)$.
Thanks to Appendix B.1, we observe that unless $\mathbf{G}$ is of type E_8 and $\ell \in \{2, 3\}$, all the cuspidal induction data $\mathfrak{m}$ such that $\mathcal{A}_{\mathfrak{m}} \in \hat{\mathbf{L}}_t$ satisfy $\breve{\chi}_{\mathcal{A}_{\mathfrak{m}}} \neq 0$ for any prime number ℓ. Moreover, we observe that the proof of Proposition 5.1.14 does not depend on ℓ good or

bad. In particular, the number of Brauer characters in $\mathscr{B}(G,t)$ is independent of ℓ in this case. Therefore, thanks to Theorem 5.1.7, this number is equal to $|\mathscr{E}(G,t)|$ and we can conclude. □

For completeness, we give in Appendix B.2 the number of Brauer characters in each union of isolated blocks of each simple exceptional group of adjoint type. When $t = 1$ the number of Brauer characters in the unipotent ℓ-blocks can be found in [GH97, 6.6].

5.1.3 A parameterisation of the modular representations

We have determined the number of Brauer characters in the union of blocks we are considering. When ℓ is good, we even know a basic set; it is a Lusztig series. In particular, thanks to Theorem 2.2.29 and the follow-up remark 2.2.30, we get a parameterisation of the basic set, firstly splitting it into families $\mathscr{U}$ and then labelling each character in a family thanks to a certain group $\bar{A}_{\mathscr{U}}$.
We would like to generalise this parameterisation to cover the case ℓ bad. To do so, we follow Chaneb [Cha21]. This allows us to determine a conjectural basic set of Brauer characters.

When ℓ is good

When ℓ is good, the unipotent characters form an ordinary basic set for the unipotent block $\mathscr{B}(G,1)$. We rephrase the parameterisation of the unipotent characters from Theorem 2.2.29 and in particular give a definition of the finite groups appearing in the theorem.
They are partitioned into families according to their unipotent support which is a special unipotent class of $\mathbf{G}$ (Definition 4.1.4 and the discussion about special conjugacy classes). Each family $\mathscr{U} \subseteq \mathrm{Uch}(G)$ is in bijection with $\overline{\mathcal{M}}(\bar{A}_{\mathscr{U}} \subseteq \tilde{A}_{\mathscr{U}})$.

Definition 5.1.16. Let $C \in \mathrm{Ucl}(\mathbf{G})$ be a unipotent class. For $\psi \in \mathrm{irr}(A_{\mathbf{G}}(u_C))$ such that there exists $\theta \in \mathrm{irr}(W)$ with $\mathfrak{Spr}_{\mathbf{G}}(\theta) = (C,\psi)$, we set $a_\psi \coloneqq a_\theta$. Here $\mathfrak{Spr}_{\mathbf{G}}$ is the map defined by the Springer correspondence (Subsection 4.1.1). If no such θ exists, we set $a_\psi \coloneqq 0$. We define the **ordinary canonical quotient** $\bar{A}_C$ as the quotient of $A_{\mathbf{G}}(u_C)$ by the intersection of the kernels of the $\psi \in \mathrm{irr}(A_{\mathbf{G}}(u_C))$ with a_ψ maximal.

Notation 5.1.17. Let $C \in \mathrm{Ucl}(\mathbf{G})$ be a unipotent class. We might write $\bar{A}_u \coloneqq \bar{A}_C$ for $u \in C$. Morever, we write $A_C^{\mathbf{G}}$ when we want to emphasise the ambient group.

By [Lus14, Thm. 0.4], if $\mathscr{U}$ is a family of unipotent characters with unipotent support C, then

$$\bar{A}_{\mathscr{U}} = \bar{A}_C.$$

For $\tilde{A}_{\mathscr{U}}$, we then take $\tilde{A}_C$, the semi-direct product of $\bar{A}_C$ with a cyclic group of order c, where c is the order of the action of F on W and the cyclic group acts on $\bar{A}_C$ by the action given by F acting on $A_{\mathbf{G}}(u_C)$. Thus, if ℓ is good,

$$|\mathscr{B}(G,1)| = \sum_C |\overline{\mathcal{M}}(\bar{A}_C \subseteq \tilde{A}_C)|,$$

where C runs over the F-stable special unipotent conjugacy classes of $\mathbf{G}$.

Remark 5.1.18. Note that if ℓ is good, there are no isolated ℓ-elements of $\mathbf{G}$. In particular, if $s \in \mathbf{G}^*$ is a semisimple ℓ-element, then W_s is a parabolic subgroup of W (seen as the Weyl group of $\mathbf{G}^*$), since $Z(\mathbf{G})$ is connected. By the description of the unipotent support and Proposition 2.2.21, the unipotent support of any character in $\mathscr{E}(G,s)$ is special. Thus,

$$|\mathscr{B}(G,1)| = \sum_C |\overline{\mathcal{M}}(\bar{A}_C \subseteq \tilde{A}_C)|,$$

where C runs over the unipotent supports of characters in $\mathscr{E}_\ell(G,1)$.

For the other unions of blocks, the Jordan decomposition (Theorem 2.2.16) leads us to a similar result.

When ℓ is bad

When ℓ is bad, the unipotent characters do not form a basic set anymore. For instance, as explained in [GH91, §1.2], the unipotent characters of $G_2(q)$ are a generating set but not a basic set when $\ell = 2$. Observe as well that there are 10 unipotent characters but only nine irreducible Brauer characters in the unipotent block (see Table B.6). We generalise the notion of *special* unipotent classes and *canonical quotient* following [Cha19].

Definition 5.1.19. An irreducible character $\psi' \in \mathrm{irr}(W)$ is ℓ-special if there is an isolated semisimple ℓ-element $s \in \mathbf{G}^*$ and a special character $\psi \in \mathrm{irr}(W_s)$ such that $j_{W_s}^W(\psi) = \psi'$.

Definition 5.1.20. A unipotent class $C \in \mathrm{Ucl}(\mathbf{G})$ is **ℓ-special** is there is an irreducible ℓ-special character $\psi \in \mathrm{irr}(W)$ such that $\mathfrak{Spr}_{\mathbf{G}}(\psi) = (C,1)$.

Lemma 5.1.21. *An F-stable unipotent class $C \in \mathrm{Ucl}(\mathbf{G})$ is ℓ-special if and only if it is the unipotent support of an irreducible character in $\mathscr{E}(G,s)$ where $s \in \mathbf{G}^*$ is an isolated ℓ-element.*

Proof. Assume that there is an isolated ℓ-element $s \in \mathbf{G}^*$ such that C is the unipotent support of a character $\chi \in \mathscr{E}(G,s)$. By the description of the unipotent support, there exists a special character $\psi \in \mathrm{irr}(W_s)$ such that $\psi' \coloneqq j_{W_s}^W(\psi)$ satisfies $\mathfrak{Spr}_G(\psi') = (C,1)$. Thus, the class C is ℓ-special.
For the other direction, $\psi' \in \mathrm{irr}(W)$ is ℓ-special and let C such that $\mathfrak{Spr}_G(\psi') = (C,1)$. By definition, there are $s \in \mathbf{G}^*$ an isolated ℓ-element and a special character $\psi \in \mathrm{irr}(W_s)$ such that $j_{W_s}^W(\psi) = \psi'$. The character ψ belongs to a family $\mathscr{F}$ of $\mathrm{irr}(W_s)$. We then consider any character in the family $\mathscr{U} \subseteq \mathscr{E}(G,s)$ corresponding to $\mathscr{F}$. By the description of the unipotent support, these characters have unipotent support C. ☐

Remark that any F-stable special class is ℓ-special. Moreover, when ℓ is bad, an F-stable unipotent class is ℓ-special if and only if it is the support of a character in $\mathscr{E}_\ell(G,1)$. The ℓ-special classes for the exceptional cases are listed in Appendix B.3. To compute them, we use the Springer correspondence.

We now extend the definition of canonical quotient.

Definition 5.1.22. Let C be an ℓ-special class. Let P be a projective indecomposable module of $\mathbf{k}[A_{\mathbf{G}}(u_C)]$ and Ψ its character. We set

$$a_\Psi := \min\{a_\theta \mid \theta \in \mathrm{irr}(W), \mathfrak{Spr}_{\mathbf{G}}(\theta) = (C,\psi), \langle \psi, \Psi \rangle \neq 0.\}$$

We define the **ℓ-canonical quotient** $\bar{A}_{\ell,C}$ as the quotient of $A_{\mathbf{G}}(u_C)$ by the intersection of the kernels of the projective indecomposable modules P with a_Ψ maximal.

The above definition is indeed a generalisation of the canonical quotient thanks to the following lemma.

Lemma 5.1.23 ([Cha19, Prop. 2.3.15]). *If ℓ is good, then $\bar{A}_C = \bar{A}_{\ell,C}$.*

The proof consists in checking that $|A_{\mathbf{G}}(u_C)|$ is divisible only by bad primes for any unipotent class $C \in \mathrm{Ucl}(\mathbf{G})$.

Finally, we count the number of Brauer characters. For any finite group A, we write $\mathcal{M}^\ell(A)$ for the set of A-conjugacy classes of pairs $[a,\phi]$ with $a \in A$ and $\phi \in \mathrm{irr}_{\mathbf{k}}(A)$. We set

$$n_{\ell,C} := |\mathcal{M}^\ell(\bar{A}_{\ell,C})|.$$

We define $\tilde{A}_{\ell,C}$, the semi-direct product of $\bar{A}_C$ with a cyclic group of order c, where c is the order of F on W and the cyclic group acts on $\bar{A}_C$ by the action given by F acting on $A_{\mathbf{G}}(u_C)$. Lastly, the set $\overline{\mathcal{M}}^\ell(\bar{A}_{\ell,C} \subseteq \tilde{A}_{\ell,C})$ consists of all $\tilde{A}_{\ell,C}$-conjugacy classes of pairs $(a,\phi) \in A' \times \mathrm{irr}_{\mathbf{k}}(C_{\bar{A}_{\ell,C}}(a))$, where $A' \subseteq \tilde{A}_{\ell,C}$ is a coset generator of $\tilde{A}_{\ell,C}/\bar{A}_{\ell,C}$

Proposition 5.1.24 ([Cha21, Thm. 3.16]). *If $\mathbf{G}$ is simple of adjoint type not of type A, then*

$$|\mathscr{B}(G,1)| = \sum_C |\overline{\mathcal{M}}^\ell(\bar{A}_{\ell,C} \subseteq \tilde{A}_{\ell,C})|,$$

where C runs over the unipotent supports of characters in $\mathscr{E}_\ell(G,1)$.

For the groups of exceptional type, the proof consists in comparing the numbers obtained through the sum with the numbers in the tables in Appendix B.2.

5.2 Candidates for the projectives: the Kawanaka modules

In this section, we focus on the third step, that is, finding some candidates for the projective modules. A first class of candidates are the generalised Gelfand–Graev characters (GGGCs). Thanks to their properties, we will see that the respective decomposition matrix is block-triangular. However, there might not be enough GGGCs. The idea of Brunat–Dudas–Taylor in [BDT20] is to decompose the GGGCs into Kawanaka characters. This will be the projectives we will choose in order to show the unitriangularity of the ℓ-decomposition matrix of the unipotent blocks.

5.2.1 Generalised Gelfand–Graev characters

We now recall the construction and the properties of the generalised Gelfand–Graev characters following the notation in [BDT20, Section II.6]. These characters were first defined in [Kaw86], and another construction was given in [Tay16].

Definition of the generalised Gelfand–Graev characters

The idea behind GGGCs is in some ways opposite to the construction behind Deligne–Lusztig characters. Instead of starting with a linear character on the maximal torus T_0 corresponding to an F^*-stable semisimple element in $\mathbf{G}^*$, we start with a linear character on some unipotent group U, corresponding to some rational unipotent element $u \in G$. To make this more precise, we use the notions introduced in Subsection 1.3.2 for unipotent conjugacy classes. The two methods to classify the unipotent conjugacy classes use a bijection between $\mathrm{Ucl}(\mathbf{G})$ and the nilpotent $\mathbf{G}$-orbits on $\mathfrak{g}$, via a Springer homeomorphism. We require additional properties for this map.

Definition 5.2.1. Let $\mathscr{K} = (\Psi_{spr}, \mathfrak{s}, \chi_p)$ with

- a Springer isomorphism $\Psi_{spr} : G_{\mathrm{uni}} \xrightarrow{\sim} \mathfrak{g}_{\mathrm{nil}}$,
- a symmetric bilinear form $\mathfrak{s} : \mathfrak{g} \times \mathfrak{g} \to \bar{\mathbb{F}}_p$ which is $\mathbf{G}$-invariant with respect to the adjoint action, and is defined over $\mathbb{F}_q$,
- and a non-trivial character $\chi_p : \mathbb{F}_p^+ \to \overline{\mathbb{Q}}_\ell^\times$.

We say $\mathscr{K} = (\Psi_{spr}, \mathfrak{s}, \chi_p)$ is a **Kawanaka datum** for $\mathbf{G}$ if the following hold:

(K1) for any $\lambda \in \check{X}$,
$$\Psi_{spr}(\mathbf{U}_\lambda(2)) = \mathrm{Lie}(\mathbf{U}_\lambda(2)),$$

(K2) for any $\lambda \in \check{X}$ and any $i \in \{1, 2\}$, there exists a constant $c_i \in \bar{\mathbb{F}}_p$ such that for any $u, v \in \mathbf{U}_\lambda(i)$,

- $\Psi_{spr}(uv) - \Psi_{spr}(u) - \Psi_{spr}(v) \in \mathrm{Lie}(\mathbf{U}_\lambda(i+1))$,
- and $\Psi_{spr}([u,v]) - c_i[\Psi_{spr}(u), \Psi_{spr}(v)] \in \mathrm{Lie}(\mathbf{U}_\lambda(2i+1))$,

(K3) for any maximal torus $\mathbf{S} \leq \mathbf{G}$ and root $\alpha \in \Phi(\mathbf{S})$ we have
$$\mathfrak{g}_\alpha^\perp \coloneqq \{x \in \mathfrak{g} \mid \mathfrak{s}(x, v) = 0 \text{ for all } v \in \mathfrak{g}_\alpha\} = \mathrm{Lie}(\mathbf{S}) \oplus \bigoplus_{\beta \in \Phi(\mathbf{S}) \setminus \{-\alpha\}} \mathfrak{g}_\beta.$$

Kawanaka data do not always exist in general. For instance, a Springer homeomorphism might not be an isomorphism of varieties. However, with certain conditions on $\mathbf{G}$, one can show their existence.

Lemma 5.2.2 ([BDT20, Lem. 6.3]). *If* $\mathbf{G}$ *is proximate (Definition 1.3.15), then there exists a Kawanaka datum* $\mathscr{K} = (\Psi_{spr}, \mathfrak{s}, \chi_p)$ *for* $\mathbf{G}$*. Moreover, we may choose it such that for any* $\lambda \in \check{X}$ *and any integer* $i \geq 2$*,*

$$\Psi_{spr}(\mathbf{U}_\lambda(i)) = \mathrm{Lie}(\mathbf{U}_\lambda(i)).$$

The second statement is a consequence of the proofs of [Tay16, Lem. 4.3 and Prop. 4.6].

From now on, we thus make the following hypothesis.

Hypothesis 3. For the rest of this thesis, we assume that $\mathbf{G}$ is proximate and we fix a Kawanaka datum $\mathscr{K} = (\Psi_{spr}, \mathfrak{s}, \chi_p)$ for $\mathbf{G}$ as in Lemma 5.2.2.

We now define a linear character on a unipotent group corresponding to some unipotent element $u \in G$. Recall from Proposition 1.3.22 that for u there is a corresponding unique parabolic subgroup P_λ for some $\lambda \in \check{X}$ associated to u. The unipotent radical of $P_{-\lambda}$, that is $U_\lambda(-1)$, is the unipotent group from which we induce a linear character.

We now define a character on $U_\lambda(-1)$. We construct from χ_p a character $\chi_q : \mathbb{F}_q^+ \to \overline{\mathbb{Q}}_\ell^\times$ as the composition of χ_p with the field trace $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$.

Definition 5.2.3. Let $u \in G_{\mathrm{uni}}$ be a rational unipotent element. We define the following map:

$$\begin{aligned} \eta_u^G : \quad & G_{\mathrm{uni}} \to \overline{\mathbb{Q}}_\ell \\ & v \mapsto \chi_q(\mathfrak{s}(\Psi_{spr}(u), \Psi_{spr}(v))). \end{aligned}$$

Remark 5.2.4. Observe that for any $x \in G$, we have

$${}^x\eta_u^G = \eta_{xux^{-1}}^G.$$

Indeed, for any $v \in G_{\mathrm{uni}}$, by the $\mathbf{G}$-invariance of $\mathfrak{s}$ and the definition of Ψ_{spr}

$$\begin{aligned} {}^x\eta_u^G(v) = \eta_u^G(x^{-1}vx) &= \chi_q\big(\mathfrak{s}\big(\Psi_{spr}(u), \Psi_{spr}(x^{-1}vx)\big)\big) \\ &= \chi_q\big(\mathfrak{s}\big(\mathrm{Ad}(x)\Psi_{spr}(u), \mathrm{Ad}(x)\Psi_{spr}(x^{-1}vx)\big)\big) \\ &= \chi_q\big(\mathfrak{s}\big(\Psi_{spr}(xux^{-1}), \Psi_{spr}(v)\big)\big) \\ &= \eta_{xux^{-1}}^G(v). \end{aligned}$$

Lemma 5.2.5. *Let* $u \in G_{\mathrm{uni}}$ *be a unipotent element. For any* $\lambda \in \check{X}_{\mathscr{D}}^{\mathbf{G}}(u)$ *an associated* F*-stable cocharacter, the character* η_u^G *restricts to a linear character* $\mathbf{U}_\lambda(-2)^F \to \overline{\mathbb{Q}}_\ell^\times$*.*

Proof. We need to check that $\eta_u^G : \mathbf{U}_\lambda(-2)^F \to \overline{\mathbb{Q}}_\ell^\times$ is a group homomorphism. Clearly it preserves the neutral element. Now, let $v, v' \in \mathbf{U}_\lambda(-2)^F = U_{-\lambda}(2)$. By (K2), we know that $\Psi_{spr}(vv') = \Psi_{spr}(v) + \Psi_{spr}(v') + x$ with $x \in U_{-\lambda}(3) = U_\lambda(-3)$. To conclude, we need to show that $U_\lambda(-3) \subseteq \ker(\eta_u^G)$. By the definition of an associated co-character $\Phi_{spr}(u) \in \mathfrak{g}(\lambda, 2)$. Thus, the kernel of η_u^G contains the F-stable elements in $\Psi_{spr}^{-1}(\mathfrak{g}(\lambda,2)^\perp)$. By (K3) and Lemma 5.2.2,

$$\Psi_{spr}^{-1}(\mathfrak{g}(\lambda,2)^\perp) = U_\lambda(1) \oplus \mathbf{L}_\lambda \oplus U_\lambda(-3).$$

Thus, $\eta_u^G : \mathbf{U}_\lambda(-2)^F \to \overline{\mathbb{Q}}_\ell$ is a group homomorphism. □

If $\mathbf{U}_\lambda(-2) = \mathbf{U}_\lambda(-1)$, then we have constructed a linear character of $\mathbf{U}_\lambda(-1)^F$. Otherwise, we need to induce the character η_u^G.

Definition 5.2.6. Let $u \in G_{\mathrm{uni}}$ be a rational unipotent element and $\lambda \in \check{X}_{\mathscr{D}}^{\mathbf{G}}(u)^F$. We define the following class function on $\mathbf{U}_\lambda(-1)^F$

$$\xi_{u,\lambda}^G := q^{-\dim(g(\lambda,-1))/2} \operatorname{Ind}_{\mathbf{U}_\lambda(-2)^F}^{\mathbf{U}_\lambda(-1)^F}(\eta_u^G).$$

Note that if $\mathbf{U}_\lambda(-2) = \mathbf{U}_\lambda(-1)$, then $\dim(g(\lambda,-1)) = 0$.

This class function is in fact a character of $\mathbf{U}_\lambda(-1)^F$.

Lemma 5.2.7 (proof of [Tay16, Lem. 5.15])**.** *Let $u \in G_{\mathrm{uni}}$ be a rational unipotent element and $\lambda \in \check{X}_{\mathscr{D}}^{\mathbf{G}}(u)^F$, then the class function $\xi_{u,\lambda}^G$ is an irreducible character of $\mathbf{U}_\lambda(-1)^F$.*

We associate to $\xi_{u,\lambda}^G$ a projective $\mathbf{k}[\mathbf{U}_\lambda(-1)^F]$-module as $\mathbf{U}_\lambda(-1)^F$ is a p-group, whence an ℓ'-group. Moreover, by Remark 5.2.4, for any $x \in G$, ${}^x\eta_u^G = \eta_{xux^{-1}}^G$ is a linear character of $U_{-{}^x\lambda}(2)$ and

$${}^x\xi_{u,\lambda}^G = \xi_{{}^xu,{}^x\lambda}^G.$$

Definition 5.2.8 (Kawanaka)**.** For $u \in G_{\mathrm{uni}}$ a rational unipotent element and $\lambda \in \check{X}_{\mathscr{D}}^{\mathbf{G}}(u)^F$ an F-stable co-character associated to u, we define the corresponding **generalised Gelfand–Graev character** (GGGC) of G as

$$\gamma_u^G := \operatorname{Ind}_{\mathbf{U}_\lambda(-1)^F}^G(\xi_{u,\lambda}^G).$$

One can show that γ_u^G does not depend on the choice of the co-character $\lambda \in \check{X}_{\mathscr{D}}^{\mathbf{G}}(u)^F$ ([BDT20, below Def. 6.6]). Moreover, by what we saw before, for any $x \in G$,

$$\gamma_u^G = \gamma_{{}^xu}^G.$$

In particular, for an F-stable unipotent conjugacy class C of $\mathbf{G}$, one could obtain at most as many different generalised Gelfand–Graev characters of the form γ_u^G for some $u \in C^F$ as there are conjugacy classes of $A_{\mathbf{G}}(u_C)$.
Observe as well that since $\mathbf{U}_\lambda(-1)^F$ is a p-group, whence an ℓ'-group and since induction preserves projectivity, we may associate to each GGGC a projective $\mathbf{k}[G]$-module. In other words, there exists a projective $\mathbf{k}[G]$-module Γ_u^G such that γ_u^G is the character associated to $(\Gamma_u^G)^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}$.

Decomposition of GGGCs and wave front set

In the previous section, we have seen that a basic set for a union of blocks $\mathscr{B}(G,t)$ ($t \in \mathbf{G}^*$ a semisimple ℓ'-element) is parameterised in terms of the unipotent supports of the ordinary characters belonging to it. Using GGGCs, we define a dual concept to the unipotent support. It will allow us to show that the decomposition matrix corresponding to the GGGCs is block-triangular.

Definition 5.2.9. Let $\chi \in \mathrm{irr}(G)$. A **wave front set** of χ is an F-stable unipotent conjugacy class C of $\mathbf{G}$ such that:

1. there is $v \in C^F$ satisfying $\langle \gamma_v^G, \chi \rangle \neq 0$ and
2. for any unipotent conjugacy class C' of $\mathbf{G}$ such that $\langle \gamma_{v'}^G, \chi \rangle \neq 0$ for some $v' \in C'$, we have $\dim(C') \leq \dim(C)$.

Similarly to the unipotent support, the wave front set is in fact unique.

Theorem 5.2.10 ([Tay16, Thm. 14.10, Thm. 15.2])**.** *Let $\chi \in \mathrm{irr}(G)$. Then χ has a unique wave front set, which we denote by C_χ^*. Moreover, for any unipotent element $u \in G$, if $\langle \gamma_u^{\mathbf{G}}, \chi \rangle \neq 0$, then $(u)_{\mathbf{G}} \subseteq \overline{C_\chi^*}$.*

For an irreducible character $\chi \in \mathrm{irr}(G)$, we write

$$\chi^* \coloneqq \pm D_{\mathbf{G}}(\chi),$$

where the sign is the unique choice making the Alvis–Curtis dual $D_{\mathbf{G}}(\chi)$ an irreducible character of G, see Definition 2.1.15. Recall that $D_{\mathbf{G}}$ fixes rational series and sends families to families (Remarks 2.2.15 and 2.2.25).
Unipotent supports and wave front sets are deeply linked:

Lemma 5.2.11 ([Tay16, Lem. 14.15])**.** *Let $\chi \in \mathrm{irr}(G)$. Then the unipotent support C_{χ^*} of χ^* is the wave front set C_χ^* of χ.*

Remark 5.2.12. Let us look at our plan we explained at the beginning of this chapter. We choose a total ordering of the ℓ-special unipotent conjugacy classes such that $C_i < C_j$ if $\dim C_i \leq \dim C_j$ for all $1 \leq i < j \leq r$.
For the ordinary irreducible modules, we choose some characters belonging to $\mathscr{B}(G,t)$ with wave front set C_i for each $1 \leq i \leq r$. Alternatively, we take the Alvis–Curtis dual of some characters with unipotent support C_i.
For the projective modules, we choose the GGGCs of the form Γ_u^G for $u \in C_i$ for each $1 \leq i \leq r$.
If there are enough GGGCs, i.e. $\sum_{i=1}^r |\mathrm{irr}(A_{\mathbf{G}}(u_{C_i}))| = \sum_{i=1}^r n_{\ell,C_i}$, then we have already partially completed Step 4 and the corresponding decomposition matrix is block-triangular by definition 5.2.9. In general, we have the number of conjugacy classes in $A_{\mathbf{G}}(u_{C_i})$ is smaller than n_{ℓ,C_i}. We will decompose the GGGCs into Kawanaka characters to overcome this issue.

5.2.2 Kawanaka characters

As we have seen, the GGGCs form an appealing class of candidates for the projective modules. Their main drawback is that in general there are more irreducible Brauer characters in a union of blocks than GGGCs.
In the unipotent case, what we would like is to have at least one projective character per element $[a,\Psi] \in \mathcal{M}^\ell(\bar{A}_{\ell,C})$ for each ℓ-special class C. An approach is to find for each $[a,\psi] \in \mathcal{M}^\ell(\bar{A}_{\ell,C})$ a finite group $A_a \subseteq G$ surjecting onto the centraliser $C_{\bar{A}_{\ell,C}}(a)$ and a character $\phi \in \operatorname{irr}_{\mathbf{k}}(A_a)$, and then instead of considering $\operatorname{Ind}_{\mathbf{U}_\lambda(-1)^F}^G(\xi_{u,\lambda}^G)$ to look at "$\operatorname{Ind}_{\mathbf{U}_\lambda(-1)^F\times A_a}^G(\xi_{u,\lambda}^G \times \phi)$". This does not work as such, but does with some technical modifications.

Definition and existence of an admissible covering

As a first step, we define the groups A_a and require certain properties such that $\mathbf{U}_\lambda(-1)^F\times A_a$, or rather $\mathbf{U}_\lambda(-1)^F \rtimes A_a$, is a subgroup of G. We follow [BDT20].

Definition 5.2.13 ([BDT20, Def. 7.1])**.** Let $u \in G_{\text{uni}}$ be a rational unipotent element. Let $A \leq C_{\mathbf{G}}(u)$ be a subgroup and $\lambda \in \check{X}_{\mathscr{D}}^{\mathbf{G}}(u)^F$ be an F-stable co-character. We say that the pair (A,λ) is **admissible** for u if the following hold:

(A1) the group A is a subgroup of $\mathbf{L}_\lambda^F$,

(A2) the subgroup A contains only semisimple elements,

(A3) and for all $a \in A$, we have $a \in C^\circ_{\mathbf{L}_\lambda}(C_A(a))$.

If $\bar{A}$ is a quotient of $A_G(u)$ on which F acts, we say that the pair (A,λ) is an **admissible covering** for $\bar{A}$ if:

(A4) the restriction of the map $C_{\mathbf{G}}(u) \to \bar{A}$ to $A \to \bar{A}$ fits into the following short exact sequence
$$1 \longrightarrow Z \longrightarrow A \longrightarrow \bar{A} \longrightarrow 1$$
where $Z \leq Z(A)$ is a central subgroup with $Z \cap [A,A] = \{1\}$.

Remark 5.2.14. Assume that (A,λ) is an admissible covering of $\bar{A}_{\ell,C}$ for $C \in \operatorname{Ucl}(\mathbf{G})$. And let any $a \in A$. Then $C_A(a)$ is sent onto $C_{\bar{A}_{\ell,C}}(\bar{a})$ under the map $A \to \bar{A}_{\ell,C}$, $a \mapsto \bar{a}$. In fact, more generally, for $\bar{A}$ any quotient of $A_G(u)$ on which F acts and (A,λ) an admissible covering of $\bar{A}$, there is a short exact sequence

$$1 \longrightarrow Z \longrightarrow C_A(a) \longrightarrow C_{\bar{A}}(\bar{a}) \longrightarrow 1$$

for each $a \in A$ with image $\bar{a} \in \bar{A}$. The surjectivity follows from the fact that $Z \cap [A,A] = \{1\}$.

In [BDT20], Brunat, Dudas and Taylor determine an admissible covering for each ordinary canonical quotient of a unipotent special class.

Proposition 5.2.15 ([BDT20, Sections 9 and 10]). *Assume that* $\mathbf{G}$ *is simple and adjoint. Let* C *be an* F*-stable special unipotent conjugacy class of* $\mathbf{G}$*. Then there always exists an admissible pair* (A_C, λ) *for* u_C *which is an admissible covering of* $\bar{A}_C$*, and such that* A_C *is abelian or* $A_C \cong \bar{A}_C$*. Moreover* $|A_C|$ *is divisible only by the bad primes for* $\mathbf{G}$*.*

We describe in more details the case of exceptional groups.

Proposition 5.2.16 ([BDT20, Section 10]). *Assume that* $\mathbf{G}$ *is a simple exceptional group of adjoint type. Let* C *be a special unipotent conjugacy class of* $\mathbf{G}$*. We distinguish between the following cases:*

1. *If* $\bar{A}_C$ *is trivial, then we choose* $A_C = \{1\} \subseteq \mathbf{G}$ *for an admissible covering.*
2. *If* $\mathbf{G}$ *is of type* E_8 *and* $C = E_8(b_6)$*, then* $A_{\mathbf{G}}(u_C) \cong S_3$ *and* $\bar{A}_C \cong A_C \cong S_2$*.*
3. *If* $\mathbf{G}$ *is of type* E_7 *and* $C = A_4 + A_1$ *or* $\mathbf{G}$ *is of type* E_8 *and* C *is one of* $E_6(a_1) + A_1, D_7(a_2), A_4 + A_1$*, then* $A_{\mathbf{G}}(u_C) \cong \bar{A}_C \cong S_2$ *and* $A_C \cong C_4$*.*
4. *Else,* $\bar{A}_C$ *is not trivial and* $A_{\mathbf{G}}(u_C) \cong \bar{A}_C \cong A_C$*.*

Definition of a Kawanaka character

Hypothesis 5.2.17. For the rest of this section, we fix an F-stable $C \in \mathrm{Ucl}(\mathbf{G})$, a rational unipotent element $u \in C^F$, and an admissible pair (A, λ) for u. We also assume that $p \neq 2$.

To define the Kawanaka characters, we need to extend the characters $\xi^G_{u',\lambda}$ to characters of $U_\lambda(-1)^G \rtimes C_A(a)$ for $u' \in C^F$ and $a \in A$.

As a first step, we need to fix representatives for the G-conjugacy classes in C^F. Note that they are in bijection with the F-conjugacy classes of $A_{\mathbf{G}}(u)$ by Theorem 1.2.5. We observe that $A \subseteq G$ stabilises each G-conjugacy class in C^F. Moreover, the group A acts on the F-conjugacy classes of $A_{\mathbf{G}}(u)$ via the quotient map $A \to A_{\mathbf{G}}(u)$, $a \mapsto \overline{a} \coloneqq aC^\circ_{\mathbf{G}}(u)$. By (A1), $C_A(a)$ normalises $\mathbf{U}_\lambda(-1)$ for any $a \in A$. To extend a character $\xi^G_{u',\lambda}$ for $u' \in C^F$ to a character of $\mathbf{U}_\lambda(-1)^F \rtimes C_A(a)$, we would like to verify that $\xi^G_{u',\lambda}$ is fixed by the action of $C_A(a)$. In other words, for any $b \in C_A(a)$, we would like

$$\xi^G_{u',\lambda} = {}^b\,\xi^G_{u',\lambda} = \xi^G_{{}^bu',{}^b\lambda}.$$

In particular, we want $u' \in C_{\mathbf{G}}(b)^F$ and $\lambda \in \check{X}^{C_{\mathbf{G}}(b)}$.
The above discussion motivates the following definition.

Definition 5.2.18. A set $\{u_a \mid a \in A\} \subseteq C^F$ is a set of **admissible representatives** if it satisfies the following conditions for all $a \in A$:

1. For all $g \in \mathbf{G}$ such that ${}^gu \in (u_a)_{\mathbf{G}^F}$, there is $x \in A_{\mathbf{G}}(u)$ such that $\overline{a} = \overline{x^{-1}g^{-1}F(g)F(x)}$,

2. for all $b \in A$, we have $bu_ab^{-1} = u_{bab^{-1}}$,

3. and for all $b \in C_A(a)$, we have $u_a \in C^\circ_{\mathbf{G}}(b)^F$ and $\lambda \in \check{X}_{\mathscr{D}}^{C^\circ_{\mathbf{G}}(b)}(u)$.

Lemma 5.2.19 ([BDT20, Lem. 7.6]). *Let $u \in G_{\mathrm{uni}}$ be a rational unipotent element and (A, λ) be an admissible pair for u. Then there always exists a set of admissible representatives.*

We are now ready to fix an extension $\tilde{\xi}^G_{u_a,\lambda} \in \mathrm{irr}(C_A(a) \ltimes \mathbf{U}_\lambda(-1)^F)$ of $\xi^G_{u_a,\lambda}$ for $a \in A$. We choose the one defined by Brunat–Dudas–Taylor in [BDT20, Def. 7.9]. The construction is very technical and we refer the reader to [BDT20, 7.3 and 7.4] for the details. However, this extension is well-understood.

Lemma 5.2.20 ([BDT20, Lem. 7.11], [Gér77]). *There exists a class function ϵ of A such that for each $a \in A$ and $t \in C_A(a)$ the following hold:*

1. $\epsilon(t) \in \{\pm 1\}$,

2. and $\tilde{\xi}^G_{u_a,\lambda}(t) = \epsilon(t) q^{\dim(\mathrm{Lie}(\mathbf{U}_\lambda^{C^\circ_G(t)}(-1)))/2}$.

We call the class function ϵ the **Weil-sign**. Finally, we define Kawanaka modules as a slight modification from the one given by Brunat–Dudas–Taylor.

Definition 5.2.21. Assume $a \in A$ and let Ψ be the character of a projective indecomposable $\mathbf{k}[C_A(a)]$-module P (i.e. the character of the $\mathbf{K}[C_A(a)]$-module $P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}$). We also write X_{u_a} for a module of $\mathbf{K}[C_A(a) \ltimes \mathbf{U}_\lambda(-1)^F]$ affording the character $\tilde{\xi}^G_{u_a,\lambda}$ for any $a \in A$. We define the **ℓ-Kawanaka module** associated to the pair (a, Ψ) to be

$$K^G_{(a,\Psi)} := \mathrm{Ind}^G_{C_A(a) \ltimes \mathbf{U}_\lambda(-1)^F}\Big(((X_{u_a})_{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{k}) \otimes \mathrm{Inf}^{C_A(a) \ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} P\Big).$$

Lemma 5.2.22. *Let $a \in A$ and Ψ be the character of a projective indecomposable* $\mathbf{k}[C_A(a)]$-*module P. Then $K^G_{(a,\Psi)}$ is a projective* $\mathbf{k}[G]$-*module.*

Proof. Since $\mathbf{U}_\lambda(-1)$ is a p-group, and $p \neq \ell$, the inflation of the module P is a projective $\mathbf{k}[C_A(a) \ltimes \mathbf{U}_\lambda(-1)^F]$-module. Tensoring and inducing preserve projectivity, thus the module $K^G_{(a,\Psi)}$ is projective. □

Definition 5.2.23. Let $a \in A$ and Ψ be the character of a projective indecomposable $\mathbf{k}[C_A(a)]$-module P. We denote by $\kappa^G_{(a,\Psi)}$ the character afforded by the $\mathbf{K}[G]$-module $(K^G_{(a,\Psi)})^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}$.

Observe that if $C_A(a)$ is an ℓ'-group, Ψ is an irreducible character of $C_A(a)$ and we get back the initial definition of Kawanaka characters.

Definition 5.2.24 ([BDT20, Def. 7.13]). Let $a \in A$ and $\phi \in \mathrm{irr}(C_A(a))$. We define the **Kawanaka character** associated to the pair (a, ϕ) to be

$$\kappa^G_{(a,\phi)} := \mathrm{Ind}^G_{C_A(a) \ltimes \mathbf{U}_\lambda(-1)^F}\Big(\tilde{\xi}^G_{u_a,\lambda} \otimes \mathrm{Inf}^{C_A(a) \ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} \phi\Big).$$

It is the character of the Kawanaka module of $\mathbf{K}[G]$,

$$K^G_{(a,\phi)} \coloneqq \mathrm{Ind}^G_{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}\Big(X_{u_a} \otimes \mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} V_\phi\Big),$$

where V_ϕ is an irreducible $\mathbf{K}[C_A(a)]$-module affording the character ϕ.

Lemma 5.2.25. *Fix $a \in A$ and Ψ the character of a projective indecomposable $\mathbf{k}[C_A(a)]$-module P. Then*

$$\kappa^G_{(a,\Psi)} = \sum_{\phi\in \mathrm{irr}(C_A(a))} d_{\phi,\Psi}\kappa^G_{(a,\phi)}.$$

Proof. For any $\phi \in \mathrm{irr}(C_A(a))$, we write V_ϕ for an irreducible $\mathbf{K}[C_A(a)]$-module affording the character ϕ. We observe that $\kappa^G_{(a,\Psi)}$ is the character of

$$\begin{aligned}
&\mathrm{Ind}^G_{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}\Big(((W_{u_a})_{\mathbf{O}} \otimes \mathbf{k}) \otimes \mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} P\Big)^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}\\
&= \mathrm{Ind}^G_{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}\Big(\Big((W_{u_a})_{\mathbf{O}} \otimes \mathbf{k} \otimes \mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} P\Big)^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}\Big)\\
&= \mathrm{Ind}^G_{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}\Big(W_{u_a} \otimes \Big(\mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} P\Big)^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}\Big)\\
&= \mathrm{Ind}^G_{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}\Big(W_{u_a} \otimes \mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}\Big),\\
&= \mathrm{Ind}^G_{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}\left(W_{u_a} \otimes \mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} \sum_{\phi\in \mathrm{irr}(C_A(a))} d_{\phi,\Psi} V_\phi\right)\\
&= \sum_{\phi\in \mathrm{irr}(C_A(a))} d_{\phi,\Psi}\, \mathrm{Ind}^G_{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}\Big(W_{u_a} \otimes \mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} V_\phi\Big).
\end{aligned}$$

Thus, $\kappa^G_{(a,\Psi)} = \sum_{\phi\in \mathrm{irr}(C_A(a))} d_{\phi,\Psi}\kappa^G_{(a,\phi)}$. ☐

We collect a few properties of the Kawanaka characters $\kappa^G_{(a,\phi)}$ for $a \in A$ and $\phi \in \mathrm{irr}(C_A(a))$ as stated in [BDT20, 7.4].

Lemma 5.2.26. *Let $a \in A$. Then,*

$$\gamma^G_{u_a} = \sum_{\phi\in \mathrm{irr}(C_A(a))} \phi(1)\kappa^G_{(a,\phi)}.$$

Proof. We write ρ for the character of the regular representation of $C_A(a)$ over $\mathbf{K}$. Then

$$\sum_{\phi\in \mathrm{irr}(C_A(a))} \phi(1)\kappa^G_{(a,\phi)} = \mathrm{Ind}^G_{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}\Big(\tilde{\xi}^G_{u_a,\lambda} \otimes \mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} \rho\Big).$$

Now we notice $\tilde{\xi}^G_{u_a,\lambda} \otimes \mathrm{Inf}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} \rho = \mathrm{Ind}^{C_A(a)\ltimes \mathbf{U}_\lambda(-1)^F}_{C_A(a)} \xi^G_{u_a,\lambda}$ and we conclude by the definition of GGGC. ☐

Lemma 5.2.27 ([BDT20, Rmk. 7.14])**.** *The Kawanaka character does not depend on the A-conjugacy class of the pairs $(a,\phi) \in A \times \mathrm{irr}(C_A(a))$. Namely, for any $a, b \in A$ and character $\phi \in \mathrm{irr}(C_A(a))$, we have ${}^b\tilde{\xi}^G_{u_a,\lambda} = \tilde{\xi}^G_{u_{bab^{-1}},\lambda}$ and $\kappa^G_{({}^ba,{}^b\phi)} = \kappa^G_{(a,\phi)}$.*

We thus denote by $\kappa^G_{[a,\phi]}$ the Kawanaka character $\kappa^G_{(a,\phi)}$ for each element $[a,\phi]$ in $\mathcal{M}(A)$ and $\kappa^G_{[a,\Psi]}$ for each element $[a,\Psi]$ in $\mathcal{M}^\ell(A)$.

Lastly, we compute the values of the Kawanaka characters on mixed conjugacy classes.

Proposition 5.2.28 ([BDT20, Prop. 7.16])**.** *Let $[a,\phi]\in\mathcal{M}(A)$. Let $sv\in G$ such that s is semisimple and $v\in C_{\mathbf{G}}(s)$ is unipotent. If s is not G-conjugate to an element in $C_A(a)$, then $\kappa^G_{[a,\phi]}(g)=0$.*
Furthermore, for $s\in C_A(a)$ and for each $t\in C_A(a)\cap(s)_G$, we fix $x_t\in G$ such that $x_tsx_t^{-1}=t$. Then

$$\kappa^G_{[a,\phi]}(g)=\frac{1}{|C_A(a)|}\sum_t \phi(t)\epsilon(t)\gamma_{u_a}^{C_{\mathbf{G}}(t)}(x_tvx_t^{-1}),$$

where t runs over all the G-conjugates of s in $C_A(a)$ and $\gamma_{u_a}^{C_{\mathbf{G}}(t)}\coloneqq \operatorname{Ind}_{C^\circ_{\mathbf{G}}(t)^F}^{C_{\mathbf{G}}(t)^F}(\gamma_{u_a}^{C^\circ_{\mathbf{G}}(t)^F})$.

The ℓ-Kawanaka modules are our candidates for the projective modules. In the next chapter, we will partially describe their decomposition in terms of irreducible ordinary modules.

Chapter 6

Unitriangularity of the decomposition matrix

This final chapter concludes our discussion about the unitriangularity of the decomposition matrix. Let us look at our plan we explained at the beginning of Chapter 5.
Step 1 consists in computing the size of $\mathscr{B}(G,t)$ for $t \in (\mathbf{G}^*)^{F^*}$ an isolated ℓ'-element. This is the content of Theorem 5.1.7 for ℓ good and Theorem 5.1.8 combined with Proposition 5.1.14 for ℓ bad.
Steps 2 and 3 concentrate on choosing candidates $V_1, \dots, V_n \in \mathscr{E}_\ell(G,t)$ and projective $\mathbf{k}[\mathbf{G}]$-modules $P_1, \dots P_n$. We let $C_1, \dots, C_r$ be the unipotent supports of the characters in $\mathscr{E}_\ell(G,t)$. We fix a total ordering $C_1 < \dots < C_r$, such that $C_i < C_j$ if $\dim C_i \leq \dim C_j$ for all $1 \leq i < j \leq r$.
Then, for each $1 \leq i \leq r$, we choose

- n_i irreducible modules $V_1^i, \dots, V_{n_i}^i \in \mathscr{E}_\ell(G,t)$ with wave front set C_i,
- and n_i projective-modules $P_1^i, \dots, P_{n_i}^i$, either ℓ-Kawanaka modules of the form $K_{[a,\Phi]}^G$ for $[a,\Phi] \in \mathcal{M}^\ell(A_{C_i})$, where A_{C_i} is an admissible covering of $\bar{A}_{C_i}$ assuming such an admissible covering exists, or GGGCs $\Gamma_u^{\mathbf{G}}$ for $u \in C_i^F$.

We require $\sum_{1 \leq i \leq r} n_i = n$. The numbers n_i are determined according to our need. For instance, for the unipotent ℓ-blocks, we fix $n_i = |\mathcal{M}^\ell(\bar{A}_{\ell,C})| = n_{\ell,C}$.
To conclude and show **Step 4**, we need to prove that for each $1 \leq i \leq r$,

(A) $(\langle V_l^i, (P_j^i)^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K} \rangle)_{1 \leq l,j \leq n_i}$ is lower unitriangular;

(B) and for all $1 \leq m \leq r$, if $m < i$, $\langle V_l^m, (P_j^i)^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K} \rangle = 0$ for all $1 \leq l \leq n_m$ and $1 \leq j \leq n_i$.

Condition (B) is automatically satisfied by the definition of the wave front set (Definition 5.2.9). In this chapter, we focus on checking the condition A. In the first section, we will state some general results about the decomposition of Kawanaka modules. The idea is to use characteristic functions of character sheaves instead of irreducible characters. The second section will focus on the unipotent blocks for simple exceptional groups of adjoint type. In the last section, we will explain how to treat some isolated blocks by considering the particular cases of $\mathbf{G}$ of type G_2 and F_4.

We recall the assumptions we made so far in Hypotheses 1, 2 and 3: the group $\mathbf{G}$ is a connected reductive proximate group defined over k with Frobenius map $F : \mathbf{G} \to \mathbf{G}$ and Weyl group W with respecto to the maximally split torus $\mathbf{T}_0$ in the Borel $\mathbf{B}_0$. The field k is algebraically closed of characteristic $p \neq \ell$. We also suppose the following.

Hypothesis 4. For the rest of this thesis, we assume that p is good for $\mathbf{G}$.

6.1 Decomposition of the Kawanaka modules

Let C be a unipotent F-stable conjugacy class of $\mathbf{G}$ and K an ℓ-Kawanaka module constructed from C, assuming an admissible covering of C exists (see Definition 5.2.21). In this section, we focus on describing the restriction of the decomposition of $K^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}$ in terms of ordinary irreducible representations with wave front set C.

Hypothesis 6.1.1. For this section, we fix an F-stable $C \in \mathrm{Ucl}(\mathbf{G})$, a rational unipotent element $u \in C^F$ and an admissible pair (A, λ) for u (assuming it exists).

Notation 6.1.2. For θ a class function of G and $g \in \mathbf{G}^*$ an F-stable special element, we denote by $\mathrm{pr}_g(\theta)$ the projection of θ on the space spanned by the Alvis–Curtis duals of irreducible characters in $\mathrm{irr}(G)_g$. We write $\mathrm{pr}_u(\theta)$ for the projection of θ on the space spanned by the Alvis–Curtis duals of irreducible unipotent characters with unipotent support C. The above space is equal to the sapce spanned by Alvis–Curtis duals of the almost characters of the R_x with $x \in \mathcal{M}(\bar{A}_{g_u}^{C_{\mathbf{G}^*}(g_s)} \subseteq \tilde{A}_{g_u}^{C_{\mathbf{G}^*}(g_s)})$ (see Remark 2.2.30). It is also equal to the space spanned by the Alvis–Curtis duals of the characteristic functions of the F-stable character sheaves in $\hat{\mathbf{G}}_g$ by Theorem 3.3.6. Therefore, if κ is a Kawanaka character, to compute $\mathrm{pr}_g(\kappa)$ we consider the restriction of a Fourier transform of κ to the Alvis–Curtis duals of the characteristic functions of the F-stable character sheaves in $\hat{\mathbf{G}}_g$.

6.1.1 Fourier transform of the Kawanaka characters

Definition 6.1.3. The **Fourier transform** of Kawanaka characters is given as follows. For $[a, \phi] \in \mathcal{M}(A)$, we set

$$\mathfrak{f}^G_{[a,\phi]} \coloneqq \sum_{[b,\psi] \in \mathcal{M}(A)} \{[a,\phi],[b,\psi]\} \kappa^G_{[b,\psi]}.$$

Here $\{-,-\}$ is the pairing for $\mathcal{M}(A)$ as defined in Definition 2.2.27

If $\mathcal{A}$ is an F-stable character sheaf in $\hat{\mathbf{G}}_g$ with unipotent support C, then to compute the scalar product $\langle \phi_{[a,\phi]}, \chi_{\mathcal{A}} \rangle$ we need the values of $\phi_{[a,\phi]}$ on mixed conjugacy classes.

Proposition 6.1.4 ([BDT20, Prop. 8.1])**.** *Let $[a,\phi] \in \mathcal{M}(A)$. Let $sv \in G$ such that s is semisimple and $v \in C_{\mathbf{G}}(s)$ is unipotent. If s is not G-conjugate to an element in a, then $\mathfrak{f}_{[a,\phi]}(sv) = 0$.*
Furthermore, if $s = a$, then

$$\mathfrak{f}_{[a,\phi]}(av) = \frac{\epsilon(a)}{|C_A(a)|} \sum_{b \in C_A(a)} \phi(b) \gamma_{u_b}^{C_{\mathbf{G}}(a)^F}(v).$$

The proof of this result relies on the similar result for Kawanaka characters, see Proposition 5.2.28.
As a corollary, we can in fact describe the Fourier transform of Kawanaka characters in terms of GGGCs of smaller group. For any $h \in G$, we write $C_G^{\circ}(h) \coloneqq C_{\mathbf{G}}^{\circ}(h)^F$. We set for each $[a,\phi] \in \mathcal{M}(A)$,

$$\gamma_{(a,\phi)} \coloneqq \frac{1}{|C_A(a)|} \sum_{b \in C_A(a)} \phi(b) \gamma_{u_b}^{C_G^{\circ}(a)}.$$

Corollary 6.1.5 ([BDT20, Cor. 8.4])**.** *Under the same assumptions as in Proposition 6.1.4,*

$$\mathfrak{f}_{[a,\phi]} = \epsilon(a) \operatorname{Ind}_{C_G^{\circ}(a)}^{G} (a^{-1} \cdot \gamma_{(a,\phi)}).$$

Here, a^{-1} denotes the translation of $\gamma_{(a,\phi)}$ by a^{-1}, i.e. $a^{-1} \cdot \gamma_{(a,\phi)}(h) = \gamma_{(a,\phi)}(a^{-1}h)$ for $h \in C_G^{\circ}(a)$.

Lemma 6.1.6. *Assume that $Z(\mathbf{G})$ is connected and p is good for $\mathbf{G}$. Let $\mathcal{A} \in \hat{\mathbf{G}}_g^F$ for a special element $g = sv = vs \in \mathbf{G}^*$ where $s \in (\mathbf{G}^*)^{F^*}$ semisimple and $v \in \mathbf{G}^*$ is unipotent such that $C_g = C$. Then for each $[a,\phi] \in \mathcal{M}(A)$,*

$$\langle \mathfrak{f}_{[a,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}}) \rangle = \pm \frac{\epsilon(a)}{|C_G^{\circ}(a)|} \sum_{u'} D_{C_{\mathbf{G}}^{\circ}(a)}(\gamma_{(a,\phi)})(u') \overline{\chi_{\mathcal{A}}(au')},$$

where u' runs over the unipotent elements of $C_G^{\circ}(a)$ which are $C_{\mathbf{G}}^{\circ}(a)$-conjugate to u.

Proof. This statement can be found in the proof of [BDT20, Prop. 8.8]. □

We now consider the specific case of unipotent character sheaves.

Lemma 6.1.7. *Assume that $\mathbf{G}$ is simple exceptional of adjoint type and that C is special, different from $A_4 + A_1$ if $\mathbf{G}$ is of type E_7 and different from $A_4 + A_1$, $E_6(a_1) + A_1$ and $D_7(a_2)$ if $\mathbf{G}$ is of type E_8. Assume furthermore that p is good for $\mathbf{G}$. There exists an F-stable unipotent character sheaf $\mathcal{A}$ of $\mathbf{G}$ with unipotent support C such that for all $[b,\phi] \in \mathcal{M}(A)$,*

$$\langle \mathfrak{f}_{[b,\phi]}^{G}, D_{\mathbf{G}}(\chi_{\mathcal{A}}) \rangle \neq 0 \iff \text{the image of } b \text{ in } \bar{A}_u \text{ is trivial and } \phi \text{ is trivial.}$$

Moreover, in this case $|\langle \mathfrak{f}_{[b,\phi]}^{G}, D_{\mathbf{G}}(\chi_{\mathcal{A}}) \rangle| = 1$.

Proof. Let $\mathscr{G}$ be a family of F-stable unipotent character sheaves of $\mathbf{G}$ with unipotent support C and $\mathscr{F}$ be the corresponding family of characters in W as in Theorem 3.1.12. Consider $\psi \in \mathscr{F}$ the unique special character of $\mathscr{F}$ and $\mathcal{A}_\psi$ the principal series character sheaf associated to it. Using CHEVIE [Mic15], we show the following claim.

Claim: *For each semisimple $s \in C_{\mathbf{G}}(u)$, the restriction $(s^*\mathcal{A}_\psi)_{(u)_{C^\circ_{\mathbf{G}}(s)}}$ is the trivial local system $\overline{\mathbb{Q}}_\ell[-\dim(C)-\dim(\mathbf{T}_0)]$ if the image of s in $\bar{A}_u$ is trivial and zero otherwise.*

To compute the restriction $(s^*\mathcal{A}_\psi)_{(su)_{C^\circ_{\mathbf{G}}(s)}}$, we apply Corollary 4.3.20. In particular, this formula does not depend on s but only on $C^\circ_{\mathbf{G}}(s)$, and there are finitely many possibilities for $C^\circ_{\mathbf{G}}(s)$ up to $\mathbf{G}$-conjugation.
The image of s in $A_{\mathbf{G}}(u)$ comes from Theorem 1.3.17: the $\mathbf{G}$-conjugacy orbits of the pairs $(u, tC^\circ_{\mathbf{G}}(u))$ with $u \in \mathbf{G}_{\mathrm{uni}}$ and $t \in C_{\mathbf{G}}(u)$ a semisimple element are in bijection with the $\mathbf{G}$-conjugacy orbits of triples $(C^\circ_{\mathbf{G}}(t'), tZ^\circ(C^\circ_{\mathbf{G}}(t')), u)$ where $t' \in \mathbf{G}$ is semisimple, the unipotent element $u \in C^\circ_{\mathbf{G}}(t')_{\mathrm{uni}}$ is distinguished in $C^\circ_{\mathbf{G}}(t')$ and $C^\circ_{\mathbf{G}}(tZ^\circ(C^\circ_{\mathbf{G}}(t'))) = C^\circ_{\mathbf{G}}(t')$. Let $\mathbf{S}$ be a maximal torus of $C^\circ_{\mathbf{G}}(u,s)$ and $\mathbf{M}$ be the pseudo-Levi subgroup $C_{\mathbf{G}}(s,\mathbf{S})$. Then the $\mathbf{G}$-conjugacy orbit of $(u, sC^\circ_{\mathbf{G}}(u))$ corresponds to the $\mathbf{G}$-conjugacy orbit of the induction datum $(\mathbf{M}, sZ^\circ(\mathbf{M}), u)$. Observe that $\mathbf{M}$ is contained in $C_{\mathbf{G}}(s)$.
If $\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$, then since $\mathbf{G}$ is adjoint, $Z(\mathbf{M})$ is connected and $s \in C^\circ_{\mathbf{G}}(u)$. On the other hand, if $\mathbf{M}$ is not a Levi subgroup, then the image of s is $A_{\mathbf{G}}(u)$ is not trivial. In other words, the image of s is trivial in $A_{\mathbf{G}}(u)$ if and only if there is a Levi subgroup $\mathbf{L}$ contained in $C^\circ_{\mathbf{G}}(s)$ such that u belongs to $\mathbf{L}$ and is distinguished in $\mathbf{L}$.
To compute the image of s in $\bar{A}_u$, we observe that either the group $A_{\mathbf{G}}(u)$ is equal to $\bar{A}_u$, or $\bar{A}_u$ is trivial, or lastly $\mathbf{G}$ is of type E_8 and C is the unipotent class $E_8(b_6)$. In this last case, we use the fact that there is a group homomorphism from $A_{C^\circ_{\mathbf{G}}(s)}(u)$ to $A_{\mathbf{G}}(u)$ and therefore from $A_{C^\circ_{\mathbf{G}}(s)}(u)$ to $\bar{A}_u$ to deduce if s is trivial in $\bar{A}_u$. The code can be found in Appendix C.3.

Since C is a special conjugacy class, that is the unipotent support of ordinary characters, the family $\mathscr{F}$ is F-stable. Hence ψ is fixed by F and the character sheaf $\mathcal{A}_\psi$ is F-stable.
We now compute the characteristic function of $\mathcal{A}_\psi$ at sv where $s \in C_{\mathbf{G}}(u)$ and v is $C^\circ_{\mathbf{G}}(s)$-conjugate to u. To simplify notation, we set $\mathcal{A} := \mathcal{A}_\psi$. First, by the claim, $\chi_{\mathcal{A},\varphi}(sv) = 0$ unless s is trivial in $\bar{A}_u$, no matter which isomorphism $\varphi : F^*\mathcal{A} \tilde{\to} \mathcal{A}$ we fix to define the characteristic function. Now, by the discussion in Subsection 3.3.2, we choose an isomorphism $\varphi_{\mathcal{A}} : F^*\mathcal{A} \to \mathcal{A}$ satisfying the condition † (Notation 3.3.3). Thanks to Equation 3.2 and since $\mathcal{A}_{(su)_{C^\circ_{\mathbf{G}}(s)}}$ is the constant local system up to a shift, for any $s \in G$ with trivial image in $\bar{A}_u$ and $x \in C^\circ_{\mathbf{G}}(s)$ such that $xsux^{-1} \in G$ we have

$$\chi_{\mathcal{A},\varphi_{\mathcal{A}}}(xsux^{-1}) = q^{d_{\mathcal{A}}}\zeta_{\mathcal{A},s},$$

where $\zeta_{\mathcal{A},s}$ is a root of unity and

$$d_{\mathcal{A}} = \frac{1}{2}(\dim(\mathbf{G}) - \dim(su)_{\mathbf{G}} - \dim \mathbf{T}_0) = \frac{1}{2}(\dim(C_{\mathbf{G}}(su)) - \dim \mathbf{T}_0).$$

Finally, we compute $\langle \mathfrak{f}^G_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}})\rangle$ for any $[b,\phi] \in \mathcal{M}(A)$. The proof follows word-for-word the proof of [BDT20, Thm. 8.8]. We reproduce it here for completeness. We set $\chi_{\mathcal{A}} \coloneqq \chi_{\mathcal{A},\varphi_{\mathcal{A}}}$.
From Lemma 6.1.6, for any $[b,\phi] \in \mathcal{M}(A)$,

$$\langle \mathfrak{f}_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}})\rangle = \pm \frac{\epsilon(b)}{|C^\circ_G(b)|} \sum_v D_{C^\circ_G(b)}(\gamma_{(b,\phi)})(v)\overline{\chi_{\mathcal{A}}(bv)},$$

where v runs over the unipotent elements in $C^\circ_G(b)$ which are $C^\circ_{\mathbf{G}}(b)$-conjugate to u. From the previous discussion, we observe that this sum is zero unless the image of b in $\bar{A}_u$ is trivial.
We now assume that the image of b in $\bar{A}_u$ is trivial. Then

$$\langle \mathfrak{f}_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}})\rangle = \pm \frac{\epsilon(b)}{|C_A(b)||C^\circ_G(b)|} q^{d_{\mathcal{A}}}\overline{\zeta_{\mathcal{A},b}} \sum_{a\in C_A(b)} \phi(a) \sum_{v\in(u)^F_{C^\circ_{\mathbf{G}}(b)}} D_{C^\circ_G(b)}(\gamma^{C^\circ_G(b)}_{u_a})(v).$$

Let θ be the class function on $C^\circ_G(b)$ defined by $\theta(g) = 1$ if $g \in (u)^F_{C^\circ_{\mathbf{G}}(b)}$ and $\theta(g) = 0$ otherwise. The class function θ is in fact the function $Y_{((u)_{C^\circ_{\mathbf{G}}(b)},1)}$ of [Gec99, (2.2)a] where 1 is the trivial character of $A_{C^\circ_{\mathbf{G}}(b)}(u)$. The scalar product becomes

$$\langle \mathfrak{f}_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}})\rangle = \pm \frac{\epsilon(b)}{|C_A(b)|} q^{d_{\mathcal{A}}}\overline{\zeta_{\mathcal{A},b}} \sum_{a\in C_A(b)} \phi(a)\langle \gamma^{C^\circ_G(b)}_{u_a}, \theta\rangle_{C^\circ_G(b)}.$$

From (2.4)b and c and (2.3)c of [Gec99], we deduce that

$$\langle \gamma^{C^\circ_G(b)}_{u_a}, \theta\rangle_{C^\circ_G(b)} = q^{-d},$$

where $d = \frac{1}{2}\big(\dim(C^\circ_{\mathbf{G}}(b)) - \dim(u)_{C^\circ_{\mathbf{G}}(b)} - \dim \mathbf{T}_0\big)$. Since $d = d_{\mathcal{A}}$, we conclude that

$$\begin{aligned}\langle \mathfrak{f}_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}})\rangle &= \pm \frac{\epsilon(b)}{|C_A(b)|} q^{d_{\mathcal{A}}}\overline{\zeta_{\mathcal{A},b}} \sum_{a\in C_A(b)} \phi(a) q^{-d} \\ &= \pm\epsilon(b)\overline{\zeta_{\mathcal{A},b}}\langle \phi, 1_{C_A(b)}\rangle,\end{aligned}$$

where $1_{C_A(b)}$ denotes the trivial character. The Weil-sign $\epsilon(b) \in \{-1,1\}$ allows us to conclude. □

Remark 6.1.8. In [Lus15, Thm. 2.4], Lusztig stated a much more general result than our claim about the restriction of character sheaves to mixed conjugacy classes. It can be summarised as follows. Let $\mathcal{A}$ be a unipotent character sheaf in a family $\mathscr{G}$ of $\hat{\mathbf{G}}$ with unipotent support C. If the character sheaf $\mathcal{A}$ is labelled by $[b,\psi] \in \mathcal{M}(\bar{A}_u)$, then $(s^*\mathcal{A})_{(u)_{C^\circ_{\mathbf{G}}(s)}}$ is zero unless the image s in $\bar{A}_u$ is conjugate to b. Moreover, if s has image b, then $(s^*\mathcal{A})_{(u)_{C^\circ_{\mathbf{G}}(s)}}$ is the shift of a local system $\mathcal{E}$ on $(u)_{C^\circ_{\mathbf{G}}(s)}$. This local system $\mathcal{E}$ comes from the inflation of ψ under the map $A_{\mathbf{G}}(bu) \to C_{\bar{A}_u}(b)$.
We observe that [Lus15, Thm. 2.4] does not hold in full generality. For instance, if we

consider $\mathbf{G}$ to be simple E_7 of adjoint type, there are two cuspidal unipotent character sheaves ([LuCS4, Prop. 20.3 c]). Their support is the closure of the $\mathbf{G}$-conjugacy class $(su)_{\mathbf{G}}$ where $s \in \mathbf{G}$ is semisimple with connected centraliser of type $\mathrm{SL}_4 \times \mathrm{SL}_4 \times \mathrm{SL}_2$ and the unipotent element $u \in C^\circ_{\mathbf{G}}(s)$ is such that $(u)_{C^\circ_{\mathbf{G}}(s)}$ is the regular class. Their associated local systems correspond to the two non-real characters of $A_{\mathbf{G}}(su) \cong C_4$. Both belong to the same family of exceptional character sheaves with unipotent support $(u)_{\mathbf{G}}$ denoted by $A_4 + A_1$ in CHEVIE notation. But, Theorem 2.4 in [Lus15] claims that the restriction of those character sheaves to their support is a local system corresponding to the lift of a character of $A_{\mathbf{G}}(u) \cong S_2$, which is necessarily real.
Similar situations occur for $\mathbf{G}$ of type E_8, when considering cuspidal characters in an exceptional family. By explicit computations in CHEVIE [Mic15] similar to the ones we did for the claim in the proof of Lemma 6.1.7, one can check when Theorem 2.4 of [Lus15] holds true in exceptional type groups.
This amounts to a huge number of case-by-case analysis, that I do not wish to reproduce in this thesis. I could not yet find a general argument for this result, but I would like to pursue such matter in future work.

Lemma 6.1.9. *Assume that $\mathbf{G}$ is simple exceptional of adjoint type E_7 or E_8 and that C is the class A_4+A_1 if $\mathbf{G}$ is of type E_7 or one of the classes A_4+A_1, $E_6(a_1)+A_1$ or $D_7(a_2)$ if $\mathbf{G}$ is of type E_8. Assume furthermore that p is good for $\mathbf{G}$ and let A be the admissible covering for $\bar{A}_C$ as in Proposition 5.2.16. There exists an F-stable unipotent character sheaf $\mathcal{A}$ with unipotent support C such that for all $[b,\phi] \in \mathcal{M}(A)$,*

$$\langle \mathfrak{f}^G_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}})\rangle = \begin{cases} \epsilon(1) \text{ if } [b,\phi]=[1,1], \\ \epsilon(b)\zeta_{\mathcal{A},b} \text{ if } b^2=1,\, b\neq 1 \text{ and } \phi \text{ is the sign character,} \\ 0 \text{ otherwise.} \end{cases}$$

where $\zeta_{\mathcal{A},b}$ is a root of unity depending only on b and $\mathcal{A}$.

Proof. Note that in all those cases $A \cong C_4$. Let $\mathscr{G}$ be a family of unipotent character sheaves with unipotent support C and $\mathscr{F}$ be the corresponding family of characters in W as in Theorem 3.1.12. Consider $\psi \in \mathscr{F}$ the unique special character of $\mathscr{F}$ and $\mathcal{A}_\psi$ the principal series character sheaf associated to it. Let $b_0 \in A$ be the non-trivial element of order 2. Using CHEVIE [Mic15] and the precise description of A from [BDT20, § 10.3], we verify that

$$(\mathcal{A}_\psi)_{(u)_{\mathbf{G}}} = \overline{\mathbb{Q}}_\ell[-\dim(C)-\dim(\mathbf{T}_0)], \quad (b_0^*\mathcal{A}_\psi)_{(u)_{C^\circ_{\mathbf{G}}(b_0)}} = \mathcal{L}_{\mathrm{sgn}}[-\dim(C)-\dim(\mathbf{T}_0)],$$

and

$$(b^*\mathcal{A}_\psi)_{(u)_{C^\circ_{\mathbf{G}}(b)}} = (b'^*\mathcal{A}_\psi)_{(u)_{C^\circ_{\mathbf{G}}(b')}} = 0$$

where $b, b' \in A$ are the two elements of order 4 and $\mathcal{L}_{\mathrm{sgn}}$ is the local system on $(u)_{C^\circ_{\mathbf{G}}(b_0)}$ corresponding to the sign character of $A_{C^\circ_{\mathbf{G}}(b_0)}(u)$.
The rest of the proof is very similar to the one of Lemma 6.1.7. We may choose the

isomorphism $\varphi : F^*\mathcal{A}_\psi \tilde{\to} \mathcal{A}_\psi$ satisfying the condition † (Notation 3.3.3) and such that the characteristic function of $\mathcal{A}_\psi$ satisfies

$$\chi_{\mathcal{A}_\psi}(v) = q^{d_{\mathcal{A}_\psi}} \text{ for all } v \in C^F,$$

and for any $x \in C^\circ_{\mathbf{G}}(b_0)$ such that $xb_0ux^{-1} \in G$

$$\chi_{\mathcal{A}_\psi}(xb_0ux^{-1}) = q^{d_{\mathcal{A}_\psi}} \mathrm{sgn}(\overline{x^{-1}F(x)})\zeta_{\mathcal{A},b_0},$$

where $\zeta_{\mathcal{A},s}$ is a root of unity and $d_{\mathcal{A}_\psi} = \frac{1}{2}(\dim(\mathbf{G}) - \dim(b_0u)_{\mathbf{G}} - \dim \mathbf{T}_0)$.
The main difference with the proof of Lemma 6.1.7 occurs when computing

$$\langle \mathfrak{f}_{[b_0,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}_\psi})\rangle,$$

for any $\phi \in \mathrm{irr}(C_A(b_0))$. Let θ be the class function on $C^\circ_G(b_0)$ such that for $g \in C^\circ_{\mathbf{G}}(b_0)$,

$$\theta(g) = \begin{cases} q^{-d_{\mathcal{A}_\psi}}\zeta^{-1}_{\mathcal{A},b_0}\chi_{\mathcal{A}_\phi}(b_0g) & \text{if } g \in (u)_{C^\circ_G(b_0)} \\ 0 & \text{otherwise.} \end{cases}$$

The class function θ is in fact the function $Y_{((u)_{C^\circ_{\mathbf{G}}(b)},\mathrm{sgn})}$ of [Gec99, (2.2)a] where sgn is the sign character of $A_{C^\circ_{\mathbf{G}}(b)}(u)$. The scalar product becomes

$$\langle \mathfrak{f}_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}_\psi})\rangle = \pm\frac{\epsilon(b)}{|C_A(b)|}q^{d_{\mathcal{A}}}\zeta_{\mathcal{A},b_0}\sum_{a\in C_A(b)}\phi(a)\langle\gamma^{C^\circ_G(b)}_{u_a},\theta\rangle_{C^\circ_G(b)}.$$

From (2.4)b and c and (2.3)c of [Gec99], we deduce that

$$\langle\gamma^{C^\circ_G(b)}_{u_a},\theta\rangle_{C^\circ_G(b)} = q^{-d}\overline{\mathrm{sgn}(a)},$$

with $d = \frac{1}{2}(\dim(\mathbf{G}) - \dim(b_0u)_{\mathbf{G}} - \dim \mathbf{T}_0) = d_{\mathcal{A}_\psi}$. Therefore,

$$\langle \mathfrak{f}_{[b_0,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}_\psi})\rangle = \pm\epsilon(b_0)\zeta_{\mathcal{A},b_0}\langle\phi,\mathrm{sgn}\rangle,$$

and we conclude the proof of the lemma. □

6.1.2 Decomposition of the Kawanaka characters

Using the decomposition of the Fourier transforms of Kawanaka characters into characteristic functions of character sheaves, we deduce results about the Kawanaka characters themselves.

Proposition 6.1.10. *Assume that* $\mathbf{G}$ *is a simple exceptional group of adjoint type. Let* $g = sv \in \mathbf{G}^*$ *be a special element where* $s \in (\mathbf{G}^*)^{F^*}$ *is semisimple and* $v \in C^*_{\mathbf{G}}(s)$ *is unipotent such that* $C_g = C$ *(that is any character sheaf in* $\hat{\mathbf{G}}_g$ *has unipotent support* C*). Assume the following:*

(KD1) (A, λ) *is an admissible covering of* $\bar{A}_C$*,*

(KD2) $A \cong A_{\mathbf{G}}(u) \cong \bar{A}_v^{C_{\mathbf{G}^*}(s)} \cong \bar{A}_C^{\mathbf{G}}$*,*

(KD3) there exists an F*-stable character sheaf* $\mathcal{A} \in \hat{\mathbf{G}}_g^F$ *such that for all* $[b, \phi] \in \mathcal{M}(A)$

$$\langle \mathfrak{f}^G_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}}) \rangle \neq 0 \iff [b, \phi] = [1, 1].$$

Then the set $\{\mathrm{pr}_g(\kappa^G_{[a,\phi]}) \mid [a, \phi] \in \mathcal{M}(A)\}$ *is an orthonormal set. Thus, the decomposition of the character* $\kappa^G_{[a,\phi]}$ *into irreducible characters of* G *contains exactly one Alvis–Curtis dual of an irreducible character in* $\mathrm{irr}(G)_g$ *with unipotent support* C *and it occurs with multiplicity one.*
Furthermore, every Alvis–Curtis dual of an irreducible character in $\mathrm{irr}(G)_g$ *occurs in exactly one* $\kappa^G_{[a,\phi]}$ *for some* $[a, \phi] \in \mathcal{M}(A)$*.*

Proof. We fix $[a, \phi] \in \mathcal{M}(A)$. We compute

$$\begin{aligned}
\kappa^G_{[a,\phi]} &= \sum_{[b,\psi] \in \mathcal{M}(A)} \{[a, \phi], [b, \psi]\} \mathfrak{f}^G_{[b,\psi]} \\
&= \{[a, \phi], [1, 1]\} \mathfrak{f}^G_{[1,1]} + \sum_{[b,\psi] \in \mathcal{M}(A) \setminus [1,1]} \{[a, \phi], [b, \psi]\} \mathfrak{f}^G_{[b,\psi]} \\
&= \frac{\phi(1)}{|C_A(a)|} \mathfrak{f}^G_{[1,1]} + \sum_{[b,\psi] \in \mathcal{M}(A) \setminus \{[1,1]\}} \{[a, \phi], [b, \psi]\} \mathfrak{f}^G_{[b,\psi]}.
\end{aligned}$$

The last lign follows from the definition of the pairing $\{-, -\}$ in $\mathcal{M}(A)$ (Definition 2.2.27). We write $\hat{G}_g$ for the set of F-stable character sheaves in $\hat{\mathbf{G}}_g$. By the assumption (KD3), we obtain

$$\mathrm{pr}_g(\kappa^G_{[a,\phi]}) = \frac{x_{\mathcal{A}}}{|C_A(a)|} D_{\mathbf{G}}(\chi_{\mathcal{A}}) + \sum_{\mathcal{A}' \in \hat{G}_g \setminus \{\mathcal{A}\}} x_{\mathcal{A}'}([a, \phi]) D_{\mathbf{G}}(\chi_{\mathcal{A}'}),$$

with $x_{\mathcal{A}} \in \mathbb{C}^\times$ and $x_{\mathcal{A}'}([a, \phi]) \in \mathbb{C}$ for all $\mathcal{A}' \in \hat{G}_g \setminus \{\mathcal{A}\}$.
The Alvis–Curtis duals of the characteristic functions of character sheaves in $\hat{G}_g$ form an orthonormal family by Theorem 3.3.5. Therefore, we get

$$\langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[a,\phi]}) \rangle = \frac{|x_{\mathcal{A}}|^2}{|C_A(a)|^2} + \sum_{\mathcal{A}' \in \hat{G}_g \setminus \{\mathcal{A}\}} x_{\mathcal{A}'}([a, \phi]) \overline{x_{\mathcal{A}'}([a, \phi])} > 0.$$

Now by construction, $\mathrm{pr}_g(\kappa^G_{[a,\phi]})$ is a character of G, non-zero since $x_{\mathcal{A}} \neq 0$. Thus for all $[a, \phi], [b, \psi] \in \mathcal{M}(A)$, we note that

$$\langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[a,\phi]}) \rangle \geq 1,$$

and

$$\langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[b,\psi]}) \rangle \geq 0.$$

By the decomposition of a GGGC into Kawanaka characters (Lemma 5.2.26), we get for all $a, b \in A$,

$$\langle \mathrm{pr}_g(\gamma_{u_a}), \mathrm{pr}_g(\gamma_{u_a}) \rangle = \sum_{\phi \in C_A(a)} \sum_{\psi \in C_A(b)} \phi(1)\overline{\psi(1)} \langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[b,\psi]}) \rangle.$$

Hence,

$$\langle \mathrm{pr}_g(\gamma_{u_a}), \mathrm{pr}_g(\gamma_{u_a}) \rangle \geq \sum_{\phi \in \mathrm{irr}(C_A(a))} \phi(1)^2 \langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[a,\phi]}) \rangle \geq \sum_{\phi \in \mathrm{irr}(C_A(a))} \phi(1)^2.$$

On the other hand, since $A_{\mathbf{G}}(u) \cong \bar{A}_v^{C_{\mathbf{G}^*}(s)}$, we may apply [GH08, Rmk 4.4],

$$\langle \mathrm{pr}_g(\gamma_{u_a}), \mathrm{pr}_g(\gamma_{u_a}) \rangle = \sum_{\phi \in \mathrm{irr}(C_{A_{\mathbf{G}}(u)}(\bar{a}))} \phi(1)^2.$$

Moreover, if u_a is not G-conjugate to u_b for $a, b \in A$, then

$$\langle \mathrm{pr}_g(\gamma_{u_a}), \mathrm{pr}_g(\gamma_{u_b}) \rangle = 0.$$

Consequently, for all $[a,\phi] \neq [b,\psi] \in \mathcal{M}(A) = \mathcal{M}(\bar{A}_C) = \mathcal{M}(A_{\mathbf{G}}(u))$,

$$\langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[a,\phi]}) \rangle = 1,$$

and

$$\langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[b,\psi]}) \rangle = 0.$$

Thus, the set $\{\mathrm{pr}_g(\kappa^G_{[a,\phi]}) \mid [a,\phi] \in \mathcal{M}(A)\}$ is orthonormal for the scalar product of characters. Since by assumption, the Frobenius map F acts trivially on $\bar{A}_C$, there are exactly $|\mathcal{M}(\bar{A}_C)| = |\mathcal{M}(A)|$ irreducible characters in $\mathrm{irr}(G)_g$ with wave front set C. Therefore, every Alvis–Curtis dual of an irreducible character in $\mathrm{irr}(G)_g$ occurs in exactly one of $\kappa^G_{[a,\phi]}$ for $[a,\phi] \in \mathcal{M}(A)$. □

Proposition 6.1.11. *Assume that* $\mathbf{G}$ *is a simple exceptional group of adjoint type and that* C *is special. We choose* (A,λ) *as in Proposition 5.2.16. In particular, it is an admissible covering of* $\bar{A}_u$.
Then, given $[a,\phi] \in \mathcal{M}(A_C)$, *the character* $\kappa^G_{[a,\phi]}$ *has at most one unipotent constituent with wave front set* C *and it occurs with multiplicity one. Furthermore, every unipotent character with wave front set* C *occurs in some* $\kappa^G_{[a,\phi]}$ *for* $[a,\phi] \in \mathcal{M}(A)$. *In particular, when* $\bar{A}_u \cong A$, $\kappa^G_{[a,\phi]} = \kappa^G_{[b,\psi]}$ *if and only if* $[a,\phi] = [b,\psi]$, *for* $[a,\phi],[b,\psi] \in \mathcal{M}(A)$.

Proof. We follow the case distinctions of Proposition 5.2.16. Assume first that $\bar{A}_u$ is trivial. By definition, $\mathcal{M}(A) = 1$ and $\mathfrak{f}_{[1,1]} = \kappa^G_{[1,1]}$. The result is then the consequence of Lemma 6.1.7.
Assume now that $A \cong A_{\mathbf{G}}(u) \cong \bar{A}_u$. The statement follows from Proposition 6.1.10 where the last condition is satisfied by Lemma 6.1.7.

We now focus on the unipotent classes such that $A_{\mathbf{G}}(u) \cong \bar{A}_u \cong S_2$ and $A \cong C_4$. In that case, we reason similarly as in the proof of Proposition 6.1.10. Let $\mathcal{A}$ be as in Lemma 6.1.9. The image of $b \in A$ is trivial in $\bar{A}_u$ if and only if b has order 2. Let $b_0 \in A$ be the non-trivial element of order 2. Since A is commutative, for any $[a, \phi] \in \mathcal{M}(A)$,

$$\mathrm{pr}_g(\kappa^G_{[a,\phi]}) = \frac{1}{|A|}(\epsilon(1) + \phi(b_0)\mathrm{sgn}(a)\zeta_{\mathcal{A},b_0}\epsilon(b_0))D_{\mathbf{G}}(\chi_{\mathcal{A}}) + \sum_{\mathcal{A}' \in \hat{G}_g \backslash \{\mathcal{A}\}} x_{\mathcal{A}'}([a,\phi])D_{\mathbf{G}}(\chi_{\mathcal{A}'}),$$

with $x_{\mathcal{A}'}([a,\phi]) \in \mathbb{C}$ for all $\mathcal{A}' \in \hat{G}_g \backslash \{\mathcal{A}\}$. Here $\hat{G}_g$ is the set of F-stable character sheaves in $\hat{\mathbf{G}}_g$. For each $a \in A$, there are at least two $\phi \in \mathrm{irr}(A)$ such that

$$\langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), D_{\mathbf{G}}(\chi_{\mathcal{A}}) \rangle \neq 0.$$

Hence, there are at least two $\phi \in \mathrm{irr}(A)$ such that $\mathrm{pr}_g(\kappa^G_{[a,\phi]}) \neq 0$. Now, since $A_{\mathbf{G}}(u) = \bar{A}_u$,

$$\begin{aligned} 2 = \langle \mathrm{pr}_g(\gamma_{u_a}), \mathrm{pr}_g(\gamma_{u_a}) \rangle &= \sum_{\phi \in C_A(a)} \sum_{\psi \in C_A(a)} \phi(1)\overline{\psi(1)} \langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[a,\psi]}) \rangle \\ &\geq \sum_{\phi \in \mathrm{irr}(C_A(a))} \langle \mathrm{pr}_g(\kappa^G_{[a,\phi]}), \mathrm{pr}_g(\kappa^G_{[a,\phi]}) \rangle \geq 2. \end{aligned}$$

Therefore, there are $\phi_1, \phi_2 \in \mathrm{irr}(A)$ such that for each $a \in A$, the projections $\mathrm{pr}_g(\kappa^G_{[a,\phi_1]})$ and $\mathrm{pr}_g(\kappa^G_{[a,\phi_2]})$ are two distinct irreducible characters. Furthermore, for any other character $\phi \in \mathrm{irr}(A) \backslash \{\phi_1, \phi_2\}$, $\mathrm{pr}_g(\kappa^G_{[a,\phi]}) = 0$. By [Tay16, Prop. 15.4], every unipotent character with wave front set C occurs in some $\gamma^G_{u_a}$ for some $a \in A$. This allows us to conclude the proof of the proposition in this case.

By the case distinctions of Proposition 5.2.16, we are left with the case where $\mathbf{G}$ is of type E_8 and C is the class labelled $E_8(b_6)$.
Thanks to CHEVIE [Mic15], we make the following observations.

- There are four unipotent character sheaves with unipotent support C.
- One of them, that we denote by $\mathcal{A}_4$, does not belong to the principal series and is not unipotently supported, thus $(\mathcal{A}_4)_C = 0$.
- The other three belong to the principal series and are labelled under the isomorphism of Proposition 3.2.17 by the characters $\phi_1 = \phi_{[2240,10]}, \phi_2 = \phi_{[1400,1]}, \phi_3 = \phi_{[840,13]} \in \mathrm{irr}(W)$.
- Using the Springer correspondence (c.f. Section 4.1), we observe that $\phi_{[1400,1]}$ corresponds to the class A_7. Therefore, the restriction $(\mathcal{A}_{\phi_2})_C = 0$.
- Moreover, $(\mathcal{A}_{\phi_3})_C = \mathcal{E}[\dim C + \dim \mathbf{T}_0]$ where $\mathcal{E}$ is the irreducible local system on C corresponding to the sign representation of $A_{\mathbf{G}}(u)$.

- Lastly, the character $\phi_{[2240,10]}$ is special, and thus $(\mathcal{A}_{\phi_1})_C = \overline{\mathbb{Q}}_\ell[\dim C + \dim \mathbf{T}_0]$ where $\overline{\mathbb{Q}}_\ell$ is the trivial local system on C and corresponds to the trivial representation of $A_{\mathbf{G}}(u)$.

Applying the same arguments as in [BDT20, Theorem 8.8] that we have explained in Lemma 6.1.7, we compute that $\langle \mathfrak{f}_{[1,1]}, \chi_{\mathcal{A}_{\phi_3}} \rangle = 0$. By Lemma 6.1.7, the same reasoning as in the proof of Proposition 6.1.10 tells us that for any $[a,\phi],[b,\psi] \in \mathcal{M}(A)$,

$$\langle \mathrm{pr}_u(\kappa^G_{[a,\phi]}), \mathrm{pr}_u(\kappa^G_{[a,\phi]}) \rangle \geq 1,$$

and

$$\langle \mathrm{pr}_u(\kappa^G_{[a,\phi]}), \mathrm{pr}_u(\kappa^G_{[b,\psi]}) \rangle \geq 0.$$

Moreover, thanks to our discussion about the character sheaves we observe that

$$\sum_{[a,\phi]\in\mathcal{M}(A)} \mathrm{pr}_u(\kappa^G_{[a,\phi]}) = 2\,\mathrm{pr}_u(\mathfrak{f}_{[1,1]}) = 2\epsilon(1) D_{\mathbf{G}}(\chi_{\mathcal{A}_{\phi_1}}).$$

Therefore,

$$4 = \langle \sum_{[a,\phi]\in\mathcal{M}(A)} \mathrm{pr}_u(\kappa^G_{[a,\phi]}), \sum_{[a,\phi]\in\mathcal{M}(A)} \mathrm{pr}_u(\kappa^G_{[a,\phi]}) \rangle \geq \sum_{\in\mathcal{M}(A)} \langle \mathrm{pr}_u(\kappa^G_{[a,\phi]}), \mathrm{pr}_u(\kappa^G_{[a,\phi]}) \rangle \geq 4.$$

Thus, for all $[a,\phi] \in \mathcal{M}(A)$, the characters $\mathrm{pr}_u(\kappa^G_{[a,\phi]})$ are irreducible characters of G and they are all distinct. □

Using [Lus15, Thm. 2.4], Brunat–Dudas–Taylor showed the same result for any finite adjoint group.

Proposition 6.1.12 ([BDT20, Thm. B and Thm. 8.9]). *Assume that* $\mathbf{G}$ *is simple and adjoint. We choose* (A,λ) *as in Proposition 5.2.15. Given* $[a,\phi] \in \mathcal{M}(A_C)$*, the character* $\kappa^G_{[a,\phi]}$ *has at most one unipotent constituent with wave front set* C *and it occurs with multiplicity one. Furthermore, every unipotent character with wave front set* C *occurs in some* $\kappa^G_{[a,\phi]}$ *for some* $[a,\phi] \in \mathcal{M}(A)$.

6.2 Unitriangularity of the unipotent blocks

In this section, we show the unitriangularity of the unipotent ℓ-blocks when $\mathbf{G}$ is a simple adjoint group of exceptional type. When ℓ is good for $\mathbf{G}$, this was already shown in more generality by Brunat–Dudas–Taylor [BDT20, Thm. A].
We come back to our plan from the introduction. Thanks to Proposition 5.1.24, we know that the unipotent classes $C_1,\dots,C_r$ which are the unipotent support of the characters in $\mathscr{E}_\ell(G,1)$ are the special classes. Moreover, we may assume that the number n_i equals $|\mathcal{M}(\bar{A}_{C_i})|$ for $1 \leq i \leq r$.
Therefore, thanks to Proposition 6.1.11 the condition (A) is satisfied when ℓ is good.

We thus assume that ℓ is bad. In order to reduce the number of cases we split our analysis in three parts. We first consider the special classes and see how and in which cases we can extend our results. In particular, we will see in Corollary 6.2.3 that the conditions we need are satisfied in most cases. We then move on to all ℓ-special classes where it suffices to choose the GGGCs as projective modules in **Step 3**. This is the case when the ℓ-canonical quotient is trivial or when $\ell = 2$ and the canonical quotient is a group with two elements (Corollary 6.2.8). Three unipotent classes for $\mathbf{G}$ of type E_8 are not covered by this discussion and we treat them separately.

Hypothesis 6.2.1. In this section, we assume that $\mathbf{G}$ is a simple exceptional group of adjoint type and that ℓ is bad for $\mathbf{G}$. Recall that we have assumed that p is good.

6.2.1 Using Kawanaka modules for the special classes

For the special conjugacy classes, we want to use what we know about the decomposition of Kawanaka characters into irreducible characters of G to deduce the decomposition of the ℓ-Kawanaka characters.

Proposition 6.2.2. *Assume that $\mathbf{G}$ is exceptional simple of adjoint type. Let C be an F-stable unipotent conjugacy class of $\mathbf{G}$. Assume that*

(Kℓ1) there exists an admissible pair (A, λ) for u_C which is an admissible covering of $\bar{A}_C$ such that for all $a \in A$ the ℓ-decomposition matrix of $C_A(a)$ is lower-unitriangular,

(Kℓ2) there is $g = sv = vs \in (\mathbf{G}^*)^{F^*}$, with s is an ℓ-element and $v \in G^*$ is a unipotent special element, such that the characters sheaves in $\hat{\mathbf{G}}_g$ have unipotent support C, and

(Kℓ3) given $[a, \phi] \in \mathcal{M}(A)$, the character $\mathrm{pr}_g(\kappa^G_{[a,\phi]})$ is either an irreducible character or zero. Furthermore, every character in $\mathrm{irr}(G)_g$ occurs in some $\kappa^G_{[a,\phi]}$ for some $[a, \phi] \in \mathcal{M}(A)$.

Let $d = |\mathcal{M}^\ell(\bar{A}_C)|$. If either ℓ does not divide $|A|$ or $A \cong \bar{A}^{C^\circ_{\mathbf{G}^*}(s)}_{(v)_{C^\circ_{\mathbf{G}^*}(s)}}$, then there exist characters $\rho_1, \dots, \rho_d \in \mathrm{irr}(G)_g \subseteq \mathcal{E}_\ell(G, 1)$ with unipotent support C and $[a_1, \Psi_1], \dots, [a_d, \Psi_d] \in \mathcal{M}^\ell(A)$ such that for $1 \le i, j \le d$,

$$\langle \rho_i^*, \kappa^G_{[a_j,\Psi_j]} \rangle = \begin{cases} 0 & i < j, \\ 1 & i = j. \end{cases}$$

Proof. By Lemma 5.2.25,

$$\mathrm{pr}_{u_C}(\kappa^G_{(a,\Psi)}) = \sum_{\psi \in \mathrm{irr}(C_A(a))} d_{\psi,\Psi} \, \mathrm{pr}_v(\kappa^G_{(a,\psi)}).$$

Let us first assume that ℓ does not divide $|A|$. Then $\mathcal{M}^\ell(A) = \mathcal{M}(A)$. Moreover, ℓ does not divide $\bar{A}_C$ either and $\mathcal{M}^\ell(\bar{A}_C) = \mathcal{M}(\bar{A}_C)$. Thus, the number d equals the number of unipotent characters with wave front set C. The statement is then just a reformulation of the third hypothesis (Kℓ3). For each $[a,\psi] \in \mathcal{M}(A) = \mathcal{M}^\ell(A)$ there is exactly one unipotent character $\rho_{[a,\psi]}$ with wave front set C such that $\mathrm{pr}_{u_C}(\kappa^G_{[a,\psi]}) = \rho^*_{[a,\psi]}$.

Suppose now that $A \cong \bar{A}^{C^\circ_{\mathbf{G}^*}(s)}_{(v)_{C^\circ_{\mathbf{G}^*}(s)}}$. For $a \in A$, we write $\bar{a}$ for its image in $\bar{A}_g := \bar{A}^{C^\circ_{\mathbf{G}^*}(s)}_{(v)_{C^\circ_{\mathbf{G}^*}(s)}}$. Similarly, for $\phi \in \mathrm{irr}(C_A(a))$ we set $\overline{\phi}$ for the character seen as a character of $C_{\bar{A}_g}(\bar{a})$. Let $\{\rho_{[b,\psi]} \mid [b,\psi] \in \mathcal{M}(\bar{A}_g)\}$ be the set of irreducible ordinary characters in $\mathrm{irr}(G)_g$ with unipotent support C. By assumption, up to reindexing, we may write

$$\mathrm{pr}_g(\kappa^G_{[a,\psi]}) = \rho^*_{[\bar{a},\overline{\psi}]},$$

for each $[a,\psi] \in \mathcal{M}(A)$. In other words, for each $a \in A$ and Ψ the character of a projective indecomposable $\mathbf{k}[C_A(a)]$-module, we have

$$\mathrm{pr}_g(\kappa^G_{(a,\Psi)}) = \sum_{\psi \in \mathrm{irr}(C_A(a))} d_{\psi,\Psi}\,\mathrm{pr}_g(\kappa^G_{(a,\psi)}) = \sum_{\psi \in \mathrm{irr}(C_A(a))} d_{\psi,\Psi}\rho^*_{[\bar{a},\overline{\psi}]}.$$

Therefore,

$$\langle \rho^*_{[a,\psi]}, \mathrm{pr}_g(\kappa^G_{[a,\Psi]})\rangle = d_{\psi,\Psi}.$$

Furthermore, for $b \in A$ not A-conjugate to a, and any $\phi \in \mathrm{irr}(C_A(b))$, we deduce that

$$\langle \rho^*_{[\bar{b},\overline{\phi}]}, \mathrm{pr}_{u_C}(\kappa^G_{[a,\Psi]})\rangle = 0.$$

For each $a \in A$, the ℓ-decomposition matrix of $C_A(a)$ is lower-unitriangular. Thus, we can fix a total ordering of $\{\Psi_j \mid 1 \le j \le s_a\}$, the set of characters of $C_A(a)$ associated to the projective indecomposable $\mathbf{k}[C_A(a)]$-modules, and an ordering of the set $\{\psi_i \mid 1 \le i \le t_a\} = \mathrm{irr}(C_A(a))$ such that for all $1 \le j \le s_a$ and for $1 \le i \le j$,

$$d_{\psi_i,\Psi_j} = \begin{cases} 0 \text{ if } i < j \\ 1 \text{ if } i = j. \end{cases}$$

Then for each $1 \le j \le s_a$, we set $\rho_j := \rho_{[a_j,\psi_j]}$ and the sets $\{\rho_j \mid 1 \le j \le |\mathcal{M}^\ell(A)|\}$ and $\{\kappa^G_{[a,\Psi]} \mid [a,\Psi] \in \mathcal{M}^\ell(A)\}$ satisfy the statement of the proposition. □

Corollary 6.2.3. *Assume that* $\mathbf{G}$ *is simple exceptional of adjoint type. Let* C *be a special* F*-stable unipotent conjugacy class of* $\mathbf{G}$ *and* (A,λ) *be the admissible covering of* $\bar{A}_C$*, as in Proposition 5.2.16.*
Assume that $\bar{A}_C \cong \bar{A}_{\ell,C}$ *and that either* ℓ *does not divide* $|A|$ *or* $A \cong \bar{A}_C$*. Then, there exist unipotent characters* $\rho_1,\dots,\rho_{n_{\ell,C}} \in \mathcal{E}(G,1)$ *with unipotent support* C *and* $[a_1,\Psi_1],\dots,[a_{n_{\ell,C}},\Psi_{n_{\ell,C}}] \in \mathcal{M}^\ell(A)$ *such that for* $1 \le i,j \le n_{\ell,C}$*,*

$$\langle \rho^*_i, \kappa^G_{[a_j,\Psi]}\rangle = \begin{cases} 0 & i < j, \\ 1 & i = j. \end{cases}$$

Proof. This is a consequence of Proposition 6.2.2. Firstly, we notice that $\bar{A}_C \cong \bar{A}_{\ell,C}$ implies that $n_{\ell,C} = |\mathcal{M}^\ell(\bar{A}_{\ell,C})| = |\mathcal{M}^\ell(\bar{A}_C)|$.
We check that the hypotheses of Proposition 6.2.2 hold. By Proposition 5.2.16, there exists an admissible covering (A,λ) for $\bar{A}_C$ and the group A is either S_2, S_3, S_4, C_4 or S_5 and the primes are $\ell \in \{2,3,5\}$. We need to check the unitriangularity of the ℓ-decomposition matrices of the following groups: S_2, S_3, S_4, S_5, $C_3, C_2 \times C_2, D_8, C_4, C_5$ and D_{12} (group with 12 elements) and C_6. We already know that the ℓ-decomposition matrix of the symmetric group is unitriangular, see [Jam78, Cor. 12.3]. Moreover, it is also trivially the case for groups of order a prime power. We can easily check that it is also true for the group D_{12}.
For the assertion (Kℓ2), we simply choose g to be a unipotent element in the class C seen in $\mathbf{G}^*$. The last hypothesis (Kℓ3) is a reformulation of Proposition 6.1.11. ☐

Lemma 6.2.4. *If* $\mathbf{G}$ *is simple exceptional of adjoint type, the only special unipotent classes of* $\mathbf{G}$ *for which we cannot apply Corollary 6.2.3 are given in Table 6.1.*

$\mathbf{G}$	$\ell = 2$	$\ell = 3$
F_4	A_2, $F_4(a_2)$	
E_7	$A_4 + A_1$, $E_7(a_4)$, $A_3 + A_2$	
E_8	$E_6(a_1)+A_1$, $D_7(a_2)$, A_4+A_1, $E_8(b_4)$, $D_7(a_1)$, $D_5 + A_2$, $E_7(a_4)$, $D_4 + A_2$, $A_3 + A_2$	$E_8(b_6)$

Table 6.1: Special unipotent conjugacy classes where we cannot apply Corollary 6.2.3.

Proof. This follows from the description of the admissible covering in Proposition 5.2.16 and from explicit computations in CHEVIE [Mic15] of the ordinary and ℓ-canonical quotients, c.f. Appendix B.3 and Appendix C.2. ☐

6.2.2 Using generalised Gelfand–Graev characters

Let C be an F-stable unipotent conjugacy class of G. We now state some general results about the restriction of the decomposition of the Γ_u^G in terms of ordinary irreducible modules with wave front set C.

Definition 6.2.5 ([Héz04, Thm. A])**.** Let C be an F-stable unipotent class of $\mathbf{G}$ and g be a special element of $\mathbf{G}^*$ with Jordan decomposition $g = sv$. We say that g satisfies the **property** (P) with respect to C if :

1. $\Phi((g)_{\mathbf{G}^*}) = C$,
2. $|\bar{A}_v^{C_{\mathbf{G}^*}(s)}| = |A_{\mathbf{G}}(u_C)|$, and
3. the image of s in the adjoint quotient of $\mathbf{G}^*$ is quasi-isolated.

If there exists such a $g \in \mathbf{G}^*$ such that s is additionally isolated and an ℓ-element, we say that C is ℓ-(P)-**special**. The list of ℓ-(P)-special classes may be found in Appendix B.3.
An element g satisfying property (P) exists for any F-stable unipotent conjugacy class of $\mathbf{G}$, even if its semisimple part might not be an ℓ-element.

Theorem 6.2.6 ([Héz04, Thm. B]). *Let C be an F-stable unipotent class of $\mathbf{G}$, then there exists a special element $g \in (\mathbf{G}^*)^{F^*}$ satisfying property (P) with respect to C.*

We know part of the restriction of the GGGCs Γ_u^G for $u \in C^F$ to irreducible characters of $\mathbf{G}^F$ with wave front set C.

Proposition 6.2.7 ([GH08, Prop. 4.3]). *Let C be an F-stable unipotent class of $\mathbf{G}$ and $u_1, \dots, u_d$ be representatives for the G-conjugacy classes contained in C. Let $g = sv = vs \in (\mathbf{G}^*)^{F^*}$ satisfying Property (P) with respect to C, with $s \in \mathbf{G}^*$ semisimple and $v \in \mathbf{G}^*$ unipotent. Assume that $\bar{A}_v^{C_{\mathbf{G}^*}(s)}$ is abelian. Then there exist $\chi_1, \dots, \chi_d \in \operatorname{irr}(\mathbf{G}^F)_g$ such that $\langle \chi_i^*, \gamma_{u_j} \rangle = \delta_{ij}$ for $1 \leq i, j \leq d$.*

As a corollary, if C as above is ℓ-(P)-special and $d = n_{\ell,C}$, then considering the generalised Gelfand-Graev characters is sufficient.

Corollary 6.2.8. *Let C be an F-stable ℓ-(P)-special unipotent class of $\mathbf{G}$. If $\bar{A}_{\ell,C}$ is trivial or $\ell = 2$ and $\bar{A}_{\ell,C} \cong S_2$, then there exist $\rho_1, \dots, \rho_{n_{\ell,C}} \in \operatorname{irr}(G)$ in the unipotent ℓ-blocks with unipotent support C and generalised Gelfand–Graev characters $\gamma_1, \dots, \gamma_{n_{\ell,C}}$ such that for $1 \leq i, j \leq n_{\ell,C}$,*

$$\langle \rho_i^*, \gamma_j \rangle = \delta_{ij}.$$

Proof. Let $u_1, \dots, u_d$ be representatives for the G-conjugacy classes contained in C^F. Since C is ℓ-(P)-special, we can choose $g \in \mathbf{G}^*$ satisfying Property (P) with respect to C, with g_s an ℓ-element. By Proposition 6.2.7, there exist $\rho_1, \dots, \rho_d \in \operatorname{irr}(G)_g$ such that $\langle \rho_i^*, \gamma_{u_j} \rangle = \delta_{ij}$ for $1 \leq i, j \leq d$.
Recall that we choose $u_C \in C^F$ such that $A_{\mathbf{G}}(u_C)$ is F-stable. Thus the number d of representatives for the G-conjugacy classes contained in C^F is equal to the number of conjugacy classes of $A_{\mathbf{G}}(u_C)$. Observe that if $\ell = 2$ and $\bar{A}_{\ell,C} \cong S_2$, then $n_{2,C} = 2$. On the other hand, if $\bar{A}_{\ell,C}$ is trivial, then $n_{\ell,C} = 1$ for any prime ℓ. In both cases, $n_{\ell,C} \leq d$ and the sets $\{\rho_1, \dots, \rho_{n_{\ell,C}}\}$ and $\{\gamma_{u_1}, \dots, \gamma_{u_{n_{\ell,C}}}\}$ satisfy the statement. □

Proposition 6.2.9. *If $\mathbf{G}$ is simple exceptional of adjoint type and ℓ is bad for $\mathbf{G}$, the only ℓ-special but not special unipotent conjugacy classes of $\mathbf{G}$ for which we cannot apply Corollary 6.2.8 are when $\mathbf{G}$ is of type E_8,*

1. *$\ell = 2$ and the unipotent conjugacy class is $E_7(a_5)$ and*
2. *$\ell = 3$ and the unipotent conjugacy class is $E_6(a_3) + A_1$.*

Moreover, the only special class for which we cannot apply Corollary 6.2.3 nor Corollary 6.2.8 is the class $E_8(b_6)$ of $\mathbf{G}$ of type E_8 when $\ell = 3$.

Proof. This follows by inspection of the tables in Appendix B.3. □

6.2.3 The leftover cases in E_8

We finally consider the cases in E_8 that we could not treat thanks to Corollary 6.2.3 or Corollary 6.2.8. In this section, we assume that $\mathbf{G}$ is of type E_8.

The special unipotent conjugacy class $E_8(b_6)$ when $\ell = 3$

We first consider the special class.

Lemma 6.2.10. *Let* $\mathbf{G}$ *be a simple group of type* E_8 *and* C *be the* F*-stable unipotent class* $E_8(b_6)$*. Then, there exist* $n_{3,C}$ *irreducible characters in the unipotent* 3*-blocks with unipotent support* C *and* $n_{3,C}$ *projective characters (either Kawanaka or GGGC) such that the decomposition matrix restricted to these rows and columns is unitriangular.*

Proof. We observe using CHEVIE [Mic15] that $A_{\mathbf{G}}(u_C) \cong \bar{A}_{3,C} \cong S_3$ and $n_{3,C} = 5$. Firstly, thanks to Proposition 5.2.16, we can find an admissible covering A of the ordinary canonical quotient associated to C. In this case, we have $A \cong \bar{A}_C \cong S_2$. We denote the elements of $\mathcal{M}(A)$ by $[1,1], [1,\mathrm{sgn}], [-1,1], [-1,\mathrm{sgn}]$, where sgn denotes the sign character. Thanks to Proposition 6.2.2 and since ℓ does not divide A, we find four unipotent characters $\rho_{[1,1]}, \rho_{[1,\mathrm{sgn}]}, \rho_{[-1,1]}, \rho_{[-1,\mathrm{sgn}]}$ with unipotent support C, and construct four Kawanaka characters with respect to A and C such that for $[b,\phi], [a,\psi] \in \mathcal{M}(A)$,

$$\langle \rho^*_{[b,\phi]}, \kappa^G_{[a,\psi]} \rangle = \begin{cases} 1 & b = a \text{ and } \phi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|\mathcal{M}^3(\bar{A}_{3,C})| = 5$, we need to find an irreducible character of G in the unipotent ℓ-blocks, which has unipotent support C but is not unipotent. As in [GH08, Proof of Prop. 4.3], for any unipotent character ρ with unipotent support C,

$$\sum_{i=1}^{3} [A_{\mathbf{G}}(u_i) : A_{\mathbf{G}}(u_i)^F] \langle \rho^*, \gamma^G_{u_i} \rangle = \frac{|A_{\mathbf{G}}(u)|}{n_\rho}, \tag{6.1}$$

where n_ρ is given by [Lus84a, 4.26.3]. In our case, since ρ is unipotent and $\bar{A}_C \cong S_2$, we have $n_\rho = 2$. Moreover, as in [GM20, Example 2.7.8 c)], we may assume that u_1 corresponds to 1 (whence $[A_{\mathbf{G}}(u_1) : A_{\mathbf{G}}(u_1)^F] = 1$), u_2 corresponds to a 2-cycle (whence $[A_{\mathbf{G}}(u_2) : A_{\mathbf{G}}(u_2)^F] = 3$), and u_3 to a 3-cycle (whence $[A_{\mathbf{G}}(u_3) : A_{\mathbf{G}}(u_3)^F] = 2$).
Let $\phi \in \mathrm{irr}(S_2)$ and $i, j \in \{\pm 1\}$. By Equation (5.2.26), there are two distinct GGGCs, say $\gamma^G_{v_1}$ and $\gamma^G_{v_{-1}}$ such that $\langle \rho^*_{[i,\phi]}, \gamma^G_{v_j} \rangle = \delta_{i,j}$. By construction, we observe that $\gamma^G_{v_1} = \gamma^G_{u_1}$ and $\gamma^G_{v_{-1}} = \gamma^G_{u_2}$. Inserting this into (6.1), we must have

$$\langle \rho^*_{[1,\phi]}, \gamma^G_{u_3} \rangle = 1 \text{ and } \langle \rho^*_{[-1,\phi]}, \gamma^G_{u_3} \rangle = 0.$$

Moreover, we can check using CHEVIE [Mic15] that the conjugacy class C is 3-(P)-special. In other words, there is $g = sv = vs \in (\mathbf{G}^*)^{F^*}$ with $s \in (\mathbf{G}^*)^{F^*}$ semisimple of order a power of 3, and $v \in (\mathbf{G}^*)^{F^*}$ unipotent such that $\bar{A}_v^{C_{\mathbf{G}^*}(s)} \cong S_3$ and $\Phi(g) = C$. Now by [Gec99, Prop. 6.7] and [Héz04, Rem. 4.4], there is a character $\mu \in \mathcal{E}(\mathbf{G}, s)_g$ such that

$$\langle \mu^*, \gamma_{u_i}^G \rangle = \begin{cases} 1 & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

We then choose the irreducible modules with characters μ, $\rho_{[1,1]}$, $\rho_{[1,\mathrm{sgn}]}$, $\rho_{[-1,1]}$, $\rho_{[-1,\mathrm{sgn}]}$ and the projective $\mathbf{k}[G]$-modules $\Gamma_{u_3}^G, K_{[1,1]}^{\mathbf{G}}, K_{[1,\mathrm{sgn}]}^{\mathbf{G}}, K_{[-1,1]}^{\mathbf{G}}, K_{[-1,\mathrm{sgn}]}^{\mathbf{G}}$ in these orders. The preceding computations show that the decomposition matrix of G restricted to these rows of irreducible $\mathbf{K}[G]$-modules and columns of projective $\mathbf{k}[G]$-modules has the following shape, where the empty entries are 0:

$$\begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

□

The two ℓ-special but not special classes

For the last two cases, since we always have $\bar{A}_C \cong \bar{A}_{\ell,C}$, it suffices to check the conditions of Proposition 6.2.2. We want to show the following lemma.

Lemma 6.2.11. *The two unipotent classes in Proposition 6.2.9 which are not special satisfy the conditions of Proposition 6.2.2.*

Thanks to Proposition 6.1.10, this amounts to checking the following:

(Kℓ1) there exists an admissible pair (A, λ) for u_C which is an admissible covering of $\bar{A}_C$ such that for all $a \in A$ the ℓ-decomposition matrix of $C_A(a)$ is lower-unitriangular,

(Kℓ2) there is $g = tv = vt \in (\mathbf{G}^*)^{F^*}$ where t is an ℓ-element and $v \in G^*$ is a unipotent special element, such that the characters sheaves in $\hat{\mathbf{G}}_g$ have unipotent support C,

(KD2) $A \cong A_{\mathbf{G}}(u) \cong \bar{A}_v^{C_{\mathbf{G}^*}(t)} \cong \bar{A}_C^{\mathbf{G}}$, and

(KD3) there exists an F-stable character sheaf $\mathcal{A} \in \hat{G}_g$ such that for all $[b, \phi] \in \mathcal{M}(A)$

$$\langle \mathfrak{f}_{[b,\phi]}^G, D_{\mathbf{G}}(\chi_{\mathcal{A}}) \rangle \neq 0 \iff [b, \phi] = [1, 1].$$

We recall some notation. For each simple root $\beta \in \Delta$, the fundamental dominant co-weight $\check{\omega}_\alpha \in \mathbb{Q} \otimes \mathbb{Z}\check{X}$ satisfies $\langle \alpha, \check{\omega}_\beta \rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$. We also fix a bijection $\mathfrak{n}$ from the semisimple elements of $\mathbf{T}_0^*$ to the Kummer local systems on $\mathbf{T}_0$, see after Proposition 3.1.11. Note as well that since E_8 is both adjoint and simply connected, as well as self-dual, the centralisers of semisimple elements in $\mathbf{G}$ and $\mathbf{G}^*$ are connected.
For the last condition (KD3), we will use CHEVIE [Mic15] to show that

Claim: *There exists* $\mathcal{A} \in \hat{G}_g$ *in the principal series such that for each* $s \in A$, *the restriction* $(s^*\mathcal{A})_{(u)_{C^\circ_{\mathbf{G}}(s)}}$ *is the trivial local system* $\overline{\mathbb{Q}}_\ell[-\dim(C) - \dim(\mathbf{T}_0)]$ *if the image of* s *in* $\bar{A}_u$ *is trivial and zero otherwise.*

The same argument as in the proof of Lemma 6.1.7 will allow us to conclude the result about the characteristic function. In order to show the claim, we reformulate Corollary 4.3.19 for the principal series. For $t \in \mathbf{T}_0^*$ corresponding to the local system $\mathcal{L}$ on $\mathbf{T}_0$, i.e. $\mathfrak{n}(t) = \mathcal{L}$, by [AA10, Prop. 4.4], we have

$$W_{\mathcal{L}} \cong N_{C_{\mathbf{G}^*}(t)}(\mathbf{T}_0^*)/\mathbf{T}_0^* \coloneqq W_t^{\mathbf{G}}.$$

Recall that we have set $d \coloneqq -\dim(\mathbf{G}) + \dim(\mathbf{G}_s) - \dim(\mathbf{T}_0)$ and $e = -\dim(\mathbf{T}_0)$.

Corollary 6.2.12. *Let* $\mathfrak{m} = (\mathbf{T}_0, \mathbf{T}_0, \mathcal{L}) \in \mathfrak{M}^{\mathbf{G}}$ *and* $t \in \mathbf{T}_0^*$ *such that* $\mathfrak{n}(t) = \mathcal{L}$. *Let* $\mathcal{A} \in \hat{\mathbf{G}}(\mathfrak{m})$ *and* $V \in \mathrm{Irr}(\overline{\mathbb{Q}}_\ell[W_t^{\mathbf{G}}])$ *such that* $\mathcal{A} \cong \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}})}(V, \mathcal{K}_{\mathfrak{m}})$. *For any* $s \in \mathbf{T}_0$, $s^*(\mathcal{A})_{s(\mathbf{G}_s)_{\mathrm{uni}}}[d]$ *is isomorphic to*

$$\bigoplus_{V' \in \mathrm{Irr}(W^{\mathbf{G}_s})} \bigoplus_{w \in W_t^{\mathbf{G}} \backslash W / W^{\mathbf{G}_s}} \langle \mathrm{Res}^{W_t^{\mathbf{G}}}_{W^s_{t,w}}(V) \otimes X^s_w, \mathrm{Res}^{W^{\mathbf{G}_s}}_{W^s_{t,w}}(V' \circ \mathrm{ad}(w^{-1})) \rangle (\mathrm{ad}(w^{-1})^* \mathcal{A}^{\overline{\mathbb{Q}}_\ell}_{V'})[e]_{(\mathbf{G}_s)_{\mathrm{uni}}},$$

where $W^s_{t,w} = {}^w W^{\mathbf{G}_s} \cap W_t^{\mathbf{G}}$, $\mathcal{A}^{\overline{\mathbb{Q}}_\ell}_{V'} = \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}_0})}(V', \mathcal{K}_{\mathfrak{m}_0})$ *with* $\mathfrak{m}_0 = (\mathbf{T}_0, \mathbf{T}_0, \overline{\mathbb{Q}}_\ell) \in \mathfrak{M}^{\mathbf{G}_s}$. *We see* V' *as an* $\mathrm{End}(\mathcal{K}_{(\mathfrak{m}_0)})$*-module under the isomorphism defined in Proposition 3.2.17 and* X^s_w *is the module of* $\overline{\mathbb{Q}}_\ell[W^{\mathbf{G}_s} \cap {}^w W_t^{\mathbf{G}}]$ *whose character is* $\chi^s_w : w_0 \mapsto \mathfrak{j}(\mu_{ww_0w^{-1}}(wsw^{-1}))$.

The code to compute the restriction as above can be found in Appendix C.3.

Notation 6.2.13. The labelling of the unipotent conjugacy classes follows CHEVIE [Mic15] notation.

The unipotent conjugacy class $E_6(a_3) + A_1$ when $\ell = 3$

We fix the setting in the case where the F-stable unipotent class is $C = E_6(a_3) + A_1$. We choose $t \in \mathbf{T}_0^*$ such that $C_{\mathbf{G}^*}(t)$ is of type E_6A_2 and $v \in C_{\mathbf{G}^*}(t)$ lies in the unipotent class $A_2, 111$ of $C_{\mathbf{G}^*}(t)$.

Admissible covering We follow [BDT20, § 10.2]. We can choose $s = \omega_{\alpha_1}(1/2)$ and $u_C \in C^\circ_{\mathbf{G}}(s)$ F-fixed. In that case $C_{\mathbf{G}}(s)$ is of type D_8. We set $\mathbf{G}_s \coloneqq C_{\mathbf{G}}(s)$. Using CHEVIE [Mic15] (see code in Appendix C.3), we compute that only one unipotent class of $\mathbf{G}_s$ fuses into C. Therefore, we know that u_C lies in the unipotent class 6631 of $\mathbf{G}_s$ which fuses into $C = E_6(a_3) + A_1$.
The group $A \coloneqq \langle s \rangle$ can be chosen as an admissible covering of $A_{\mathbf{G}}(u_C)$ for a fixed co-character. Observe that

$$A \cong A_{\mathbf{G}}(u_C) \cong \bar{A}_v^{C_{\mathbf{G}^*}(s)}$$

and if $\ell = 3$, then $A \cong \bar{A}_{\ell,C}$. Thus the conditions (Kℓ1), (Kℓ2) and (KD2) are satisfied. We are left to check the last condition of Proposition 6.1.10.

Character sheaves We fix $\mathcal{L} \coloneqq \mathfrak{n}(t) \in \mathcal{S}(\mathbf{T}_0)$. We now consider a principal series character sheaf of $\mathbf{G}$ coming from $\mathcal{L}$ with unipotent support $E_6(a_3) + A_1$. In CHEVIE notation, we choose the one corresponding to the character $\phi_{30,15}, 111$ of $W_{\mathcal{L}}$. Observe that $\mathcal{A} \coloneqq \mathcal{A}^{\mathcal{L}}_{\phi_{30,15},111}$ is F-stable.

We detail how we use CHEVIE [Mic15] to check that $\mathcal{A}$ satisfies the claim. This is the function RestrictionMixedSupport in Appendix C.3.
We compute that $W^{\mathbf{G}_s}\backslash W/W_{\mathcal{L}} = \{1, g\}$ for some $g \in W$. Moroever, we check that the groups $W^{\mathbf{G}_s} \cap W_{\mathcal{L}}$ and $W^{\mathbf{G}_s^g} \cap W_{\mathcal{L}}$ are both Weyl groups. Therefore χ_1^s and χ_g^s from Corollary 6.2.12 are trivial, by Lemma 4.2.6.
We consider the restriction to $(su_C)_{\mathbf{G}}$. By the same argument as in the proof of [Lus15, Thm. 2.4] (that we have reproduced in Proposition 4.3.6), we need to consider only the character sheaves of $\mathbf{G}_s$ which correspond under the Springer correspondence to the unipotent class 6631, that is character sheaves such that

$$(\mathcal{A}^{\overline{\mathbb{Q}}_\ell}_{E'})_{(\mathbf{G}_s)_{\mathrm{uni}}} = IC(\overline{(u)_{\mathbf{G}_s}}, \mathcal{E}')[\dim(\mathbf{T}_0) - \dim((u)_{\mathbf{G}_s})]$$

for $E' \in \mathrm{irr}(W^{\mathbf{G}_s})$ and $\mathcal{E}'$ a local system on $(u)_{\mathbf{G}_s}$. Indeed, let $v \in \mathbf{G}_s$ be a unipotent element and $E' \in \mathrm{irr}(W^{\mathbf{G}_s})$ such that $(\mathcal{A}^{\overline{\mathbb{Q}}_\ell}_{E'})_{\mathbf{G}_{suni}} = IC(\overline{(v)_{\mathbf{G}_s}}, \mathcal{E}')[\dim(\mathbf{T}_0) - \dim((v)_{\mathbf{G}_s})]$ for $\mathcal{E}'$ a local system on $(v)_{\mathbf{G}_s}$. If $(u)_{\mathbf{G}_s} \notin \overline{(v)_{\mathbf{G}_s}}$, then

$$(\mathcal{A}^{\overline{\mathbb{Q}}_\ell}_{E'})_{(u)_{\mathbf{G}_s}} = IC(\overline{(v)_{\mathbf{G}_s}}, \mathcal{E}')_{(u)_{\mathbf{G}_s}} = 0.$$

On the other hand, if $(u)_{\mathbf{G}_s} \subseteq \overline{(v)_{\mathbf{G}_s}} - (v)_{\mathbf{G}_s}$, then $(u)_{\mathbf{G}} \subseteq \overline{(v)_{\mathbf{G}}} - (v)_{\mathbf{G}}$. By definition of the unipotent support, we must have $s^*((\mathcal{A}_E)_{s(v)_{\mathbf{G}_s}}) = 0$. Thus the character sheaf $\mathcal{A}^{\overline{\mathbb{Q}}_\ell}_{E'}$ cannot appear in the decomposition of $s^*((\mathcal{A}_E)_{|s(\mathbf{G}_s)_{uni}})$.

By the Springer correspondence, there is only one character $E' \in \mathrm{irr}(W^{\mathbf{G}_s})$ whose image is $(\overline{(u)_{\mathbf{G}_s}}, \mathcal{E}')$ for $\mathcal{E}'$ a local system on $(u)_{\mathbf{G}_s}$. In that case, $\mathcal{E}'$ is the trivial local system on $(u)_{\mathbf{G}_s}$ corresponding to the trivial character of $A_{\mathbf{G}_s}(u)$. Using CHEVIE, we conclude that the coefficient is zero, whence

$$\mathcal{A}_{(su)_{\mathbf{G}_s}} = 0.$$

We see that there is an F-stable character sheaf $\mathcal{A}$ with unipotent support C such that

- $\mathcal{A}_{|(su_C)_{C^\circ_{\mathbf{G}}(s)}} = 0$, and
- $\mathcal{A}_{|(u_C)_{\mathbf{G}}}[-\dim(C) - \dim(\mathbf{T}_0)]$ is a local system corresponding to the trivial character of $A_{\mathbf{G}}(u_C)$.

Therefore, for all $[b, \phi] \in \mathcal{M}(A)$

$$\langle \mathfrak{f}^G_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}}) \rangle \neq 0 \iff [b, \phi] = [1, 1].$$

This concludes the analysis of the first case.

The unipotent conjugacy class $E_7(a_5)$ when $\ell = 2$

We fix the setting in the case where the F-stable unipotent class is $C = E_7(a_5)$. We choose $t \in \mathbf{T}_0^*$ such that $C_{\mathbf{G}^*}(t)$ is of type E_7A_1 and $v \in C_{\mathbf{G}^*}(t)$ lies in the unipotent class $D_4(a_1), 11$ of $C_{\mathbf{G}^*}(t)$.

Admissible covering We fix $\mathbf{M}$ the standard Levi subgroup of $\mathbf{G}$ of type E_7. We fix an element $u_C \in \mathbf{M}^F$ such that $(u_C)_{\mathbf{G}}$ is the unipotent conjugacy class $E_7(a_5)$ and F acts trivially on $A_{\mathbf{M}}(u_C) \cong S_3$. We write $C_{\mathbf{M}} \coloneqq (u_C)_{\mathbf{M}}$. Using CHEVIE, we check that the unipotent conjugacy class $C_{\mathbf{M}}$ is distinguished in $\mathbf{M}$. We fix a co-character $\lambda \in Y_{\mathcal{D}}^{\mathbf{M}}(u)^F$. By the same reasoning as in [BDT20, § 10.4], the group $A = C_{\mathbf{L}_{\mathbf{M}}(\lambda)}(u_C)$ is an admissible covering of $A_{\mathbf{M}}(u_C)$. Then, by the argument at the end of [BDT20, § 10.5], where they apply [BDT20, Lem. 4.4], the admissible pair (A, λ) is also an admissible covering for $A_{\mathbf{G}}(u_C)$. Observe that

$$A \cong A_{\mathbf{G}}(u_C) \cong \bar{A}_v^{C_{\mathbf{G}^*}(t)}$$

and if $\ell = 2$, then $A \cong \bar{A}_{\ell,C}$.

Remark 6.2.14. Observe that by [MS03, Thm. 1] there are some $h_1, h_2 \in \mathbf{G}$ such that

$$A_{\mathbf{G}}(u_C) \cong \langle \omega_{\alpha_1}(1/2)^{h_1} C^\circ_{\mathbf{G}}(u_C), \omega_{\alpha_2}(1/3)^{h_2} C^\circ_{\mathbf{G}}(u_C) \rangle.$$

Character sheaves We fix $\mathcal{L} \coloneqq \mathfrak{n}(t) \in \mathcal{S}(\mathbf{T}_0)$. We consider a principal series character sheaf of $\mathbf{G}$ coming from $\mathcal{L}$ with unipotent support $E_7(a_5)$. Thanks to Lusztig's map ([Lus92, Thm. 10.7]), we choose the one corresponding to the character of $W_{\mathcal{L}}$ denoted by $\phi_{315,16}, 11$. We now check that $\mathcal{A} \coloneqq \mathcal{A}^{\mathcal{L}}_{\phi_{315,16},11}$ satisfies the claim.

Let us look at the case where $s \in A$ is an involution. Then, there exists $x \in \mathbf{G}$ such that $a^x = \omega_{\alpha_1}(1/2) \in \mathbf{T}_0$. We fix $s \coloneqq \omega_{\alpha_1}(1/2) \in \mathbf{T}_0$ such that $\mathbf{G}_s \coloneqq C_{\mathbf{G}}(s)$ is of type D_8. Using CHEVIE, since only one unipotent class of $\mathbf{G}_s$ fuses into C, we know that $u \coloneqq u_C^x$ lies in the unipotent class 7522 of $\mathbf{G}_s$ which fuses into $E_7(a_5)$.
We want to compute the restriction of the previous character sheaf to the mixed conjugacy class $(su)_{\mathbf{G}_s}$. We compute that $W^{\mathbf{G}_s}\backslash W/W_{\mathcal{L}} = \{1, g\}$ for some $g \in W$. The group $W^{\mathbf{G}_s} \cap W_{\mathcal{L}}$ is a Coxeter group. On the other hand, $W^g_{\mathcal{L}} \coloneqq W^{\mathbf{G}_s^g} \cap W_{\mathcal{L}}$ is not a Coxeter group and we have $W^g_{\mathcal{L}}/(W^g_{\mathcal{L}})^\circ \cong C_2 \cong Z(\mathbf{G}_s^g)$. Thus, by Lemma 4.2.6, χ^s correspond to the lift of the sign character of $W^g_{\mathcal{L}}/(W^g_{\mathcal{L}})^\circ$.
Using CHEVIE, we compute that[1]

$$(\mathcal{A})_{(su)_{\mathbf{G}_s}} = 0.$$

Lastly, we consider the case where $s \in A$ has order 3. By similar arguments as before, using CHEVIE, we compute that

$$\mathcal{A}_{(su_C)_{C_{\mathbf{G}}(s)}} = 0.$$

[1]if we did not tensor by the sign character when applying Corollary 6.2.12, we would have had $(\mathcal{A})_{(su)_{\mathbf{G}_s}} \neq 0$

We see that there is an F-stable character sheaf $\mathcal{A}$ with unipotent support C such that

- $\mathcal{A}_{|(au_C)_{C^\circ_{\mathbf{G}}(s)}} = 0$ if $a \neq 1$ for any $a \in A$.
- $\mathcal{A}_{|(u_C)_{\mathbf{G}}}[-\dim(C)-\dim(\mathbf{T})]$ is a local system corresponding to the trivial character of $A_{\mathbf{G}}(u_C)$.

Therefore, for all $[b,\phi] \in \mathcal{M}(A)$

$$\langle \mathfrak{f}^G_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}})\rangle \neq 0 \iff [b,\phi] = [1,1].$$

This concludes our discussion about the last case.

6.2.4 The proof

We are now ready to prove our main result.

Theorem 6.2.15. *Let* $\mathbf{G}$ *be a simple exceptional group of adjoint type defined over* k*, an algebraically closed field of characteristic* p *with Frobenius endomorphism* F*. Assume that* p *is good for* $\mathbf{G}$*. If* ℓ *is bad for* $\mathbf{G}$*, then the decomposition matrix of the unipotent* ℓ*-blocks of* G *is unitriangular.*

Proof. We fix a total ordering of the ℓ-special unipotent conjugacy classes of $\mathbf{G}$, $C_1,\dots,C_r$ such that for all $1 \leq l,m \leq r$, we fix $l < m$ if $\dim(C_l) < \dim(C_m)$.

Let C_l be a unipotent ℓ-special conjugacy class and $n_{\ell,C_l} \coloneqq |\mathcal{M}^\ell(\bar{A}_{\ell,C_l})|$. Thanks to our previous discussion, we can find projective $\mathbf{k}G$-modules $P^l_1,\dots,P^l_{n_{\ell,C_l}}$ with characters π^l_j associated to their lift to $\mathbf{K}[G]$-modules and irreducible characters of G in the unipotent ℓ-blocks with unipotent support C_l, $\rho^l_1,\dots,\rho^l_{n_{\ell,C_l}}$, such that for all $1 \leq i,j \leq n_{\ell,C_l}$

$$\langle (\rho^l_i)^*, \pi^l_j\rangle = \begin{cases} 0 & i<j, \\ 1 & i=j. \end{cases}$$

In particular, for a fixed l the P^l_i are all distinct.

Let $C_m \neq C_l$ be another unipotent ℓ-special conjugacy class of $\mathbf{G}$ and ρ' be an irreducible character of G with unipotent support C_m. Suppose that there is $1 \leq j \leq n_{\ell,C_l}$, such that $\langle (\rho')^*, \pi^l_j\rangle \neq 0$.
We observe that if $\langle (\rho')^*, \pi^l_j\rangle \neq 0$, then there exists $v \in C_l^F$ and a generalised Gelfand–Graev character γ_v, such that $\langle (\rho'), \gamma_v\rangle \neq 0$. If P^l_j is itself a GGGC, then this is obvious. Otherwise it is a consequence of Lemmas 5.2.25 and 5.2.26. In any case, since $(\rho')^*$ has wave front set C_m, we conclude by the unicity of the wave front set ([Tay16, Thm. 15.2]) that $(v)_{\mathbf{G}} = C_l \subseteq \overline{C_m}$, whence $\dim(C_l) < \dim(C_m)$ and thus $l < m$.

Now for each $1 \leq l \leq r$ and $1 \leq i \leq n_{\ell,C_l}$, we set $\mu^l_i \coloneqq (\rho^l_i)^*$. The irreducible character μ^l_i lies in the unipotent ℓ-blocks. Moreover, for $1 \leq m \leq r$ and $1 \leq j \leq n_{\ell,C_m}$, we have

$$\langle \mu^l_i, \pi^m_j\rangle = \begin{cases} 0 \text{ if } n<m \text{ or } (n=m \text{ and } i<j), \\ 1 \text{ if } n=m \text{ and } i=j. \end{cases}$$

Therefore, summing over all the ℓ-special unipotent conjugacy classes, we obtain the exact number of indecomposable projective $\mathbf{k}[G]$-modules in the unipotent ℓ-blocks. We conclude thanks to Proposition 5.0.1. □

The assumption that p is good is crucial as we do use a lot of properties for GGGC which are not yet proven for the extension to bad primes as defined by Geck [Gec21a]. Since we do not know yet a basic set for the unipotent ℓ-blocks for groups with a non-trivial centre as described by Chaneb, we cannot extend our result to any finite reductive group of exceptional type. For instance, it is not clear how to treat the case of simply connected groups of type E_6 or E_7.

Combining our result with the theorem of Brunat–Dudas–Taylor for ℓ good ([BDT20, Thm. A]) and the theorem of Chaneb for classical groups when $\ell = 2$ [Cha21, Thm. 2.8], we obtain

Theorem 6.2.16. *Let $\mathbf{G}$ be a simple group of adjoint type defined over k, an algebraically closed field of characteristic p with Frobenius endomorphism F. Assume that p is good for $\mathbf{G}$. Let ℓ be a prime different from p. The decomposition matrix of the unipotent ℓ-blocks of G is unitriangular.*

Remark 6.2.17. Note that all the proofs of this chapter and the preceding one that apply for G of type E_6 also for G of type ${}^2E_6$. Firstly, the number of projective indecomposable modules in $\mathcal{E}_\ell(G,1)$ is independent of whether F is of split or non-split type (see [GH97, 6.6]). Similarly for Proposition 5.1.24, the parametrisation in terms of ℓ-special classes is independent of F since, in the non-split case, the map F acts on W by conjugation, and thus trivially on $\mathrm{irr}(W)$. Therefore, the choice of candidates for the irreducible ordinary modules in $\mathcal{E}_\ell(G,1)$ does not depend on whether F is split or not.
The definitions of the GGGRs and Kawanaka modules hold in both cases. In particular, the admissible covering the admissible coverings fixed by Brunat–Dudas–Taylor for the special classes does not depend on whether F is split. To compute the restriction of the Kawanaka modules to unipotent characters as in Corollary 6.2.3, we use unipotent character sheaves. By [LuCS4, Cor. 20.4], the unipotent character sheaves are F-stable independently of F. Moreover, Consequently, Corollary 6.2.3 also holds if F is non-split. Lastly, the cases covered by Corollary 6.2.8 are also independent of whether F is split or not.

6.3 Unitriangularity of the isolated blocks

In this final section, we explore how one can extend and apply the methods developed in the rest of this thesis to consider non-unipotent isolated ℓ-blocks of simple exceptional groups of adjoint type.
We first gather some general arguments that we then apply to show the unitriangularity of the decomposition matrices of the other isolated ℓ-blocks for $\mathbf{G}$ of type G_2 and F_4. We believe that similar methods will be sufficient to treat the case of the groups of type E_n for $n = 6, 7, 8$ and intend to tackle these cases in the future.

6.3.1 Some general arguments

In this part, we summarise three techniques used to show the unitriangularity of the decomposition matrices when ℓ is bad for the isolated blocks. We fix $t \in G^*$ an isolated non-trivial ℓ'-element.
We follow the strategy explained at the beginning of this chapter.

Step 1 The number n of projective indecomposable $\mathbf{k}[G]$-modules in $\mathscr{B}(G,t)$ may be found in Appendix B.2.

Steps 2 and 3 We determine the unipotent supports $C_1, \dots, C_r$ of the characters in $\mathscr{E}_\ell(G,t)$ with a total ordering $C_1 < \cdots < C_r$, such that $C_i < C_j$ if $\dim C_i \leq \dim C_j$ for all $1 \leq i < j \leq r$.
Then, for each $1 \leq i \leq r$,

- we choose n_i irreducible modules $V_1^i, \dots, V_{n_i}^i \in \mathscr{E}_\ell(G,t)$ with wave front set C_i
- and n_i projective-modules $P_1^i, \dots, P_{n_i}^i$, either ℓ-Kawanaka modules of the form $K_{[a,\Phi]}^G$ for $[a,\Phi] \in \mathcal{M}^\ell(A_{C_i})$, where A_{C_i} is an admissible covering of $\bar{A}_{C_i}$ assuming such an admissible covering exists, or GGGCs $\Gamma_u^{\mathbf{G}}$ for $u \in C_i^F$, or sometimes projective induced from a Levi subgroup.

We require $\sum_{1\leq i\leq r} n_i = n$.

Step 4 Check the unitriangularity of the decomposition matrix of $\mathscr{B}(G,t)$.

When $\mathbf{G}$ is of type G_2 or F_4, we notice that the number n determined in **Step 1** does not depend on ℓ. Moreover, thanks to Proposition 5.1.15, we choose in **Step 2** the ordinary modules in $\mathscr{E}(G,t)$. Therefore our arguments apply to any ℓ, good or bad for $\mathbf{G}$.
For **Step 4**, as we have already discussed, we only need to verify that $([(V_l^i)^*, P_j^i])_{1\leq l,j\leq n_i}$ is lower unitriangular for each $1 \leq i \leq r$ if the chosen projective $\mathbf{k}[G]$-modules are summands of GGGCs. If we can show that $([(V_l^i)^*, P_j^i])_{1\leq l,j\leq n_i}$ is lower triangular (but not necessarily with ones on the diagonal), then the decomposition matrix of the union of blocks $\mathscr{B}(G,t)$ will also be lower triangular. Since $\mathscr{E}(G,t)$ is a basic set (Proposition 5.1.15), it implies that the decomposition matrix is lower unitriangular.

We present different methods to check if $([(V_l^i)^*, P_j^i])_{1\leq l,j\leq n_i}$ is lower triangular. We describe them in an example once and then will just quote the arguments.

General arguments

We fix $1 \leq i \leq r$ and consider $C \coloneqq C_i$ and $n_C \coloneqq n_i$. Most of our arguments use the theory of character sheaves.
For $\mathcal{A} \in \hat{\mathbf{G}}_t$ an F-stable character sheaf, we fix an isomorphism $\varphi_{\mathcal{A}} : F^*\mathcal{A} \xrightarrow{\sim} \mathcal{A}$ satisfying condition † of Subsection 3.3.2. Assume that $\mathcal{A}$ has unipotent support C and comes from the F^*-stable induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$. Let D be an F-stable conjugacy

class of $\mathbf{G}$ with unipotent part equal to C. By Proposition 4.3.6, $\mathcal{A}_D$ is a local system $\mathcal{L}[\dim(D)+\dim(Z^\circ(\mathbf{L}))]$. In particular, thanks to Equation 3.2 and Lemma 3.3.4, for any $h \in D^F$,

$$\chi_{\mathcal{A}}(h) = q^{(\dim(\mathbf{G})-\dim(D)-\dim(Z^\circ(\mathbf{L}))/2)}\chi_{\mathcal{L},\psi}(h).$$

Here $\chi_{\mathcal{L},\psi}$ comes from an isomorphism $\psi : F^*\mathcal{L} \tilde{\to} \mathcal{L}$ of local systems on the class D such that $\psi_h : \mathcal{L}_h \to \mathcal{L}_h$ is of finite order.

***(GGGC)* using GGGCs when $n_C \leq |\mathrm{irr}(A_{\mathbf{G}}(u_C))|$.** There are $|\mathrm{irr}(A_{\mathbf{G}}(u_C))|$ different GGGCs associated to C. Suppose that we can choose n_C of them $(\gamma_1, \dots, \gamma_{n_C})$ and n_C characters sheaves $\mathcal{A}_1, \dots, \mathcal{A}_{n_C} \in \hat{\mathbf{G}}_t$ with unipotent support C, which are F-stable, such that the matrix

$$(\langle \gamma_i^{\mathbf{G}}, D_G(\chi_{\mathcal{A}_j})\rangle)_{1\leq i,j\leq n_C}$$

is lower triangular with diagonal entries of norm 1. Assume furthermore that there is a special $g \in \mathbf{G}^*$ with Jordan decomposition $g = tv$ with v unipotent such that the character sheaves $\mathcal{A}_1, \dots \mathcal{A}_{n_C} \in \hat{\mathbf{G}}_g$ and the number $|\mathcal{M}(\bar{A}_v^{C^\circ_{\mathbf{G}^*}(t)})|$ of F-stable character sheaves in $\hat{\mathbf{G}}_g$ equals n_C. By simple inductive arguments, we will show that there exist characters $\chi_1, \dots, \chi_{n_C} \in \mathrm{irr}(G)_g \subseteq \mathscr{E}(G,t)$ (with unipotent support C) such that the matrix

$$(\langle \gamma_i^{\mathbf{G}}, \chi_j^*\rangle)_{1\leq i,j\leq n_C}$$

is lower unitriangular. We give examples of such arguments in our case analysis for G_2 and F_4, see for instance the isolated blocks A_1A_1 of G_2.

We describe how to compute the scalar product between a character sheaf and a GGGC.
Let $\mathcal{A} \in \hat{\mathbf{G}}_t$ be an F-stable character sheaf as above. In particular, we assume that $\mathcal{A}$ has unipotent support C and comes from the F^*-stable induction datum $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E}) \in \mathfrak{M}^{\mathbf{G}}$. Suppose that $\mathcal{A}_C \neq 0$. Then $\mathcal{A}$ is unipotently supported and $\Sigma = C_0 Z^\circ(\mathbf{L})$ where C_0 is a unipotent conjugacy class of $\mathbf{L}$. We write $\mathcal{E} = \mathcal{E}_0 \boxtimes \mathcal{Z}$ with $\mathcal{E}_0$ an irreducible $\mathbf{L}$-equivariant local system on C_0 and $\mathcal{Z} \in \mathcal{S}(Z^\circ(\mathbf{L}))$. Therefore, by Proposition 3.2.17, there exists an irreducible $\mathbf{K}[W_{\mathfrak{m}}]$-module V such that

$$\mathcal{A} = \mathrm{Hom}_{\mathrm{End}(\mathcal{K}_{\mathfrak{m}})}(V, \mathcal{K}_{\mathfrak{m}}).$$

Thanks to the discussion in Subsection 4.1.1 on the generalised Springer correspondence, and by the same argument as in Proposition 4.3.6, we have

$$\mathcal{A}_C[-\dim Z^\circ(\mathbf{L})] \cong \bigoplus_{\substack{V'\in\mathrm{Irr}(W_{\mathbf{L}}),\\ C_{V'}=C}} \langle \mathrm{Ind}_{W_{\mathfrak{m}}}^{W_{\mathbf{L}}}(V), V'\rangle \mathcal{E}_{V'}[\dim C],$$

where $\mathfrak{Spr}(V') = (C_{V'}, \mathcal{E}_{V'})$ for $V' \in W_{\mathbf{L}}$. The local system $\mathcal{E}_{V'}$ is irreducible and corresponds to an irreducible ordinary character $\theta_{V'}$ of the component group $A_{\mathbf{G}}(u_C)$.

Fix $\psi_{V'} : F^*\mathcal{E}_{V'} \xrightarrow{\sim} \mathcal{E}_{V'}$, an isomorphism of local systems, such that $(\psi_{V'})_u : (\mathcal{E}_{V'})_u \xrightarrow{\sim} (\mathcal{E}_{V'})_u$ is an automorphism of finite order for any $u \in C^F$. Then, for any element $g \in \mathbf{G}$ such that ${}^g u_C \in C^F$, $\chi_{\mathcal{E}_{V'},\psi_{V'}}(g u_C g^{-1})$ is equal to $\theta_{V'}(g^{-1}F(g)C^\circ_{\mathbf{G}}(u_C))$ up to a root of unity, see for instance [DLM97, § 1.4]. Recall that we have fixed $u_C \in C$ such that F acts trivially on $A_{\mathbf{G}}(u_C)$.
Since $\mathcal{A}$ restricted to C is a (shifted) local system, we can describe its characteristic function: for any $u \in C^F$,

$$\chi_{\mathcal{A}}(u) = q^{(\dim(\mathbf{G})-\dim(C)-\dim(Z^\circ(\mathbf{L}))/2)} \sum_{\substack{V'\in\mathrm{Irr}(W_{\mathbf{L}}),\\ C_{V'}=C}} \langle \mathrm{Ind}_{W_{\mathrm{m}}}^{W_{\mathbf{L}}}(V), V'\rangle \chi_{\mathcal{E}_{V'},\psi_{V'}}(u)\zeta'_{V'}$$

where $\zeta'_{V'}$ are roots of unity. Thanks to our previous discussion for each $V' \in \mathrm{irr}_{\mathbf{K}}(W_{\mathbf{L}})$, there is a root of unity ζ'_V such that

$$\chi_{\mathcal{A}}(g u_c g^{-1}) = q^{(\dim(\mathbf{G})-\dim(C)-\dim(Z^\circ(\mathbf{L}))/2)} \sum_{\substack{V'\in\mathrm{Irr}(W_{\mathbf{L}}),\\ C_{V'}=C}} \langle \mathrm{Ind}_{W_{\mathrm{m}}}^{W_{\mathbf{L}}}(V), V'\rangle \zeta_{V'}\theta_{V'}(g^{-1}F(g)C^\circ_{\mathbf{G}}(u_C)),$$

for any $g \in \mathbf{G}$ such that ${}^g u_C \in C^F$.
Now let $u :=^g u_C \in C^F$ for some $g \in \mathbf{G}$. Thanks to [Gec99, 2.3 and 2.4],

$$\langle \gamma_u, D_G(\chi_{\mathcal{A}})\rangle = \sum_{\substack{V'\in\mathrm{Irr}(W_{\mathbf{L}}),\\ C_{V'}=C}} \langle \mathrm{Ind}_{W_{\mathrm{m}}}^{W_{\mathbf{L}}}(V), V'\rangle \overline{\zeta'_{V'}}\theta_{V'}(g^{-1}F(g)C^\circ_{\mathbf{G}}(u_C)), \tag{6.2}$$

where $\zeta_{V'}$ is a root of unity.
To compute the restriction of a character sheaf to its unipotent support, we use CHEVIE [Mic15] and the code in Appendix C.3.

***(Kaw)* using Kawanaka characters when $n_c = |A_{\mathbf{G}}(u_C)|$.** In these cases, we use the same arguments as for the two ℓ-special but not special classes of E_8. We rewrite the conditions to check:

(Kℓ1) there exists an admissible pair (A, λ) for u_C which is an admissible covering of $\bar{A}_C$ such that for all $a \in A$ the ℓ-decomposition matrix of $C_A(a)$ is lower-unitriangular,

(Kℓ2) there is $g = tv = vt \in (\mathbf{G}^*)^{F^*}$ where t is an ℓ-element and $v \in G^*$ is a unipotent special element, such that the characters sheaves in $\hat{\mathbf{G}}_g$ have unipotent support C,

(KD2) $A \cong A_{\mathbf{G}}(u) \cong \bar{A}_v^{C_{\mathbf{G}^*}(t)} \cong \bar{A}_C^{\mathbf{G}}$, and

(KD3) there exists an F-stable character sheaf $\mathcal{A} \in \hat{\mathbf{G}}_g$ such that for all $[b, \phi] \in \mathcal{M}(A)$

$$\langle \mathfrak{f}^G_{[b,\phi]}, D_{\mathbf{G}}(\chi_{\mathcal{A}})\rangle \neq 0 \iff [b, \phi] = [1, 1].$$

***(HC)* using Harish-Chandra induction.** We let $\mathbf{L}$ be a F-stable Levi subgroup of $\mathbf{G}$ sitting in an F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. If R is a projective module of $\mathbf{k}[L]$, with character ψ, then the Harish-Chandra induction $I_{\mathbf{L}}^{\mathbf{G}}(R)$ is a projective $\mathbf{k}[G]$-module (Corollary 2.1.11). Moreover, we can write $I_{\mathbf{L}}^{\mathbf{G}}(R^{\mathbf{O}}) \otimes_{\mathbf{O}} \mathbf{K} \cong I_{\mathbf{L}}^{\mathbf{G}}(R^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K})$, and

$$I_{\mathbf{L}}^{\mathbf{G}}(R^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}) = \sum_{V \in \mathrm{Irr}_{\mathbf{K}}(L)} [R, V] I_{\mathbf{L}}^{\mathbf{G}}(V).$$

Since Harish-Chandra induction commutes with induction from the relative Weyl groups (by the Howlett–Lehrer Comparison Theorem 2.1.14), we obtain

$$I_{\mathbf{L}}^{\mathbf{G}}(R^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}) = \sum_{(\mathbf{M}, V_0)} \sum_{E \in \mathrm{irr}_{\mathbf{K}}(W^L_{\mathbf{M},V_0})} [R, H^{\mathbf{L}}_{\mathbf{M},V_0}(E)] H^{\mathbf{G}}_{\mathbf{M},V_0}(\mathrm{Ind}_{W^L_{\mathbf{M},V_0}}^{W^G_{\mathbf{M},V_0}} E), \tag{6.3}$$

where the first sum runs over the cuspidal pairs of L and the maps $H^{\mathbf{G}}_{\mathbf{M},V_0}$ and $H^{\mathbf{L}}_{\mathbf{M},V_0}$ come from the Howlett–Lehrer Comparison Theorem.

This way, we get more information about the decomposition matrix of G by the knowledge of the decomposition matrix of L and of the relative Weyl groups. In particular, in our case, we will need to compute the projection of $I_{\mathbf{L}}^{\mathbf{G}}(R^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K})$ on the chosen basic set which is included in $\mathscr{E}(G,t)$. Since Harish-Chandra induction preserves Lusztig series (Proposition 2.2.14), we will restrict ourselves to the cuspidal pairs $(\mathbf{M}, V_0)$ such that $V_0 \in \mathcal{E}(M,t)$.

Remark 6.3.1. Note that here the relative Weyl groups depend on the action of F.

6.3.2 The isolated blocks of G_2

In this subsection, we assume that $\mathbf{G}$ is simple, adjoint of type G_2 and that F acts trivially on the Weyl group W.

The isolated blocks A_2

We choose a representative $t = \omega_{\alpha_1}(2/3)$ for the unique conjugacy class of isolated element such that $C_{\mathbf{G}^*}(t)$ is of type A_2. As we have computed in Table B.6, we need to find three different characters of G. So in **Step 1**, we have $n = 3$.
Now for **Step 2**, we determine the unipotent supports of the characters in $\mathscr{E}(G,t)$. They are labelled $A_2, G_2(a_1)$ and G_2. We use CHEVIE [Mic15] and the function UnipSupportG in Appendix C.3. We also observe that there are three characters in $|\mathscr{E}(G,t)|$ each with a different unipotent support. So we set $n_C = 1$ for each unipotent class $A_2, G_2(a_1)$ and G_2. We have $\sum_{1 \le i \le r} n_i = 1 + 1 + 1 = 3 = n$ and the condition is satisfied.
Moreover, we are in the setting of (GGGC). Therefore, in **Step 3**, we choose one generalised Gelfand–Graev character for each unipotent class $A_2, G_2(a_1)$ and G_2. Lastly, we use CHEVIE for **Step 4**. We compute the scalar product of the GGGC associated to the class C with the Alvis–Curtis dual of the characteristic function of the character sheaf in $\hat{\mathbf{G}}_t$ with unipotent support C, applying Equation 6.2. For each class C, the scalar product has norm 1. Here Lusztig's non-linear Fourier transform is trivial. We

can easily deduce that the scalar product of the GGGC associated to the class C with the dual of the character in $\mathscr{E}$ with unipotent support C has norm 1. We are done thus with the last step.

The isolated blocks A_1A_1

We choose a representative $t = \omega_{\alpha_2}(1/2)$ for the unique conjugacy class of isolated elements such that $C_{\mathbf{G}^*}(t)$ is of type A_1A_1. As we have computed in Table B.6, we need to find four different characters of G. We summarise the unipotent support C, the number n_C of characters with the same unipotent support as well as $A_{\mathbf{G}}(u_C)$ for the characters in $\mathcal{E}(G,t)$. We have obtained this data using CHEVIE in the same way we explained for the previous block.

C	$\tilde{A}_1$	$G_2(a_1)$	G_2
n_C	1	2	1
$A_{\mathbf{G}}(u_C)$	1	S_3	1

When C is the class $\tilde{A}_1$ or G_2, we apply (GGGC) exactly in the same manner as for the previous block.

We now describe the case where C is the unipotent class $G_2(a_1)$. We are also in the setting of (GGGC). Using CHEVIE, we compute that one character sheaf $\mathcal{A}_1 \in \hat{\mathbf{G}}_t$ with unipotent support $G_2(a_1)$ has restriction to $G_2(a_1)$ corresponding to the trivial character. The other character sheaf $\mathcal{A}_2 \in \hat{\mathbf{G}}_t$ with unipotent support $G_2(a_1)$ has restriction to $G_2(a_1)$ corresponding to the direct sum of the trivial character and the reflection representation of $S_3 \cong A_{\mathbf{G}}(u_C)$. Applying Equation 6.2, we compute the value of $\langle \chi^*_{\mathcal{A}_i}, \gamma_j \rangle_{\mathbf{K}}$ for $1 \leq i \leq 2$ and $j \in \{1, (12), (123)\}$ a system of representatives of the conjugacy classes of $A_{\mathbf{G}}(u_C)$. This is summarised in the following table

	γ_1	$\gamma_{(12)}$	$\gamma_{(123)}$
$\mathcal{A}_1$	a	a	a
$\mathcal{A}_2$	$b+2c$	b	$b-c$

where a, b, c are some roots of unity. Let $g \in (\mathbf{G}^*)^{F^*}$ be special with semisimple part equal to t and such that $C_g = C$, i.e. the F-stable character sheaves in $\hat{\mathbf{G}}_g$ are $\mathcal{A}_1$ and $\mathcal{A}_2$. Firstly, we observe that

$$\langle \mathrm{pr}_g(\gamma_{(12)}), \mathrm{pr}_g(\gamma_{(12)}) \rangle_{\mathbf{K}} = |a|^2 + |b|^2$$

and hence $\mathrm{pr}_g(\gamma_{(12)}) = \chi_1^* + \chi_2^*$ where χ_1 and χ_2 are the two ordinary irreducible characters in $\mathrm{irr}(G)_g$. Now, $\langle \mathrm{pr}_g(\gamma_{(12)}), \mathrm{pr}_g(\gamma_{(123)}) \rangle_{\mathbf{K}} = 2 - b\bar{c}$ is a non-negative integer, whence $b\bar{c} = \pm 1$. Suppose that $b\bar{c} = -1$. Then on one hand,

$$\langle \mathrm{pr}_g(\gamma_{(12)}), \mathrm{pr}_g(\gamma_{(1)}) \rangle_{\mathbf{K}} = 1 + 1 + 2b\bar{c} = 0$$

and thus $\mathrm{pr}_g(\gamma_{(1)}) = 0$. On the other hand,

$$\langle \mathrm{pr}_g(\gamma_{(1)}), \mathrm{pr}_g(\gamma_{(1)}) \rangle_{\mathbf{K}} = 1 + (b + 2c)\overline{(b + 2c)} = 1 + 1 + 4 + 2b\overline{c} + 2c\overline{b} = 2,$$

which leads to a contradiction. Thus, $b\overline{c} = 1$ and $\mathrm{pr}_g(\gamma_{(123)})$ is an irreducible character. In particular, there are two different GGGCs corresponding to C and two ordinary characters in $\mathscr{E}(G,t)$ with wave front set C such that the decomposition matrix restricted to these rows and columns is lower-unitriangular.

By the same arguments as in the proof of Theorem 6.2.15, we conclude the proof of the unitriangularity of the ℓ-decomposition matrix of the isolated blocks.

Proposition 6.3.2. *Let* $\mathbf{G}$ *be a simple group of type* G_2 *defined over* k*, an algebraically closed field of characteristic* p *with Frobenius endomorphism* F*. Assume that* p *is good for* $\mathbf{G}$ *and* $p \neq \ell$*. The decomposition matrix of the isolated but non-unipotent* ℓ*-blocks of* G *is lower-unitriangular.*

6.3.3 The isolated blocks of F_4

In this subsection, we assume that $\mathbf{G}$ is simple, adjoint of type F_4 and that F acts trivially on W. The numbers of projective indecomposable modules in each block can be found in Table B.7.

The isolated blocks $\tilde{A}_2 A_2$

We choose a representative $t = \omega_{\alpha_1}(2/3)\omega_{\alpha_2}(1/3)$ for the unique conjugacy class of isolated elements in $\mathbf{G}^*$ such that $C_{\mathbf{G}^*}(t)$ is of type $\tilde{A}_2 A_2$. In this case, we need to find 9 projective characters. Using CHEVIE, we summarise the unipotent supports C, the number n_C of characters with the same unipotent support as well as $A_{\mathbf{G}}(u_C)$ for the characters in $\mathcal{E}(G,t)$.

C	$\tilde{A}_2 + A_1$	$F_4(a_3)$	C_3	$F_4(a_2)$	$F_4(a_1)$	B_3	F_4
n_C	1	2	1	1	2	1	1
$A_{\mathbf{G}}(u_C)$	1	S_4	1	S_2	S_2	1	1

In all the cases where $n_C = 1$, the argument (GGGC) applies.
We now consider the case $C = F_4(a_3)$, where we would like to apply (GGGC). The two character sheaves are parameterised by the characters $111, 21$ and $111, 21$ of $W_{C_{\mathbf{G}^*}(t)}$. Using CHEVIE and applying Equation 6.2, we compute the values $\langle \chi^*_{\mathcal{A}_i}, \gamma_j \rangle_{\mathbf{K}}$ for $i \in \{(111,21),(21,111)\}$ and $j \in \{1,(12),(123),(12)(34),(1234)\}$, a system of representatives of the conjugacy classes of $A_{\mathbf{G}}(u_C)$. These data are summarised in the following table:
where a, b, c are some roots of unity. By the same type of arguments as we did for the class $G_2(a_1)$ in G_2 in the isolated blocks $A_1 A_1$, we choose two different GGGCs corresponding to C and two ordinary characters in $\mathscr{E}(G,t)$ with wave-front set C such that

	γ_1	$\gamma_{(12)}$	$\gamma_{(12)(34)}$	$\gamma_{(123)}$	$\gamma_{(1234)}$
$\mathcal{A}_{111,21}$	$a+3b$	$a+b$	$a-b$	a	$a-b$
$\mathcal{A}_{21,111}$	c	c	c	c	c

the decomposition matrix restricted to these rows and columns is unitriangular.

Lastly, we consider the case $C = F_4(a_1)$. The two character sheaves are parameterised by the characters $21,3$ and $3,21$ of $W_{C_{\mathbf{G}^*}(t)}$. Firstly using CHEVIE and applying Equation 6.2, we compute the values $\langle \chi^*_{\mathcal{A}_i}, \gamma_j \rangle_{\mathbf{K}}$ for $i \in \{(21,3),(3,21)\}$ and $j \in \{1,(12)\}$, a system of representatives of the conjugacy classes of $A_{\mathbf{G}}(u_C)$. These data are summarised in the following table:

	γ_1	$\gamma_{(12)}$
$\mathcal{A}_{21,3}$	$a+b$	$a-b$
$\mathcal{A}_{3,21}$	c	c

where a,b,c are some roots of unity. By similar analysis, we deduce that either $a=b$ or $a=-b$. Therefore, the decomposition matrix is lower triangular.

The isolated blocks B_4

We choose a representative $t = \omega_{\alpha_2}(1/2)$ for the unique conjugacy class of isolated element such that $C_{\mathbf{G}^*}(t)$ is of type B_4. In this case, we need to find

$$|\operatorname{irr}(W_{C_{\mathbf{G}^*}(t_1)})| + |\operatorname{irr}(N_{C_{\mathbf{G}^*}}(t_1)(C_{\mathbf{L}^*}(s))/C_{\mathbf{L}^*}(t_1))| = 20+5$$

characters where $\mathbf{L}$ is a Levi subgroup of type B_2. This is a consequence of [GH97, Thm. 6.4] and Proposition 5.1.14. Using CHEVIE, we summarise the unipotent supports C, the number n_C of characters with the same unipotent support as well as $A_{\mathbf{G}}(u_C)$ for the characters in $\mathcal{E}(G,t)$.

C	A_1	A_2	$F_4(a_3)$	$C_3(a_1)$	B_2	$F_4(a_2)$	$F_4(a_1)$	B_3	F_4
n_C	1	4	$4+1$	1	4	4	4	1	1
$A_{\mathbf{G}}(u_C)$	1	S_2	S_4	S_2	S_2	S_2	S_2	1	1

In all the cases where $n_C = 1$, we apply (GGGC).

We now consider the cases where $n_C = 4$, where we would like to apply (Kaw). Using the same notation and the same reasoning as in [BDT20, § 10.2], we determine an admissible covering in each case. For each unipotent class C, a subset $K_C \subseteq \tilde{\Delta}$ such that there exists an involution $a_C \in \mathbf{G}$ such that $\Delta(a_c) = K_C$, see below Remark 1.3.4. We write the index of the roots in K_C, writing 0 for the root $-\alpha_0$. We then fix $A_C = \langle a_C \rangle$. Using the same methods as for the two non-special classes in Subsection 6.2.3, we are able to check all the conditions.

C	A_2	B_2	$F_4(a_2)$	$F_4(a_1)$
K_C	$[0,2,3,4]$	$[0,1,2,3]$	$[0,2,3,4]$	$[0,1,2,3]$

We now consider the case $C = F_4(a_3)$. Using CHEVIE, we compute the values of $\langle \chi^*_{\mathcal{A}_i}, \gamma_j \rangle_{\mathbf{K}}$ for $1 \leq i \leq 2$ and $j \in \{1, (12), (123), (12)(34), (1234)\}$, a system of representatives of the conjugacy classes of $A_{\mathbf{G}}(u_C) \cong S_4$. We gather the results in the following table

	γ_1	$\gamma_{(12)}$	$\gamma_{(12)(34)}$	$\gamma_{(123)}$	$\gamma_{(1234)}$
$\mathcal{A}_{211.}$	0	0	0	0	0
$\mathcal{A}_{2.11}$	$2b+c$	c	$2b+c$	$b-c$	c
$\mathcal{A}_{.31}$	$3a$	a	$-a$	0	$-a$
$\mathcal{A}_{B2:1.1}$	0	0	0	0	0
$\mathcal{A}_{11.2}$	d	d	d	d	d

where a, b, c are some roots of unity. By further analysis, we obtain

	γ_1	$\gamma_{(12)}$	$\gamma_{(12)(34)}$	$\gamma_{(123)}$	$\gamma_{(1234)}$
ρ_1	0	0	2	0	1
ρ_2	0	0	2	0	1
ρ_3	3	1	1	0	0
ρ_4	3	1	1	0	0
$\rho^*_{11.2}$	1	1	1	1	1

where $\{\rho_i \mid 1 \leq i \leq 4\} = \{\rho^*_{211.}, \rho^*_{2.11}, \rho^*_{.31}, \rho^*_{B2:1.1}\}$.
Next, we use the Kawanaka characters and their Fourier transforms. Here the admissible covering A was computed in [BDT20, 10.7]. We write down the values $\langle \chi^*_{\mathcal{A}}, \mathfrak{f}_{[a,\psi]} \rangle_{\mathbf{K}}$ for $[a, \psi] \in \mathcal{M}(A)$ and $\mathcal{A}$ a character sheaf in $\hat{\mathbf{G}}_t$ with unipotent support $F_4(a_3)$ in the family indexed by $g \in \mathbf{G}^*$ with four character sheaves. The computations are made thanks to the code in Appendix C.3. We use CHEVIE notation. If $\mathfrak{f}_{[x,\phi]}$ does not appear in the table, then its projection $\mathrm{pr}_g(\mathfrak{f}_{[x,\phi]})$ is equal to 0. The variables written in small cases have norm one.

	$\mathfrak{f}_{[1,1]}$	$\mathfrak{f}_{[1,\sigma]}$	$\mathfrak{f}_{[1,\lambda^2]}$	$\mathfrak{f}_{[2,1]}$	$\mathfrak{f}_{[2,\epsilon'']}$	$\mathfrak{f}_{[2,\epsilon]}$	$\mathfrak{f}_{[2',1]}$	$\mathfrak{f}_{[2',\epsilon]}$	$\mathfrak{f}_{[2',\epsilon']}$	$\mathfrak{f}_{[2',r]}$	$\mathfrak{f}_{[4,1]}$	$\mathfrak{f}_{[4,-1]}$
$\mathcal{A}_{211.}$	0	0	0	0	0	0	$a_{2'}$	0	$a'_{2'}$	0	a_4	0
$\mathcal{A}_{2.11}$	b_1	b'_1	0	b_2	0	0	$b_{2'}$	0	0	0	0	0
$\mathcal{A}_{.31}$	0	0	c_1	c_2	0	0	0	$c_{2'}$	0	0	0	0
$\mathcal{A}_{B2:1.1}$	0	0	0	0	X_2	Y_2	0	0	0	$X_{2'}$	0	X_4

Applying the Fourier transform, we obtain the following table

	$4\kappa^{\mathbf{G}}_{[4,1]}$	$4\kappa^{\mathbf{G}}_{[4,-1]}$	$4\kappa^{\mathbf{G}}_{[4,I]}$	$4\kappa^{\mathbf{G}}_{[4,-I]}$
$\mathcal{A}_{211.}$	$2a_4$	$-2a_4$	0	0
$\mathcal{A}_{2.11}$	$b_1+b_{2'}$	$b_1+b_{2'}$	$b_1-b_{2'}$	$b_1-b_{2'}$
$\mathcal{A}_{.31}$	$c_1+c_{2'}$	$c_1+c_{2'}$	$c_1-c_{2'}$	$c_1-c_{2'}$
$\mathcal{A}_{B2:1.1}$	$-2X_4$	$2X_4$	0	0

Thanks to the decomposition of $\gamma_{(1234)}$ into Kawanaka characters (Lemma 5.2.26), we deduce that

$$\mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,I]}) = 0 = \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,-I]})$$

and

$$\begin{aligned}
\langle \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,1]}), \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,1]})\rangle &= \frac{1}{16}(4+4+4+4|X_4|^2) \geq 1 \\
\langle \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,-1]}), \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,-1]})\rangle &= \frac{1}{16}(4+4+4+4|X_4|^2) \geq 1 \\
\langle \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,1]}), \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,-1]})\rangle &= \frac{1}{16}(-4+4+4-4|X_4|^2).
\end{aligned}$$

Hence $|X_4| = 1$ and up to renaming, we may assume that $\mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,1]}) = \rho_1$ and $\mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,-1]}) = \rho_2$.

We repeat this process and we get

	$8\kappa^{\mathbf{G}}_{[2',1]}$	$8\kappa^{\mathbf{G}}_{[2',\epsilon]}$	$8\kappa^{\mathbf{G}}_{[2',\epsilon']}$
$\mathcal{A}_{211.}$	$6a_{2'}+2a_4$	$-2a_{2'}+2a_4$	$6a_{2'}-2a_4$
$\mathcal{A}_{2.11}$	$b_1+2b'_1+2b_2+3b_{2'}$	$b_1+2b'_1-2b_2-b_{2'}$	$b_1+2b'_1-2b_2+3b_{2'}$
$\mathcal{A}_{.31}$	$-c_1+2c_2-c_{2'}$	$-c_1-2c_2+3c_{2'}$	$-c_1-2c_2-c_{2'}$
$\mathcal{A}_{B2:1.1}$	$-2X_{2'}+2X_4$	$-2X_{2'}+2X_4$	$-2X_{2'}-2X_4$

	$8\kappa^{\mathbf{G}}_{[2',\epsilon'']}$	$8\kappa^{\mathbf{G}}_{[2',r]}$
$\mathcal{A}_{211.}$	$-2a_{2'}-2a_4$	$-4a_{2'}$
$\mathcal{A}_{2.11}$	$b_1+2b'_1+2b_2-b_{2'}$	$2b_1+4b'_1-2b_{2'}$
$\mathcal{A}_{.31}$	$-c_1+2c_2+3c_{2'}$	$-2c_1-2c_{2'}$
$\mathcal{A}_{B2:1.1}$	$-2X_{2'}-2X_4$	$4X_{2'}$

We observe that $\langle \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[2',r]}), \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[4,1]})\rangle = 0$, whence $|X_{2'}| = 1$ and

$$\langle \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[2',r]}), \mathrm{pr}_g(\kappa^{\mathbf{G}}_{[2',r]})\rangle = 1.$$

We conclude that $\mathrm{pr}_g(\kappa^{\mathbf{G}}_{[2',r]}) \in \{\rho_3, \rho_5\}$. Therefore, up to reordering of the ρ_i, we have found Kawanaka characters and GGGCs (which are all projective as $\ell \neq 2$) such that the decomposition matrix restricted to these rows and columns has the following form:

$$\begin{pmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&1&0&1 \end{pmatrix}.$$

The isolated blocks C_3A_1

We choose a representative $t = \omega_{\alpha_1}(1/2)\omega_{\alpha_3}(1/2)$ for the unique conjugacy class of isolated element such that $C_{\mathbf{G}^*}(t)$ is of type C_3A_1. In this case, we need to find

$$|\operatorname{irr}(W_{C_{\mathbf{G}^*}(t_1)})| + |\operatorname{irr}(N_{C_{\mathbf{G}^*}}(t_1)(C_{\mathbf{L}^*}(s))/C_{\mathbf{L}^*}(t_1))| = 20 + 4$$

characters where $\mathbf{L}$ is a Levi subgroup of type B_2. Using CHEVIE, we summarise the unipotent supports C, the number n_C of characters with the same unipotent support as well as $A_{\mathbf{G}}(u_C)$ for the characters in $\mathcal{E}(G,t)$.

C	$A_1+\tilde{A}_1$	$\tilde{A}_2$	$C_3(a_1)$	$F_4(a_3)$	$F_4(a_2)$	$F_4(a_1)$	B_3	C_3	F_4
n_C	1	1	4	4+1	4+1	4+1	1	1	1
$A_{\mathbf{G}}(u_C)$	1	1	S_2	S_4	S_2	S_2	1	1	1

In all the cases where $n_C = 1$, we apply (GGGC). When $C = C_3(a_1)$, we choose the admissible covering $A_C = \langle a_C \rangle$ such that $\Delta(a_C) = [0,1,2,3]$ and check the conditions of (Kaw).

We consider the other cases individually, starting with $C = F_4(a_3)$. Thanks to CHEVIE [Mic15] and the Equation 6.2, we compute the values $\langle \chi^*_{\mathcal{A}_i}, \gamma_j \rangle_{\mathbf{K}}$ for $1 \le i \le 2$ and $j \in \{1,(12),(123),(12)(34),(1234)\}$, a system of representatives of the conjugacy classes of the finite group $A_{\mathbf{G}}(u_C) \cong S_4$. We summarise the results in the following table

	γ_1	$\gamma_{(12)}$	$\gamma_{(12)(34)}$	$\gamma_{(123)}$	$\gamma_{(1234)}$
$\mathcal{A}_{111.,2}$	$3a_1+3a_2$	$-a_1+a_2$	$-a_1-a_2$	0	a_1+a_2
$\mathcal{A}_{1.11,2}$	$3b_1+2b_2+b_3$	b_1+b_3	$-b_1+2b_2+b_3$	$-b_2+b_3$	$-b_1+b_3$
$\mathcal{A}_{.21,2}$	0	0	0	0	0
$\mathcal{A}_{B2:11,2}$	0	0	0	0	0
$\mathcal{A}_{11.1,11}$	$3c_1+c_2$	c_1+c_2	$-c_1+c_2$	c_1	$-c_1+c_2$

In particular $\langle \chi^*_{\mathcal{A}_{11.1,11}}, \gamma_{(123)} \rangle_{\mathbf{K}} = 1$. Next, we need to use the Kawanaka characters and their Fourier transform. Here the admissible covering A was computed in [BDT20, 10.7]. We write down the values $\langle \chi^*_{\mathcal{A}}, \mathfrak{f}_{[a,\psi]} \rangle_{\mathbf{K}}$ for $[a,\psi] \in \mathcal{M}(A)$ and $\mathcal{A}$ a character sheaf in $\hat{\mathbf{G}}_t$ with unipotent support $F_4(a_3)$ in the family indexed by $g \in \mathbf{G}^*$ with four character sheaves. The computations are made thanks to the code in Appendix C.3. We keep the same conventions as in the previous case.

	$\mathfrak{f}_{[1,1]}$	$\mathfrak{f}_{[1,\sigma]}$	$\mathfrak{f}_{[1,\lambda^2]}$	$\mathfrak{f}_{[1,\lambda]}$	$\mathfrak{f}_{[2,1]}$	$\mathfrak{f}_{[2,\epsilon']}$	$\mathfrak{f}_{[2,\epsilon'']}$	$\mathfrak{f}_{[2,\epsilon]}$	$\mathfrak{f}_{[2',\epsilon]}$	$\mathfrak{f}_{[2',\epsilon']}$	$\mathfrak{f}_{[2',r]}$	$\mathfrak{f}_{[4,-1]}$
$\mathcal{A}_{111.,2}$	0	0	a_1	a_1'	0	a_2	0	0	0	0	0	0
$\mathcal{A}_{1.11,2}$	b_1	b_1'	0	b_1''	b_2	0	0	0	0	0	0	0
$\mathcal{A}_{.21,2}$	0	0	0	0	c_2	0	0	0	$c_{2'}$	$c_{2'}'$	0	0
$\mathcal{A}_{B2:11,2}$	0	0	0	0	0	0	X_2	Y_2	0	0	$X_{2'}$	X_4

By the same kind of analysis as before, we find some Kawanaka characters such that the decomposition matrix is lower-unitriangular.

We now consider the case where $C = F_4(a_2)$. We cannot only take Kawanaka characters because then we would have only four projectives characters. Therefore, we apply (HC) and induce projective characters from a standard Levi $\mathbf{M} \coloneqq \mathbf{L}_{[1,2,3]}$ of $\mathbf{G}$ such that $\mathbf{M}^* \subseteq C_{\mathbf{G}^*}(t)$. We want projective characters P of $\mathbf{k}[M]$ such that for each character $\rho \in \mathcal{E}(\mathbf{G}, t)$, we have $[P : \rho^*] \neq 0$ if and only if ρ has unipotent support equal or bigger to $F_4(a_2)$ (i.e. $F_4(a_2)$, $F_4(a_1)$ or F_4). By [BDT20, Thm. A], we know that the decomposition matrix of the ℓ-unipotent blocks of $\mathbf{M}$ is lower-unitriangular. Since $t \in Z(\mathbf{M}^*)$, we know that it is also the case of $\mathscr{B}_\ell(M, t)$. We parameterise the projective characters in $\mathcal{E}_\ell(M, t)$ with the ordinary characters of $\mathcal{E}(M, t)$. Using CHEVIE [Mic15], we check that the projective modules $P_{111.}, P_{1.11}, P_{.21}$ and $P_{B2:11}$ satisfy the conditions and that the decomposition matrix restricted to $\{\mathrm{Ind}_M^G(P_{111.}), \mathrm{Ind}_M^G(P_{1.11}), \mathrm{Ind}_M^G(P_{.21}), \mathrm{Ind}_M^G(P_{B2:11})\}$ and $\{\rho_{111.2}, \rho_{1.11,2}, \rho_{.21,2}, \rho_{B2:11,2}\}$ (the duals of some characters in $\mathcal{E}(M, t)$ with unipotent support C) is diagonal. Moreover, their restriction to $\{\rho_{11.1,11}\} = \{\rho_{1.2,2}^*\}$ is zero. Lastly, we check that the projection of Γ_1 has non-zero restriction to $\rho_{1.2,2}^*$.

Finally, we consider the case where $C = F_4(a_1)$. This time, we use Kawanaka characters to find four projective characters, applying (Kaw) and the induction of the projective character $P_{.111}$ of M as before for the last projective character.

The isolated blocks $A_3\tilde{A}_1$

We choose a representative $t = \omega_{\alpha_1}(1/2)\omega_{\alpha_3}(1/2)$ for the unique conjugacy class of isolated element such that $C_{\mathbf{G}^*}(t)$ is of type $A_3\tilde{A}_1$. In this case, we need to find 10 characters. Using CHEVIE, we summarise the unipotent supports C, the number n_C of characters with the same unipotent support as well as $A_{\mathbf{G}}(u_C)$ for the characters in $\mathcal{E}(G, t)$.

C	$A_2 + \tilde{A}_1$	$F_4(a_3)$	B_3	$F_4(a_2)$	$F_4(a_1)$	B_2	F_4
n_C	1	1	$1+1$	$1+1$	$1+1$	1	1
$A_{\mathbf{G}}(u_C)$	1	S_4	1	S_2	S_2	S_2	1

When $n_C = 1$, we apply (GGGC). For $C = F_4(a_1)$ and $C = F_4(a_2)$ we use the Kawanaka characters similarly as we did before. For $C = B_3$, we need to use (HC) with $\mathbf{M} = \mathbf{L}_{[1,3,4]} \subseteq \mathbf{G}$ and the projective character $P_{11,111}$ as well as the GGGC corresponding to C.

By the same proof as for the unipotent blocks in Theorem 6.2.15, we conclude this last chapter by the following theorem.

Theorem 6.3.3. *Let* $\mathbf{G}$ *be a simple group of type* F_4 *defined over* k*, an algebraically closed field of characteristic* p *with Frobenius endomorphism* F*. Assume that* p *is good for* $\mathbf{G}$ *and* $p \neq \ell$*. The decomposition matrix of the isolated non-unipotent* ℓ*-blocks of* G *is lower-unitriangular.*

Appendices

Appendix A

Prerequisites on the representation theory of finite groups

We recall a few facts on the representation theory of finite groups. A nice reference is the book of P. Webb [Web16]. The lecture notes of C. Lassueur [Las23] are also very clear and complete.

Let G be a finite group and $\mathbf{\Lambda}$ a commutative ring. We write $\mathbf{\Lambda}[G]$-mod for the category of $\mathbf{\Lambda}[G]$-modules. An important class of $\mathbf{\Lambda}[G]$-modules are the irreducible modules, i.e. the ones who do not have any non-trivial proper submodules. We denote by $\mathrm{Irr}_{\mathbf{\Lambda}}(G)$ the irreducible $\mathbf{\Lambda}[G]$-modules of the group G up to isomorphism.
Two isomorphic $\mathbf{\Lambda}[G]$-modules have the same character: let $\rho : G \to \mathrm{GL}(V)$ be given by the action of G on V, the character of V is the class function

$$\begin{aligned} \chi_V : G &\to \mathbb{F} \\ g &\mapsto \mathrm{Tr}(\rho(g), V). \end{aligned}$$

To denote the ordinary irreducible characters of the group G, we use $\mathrm{irr}_{\mathbb{F}}(G)$. When the underlying field is clear, we might sometimes drop the subscript $\mathbb{F}$.
Assume that $\Lambda = \mathbb{F}$ is a field of characteristic zero, then the isomorphism class of an $\mathbb{F}[G]$-module V is completely determined by its character, [Web16, Cor. 3.3.3]. If ϕ is a character of G, we write V_ϕ for an $\mathbb{F}[G]$-module with character ϕ.
We define the scalar product of two $\mathbb{F}[G]$-modules V and W:

$$\langle V, W \rangle_{\mathbb{F}} \coloneqq \dim_{\mathbb{F}}(\mathrm{Hom}_{\mathbb{F}[G]}(V, W)).$$

When $\mathbb{F} \subseteq \mathbb{C}$, and χ, ψ are two characters of G, we define a scalar product of characters:

$$\langle \chi, \psi \rangle \coloneqq \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)} = \langle V_\chi, V_\psi \rangle.$$

We now move on to the modular representation theory. We fix a splitting ℓ-modular system $(\mathbf{O}, \mathbf{K}, \mathbf{k})$ where

- $\mathbf{O}$ is a complete discrete valuation ring of characteristic 0 with a unique maximal ideal M,
- $\mathbf{K}$ is the field of fractions of $\mathbf{O}$, also of characteristic zero. We assume that $\mathbf{K}$ is big enough for the group G we are considering, that is it contains all the $|G|$th roots of unity. In particular, with respect to the inclusion $\mathbf{K} \subseteq \mathbb{C}$, we have $\mathrm{irr}_{\mathbf{K}}(G) = \mathrm{irr}_{\mathbb{C}}(G)$.
- $\mathbf{k} = \mathbf{O}/M$ is a field of characteristic ℓ. We assume $\mathbf{k} = \overline{\mathbb{F}_\ell}$.

If $W \in \mathrm{Irr}_{\mathbf{k}}(G)$, we set P_W for its projective cover ([Las23, Def. 23.3]). It is a projective indecomposable module. Recall that to any projective $\mathbf{k}[G]$-module P corresponds a projective $\mathbf{O}[G]$-module $P^{\mathbf{O}}$ such that $P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{k} \cong P$ ([Las23, Cor. 32.6]), and the $\mathbf{K}[G]$-module $P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}$ is unique up to isomorphism.
On the other hand, to any $\mathbf{K}[G]$-module V corresponds at least one $\mathbf{O}[G]$-module $V_{\mathbf{O}}$, free over $\mathbf{O}$ such that $V_{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K} \cong V$. Then $V_{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{k}$ is an $\mathbf{k}[G]$-module ([Las23, Prop. 14.6]). For any $\mathbf{K}[G]$-module V and any projective $\mathbf{k}[G]$-module P, we have by Brauer reciprocity ([Las23, Thm. 34.2]),

$$\langle P, V_{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{k} \rangle_{\mathbf{k}} = \langle P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}, V \rangle_{\mathbf{K}} =: [P, V].$$

We denote the decomposition matrix of G by $D^G = (d^G_{V,W})_{V \in \mathrm{irr}_{\mathbf{K}}(G), W \in \mathrm{irr}_{\mathbf{k}}(G)}$ with entries

$$d^G_{V,W} := [P_W, V].$$

For a projective $\mathbf{k}[G]$-module P, let Ψ_P denote the character associated to the $\mathbf{K}[G]$-module $P^{\mathbf{O}} \otimes_{\mathbf{O}} \mathbf{K}$. We say that Ψ_P is a projective character. We may sometimes write

$$d_{\chi_V, \Psi_{P_W}} = [P_W, V] = \langle \Psi_{P_W}, \chi_V \rangle$$

for $V \in \mathrm{Irr}_{\mathbf{K}}(G)$ and $W \in \mathrm{Irr}_{\mathbf{k}}(G)$. For $W \in \mathrm{irr}_{\mathbf{k}}(G)$, we write ψ_W for its Brauer character. This class function on the ℓ'-elements of G is defined as follows. If $g \in G$ is a ℓ'-element (that is the prime ℓ does not divide the order of g), then the trace of the action of g on W is a sum of roots of unity of $\mathbf{k}$. The lift of this sum to $\mathbf{O} \subseteq \mathbf{K}$ gives then the value of $\psi_W(g)$. We may sometimes extend this function by zero to all of G. All these facts and more can be found in [Web16, Chapt. 10].

The group algebra $\mathbf{k}[G]$ is partitioned into ℓ-blocks

$$\mathbf{k}[G] = \mathscr{B}_1 \oplus \cdots \oplus \mathscr{B}_n,$$

which corresponds to a set of central orthogonal primitive idempotents $\{e_1, \ldots, e_n\}$ with $\mathscr{B}_i = \mathbf{k}[G]e_i$. An indecomposable module $W \in \mathrm{Irr}_{\mathbf{k}}(G)$ belongs to a block $\mathscr{B}_i$ if $e_i W = W$, and we write $W \in \mathrm{Irr}_{\mathbf{k}}(\mathscr{B})$. This block is unique. It leads to the block-diagonal shape of the ℓ-decomposition matrix. Two ordinary irreducible modules V and V' of G belong to the same ℓ-block $\mathscr{B}$ if there exist $W_1, \ldots, W_{n-1} \in \mathrm{Irr}_{\mathbf{k}}(\mathscr{B})$ and $V = V_1, \ldots, V_n = V' \in \mathrm{Irr}_{\mathbf{K}}(G)$ such that

$$\langle V_i, P_{W_i} \otimes_{\mathbf{O}} \mathbf{K} \rangle \neq 0 \quad \text{and} \quad \langle V_{i+1}, P_{W_i} \otimes_{\mathbf{O}} \mathbf{K} \rangle \neq 0 \text{ for } 1 \leq i \leq n-1.$$

We write $V \in \mathrm{Irr}_{\mathbf{K}}(\mathscr{B})$. We refer the reader to [Web16, Section 12.1] for all these facts.

Appendix B

Tables

B.1 Induction data of exceptional adjoint groups

In the following tables, we collect some information about the various cuspidal induction data $\mathfrak{m} = (\mathbf{L}, \Sigma, \mathcal{E})$ for $\mathbf{G}$ a simple exceptional group of adjoint type. This information comes from the summary in [DLM14, App. A] and [AA10, Table 1], which themselves come from [LuCS4] and [LuCS5]. We only consider the cases where $\mathbf{L}$ is not a maximal torus. In the first column, we describe the series $\hat{\mathbf{G}}_t$ such that $\mathcal{E} \in \hat{\mathbf{L}}_t$. Let $s \in \mathbf{L}$ such that s belongs to the semisimple part of Σ. In the third column, we write $C_{\mathbf{L}}(s)$. In the fourth column, we describe a representative of s as follows. We choose $\mathbf{L}$ such that it is a standard Levi subgroup and $s \in \mathbf{G}$ is written using the "additive" notation of CHEVIE [Mic15]. The following column describes, if known, the unipotent part of Σ as a unipotent conjugacy class in $C_{\mathbf{L}}(s)$ using CHEVIE notation. In the last column, we give the number n of cuspidal character sheaves on $\mathbf{L}$ with unipotent support Σ belonging to the series $\hat{\mathbf{L}}_t$.

G_2

All the cuspidal character sheaves of $\mathbf{G}$ of type G_2 are unipotent. Moreover, if $\mathbf{L}$ is a proper Levi subgroup which is not a torus, then $\mathbf{L}$ is adjoint of type A_1 and does not have any cuspidal character sheaf.

$C_{\mathbf{G}^*}(t)$	$\mathbf{L}$	$C_{\mathbf{L}}(s)$	s	unipotent class in $C_{\mathbf{L}}(s)$	n
		G_2	1	$G_2(a_1)$	1
G_2	G_2	A_2	$\langle 0, 1/3 \rangle$	3	2
		A_1A_1	$\langle 1/2, 0 \rangle$	$(2, 2)$	1

Table B.1: Induction data of G_2

F_4

All the cuspidal character sheaves of $\mathbf{G}$ of type F_4 are unipotent. The only Levi subgroup $\mathbf{L}$ with cuspidal character sheaves is of type B_2. For $\mathbf{L}_{\text{ad}}$ there is one unipotent cuspidal character sheaf and another one in the series indexed by a central element of $(\mathbf{L}_{\text{ad}})^*$. Thus, $\mathbf{L}$ has cuspidal character sheaves which belong to the unipotent and the central series of $\mathbf{L}$ tensored by a local system pulled back from a Kummer local system on the abelianisation $\mathbf{L}/[\mathbf{L},\mathbf{L}]$, c.f. [DLM14, facts p. 493] or [LuCS4, 17.9, 17.10]. In particular, there is one cuspidal character sheaf in the unipotent series, one coming from the central series of $\mathbf{L}/Z^\circ(\mathbf{L})$ and one indexed by another central element of $\mathbf{L}^*$.

$C_{\mathbf{G}^*}(t)$	$\mathbf{L}$	$C_{\mathbf{L}}(s)$	s	unipotent class in $C_{\mathbf{L}}(s)$	n
F_4	F_4	F_4	1	$F_4(a_3)$	1
		C_3A_1	$\langle 1/2,0,0,0\rangle$	$(2,4)\times 2$	1
		B_4	$\langle 0,0,0,1/2\rangle$	$(1,3,5)$	1
		A_2A_2	$\langle 0,1/3,0,0\rangle$	reg	2
		A_3A_1	$\langle 0,0,1/4,0\rangle$	reg	2
	B_2	A_1A_1	$\langle 1/2,0,1/2,1/2\rangle$		1
B_4	B_2	A_1A_1	$\langle 1/2,0,1/2,1/2\rangle$		1
C_3A_1	B_2	A_1A_1	$\langle 1/2,0,1/2,1/2\rangle$		1

Table B.2: Induction data of F_4

E_6

There are two cuspidal character sheaves per central series when $\mathbf{G}$ is of type E_6; they all have the same support. The only Levi subgroup $\mathbf{L}$ with cuspidal character sheaves is of type D_4. For $\mathbf{L}_{\text{ad}}$ there is one cuspidal character sheaf per central series of $\mathbf{L}_{\text{ad}}$. There is no cuspidal character sheaf of $\mathbf{L}$ which belongs to an isolated non-unipotent series of $\mathbf{G}$.

$C_{\mathbf{G}^*}(t)$	$\mathbf{L}$	$C_{\mathbf{L}}(s)$	s	unipotent class in $C_{\mathbf{L}}(s)$	n
E_6	E_6	$A_2A_2A_2$	$\langle 0,0,0,1/3,0,0\rangle$	reg	2
	D_4	$A_1A_1A_1A_1$	$\langle 1/2,0,0,1/2,0,1/2\rangle$		1

Table B.3: Induction data of E_6

E_7

There are two cuspidal character sheaves per central series when $\mathbf{G}$ is of type E_7; they all have the same support. The Levi subgroups with cuspidal character sheaves are of type D_4 and E_6. For $\mathbf{L}$ of type E_6 no isolated non-unipotent series contains character sheaves coming from a cuspidal character sheaf of $\mathbf{L}$. The isolated series D_6A_1 is the only one containing character sheaves coming from a cuspidal induction datum with $\mathbf{L}$ of type D_4.

$C_{\mathbf{G}^*}(t)$	$\mathbf{L}$	$C_{\mathbf{L}}(s)$	s	unipotent class in $C_{\mathbf{L}}(s)$	n
E_7	E_7	$A_3A_3A_1$	$\langle 0,0,0,1/4,0,0,0\rangle$	reg	2
	E_6	$A_2A_2A_2$	$\langle 0,0,0,1/3,0,0,1/3\rangle$	reg	2
	D_4	$A_1A_1A_1A_1$	$\langle 1/2,0,0,1/2,0,1/2,0\rangle$		1
D_6A_1	D_4	$A_1A_1A_1A_1$	$\langle 1/2,0,0,1/2,0,1/2,0\rangle$		1

Table B.4: Induction data of E_7

E_8

All the cuspidal character sheaves of $\mathbf{G}$ of type E_8 are unipotent. The Levi subgroups with cuspidal character sheaves are of type D_4, E_6, and E_7. If $\mathbf{L}$ is of type E_7 then $\mathbf{L}_{\text{ad}}$ has cuspidal character sheaves in the central series for E_7, in particular in the series E_7A_1.

$C_{\mathbf{G}^*}(t)$	$\mathbf{L}$	$C_{\mathbf{L}}(s)$	s	unipotent class in $C_{\mathbf{L}}(s)$	n
E_8	E_8	E_8	1	$F_4(a_3)$	1
		A_1E_7	$\langle 0,0,0,0,0,0,0,1/2\rangle$	reg $\times$ $(A_1+D_6(a_2))$	1
		A_2E_6	$\langle 0,0,0,0,0,0,1/3,0\rangle$	reg $\times$ (A_1+A_5)	2
		D_5A_3	$\langle 0,0,0,0,0,1/4,0,0\rangle$	$(3,7)\times$ reg	2
		A_4A_4	$\langle 0,0,0,0,1/5,0,0,0\rangle$	reg	4
		$A_1A_2A_5$	$\langle 0,0,0,1/6,0,0,0,0\rangle$	reg	2
		D_8	$\langle 1/2,0,0,0,0,0,0,0\rangle$	$(1,3,5,7)$	2
	E_7	$A_3A_3A_1$	$\langle 0,0,0,1/4,0,0,0,1/4\rangle$	reg	2
	E_6	$A_2A_2A_2$	$\langle 0,0,0,1/3,0,0,1/3,0\rangle$	reg	2
	D_4	$A_1A_1A_1A_1$	$\langle 1/2,0,0,1/2,0,1/2,0,0\rangle$		1
E_7A_1	E_7	$A_3A_3A_1$	$\langle 0,0,0,1/4,0,0,0,1/4\rangle$	reg	2
	E_6	$A_2A_2A_2$	$\langle 0,0,0,1/3,0,0,1/3,0\rangle$	reg	2
	D_4	$A_1A_1A_1A_1$	$\langle 1/2,0,0,1/2,0,1/2,0,0\rangle$		1
E_6A_2	E_6	$A_2A_2A_2$	$\langle 0,0,0,1/3,0,0,1/3,0\rangle$	reg	2
	D_4	$A_1A_1A_1A_1$	$\langle 1/2,0,0,1/2,0,1/2,0,0\rangle$		1
D_5A_3	D_4	$A_1A_1A_1A_1$	$\langle 1/2,0,0,1/2,0,1/2,0,0\rangle$		1
D_8	D_4	$A_1A_1A_1A_1$	$\langle 1/2,0,0,1/2,0,1/2,0,0\rangle$		1

Table B.5: Induction data of E_8

B.2 On the number of Brauer characters

We collect the number of Brauer characters in each isolated union of blocks $\mathscr{B}(G,t)$ for $\mathbf{G}$ simple exceptional of adjoint type, F acting trivially on W and $t \in \mathbf{G}^*$ isolated. The rows of the tables are indexed by the type of $C_{\mathbf{G}^*}(t)$. If t is not an ℓ'-element, we write nothing in the corresponding cell. This information has been obtained using CHEVIE [Mic15] and the code in Appendix C.2. Note that it depends on the action of the Frobenius map on $C_{\mathbf{G}^*}(t)$.

	ℓ good	$\ell = 2$	$\ell = 3$
G_2	10	9	8
$\tilde{A}_2$	3	3	–
$\tilde{A}_1 A_1$	4	–	4

Table B.6: Number of Brauer characters in the isolated blocks for $G_2(q)$

	ℓ good	$\ell = 2$	$\ell = 3$
F_4	37	28	35
B_4	25	–	25
$C_3 A_1$	24	–	24
$\tilde{A}_2 A_2$	9	9	–
${}^2\tilde{A}_2 {}^2A_2$	9	9	–
$A_3 \tilde{A}_1$	10	–	10
${}^2A_3 \tilde{A}_1$	10	–	10

Table B.7: Number of Brauer characters in the isolated blocks for $F_4(q)$

	ℓ good	$\ell = 2$	$\ell = 3$
E_6	30	27	28
$A_5 A_1$	22	–	22
$A_2 A_2 A_2$	27	27	–
$A_2(q^3)$	3	3	–
${}^2A_2 A_2(q^2)$	9	9	–

Table B.8: Number of Brauer characters in the isolated blocks for $E_6(q)$

	ℓ good	$\ell = 2$	$\ell = 3$
E_7	76	64	72
A_7	22	–	22
${}^2A_7$	22	–	22
A_5A_2	33	33	–
${}^2A_5{}^2A_2$	33	33	–
$A_3A_1A_3$	50	–	50
${}^2A_3A_1{}^2A_3$	50	–	50
$A_3(q^2)A_1$	10	–	10
D_6A_1	84	–	84

Table B.9: Number of Brauer characters in the isolated blocks for $E_7(q)$

	ℓ good	$\ell = 2$	$\ell = 3$	$\ell = 5$
E_8	166	131	150	162
E_7A_1	152	–	144	152
E_6A_2	90	81	–	90
${}^2E_6{}^2A_2$	90	81	–	90
D_5A_3	100	–	100	100
${}^2D_5{}^2A_3$	100	–	100	100
A_4A_4	49	49	49	–
${}^2A_4{}^2A_4$	49	49	49	–
${}^2A_4(q^2)$	7	7	7	–
$A_2A_1A_5$	66	–	–	66
${}^2A_2A_1{}^2A_5$	66	–	–	66
A_1A_7	44	–	44	44
$A_1{}^2A_7$	44	–	44	44
A_8	30	30	–	30
${}^2A_8$	30	30	–	30
D_8	120	–	120	120

Table B.10: Number of Brauer characters in the isolated blocks for $E_8(q)$

B.3 The ℓ-special classes for simple exceptional groups of adjoint type

The following tables collect the ℓ-special classes for simple exceptional groups of adjoint type. The first column contains the name of the unipotent class C, the second column the group $A_{\mathbf{G}}(u_C)$, the third the ordinary canonical quotient if the class is special. We then compute the ℓ-canonical quotient for the ℓ-special classes for each bad prime ℓ. In the last columns, one can read if the class is ℓ-P-special or not, see Definition 6.2.5. For ℓ good for $\mathbf{G}$, it is a consequence of [GH91, Thm. 5.1], see Theorem 5.1.7. When ℓ is bad for $\mathbf{G}$, this information when ℓ is bad has been obtained thanks to Proposition 5.1.14

and the discussion below.

C	$A_G(u_C)$	$\bar{A}_C$	$\bar{A}_{2,C}$	$\bar{A}_{3,C}$	2-P-special	3-P-special
1	1	1	1	1	*true*	*true*
$G_2(a_1)$	S_3	S_3	S_3	S_3	*true*	*true*
G_2	1	1	1	1	*true*	*true*
$\tilde{A}_1$	1	–	1	–	*true*	–
A_1	1	–	–	1	–	*true*

Table B.11: The ℓ-special classes of G_2

C	$A_G(u_C)$	$\bar{A}_C$	$\bar{A}_{2,C}$	$\bar{A}_{3,C}$	2-P-special	3-P-special
1	1	1	1	1	*true*	*true*
$\tilde{A}_1$	S_2	S_2	S_2	S_2	*true*	*true*
$A_1 + \tilde{A}_1$	1	1	1	1	*true*	*true*
$\tilde{A}_2$	1	1	1	1	*true*	*true*
A_2	S_2	1	S_2	1	*true*	*false*
$F_4(a_3)$	S_4	S_4	S_4	S_4	*true*	*true*
C_3	1	1	1	1	*true*	*true*
B_3	1	1	1	1	*true*	*true*
$F_4(a_2)$	S_2	1	S_2	1	*true*	*false*
$F_4(a_1)$	S_2	S_2	S_2	S_2	*true*	*true*
F_4	1	1	1	1	*true*	*true*
A_1	1	–	1	–	*true*	–
$A_2 + \tilde{A}_1$	1	–	1	–	*true*	–
B_2	S_2	–	S_2	–	*true*	–
$C_3(a_1)$	S_2	–	S_2	–	*true*	–
$\tilde{A}_2 + A_1$	1	–	–	1	–	*true*

Table B.12: The ℓ-special classes of F_4

C	$A_G(u_C)$	$\bar{A}_C$	$\bar{A}_{2,C}$	$\bar{A}_{3,C}$	2-P-special	3-P-special
E_6	1	1	1	1	*true*	*true*
$E_6(a_1)$	1	1	1	1	*true*	*true*
D_5	1	1	1	1	*true*	*true*
$E_6(a_3)$	S_2	S_2	S_2	S_2	*true*	*true*
$D_5(a_1)$	1	1	1	1	*true*	*true*
$A_4 + A_1$	1	1	1	1	*true*	*true*
D_4	1	1	1	1	*true*	*true*
A_4	1	1	1	1	*true*	*true*
$D_4(a_1)$	S_3	S_3	S_3	S_3	*true*	*true*
A_3	1	1	1	1	*true*	*true*
$2A_2$	1	1	1	1	*true*	*true*
$A_2 + 2A_1$	1	1	1	1	*true*	*true*
$A_2 + A_1$	1	1	1	1	*true*	*true*
A_2	S_2	S_2	S_2	S_2	*true*	*true*
$2A_1$	1	1	1	1	*true*	*true*
A_1	1	1	1	1	*true*	*true*
1	1	1	1	1	*true*	*true*
A_5	1	–	1	–	*true*	–
$A_3 + A_1$	1	–	1	–	*true*	–
$3A_1$	1	–	1	–	*true*	–
$2A_2 + A_1$	1	–	–	1	–	*true*

Table B.13: The ℓ-special classes of E_6

C	$A_G(u_C)$	$\bar{A}_C$	$\bar{A}_{2,C}$	$\bar{A}_{3,C}$	2-P-special	3-P-special
D_6	1	–	1	–	*true*	–
$D_6(a_2)$	1	–	1	–	*true*	–
A_5'	1	–	1	–	*true*	–
$D_4 + A_1$	1	–	1	–	*true*	–
$A_3 + 2A_1$	1	–	1	–	*true*	–
$(A_3 + A_1)'$	1	–	1	–	*true*	–
$4A_1$	1	–	1	–	*true*	–
$3A_1'$	1	–	1	–	*true*	–
$A_5 + A_1$	1	–	–	1	–	*true*
$2A_2 + A_1$	1	–	–	1	–	*true*

Table B.14: The ℓ-special but not special classes of E_7

C	$A_G(u_C)$	$\bar{A}_C$	$\bar{A}_{2,C}$	$\bar{A}_{3,C}$	2-P-special	3-P-special
E_7	1	1	1	1	*true*	*true*
$E_7(a_1)$	1	1	1	1	*true*	*true*
$E_7(a_2)$	1	1	1	1	*true*	*true*
E_6	1	1	1	1	*true*	*true*
$E_7(a_3)$	S_2	S_2	S_2	S_2	*true*	*true*
$E_6(a_1)$	S_2	S_2	S_2	S_2	*true*	*true*
$E_7(a_4)$	S_2	1	S_2	1	*true*	*false*
A_6	1	1	1	1	*true*	*true*
$D_5 + A_1$	1	1	1	1	*true*	*true*
$D_6(a_1)$	1	1	1	1	*true*	*true*
$E_7(a_5)$	S_3	S_3	S_3	S_3	*true*	*true*
D_5	1	1	1	1	*true*	*true*
$E_6(a_3)$	S_2	S_2	S_2	S_2	*true*	*true*
$D_5(a_1) + A_1$	1	1	1	1	*true*	*true*
$A_4 + A_2$	1	1	1	1	*true*	*true*
A_5''	1	1	1	1	*true*	*true*
$D_5(a_1)$	S_2	S_2	S_2	S_2	*true*	*true*
$A_4 + A_1$	S_2	S_2	S_2	S_2	*true*	*true*
D_4	1	1	1	1	*true*	*true*
$A_3 + A_2 + A_1$	1	1	1	1	*true*	*true*
A_4	S_2	S_2	S_2	S_2	*true*	*true*
$A_3 + A_2$	S_2	1	S_2	1	*true*	*false*
$D_4(a_1) + A_1$	S_2	S_2	S_2	S_2	*true*	*true*
$D_4(a_1)$	S_3	S_3	S_3	S_3	*true*	*true*
$(A_3 + A_1)''$	1	1	1	1	*true*	*true*
$2A_2$	1	1	1	1	*true*	*true*
A_3	1	1	1	1	*true*	*true*
$A_2 + 3A_1$	1	1	1	1	*true*	*true*
$A_2 + 2A_1$	1	1	1	1	*true*	*true*
$A_2 + A_1$	S_2	S_2	S_2	S_2	*true*	*true*
A_2	S_2	S_2	S_2	S_2	*true*	*true*
$3A_1''$	1	1	1	1	*true*	*true*
$2A_1$	1	1	1	1	*true*	*true*
A_1	1	1	1	1	*true*	*true*
1	1	1	1	1	*true*	*true*

Table B.15: The special classes of E_7

C	$A_G(u_C)$	$\bar{A}_C$	$\bar{A}_{2,C}$	$\bar{A}_{3,C}$	$\bar{A}_{5,C}$	2-P-special	3-P-special	5-P-special
E_7	1	–	1	–	–	*true*	–	–
D_7	1	–	1	–	–	*true*	–	–
$E_7(a_2)$	1	–	1	–	–	*true*	–	–
D_6	1	–	1	–	–	*true*	–	–
A_7	1	–	1	–	–	*true*	–	–
$D_5 + A_1$	1	–	1	–	–	*true*	–	–
$E_7(a_5)$	S_3	–	S_3	–	–	*true*	–	–
$D_6(a_2)$	S_2	–	S_2	–	–	*true*	–	–
$D_5(a_1) + A_2$	1	–	1	–	–	*true*	–	–
A_5	1	–	1	–	–	*true*	–	–
$D_4 + A_1$	1	–	1	–	–	*true*	–	–
$2A_3$	1	–	1	–	–	*true*	–	–
$A_3 + A_2 + A_1$	1	–	1	–	–	*true*	–	–
$A_3 + 2A_1$	1	–	1	–	–	*true*	–	–
$A_3 + A_1$	1	–	1	–	–	*true*	–	–
$A_2 + 3A_1$	1	–	1	–	–	*true*	–	–
$4A_1$	1	–	1	–	–	*true*	–	–
$3A_1$	1	–	1	–	–	*true*	–	–
$E_6 + A_1$	1	–	–	1	–	–	*true*	–
$E_6(a_3) + A_1$	S_2	–	–	S_2	–	–	*true*	–
$2A_2 + 2A_1$	1	–	–	1	–	–	*true*	–
$2A_2 + A_1$	1	–	–	1	–	–	*true*	–
$A_4 + A_3$	1	–	–	–	1	–	–	*true*

Table B.16: The ℓ-special but not special classes of E_8

C	$A_G(u_C)$	$\bar{A}_C$	$\bar{A}_{2,C}$	$\bar{A}_{3,C}$	$\bar{A}_{5,C}$	2-P-special	3-P-special	5-P-special
E_8	1	1	1	1	1	*true*	*true*	*true*
$E_8(a_1)$	1	1	1	1	1	*true*	*true*	*true*
$E_8(a_2)$	1	1	1	1	1	*true*	*true*	*true*
$E_8(a_3)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
$E_8(a_4)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
$E_8(b_4)$	S_2	1	S_2	1	1	*true*	*false*	*false*
$E_7(a_1)$	1	1	1	1	1	*true*	*true*	*true*
$E_8(a_5)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
$E_8(b_5)$	S_3	S_3	S_3	S_3	S_3	*true*	*true*	*true*
$E_8(a_6)$	S_3	S_3	S_3	S_3	S_3	*true*	*true*	*true*
$D_7(a_1)$	S_2	1	S_2	1	1	*true*	*false*	*false*
E_6	1	1	1	1	1	*true*	*true*	*true*
$E_7(a_3)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
$E_8(b_6)$	S_3	S_2	S_2	S_3	S_2	*false*	*true*	*false*
$E_6(a_1)+A_1$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
$D_7(a_2)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
$E_6(a_1)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
D_5+A_2	S_2	1	S_2	1	1	*true*	*false*	*false*
$E_7(a_4)$	S_2	1	S_2	1	1	*true*	*false*	*false*
A_6+A_1	1	1	1	1	1	*true*	*true*	*true*
$D_6(a_1)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
A_6	1	1	1	1	1	*true*	*true*	*true*
$E_8(a_7)$	S_5	S_5	S_5	S_5	S_5	*true*	*true*	*true*
D_5	1	1	1	1	1	*true*	*true*	*true*
$E_6(a_3)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
D_4+A_2	S_2	1	S_2	1	1	*true*	*false*	*false*
$A_4+A_2+A_1$	1	1	1	1	1	*true*	*true*	*true*
$D_5(a_1)+A_1$	1	1	1	1	1	*true*	*true*	*true*
$D_5(a_1)$	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
A_4+A_2	1	1	1	1	1	*true*	*true*	*true*
A_4+2A_1	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
A_4+A_1	S_2	S_2	S_2	S_2	S_2	*true*	*true*	*true*
D_4	1	1	1	1	1	*true*	*true*	*true*

Table B.17: The special classes of E_8

C	$A_G(u)$	$\bar{A}_C$	$\bar{A}_{2,C}$	$\bar{A}_{3,C}$	$\bar{A}_{5,C}$	2-P-special	3-P-special	5-P-special
A_4	S_2	S_2	S_2	S_2	S_2	$true$	$true$	$true$
$D_4(a_1) + A_2$	S_2	S_2	S_2	S_2	S_2	$true$	$true$	$true$
$A_3 + A_2$	S_2	1	S_2	1	1	$true$	$false$	$false$
$D_4(a_1) + A_1$	S_3	S_3	S_3	S_3	S_3	$true$	$true$	$true$
$D_4(a_1)$	S_3	S_3	S_3	S_3	S_3	$true$	$true$	$true$
A_3	1	1	1	1	1	$true$	$true$	$true$
$2A_2$	S_2	S_2	S_2	S_2	S_2	$true$	$true$	$true$
$A_2 + 2A_1$	1	1	1	1	1	$true$	$true$	$true$
$A_2 + A_1$	S_2	S_2	S_2	S_2	S_2	$true$	$true$	$true$
A_2	S_2	S_2	S_2	S_2	S_2	$true$	$true$	$true$
$2A_1$	1	1	1	1	1	$true$	$true$	$true$
A_1	1	1	1	1	1	$true$	$true$	$true$
1	1	1	1	1	1	$true$	$true$	$true$

Table B.18: The special classes of E_8 continued

Appendix C

Code

C.1 Induction data of exceptional adjoint groups

This is the code for the proof of Lemma 3.2.22.

```
####################################################################
# Check if N_G(L)/L = N_{C^o_\G(s)}(C^o_L(s))/C^o_L(s).

Same_relative_group := function(G,L,s)
local Gs, Ls, WL, WGsLs;
Gs := Centralizer(G,s).group;
Ls := Centralizer(L,s).group;
WL := Normalizer(G,L)/L;
WGsLs := Normalizer(Gs,Ls)/Ls;
return Size(WL)= Size(WGsLs);
end;

####################################################################
# For s isolated in L, find z in Z^o(L) such that
# N_G(L)/L = N_{C^o_\G(sz)}(C^o_L(sz))/C^o_L(sz) and sz isolated in G
# Return sz.

Find_sz_with_same_relative_group_and_isolated := function(G,L,s)
local hyp, sz, y,c, possible_sz;
hyp := Same_relative_group(G,L,s) and IsIsolated(G,s);
sz := "still looking for sz";
if hyp = true then
  sz := s;
else
# we try to find the correct z by multiplying s by some elements in the centre
  for y in AlgebraicCentre(L).Z0.generators do
    c := 1;
    while hyp = false and c <100 do
      possible_sz := s*SemisimpleElement(G,(1/c)*y);
      hyp := Same_relative_group(G,L,possible_sz) and IsIsolated(G, possible_sz);
      c := c+1;
```

```
      od;
    od;
    if hyp = true then
      sz := possible_sz;
    fi;
  fi;
  return sz;
  end;

  ####################################################################
  # For all the exceptional adjoint groups and all their cuspidal
  # data (L,(su)Z^o(L), E) s isolated in L, find z in Z^o(L) such
  # that N_G(L)/L = N_{C^o_\G(sz)}(C^o_L(sz))/C^o_L(sz). Return sz.

  Find_sz_with_same_relative_group_all_cases := function()
  local cusp_data_F4,cusp_data_E6,cusp_data_E7,cusp_data_E8,groups_and_data,All_sz,group_datum,G,cusp,All_sz_G,L,s;
  # we write down the data in CHEVIE
  cusp_data_F4 := [[[2,3],[1/2,0,1/2,1/2]]];
  cusp_data_E6 := [[[2,3,4,5], [1/2,0,0,1/2,0,1/2]]];
  cusp_data_E7 := [[[2,3,4,5], [1/2,0,0,1/2,0,1/2,0]],[[1,2,3,4,5,6],[0,0,0,1/3,0,0,1/3]]];
  cusp_data_E8 := [[[2,3,4,5], [1/2,0,0,1/2,0,1/2,0,0]],[[1,2,3,4,5,6],[0,0,0,1/3,0,0,1/3,0]],[[1,2,3,4,5,6,7],[0,0,0,1/4,0,0,0,1/4]]];
  groups_and_data := [[CoxeterGroup("F",4),cusp_data_F4], [CoxeterGroup("E",6),cusp_data_E6], [CoxeterGroup("E",7),cusp_data_E7], [CoxeterGroup("E",8),cusp_data_E8]];

  # we check for each group G and each cuspidal datum with L proper and not a maximal torus
  All_sz := [];
  for group_datum in groups_and_data do
    G := group_datum[1];
    All_sz_G := [];
    for cusp in group_datum[2] do
      L := ReflectionSubgroup(G,cusp[1]);
      s := SemisimpleElement(G,cusp[2]);
      Add(All_sz_G, Find_sz_with_same_relative_group_and_isolated(G,L,s));
    od;
    Add(All_sz,All_sz_G);
  od;
  return All_sz;
  end;
```

C.2 ℓ-special classes and number of Brauer characters

This is the code to determine the ℓ-special classes and to prove Proposition 5.1.24. This code is a slight modification of the one in [Cha19, Appendix B].

```
######################################################################
# Computing the l-special classes
######################################################################

######################################################################
# Take a character E of the dual of W
# Return a character dualE of W via the isomorphism W^* to W.

Dualize:= function(W,E)
local dualE;
dualE := E;
if IsomorphismType(W) = IsomorphismType(CoxeterGroup("G",2)) then
  if E =3 then dualE :=4; elif E = 4 then dualE :=3;fi;
elif IsomorphismType(W) = IsomorphismType(CoxeterGroup("F",4)) then
if E =2 then dualE :=3; elif E = 3 then dualE :=2;fi;
  if E =5 then dualE :=7; elif E = 7 then dualE :=5;fi;
  if E =6 then dualE :=8; elif E = 8 then dualE :=6;fi;
  if E =11 then dualE :=12; elif E = 12 then dualE :=11;fi;
  if E =18 then dualE :=19; elif E = 19 then dualE :=18;fi;
  if E =21 then dualE :=23; elif E = 23 then dualE :=21;fi;
  if E =22 then dualE :=24; elif E = 24 then dualE :=22;fi;
fi;
return dualE;
end;

Dualize_list := function(W,listchar)
return List(listchar, E->Dualize(W,E));
end;

######################################################################
# Give the order of a semisimple element in G.

Order_semisimple := function(G,s)
local id,n,t;
id := List([1..G.rank], i-> 0);
n := 1;
t := s^n;
while t.v <> id do
  n := n+1;
  t := s^n;
od;
return n;
end;

######################################################################
# Determine if a semisimple element is an l-element.

Is_l_element := function(G,s,l)
```

```
local order;
order := Order_semisimple(G,s);
return ((order mod l =0) and IsPrimePower(order)) or order = 1;
end;


###################################################################
# Give the list of l-special unipotent classes of G.

lspec:=function(G, l)
local lspec,Gstar,lIsolated,s,Ws,specWs,jind,M,CharinWstar,CharinW;
lspec :=[];
Gstar := Dual(G);
lIsolated := Filtered(QuasiIsolatedRepresentatives(Gstar), s-> IsIsolated(Gstar,s) and Is_l_element(Gstar,s,l));
for s in lIsolated do
  Ws := Centralizer(Gstar,s).group;
  specWs := Filtered([1..Length(ChevieCharInfo(Ws).a)], i-> ChevieCharInfo(Ws).a[i]=ChevieCharInfo(Ws).b[i]);
  jind:=jInductionTable(Ws,Gstar);
  M:=Transposed(Transposed(jind.scalar){specWs});
  CharinWstar:=Filtered([1..Length(M)],i->Sum(M[i])>0);
  CharinW := Dualize_list(G,CharinWstar);
  Add(lspec, List(UnipotentClasses(G).springerSeries[1].locsys{CharinW},i -> i[1]));
od;
return Set(Flat(lspec));
end;

###################################################################
# Computing the l-canonical quotient
###################################################################

###################################################################
# Return the list of conjugacy classes of  l'-elements
# in the finite group A.

lprime_conj_classes:=function(A,l)
return Filtered(ConjugacyClasses(A), i -> Gcd(Order(A,Representative(i)),l)=1);
end;

###################################################################
# Count the number of l-modular representations of centralisers of
# l'-elements in the finite group A, i.e, |M^l(A)|.

lnumber:=function(A,l)
return Sum(ConjugacyClasses(A),i -> Length(lprime_conj_classes(Centralizer(A,Representative(i)),l)));
end;

###################################################################
```

```
# A group and R a list of representations parameterised
# as in CharTable.
# Return the intersections of kernels of the representations in R.

Intersection_kernels := function(A,R)
local C;
C := Intersection(List(Rep, r->KernelChar(r)));
return Subgroup(A,Flat(List(C, i->Elements(ConjugacyClasses(A)[i]))));
end;

#######################################################################
# Return the group \bar{A}_{1,C} for C a unipotent conjugacy
# class of G.

Canonical_quotient := function(G,C, l)
local locsys, Au, Pos, DecMat, a, i, j, Fil, PIMs, PIMs_as_characters,p,j,P;
locsys:= UnipotentClasses(G).springerSeries[1].locsys;
Au := C.Au;
# Give the position of all the Springer correspondents E_{C,phi} for each phi in irr(A_G(u_C)).
Pos := List([1..Length(CharTable(Au).irreducibles)], i -> PositionProperty(locsys, j -> UnipotentClasses(G).classes[j[1]] = C and j[2] = i));

# Create the list of all the a-values for the PIMs of A_G(u_C)
DecMat:=Transposed(DecompositionMatrix(Au,l));
a:=List([1..Length(DecMat)], i -> -1);
for i in [1..Length(DecMat)] do
  Fil:=Filtered([1 .. Length(Pos)], j -> Pos[j] <> false and DecMat[i][j]<>0);
  if Fil <> [] then
    a[i]:=Minimum(List(Pos{Fil}, j->ChevieCharInfo(G).a[j]));
  fi;
od;

# Find the PIMS with the a-value maximal
PIMs := DecMat{Filtered([1..Length(DecMat)], i -> a[i]=Maximum([0, Maximum(a)]))};
PIMs_as_characters := [];
for p in PIMs do
  P := 0*CharTable(Au).irreducibles[1];
  for j in [1..Length(p)] do
    P := P + p[j]*CharTable(Au).irreducibles[j];
  od;
  Add(PIMs_as_characters,P);
od;
if Size(Intersection_kernels(Au,PIMs_as_characters)) = 1 then return Au;
else return Au/Intersection_kernels(Au,PIMs_as_characters); fi;
end;
```

```
#################################################################
# Return the number unipotent lmodular representations of the simple
# adjoint exceptionnal group of type G.

Size_unipotent_block:= function(G,l)
local ls;
ls:=lspec(W,l);
return Sum(ls, i -> lnumber(Canonical_quotient(G,UnipotentClasses(G).classes[i],l),l));
end;
```

C.3 Mixed support of character sheaves

This is the code used in the proof of the unitriangularity to compute the restriction of a character sheaf of the principal series to a mixed conjugacy class, see Chapter 6. The second to last function allows us to prove the claim in Lemma 6.1.7. The last function was used in the last non-special but ℓ-special cases of E_8 in Subsection 6.1.7. Both cases use the formula for the restriction of character sheaves in the principal series, c.f. Corollaries 4.3.20 and 6.2.12.

```
#################################################################
# Return the fusion of the unipotent classes of H to G.

UnipotentFusion := function(G,H)
local Ucl_G, Ucl_H, Ucl_G_dynkin, fusion, uH, uG_dynkin;
Ucl_G := UnipotentClasses(G).classes;
Ucl_H := UnipotentClasses(H).classes;
Ucl_G_dynkin := List(Ucl_G, c -> c.dynkin);
fusion := List(Ucl_G, x -> []);
for uH in [1..Length(Ucl_H)] do
  uG_dynkin := InducedLinearForm(G,H,Ucl_H[uH].dynkin);
  Add(fusion[Position(Ucl_G_dynkin, uG_dynkin)], uH);
od;
return fusion;
end;

#################################################################
# Find the family of Uch(G) to which F belongs.

FindFamily := function(G,F)
return Filtered(UnipotentCharacters(G).families, f-> F in f.charNumbers)[1];
end;

#################################################################
# Find the special character of Irr(W) in a family f of Uch(G).

FindSpecInFamily := function(W,f)
local C, R, spec, i,r;
```

```
C := ChevieCharInfo(W);
R := f.charNumbers;
spec := Filtered([1..Length(C.a)],i->C.a[i]=C.b[i]);
return Filtered(R, r->r in spec)[1];
end;

######################################################################
# Give the unipotent support of a character V coming
# from a ReflectionSubgroup H of the dual group of G.

UnipSupportG := function(G,H,V)
local Vspec, V0;
Vspec := FindSpecInFamily(H,FindFamily(H,V));
V0 := Dualize(G, PositionProperty(Transposed(jInductionTable(H,Dual(G))).scalar)[Vspec], l-> l<>0));
return UnipotentClasses(G).springerSeries[1].locsys[V0][1];
end;

######################################################################
# Find all the Gs=C^\circ_G(s) such that s commutes with a
# conjugate huh^{-1} and whether the image of s in \bar{A}_(huh^{-1})
# is trivial or not. This function uses the properties of simple
# adjoint groups of exceptional type.

List_pseudoLevi_commute_with_u := function(G,u)
local Ls,list_Gs_u, list_Gs,image, Gs, list_u, Abar, u_in_Gs,uGs, Pos,p, Lpos, L,uLs,uLs_list, uGsdyn, Image_is_trivial;

# create the list of all connected centralisers of semisimple elements of which a unipotent clall fuses into (u)_G
list_Gs_u :=[];
list_Gs := List(SemisimpleCentralizerRepresentatives(G), h-> ReflectionSubgroup(G,h));

Abar := Canonical_quotient(G,UnipotentClasses(G).classes[u],11);
image := Size(Abar) =1;
for Gs in list_Gs do
# look if there is a unipotent class of Gs fusionning into (u)_G
list_u :=[];
for u_in_Gs in UnipotentFusion(G,Gs)[u] do
  # for each u, add u to the list and also check if u is distinguished in Gs
  Add(list_u,[u_in_Gs, Rank(AlgebraicCentre(Gs).Z0) = Rank(UnipotentClasses(Gs).classes[u_in_Gs].red),image]);
od;
if list_u <> [] then
  Add(list_Gs_u,[Gs,list_u]);
fi;
od;

# determine whether the image of s in \bar{A}_(huh^{-1}) is trivial or not.
```

```
if Size(Abar) <> 1 then
  # find the unique Levi L up to conjugation such that u is distinguished in L
  Lpos := PositionProperty(list_Gs_u, Gs -> ForAny(Gs[2], us->us[2]) = true and IsParabolic(Gs[1]));
  L := list_Gs_u[Lpos][1];

  # determine whether the image of s in A_G(huh^{-1}) is trivial or not
  for Gs in list_Gs_u do
    for u_in_Gs in Gs[2] do
      uGs := UnipotentClasses(Gs[1]).classes[u_in_Gs[1]];
      # use the fact that there is a homomorphism from A_{Gs}(u) to A_G(u)
      if GcdInt(Size(uGs.Au), Size(Abar)) =1 then
        u_in_Gs[3] := true;
      else
        uGsdyn := uGs.dynkin;
        Image_is_trivial := u_in_Gs[3];
        Pos := Filtered(ParabolicRepresentatives(Gs[1]), p->Length(p) =Length(L.callarg[1]));
        p := 0;
        while Image_is_trivial <> true and p < Length(Pos) do
          p := p+1;
          Ls := ReflectionSubgroup(Gs[1],Pos[p]);
          if IsParabolic(Ls) then
            uLs_list := UnipotentFusion(G,Ls)[u];
            if Length(uLs_list) = 1 then
              uLs := UnipotentClasses(Ls).classes[uLs_list[1]];
              if Rank(AlgebraicCentre(Ls).Z0) = Rank(uLs.red) then
                Image_is_trivial := InducedLinearForm(Gs[1],Ls,uLs.dynkin) = uGsdyn;
              fi;
            fi;
          fi;
        od;
        u_in_Gs[3] := Image_is_trivial;
      fi;
    od;
  od;
fi;
return list_Gs_u;
end;

##########################################################################
# For a unipotent character sheaf A_V in the principal series
# for V in irr(W), compute the restriction (s^*A_V)_{(uGs)_Gs}
# where (uGs)_G is the unipotent support of A_V for s
# commuting with uGs.

Unip_principal_series_restriction_at_Gs := function(G,V,Gs,uGs)
local u,restV,Gs,indTable,restV_at_Gs,V2;
```

```
indTable := InductionTable(Gs,G);
restV_at_Gs := [];
for V2 in PositionsProperty(UnipotentClasses(Gs).springerSeries[1].
 ↪ locsys, L ->L[1]=uGs) do
  Add(restV_at_Gs,[V2, indTable.scalar[V][V2]]);
od;
return restV_at_Gs;
end;

####################################################################
# For a unipotent character sheaf A_V in the principal series
# for V in irr(W), for  u is the unipotent support of A_V
# check that the restriction (s^*A_V)_{(u)_Gs} is trivial if s
# commutes with u and s has trivial image in the canonical
# quotient \bar{A}_u and that is zero otherwise.

Check_restriction_V_is_trivial_at_suGs := function(G,V,Gs)
local u, non_correct_rest, uGs,restV_at_Gs_uGs, check, restV,
 ↪ Image_springer;
non_correct_rest := [];
for uGs in Gs[2] do
  restV_at_Gs_uGs := Unip_principal_series_restriction_at_Gs(G,V,Gs
 ↪ [1],uGs[1]);
  if uGs[3] = false then
    check := (Sum(List(restV_at_Gs_uGs, V2->V2[2])) = 0);
  else
    if Sum(List(restV_at_Gs_uGs, V2->V2[2]))<>1   then
      check := false;
    else
      restV := Filtered(restV_at_Gs_uGs, V2 ->V2[2]=1)[1][1];
      Image_springer := UnipotentClasses(Gs[1]).springerSeries[1].
 ↪ locsys[restV];
      check := ChevieCharInfo(UnipotentClasses(Gs[1]).classes[
 ↪ Image_springer[1]].Au).positionId = Image_springer[2];
    fi;
  fi;
  if not check then Add(non_correct_rest, Gs); fi;
od;
return non_correct_rest;
end;

####################################################################
# Same function as before but for any s and
# any uGs such that (u)_G = (uGs)_G.

Check_restriction_V_is_trivial:= function(G,V,u)
local non_correct_rest, Gs,i;
non_correct_rest := [];
for Gs in List_pseudoLevi_commute_with_u(G,u) do
  if Check_restriction_V_is_trivial_at_suGs(G,V,Gs) <> [] then
    Add(non_correct_rest, Check_restriction_V_is_trivial_at_suGs(G,V,
 ↪ Gs));
```

```
    fi;
  od;
  return non_correct_rest;
  end;

  ####################################################################
  # For each family f of unipotent character sheaves, let u be the
  # unipotent support and let A_V be the character sheaf
  # for V in irr(W) the special character such that Spr(V) = (u,1),
  # check that the restriction (s^*A_V)_{(u)_Gs} is trivial for all s
  # commuting with u. Return the families and s and the restriction
  # which are not trivial.

  Char_sheaves_with_non_trivial_restriction := function(G)
  local u, not_trivial_restriction, f,V, Check;
  not_trivial_restriction := [];
  for f in UnipotentCharacters(G).families do
    V := FindSpecInFamily(G,f);
    u := UnipotentClasses(G).springerSeries[1].locsys[V][1];
    Check := Check_restriction_V_is_trivial(G,V,u);
    if Check <> [] then
      Add(not_trivial_restriction, Check);
    fi;
  od;
  return not_trivial_restriction;
  end;

  ####################################################################
  # Let A be a character sheaf in the principal series coming from the
  # induction datum m=(T_0,T_0, loc) where loc is a Kummer local
  # system on T_0. Let V be the character of Wm which corresponds to A.
  # Note that Wm = Wloc =Wt where t in G* corresponds to loc.
  # Compute the restriction of restriction (s^*A_V)_{(u)_Gs} where u is
  # the unipotent support of A. Return a list of triples where the
  # first element is the number of times the local system indexed by
  # the character appears.

  RestrictionMixedSupport := function(G, Wt,Wloc, Gs,V)
  local formula, u, uGs, D, lambda, Ws_lambda, Ws_loc_lambda,
   ↪ Ws_loc_lambda_Weyl, resV, sign,t,Ten,Tensign, res, uGs_lambda,
   ↪ Ucl_Ws_lambda, v, F;
  formula := [];
  u := UnipSupportG(G,Wt,V);
  uGs := UnipotentFusion(G,Gs)[u];
  if uGs = [] then
    Print("Error: The unipotent support of the character sheaf A
   ↪ indexed by", V, " doesn't commute with C_G(s).");
  else
    D := DoubleCosets(G,Gs,Wloc);
    for lambda in D do
    Ws_lambda := ConjugateSubgroup(Gs,lambda.representative);
    Ws_loc_lambda := Intersection(Wloc,Ws_lambda);
```

```
    Ws_loc_lambda_Weyl := ReflectionSubgroup(G,Intersection(Ws_lambda.
  ↪ rootInclusion, Wloc.rootInclusion));
    resV := InductionTable(Ws_loc_lambda,Wloc).scalar[V];

    # twist the restriction by the character chi_lambda of
  ↪ Ws_loc_lambda if we know how to do it.
    if Size(Ws_loc_lambda) = 2*Size(Ws_loc_lambda_Weyl) then
      # find the sign character of Ws_loc_lambda/Ws_loc_lambda_Weyl
      sign := Filtered(PositionsProperty(Transposed(InductionTable(
  ↪ Ws_loc_lambda_Weyl, Ws_loc_lambda).scalar)[ChevieCharInfo(
  ↪ Ws_loc_lambda_Weyl).positionId], x->x<>0), y-> y<>1)[1];
      t := CharTable(Ws_loc_lambda);
      Ten := Tensored(t.irreducibles, [t.irreducibles[sign]]);
      Tensign := List(Ten, l-> PositionProperty(t.irreducibles, z->z=l)
  ↪ );
      resV := resV{Tensign};
    fi;

      if Size(Ws_loc_lambda) > 2*Size(Ws_loc_lambda_Weyl) then
        Print("Error: we cannot easily compute chi^d.");
      else
      # Compute the restriction of the possible characters of Ws_lambda
        res := InductionTable(Ws_loc_lambda,Ws_lambda).scalar;
        uGs_lambda := UnipotentFusion(G,Ws_lambda)[u];
        Ucl_Ws_lambda := UnipotentClasses(Ws_lambda);
        for v in uGs_lambda do
          for F in PositionsProperty(Ucl_Ws_lambda.springerSeries[1].
  ↪ locsys, l-> l[1] = v) do
            Add(formula,[resV*res[F], Ucl_Ws_lambda.springerSeries[1].
  ↪ locsys[F], Ucl_Ws_lambda.classes[v].Au]);
          od;
        od;
      fi;
    od;
  fi;
  return formula;
  end;
```

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