Fixed Cardinality Combinatorial Optimization Problems – A Survey

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Abstract

In this paper, we consider combinatorial optimization problems with cardinality constraints. In k-cardinality combinatorial optimization problems, a cardinality constraint requires feasible solutions to contain exactly k elements of a finite set E. We formally define the problem and give several examples. For the case of the minimax – or bottleneck – objective function we give a necessary and sufficient condition, when the k-cardinality combinatorial optimization problem can be reduced to a feasibility problem. We also present some new complexity results. The main part of the paper is a survey of the existing literature on the topic.

Contents

1 Introduction

A combinatorial optimization problem is given by a finite set E with cardinality $|E| = m$, the set of **feasible solutions** $\mathcal{X}$, i.e. a family $\mathcal{X} \subseteq 2^E$ of subsets of E, and an **objective function** $f : \mathcal{X} \to \mathbb{N}$ assigning to each feasible solution $x \in \mathcal{X}$ its non-negative integer **objective value** $f(x)$. The mathematical program

$$\min_{x \in \mathcal{X}} f(x)$$

is the general form of a **combinatorial optimization problem (COP)**.

If the context is clear, we often identify x with its incidence vector $x \in \mathbb{R}^m$ defined by

$$x(e) \quad = \quad \begin{cases} 1 & \text{if } e \in E, \\ 0 & \text{otherwise} \end{cases}$$

In this paper additional **cardinality constraints** are considered. If k is any natural number with $1 \leq k \leq m$ we denote with

$$\mathcal{X}_k := \{x \in \mathcal{X} : |x| = k\}$$

the family of feasible solutions with cardinality k. Then, the k-cardinality combinatorial optimization problem is given as

$$\min_{x \in \mathcal{X}_k} f(x). \qquad (k\text{CardCOP})$$

We focus on three types of problems. In the **existence problem** we want to decide whether the feasible set $\mathcal{X}_k$ is non-empty. In the **sum problem** and the **max** or **bottleneck problem** we want to find minimizers of kCardCOP, where the objective function is defined using a weight function $w : E \to \mathbb{N}$ as

$$f(x) = \sum_{e \in x} w(e) = \sum_{e \in E} w(e)x(e)$$

or

$$f(x) = \max_{e \in x} w(e) = \max_{e \in E} w(e)x(e).$$

In Section 2 we will discuss examples of kCardCOP, their interrelation and some easily solvable cases of kCardCOP. In Section 3 it is shown that bottleneck problems are as easy or difficult as existence problems. Section 4 contains a survey of existing literature on k-cardinality combinatorial optimization problems, and some new results. The survey

shows in particular that kCardCOP is $\mathcal{NP}$-hard for a large class of problems although their unconstrained versions are polynomially solvable.

2 Examples of Cardinality Constrained COP

As Table 1 shows, there is a large class of problems – some well known, some not – which can be formulated as kCardCOP. A detailed discussion on relevant literature is given in Section 4.

E	$\mathcal{X}$	k	Problem
set of locations	2^E	p	p-facility location
edge set of $G=(V,E)$	simple cycles	$\|V\|$	TSP
edge set of $G=(V,E)$	matchings	$\lfloor\frac{\|V\|}{2}\rfloor$	min weight matching
edge set of $G=(V,E)$	cuts	k	kCard cut
edge set of $G=(V,E)$	acyclic subgraphs	k	kCard acyclic subgraph
edge set of $G=(V,E)$	connected subgraphs	k	kCard connected subgraph
edge set of $G=(V,E)$	trees	k	kCard tree
element set of matroid M	bases	rank of M	min weight matroid base
element set of matroid M	independent sets	k	kCard independent set
element set of 2 matroids	matroid intersections	k	kCard matroid intersection
assets	2^E	k	k-portfolio problem

Table 1: Examples of k-Cardinality Combinatorial Optimization Problems

As far as the complexity of kCardCOP is concerned, we have to make a distinction between problems with bottleneck objective (addressed in Section 3) and those with sum objective. Few examples of polynomially solvable kCardCOP with sum objective are known. One such class are problems where the COP without cardinality constraints is solved by a primal-dual algorithm (or augmenting structure algorithm). Corresponding kCardCOP include the cardinality constrained versions of the following problems:

- assignment
- non-bipartite matching
- acyclic subgraph
- matroid base

- independent set
- matroid intersection.

Conversely, every algorithm to solve kCardCOP can obviously be used to solve COP without cardinality constraints by

$$\min_{x \in \mathcal{X}} f(x) = \min_{k=1,..,|E|} \{f(x) : x \in \mathcal{X}_k\}.$$

Thus, k-cardinality COPs are always at least as hard as their counterparts without fixed cardinality. However, we have to note that any k-cardinality combinatorial optimization problem can be solved in polynomial time, if k is fixed, i.e. not part of the input. A simple brute force enumeration of the at most $O(n^k)$ feasible sets with cardinality k will work. In this sense, kCardCOP is a special case of fixed parameter optimization as discussed in [Downey and Fellows, 1995a] and [Downey and Fellows, 1995b]. In the latter theory problem parameters are fixed and the reulting impact on tractability and complexity is studied.

All $\mathcal{NP}$-completeness results mentioned in this paper assume k to be part of the input.

Another class of polynomially solvable kCardCOP consists of bottleneck problems, for which the existence problem can be solved efficiently. Such problems are considered in the next section.

3 Interrelation Between Bottleneck and Existence Problems

A well-known method to solve (unconstrained) bottleneck COP is the threshold method, introduced in [Garfinkel, 1971]. This method can also be applied to solve kCardCOP as the following result shows.

Theorem 1 *Given an instance of kCardCOP with finite set E and weight function $w : E \to \mathbb{N}$. Then the bottleneck problem is solvable in polynomial time if and only if the existence problem $Ex(E')$*

$$\{x \in \mathcal{X} : |x| = k \text{ and } x \subseteq E'\} \neq \emptyset$$

is polynomially solvable for any subset $E' \subseteq E$.

Proof:

If bottleneck kCardCOP is polynomially solvable we replace, for any given $E' \subseteq E$, the original weight function by

$$w'(e) \quad = \quad \begin{cases} 0 & \text{if } e \in E' \\ 1 & \text{otherwise.} \end{cases}$$

The answer to the existence problem with respect to E' is YES if and only if the optimal objective value of the bottleneck problem is 0.

If on the other hand the $Ex(E')$ existence kCardCOP can be solved in polynomial time, the following algorithm will solve the bottleneck kCardCOP.

Threshold Algorithm

- Sort the set $\{w(e) : e \in E\}$ such that $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_m)$
- For $i = 1, \ldots, m$ do
  - If x^* solves the existence problem $Ex(E')$ with respect to $E' := \{e \in E : w(e) \leq w(e_i)\}$, output x^* as a solution of the bottleneck kCardCOP with optimal objective value $w(e_i)$, STOP. Otherwise iterate.

□

Obviously, an efficient implementation of the threshold algorithm would use binary search. This is not included in the preceding presentation in order to focus on the essential idea of the algorithm.

An example of a bottleneck kCardCOP which can be solved in polynomial time using Theorem 1 is the kCard tree problem. The existence problem is the check for a subtree in $G' = (V, E')$ with cardinality k. This can obviously be done in polynomial time.

4 Survey of *k*-Cardinality Optimization Problems

In this section we present a review of the existing literature on fixed cardinality optimization problems. We include only references, where an explicit or implicit (fixed) cardinality constraint on the feasible solutions is given. We could not incorporate the huge amount of literature, that deals with closely related problems, such as additional linear constraints (e.g. the shortest path problem with a knapsack type constraint), or other types of fixed cardinalities (e.g. solutions containing a fixed number of elements of a certain set). We also left out COP where the cardinality is bounded by some $k \in I\!N$ ($\leq k$CardCOP). The description below is organized chronologically according to the evolution of the results in research reports. We cite, however, the correponding articles as published in journals for ease of reference.

4.1 The k-Cardinality Tree Problem

The best understood problem with a fixed cardinality constraint is the k-cardinality tree problem. All the relevant aspects have been subject of research, such as complexity, approximation algorithms, integer programming approaches, and heuristics. This fact may be due to the wide variety of applications. We found references to such diverse fields as oil-field leasing [Hamacher and Joernsten, 1993], facility layout [Foulds and Hamacher, 1992] and [Foulds et al., 1998], open pit mining [Philpott and Wormald, 1997], quorum-cast routing [Cheung and Kumar, 1994] and telecommunications [Garg and Hochbaum, 1997]. But it also has been applied within combinatorial optimization itself, e.g. as a subproblem in matrix decomposition, see [Borndörfer et al., 1998, Borndörfer et al., 1997].

Given a graph $G = (V, E)$ with a weight function on the edges or the nodes, the objective is to find a subtree of G containing exactly k edges (or, equivalently, $k + 1$ nodes) such that the sum of the weights is minimal. A majority of the existing articles have considered the edge-weighted case. We will therefore assume this case below, and note specifically if the node-weighted case has been considered.

Several authors have proved independently that the k-cardinality tree problem is $\mathcal{NP}$-hard: See [Hamacher et al., 1991], [Zelikovsky and Lozevanu, 1993], [Fischetti et al., 1994], [Marathe et al., 1996]. In the latter paper it has been shown that the problem is still $\mathcal{NP}$-hard if $c(e) \in \{1, 2, 3\}$ for all edges e and $G = K_n$, but polynomially solvable if there are only two distinct weights. For the node-weighted case, $\mathcal{NP}$-completeness has been shown independently in [Faigle and Kern, 1994] and [Ehrgott, 1992].

Several authors have considered special types of graphs. Note first that the node-weighted problem is trivially solved when $G = K_n$. The problem is polynomially solvable if G is a tree, see [Faigle and Kern, 1994] for the node-weighted case and [Maffioli, 1991] for the edge weighted case. It should also be noted that when G is a tree, both cases are equivalent (see [Ehrgott et al., 1997]). Polynomial time algorithms for the node weighted problem exist for interval graphs and co-graphs, [Woeginger, 1992]. Recently another polynomially solvable special case has been discovered in [Blum, 1998] and [Blum and Ehrgott, 1999]. Using the node weights, the authors defined hurdles and troughs and showed that k-Card tree is polynomially solvable, if G contains a single trough, or no hurdles. On the other hand $\mathcal{NP}$-completeness results for the node-weighted case have been obtained for grid and split graphs by [Woeginger, 1992].

The edge-weighted problem is $\mathcal{NP}$-complete for planar graphs and for Euclidean graphs, i.e. complete graphs, where the nodes are points in the plane and edge weights correspond to Euclidean distances (see [Marathe et al., 1996]). In the same paper polynomial algorithms for decomposable graphs and graphs with bounded tree-width have been given. The same holds for the problem in the plane, when all points lie on the boundary of a convex region. In [Dudas et al., 1998], the authors have focussed on graded distance matrices. They have assumed that $G = K_n$ and have proved that k-Card tree is $\mathcal{NP}$-complete on matrices graded up its rows or columns, whereas it is solvable in polynomial time if the matrix is

graded down its rows (columns) or both graded up its rows and columns.

Concerning methodology, both exact and heuristic algorithms have been developed, with a general focus on approximation algorithms. We first note that integer programming formulations have been presented in [Fischetti et al., 1994] and later in [Garg, 1996]. Based on detailed studies of the associated polyhedron in the former paper a Branch and Cut algorithm has been developed and implemented in [Freitag, 1993]. The code and also implementations of most of the heuristics in [Ehrgott et al., 1997] are documented in [Ehrgott and Freitag, 1996], and are available as public domain software at http://www.mathematik.uni-kl.de/ wwwwi/WWWWI/ORSEP/contents.html. A Branch and Bound method is described in [Cheung and Kumar, 1994].

The heuristics mentioned are based on greedy and dual greedy strategies and also make use of dynamic programming approaches. More recently, authors have successfully applied local search methods to the k-cardinality tree problem: [Joernsten and Lokketangen, 1997] have applied a tabu search strategy, [Catanas, 1997] has presented both a genetic algorithm and a tabu search method. Other constructive heuristics have been presented in [Cheung and Kumar, 1994]. A comparison of generic local search, genetic algorithms, and tabu search for the node-weighted problem has been undertaken in [Blum, 1998]. The results reported are often better than those obtained by constructive heuristics of [Ehrgott et al., 1997].

A large body of literature is available on approximation algorithms for the problem. The papers published on this topic represent an ongoing improvement until finally a constant approximation factor could be obtained. A first result appeared in [Woeginger, 1992]. He has proved that there is an $O(\sqrt{k})$-approximation algorithm for the node-weighted problem on grid graphs. Later there were two streams of research articles: one focusing on the problem on general graphs G, the other dealing with Euclidean graphs.

In the former, [Marathe et al., 1996] could obtain a $2\sqrt{k}$-approximation, which has been improved in [Awerbuch et al., 1995]([Awerbuch et al., 1998]) to $O(\log^2 k)$. A first constant factor approximation with factor 17 has been derived in [Blum et al., 1996]. The currently best approximation guarantee is 3, proved in [Garg, 1996].

For the problem in the plane, [Marathe et al., 1996] has given an $O(k^{\frac{1}{4}})$-approximation. Improved algorithms, with ratio of $O(\log k)$ are due to [Garg and Hochbaum, 1997], [Rajagopalan and Vazirani, 1995], and [Mata and Mitchell, 1995]. Further decrease to $O(\frac{\log k}{\log \log n})$ [Eppstein, 1997] and a first constant factor approximation ([Blum et al., 1995], with the constant not specified) followed. Using guillotine subdivisions [Mitchell, 1996] and [Mitchell et al., 1998] could develop a $2\sqrt{2}$-approximation for the l_2 metric, and a 2-approximation for the l_1 metric. Finally, a polynomial time approximation scheme has been given in [Arora, 1996]. The same author recently proposed nearly linear time approximation schemes that also work in higher dimensions in an abstract (see [Arora, 1998]).

The following two sections are devoted to two problems which are closely related to the k-cardinality tree problem, namely the k-cardinality subgraph and travelling salesman

problems.

4.2 The k-Cardinality TSP and Related Routing Problems

In this section we assume, as usual in TSP and routing, that the problems are defined on complete graphs. Because a k-cardinality tree can be used to construct a circle containing exactly the k nodes of the subtree analogous to the problem without cardinality constraints (see e.g. [Lawler et al., 1985]), many of the references cited in Section 4.1 have treated both k-Cardinality Tree as well as k-Cardinality TSP.

Many authors have observed that the problem is obviously $\mathcal{NP}$-hard, because, for $k = n$ it is the TSP. We again distinguish between the planar and the general case.

For the problem where nodes are points in the plane, heuristics known from the TSP (r-opt and savings heuristic) have been used for the k-cardinality TSP in a paper by [Hamacher and Moll, 1996]. The paper also contains a geometric method based on clustering and a Branch and Bound algorithm. The details are in the first author's diploma thesis [Hamacher, 1993]. Much of the research has been focussed on approximation algorithms. [Mata and Mitchell, 1995] have presented a constant factor approximation and [Arora, 1996] has given a polynomial time approximation scheme.

Concerning the problem on graphs, we mention [Garg, 1996], with a 3-approximation. This has been used to obtain a 10.77-approximation for the minimum latency problem. The ideas of the $O(\log^2 k)$-approximation algorithm for k-Card Tree in [Awerbuch et al., 1995] have been used to obtain similar results for related problems: $O(\log^2 \min(R, n))$ for the quota-driven TSP and the price-collecting TSP, where R is the required node weight to be visited. A similar bound for the orienteering problem has also been given.

4.3 The k-Cardinality Subgraph Problem

This problem is perhaps even closer to the kCard tree problem. The only change is, that the requirement of acyclicity is dropped. Thus, the objective is to find a connected subgraph of G containing exactly either k edges or k nodes and having a minimal (maximal) total edge or node weight.

That all four variants of the (minimization) problem are $\mathcal{NP}$-hard has been proved in a diploma thesis by [Ehrgott, 1992]. An integer programming formulation, discussion of the polyhedral structure and a Branch and Cut algorithm for the problem can be found in [Ehrgott, 1994]. The codes are available together with the above mentioned codes for kCard tree, we refer to [Ehrgott and Freitag, 1996] again.

Most authors have considered k-cardinality subgraph in the maximization version. When the cardinality constraint is put on the nodes, the subgraph is usually assumed to be induced by the set of selected nodes. A greedy algorithm with detailed analysis (including proofs of tight lower and upper bounds) of its worst case performance ratio has been pro-

posed in [Asahiro et al., 1996]. [Ravi et al., 1991] and [Ravi et al., 1994] have considered the case $G = K_n$ and two types of objectives, namely maximizing the minimal weight (bottleneck objective) and maximizing the sum of the weights between the selected nodes. For the bottleneck objective they have shown that if the weights do not satisfy the triangle inequality, there is no polynomial time fixed ratio approximation algorithm, unless $I\!P = I\!NP$. If they do, a 2-approximation algorithm has been presented, as well as a proof of $I\!NP$-hardness of obtaining a better performance ratio. For the sum problem, a 4-approximation has been given, provided the triangle inequality is satisfied. A similar analysis has been carried out in [Krumke et al., 1997] for a minimization of the largest weight (bottleneck), the sum, and the variance of the weights. $I\!NP$-completeness in general, non-existence of fixed ratio approximation algorithms in the absence of the triangle inequality and $I\!NP$-completeness of approximation with a ratio less than 2 in its presence have been proved. Moreover, for trees, a polynomial time algorithm has been developed.

[Kortsarz and Peleg, 1993] have observed that the problem is $I\!NP$-complete, even if $w(e) = 1$ for all edges e. They have proved an $O(n^{0.3885})$-approximation for the problem, which they have called the densest k-vertex subgraph problem. $I\!NP$-completeness of this problem when the maximum degree is 3 has been proved in [Feige and Seltser, 1997]. They have also given an algorithm which finds a subgraph with at least $\binom{(1-\varepsilon)k}{2}$ edges in $n^{O((1+\log\frac{n}{k})/\varepsilon)}$ time.

A problem with cardinality constraints on both node and edge set has been considered in [Asahiro and Iwama, 1995]. The authors have given bounds on performance ratios for the feasibility problem.

A more general setting has been discussed in [Nehme and Yu, 1997], where a subset of k nodes of a hypergraph is to be found, maximizing the weight of the induced subhypergraph and subject to additional precedence constraints. For several cases polynomial time algorithms, respectively $I\!NP$-completeness results have been given.

In [Goldschmidt and Hochbaum, 1997] the cardinality constraint has been put on the edges and the weights on the nodes. $I\!NP$-completeness has been proved even for $w(v) = 1$ for all nodes v or for node degrees not bigger than 3. For the problem of finding a maximal weight subgraph with k edges an $O(kn)$-time 3-approximation algorithm has been given as well as an $O(n+m)$-time 2-approximation for the unweighted case. Again, for the case of G being a tree the problem can be solved in polynomial time. Note that this fact also follows from results mentioned in Section 4.1.

4.4 The k-Cardinality Cut Problem

Another interesting problem in graphs, is to find a cut of given cardinality. As far as we could determine, this problem has never been treated in the literature.

Let $G = (V, E)$ be a graph and let $w : E \to I\!N$ be a positive integral weight function on the edge set. The k cardinality cut problem (kCard cut) is to find a cut, i.e. a partition of

the node set $V = V_1 \cup V_2$ such that $C := \{[v_i, v_j] : v_i \in V_1, v_j \in V_2\}$ has cardinality k and minimal weight $\sum_{e \in C} w(e)$. The k cardinality s-t-cut problem (kCard s-t-cut) is defined analogously, but with the additional specification of two nodes s and t such that $s \in V_1$ and $t \in V_2$. Theorem 2 below shows that both the existence and the sum version of these kCard COP are *INP*-complete.

Theorem 2 *The following problems are INP-complete.*

1. *kCard cut, even if $w(e) = 1$ for all $e \in E$.*
2. *kCard s-t-cut, even if $w(e) = 1$ for all $e \in E$.*

.

Proof:

1. Reduction from MAX-CUT. Solving kCard cut for $k = 1, \ldots, n$ solves MAX-CUT, which is *INP*-complete, see [Garey and Johnson, 1979, p. 210]. kCard cut is certainly contained in *INP*.
2. Reduction from kCard cut. Solving kCard s-t-cut for all pairs of nodes s, t provides a solution for kCard cut. Or: Let $G = (V, E)$ be a graph, define $G' = (V \cup \{s, t\}, E \cup \{[s, v_1], [v_2, t]\})$, where v_1 and v_2 are arbitrary, different nodes in G. A kCard s-t-cut in G' is a kCard cut in G and vice versa.

□

Despite this result, there are some special cases for which the problem can be solved in polynomial time. One is the unweighted problem, when G is a complete graph. Then there exists a cut containing k edges if and only if $k = i(n - i)$ for some $i \in \left\{1, \ldots, \lfloor \frac{n}{2} \rfloor\right\}$. With Theorem 1 this implies that the bottleneck kCard cut problem can be solved polynomially, if G is complete. Another case is G being a tree. Since every edge is a cut edge one might simply select the edges with the k smallest weights and define a cut appropriately, to solve even the sum kCard cut problem.

4.5 Other Fixed Cardinality Problems on Graphs

We could only find a few references to other graph theoretical optimization problems. Interestingly, to the authors' knowledge, the problem of finding a shortest path with a fixed number of nodes has not been treated, despite a general interest in the shortest path problem with an additional linear constraint, documented by quite many papers on this topic.

For other problems, we mention [Minoux, 1976], with the historically first appearance of a fixed cardinality combinatorial optimization problem, namely the weighted matching problem. A necessary and sufficient condition for a k-cardinality matching to be optimal has been given.

The problem of finding a minimal weight k-cardinality clique in a complete graph has been considered in [Späth, 1985]. Simple exchange heuristics have been presented and numerical results reported.

Finally, in [Halldórsson et al., 1995] the following rather general problem has been discussed. Find a subset of k nodes such that the weight of the minimal spanning tree (minimal Hamiltonian cycle, Steiner tree) on this set is maximized. For complete graphs these problems are $\mathcal{NP}$-hard. The spanning tree and TSP version cannot be approximated within a factor of $n^{\frac{1}{6}}$ unless $\mathcal{P} = \mathcal{NP}$. A $k-1$-approximation algorithm is given for both. On metric graphs there are a 4-approximation for tree, a 3-approximation for TSP and Steiner tree, which are tight. Approximation is $\mathcal{NP}$-hard for a factor less than 2 (tree and TSP), respectively $\frac{4}{3}$ (Steiner tree). In the plane, approximation factors 2.25 (tree), 2.16 (Steiner tree) have been obtained.

4.6 Location Problems

A classical field in which cardinality constraints are imposed is location theory: Choose among a given set of points (nodes of a graph) a subset of a given cardinality, such that some measure related to the distance of the given points is maximized or minimized. In fact, several of the papers listed in Sections 4.3 and 4.5 also belong to this category.

In [Späth, 1984] 4 heuristics have been compared empirically for the problem of selecting k out of n points, such that their total distance to their optimal location is minimized. In [Aggarwal et al., 1991] the selection of such points under some objectives related only to the distance of the selected points has been investigated. For several types of such objectives polynomial time algorithms have been presented. For the same problem, but with the minimal distance between any two of the k points to be maximized it has been shown that a heuristic achieves an approximation factor of 3 in general and of 2 for certain k. A similar problem is dicussed in [White, 1991].

In our context of combinatorial optimization we have to mention the k-median and k-center problem on networks, and the k-facility location problem as a discrete location problem. Because excellent surveys on both the network and discrete multifacility location problem exist, we refer the interested reader to these, see [Labbé et al., 1995, Sections 3.5, 4.2.3], [Labbé, 1998, pp. 232-234], and [Labbé, 1997, Section 3] and references therein.

[Yamaguchi et al., 1998] and [Hamacher and Schöbel, 1998] consider the problem to locate a cycle with given cardinality k in a network such that the distance between cycle and nodes is minimized. For general graphs the problem is $\mathcal{NP}$-hard, while it can be solved in polynomial time if G is a grid graph.

4.7 Other Combinatorial Optimization Problems with Cardinality Constraints

[Babel et al., 1998] consider the k-partitioning problem. A set of $n = km$ items, each having a nonnegative weight is to be partitioned into m subsets of exactly k items, such that the largest weight of all subsets is minimal. The problem is $\mathcal{NP}$-complete. Approximation algorithms, the best having a performance bound of $\frac{4}{3}$, have been reported.

For the $\mathcal{NP}$-complete k-cardinality bin-packing problem, an $O(n \log n)$-time $\frac{3}{2}$ worst case ratio algorithm has been presented in [Kellerer and Pferschy, 1997]. The online version of the same problem has been investigated in [Babel and Kellerer, 1998]. This paper contains an approximation algorithm, the performance ratio of which tends to 2, as k increases.

The authors of the former paper have also been involved in a study of the k-cardinality knapsack problem. [Caprara et al., 1998] have reported a linear storage, polynomial time approximation scheme and a dynamic programming based fully polynomial time approximation scheme for this problem.

We also found one reference for a fixed cardinality problem outside the general combinatorial optimization problems: [Beasley et al., 1998] has listed heuristics based on genetic algorithms, tabu search and simulated annealing for a portfolio optimization problem, where the number of assets to be contained in the portfolio is specified. Numerical tests have been provided, too.

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